EIGENVALUES AND EIGENVECTORS

- Linear equations Ax = b come from steady state problems. Eigen values have their greatest importance in dynamic problems. The solution of du/dt= Au is changing with time—growing or decaying or oscillating.
- Almost all vectors change direction, when they are multiplied by square matrix A.
- Certain exceptional vectors "x" are in the same direction as Ax. Those are the "Eigen vectors".
- The basic equation is $Ax = \lambda x$. The number λ is an "Eigen value" of A.
- The eigen value tells whether the special vector "x" is stretched or shrunk or reversed or left unchanged—when it is multiplied by A.

Note:

The prefix *eigen-* is adopted from the German word "eigen" for "own" in the sense of a characteristic description (that is why the eigenvectors are sometimes also called characteristic vectors, and, similarly, the eigenvalues are also known as characteristic values).

EIGENVALUES AND EIGENVECTORS

- Eigenvalue problem:

If A is an $n \times n$ matrix, do there exist nonzero vectors $\boldsymbol{x}$ in R^n such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$?

- Eigenvalue and eigenvector:

A : an $n \times n$ matrix

λ : a scalar

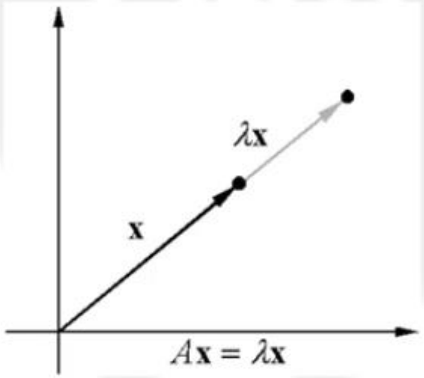
$\boldsymbol{x}$: a nonzero vector in R^n

Eigenvalue

$$Ax = \lambda x$$

Eigenvector

- Geometrical Interpretation



- Ex 1: Verifying eigenvalues and eigenvectors

$$A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix} \quad x_1=\begin{bmatrix}1\\ 0\end{bmatrix} \quad x_2=\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$Ax_1=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}2\\ 0\end{bmatrix}=\overset{\text{Eigenvalue}}{2}\underset{\text{Eigenvector}}{\begin{bmatrix}1\\ 0\end{bmatrix}}=2x_1$$

$$Ax_2=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}=-1\begin{bmatrix}0\\ 1\end{bmatrix}=(-1)x_2$$

Theorem 1: (The eigenspace of A corresponding to λ)

If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ together with the zero vector is a subspace of R^n. This subspace is called the eigenspace of λ.

Pf:

x_1 and x_2 are eigenvectors corresponding to λ

$$(i.e.\ \ Ax_1 = \lambda x_1,\ \ Ax_2 = \lambda x_2)$$

$$(1)\ A(x_1 + x_2) = Ax_1 + Ax_2 = \lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2)$$

$(i.e.\ \ x_1 + x_2$ is an eigenvector corresponding to $\lambda)$

$$(2)\ A(cx_1) = c(Ax_1) = c(\lambda x_1) = \lambda(cx_1)$$

$(i.e.\ \ cx_1$ is an eigenvector corresponding to $\lambda)$

AN EXAMPLE OF EIGENSPACES IN THE PLANE

Ex 2:

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol:

If $\quad \mathrm{v} = (x, y)$

$$A\mathrm{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

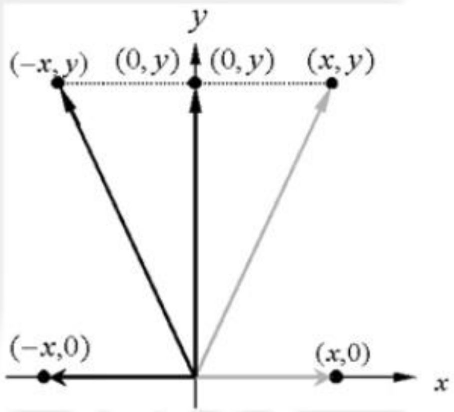
For a vector on the x-axis Eigenvalue $\lambda = -1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1 \begin{bmatrix} x \\ 0 \end{bmatrix}$$

For a vector on the y-axis   Eigenvalue $\lambda = 1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1\begin{bmatrix} 0 \\ y \end{bmatrix}$$

Geometrically, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection in the y-axis.



The eigenspace corresponding to $\lambda_1 = -1$ is the x-axis.
The eigenspace corresponding to $\lambda_2 = 1$ is the y-axis.

THEOREM-2

Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$

Let A is an $n\times n$ matrix.

(1) An eigenvalue of A is a scalar λ such that $\det(\lambda \mathrm{I} - A) = 0$

(2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda \mathrm{I} - A)x = 0$.

- Note:

$$Ax = \lambda x \quad \Rightarrow \quad (\lambda \mathrm{I} - A)x = 0 \qquad \text{(homogeneous system)}$$

$(\lambda \mathrm{I} - A)x = 0$ has nonzero solutions iff $\det(\lambda \mathrm{I} - A) = 0$

- Characteristic equation of A:

$$\det(\lambda \mathrm{I} - A) = 0$$

FINDING EIGENVALUES AND EIGENVECTORS

Ex 3:

Find the eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow \lambda = -1, -2$$

Eigenvalue: $\lambda_1 = -1, \lambda_2 = -2$

In Augmented Matrix

$$(\lambda I - A)x = 0 \quad \Rightarrow \begin{bmatrix} \lambda - 2 & 12 & 0 \\ -1 & \lambda + 5 & 0 \end{bmatrix} \text{.......(a)}$$

$(1)\lambda_1 = -1$

$$\text{(a)} \Rightarrow \begin{bmatrix} -3 & 12 & 0 \\ -1 & 4 & 0 \end{bmatrix} \xrightarrow{G-J} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, t \neq 0$$

$(2)\lambda_2 = -2$

$$\text{(a)} \Rightarrow \begin{bmatrix} -4 & 12 & 0 \\ -1 & 3 & 0 \end{bmatrix} \xrightarrow{G-J} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, t \neq 0$$

FINDING EIGENVALUES AND EIGENVECTORS

Ex 4:

Find the eigenvalues and corresponding eigenvectors for the matrix A.

What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol: : Characteristic equation

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

In Augmented Matrix

$$(\lambda I - A)x = 0 \quad \Rightarrow \begin{bmatrix} \lambda - 2 & -1 & 0 & 0 \\ 0 & \lambda - 2 & 0 & 0 \\ 0 & 0 & \lambda - 2 & 0 \end{bmatrix} \text{.......(a)}$$

The eigenspace of A corresponding to : $\lambda = 2$

$$(a)\text{?}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{G-J} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,s,t \neq 0$$

$$\left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s,t \in R \right\} \textit{the eigen space A corresponds to ? } = 2$$

- Ex 5: Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Sol: Characteristic equation

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-1 & -5 & 10 \\ -1 & 0 & \lambda-2 & 0 \\ -1 & 0 & 0 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-1)^2(\lambda-2)(\lambda-3) = 0$$

Eigenvalue: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

$$(1)\lambda_1 = 1$$

$$\Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \ s,t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 1$

$$(2)\lambda_2 = 2$$

$$\Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \quad t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 2$

$(3)\lambda_3 = 3$

$$\Rightarrow (\lambda_3 \mathrm{I} - A)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

Theorem 3: (Eigenvalues for triangular matrices)

If A is an $n\times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- Ex 6: (Finding eigenvalues for diagonal and triangular matrices)

$$(a) A=\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \quad (b) A=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$(a)\ |\lambda \mathrm{I}-A|=\begin{vmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ -5 & -3 & \lambda+3 \end{vmatrix}=(\lambda-2)(\lambda-1)(\lambda+3)$$

$$\lambda_1=2,\ \lambda_2=1,\ \lambda_3=-3$$

$$(b)\ \lambda_1=-1,\ \lambda_2=2,\ \lambda_3=0,\ \lambda_4=-4,\ \lambda_5=3$$

Eigenvalues and eigenvectors of linear transformations:

A number λ is called an eigenvalue of a linear transformation $T : V \to V$ if there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda \mathbf{x}$. The vector $\mathbf{x}$ is called an eigenvector of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is called the eigenspace of λ.

▪ Ex 7: (Finding eigenvalues and eigenspaces)

Find the eigenvalues and corresponding eigenspaces

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Sol:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda + 2)^2(\lambda - 4)$$

eigenvalues $\lambda_1 = 4, \lambda_2 = -2$

The eigenspaces for these two eigenvalues are as follows.

$B_1 = \{(1, 1, 0)\}$ Basis for $\lambda_1 = 4$

$B_2 = \{(1, -1, 0), (0, 0, 1)\}$ Basis for $\lambda_2 = -2$

- Notes:

(1) Let $T: R^3 \rightarrow R^3$ be the linear transformation whose standard matrix is A in Ex. 7 and let B' be the basis of R^3 made up of three linear independent eigenvectors found in 7. Then A', the matrix of T relative to the basis B', is diagonal.

$$B' = \{(1,1,0), (1,-1,0), (0,0,1)\}$$

Eigenvectors

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues

(2) The main diagonal entries of the matrix A' are the eigenvalues of A.

- Notes:

(1) Let $T: R^3 \to R^3$ be the linear transformation whose standard matrix is A in Ex. 7 and let B' be the basis of R^3 made up of three linear independent eigenvectors found in 7 . Then A', the matrix of T relative to the basis B', is diagonal.

$$B' = \{(1,1,0),(1,-1,0),(0,0,1)\}$$

Eigenvectors

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues

(2) The main diagonal entries of the matrix A' are the eigenvalues of A.

EIGENVALUES AND INVERTIBILITY

Theorem 4:

A square matrix A is invertible if and only if $\lambda=0$ is not an eigenvalue of A.

EIGENVALUES OF THE POWER OF A MATRIX

Theorem 5:

If k is a positive integer, λ is an eigenvalue of a matrix A, and x is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and x is a corresponding eigenvector.

DIAGONALIZATION

Given a linear transformation, we are interested in how to write it as a matrix. We are especially interested in the case that the matrix is written with respect to a basis of eigenvectors, in which case it is a particularly nice matrix

Now suppose we are lucky, and we have $L: V \rightarrow V$, and the basis $\{v_1, v_2, \ldots, v_n\}$ is a set of linearly independent eigenvectors for L, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Then: $L(v_1) = \lambda_1 v_1, \qquad L(v_2) = \lambda_2 v_2, \ldots, L(v_n) = \lambda_n v_n$

As a result, the matrix of L in the basis of eigenvectors is diagonal

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

Suppose that V is any n-dimensional vector space. We call a linear transformation $L: V \rightarrow V$ diagonalizable if there exists a collection of n linearly independent eigenvectors for L. In other words, L is diagonalizable if there exists a basis for V of eigenvectors for L.

Diagonalization

- Diagonalization problem:

For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix:

A square matrix A is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(P diagonalizes A)

- Notes:

(1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called similar.

(2) The eigenvalue problem is related closely to the diagonalization problem