STOCHASTIC PROCESSES

STATIONARY DISTRIBUTION

Let $\{X_n, n \geq 0\}$ be a homogeneous Markov Chain with TPM$[P]$. If there exists a probability vector q, such that

$$qP = q.$$

Then q is called a stationary probability or steady state or limiting distribution of Markov chain.

NOTE: The stationary distribution q, if it exists, is unique because it is a probability vector.

$$q \geq 0$$
$$\Sigma q = 1.$$

SIGNIFICANCE OF STATIONARY DISTRIBUTION

QUESTION 1: Consider three big markets A, B, C and TPM corresponding to them is given as below

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.6 & 0.2 \end{bmatrix} \end{array}$$

At present, its market shares are 30%, 40% and 30% respectively

After 1 time period, its market shares will be

$$q_1 = q_0 P$$

$$q_1 = \overset{3\times 1}{\begin{bmatrix} 0.3 & 0.4 & 0.3 \end{bmatrix}} \times \overset{(3\times 3)}{\begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}}$$

= No. of columns in A = No. of row in B

$$q_1 = \begin{bmatrix} (0.3)(0.5) + (0.4)(0.4) + (0.3)(0.2) \\ (0.3)(0.2) + (0.4)(0.5) + (0.3)(0.6) \\ (0.3)(0.3) + (0.4)(0.1) + (0.3)(0.2) \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0.37 & 0.44 & 0.19 \end{bmatrix}$$

⇒ market share of A and B have increased but C has decreased.

After 2 time period, its market share will be

$$q_2 = q_1 P^2$$

$$q_2 = \begin{bmatrix} 0.37 & 0.44 & 0.19 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0.399 & 0.408 & 0.193 \end{bmatrix}$$

After 3 time period, its market share will be

$$q_3 = q_0 P^3 = \begin{bmatrix} 0.4013 & 0.3996 & 0.1991 \end{bmatrix}$$

After 7 time period, its market shares will be

$$q_7 = q_0 P^7 = \begin{bmatrix} 0.4 & 0.4 & 0.2 \end{bmatrix}$$

After 12 time period, its market shares will be:

$$q_{12} = q_0 P^{12} = [0.4 \quad 0.4 \quad 0.2]$$

At 13 times it will remain same as q_{12}.

This is steady state.

EXAMPLE 1: (Question on stationary probability)

Given that a person last cola purchase was COKE there is 90% chance that his next cola purchase will also be COKE. If a person last cola purchase was PEPSI, there is a 80% chance that his next cola purchase will be PEPSI. The present market share of COKE and PEPSI is 55% and 45% respectively. Construct the TPM. In the long run, what is the market share of each cola.

Solution: The required TPM is

$$P = \begin{array}{c c} & \begin{array}{cc} C & P \end{array} \\ \begin{array}{c} C \\ P \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

Let $q = [p_1, p_2]$ with $p_1 + p_2 = 1$ be the stationary distribution such that

$$qP = q.$$

That is $[p_1 \quad p_2] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = p_1 \, p_2 \rightarrow (1).$

you are asked to find the market share extracting from eq 1.

$$[p_1 \; p_2] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = [p_1, p_2]$$

eq 2 ← $0.9p_1 + 0.2p_2 = p_1 \Rightarrow -0.1p_1 + 0.2p_2 = 0$

eq 3 ← $0.1p_1 + 0.8p_2 = p_2 \Rightarrow 0.1p_1 - 0.2p_2 = 0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0 \quad 0 \quad 0$

And $\boxed{p_1 + p_2 = 1} \rightarrow$ eq 5

when you solve eq 2 and 3 the answer would always be $p_1 = 0$ & $p_2 = 0$. However, this does not fulfil $p_1 + p_2 = 1$ equation. using any of eq 2 or 3, we are going to find out values of p_1 and p_2. (you always use eq 5 to solve)

using eq 2.

$-0.1p_1 + 0.2p_2 = 0 \rightarrow$ eq 4.

$p_2 = 1 - p_1 \rightarrow$ put in 4

$-0.1p_1 + 0.2(1 - p_1) = 0$

$-0.1p_1 + 0.2 - 0.2p_1 = 0$

$0.2 - 0.3p_1 = 0$

$0.2 = 0.3p_1$

$p_1 = 0.2/0.3$

$\boxed{p_1 = 2/3} \rightarrow$ putting in 5

$\frac{2}{3} + p_2 = 1$

$2 + 3p_2 = 3$

$3p_2 = 1$

$\boxed{p_2 = 1/3}$

Hence in the long run, the market share of COKE and PEPSI will be 66.67% and 33.33% respectively

EXAMPLE 2: In a certain market, there are three brands of LIPSTICKS A, B and C. Given that a lady purchase lipstick of brand A there are 70% chance that she would continue with brand A, 20% and 10% chances that she would shift to brand B and C respectively. Given that a lady last purchased lipstick of brand B, there is 50% chances that she would shift to brand A and 10% to brand C. If she purchase brand C, there is 60% 20% chance that she would shift to brand A and B resp. The present market share of 3 brands is 60% 30% and 10% resp. Using this information find market share of the brands A, B and C in steady state.

Solution

let $q = [p_1\ p_2\ p_3]$ with $p_1+p_2+p_3=1$ be the stationary distribution such that

$$qP = q.$$

$$[p_1\ p_2\ p_3]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.2 \end{bmatrix} = [p_1\ p_2\ p_3]$$

STOCHASTIC PROCESSES.

$0.7p_1 + 0.5p_2 + 0.6p_3 = p_1 \Rightarrow -0.3p_1 + 0.5p_2 + 0.6p_3 = 0$

$0.2p_1 + 0.4p_2 + 0.2p_3 = p_2 \Rightarrow 0.2p_1 - 0.6p_2 + 0.2p_3 = 0$

$0.1p_1 + 0.1p_2 + 0.2p_3 = p_3 \Rightarrow 0.1p_1 + 0.1p_2 - 0.8p_3 = 0$

$0 \quad 0 \quad 0$

$\boxed{p_1 + p_2 + p_3 = 1} \Rightarrow p_1 = 1 - p_2 - p_3 \rightarrow eq\ 1$

solving the eqs.

$-0.3(1 - p_2 - p_3) + 0.5p_2 + 0.6p_3 = 0$

$-0.3 + 0.3p_2 + 0.3p_3 + 0.5p_2 + 0.6p_3 = 0$

$0.8p_2 + 0.9p_3 = 0.3$

$p_3 = \frac{0.3 - 0.8p_2}{0.9} \rightarrow 2.$

putting in eqs.

$-0.3(1 - p_2 - p_3) + 0.5p_2 + \cancel{0.6}^{0.2}\left(\frac{0.3 - 0.8p_2}{\cancel{0.9}\,0.3}\right) = 0.$

$-0.3 + 0.3p_2 + 0.3p_3 + 0.5p_2 + \frac{0.06 - 0.16p_2}{0.3} = 0.$

$-0.3 + 0.3p_2 + 0.\cancel{3}^{1}\left(\frac{0.3 - 0.8p_2}{\cancel{0.9}\,0.3}\right) + 0.5p_2 + \frac{0.06 - 0.16p_2}{0.3} = 0$

$-0.3 + 0.3p_2 + \frac{0.03 - 0.08p_2}{0.3} + 0.5p_2 + \frac{0.06 - 0.16p_2}{0.3} = 0$

$-0.09 + 0.09p_2 + 0.03 - 0.08p_2 + 0.15p_2 + 0.06 - 0.16p_2 = 0.$

$+ 0.16 p_2 = 0$

$p_2 =$

$p_2 = 0.22 / 0.36$

$p_2 =$

$P_1 = 0.7p_1 + 0.5p_2 + 0.6p_3 \Rightarrow -0.3p_1 + 0.5p_2 + 0.6p_3 = 0$

$P_2 = 0.2p_1 + 0.4p_2 + 0.2p_3 \Rightarrow 0.2p_1 - 0.6p_2 + 0.2p_3 = 0$

$P_3 = 0.1p_1 + 0.1p_2 + 0.2p_3 \Rightarrow 0.1p_1 + 0.1p_2 - 0.8p_3 = 0.$

$P_1 + P_2 + P_3 = 1$

$\boxed{P_1 = 1 - P_2 - P_3}$

substituting in $0.2p_1 - 0.6p_2 + 0.2p_3 = 0$

$\Rightarrow 0.2(1 - p_2 - p_3) - 0.6p_2 + 0.2p_3 = 0.$

$\Rightarrow 0.2 - 0.2p_2 - \cancel{0.2p_3} - 0.6p_2 + \cancel{0.2p_3} = 0.$

$\Rightarrow 0.2 - 0.8p_2 = 0$

$\frac{0.2}{0.8} = p_2.$

$\boxed{P_2 = 0.25}$

putting values in $0.1p_1 + 0.1p_2 - 0.8p_3 = 0.$

$\Rightarrow 0.1(1 - 0.25 - p_3) + 0.1(0.25) - 0.8p_3 = 0.$

$\Rightarrow 0.1 - 0.025 - 0.1p_3 + 0.025 - 0.8p_3 = 0$

$\Rightarrow 0.1 - 0.9p_3 = 0$

$\Rightarrow \frac{0.1}{0.9} = p_3$

$\Rightarrow \boxed{P_3 = 0.111}$

putting values of P_2 & P_3 in $P_1 = 1 - P_2 - P_3$

$P_1 = 1 - 0.25 - 0.111$

$P_1 = 0.638$

Hence in steady state the market share of brands A, B and C will be 63.889%, 25% & 11.11%

Practice Questions

Ex 3 A professor tried not to be late for class too often. If he is one day late, he is 90% sure to be on time next day. If he is on time, then the next day there is a 30% chance of his being late. In the long run, how often he is late for class?

$P_1 = 0.25$ $P_2 = 0.75$

Hence in the long run, there is 25% chance that he comes late for class.