Random Number Generation
&
Random Variate Generation

Lecture # 26

Lecture by Engr. Sidra

Approaches to Generate Random Variates

Inverse Transform

- Simplest
- Algorithm:
  1. Generate U~U(0,1)
  2. Set $X = F^{-1}(U)$ and return
- Inverting F might be easy or difficult in which case numerical methods might be necessary.
- Problems with I.T approach
  - Must invert CDF, which may be difficult
  - May not be fastest or simplest for given distribution.

Advantages

- Reduce variability in estimates of effects in faster machine
- It makes correlation as strong as possible in comparison with other variate generation methods.

I.T for Exponential Distribution

- **Step 1**: Compute the cdf of the desired random variable *X*.
  - For the exponential distribution, cdf $F(x) = 1 - e^{-\lambda x}$, $x \ge 0$.
- **Step 2**: Set *F*(*x*)=*R* on the range of *X*.
  - For the exponential distribution, $1 - e^{-\lambda X} = R$, $x \ge 0$.
- **Step 3**: Solve the equation *F*(*x*)=*R* for *X* in terms of *R*.
  - For the exponential distribution, the solution proceeds as follows:

$$1 - e^{-\lambda X} = R$$

$$e^{-\lambda X} = 1 - R$$

$$-\lambda X = \ln(1 - R)$$

$$X = -\frac{1}{\lambda}\ln(1 - R)$$

- **Step 4**: Generate (as needed) random variables R1,R2,... and compute the desired random variates by $X_i = F^{-1}(R_i)$
  - For the exponential case

$$X_i = -\frac{1}{\lambda}\ln(1 - R_i) \quad \text{or} \quad X_i = -\frac{1}{\lambda}\ln R_i$$

Approaches to Generate Random Variates

Composition

- Used when distribution function F from which we wish to sample can be expressed as a convex combination of other distribution functions.
- Convex combination: linear combination of points where all coefficients are non-negative and sum to 1.
- Algorithm:
  1. Generate positive random integer J

     such that P(J=j) = Pj for j=1,2,3...
  2. Given that J=j, generate X with distribution function Fj and return.

     F(x) = $\sum_{j=1}^{\infty} Fj(x)Pj$

Composition

- To find Fjs from which generation is easy and fast.
- Sometimes can use geometry of distribution to suggest a decomposition
- Distribution function as weighted sum of other distribution functions.

Approaches to Generate Random Variates

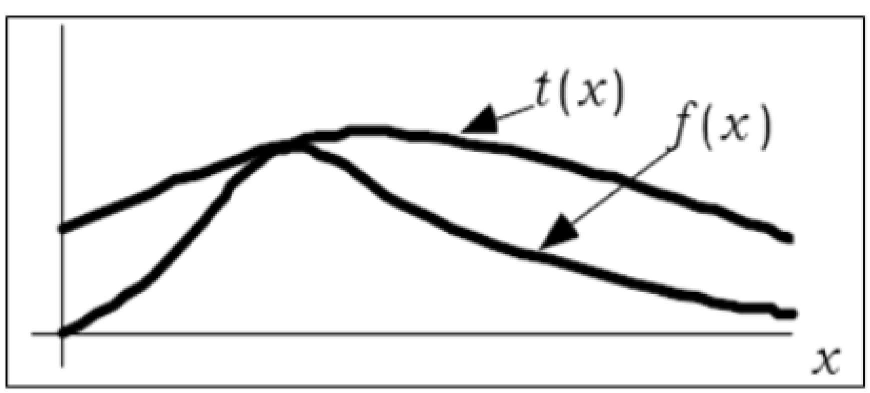
Convolution

- X can be expressed as a sum of other random variables.
- Algorithm:
  1. Generate $Y_1, Y_2, \ldots Y_m$ independently from their distribution
  2. Return $X = Y_1 + Y_2 \ldots Y_m$, where Y_j's are IID and m is fixed and finite.
- Random variables itself as a sum of other random variables.
- Depending on particular parameters of distribution, it may not be an efficient way.

Approaches to Generate Random Variates

Acceptance- Rejection

- Specify a function t(x) that majorize f(x) i.e. t(x) ≥ f(x) for all x.
  - c = $\int_{-\infty}^{\infty} t(x)dx \ \geq 1$
  - r(x) = $\frac{t(x)}{c}$ $for\ all\ c\ (\ c\ is\ density\ function)$



- Algorithm:
  1. Generate Y having density r.
  2. Generate U ~ U(0, 1)
  3. If U ≤ $\frac{f(y)}{t(y)}$, set X = Y and return or go back.
- Efficiency of this technique depends on being able to minimize number of rejections.

Acceptance- Rejection

Suppose that an analyst needed to devise a method for generating random variates, X, uniformly distributed between 1/4 and 1. One way to proceed would be to follow these steps:

Step 1. Generate a random number R.

Step 2a. If $R \geq 1/4$, accept $X = R$, then go to Step 3.

Step 2b. If $R < 1/4$, reject R, and return to Step 1.

Step 3. If another uniform random variate on $[1/4, 1]$ is needed, repeat the procedure beginning at Step 1. If not, stop.