Variable Seperable

1) $x \frac{dy}{dx} + \cot y = 0$ $\qquad = \frac{1}{2(2)} \ln \left| \frac{y-2}{y+2} \right| = \frac{x^3}{3} + c$

$x \frac{dy}{dx} = -\cot y$

$\frac{dy}{-\cot y} = \frac{dx}{x}$

$-\tan y \, dy = \frac{dx}{x}$

Integrati

$-\int \frac{\sin y}{\cos y} dy = \int \frac{dx}{x}$

$\ln \cos y = \ln x + c$

Q2 $\frac{dy}{dx} = x^2 y^2 - 4x^2$

$\frac{dy}{dx} = x^2(y^2 - 4)$

$\frac{1}{y^2 - 4} = x^2 dx$

$\int \frac{1}{y^2 - (2)^2} dy = \int x^2 dx$

$\frac{x^3}{3} + c$

Ans

Variable Seperable.

Q3 $3e^x \tan y \, dx + (1 - e^x)\sec^2 y \, dy = 0$

$(1 + e^x)\sec^2 y \, dy = -3e^x \tan y \, dx$

$$\frac{\sec^2 y}{\tan y} dy = \frac{-3e^x}{1 + e^x} dx$$

Integrate

$$\int \frac{\sec^2 y}{\tan y} dy = -3\int \frac{e^x}{1+e^x} dx$$

$$\ln(\tan y) = -3 \ln(1 + e^x) + c$$

Ans

Q4 $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$$\frac{dy}{dx} = e^{-y}\left[e^x + x^2\right]$$

$e^y \, dy = \left[e^x + x^2\right] dx$

Integrate

$\int e^y \, dy = \int (e^x + x^2) \, dx$

$e^y = e^x + \frac{x^3}{3} + C$ Ans

Reduceable to Separable.

(1) $\frac{dy}{dx} = \sec(x+y)$ — (1)

let $x+y = t$

$\frac{dx}{dx} + \frac{dy}{dx} = \frac{dt}{dx}$

$1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{dt}{dx} - 1$

(1) $\Rightarrow$

$\frac{dt}{dx} - 1 = \sec(t)$

$\frac{dt}{dx} = \sec t + 1$

$\frac{dt}{\sec t + 1} = dx$

Integrate-

$\int \frac{1}{1+\frac{1}{\cos t}} dt = \int dx$

$\int \frac{\cos t}{1+\cos t} = x + c$

$\int \frac{\cos t + 1 - 1}{1+\cos t} dt = x + c$

$\int 1 - \frac{1}{1+\cos t} dt = x + c$

$\int 1 - \frac{1}{2\cos^2\frac{t}{2}} dt = x + c$

$\int 1 - \frac{1}{2}\sec^2\frac{t}{2} dt = x + c$

$t - \frac{1}{2}\tan\frac{t}{2} = x + c$

$x + y - \frac{1}{2}\tan\left(\frac{x+y}{2}\right) = x + c$

Ans

Reduceable to Seperable

(2) $(x+y)^2 \dfrac{dy}{dx} = a^2$ — (1)

let $t = x+y$

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

(1) =>

$$(t)^2\left(\frac{dt}{dx} - 1\right) = a^2$$

$$t^2(dt - dx) = a^2 dx$$

$$t^2 dt - t^2 dx = a^2 dx$$

$$t^2 dt = t^2 dx + a^2 dx$$

$$t^2 dt = (t^2 + a^2) dx$$

$$\frac{t^2}{t^2+a^2} dt = dx$$

$$\begin{array}{r} 1 \\ t^2+a^2 \overline{) \cancel{t^2}} \\ \underline{-\cancel{t^2} \pm a^2} \\ -a^2 \end{array}$$

$$1 - \frac{a^2}{t^2+a^2}$$

$$1 - \frac{a^2}{t^2+a^2} dt = dx$$

Apply integration

$$\int 1\, dt - \int \frac{a^2}{t^2+a^2} dt = \int dx$$

$$t - \frac{a^2}{a}\left[\tan^{-1}\frac{t}{a}\right] = x + c$$

$$t - a\left[\tan^{-1}(t/a)\right] = x + c$$ Ans

$$\frac{dy}{dx} = \frac{2x+y+1}{2(2x+y)-1}$$

Reduce able to Seperable.

(3) $$\frac{dy}{dx} = \frac{4x+6y+5}{2x+3y+4} \quad (1)$$

~~let dx~~

let $2x+3y=t$.

$$\frac{dt}{dx} = 2+3\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dt}{dx}-2\right)$$

(1) $\Rightarrow \frac{1}{3}\left(\frac{dt}{dx}-2\right) = \frac{2t+5}{t+4}$

$$\frac{dt}{dx} = 3\left(\frac{2t+5}{t+4}\right)+2$$

$$\cancel{\frac{dt}{dx} = \frac{t+4}{8t+23}}$$

$$\frac{dt}{dx} = \frac{8t+23}{t+4}$$

$$\frac{t+4}{8t+23}\,dt = dx$$

Integrate.

$$= \int \frac{t+4}{8t+23}\,dt = \int dx$$

$$= \int \frac{8(t+4)}{8(8t+23)}\,dt = x+c$$

$$= \int \frac{8t+32}{8(8t+23)}\,dt = x+c$$

$$= \int \frac{8t+23+9}{8(8t+23)}\,dt = x+c$$

$$= \frac{1}{8}\int \frac{\cancel{8t+23}}{\cancel{8t+23}} + \frac{9}{8t+23}\,dt = x+c$$

$$= \frac{1}{8}\int 1+\frac{9}{8t+23}\,dt = x+c$$

$$= \frac{1}{8}\left[t+9\ln(8t+23)\right] = x+c$$

$$= \frac{1}{8}\left[2x+3y+9\ln(16x+24y+23)\right] = x+c. \text{ Ans}$$

Homogenous.

Q1 $\frac{dy}{dx} = \frac{2xy}{x^2+y^2}$ —(1)

Let $y = vx$.

$\frac{dy}{dx} = \frac{d}{dx}(vx)$

$\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \cdot \frac{dx}{dx}$.

$\frac{dy}{dx} = x \cdot \frac{dv}{dx} + v$

Put it in Eqn (1)

$v + x \cdot \frac{dv}{dx} = \frac{2x(vx)}{x^2+(vx)^2}$.

$v + x\frac{dv}{dx} = \frac{2v\cancel{x^2}}{\cancel{x^2}(1+v^2)}$

$v + x\frac{dv}{dx} = \frac{2v}{(1+v^2)}$.

$x\frac{dv}{dx} = \frac{2v - v - v^3}{1+v^2}$

$x\frac{dv}{dx} = \frac{v-v^3}{1+v^2}$

$\frac{1+v^2}{v-v^3} dv = \frac{1}{x} dx$.

$\frac{1+v^2+1-1}{v(1-v^2)} dv = \frac{1}{x} dx$

$v - v^3 \overline{)\ 1+v}$

$\frac{v+v^3}{(v)^2-(v^3)^2} = \frac{v+v^3}{v^2-v^6}$

$= \frac{\cancel{v}(1+v^2)}{\cancel{v}(v-v^5)_v}$.

$v - v^3 \overline{)\ v^2}$

$\frac{v^2 - v^4}{v^4}$

$v + \frac{v^4}{v-v^3}$

$\frac{v^3}{1-v^2}$

$\frac{2 + v^2 - 1}{v(1-v^2)}$

$\frac{2}{v(1-v^2)} - \frac{\cancel{1-v^2}}{v\cancel{(1-v^2)}}$

$\frac{2}{v(1-v^2)} - \frac{1}{v}$.

$\frac{1}{v(1-v^2)} = \frac{A}{v} + \frac{B}{1+v} + \frac{C}{1-v}$.

$1 = A(1+v)(1-v) + B(v)(1-v) + C(v)(1+v)$

for $v = 1 \rightarrow C = \frac{1}{2}$.

for $v = 0$. $A \rightarrow A = 1$

for $v = -1$. $\rightarrow B = \frac{1}{2}$

$2\left[\frac{1}{v} + \frac{1}{2(1+v)} + \frac{1}{2(1-v)}\right] - \frac{1}{v}$

$2\left[\ln v + \frac{1}{2}\ln(1+v) + \frac{1}{2}(-1)\ln(1-v)\right] - \ln v$

$= \ln x + c$

Ans

Homogenous.

2) $(x^2+3y^2)\,dx - 2xy \cdot dy = 0$

$(x^2+3y^2)\,dx = 2xy\,dy$

$dy/dx = (x^2+3y^2)/2xy$ — ①

Let $y = vx$

$$\frac{dy}{dx} = x \cdot \frac{dv}{dx} + v$$

① ⇒

$$x \cdot \frac{dv}{dx} + v = \frac{x^2 + 3(vx)^2}{2(x)(vx)}$$

$$x \cdot \frac{dv}{dx} + v = \frac{\cancel{x^2}(1+3v^2)}{2\cancel{x^2}\,v}$$

$$x \cdot \frac{dv}{dx} = \frac{1+3v^2}{2v} - v$$

$$x \cdot \frac{dv}{dx} = \frac{3v^2+1-2v^2}{2v}$$

$$x \cdot \frac{dv}{dx} = \frac{v^2+1}{2v}$$

$$\frac{2v}{v^2+1}\,dv = \frac{1}{x}\,dx$$

Integrate.

$$\int \frac{2v}{v^2+1}\,dv = \int \frac{1}{x}\,dx$$

$\ln(v^2+1) = \ln x + c$

Back substitute

$$\ln\left[\left(\frac{y}{x}\right)^2 + 1\right] = \ln x + c$$

$$\ln\left[\frac{y^2+x^2}{x^2}\right] = \ln x + c$$

Ans

Homogenous.

Q3.

$$x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \quad -①$$

Let $y = vx \Rightarrow v = \frac{y}{x}$.

$$\frac{dy}{dx} = x\cdot\frac{dv}{dx} + v.$$

① ⇒

$$x\cdot\frac{dv}{dx} + v = \frac{vx + \sqrt{(vx)^2+x^2}}{x}$$

$$x\cdot\frac{dv}{dx} + v = \frac{vx+\sqrt{x^2(v^2+1)}}{x}$$

$$x\cdot\frac{dv}{dx} + v = \frac{\cancel{x}\left[v+\sqrt{v^2+1}\right]}{\cancel{x}}$$

$$x\cdot\frac{dv}{dx} = \cancel{v} + \sqrt{v^2+1} - \cancel{v}$$

$$x\cdot\frac{dv}{dx} = \sqrt{v^2+1}.$$

$$\frac{1}{\sqrt{v^2+1}}dv = \frac{1}{x}dx.$$

Integrate.

$$\ln\left|v+\sqrt{v^2+1}\right| = \ln x + c$$

Back substitute

$$\ln\left|\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1}\right| = \ln x + c$$

$$\ln\left|\frac{y}{x}+\sqrt{\frac{y^2+x^2}{x^2}}\right| = \ln x + c$$

$$\ln\left|\frac{y+\sqrt{y^2+x^2}}{x}\right| = \ln x + c$$

Ans

Homogenous

Q4

$(4y+3x)dy + (y-2x)dx = 0$

$$\frac{dy}{dx} = \frac{-(y-2x)}{4y+3x}$$

$$\frac{dy}{dx} = \frac{2x-y}{4y+3x} \quad (1)$$

let $y = vx$

$$\frac{dy}{dx} = x \cdot \frac{dv}{dx} + v$$

(1) =>

$$x \cdot \frac{dv}{dx} + v = \frac{2x - vx}{4vx+3x}$$

$$x \cdot \frac{dv}{dx} + v = \frac{\cancel{x}(2-v)}{\cancel{x}(4v+3)}$$

$$x \cdot \frac{dv}{dx} = \frac{2-v}{4v+3} - v$$

$$x \cdot \frac{dv}{dx} = \frac{2-v-v(4v+3)}{4v+3}$$

$$x \cdot \frac{dv}{dx} = \frac{-4v^2-4v+2}{4v+3}$$

$$\frac{4v+3}{-4v^2-4v+2}\, dv = \frac{1}{x} dx$$

Integrate

$$\int \frac{4v+3}{-4v^2-4v+2}\, dv = \int \frac{1}{x}\, dx$$

$$\frac{-1}{2}\int \frac{-8v+6}{-4v^2-4v+2}\, dv = \ln x + c$$

$$\frac{-1}{2}\left\{\int \frac{-8v-4}{-4v^2-4v+2} - \int \frac{2}{-4v^2-4v+2}\, dv\right\} = \ln x + c \quad (11)$$

$$\int \frac{2}{-4v^2-4v+2} = \int \frac{-2}{4v^2+4v-2}$$

$$= \int \frac{-2}{4v^2+4v+1-3} = \int \frac{-2}{(2v+1)^2-(\sqrt{3})^2}$$

$$= \int \frac{-2}{(2v+1-\sqrt{3})(2v+1+\sqrt{3})}$$

$$-2\int \frac{A}{2v+1-\sqrt{3}} + \frac{B}{2v+1+\sqrt{3}}$$

for $v = \frac{-1-\sqrt{3}}{2}$, $B = \frac{1}{-2\sqrt{3}}$

for $v = \frac{-1+\sqrt{3}}{2}$, $A = \frac{1}{2\sqrt{3}}$

(11) $\Rightarrow$ $\frac{-1}{2}\int \frac{-8v-4}{-4v^2-4v+2} + \frac{1}{\cancel{2}}\int \frac{\cancel{2}}{4v^2+4v+2}\, dv = \ln x + c$

$\frac{-1}{2}\ln(-4v^2-4v+2) + \int \frac{1}{2\sqrt{3}(2v+1-\sqrt{3})} - \frac{1}{2\sqrt{3}(2v+1+\sqrt{3})}\, dv = \ln x +$

$\frac{-1}{2}\ln(-4v^2-4v+2) + \frac{1}{4\sqrt{3}}\int \frac{2}{2v+1-\sqrt{3}} - \frac{2}{2v+1+\sqrt{3}}\, dv = \ln x + c$

$\frac{-1}{2}\ln(-4v^2-4v+2) + \frac{1}{4\sqrt{3}}\left[\ln(2v+1-\sqrt{3}) - \ln(2v+1+\sqrt{3})\right] = \ln x + c$

After back substitution

$\frac{-1}{2}\ln\left(-4\frac{y^2}{x^2} - \frac{4y}{x} + 2\right) + \frac{1}{4\sqrt{3}}\left[\ln\left(2\frac{y}{x}+1-\sqrt{3}\right) - \ln\left(2\frac{y}{x}+1+\sqrt{3}\right)\right] =$

$\ln x + c$

Ans

## Non-Homogenous

2) $\frac{dy}{dx} + \frac{x-y-2}{x-2y-3} = 0$

$$\frac{dy}{dx} = \frac{2-x+y}{x-2y-3} \quad \text{—(i)}$$

let $x = u+h$ , $y = v+k$

$dx = du$ , $dy = dv$

(i) $\Rightarrow$

$$\frac{dv}{du} = \frac{2-(u+h)+v+k}{u+h-2(v+k)+3}$$

$$\frac{dv}{du} = \frac{v-u+2-h+k}{u-2v+h-2k+3} \quad \text{—(ii)}$$

$2-h+k=0$

$h-2k+3=0$

$\Rightarrow h = 7$

$k = 5$

at $h=7$ & $k=5$ Eq (ii) becomes

$$\frac{dv}{du} = \frac{v-u}{u-2v} \quad \text{—(iii)}$$

let

$v = ut$

$$\frac{dv}{du} = t + u\cdot\frac{dt}{du}$$

(iii) $\Rightarrow$

$$t + u\cdot\frac{dt}{du} = \frac{ut-u}{u-2ut}$$

$$t + u\cdot\frac{dt}{du} = \frac{t-1}{1-2t}$$

$$u\cdot\frac{dt}{du} = \frac{t-1}{1-2t} - t$$

$$u\cdot\frac{dt}{du} = \frac{t-1-t(1-2t)}{1-2t}$$

$$u\cdot\frac{dt}{du} = \frac{\cancel{t}-1-\cancel{t}+2t^2}{1-2t}$$

$$\frac{1-2t}{2t^2-1}\,dt = \frac{1}{u}\,du$$

$$\frac{1}{-2}\left[\frac{4t-2}{2t^2-1}\right]dt = \frac{1}{u}\,du$$

$$-\frac{1}{2}\left[\frac{4t}{2t^2-1} - \frac{2}{2t^2-1}\right]dt = \frac{1}{u}\,du$$

$$-\frac{1}{2}\left[\frac{4t}{2t^2-1} - \frac{1}{t^2-\frac{1}{2}}\right]dt = \frac{1}{u}\,du$$

$$-\frac{1}{2}\left[\frac{4t}{2t^2-1} - \frac{1}{(t)^2-\left(\frac{1}{\sqrt{2}}\right)^2}\right]dt = \frac{1}{u}\,du$$

On Integration

$$-\frac{1}{2}\left[\ln(2t^2-1) - \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)}\ln\left|\frac{t-\frac{1}{\sqrt{2}}}{t+\frac{1}{\sqrt{2}}}\right|\right] = \ln u + \ln c$$

$$-\frac{1}{2}\left[\ln\left(2\frac{v^2}{u^2}-1\right) - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}\frac{v}{u}-1}{\sqrt{2}\frac{v}{u}+1}\right|\right]$$

$= \ln u + \ln c$

Non Homogenous

put

$u = x - h, \quad v = y - k$

$u = x - 7, \quad v = y - 5$

$$-\frac{1}{2}\left[\ln\left\{2\left(\frac{y-5}{x-7}\right)^2 - 1\right\} - \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{2}(y-5) - (x-7)}{\sqrt{2}(y-5) + (x-7)}\right|\right] = \ln(x-7) + \ln c$$

Ans

Q3,,

$(x - 2y - 2)dx + (2x - y + 3)dy = 0$

$$\frac{dy}{dx} = \frac{2 + 2y - x}{2x - y + 3} \quad \text{— (1)}$$

let $x = u + h, \quad y = v + k$

$dx = du \qquad dy = dv$

(1) =>

$$\frac{dv}{du} = \frac{2 + 2(v+k) - (u+h)}{2(u+h) - (v+k) + 3}$$

$$\frac{dv}{du} = \frac{2v - u + 2k - h + 2}{2u - v + 2h - k + 3} \quad \text{— (11)}$$

$2k - h + 2 = 0$ — (A)

$2h - k + 3 = 0$ — (B)

After solving

$h = \frac{-8}{3}, \quad k = \frac{-7}{3}$

After putting these values in Eq (111) we get

$$\frac{dv}{du} = \frac{2v - u}{2u - v} \quad \text{— (111)}$$

Now let $v = ut$

$$\frac{dv}{du} = t + u\frac{dt}{du}$$

Eq (111) =>

$$t + u\frac{dt}{du} = \frac{2(ut) - u}{2u - ut}$$

$$t + u\frac{dt}{du} = \frac{2t - 1}{2 - t}$$

$$u\frac{dt}{du} = \frac{t^2 - 1}{2 - t}$$

$$\frac{2 - t}{t^2 - 1}dt = \frac{1}{u}du$$

$$\left[\frac{2}{t^2 - 1} - \frac{t}{t^2 - 1}\right]dt = \frac{1}{u}du$$

Intergrate

$$2\left[\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right] - \frac{1}{2}\ln(t^2 - 1)$$

$$= \ln u + \ln c$$

Non homogenous.

$t = \frac{v}{u}$

$\cancel{t = (x + 8/3)}$

$t = \frac{y + 7/3}{x + 8/3}$

$t = \frac{3y+7}{3x+8}$

$\ln \left| \frac{\frac{3y+7}{3x+8} - 1}{\frac{3y+7}{3x+8} + 1} \right| - \frac{1}{2} \ln \left\{ \left( \frac{3y+7}{3x+8} \right)^2 - 1 \right\} = \ln \left( x + \frac{8}{3} \right) + \ln c$

Ans.

Q4

$(2x + 3y - 5) \frac{dy}{dx} + (3x + 2y - 5) = 0$

$\frac{dy}{dx} = \frac{5 - 3x - 2y}{2x + 3y - 5}$ — (I)

let $x = u + h$, $y = v + k$

$dx = du$, $dy = dv$

(I) =>

$\frac{dv}{du} = \frac{5 - 3(u+h) - 2(v+k)}{2(u+h) + 3(v+k) - 5}$

$\frac{dv}{du} = \frac{-3u - 2v - 2k - 3h + 5}{2u + 3v + 2h + 3k - 5}$ — (II)

$-2k - 3h + 5 = 0$ — (A)

$2h + 3k - 5 = 0$ — (B)

After solving (A) & (B) we get $h = -5, k = -5$

Eq (II) becomes

$\frac{dv}{du} = \frac{-3u - 2v}{2u + 3v}$ — (III)

Now let $v = ut$

$\frac{dv}{du} = t + u\frac{dt}{du}$

Eq (III) =>

$t + u\frac{dt}{du} = \frac{-3u - 2(ut)}{2u + 3(ut)}$

$u\frac{dt}{du} = \frac{-3 - 4t - 3t^2}{2 + 3t}$

$\frac{2+3t}{3t^2 + 4t + 3} dt = -\frac{1}{u} du$

Non Homogenous.

$$\frac{1}{2}\left[\frac{6t+4}{3t^2+4t+3}\right]dt = -\frac{1}{u}du$$

On Integration

$$\frac{1}{2}\ln(3t^2+4t+3) = -\ln u + \ln c$$

$$t = \frac{v}{u} \Rightarrow \frac{y+5}{x+5}$$

$$\frac{1}{2}\ln\left[3\left(\frac{y+5}{x+5}\right)^2 + 4\left(\frac{y+5}{x+5}\right)+3\right] = -\ln(x+5) + \ln c$$

Ans

## Q6

$$(6x+2y-10)\frac{dy}{dx} - 2x - 9y + 20 = 0$$

$$\frac{dy}{dx} = \frac{2x+9y-20}{6x+2y-10} \quad \text{— (1)}$$

let

$x = u+h$ , $y = v+k$

$dx = du$ , $dy = dv$

(1) ⇒

$$\frac{dv}{du} = \frac{2(u+h)+9(v+k)-20}{6(u+h)+2(v+k)-10}$$

$$\frac{dv}{du} = \frac{2u+2h+9v+9k-20}{6u+6h+2v+2k-10} \quad \text{— (ii)}$$

$2h+9k-20=0$ — (A)

$6h+2k-10=0$ — (B)

After solving A & B
we get $h = 1$ , $k = 2$
After putting values in Eq (ii)
we get

$$\frac{dv}{du} = \frac{2u+9v}{6u+2v} \quad \text{— (iii)}$$

let $v = ut$

$$\frac{dv}{du} = t + u\frac{dt}{du}$$

Eq (iii) ⇒

$$t + u\frac{dt}{du} = \frac{2u+9(ut)}{6u+2(ut)}$$

$$u\frac{dt}{du} = \frac{2+9t}{6+2t} - t$$

$$u\frac{dt}{du} = \frac{2+9t-t(6+2t)}{6+2t}$$

$$u\frac{dt}{du} = \frac{2+9t-6t-2t^2}{6+2t}$$

$$\frac{6+2t}{2+3t-2t^2}\,dt = \frac{1}{u}\,du$$

$$\frac{1}{-2}\left[\frac{4t+12}{2t^2-3t-2}\right]dt = \frac{1}{u}\,du$$

$$\frac{-1}{2}\left[\frac{4t-6}{2t^2-3t-2} + \frac{18}{2t^2-3t-2}\right]dt = \frac{1}{u}\,du$$