LECTURE-4(A)

COURSE TITLE

LINEAR ALGEBRA

&

GEOMETRY

(MT-272)

DEPARTMENT OF TELECOMMUNICATION ENGINEERING

COURSE TEACHER

DR. FAREED AHMAD

LINEAR COMBINATION IN A VECTOR SPACE

A vector $\mathbf{u}$ in a vector space V is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ in V if $\mathbf{u}$ can be written in the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k,$$

where $c_1, c_2, \ldots, c_k$ are real-number scalars

EXAMPLE-1

Ex 1: Every vector **v** = (a, b, c) in R^3 is expressible as a linear combination of the standard basis vectors

$$\hat{\imath} = (1, 0, 0),\ \hat{\jmath} = (0, 1, 0),\ \hat{k} = (0, 0, 1)$$

Since **v** = (a, b, c)

or **v** = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)

or **v** = $a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

It means **v** is a linear combination of $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$

EXAMPLE-2

Ex 2: Finding a linear combination

$$\mathbf{v}_1=(1,2,3)\quad \mathbf{v}_2=(0,1,2)\quad \mathbf{v}_3=(-1,0,1)$$

Prove (a) $\mathbf{w}=(1,1,1)$ is a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

(b) $\mathbf{w}=(1,-2,2)$ is not a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

Sol: (a) $\mathbf{w}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$ ......(1)

$$(1,1,1)=c_1(1,2,3)+c_2(0,1,2)+c_3(-1,0,1)$$
$$=(c_1-c_3,\ 2c_1+c_2,\ 3c_1+2c_2+c_3)$$

On comparing both sides we get

$$\begin{aligned} c_1 \qquad\quad -c_3 &= 1\\ \Rightarrow\quad 2c_1+\ c_2 \qquad\quad &= 1\\ 3c_1+2c_2+c_3 &= 1 \end{aligned}$$

EXAMPLE-2 ...Con'd

The Augmented Matrix $\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$

$$\xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Retransform into linear equations,

$$c_1 - c_3 = 1 \quad \& \quad c_2 + 2c_3 = -1$$

$$\Rightarrow c_1 = 1 + t,\ c_2 = -1 - 2t,\ c_3 = t$$ (infinitely many solutions)

Putting values of c's in eq.(1) , $\mathbf{w} = (1+t)\,\mathbf{v}_1 + (-1-2t)\,\mathbf{v}_2 + t\,\mathbf{v}_3$

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 \qquad \overset{t=2}{\Rightarrow} \mathbf{w} = 3\mathbf{v}_1 - 5\mathbf{v}_2 + 2\mathbf{v}_3$$

$$\vdots$$

EXAMPLE-2 ...Con'd

Sol: (b) For linear combination, vector equation is $\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$ (1)

Since $\mathbf{w} = (1,-2,2)$

$\therefore (1,-2,2) = c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1,0,1)$

$(1,-2,2) = (c_1 - c_3,\ 2c_1 + c_2,\ 3c_1 + 2c_2 + c_3)$

On comparing both sides we get system of linear eq.

$$\begin{aligned} c_1 \quad\quad - c_3 &= 1 \\ 2c_1 + c_2 \quad\quad &= -2 \\ 3c_1 + 2c_2 + c_3 &= 2 \end{aligned}$$

The Augmented Matrix $\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{bmatrix}$

$$\xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 2 & 4 & -1 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

EXAMPLE-2 ...Con'd

From last row of matrix we have

$$0c_1+0c_2+0c_3 = 7$$

$\Rightarrow \quad 0 = 7$ Mathematically it is false statement.

$\Rightarrow$ System is inconsistent.

$\Rightarrow$ eq. (1) is false

Consequently , '**w**' can not be written as a linear combination of vectors $\mathbf{v_1}$, $\mathbf{v_2}$, & $\mathbf{v_3}$.

EXAMPLE-3

Ex 3: Express **w** = -9 -7x -15x^2 as linear combination of $\mathbf{p_1}$=2+x+4x^2, $\mathbf{p_2}$=1-x+3x^2, and $\mathbf{p_3}$=3+2x+5x^2

Sol: For linear combination, the vector equation is $\mathbf{w} = c_1\mathbf{p_1} + c_2\mathbf{p_2} + c_3\mathbf{p_3}$……(1)

Since **w** = -9 -7x -15x^2

$$\therefore -9 -7x -15x^2 = c_1(2+x+4x^2) + c_2(1-x+3x^2) + c_3(3+2x+5x^2)$$

$$\text{Or } -9 -7x -15x^2 = (2c_1+c_2+3c_3) + (c_1-c_2+2c_3)x + (4c_1+3c_2+5c_3)x^2$$

On comparing both sides we get system of linear eq.

$$2c_1 + c_2 + 3c_3 = -9$$
$$c_1 - c_2 + 2c_3 = -7$$
$$4c_1 + 3c_2 + 5c_3 = -15$$

The Augmented Matrix $\Rightarrow \begin{bmatrix} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{bmatrix}$

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{bmatrix} \xrightarrow[-4R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 0 & -\frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

EXAMPLE-3 ...Con'd

$$\xrightarrow{-\frac{2}{3}R_2} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 0 & 1 & -\frac{1}{3} & -\frac{5}{3} \\ 0 & 1 & -1 & 3 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 0 & 1 & -\frac{1}{3} & -\frac{5}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{4}{3} \end{bmatrix} \xrightarrow{-\frac{3}{2}R_3} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{2} \\ 0 & 1 & -\frac{1}{3} & -\frac{5}{3} \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[\substack{-\frac{1}{3}R_3+R_2}]{-\frac{3}{2}R_3+R_1} \begin{bmatrix} 1 & \frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Retransform into linear equations,

$$c_1 = -2 \quad , c_2 = 1 \quad \& \quad c_3 = -2$$

Therefore, $\mathbf{w} = -2\mathbf{p_1} + \mathbf{p_2} + -2\mathbf{p_3}$ Or $-9 - 7x - 15x^2 = -2\mathbf{p_1} + \mathbf{p_2} + -2\mathbf{p_3}$

Hence, **w** is linear combination of $\mathbf{p_1}$, $\mathbf{p_2}$, & $\mathbf{p_3}$.

EXAMPLE-4

Ex 4: Determine whether $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a linear combination of $\mathbf{A} = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, & $\mathbf{C} = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$ or not?

Sol: For linear combination, the vector equation is

$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = c_1\mathbf{A} + c_2\mathbf{B} + c_3\mathbf{C} \quad ......(1)$$

or $$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = c_1\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + c_2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c_3\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

or $$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = \begin{bmatrix} 4c_1 & 0 \\ -2c_1 & -2c_1 \end{bmatrix} + \begin{bmatrix} c_2 & -c_2 \\ 2c_2 & 3c_2 \end{bmatrix} + \begin{bmatrix} 0 & 2c_3 \\ c_3 & 4c_3 \end{bmatrix}$$

or $$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = \begin{bmatrix} 4c_1 + c_2 & -c_2 + 2c_3 \\ -2c_1 + 2c_2 + c_3 & -2c_1 + 3c_2 + 4c_3 \end{bmatrix}$$

EXAMPLE-4 ...Con'd

On comparing corresponding elements of matrices both sides we get system of linear eq.

$$\begin{aligned} 4c_1 + c_2 \quad &= 6 \\ -c_2 + 2c_3 &= -8 \\ -2c_1 + 2c_2 + c_3 &= -1 \\ -2c_1 + 3c_2 + 4c_3 &= -8 \end{aligned}$$

The Augmented Matrix is

$$\begin{bmatrix} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{bmatrix} \xrightarrow{G.-J.\,Elim.} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From last row of matrix we have $c_1 = 1$, $c_2 = 2$ & $c_3 = -3$

Eq.(1)$\Rightarrow \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = \mathbf{A} + 2\mathbf{B} - 3\mathbf{C}$

Hence, $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is linear combination of **A**, **B**, & **C.**