GEOMETRIC VECTORS

Vectors can be represented geometrically as directed line segments or arrows in 2-space or 3-space.

The direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude.

The tail of the arrow is called the ***initial point*** of the vector, and the tip of the arrow the ***terminal point***.

If, as in Figure *a*, the initial point of a vector ***v*** is *A* and the terminal point is *B*, we write $\mathbf{v} = \overrightarrow{AB}$

(*a*) The vector $\overrightarrow{AB}$

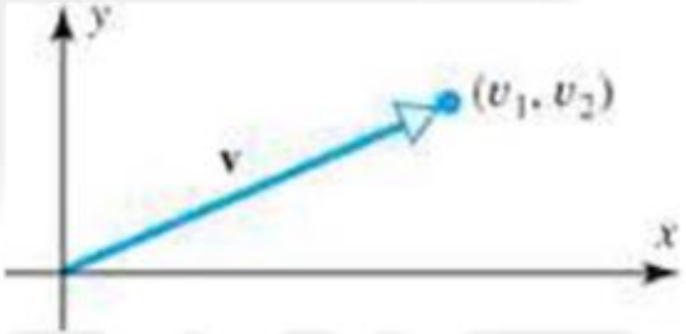
Vectors with the same length and same direction, such as those in Figure *b*, are called ***equivalent***.


(*b*) Equivalent vectors

Equivalent vectors are regarded as ***equal*** even though they may be located in different positions. If ***v*** and ***w*** are equivalent, we write **v** = **w.**

VECTORS IN COORDINATE SYSTEM

Let ***v*** be any vector in the plane, and assume, as in Figure , that ***v*** has been positioned so that its initial point is at the origin of a rectangular coordinate system of two dimension.



The coordinates $(v_1, v_2)$of the terminal point of ***v*** are called the ***components of v***, and we write **v** $=(v_1,v_2)$.

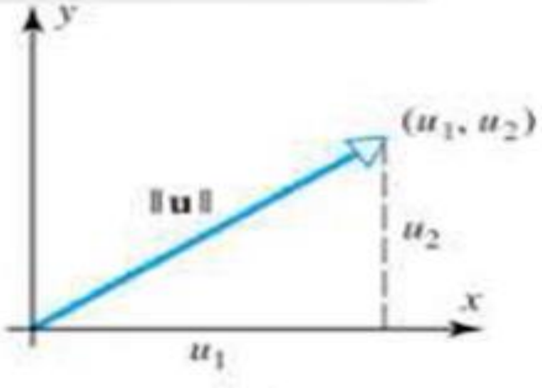
If equivalent vectors, ***v*** and ***w***, are located so that their initial points fall at the origin, then it is obvious that their terminal points must coincide (since the vectors have the same length and direction). Thus the vectors have the same components.

Two vectors **v** $= (v_1,v_2)$ & **w** $= (w_1,w_2)$ are equivalent if and only if

NORM OF A VECTOR

The ***length*** of a vector ***u*** is often called the ***norm*** of ***u*** and is denoted by $\|\mathbf{u}\|$. It follows from the Theorem of Pythagoras that the norm of a vector $\mathbf{u}=(u_1,u_2)$ in 2-space is (as given in fig.(a))

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2} \qquad \text{.....(a)}$$



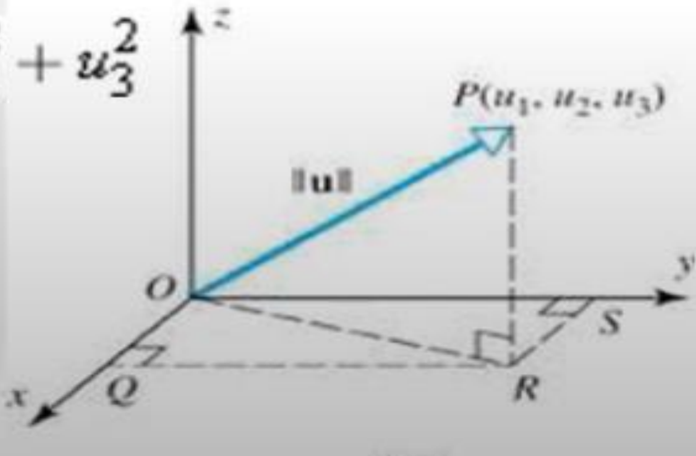
(a)

Let $\mathbf{u}=(u_1,u_2,u_3)$ be a vector in 3-space. Using fig. *(b)* and two applications of the Theorem of Pythagoras, we obtain

$$\|\mathbf{u}\|^2 = (OR)^2 + (RP)^2 = (OQ)^2 + (OS)^2 + (RP)^2 = u_1^2 + u_2^2 + u_3^2$$

Thus,

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2} \qquad \text{.....(b)}$$

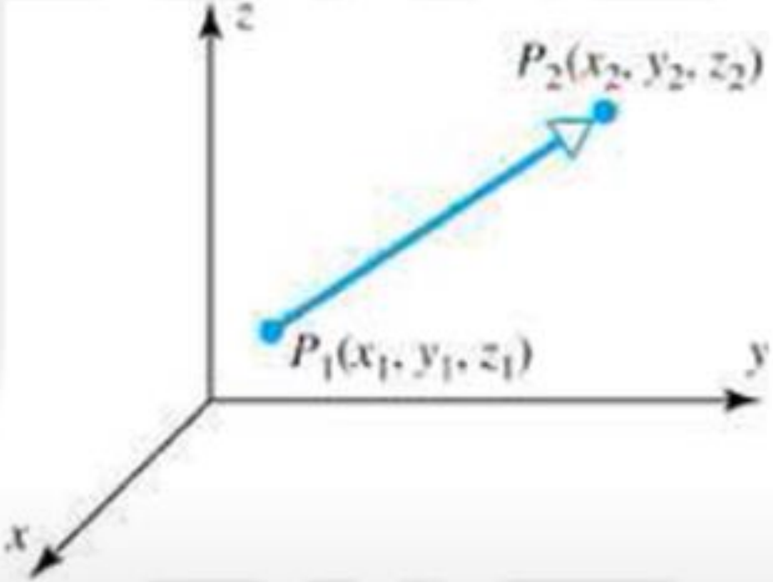


(b)

DISTANCE BETWEEN TWO POINTS

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two points in 3-space as shown in figure, then the **distance** d between them is the norm of the vector $\overrightarrow{P_1P_2}$. Since,

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$



From eq.(b) we have

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

FINDING NORM AND DISTANCE

Example-1 The norm of the vector **u**=(-3,2,1) is

$$\|\mathbf{u}\| = \sqrt{(-3)^2 + (2)^2 + (1)^2} = \sqrt{14}$$

The distance *d* between the points $P_1(2,-1,-5)$ and $P_2(4,-3,1)$ is

$$d = \sqrt{(4-2)^2 + (-3+1)^2 + (1+5)^2} = \sqrt{44} = 2\sqrt{11}$$

PROPERTIES

- Properties of norm:

  (1) $\|\boldsymbol{u}\| \geq 0$

  (2) $\|\boldsymbol{u}\| = 0$ *if and only if* $\boldsymbol{u} = 0$

  (3) $\|c\boldsymbol{u}\| = |c|\|\boldsymbol{u}\|$

- Properties of distance:

  (1) $d(\mathbf{u},\mathbf{v}) \geq 0$

  (2) $d(\mathbf{u},\mathbf{v}) = 0$ *if and only if* $\mathbf{u} = \mathbf{v}$

  (3) $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$

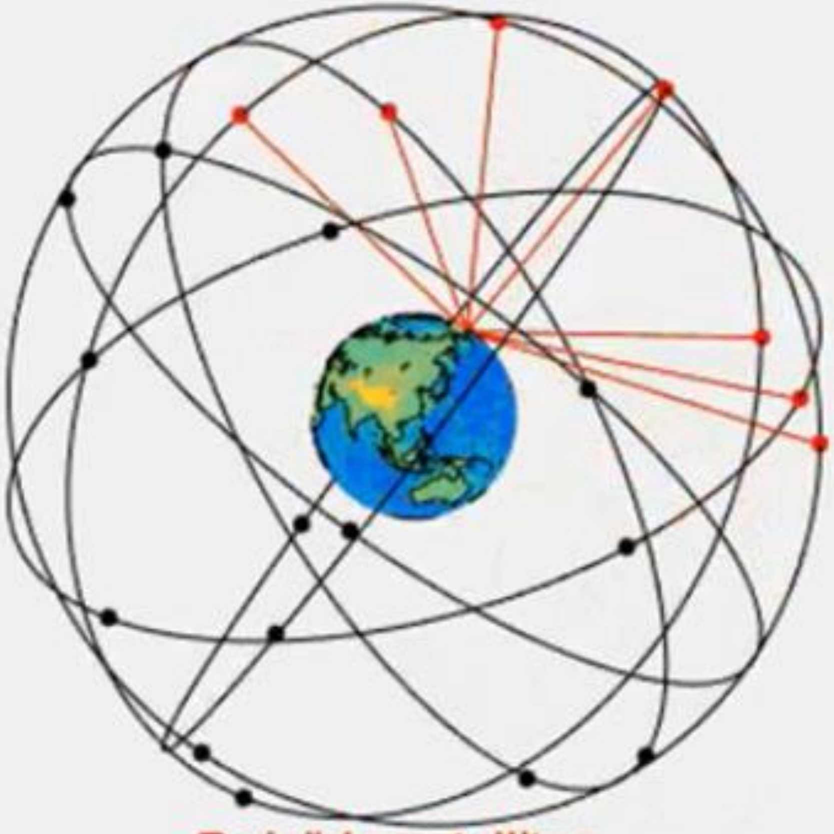
GLOBAL POSITIONING SYSTEM

Global Positioning System (GPS) is the system used by the

- MILITARY
- SHIPS
- AIRPLANE PILOTS
- SURVEYORS
- AUTOMOBILES
- HIKERS etc.

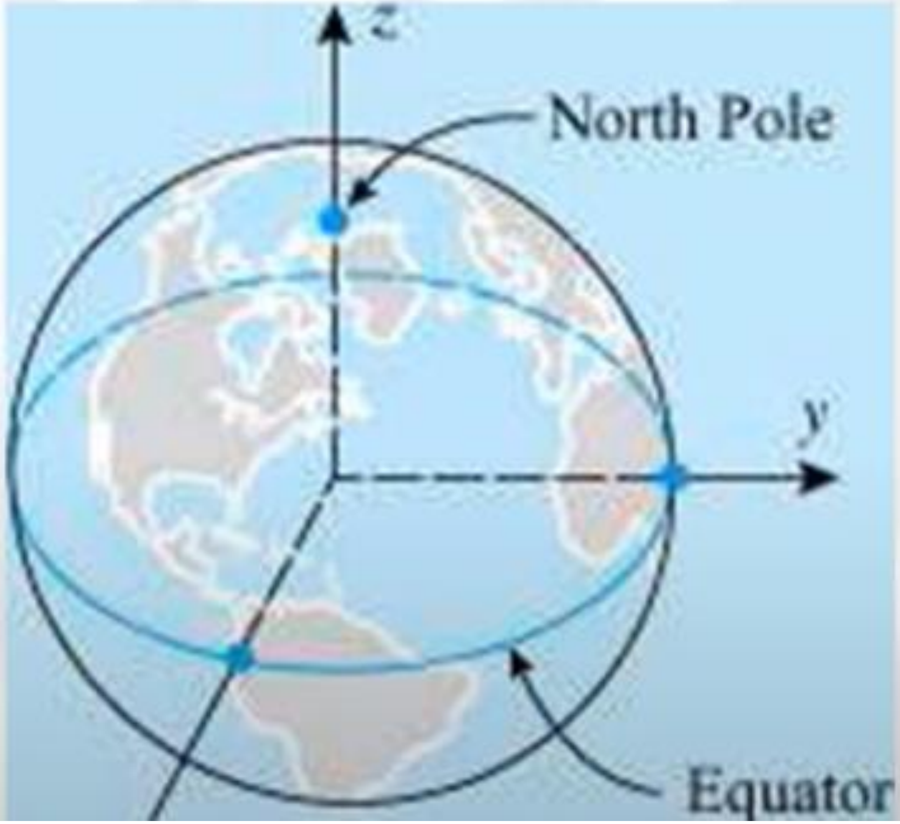
to locate current positions by communicating with a system of satellites.

The system, which is operated by the United States Space Force, nominally uses 31 satellites that orbit the Earth every 12 hours at a height of about 12,540 miles.



An example of a 24 satellite GPS constellation in motion with the Earth rotating. Notice how the number of *satellites in view* from a given point on the Earth's surface changes with time. The point in this example is in Golden, Colorado, USA.

Assume that the Earth is a sphere, and suppose that there is an xyz-coordinate system with its origin at the Earth's center and its z-axis through the North Pole.



GLOBAL POSITIONING SYSTEM….Con'd

Let us assume that relative to this coordinate system a ship is at an unknown point (x, y, z) at some time t.

For simplicity, assume that distances are measured in units equal to the Earth's radius, so that the coordinates of the ship always satisfy the equation

$$x^2 + y^2 + z^2 = 1$$

The GPS identifies the ship's coordinates (x, y, z) at a time t using a triangulation system and computed distances from four satellites. These distances are computed using the speed of light (approximately 0.469 Earth radii per hundredth of a second) and the time it takes for the signal to travel from the satellite to the ship.

For example, if the ship receives the signal at time t and the satellite indicates that it transmitted the signal at time t_0, then the distance d traveled by the signal will be $d = 0.469(t - t_0)$

GLOBAL POSITIONING SYSTEM….Con'd

Suppose that in addition to transmitting the time t_0 , each satellite also transmits its coordinates (x_0,y_0,z_0) at that time, thereby allowing *d* to be computed as

$$d = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

equate the squares of *d* from both equations and round off to three decimal places, then we obtain the second-degree equation

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2= 0.22(t-t_0)$$

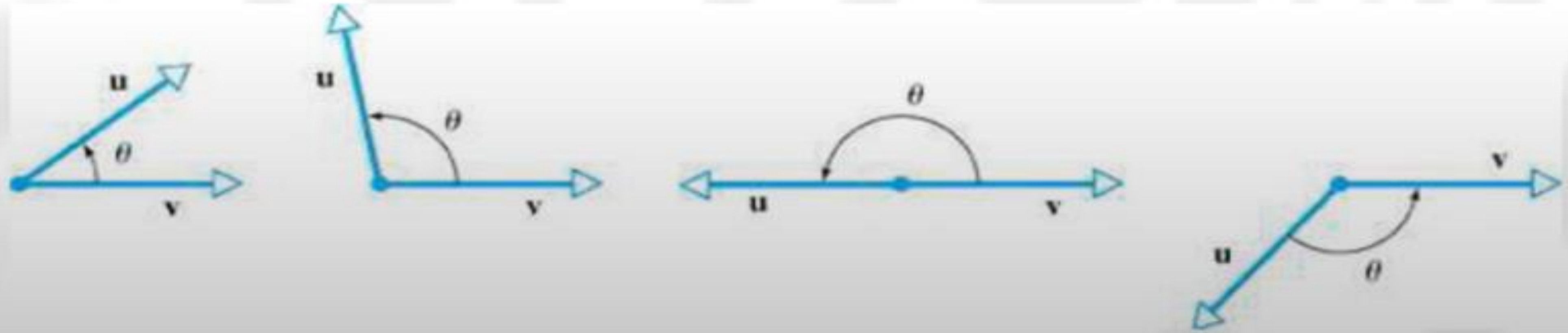
Since there are four different satellites, and we can get an equation like this for each one, we can produce four equations in the unknowns *x*, *y*, *z*, and t_0 . Although these are second-degree equations, it is possible to use these equations and some algebra to produce a system of linear equations that can be solved for the unknowns.

DOT PRODUCT

If **u** and **v** are vectors in 2-space or 3-space and θ is the angle between **u** and **v**, then the dot product or Euclidean inner product **u.v** is defined by

$$\mathbf{u}\cdot\mathbf{v}=\begin{cases}\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta & \text{if } \mathbf{u}\neq\mathbf{0} \text{ and } \mathbf{v}\neq\mathbf{0}\\ 0 & \text{if } \mathbf{u}=\mathbf{0} \text{ or } \mathbf{v}=\mathbf{0}\end{cases}$$

The angle θ between ***u*** and ***v*** satisfies $0 \leq \theta \leq \pi$.



EXAMPLE-1

Example-1 Find **u.v** if the angle between the vectors **u**=(0,0,1) and **v**=(0,2,2) is 45°. Thus

$$\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta=(\sqrt{0^2+0^2+1^2})(\sqrt{0^2+2^2+2^2})\left(\frac{1}{\sqrt{2}}\right)=2$$

COMPONENT FORM OF THE DOT PRODUCT

Let $\mathbf{u}=(u_1,u_2,u_3)$ and $\mathbf{v}=(v_1,v_2,v_3)$ be two nonzero vectors. If θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, then the law of cosines yields

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

Since $\overrightarrow{PQ} = \boldsymbol{v} - \boldsymbol{u}$, we can rewrite above eq. as

$$\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2)$$

or

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2)$$

Substituting $\|\mathbf{u}\|^2 = u_1^2 + u_3^2 + u_3^2, \qquad \|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2$.

COMPONENT FORM OF THE DOT PRODUCT

and $$\|\mathbf{v}-\mathbf{u}\|^2 = (v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2$$

we obtain, after simplifying,

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

FINDING THE ANGLE BETWEEN VECTORS

If ***u*** and ***v*** are nonzero vectors, then angle 'θ' between vectors **u** & **v** can be written as

$$cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}$$

EXAMPLE-1

Example-1 Consider the vectors **u**=(2,-1,1)and **v**=(1,1,2). Find **u.v** and determine the angle θ between ***u*** and ***v***.

Sol: Since

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+u_3v_3=(2)(1)+(-1)(1)+(1)(2)=3$$

For the given vectors we have $||\boldsymbol{u}|| = ||\boldsymbol{v}|| = \sqrt{6}$, therefore,

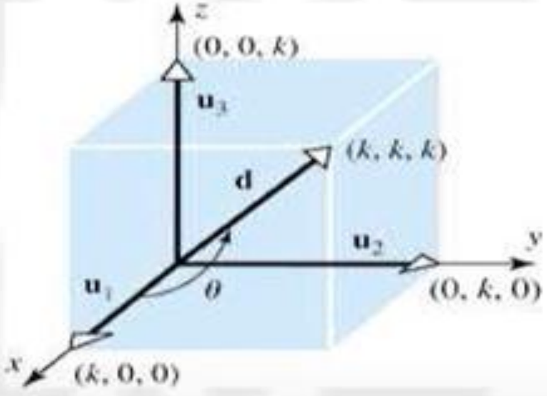
$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||}=\frac{3}{\sqrt{6}\sqrt{6}}=\frac{1}{2}$$

Thus, $\theta = 60°$.

EXAMPLE-2

Example-2 Find the angle between a diagonal of a cube and one of its edges.

Sol: Let k be the length of an edge and introduce a coordinate system as shown in figure



If we let $\mathbf{u}_1$=(k,0,0), $\mathbf{u}_2$=(0,k,0), and $\mathbf{u}_3$(0,0,k), then the vector

$$\mathbf{d} = (k, k, k) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

is a diagonal of the cube.

The angle θ between $\boldsymbol{d}$ and the edge $\mathbf{u}_1$satisfies

EXAMPLE-2 ....Con'd

$$\cos\theta = \frac{\boldsymbol{u}_1 \cdot \boldsymbol{d}}{\|\boldsymbol{u}_1\|\|\boldsymbol{d}\|} = \frac{k^2}{k\sqrt{3k^2}} = \frac{1}{\sqrt{3}}$$

Therefore, $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74°$

THEOREM

Theorem statement

Let **u** and **v** be vectors in 2- or 3-space.

(a) $\boldsymbol{v} \cdot \boldsymbol{v} = ||\boldsymbol{v}||^2$ or $||\boldsymbol{v}|| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}}$

(b)If the vectors **u** and **v** are nonzero and θ is the angle between them, then

θ is acute	if and only if	$\mathbf{u} \cdot \mathbf{v} > 0$
θ is obtuse	if and only if	$\mathbf{u} \cdot \mathbf{v} < 0$
$\theta = \pi / 2$	*if and only if*	$\mathbf{u} \cdot \mathbf{v} = 0$

ORTHOGONAL VECTORS

- Perpendicular vectors are also called ***orthogonal*** vectors.
- Two *nonzero* vectors are orthogonal if and only if their dot product is zero.
- If we agree to consider ***u*** and ***v*** to be perpendicular when either one or both of these vectors is **0**, then *two vectors **u** and **v** are orthogonal* (*perpendicular*) *if and only if* $\mathbf{u}.\mathbf{v} = 0$.
- To indicate that ***u*** and ***v*** are orthogonal vectors, we write $\mathbf{u} \perp \mathbf{v}$.

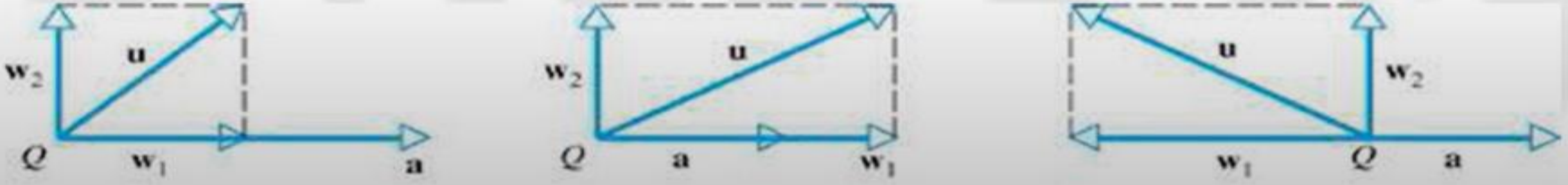
AN ORTHOGONAL PROJECTION

If **u** and **a** are non-zero vectors in which **a** is specific vector as shown in figure (a) below. If they are positioned so that their initial points coincide at a point *Q*, we can decompose the vector ***u*** as follows (Figure below):

Drop a perpendicular from the tip of ***u*** to the line through ***a***, and construct the vector $\mathbf{w}_1$ from *Q* to the foot of this perpendicular. Next form the difference

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$$

As indicated in Figure, the vector $\mathbf{w}_1$ is parallel to ***a***, the vector $\mathbf{w}_2$ is perpendicular to ***a***.



(a) (b) (c)

AN ORTHOGONAL PROJECTION

The vector $\mathbf{w}_1$ is called the ***orthogonal projection of u on a*** or sometimes the ***vector component of u along a***. It is denoted by

$$proj_{\boldsymbol{a}}\boldsymbol{u}$$

The vector $\mathbf{w}_2$ is called the ***vector component of u orthogonal to a***. Since we have $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$, this vector can be written as

$$\boldsymbol{w}_2 = \boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u}$$

EXAMPLE-1

Example-1 Let **u**=(2,-1,3) and **a**=(4,-1,2). Find the vector component of ***u*** along ***a*** and the vector component of ***u*** orthogonal **a**.

Sol: Since $\mathbf{u}\cdot\mathbf{a} = (2)(4) + (-1)(-1) + (3)(2) = 15$

$\|\mathbf{a}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$

Thus the vector component of ***u*** along ***a*** is

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{15}{21}(4,\ -1,\ 2) = \left(\frac{20}{7},\ -\frac{5}{7},\ \frac{10}{7}\right)$$

and the vector component of ***u*** orthogonal to ***a*** is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (2,\ -1,\ 3) - \left(\frac{20}{7},\ -\frac{5}{7},\ \frac{10}{7}\right) = \left(-\frac{6}{7},\ -\frac{2}{7},\ \frac{11}{7}\right)$$

EXAMPLE-2

Example-2 Find a unit vector that is orthogonal to both $\mathbf{u}=(1,0,1)$ and $\mathbf{v}=(0,1,1)$.

Sol: Let $\mathbf{w}=(x,y,z)$ be orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Therefore, $\mathbf{u} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$.

If $\mathbf{u} \cdot \mathbf{w} = 0 \quad \Rightarrow (1,0,1) \cdot (x,y,z) = 0 \quad \Rightarrow x+z = 0 \quad .....(1)$

If $\mathbf{v} \cdot \mathbf{w} = 0 \quad \Rightarrow (0,1,1) \cdot (x,y,z) = 0 \quad \Rightarrow y+z = 0 \quad .....(2)$

$(1) \Rightarrow x = -z$ and $(2) \Rightarrow y = -z$

Let $z = -s \quad \Rightarrow x = s$ and $y = s$

$\therefore \mathbf{w}=(s,s,-s)$

EXAMPLE-2 ....Con'd

Since, $\|\boldsymbol{w}\| = \sqrt{s^2 + s^2 + (-s)^2} \quad \Rightarrow \|\boldsymbol{w}\| = \sqrt{3}s$

Therefore, the unit vector can be written as

$$\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = either \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) or \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

The minus sign in the above equation is extraneous because it yields an angle of $\frac{2\pi}{3}$

EUCLIDEAN *n*-SPACE

- If n is a positive integer, then an ***ordered n-tuple*** is a sequence of n real numbers $(a_1, a_2, \ldots\ldots, a_n)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by R^n .

- Two vectors $\mathbf{u}=(u_1, u_2, \ldots., u_n)$ and $\mathbf{v}=(v_1, v_2, \ldots.., v_n)$ in R^n are called ***equal*** if

$$u_1 = v_1, \quad u_2 = v_2, \ldots, \quad u_n = v_n$$

- The *sum* ***u+v*** is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

- If k is any scalar, the ***scalar multiple*** $k\mathbf{u}$ is defined by

$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

EXAMPLES OF VECTORS IN HIGHER-DIMENSIONAL SPACES

Experimental Data

- A scientist performs experiment and makes *n* numerical measurements each time the experiment is performed.

- Result of each experiment can be regarded as a vector $\mathbf{y} = (y_1, y_2, \ldots\ldots, y_n)$ in R^n.

- $y_1, y_2, \ldots, y_n$ are the measured values.

EXAMPLES OF VECTORS IN HIGHER-DIMENSIONAL SPACES

Electrical Circuits

- A certain kind of processing chip is designed to receive four input voltages and produces three output voltages in response.
- The input voltages can be regarded as vectors in R^4 and the output voltages as vectors in R^3.
- Chip can be viewed as a device that transforms each input vector $\mathbf{v}=(v_1,v_2,\ldots,v_4)$ in R^4 into some output vector $\mathbf{w}=(w_1,w_2,w_3)$ in R^3.

EXAMPLES OF VECTORS IN HIGHER-DIMENSIONAL SPACES

Graphical Images

- Color images are created on computer screens by assigning each pixel (an addressable point on the screen) three numbers that describe the ***hue, saturation***, and ***brightness*** of the pixel.

- A complete color image can be viewed as a set of 5-tuples of the form $v=(x,y,h,s,b)$ in which x and y are the screen coordinates of a pixel and h, s, and b are its hue, saturation, and brightness.

EXAMPLES OF VECTORS IN HIGHER-DIMENSIONAL SPACES

Mechanical Systems

- If six particles move along the same coordinate line so that at time *t* their coordinates can be considered as $x_1, x_2, \ldots, x_6$ and their velocities are as $v_1, v_2, \ldots, v_6$, respectively.
- This information can be represented by the vector

$$\mathbf{v} = (x_1, x_2, \ldots, x_6, v_1, v_2, \ldots, v_6, t) \text{ in } R^{13}.$$

- This vector is called the ***state*** of the particle system at time *t*.

EUCLIDEAN INNER PRODUCT

If $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ and $\mathbf{v}=(v_1,v_2,\ldots,v_n)$ are any vectors in R^n, then the ***Euclidean inner product*** ***u.v*** is defined by

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+\ldots+u_nv_n$$

EXAMPLE 1 The Euclidean inner product of the vectors

$$\mathbf{u}=(-1,3,5,7) \quad \text{and} \quad \mathbf{v}=(5,-4,7,0)$$

in R^4 is

$$\mathbf{u}\cdot\mathbf{v}=(-1)(5)+(3)(-4)+(5)(7)+(7)(0)=18$$

APPLICATION OF DOT PRODUCTS TO ISBNs

Most books published since 1970 have been assigned a unique 10-digit number called an ***International Standard Book Number*** or ISBN.

- The first nine digits of this number are split into three groups
  - First group representing the country or group of countries in which the book originates.
  - Second identifying the publisher.
  - Third assigned to the book title itself.
- The tenth and final digit, called a ***check digit***, is computed from the first nine digits and is used to ensure that an electronic transmission of the ISBN, say over the Internet, occurs without error.

PROCEDURE TO DETERMINE CHECK DIGIT

Consider the first nine digits of the ISBN as a vector ***b*** in R^9 , and let ***a*** be the vector **a** = (1,2,3,...,9)

Then the check digit *c* is computed using the following procedure:

1). Form the dot product **a**.**b**.

2). Divide **a**.**b** by 11, thereby producing a remainder *c* that is an integer between 0 and 10, inclusive. The check digit is taken to be *c*, with the provision that c=10 is written as X to avoid double digits.

EXAMPLE-1 Consider the ISBN of the brief edition of *Calculus*, sixth edition, by Howard Anton is

0-471-15307-9

which has a check digit of 9.

The check digit is consistent with the first nine digits of the ISBN, since

$$\mathbf{a} \cdot \mathbf{b} = (1, 2, 3, 4, 5, 6, 7, 8, 9) \cdot (0, 4, 7, 1, 1, 5, 3, 0, 7) = 152$$

Dividing 152 by 11 produces a quotient of 13 and a remainder of 9, so the check digit is c=9.

If an electronic order is placed for a book with a certain ISBN, then the warehouse can use the above procedure to verify that the check digit is consistent with the first nine digits, thereby reducing the possibility of a costly shipping error.

NORM AND DISTANCE IN EUCLIDEAN *N*-SPACE

The ***Euclidean norm*** (or ***Euclidean length***) of a vector $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ in R^n is defined by

$$\|\mathbf{u}\| = (\mathbf{u}\cdot\mathbf{u})^{1/2} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Similarly, the ***Euclidean distance*** between the points $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ and $\mathbf{v} = (v_1,v_2,\ldots,v_n)$ in R^n is defined by

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2}$$

FINDING NORM AND DISTANCE IN EUCLIDEAN SPACE

EXAMPLE-1 If **u**=(1,3,-2,7) and **v**=(0,7,2,2), then in the Euclidean space R^4,

$$\|\mathbf{u}\| = \sqrt{(1)^2 + (3)^2 + (-2)^2 + (7)^2} = \sqrt{63} = 3\sqrt{7}$$

and

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

PROPERTIES OF LENGTH IN R^n

If u and v are vectors in R^n , and k is any scalar then

(1) $\|\boldsymbol{u}\| \geq 0$

(2) $\|\boldsymbol{u}\| = 0 \; if \;\; and \; only \; if \; \boldsymbol{u} = 0$

(3) $\|k\boldsymbol{u}\| = |k|\|\boldsymbol{u}\|$

(4) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ (Triangular Inequality)

PROPERTIES OF DISTANCE IN R^n

If **u**, **v** and **w** are vectors in R^n , then

(1) $d(\mathbf{u},\mathbf{v}) \geq 0$

(2) $d(\mathbf{u},\mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$

(4) $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$ (Triangular Inequality)