LECTURE-3(C)

COURSE TITLE

LINEAR ALGEBRA

&

GEOMETRY

(MT-272)

DEPARTMENT OF TELECOMMUNICATION ENGINEERING

COURSE TEACHER

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VECTOR SPACES

Let V be a set on which two operations (addition and scalar multiplication) are defined. **If the following ten axioms are satisfied** for every element **u**, **v**, and **w** in V and every scalar (real number) c and d, then ***V* is called a vector space**, and the **elements** in ***V*** are called **vectors**

Addition:

(1) $\mathbf{u}+\mathbf{v}$ is in V (closure property under vector addition)

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ (commutative property of vector addition)

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$ (associative property of vector addition)

(4) V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u}+\mathbf{0} = \mathbf{u}$

(additive identity property)

VECTOR SPACES Con'd

(5) For every **u** in V, there is a vector in V denoted by –**u** such that **u**+(–**u**) = **0** (additive inverse property)

Scalar multiplication:

(6) $c\mathbf{u}$ is in V (closure under scalar multiplication)

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$ (distributive property of scalar multiplication over vector addition)

(8) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$ (distributive property of scalar multiplication over real-number addition)

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$ (associative property of multiplication)

(10) $1(\mathbf{u}) = \mathbf{u}$ (multiplicative identity property)

VECTOR SPACES Con'd

- Notes:

A vector space consists of four entities:

a set of vectors, a set of real-number scalars, and two operations

V: nonempty set

c: scalar

$+(\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v}$: vector addition

$\cdot(c, \mathbf{u}) = c\mathbf{u}$: scalar multiplication

$(V, \ +, \ \cdot)$ is called a vector space

※ The set V together with the definitions of vector addition and scalar multiplication satisfying the above ten axioms is called a vector space

EXAMPLES OF VECTOR SPACES

(1) n-tuple space: R^n

$$(u_1, u_2, \cdots u_n) + (v_1, v_2, \cdots v_2) = (u_1 + v_1, u_2 + v_2, \cdots u_n + v_n)$$ (standard vector addition)

$$k(u_1, u_2, \cdots u_n) = (ku_1, ku_2, \cdots ku_n)$$ (standard scalar multiplication for vectors)

(2) Matrix space : $V = M_{m\times n}$

(the set of all $m\times n$ matrices with real-number entries)

Ex: $(m = n = 2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ (standard matrix addition)

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ (standard scalar multiplication for matrices)

EXAMPLES OF VECTOR SPACES

(3) *n*-th degree or less polynomial space : $V = P_n$

(the set of all real-valued polynomials of degree n or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$ (standard polynomial addition)

$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$ (standard scalar multiplication for polynomials)

※ By the fact that the set of real numbers is closed under addition and multiplication, it is straightforward to show that P_n satisfies the ten axioms and thus is a vector space

(4) Continuous function space : $V = C(-\infty,\infty)$

(the set of all real-valued continuous functions defined on the entire real line)

$$(f+g)(x)=f(x)+g(x)$$ (standard addition for functions)

$$(kf)(x)=kf(x)$$ (standard scalar multiplication for functions)

※ By the fact that the sum of two continuous function is continuous and the product of a scalar and a continuous function is still a continuous function, $C(-\infty,\infty)$ is a vector space

EXAMPLE-1

Example-1: Show that the set *V* of all 2×2 matrices with real entries is a vector space with the following operations.

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} + \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 & u_2 + v_2 \\ u_3 + v_3 & u_4 + v_4 \end{bmatrix}$$

Addition Operation

$$\boldsymbol{cu} = \boldsymbol{c} \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} = \begin{bmatrix} cu_1 & cu_2 \\ cu_3 & cu_4 \end{bmatrix}$$

Scalar Multiplication Operation

Solution: In this example we will find it convenient to verify the axioms in the following order: 1, 6, 2, 3, 7, 8, 9, 4, 5, and 10. Let,

$$\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

To prove Axiom 1, we must show that **u+v** is an object in *V*; that is, we must show that **u+v** is a 2×2 matrix. But this follows from the definition of matrix addition, since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

Axiom-1 Holds

EXAMPLE-1 Con'd

Similarly, Axiom-6 holds because for any real number k, we have

$$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

so **ku** is a 2 × 2 matrix and consequently is an object in V.

Axiom 2 also holds since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

Similarly, Axiom 3, 7, 8, and 9 also hold.

To prove Axiom 4, let us consider an object in V i.e. $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

Axiom-4 Holds

and similarly **u**+0=u holds

EXAMPLE-1 Con'd

To prove Axiom 5, we must show that each object ***u*** in *V* has a negative **–u** such that **u**+(-**u**)=**0** and (-**u**)+**u**=**0**. This can be done by defining the negative of **u** to be

$$-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$

With this definition,

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

Axiom-5 Holds

and similarly (-**u**)+**u**=**0** also holds.

Finally, Axiom 10 is a simple computation:

$$1\mathbf{u} = 1\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

Axiom-10 Holds

Since all axioms are true, hence, all 2×2 matrices with real entries is a vector space.

SUBSPACES

- Subspace:

$(V,+,\cdot)$: a vector space

$$\left.\begin{array}{l} W \neq \Phi \\ W \subseteq V \end{array}\right\} : \text{a nonempty subset of } V$$

$(W,+,\cdot)$: The nonempty subset W is called a subspace **if** W **is a vector space** under the operations of addition and scalar multiplication defined on V

- Trivial subspace:

Every vector space V has at least two subspaces

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V (It satisfies the ten axioms)

(2) V is a subspace of V

※ Any subspaces other than these two are called proper (or nontrivial) subspaces

SUBSPACES Con’d

- Theorem 1: Test whether a nonempty subset being a subspace

  If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold

  (1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W

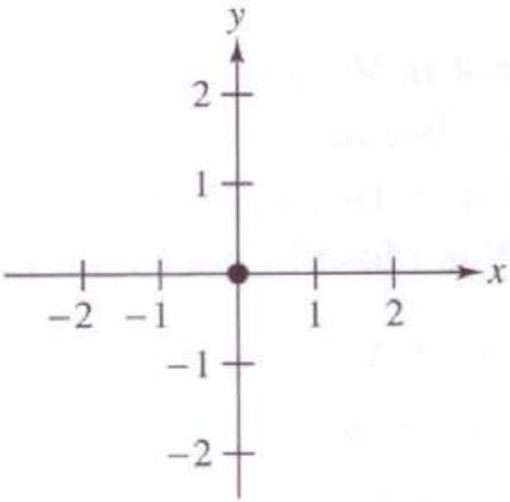
  (2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W

SUBSPACES OF R^2

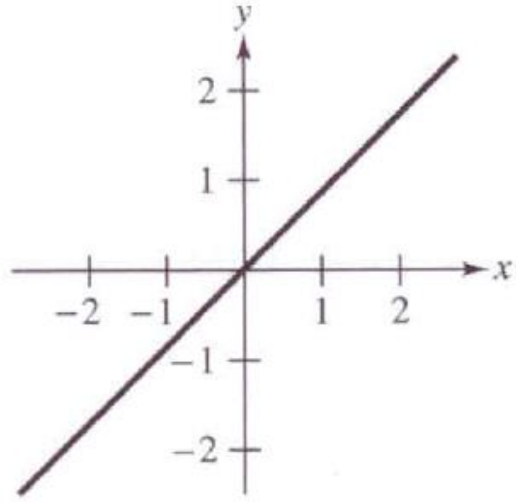
(1) W consists of the *single point* $\mathbf{0} = (0, 0)$

(2) W consists of all points on a *line* passing through the origin

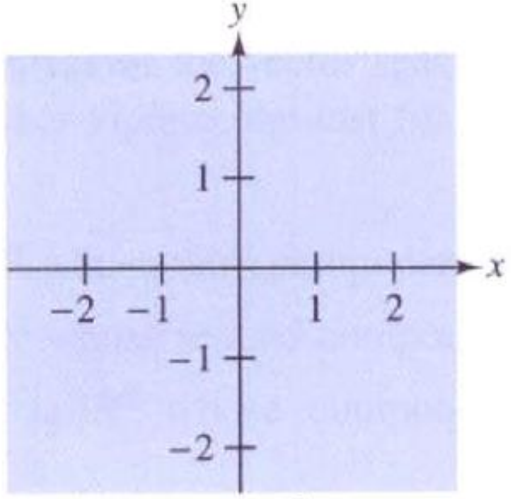
(3) R^2



$W = \{(0, 0)\}$



$W =$ all points on a line passing through the origin



$W = R^2$

SUBSPACES OF R^3

(1) W consists of the *single point* $\mathbf{0}=(0,0,0)$

(2) W consists of all points on a *line* passing through the origin

(3) W consists of all points on a *plane* passing through the origin

(4) R^3