LECTURE-4(B)

COURSE TITLE

LINEAR ALGEBRA

&

GEOMETRY

(MT-272)

DEPARTMENT OF TELECOMMUNICATION ENGINEERING

COURSE TEACHER

DR. FAREED AHMAD

SPANNING SETS

- Spanning Set:

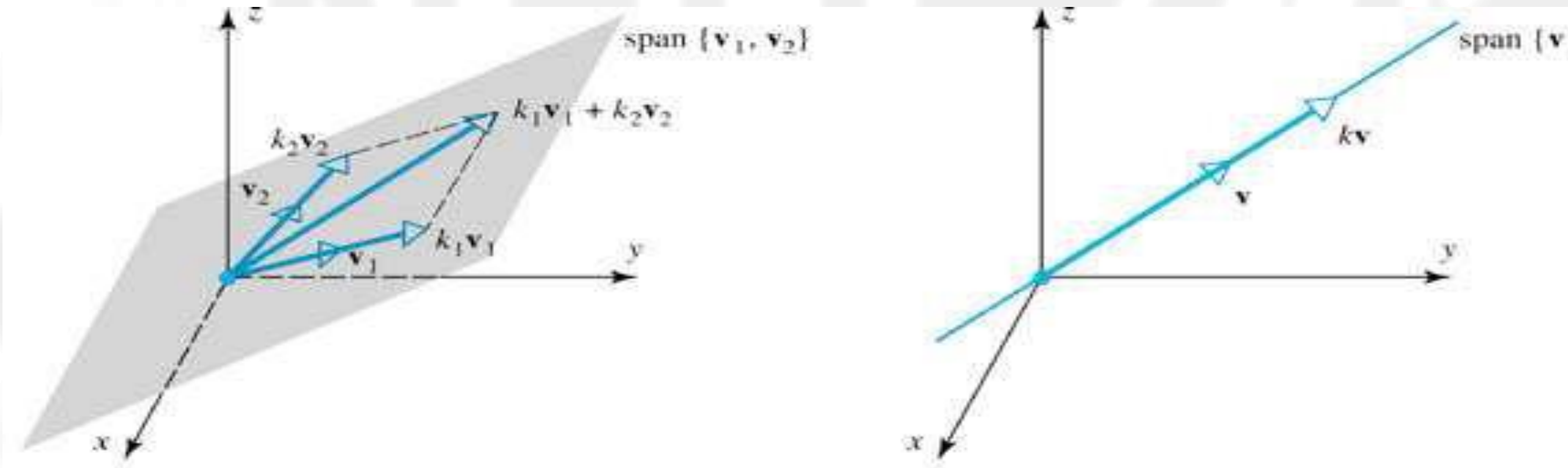
  If S={$\mathbf{v}_1,\mathbf{v}_2,\ldots\ldots, \mathbf{v}_k$} is a set of vectors in a vector space W of V consisting of all linear combinations of the vectors in S is called space spanned by $\mathbf{v}_1, \mathbf{v}_2,\ldots\ldots,\mathbf{v}_k$ and we say that the vectors $\mathbf{v}_1, \mathbf{v}_2,\ldots\ldots\ldots,\mathbf{v}_k$ span W. It is denoted by

$$W = \text{Span}(S) \text{ or } W = \text{Span}\{\mathbf{v}_1,\mathbf{v}_2,\ldots\ldots\ldots,\mathbf{v}_k\}$$

SPANNING SETS …Con'd

- The span of a set: span(S)

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the span of S is the set of all linear combinations of the vectors in S.



(*a*) Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(*b*) Span $\{\mathbf{v}\}$ is the line through the origin determined by $\mathbf{v}$.

SPANNING SETS …Con'd

- Alternative definition of a spanning set of a vector space:

  If every vector in a given vector space V can be written as a linear combination of vectors in a set S, then S is called a **spanning set** of the vector space V.

- Note: The above statement can be expressed as follows

$$
\begin{aligned}
&\operatorname{span}(S) = V \\
\Leftrightarrow\ & S \text{ spans (generates) } V \\
\Leftrightarrow\ & V \text{ is spanned (generated) by } S \\
\Leftrightarrow\ & S \text{ is a spanning set of } V
\end{aligned}
$$

EXAMPLE-1

Ex 1:

(a) The set $S=\{(1,0,0),(0,1,0),(0,0,1)\}$ spans R^3 because any vector $\mathbf{u}=(u_1,u_2,u_3)$ in R^3 can be written as

$$\mathbf{u}=u_1(1,0,0)+u_2(0,1,0)+u_3(0,0,1)$$

(b) The set $S=\{1,x,x^2\}$ spans P_2 because any polynomial function $p(x)=a+bx+cx^2$ in P_2 can be written as

$$p(x)=a(1)+b(x)+c(x^2)$$

EXAMPLE-2

Ex 2: A spanning set for R^3

Show that the set $S=\{(1,2,3),(0,1,2),(-2,0,1)\}$ spans R^3

Sol:

We must examine whether any vector $\mathbf{u}=(u_1,u_2,u_3)$ in R^3 can be expressed as a linear combination of $\mathbf{v}_1=(1,2,3)$, $\mathbf{v}_2=(0,1,2)$, and $\mathbf{v}_3=(-2,0,1)$

If $\mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 \Rightarrow (u_1,u_2,u_3)=c_1(1,2,3)+c_2(0,1,2)+c_3(-2,0,1)$

$\Rightarrow (u_1,u_2,u_3)=(c_1-2c_3, 2c_1+c_2, 3c_1+2c_2+c_3)$

$$\Rightarrow \begin{array}{l} c_1 \qquad -2c_3=u_1 \\ 2c_1+c_2 \qquad =u_2 \\ 3c_1+2c_2+c_3=u_3 \end{array}$$

EXAMPLE-2 …Con'd

Note: (1) If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution ($\mathbf{x} = A^{-1}\mathbf{b}$) given any $\mathbf{b}$

(2) From , a square matrix A is invertible (nonsingular) if and only if det $(A) \neq 0$

The above problem thus reduces to determine whether this system is consistent for all values of u_1, u_2, and u_3

$$\because |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

$\therefore A\mathbf{x} = \mathbf{u}$ has exactly one solution for every $\mathbf{u}$

$$\Rightarrow \text{span}(S) = R^3$$

EXAMPLE-3

Ex 3: Determine whether the given vectors spans R^3

$\mathbf{v}_1=(3,1,4)$, $\mathbf{v}_2=(2,-3,5)$, $\mathbf{v}_3=(5,-2,9)$, $\mathbf{v}_4=(1,4,-1)$

Sol:

We must examine whether any vector $\mathbf{u}=(u_1,u_2,u_3)$ in R^3 can be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$

If $u=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3+c_4\mathbf{v}_4 \Rightarrow$

$$\begin{aligned} 3c_1+2c_2+5c_3+c_4&=u_1\\ c_1-3c_2-2c_3+4c_4&=u_2\\ 4c_1+5c_2+9c_3-c_4&=u_3 \end{aligned}$$

The Augmented Matrix is

$$\begin{bmatrix} 3 & 2 & 5 & 1 & u_1 \\ 1 & -3 & -2 & 4 & u_2 \\ 4 & 5 & 9 & -1 & u_3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3 & -2 & 4 & u_2 \\ 3 & 2 & 5 & 1 & u_1 \\ 4 & 5 & 9 & -1 & u_3 \end{bmatrix}$$

EXAMPLE-3 …Con'd

$$\xrightarrow{By\ row\ operations} \begin{bmatrix} 1 & -3 & -2 & 4 & u_2 \\ 0 & 1 & 1 & -1 & \dfrac{u_1 - 3u_2}{11} \\ 0 & 0 & 0 & 0 & \dfrac{u_3 - 4u_2}{17} - \dfrac{u_1 - 3u_2}{11} \end{bmatrix}$$

Thus the system is inconsistent unless the last entry in the last row of the above matrix is zero. Since this is not the case for all values of u_1, u_2, and u_3, the given vectors do not span R^3.