LECTURE-4(C)

COURSE TITLE

LINEAR ALGEBRA

&

GEOMETRY

(MT-272)

DEPARTMENT OF TELECOMMUNICATION ENGINEERING

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LINEAR INDEPENDENCE AND LINEAR DEPENDENCE

- Definitions :

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ is a nonempty set of vectors, then the vector equation

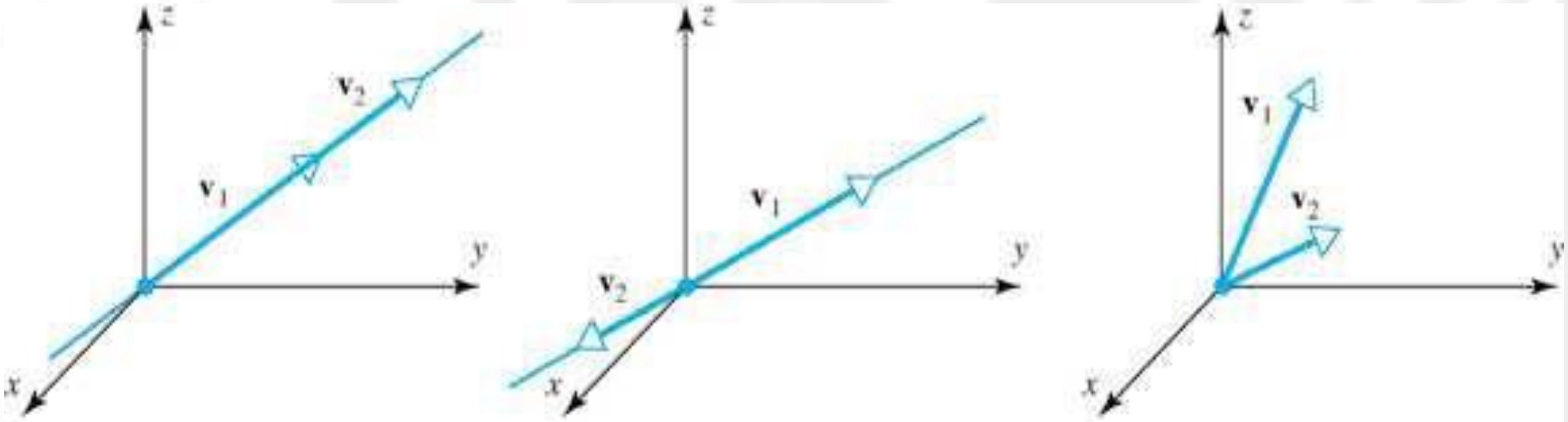
$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots.+c_k\mathbf{v}_k=\mathbf{0}$$

has at least one solution, namely $c_1=0$, $c_2=0,\ldots,c_k=0$.

If this is the only solution, then S is called a linearly independent set. If there are other solutions, then S is called a linearly dependent set.

GEOMETRIC INTERPRETATION OF LINEAR INDEPENDENCE

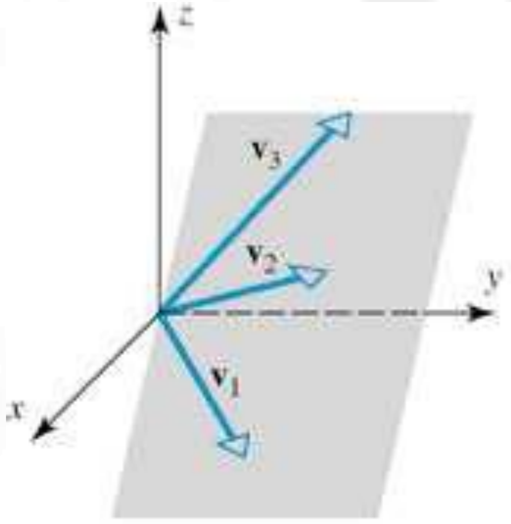
In R^2 or R^3, a set of two vectors is linearly independent if and only if the vectors do not lie on the same line when they are placed with their initial points at the origin.



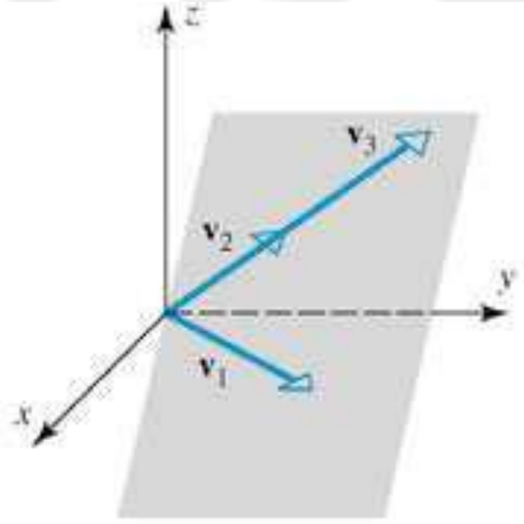
(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

GEOMETRIC INTERPRETATION OF LINEAR INDEPENDENCE

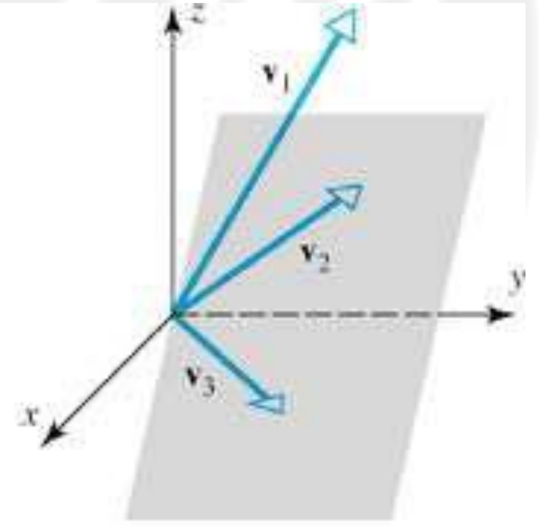
In R^3, a set of three vectors is linearly independent if and only if the vectors do not lie in the same plane when they are placed with their initial points at the origin.



(a) Linearly dependent



(b) Linearly dependent



(c) Linearly independent

EXAMPLE-1

Ex 1: Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

Sol: For L.I. or L.D. the vector eq. is

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Rightarrow \begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + c_2 + \qquad\quad &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

(or $\det(A) = -1 \neq 0$, so there is only the trivial solution)

$\Rightarrow S$ is (or $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are) linearly independent

EXAMPLE-2

Ex 2: Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\{1+x-2x^2, 2+5x-x^2, x+x^2\}$$

Sol: For L.I. or L.D. the vector eq. is $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$

i.e., $c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2)=0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1+2c_2\quad\; &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{G.-J.E.} \left[\begin{array}{cccc} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\therefore c_1=-\frac{2}{3}c_3 \qquad \& \; c_2=-\frac{1}{3}c_3$$

$\Rightarrow$ This system has infinitely many solutions

(i.e., this system has nontrivial solutions, e.g., $c_1=2$, $c_2=-1$, $c_3=3$)

$\Rightarrow$ S is (or $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are) linearly dependent

EXAMPLE-3

Ex 3: Determine whether the following set of vectors in the 2×2 matrix space is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$$

Sol: For L.I. or L.D. the vector eq. is $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$

or $$c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}$$

EXAMPLE-3 …Con’d

The Augmented Matrix is

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution)

$\Rightarrow$ S is linearly independent

EXAMPLE-4

Ex 4: For which real values of λ do the following vectors form a L.D set in R^3.

$v_1 = (\lambda, -0.5, -0.5)$, $v_2 = (-0.5, \lambda, -0.5)$, $v_3 = (-0.5, -0.5, \lambda)$

Sol: For L.I. or L.D. the vector eq. is

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Rightarrow \begin{array}{l} \lambda c_1 - 0.5c_2 - 0.5c_3 = 0 \\ -0.5c_1 + \lambda c_2 - 0.5c_3 = 0 \quad \ldots.(a) \\ -0.5c_1 - 0.5c_2 + \lambda c_3 = 0 \end{array}$$

Since, for Ax=0 , if $|A| = 0$ implies that system Ax=0 has non trivial solution.

From system (a) we have

$$|A| = \begin{vmatrix} \lambda & -0.5 & -0.5 \\ -0.5 & \lambda & -0.5 \\ -0.5 & -0.5 & \lambda \end{vmatrix}$$

EXAMPLE-4 …..Con’d

For L.D. set of vectors $\Rightarrow \quad |A| = \begin{vmatrix} \lambda & -0.5 & -0.5 \\ -0.5 & \lambda & -0.5 \\ -0.5 & -0.5 & \lambda \end{vmatrix} = 0$

$$\therefore\ |A| = \lambda^3 - \frac{3}{4}\lambda - \frac{1}{4} = 0$$

or $(\lambda - 1)(\lambda + \frac{1}{2})^2 = 0$

$$\therefore\ \lambda = 1 \ or\ \lambda = -\frac{1}{2}$$

Thus the vectors are linearly dependent for these two values of λ and linearly independent for all other values.

THEOREM

Statement

Let $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots.,\mathbf{v}_k\}$ be a set of vectors in R^n. If $k>n$, then S is linearly dependent.