LECTURE-5(A)

COURSE TITLE

LINEAR ALGEBRA

&

GEOMETRY

(MT-272)

DEPARTMENT OF TELECOMMUNICATION ENGINEERING

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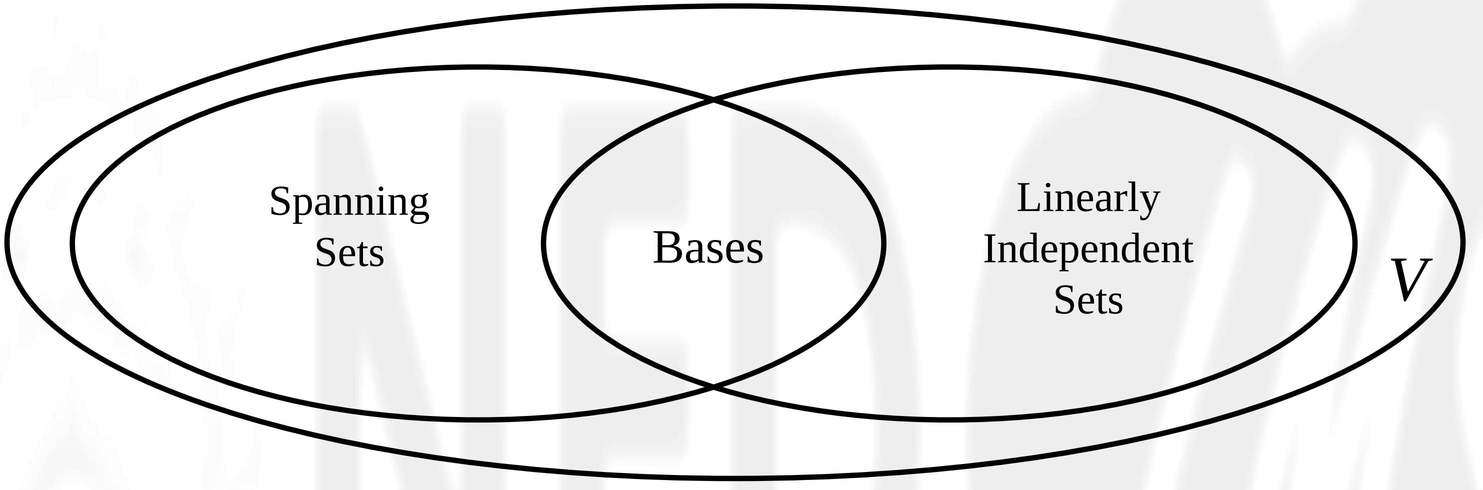
BASIS FOR A VECTOR SPACE

If *V* is any vector space and S= $\{ \mathrm{v}_1, \mathrm{v}_2, \dots \dots, \mathrm{v}_\mathrm{n}\}$ is a set of vectors in *V* , then *S* is called a **basis** for *V* if the following two conditions hold:

- *S* spans *V*.
- *S* is linearly independent.

BASIS FOR A VECTOR SPACE



- Notes:

  A basis S must have enough vectors to span V, but not so many vectors that one of them could be written as a linear combination of the other vectors in S

STANDARD BASES

(1) the **standard basis** for R^3:

$\{i, j, k\}$, for $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

(2) the **standard basis** for R^n :

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, for $\mathbf{e}_1=(1,0,\ldots,0)$, $\mathbf{e}_2=(0,1,\ldots,0)$,…, $\mathbf{e}_n=(0,0,\ldots,1)$

Ex: For R^4, $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

STANDARD BASES

(3) the **standard basis** for $m \times n$ matrix space:

$\{ E_{ij} \mid 1 \le i \le m,\ 1 \le j \le n \}$, and in E_{ij} $\begin{cases} a_{ij} = 1 \\ \text{other entries are zero} \end{cases}$

Ex: 2×2 matrix space:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

(4) the **standard basis** for $P_n(x)$:

$\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x) = \{1, x, x^2, x^3\}$

EXAMPLE-1

Example-1: The nonstandard basis for R^2

Show that $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1,1),\ (1,-1)\}$ is a basis for R^2

(1) For any $\mathbf{u} = (u_1, u_2) \in R^2$, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{u} \ \Rightarrow \begin{cases} c_1 + c_2 = u_1 \\ c_1 - c_2 = u_2 \end{cases}$

Because the coefficient matrix of this system has a **nonzero determinant**, the system has a unique solution for each **u**. Thus you can conclude that S spans R^2

(2) For $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \ \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{cases}$

Because the coefficient matrix of this system has a **nonzero determinant**, you know that the system has only the trivial solution. Thus you can conclude that S is linearly independent

EXAMPLE-2

Example-2: Let $\mathbf{v}_1=(1,2,1)$, $\mathbf{v}_2=(2,9,0)$, $\mathbf{v}_3=(3,3,4)$.

Show that the set S= $\{\boldsymbol{v}_{1,}\boldsymbol{v}_{2,}\boldsymbol{v}_3\}$ is a basis for R^3

Sol: For basis , we must show that S spans R^3 & also show that S is linearly independent set.

For Spanning :

Let $\mathbf{u}=(u_1,u_2,u_3)$ be an arbitrary vector in R^3, it can be expressed as a linear combination

$$\mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 \quad ....(a)$$

of the vector in S .

Putting values of $\mathbf{u}$, $\mathbf{v}_{1,}\mathbf{v}_2, \mathbf{v}_3$ in (a) we have

$$(u_1,u_2,u_3) = c_1(1,2,1) + c_2(2,9,0)+c_3(3,3,4)$$

or $(u_1, u_2 , u_3) = (c_1+2c_2+3c_3 ,2c_1+9c_2+3c_3 ,c_1+5c_3)$

EXAMPLE-2 ….Con'd

On equating corresponding components

$$\begin{aligned} c_1+2c_2+3c_3&=u_1 \\ 2c_1+9c_2+3c_3&=u_2 \quad \text{.....(1)} \\ c_1\qquad +4c_3&=u_3 \end{aligned}$$

For L.I.: For L.I. the vector equation

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = \mathbf{0} \quad \text{.....(b)}$$

By putting values we get

$$c_1(1,2,1) + c_2(2,9,0)+c_3(3,3,4) = (0,0,0)$$

or $(c_1+2c_2+3c_3\ ,2c_1+9c_2+3c_3\ ,c_1+4c_3) =(0,0,0)$

On comparing both sides

$$\begin{aligned} c_1+2c_2+3c_3&=0 \\ 2c_1+9c_2+3c_3&=0 \quad \text{.....(2)} \\ c_1\qquad +4c_3&=0 \end{aligned}$$

EXAMPLE-2 ....Con’d

Observe that systems (1) and (2) have same coefficient matrix . System (1) is non-homogeneous system of linear equation that is AX = b while system (2) is homogenous system of linear equation that is AX = 0

NOTE : For system AX = b

If $|A| \neq 0$ then system is called consistent and has some solution.

For system AX = 0

If $|A| \neq 0$ then system has trivial solution.

Therefore, from coefficient matrix of system (1) and (2) we can decide whether (1) has some solution and (2) has trivial solution or not .

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$

EXAMPLE-2 ….Con'd

$$\therefore \quad |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1$$

This implies that system (1) has some solution means arbitrary vector ' **u** ' spanned by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$, and system (2) has trivial solution means that $\{\, \mathbf{v_1}, \mathbf{v_2}, \mathbf{v}_3 \,\}$ is linearly independent set .

Since both conditions are true, therefore, S is a basis for R^3 .

EXAMPLE-3

Example-3: Determine whether the following vectors form basis for P_2 or not.

$$\mathbf{p}_1=4+6x+x^2,\ \ \mathbf{p}_2=-1+4x+2x^2,\ \ \mathbf{p}_3=5+2x-x^2$$

Sol: Let **S=** $\{\mathbf{p_1},\mathbf{p_2},\mathbf{p_3}\}$

For basis , we must show that *S* spans P_2 and also show that S is linearly independent set.

For Spanning :

Let $\mathbf{u}=u_1+u_2x+u_3x^2$ be an arbitrary vector in P_2 it can be expressed as a linear combination

$\mathbf{u}=c_1\mathbf{p}_1+c_2\mathbf{p}_2+c_3\mathbf{p}_3$ ....(a) of the vector in S .

Putting values of **u** , $\boldsymbol{p_1},\boldsymbol{p}_2,\boldsymbol{p_3}$ in (a) we have

$$u_1+u_2x+u_3x^2 = c_1(4+6x+x^2) + c_2(-1+4x+2x^2)+c_3(5+2x-x^2)$$

$$u_1+u_2x+u_3x^2 =(4c_1-c_2+5c_3)+(6c_1+4c_2+2c_3)x+(c_1+2c_2-c_3)x^2$$

EXAMPLE-3 …Con'd

Comparing on both sides

$$\begin{aligned} 4c_1 - c_2 + 5c_3 &= u_1 \\ 6c_1 + 4c_2 + 2c_3 &= u_2 \quad ....(1) \\ c_1 + 2c_2 - c_3 &= u_3 \end{aligned}$$

For L.I: For L.I. the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = 0 \; .....(b)$$

By putting values in (b) we get

$$c_1(4+6x + x^2) + c_2(-1+4x+2x^2) + c_3(5+2x-x^2) = 0+0x+0x^2$$

$$(4c_1 - c_2 + 5c_3) + (6c_1 + 4c_2 + 2c_3)x + (c_1 + 2c_2 - c_3)x^2 = 0+0x+0x^2$$

EXAMPLE-3 …Con'd

Comparing on both sides

$$4c_1 - c_2 + 5c_3 = 0$$
$$6c_1 + 4c_2 + 2c_3 = 0 \quad \ldots(2)$$
$$c_1 + 2c_2 - c_3 = 0$$

Observe that systems (1) and (2) have same coefficient matrix . System (1) is non-homogeneous system of linear equation that is AX = b while system (2) is homogenous system of linear equation that is AX = 0

Therefore from coefficient matrix of system (1) and (2) we can decide whether (1) has some solution and (2) has trivial solution or not .

The coefficient matrix is $A = \begin{bmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

EXAMPLE-3 …Con'd

$$\therefore |A| = \begin{vmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 0$$

This implies that system (1) has no solution, consequently ' u ' does not span by $\boldsymbol{P_2}$. In addition system (2) has nontrivial solution means that from equation (b), $\{\ \mathbf{p_1}, \mathbf{p_2}, \mathbf{p}_3\ \}$ is linearly dependent set. Hence, Set S is not basis for P_2.