Differential Equation

DEF: An equation involving derivatives is called differential equation

There are two types of DE

1- ODE ✓

2- PDE ✓

1- $\frac{dy}{dx} = x + \sin y$ ✓

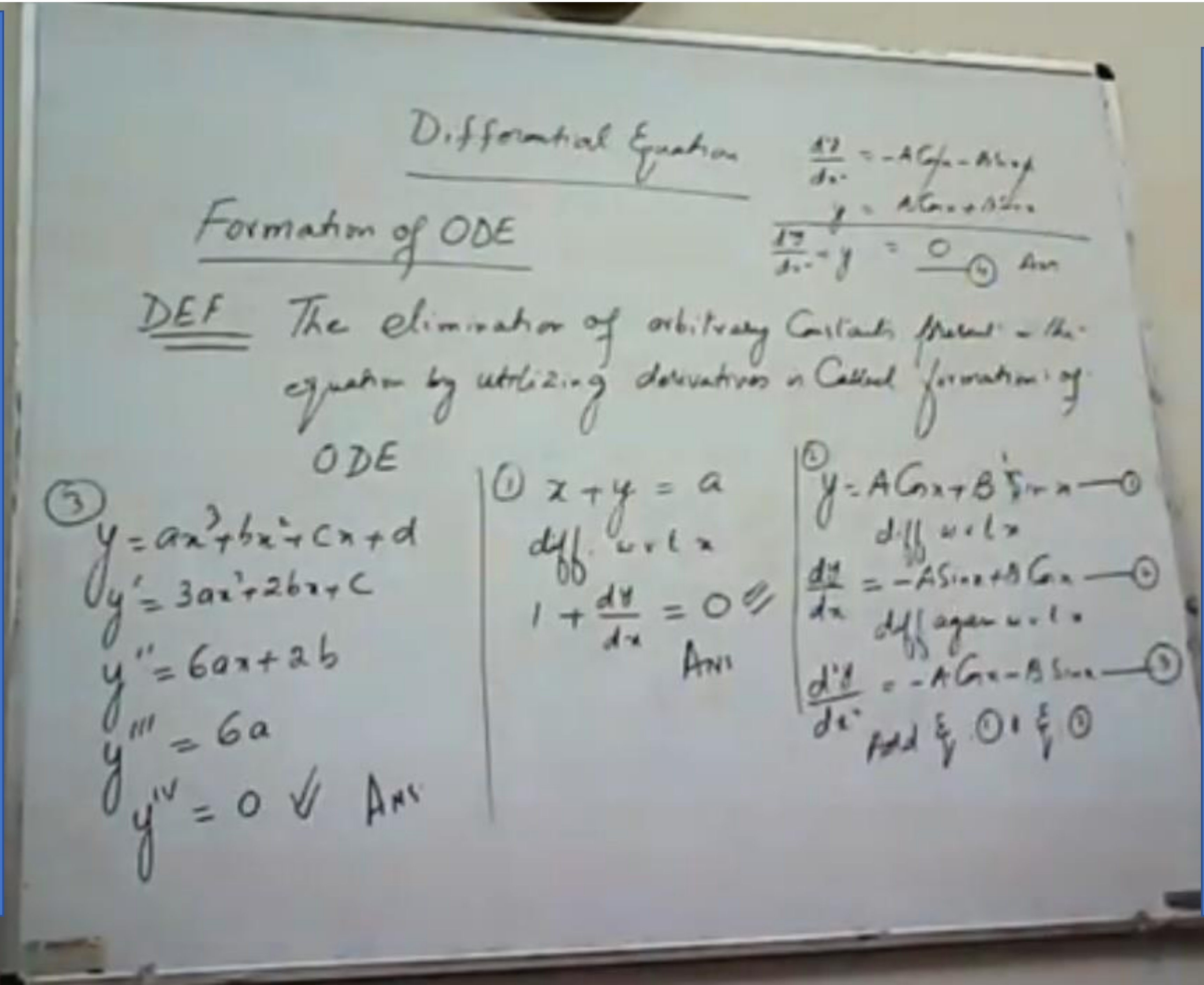
2- $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$ ✓

3- $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

4- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

1. Chek all the orbitrary costants
2.Total arbitrary constants in the example , will have to differentiate that number oif times
3.There are scenatrios in getting answers after differentiation

After differenciating properly , no more diff. afterwards instead simplificatoion takes place



In ex. 1 and 3 after proper diff. we get the ans..and that is the ans ..instead in ex 2 after diff. that’s not the ans ,we have to simplify it on order to get

(4) $Ax^2 + By^2 = 1$ —— (1)

$2Ax + 2Byy' = 0$ —— (2)

$2A + 2By'y' + 2Byy'' = 0$ —— (3)

from eq (2)

$2A = -2By'^2 - 2Byy''$ —— (4)

Substitute this value in eq (2)

$(-2By'^2 - 2Byy'')x + 2Byy' = 0$ —— (5)

$-2B\left[(y'^2 + y \cdot y'')x - yy'\right] = 0$

dividing by $-2B$

$$\boxed{(y'^2 + y \cdot y'')x - y \cdot y' = 0}$$

(1) $y = A\cos x + B\sin x$ —— (1)

diff w.r.t. x

$\frac{dy}{dx} = -A\sin x + B\cos x$ —— (2)

diff again w.r.t. x

$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$ —— (3)

Add eq (1) & eq (3)

$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$

$y = A\cos x + B\sin x$

$\frac{d^2y}{dx^2} + y = 0$ —— (4) Ans

USE SUBSTITUTION METHOD INCASE WHEN NO MORE SIMPLIFICATION LEFT AFTER DOING DIFF.

(4) $Ax^2+By^2=1$ —— (1)

$2Ax+2Byy'=0$ —— (2)

$2A+2By'y'+2Byy''=0$ —— (3)

from eq (2)

$2A=-2By'^2-2Byy''$ —— (4)

Substitute this value in (2)

$(-2By'^2-2Byy'')x+2Byy'=0$ —— (5)

$-2B[(y'^2+yy'')x-yy']=0$

dividing by $-2B$

$$\boxed{(y'^2+yy'')x-y\cdot y'=0}$$

(5) $y=a(x-a)^2$ —— (1)

$y'=a\,2(x-a)$ —— (2)

Dividing (1) & (2)

$$\frac{y}{y'}=\frac{a(x-a)^2}{2a(x-a)}$$

$$\frac{y}{y'}=\frac{x-a}{2}$$

$$\frac{2y}{y'}=x-a$$

$$\Rightarrow a=x-\frac{2y}{y'}$$

Substitute a in (1)

$$y=\left(x-\frac{2y}{y'}\right)\left(\frac{2y}{y'}\right)^2$$ Ans

NED LMS

Practice Problem

1) $y = A\sin^{-1}x$

2) $y = ae^{x} + be^{2x}$

3) $y^{2} = Ax^{2} + Bx + C$

4) $x^{2} + y^{2} + 2gx + 2fy + C = 0$

5) $(x-h)^{2} + (y-k)^{2} = 0$

Some Practise solutions>>>:

lecture 1(b)

Qs

$$(x-h)^2+(y-k)^2=0 \quad \text{— (I)}$$

diff w.r to x

$$2(x-h)(1)+2(y-k)\frac{dy}{dx}=0 \quad \text{— (II)}$$

$$2(1)+2[y'y'+(y-k)y'']=0$$

$$2+2y'^2+(y-k)y''=0$$

$$y-k=\frac{-2-2y'^2}{y''} \quad \text{— (III)}$$

Put value in Eqn (II)

$$2(x-h)+2\left(\frac{-2-2y'^2}{y''}\right)y'=0$$

$$2(x-h)-4\frac{(1+y'^2)}{y''}y'=0$$

$$(x-h)=\frac{{}^{2}\cancel{4}(1+y'^2)y'}{\cancel{2}\,y''}$$

$$(x-h)=\frac{2y'(1+y'^2)}{y''} \quad \text{— (IV)}$$

Put values from Eq (III) and (IV) in Eq (I)

$$\left[\frac{2y'(1+y'^2)}{y''}\right]^2+\left[\frac{-2-2y'^2}{y''}\right]^2=0$$

$$4y'^2(1+y'^2)^2 + 4(1+y'^2)^2 = 0$$

$$4y'^2(1+2y'^2+y'^4) + 4(1+2y'^2+y'^4) = 0$$

$$4y'^2 + 8y'^4 + 4y'^6 + 4 + 8y'^2 + 4y'^4 = 0$$

$$4y'^6 + 12y'^4 + 12y'^2 + 4 = 0 \quad \text{Ans}$$

$$y^2 = Ax^2 + Bx + C$$

Diff. w.r.t to x.

$$2yy' = 2Ax + B$$

Diff w.r.t to x

$$2[y'y' + yy''] = 2A$$

$$2[y'^2 + yy''] = 2A$$

Diff w.r.t to x

$$2[2y'y'' + yy''' + y'y''] = 0$$

$$2y'y'' + yy''' + y'y'' = 0$$

Solution of ODE

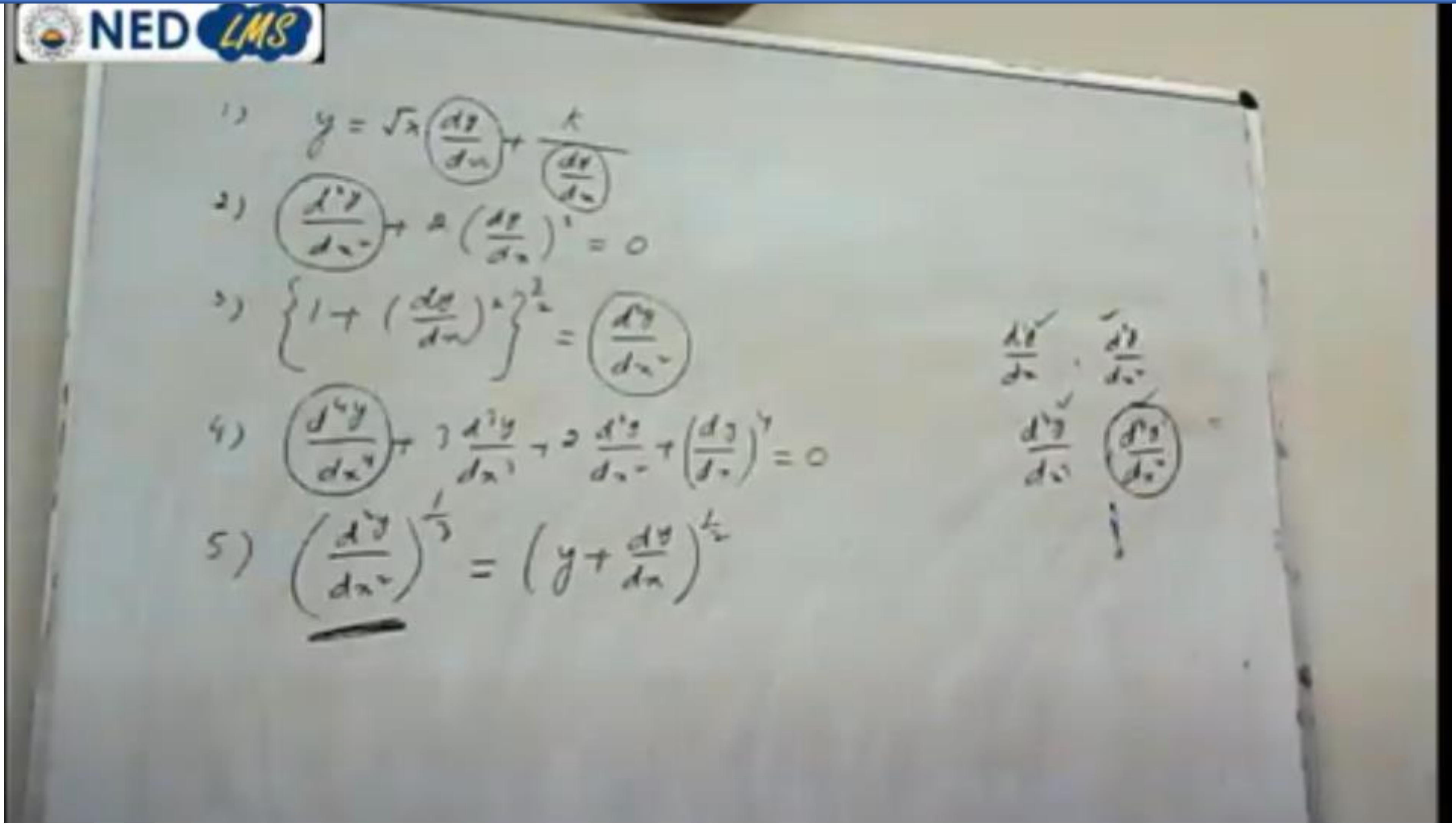
Order & degree of Differential Eq (ODE & PDE)

ORDER :- The order of any differential Eq is the order of the highest derivative term that appears in the diff Eq.

DEGREE The degree of any diff Eq is the power of the highest derivative term that appears in the diff Eq

NOTE :- The diff. Eq. must be made free from radicals and fractions as far as derivative terms are concerned

Q1 : IDENTIFY THE HIGHEST ORDER DERIVATIVE TERM IN EACH PART



NOTE part applied here in this slide

1) $y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}}$ $\Rightarrow y\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + k$

2) $\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^3 = 0$

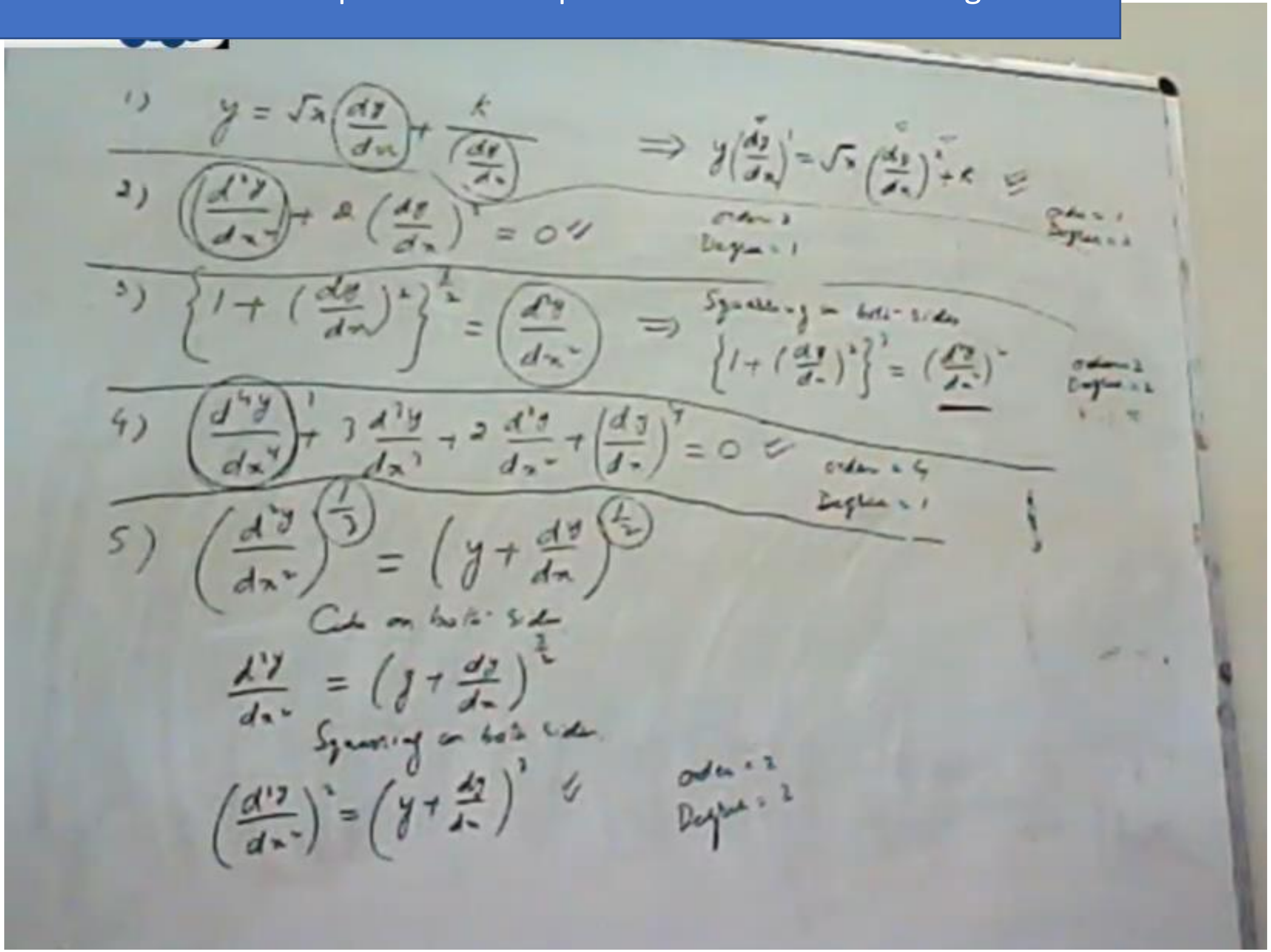
3) $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \frac{d^2y}{dx^2}$

4) $\frac{d^4y}{dx^4} + 3\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = 0$

5) $\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{3}} = \left(y + \frac{dy}{dx}\right)^{\frac{1}{2}}$

Q : remove fractional radical where required in the eq. and at last findout the degree

In the case of finding the degree, the differential having the largest order its degree would be measured only



More ex. Of same like previous slide

6) $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right)^2 = nz$ $\Rightarrow$ order = 1, degree = 2

7) $\left(\frac{\partial^2 u}{\partial x^2}\right) + \left(\frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial^2 u}{\partial z^2}\right) = 0$ $\Rightarrow$ order = 2, degree = 1

8) $\left(\frac{\partial^2 y}{\partial t^2}\right) = c^2\left(\frac{\partial^2 y}{\partial x^2}\right)$ $\Rightarrow$ Order = 2, degree = 1

9) $\left(\frac{\partial^2 z}{\partial x^2}\right) + 2\frac{\partial z}{\partial y} + 3z = 0$ $\Rightarrow$ order = 2, degree = 1

10) $\left(\frac{\partial^3 z}{\partial x^3}\right) = \frac{1}{3}\left(\frac{\partial^2 z}{\partial y^2}\right)^{2/3}$ $\Rightarrow$ Order = 3, degree = 1

Cubing on both sides

$$\left(\frac{\partial^3 z}{\partial x^3}\right)^3 = \left(\frac{1}{3}\right)^3\left(\frac{\partial^2 z}{\partial y^2}\right)^2$$

(6) $$\frac{\partial^2 u}{\partial t^2} = k\left(\frac{\partial u}{\partial x}\right)^{\frac{1}{2}}$$

(7) $$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

(8) $$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial z^3} = nu$$

(9) $$\frac{\partial^4 t}{\partial x^4} - 2\frac{\partial^4 t}{\partial y^4} = 0$$

(10) $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} = 0$$

Practice Problems (Find order & degree)

(1) $$\frac{dy}{dx} = x + \sin y$$

(2) $$\frac{d^4 x}{dt^4} + \frac{d^2 x}{dt^2} + \left(\frac{dx}{dt}\right)^3 = 0$$

(3) $$\left\{y + x\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{3}} = x\frac{d^2 y}{dx^2}$$

(4) $$\left(\frac{d^2 y}{dx^2}\right)^{\frac{2}{3}} + a^2 x = 0$$

(5) $$\left(\frac{dy}{dx}\right)^{\frac{4}{3}} + x\sin y = \cos x$$

(6) $\frac{\partial^2 u}{\partial t^2} = k\left(\frac{\partial u}{\partial x}\right)^2$

(7) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

(8) $\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial z^3} = nu$

(9) $\frac{\partial^4 t}{\partial x^4} - 2\frac{\partial^4 t}{\partial y^4} = 0$

(10) $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} = 0$

Solution of ODE:

1) First order & first degree ODE will be covered (order = 1, degree = 1)

2) If the order or degree or both are more than one, then we will cover in later part

Solution of ODE

Order 1 and degree 1

1- Separable Variables
2- Reducible to Separable
3- Homogeneous
4- Reducible to homogeneous
5- Linear
6- Reducible to linear (Bernoulli's)

(1) Separable Variables

The general form is

$$N(x)\,dx = M(y)\,dy$$

OR

$$N(y)\,dy = M(x)\,dx$$

Ex1

$$\frac{dy}{dx} = \frac{2x}{y + x^2 y}$$

$$\frac{dy}{dx} = \frac{2x}{y(1+x^2)}$$

$$y \cdot dy = \frac{2x}{1+x^2}\,dx$$

Integrate both sides

$$\int y \cdot dy = \int \frac{2x}{1+x^2}\,dx$$

$$\frac{y^2}{2} = \ln(1+x^2) + C$$

(1) Separable ...

The general form is

$$N(x)\,dx = M(y)\,dy$$

OR $\quad N(y)\,dy = M(x)\,dx$

Ex1 $\quad \dfrac{dy}{dx} = \dfrac{2x}{y + x^2 y}$

$$\frac{dy}{dx} = \frac{2x}{y(1+x^2)}$$

$$y \cdot dy = \frac{2x}{1+x^2}\,dx$$

Integrate both side

$$\int y \cdot dy = \int \frac{2x}{1+x^2}\,dx$$

$$\frac{y^2}{2} = \ln(1+x^2) + C$$

Ex2 $\quad y(e^x+1) - e^x \dfrac{dy}{dx} = 0$

$$y(e^x+1) = e^x \frac{dy}{dx}$$

$$\frac{e^x+1}{e^x}\,dx = \frac{1}{y}\,dy$$

Integrate

$$\int \left(\frac{e^x+1}{e^x}\right) dx = \int \frac{1}{y}\,dy$$

$$\int (1+e^{-x})\,dx = \ln y + C$$

$$x - e^{-x} = \ln y + C$$

By using Partial fractions

$$\frac{1}{y(1-ay)} = \frac{A}{y} + \frac{B}{1-ay} \quad \text{--- (1)}$$

$$1 = A(1-ay) + By \quad \text{--- (2)}$$

Substitute $y = 0$ in eq (2)

$$1 = A(1-0) + B(0)$$

$$\Rightarrow \boxed{A = 1}$$

Substitute $y = \frac{1}{a}$ in eq (2)

$$1 = A\left(1 - a\frac{1}{a}\right) + B\cdot\frac{1}{a}$$

$$1 = A(1-1) + \frac{B}{a}$$

$$1 = 0 + \frac{B}{a}$$

$$\boxed{a = B}$$

Ex 3

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

$$y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$$

$$y - ay^2 = a\frac{dy}{dx} + x\frac{dy}{dx}$$

$$y - ay^2 = (a+x)\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{a+x}dx = \frac{1}{y-ay^2}dy$$

$$\int\frac{1}{a+x}dx = \int\frac{1}{y-ay^2}dy$$

$$\ln(a+x) = \int\frac{1}{y(1-ay)}dy$$

By using Partial fractions

$$\frac{1}{y(1-ay)} = \frac{A}{y} + \frac{B}{1-ay} \quad \text{— (1)}$$

$$1 = A(1-ay) + By \quad \text{— (2)}$$

Substitute $y = 0$ in eq (2)

$$1 = A(1-0) + B(0)$$

$$\Rightarrow \boxed{A = 1}$$

Substitute $y = \frac{1}{a}$ in eq (2)

$$1 = \ldots(1 - a\tfrac{1}{a}) + B\tfrac{1}{a}$$

$$\ldots 0 + \frac{\ldots}{a}$$

$$\boxed{\ldots = B}$$

Ex

Hence

$$\frac{1}{y(1-ay)} = \frac{1}{y} + \frac{a}{1-ay}$$

from eq (A)

$$\ln(a+x) = \int\left(\frac{1}{y} + \frac{a}{1-ay}\right)dy$$

$$\ln(a+x) = \ln y - \ln(1-ay) + \underbrace{\ln C}_{C}$$

$$\Rightarrow \frac{1}{a+x}\,dx = \frac{1}{y-ay^2}\,dy$$

$$\int \frac{1}{a+x}\,dx = \int \frac{1}{y-ay^2}\,dy$$

$$\ln(a+x) = \int \frac{1}{y(1-ay)}\,dy \quad \text{— (A)}$$

Ex 4

$$\sqrt{1+x^2+y^2+x^2y^2} - xy\frac{dy}{dx} = 0$$

$$\sqrt{1+x^2+y^2+x^2y^2} = xy\frac{dy}{dx}$$

$$\sqrt{1+x^2+y^2(1+x^2)} = xy\frac{dy}{dx}$$

$$\sqrt{(1+x^2)(1+y^2)} = xy\frac{dy}{dx}$$

$$\sqrt{1+x^2}\,\sqrt{1+y^2} = xy\frac{dy}{dx}$$

$$\frac{\sqrt{1+x^2}}{x}\,dx = \frac{y}{\sqrt{1+y^2}}\,dy$$

$$\int\frac{\sqrt{1+x^2}}{x}\,dx = \int\frac{y}{\sqrt{1+y^2}}\,dy$$

1- Algebraic

2. Trigonometric

$1+x^2$

$1-x^2$

x^2-1

$$= \int(1+y^2)^{-\frac{1}{2}}\,y\,dy$$

$$= \frac{1}{2}\int(1+y^2)^{-\frac{1}{2}}\,2y\,dy$$

$$= \frac{1}{2}\,\frac{(1+y^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= (1+y^2)^{\frac{1}{2}} + C$$

Used Trigonnometric substituion

$$x^2+y^2+x^2y^2 - xy\frac{dy}{dx} = 0$$

$$+y^2+x^2y^2 = xy\frac{dy}{dx}$$

$$+y^2(1+x^2) = xy\frac{dy}{dx}$$

$$)(1+y^2) = xy\frac{dy}{dx}$$

$$\sqrt{1+y^2} = xy\frac{dy}{dx}$$

$$\frac{\sqrt{1+x^2}}{x}\,dx = \frac{y}{\sqrt{1+y^2}}\,dy$$

$$\int\frac{\sqrt{1+x^2}}{x}\,dx = \int\frac{y}{\sqrt{1+y^2}}\,dy$$

$x = \operatorname{Tan}\theta$

$dx = \operatorname{Sec}^2\theta\, d\theta$

$$= \int (1+y^2)^{-\frac{1}{2}}\, y\, dy$$

$$\int\frac{\sqrt{1+\operatorname{Tan}^2\theta}}{\operatorname{Tan}\theta}\operatorname{Sec}^2\theta\, d\theta \qquad = \frac{1}{2}\int (1+y^2)^{-\frac{1}{2}}\, 2y\, dy$$

$$\int\frac{\operatorname{Sec}\theta}{\operatorname{Tan}\theta}\operatorname{Sec}^2\theta\, d\theta \qquad = \frac{1}{2}\,\frac{(1+y^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

Try to do yourself the rest of integration

$$= (1+y^2)^{\frac{1}{2}} + C$$