Stochastic Processes

A stochastic process $N(t)$ is said to be a counting process if $N(t)$ counts the total number of events that have occurred up to time t.

i.e. it must satisfy;

1.) $N(t) \geq 0 \quad \forall t \geq 0$.

2) $N(t)$ takes only integer values

3) $N(t)$ is monotonically increasing i.e, if $s < t$, then $N(s) \leq N(t)$

4) $N(t) - N(s)$ is equal to the number of events that have occurred in the interval (s, t)

A counting process is said to have

1) <u>Independent increments</u>; if the no. of events that occur in disjoint time intervals are independent.

2) <u>Stationary increments</u>; if the no of events in the interval $(s, s+t)$ i.e. $N(s, s+t)$ has the same distribution as $N(0, t)$ $\forall s, t \geq 0$.

which means

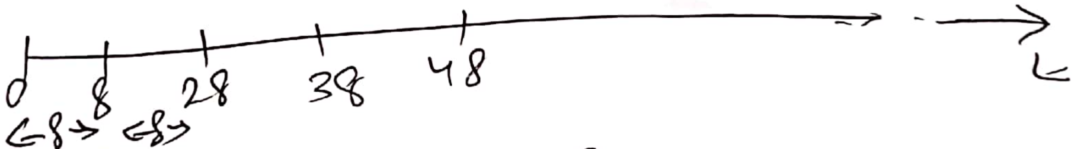
$$\left.\begin{array}{l} \text{length of } (s, [s+t]) \to t \\ \text{length of } (0, t) \to t \end{array}\right\} \text{same value}$$

Counting Process Phenomenon:

suppose that we would like to model the arrival of events that happen completely at random at a rate λ per unit time.

At time $t=0$ we have no arrivals yet so $N(0)=0$

we now divide the ~~half~~ line $(0,\infty)$ to tiny sub-intervals to a time slot of length δ.



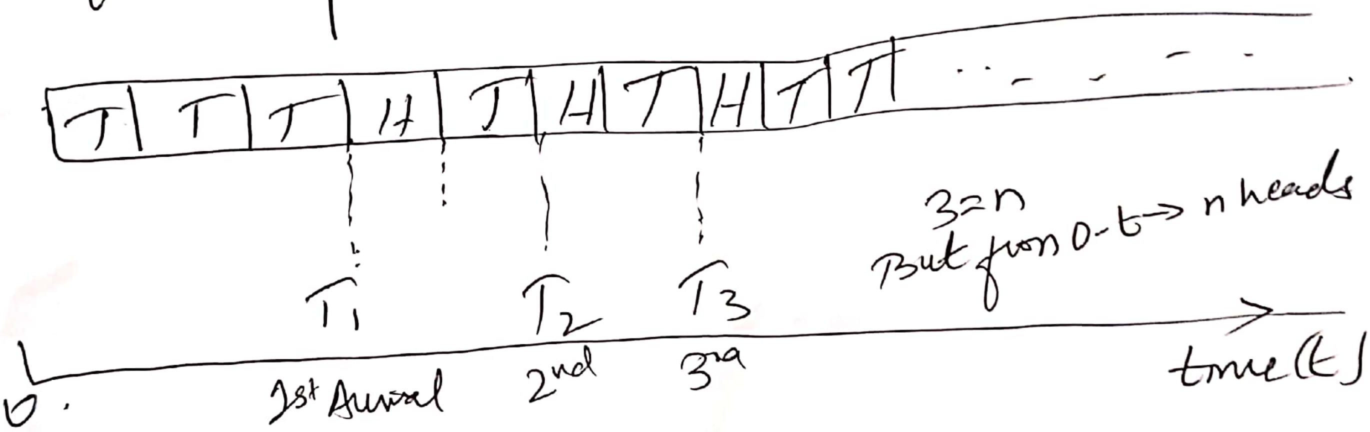
Thus the k^{th} interval is $[(k-1)\delta, k\delta]$

The total length of the line is $n\delta$.

So $\boxed{n\delta = t}$.

Hence there are $n \approx \frac{t}{\delta}$ time slots in the interval of $(0,t)$

Now, we assume that in each time slot, we toss a coin for which $(P_H = p = \lambda\delta)$
If the coin lands heads up, we say that we have an arrival in the sub interval
If tail up, then no arrival in that interval



Now, let $N(t)$ be defined as the no of arrivals (Number of heads) from time 0 to time t.
Thus, $N(t)$ is the no. of heads in n coin flips.

$$N(t) \sim \text{Binomial}(n, \lambda\delta)$$

<u>Mean of Binomial Distribution</u>

$$\text{Mean} = np$$

$$\text{mean} = n \cdot \lambda\delta$$

But $n = \frac{t}{\delta}$

So → $\text{mean} = \frac{t}{\cancel{\delta}} \cdot \lambda \cancel{\delta}$

$$\boxed{\text{mean} = \lambda t}$$

Now if $\delta \to 0$ (very small) Then $n \to \infty$.
and as this condition arises we say that

The counting process converges to poisson distribution with rate λt

So we can also say that no. of arrivals in any interval of length (t) follows a Poisson distribution as $\delta \to 0$.

→ The no. of arrivals in each interval is determined by the results of the coin flips for that interval.

Since different coin flips are independent we conclude the stated counting process has independent increments.

→ Based on the counting process, we can define poisson process.

One of the most important types of counting process is Poisson process.

→ It is usually used in scenarios where we are counting the occurrences of certain events that appear to happen at a certain rate but completely at random.

Such as,

1) The no. of car accidents at a site or an area
2) The no. of ~~dog~~ bites in a city
3) location of users in a wireless network
4) timing of ~~eets~~ earthquakes in an area
5) outbreak of viruses.

DEFINITION OF POISSON PROCESS

2 → ways

1) The AXIOMATIC WAY:

~~Let λ > 0 be fixed~~. The counting process $\{N(t), t \in (0,\infty)\}$ is called a poisson process with rates λ, if all the following conditions hold

1) $N(0) = 0$
2) $N(t)$ has independant increments
3) The no. of arrivals in any length $t > 0$ has Poisson Distribution

i.e. $P(N(s,t)) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n = 0, 1, 2 \ldots$

If $\lambda = 1$, then $N(t)_{t \geq 0}$ is also called standard poisson process

$$P(N(s,t)) = \frac{e^{-t} \cdot t^n}{n!}$$

2) Infinitesimal Description

→ we need to first understand Small-o notation

→ A function $f: R \to R$ is said to be $o(h)$ for $h \to 0$ if $\lim_{h \to 0} \frac{f(h)}{h} = 0$.