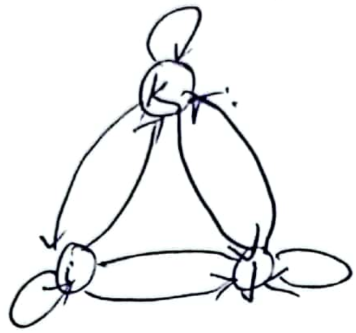
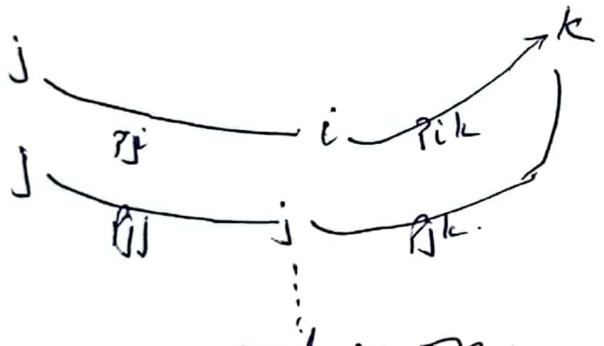
CHAPMAN KOLMOGOROV EQUATION

→ Higher Transition probability calculation is carried out in CK Equation.

Suppose we want to go from state j to state k. But in 2 steps



There are multiple paths for this transition.



and so on.

$$P_{jk}^{(2)} = P_{ji}^{(1)} \cdot P_{ik}^{(1)} + P_{jj}^{(1)} \cdot P_{jk} + \cdots \; P_{jn}^{(1)} \cdot P_{nk}^{(1)}$$

$$\boxed{P_{jk}^{(2)} = \sum_{n} P_{jn}^{(1)} \cdot P_{nk}^{(1)}}$$

we can also create a matrix for the same equation

$$P_{jk}^{(2)} = \begin{bmatrix} P_{ji}^{(1)} \cdot P_{jj}^{(1)} \cdots \end{bmatrix} \begin{bmatrix} P_{ik}^{(1)} \\ P_{jk}^{(1)} \\ \vdots \end{bmatrix}$$

$= j^{th}$ row of P. $\times$ k^{th} column of P

$(j,k)^2 = (j,k)^{th}$ element of P^2.

Hence we can assume or say that
$(j,k)^{th}$ element of $P^{(2)} = (j,k)^{th}$ element of P^2

$$\boxed{P^{(2)} = P^2}$$

Further explanation:

Suppose X_n, $n = 0, 1, 2 \ldots$ is a homogeneous markov Chain.

A markov chain is called homogeneous if and only if the transition probabilities are independent of the time t.

Then,
$$P_{ij}^{(m+n)} = \sum_{k} P_{ik}^{(m)} \cdot P_{kj}^{(n)} \longrightarrow (1)$$

$P_{ij}^{(m+n)}$ = conditional prob. that the markov chain goes from state i to j in $m+n$ steps.

$P_{ik}^{(m)}$ = conditional probabilities of reaching an intermediary state k in 'm' steps.

$P_{kj}^{(n)}$ = conditional probabilities from k reaching state j in 'n' steps

Now

(3).

∴ we know that n-step transition probability shown as

$$P_{ij}^{(n)} = P(X_{n+1} = j \mid X_1 = i)$$

Difference of both the steps is $n+1-1 = n$

Writing eq (1) in n-step probability form

$$P(X_{m+n} = j \mid X_0 = i) = \sum_{K} P(X_m = k \mid X_0 = i) \cdot P(X_{m+n} = j \mid X_m = k)$$

$\boxed{m+n-0 = m+n}$ $\boxed{m-0 = m}$ $\boxed{m+n-m = n}$

PROOF OF CK EQUATION:

The (i,j)th entry of matrix $P^{(m+n)}$ is given by

$$P_{ij}^{(m+n)} = P(X_{m+n} = j \mid X_0 = i)$$

There are intermediate step

$$P_{ij}^{(m+n)} = \sum_{K} P(X_{m+n} = j, X_m = k \mid X_0 = i)$$

using probability of A condition B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P_{ij}^{(m+n)} = \sum_{K} \frac{P(X_{m+n} = j, X_m = k, X_0 = i)}{P(X_0 = i)}$$

Now since they are moving from i to k, we

2. n step probability here and multiply (4)
and divide the probability with the equation

$$P_{ij}^{(m+n)} = \sum_{K} \underbrace{\frac{P(X_{m+n}=j, X_m=k, X_0=i)}{P(X_m=k, X_0=i)}}_{P(A/B)} \cdot \underbrace{\frac{P(X_m=k, X_0=i)}{P(X_0=i)}}_{P(A/B)}$$

Since the above equation equates to conditional probability, we can write it as

$$P_{ij}^{(m+n)} = \sum_{K} (P X_{m+n}=j \mid X_m=k, X_0=i) \cdot P(X_m=k / X_0=i)$$

Now As we know that any step t is only dependends on (t-1) and not on any past position.

→ future depends only on present and not past states

So now we write.

$$P_{ij}^{(m+n)} = \sum_{K} (P X_{m+n}=j \mid X_m=k) \cdot P(X_m=k / X_0=i)$$

$$P_{ij}^{(m+n)} = \sum_{K} P_{kj}^{(n)} \cdot P_{ik}^{(m)}$$

$$\boxed{P_{ij}^{m+n} = \sum_{K} P_{ik}^{m} \cdot P_{kj}^{n}}$$ Rearranged

This equation holds for all states i, j and $m, n \geq 0$.

STIONS ON CHAPMAN KOLMOGOROV EQUATION:

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Ex 1: The TPM of the markov chain with three states 1, 2, 3 is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

Compute 2 step TPM.

Simply computing

$P^2 = P \cdot P$

$$P^2 = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$ Answer.

Ex 2: Three boys A, B, C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is as likely to throw the ball to B as to A. Find

i) Transition matrix ii) 2-step probability

i) $$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

ii) $$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$ Ans