28.1.23

APPLICATIONS OF D.E

There are two applications:

1. Electric circuit
2. Simple Harmonic Motion

* choose these Q

* Do not choose simultaneous sum

$\frac{dt}{}$; eqn

1. ELECTRIC CIRCUIT

It is series in which a source of electric energy such as battery or generator and a resistor which utilise electrical energy.

There are 3 basic components of Electric Circuit

- Resistor
- Inductor
- Capacitor.

KIRCHOFF'S LAW

The algebraic sum of all the voltage drop through any closed loop is zero

NAME OF ELEMENT	SYMBOL	NOTATION	UNIT	VOLTAGE DROP
RESISTOR	—/\/\/\—	R	Ω ohm	RI
CAPACITOR	—||—	C	Farad (F)	$\frac{Q}{C}$ (or) $\frac{V}{C}$
INDUCTOR	—ooooo—	L	Henry (H)	$L, \frac{dI}{dt}$

RL circuit

$$RI + L\frac{dI}{dt} = E(t) = E_0 \sin \omega t$$

RC Circuit

$$RI + \frac{Q}{C} = E(t) = E_0 \sin \omega t$$

RLC circuit

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E(t)}{L}$$

Problem ⇒ CAT 2, sem

1. In the LCR circuit the charge 'q' is given by the eqn,

Repeated university Q

$$\frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{q}{LC} = 0$$

If at time $t = 0$, the charge $q = Q$ and $i = 0$ and $L = 2CR^2$.

P.T $q = Q e^{-kt}(\cos kt + \sin kt)$ if $2KRC = 1$

Solution: here diff variable = t

G.T.,

$$\frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{q}{LC} = 0$$

take as constant

$$\Rightarrow \left(\frac{d^2}{dt^2} + \frac{1}{RC}\frac{d}{dt} + \left(\frac{1}{LC}\right)\right) = 0$$

should → 6 TYPES

$$\Rightarrow \left(D^2 + \frac{1}{RC}D + \frac{1}{LC}\right)\underset{(y)}{q} = 0$$

$\frac{d}{dt} = D$

↳ AE: Replace 'D' by 'm'

$$\left(m^2 + \frac{m}{RC} + \frac{1}{LC}\right) = 0$$

$a = 1$; $b = \frac{1}{RC}$; $c = \frac{1}{LC}$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{\frac{-1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{LC}}}{2}$$

G.T $L = 2CR^2$

$$= \frac{\frac{-1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{\cancel{4}2}{\cancel{2}CR^2 . C}}}{2a}$$

$$= \frac{\frac{-1}{RC} \pm \sqrt{-\frac{1}{C^2R^2}}}{2}$$

$$= \frac{\frac{-1}{RC} \pm \frac{i}{RC}}{2}$$

$\frac{1}{} \times \frac{1}{2}$

$$= \frac{-1}{2RC} \pm \frac{i}{2RC} = \frac{1}{2RC}(-1 \pm i)$$

↳ Nature of roots: complex and distinct

$m = \alpha \pm i\beta$

$\alpha = -\frac{1}{2RC}$ $\quad \beta = \frac{1}{2RC}$

↳ CF: $e^{\alpha t}[C_1 \cos \beta t + C_2 \sin \beta t]$ (GIF)

$$= e^{-\frac{1}{2RC}t}\left[C_1 \cos \frac{1}{2RC} t + C_2 \sin \frac{1}{2RC} t\right]$$

↳ Thus, the general solution

$q = CF$

$$= e^{-\frac{1}{2RC}t}\left[C_1 \cos \frac{1}{2RC} t + C_2 \sin \frac{1}{2RC} t\right]$$

Given initial condition ⇒

G.T $2KRC = 1$

$2RC = \frac{1}{K}$

$$\boxed{q = e^{-Kt}[C_1 \cos Kt + C_2 \sin Kt]} \rightarrow (*)$$

Given initial condition (I.C)

1) At $t=0$, $q=Q$

2) At $t=0$, $i=0$

* If itial condition is no given find only eqn(*)

To find the current (i),

different eqn (*) w.r.t time (t)

$uv = uv' + vu'$

$$i = \frac{dq}{dt} = \frac{d}{dt}\left[e^{-kt}\left(C_1\cos kt + C_2\sin kt\right)\right]$$

$$i = e^{-kt}\left[-kC_1\sin kt + C_2 k\cos kt\right]$$

$$-ke^{-kt}\left[C_1\cos kt + C_2\sin kt\right] \quad \text{—(**)}$$

Apply the I.C in eqn (*)

$$Q = e^{0}\left[C_1\overset{1}{\cos 0} + C_2\overset{0}{\sin 0}\right]$$

$$= 1\,(C_1+0)$$

$$\boxed{Q = C_1}$$

Apply the IC in eqn (**)

$$i = \overset{1}{e^{0}}\left[-kC_1\overset{0}{\sin 0} + C_2 k\overset{1}{\cos 0}\right]$$

$$-k\overset{1}{e^{0}}\left[C_1\overset{1}{\cos 0} + C_2\overset{0}{\sin 0}\right]$$

$$i = 1\,[C_2 k] - k\,[C_1]$$

$$0 = C_2 k - C_1 k$$

$$0 = k\,[C_2 - C_1]$$

$$0 = -k\,[C_1 - C_2]$$

$$\frac{0}{-k} = [Q - C_2]$$

$$0 = Q - C_2$$

$$\boxed{Q = C_2}$$

Apply c_1, c_1, c_2 in (*)

$$q = e^{-kt}\left[c_1 \cos kt + c_2 \sin kt\right]$$

$$= e^{-kt}\left[Q \cos kt + Q \sin kt\right]$$

$$\boxed{q = Q e^{-kt}\left[\cos kt + \sin kt\right]}$$

Hence proved

3.2.23 A charge q on the plate of a capacitor is given by the eqn.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin\omega t$$

The circuit is turned so that

$$\omega^2 = \frac{1}{LC}, \quad R^2 < \frac{4L}{C} \text{ and}$$

$$q = \frac{dq}{dt} = 0, \text{ at } t = 0,$$

P.T

$$q = \frac{E}{R\omega}\left[e^{-\frac{RT}{L}}\left(\cos pt + \frac{R}{2LP}\sin pt\right) - \cos\omega t\right],$$

where $P^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$

Sol:

G.T

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin\omega t \quad \text{--- (1)}$$

$\div$ eqn (1) throughout by L

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E}{L}\sin\omega t$$

$$\Rightarrow \left(\frac{d^2}{dt^2} + \frac{R}{L}\frac{d}{dt} + \frac{1}{LC}\right) q = \frac{E}{L}\sin\omega t$$

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)\underset{y}{q} = \frac{E}{L}\sin\omega t$$

$\frac{E}{L}$ ↳ constant

$\frac{d}{dt} = D$

$\frac{d}{dx} = D$

↳ AE : Replace D by m

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

$$a = 1$$

$$b = \frac{R}{L}$$

$$c = \frac{1}{LC}$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

G.T , $\omega^2 = \frac{1}{LC}$

$p^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$

$$= \frac{1}{2}\left[-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\right]$$

$$= \frac{1}{2}\left[-\frac{R}{L} \pm \sqrt{4\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)}\right]$$

$$= \frac{1}{2}\left[-\frac{R}{L} \pm \sqrt{-4\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}\right]$$

$$= \frac{1}{2}\left[-\frac{R}{L} \pm \sqrt{4p^2i^2}\right]$$

$$m = \frac{1}{2}\left[-\frac{R}{L} \pm 2pi\right] = \frac{-R}{2L} \pm \frac{2pi}{2}$$

↳ Nature of roots : complex and distinct

$$\alpha = -\frac{R}{2L} \quad ; \quad \beta = p \qquad (\alpha \pm i\beta)$$

↳ CF : $e^{\alpha t}\left[c_1 \cos\beta t + c_2 \sin\beta t\right]$ (G.F)

$$= e^{\frac{-Rt}{2L}}\left[c_1 \cos pt + c_2 \ldots\right]$$

$\hookrightarrow$ PI $= \frac{1}{\phi(D)} \sin at$ (G.F)

$$= \frac{1}{D^2 + \frac{R}{L}D + \frac{1}{LC}} \frac{E}{L} \sin \omega t$$

$a = \omega$

$D^2 = -a^2$

$= -\omega^2$

$$= \frac{1}{-\omega^2 + \frac{R}{L}D + \frac{1}{LC}} \frac{E}{L} \sin \omega t$$

$$= \frac{1}{-\cancel{\omega^2} + \frac{R}{L}D + \cancel{\omega^2}} \cdot \frac{E}{L} \sin \omega t$$

$$= \frac{1}{\frac{R}{L}D} \frac{E}{L} \sin \omega t$$

constant

$$= \frac{\cancel{L}}{R} \cdot \frac{1}{D} \cdot \frac{E}{\cancel{L}} \sin \omega t$$

$$= \frac{E}{R} \frac{1}{D} (\sin \omega t)$$

$$= \frac{E}{R} \cdot \left(-\frac{\cos \omega t}{\omega}\right) = -\frac{E}{R\omega} (\cos \omega t)$$

Thus, the general solution,

$q = CF + PI$

$$q = e^{\frac{-Rt}{2L}} [C_1 \cos Pt + C_2 \sin Pt] - \frac{E}{R\omega} \cos \omega t$$

$\llcorner$ (*)

Diff eqn (*) to find i

$uv' + vu'$

$$i = \frac{dq}{dt} = \frac{d}{dt} \left[\underbrace{e^{\frac{-Rt}{2L}}}_{u} \underbrace{(C_1 \cos Pt + C_2 \sin Pt)}_{v} - \frac{E}{R\omega} \cos \omega t \right]$$

$$i = e^{-\frac{Rt}{2L}} [-PC_1 \sin Pt + PC_2 \cos Pt] + \left(-\frac{R}{2L} e^{\frac{-R}{2L}}\right)$$

$$[C_1 \cos Pt + C_2 \sin Pt] + \frac{E\omega}{R\omega} \sin \omega t$$

$\llcorner$ (**)

↳ I.C

1) At $t=0,\ q=0$

2) At $t=0,\ i=\frac{dq}{dt}=0$

Apply the I.C(1) in eqn (*)

$$q = e^{\frac{-Rt}{2L}}\left(C_1\cos pt + C_2\sin pt\right) - \frac{E}{R\omega}\cos\omega t$$

$$0 = e^{0}\left[C_1\cos 0 + C_2\sin 0\right] - \frac{E}{R\omega}\cos 0$$

$$= C_1 - \frac{E}{R\omega}$$

$$\Rightarrow C_1 = \frac{E}{R\omega}$$

Apply the I.C (2) in eqn (**)

$$i = e^{\frac{-Rt}{2L}}\left[-pC_1\sin pt + pC_2\cos pt\right]$$
$$+\left(\frac{-R}{2L}e^{\frac{-Rt}{2L}}\right)\left[C_1\cos pt + C_2\sin pt\right] + \frac{E}{R}\sin\omega t$$

$$= \underbrace{e^{0}}_{1}\left[-pC_1\overbrace{\sin 0}^{0} + pC_2\cos 0\right]$$
$$+\left(\frac{-R}{2L}\underbrace{e^{0}}_{1}\right)\left[C_1\overbrace{\cos 0}^{1} + C_2\sin 0\right] + \frac{E}{R}\sin 0$$

$$= pC_2 + \left(\frac{-R}{2L}\right)(C_1)$$

$$0 = C_2 p - \frac{R}{2L}C_1$$

$$= C_2 p - \frac{\cancel{R}}{2L}\cdot\frac{E}{\cancel{R}\omega}$$

$$0 = C_2 p - \frac{E}{2L\omega}$$

$$\frac{E}{2L\omega p} = C_2$$

Substitute the values of C_1 & C_2 in eqn (*)

$$q = e^{\frac{-Rt}{2L}} \left[\frac{E}{R\omega} \cos pt + \frac{E \times R}{2\omega L p \times R} \sin pt \right]$$

$$- \frac{E}{R\omega} \cos \omega t$$

$$= \frac{E}{R\omega} \left\{ e^{\frac{-Rt}{2L}} \left(\cos pt + \frac{R}{2Lp} \sin pt \right) - \cos \omega t \right\}$$

Hence Proved

4.2.23

8. An emf $E\sin pt$ applied at $t=0$ to a circuit containing capacitance and inductance. The charge q is given by

$$L \frac{d^2q}{dt^2} + \frac{q}{c} = E \sin pt$$. If $p^2 = \frac{1}{LC}$ and initially, the current and the charge are 0, Find the current I ~~on~~ at time t

Sol:

$$L \frac{d^2q}{dt^2} + \frac{q}{C} = E\sin pt \quad \text{---(1)} \qquad \frac{d}{dt} = D$$

$\div$ (1) by L

$$\left(D^2 + \frac{1}{LC}\right) q = \frac{E}{L} \sin pt$$

↳ AE: $m^2 + \frac{1}{LC} = 0$

$$m^2 = -\frac{1}{LC}$$

$$m = \pm \frac{i}{\sqrt{LC}} = \pm ip$$

↳ Nature of roots: Complex and distinct

$$\alpha = 0 \qquad \beta = p$$

$$\text{PI} = \frac{1}{D^2+\frac{1}{LC}} \frac{E}{L} \sin pt$$

$a = p$

$D^2 = -p^2$

$$= \frac{1}{D^2+p^2} \frac{E}{L} \sin pt$$

$p^2 \rightarrow$ constant

$$= \frac{1}{-p^2+p^2} \frac{E}{L} \sin pt$$

$$= \frac{1}{0} \frac{E}{L} \sin pt$$

$$= \frac{t}{2D+0} \frac{E}{L} \sin pt$$

$\rightarrow$ should not multiply with $\sin pt$

$$= \frac{Et}{2L} \frac{1}{D} (\sin pt)$$

$\rightarrow$ now multiply $t \Rightarrow$ after integration or differentiation

$$= \frac{Et}{2L} \left(- \frac{\cos pt}{p}\right)$$

$\rightarrow$ constant

$$PI = \underbrace{-\frac{Et}{2Lp}}_{u} \underbrace{(\cos pt)}_{v}$$

$uv' + vu'$

$$q = c_1 \cos pt + c_2 \sin pt - \frac{Et \cos pt}{2Lp} \quad \text{---} (*)$$

Differentiate eqn $(*)$ w.r.t 't' to find i.

$$i = \frac{dq}{dt} = -pc_1 \sin pt + pc_2 \cos pt - \frac{E}{2Lp}\left[\cos pt + t(-p \sin pt)\right] \quad \text{---} (**)$$

Given that I.C

1) At $t=0$ $q=0$

2) At $t=0$ $i=0$

Apply IC in $(*)$

$$0 = c_1 \cos 0 + c_2 \sin 0 - \frac{E(0) \cos 0}{2Lp}$$

$\sin 0 \rightarrow 0$

$$0 = c_1 - 0$$

$$c_1 = 0$$

Apply IC in (**)

$$0 = -PC_1 \sin 0 + PC_2 \cos 0 - \frac{E}{2LP}\left[\cos 0 - tp \sin 0\right]$$

$$0 = PC_2 - \frac{E}{2LP}$$

$$PC_2 = \frac{E}{2LP}$$

$$C_2 = \frac{E}{2LP^2}$$

sub C_1 & C_2 in (**) = to find i

$$i = \cancel{P}\frac{E}{2LP\cancel{^2}}\cos pt - \frac{E}{2LP}\left[\cos pt - pt \sin pt\right]$$

$$= \frac{E}{2LP}\left[\cancel{\cos pt} - \cancel{\cos pt} + pt \sin pt\right]$$

$$= \frac{E}{2L\cancel{P}} \cdot Pt \sin pt$$

$$\boxed{i = \frac{Et}{2L}\sin pt}$$

Q The equation of RC circuit is

(x) 2m $R\frac{dI}{dt} + \frac{I}{C} = E_0 \omega \cos \omega t$. Find the current I

$$R\frac{dI}{dt} + \frac{I}{C} = E_0 \omega \cos \omega t \quad \text{---(1)} \qquad D = \frac{d}{dt}$$

$$\left(RDI + \frac{I}{C}\right) = E_0 \omega \cos \omega t$$

Divide eqn (1) by R

$$\left(D+\frac{1}{RC}\right)I = \frac{E_0\omega}{R}\cos\omega t$$

↳ AE = $m+\frac{1}{RC}=0$

$$m = -\frac{1}{RC}$$

↳ Nature of Root : Real ~~& distinct~~

↳ CF : $C_1 e^{-\frac{1}{RC}t} = C_1 e^{-\frac{t}{RC}}$

↳ PI : $\frac{1}{\left(D+\frac{1}{RC}\right)} \cdot \frac{E_0\omega}{R}\cos\omega t$ $\quad a=\omega$, $D^2=-\omega^2$

$$= \frac{1}{\left(D+\frac{1}{RC}\right)} \times \frac{\left(D-\frac{1}{RC}\right)}{\left(D-\frac{1}{RC}\right)} \cdot \frac{E_0\omega}{R}\cos\omega t$$

$$= \frac{D-\frac{1}{RC}}{D^2-\frac{1}{R^2C^2}} \cdot \frac{E_0\omega}{R}\cos\omega t$$ → constant $\quad D^2-\omega^2$

$$= \left(\frac{E_0\omega}{R}\right)\left(\frac{D-\frac{1}{RC}}{-\omega^2-\frac{1}{R^2C^2}}\right)\cos\omega t$$

$$= \frac{E_0\omega}{R} \cdot \frac{1}{-\left[\frac{\omega^2R^2C^2+1}{R^2C^2}\right]}\left(-\omega\sin\omega t - \frac{\cos\omega t}{CR}\right)$$

$$= +\frac{E_0\omega}{\cancel{R}} \times \frac{R^{\cancel{2}}C^{\cancel{2}}}{1+\omega^2R^2C^2} \times \frac{CR\omega\sin\omega t+\cos\omega t}{\cancel{CR}}$$

$$= \frac{E_0\omega C}{1+\omega^2R^2C^2}\left[\omega CR\sin\omega t + \cos\omega t\right]$$

(SHM) SIMPLE HARMONIC MOTION

2.

Definition of SHM :

* When a particles moves in a straight line so that its acceleration is always towards a fixed point on a line and is proportional to the distance from that point.

* Then the motion is called simple harmonic motion - SHM

EQUATION OF SHM is

* $$\boxed{\frac{d^2x}{dt^2} + \mu^2 x = 0}$$

* where $\mu \Rightarrow$ constant of proportionality

SOLUTION

* By solving the above equation we get $\boxed{x = a\cos \mu t - \epsilon}$

* where $x \Rightarrow$ displacement

ϵ is very small (negligible)

* So the the above displacement can be written as

$$\boxed{x = a\cos \mu t}$$

* At time $t = 0$, $x = a$ and $v = 0$

displacement

$$v = \frac{dx}{dt} = -\mu a \sin \mu t = 0$$

where v is velocity

$$\left(a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}(-a\mu \sin \mu t)\right.$$

$$a = -a\mu^2 \cos \mu t$$

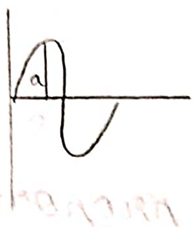
when $t = 0$

$$a = -a\mu^2$$

↳ acceleration) (no need)

AMPLITUDE

The distance a from the centro, to either extreme position is called amplitude of the motion.



* Amplitude
* F * Time Peri

PERIOD (OR) PERIODIC TIME

Time taken to complete 1 oscillation is known is periodic time of the SHM

$$\boxed{T = \frac{2\pi}{\mu}}$$

FREQUENCY

* The no. of oscillations per second is called frequency of SHM

[• Simple example of SHM is simple Pendulum]

*That is the reciprocal of Periodic Time (frequency)

$$\boxed{\text{frequency} = \frac{1}{T} = \frac{\mu}{2\pi}}$$

PROBLEMS (1) CAT 3, SEM

6.2.23 Q. A particle is executing an SHM about the origin O from which the distance x of the particle is measured. Initially $t=0$ i) $x=20$ and velocity ii) $v=0$. The eqn of motion is

$x \Rightarrow$ displacement

$$\ddot{x} + x = 0.$$

Solve for x and find periodic time and amplitude of SHM.

Sol: $\rightarrow$ Dependent variable

$$\frac{d^2x}{dt^2} + \mu^2 x = 0 \quad (\text{G.I.F})$$

$\hookrightarrow$ Independent variable

$$x = a\cos\mu t \quad (\text{G.F.S})$$

(General solution)

have only 1 dependent variable

first o.d.e

$\therefore$ we use $\frac{d}{dt}$

G.T., $\ddot{x} + x = 0$

$$\frac{d^2x}{dt^2} + x = 0$$

$$\left(\frac{d^2}{dt^2} + 1\right)x = 0$$

$$(D^2+1)x = 0$$

$\hookrightarrow$ AE : $m^2+1=0$

$m^2 = -1$

$m = \pm i$

$\alpha = 0 \quad \beta = 1$

$\hookrightarrow$ Nature of roots : complex & distinct

$\hookrightarrow$ CF: $e^{0t}[c_1\cos t + c_2 \sin t]$

$\hookrightarrow$ Complete solution, C.S.

$$x = c_1\cos t + c_2\sin t \quad \text{---}(*)$$

I.C

1) $t=0$, $x=20$

2) $t=0$, $v=0$

↳ To find : v

Differentiate eqn (*) w.r.t 't'

$$\frac{dx}{dt} = v = -c_1 \sin t + c_2 \cos t \quad \text{---} (**)$$

↳ Apply IC in (*)

$$x = c_1 \cos t + c_2 \sin t$$

$$20 = c_1 \cos 0 + c_2 \cdot \underbrace{\sin 0}_{=0}$$

$$20 = c_1$$

$$\Rightarrow c_1 = 20$$

↳ Apply IC in (**)

$$v = -c_1 \sin t + c_2 \cos t$$

$$0 = c_2$$

$$\Rightarrow c_2 = 0.$$

↳ Sub. c_1 & c_2 in (*)

$$x = c_1 \cos t + c_2 \sin t$$

$$x = 20 \cos t + (0) \sin t$$

$$x = 20 \cos t$$

$$\frac{d^2x}{dt^2} + \mu^2 x = 0$$

↳ To find : T

(G.S) $x = a \cos \mu t$

$$T = \frac{2\pi}{\mu} = \frac{2\pi}{1} = 2\pi$$

$\Rightarrow \mu = 1$

compare $x = 20 \cos t$ with G.S of SHM

$$x = a \cos \mu t$$

$$\therefore a = 20 \; ; \; \mu = 1.$$

Ans : $t = 2\pi$

$a = 20.$

2. A particle is moving in SHM to an additional force if the eqn of the motion is $\frac{d^2y}{dt^2} + w^2 y = b\cos nt$.

solve for y.

Sol:

$$\frac{d^2y}{dt^2} + w^2 y = b\cos nt$$

(G.F)

$$\frac{d^2x}{dt^2} + M^2 x = 0$$

$M = w$

$x = y$.

$$(D^2 + w^2) y = b\cos nt$$

↳ AE : $m^2 + w^2 = 0$

$$m^2 = -w^2$$

$$m = \pm wi$$

$\alpha = 0 \qquad \beta = w$

↳ Nature of roots : complex & distinct

↳ CF : $e^{0t} [c_1 \cos wt + c_2 \sin wt]$

$$= c_1 \cos wt + c_2 \sin wt$$

↳ PI $= \frac{1}{f(D)} \cos at$

constant

$$= \frac{1}{D^2 + w^2} b\cos nt$$

$\cos at = \cos nt$

$a = n$

$D^2 = -n^2$

$$PI = \frac{1}{-n^2 + w^2} b\cos nt$$

CASE 1 : If $n = w$

the $PI = \frac{1}{-n^2 + n^2} b\cos nt$

$$= \frac{1}{0} b\cos nt$$

$$PI = \frac{t}{2D+0} b\cos nt$$

$$= \frac{t}{2D} b.\cos nt$$

$$= \frac{tb}{2} \cdot \frac{1}{D} (\cos nt)$$

$$= \frac{tb}{2} \cdot \frac{\sin nt}{n}$$

$$PI = \frac{bt}{2} \frac{\sin nt}{n}$$

CASE (2) If $n \neq \omega$

then $P.I = \frac{1}{-n^2+\omega^2} b\cos nt.$

Thus, the complete solution for case (1)

$$y = c_1\cos\omega t + c_2\sin\omega t + \frac{bt}{2n}\sin nt$$

Case (2)

$$y = c_1\cos\omega t + c_2\sin\omega t + \frac{1}{-n^2+\omega^2} b\cos nt$$

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3. (*) CAT 3 SEM

An particle is executing SHM for with amplitude a and period $\frac{2\pi}{\mu}$. The eqn of motion is given by $\frac{d^2x}{dt^2} = -\mu^2 x$. If x_1, x_2, x_3 are the distance of the particle from the main position at 3 consecutive seconds. S.T the complete oscillation is

$$\frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{(2x_2)}\right)}$$ where μ

* here IC - not given

To prove : $\mu = \cos^{-1}\frac{x_1+x_3}{2x_2}$

Soln

G.T. $\frac{d^2x}{dt^2} = -\mu^2 x$

$$\frac{d^2x}{dt^2} + \mu^2 x = 0$$

$$\boxed{(D^2 + \mu^2)x = 0}$$

↳ AE : $m^2 + \mu^2 = 0$

$m^2 = -\mu^2$

$m = \pm \mu i$ $\quad\quad \alpha = 0 \quad \beta = \mu$

↳ Nature of roots: distinct & complex

↳ CF : $e^{0t}(C_1 \cos \mu t + C_2 \sin \mu t)$

I.C (At time $t = 0$, $x = a$, $v = 0$)

↳ G.S : $x = CF$

$$\boxed{x = C_1 \cos \mu t + C_2 \sin \mu t} \quad -(*)$$

Here, we take I.C (not given in Q)

I.C.

1) At time $t = 0$; $x = a$

2) At time $t = 0$, $v = 0$

↳ To find 'v' :

Diff 'x' ~~to find 'v'~~ w.r.t time 't'
(eqn (*))

$$\frac{dx}{dt} = \boxed{v = -C_1 \mu \sin \mu t + C_2 \mu \cos \mu t} \quad -(**)$$

↳ Apply I.C in (*)

$$a = C_1 \cos 0 + C_2 \sin 0$$

$$\boxed{C_1 = a}$$

↳ Apply I.C in (**)

$$v = -c_1 \mu \sin \mu t + c_2 \mu \cos \mu t$$

$$0 = -c_1 \mu \sin 0 + c_2 \mu \cos \mu t$$

$$0 = c_2 \mu$$

$$\boxed{c_2 = 0}$$

↳ Sub c_1 & c_2 in eqn (*)

$$x = c_1 \cos \mu t + c_2 \sin \mu t$$

$$x = a \cos \mu t + 0 \cdot \sin \mu t$$

G.S $\boxed{x = a \cos \mu t}$

Let us take x_1 be the position of the particle at 't' seconds

$$\Rightarrow \boxed{x_1 = a \cos \mu t}$$

Let x_2 be the position of the particle 't+1' seconds

$$\Rightarrow \boxed{x_2 = a \cos \mu (t+1)}$$

Let x_3 be the position of the particle at 't+2' seconds

$$\Rightarrow \boxed{x_3 = a \cos \mu (t+2)}$$

CONTINUATION ⇒

8.2.23.

$x_1 = a\cos\mu t$

① $x_2 = a\cos\mu(t+1)$

② $x_3 = a\cos\mu(t+2)$

Consider $\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)$

Let us take $\frac{x_1+x_3}{2x_2} = \frac{a\cos\mu t + a\cos\mu(t+2)}{2[a\cos\mu(t+1)]}$

$$= \frac{\cancel{a}[\cos\mu t + \cos(\mu t + 2\mu)]}{2\cancel{a}\cos(\mu t+\mu)}$$

$\because \cos A + \cos B = \cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$

$$\frac{x_1+x_3}{2x_2} = \frac{\cancel{2}\cos\left(\frac{\mu t+\mu t+2\mu}{2}\right)\cos\left(\frac{\cancel{\mu t}-\cancel{\mu t}-2\mu}{2}\right)}{\cancel{2}\cos(\mu t+\mu)}$$

$$= \frac{\cos\left(\frac{2\mu t+2\mu}{2}\right)\cos\left(\frac{-2\mu}{2}\right)}{\cos(\mu t+\mu)}$$

$$= \frac{\cos\left[\frac{\cancel{2}(\mu t+\mu)}{\cancel{2}}\right]\cos(-\mu)}{\cos(\mu t+\mu)}$$

$\because \cos(-\theta) = \cos\theta$

$$= \frac{\cancel{\cos(\mu t+\mu)}.\cos\mu}{\cancel{\cos(\mu t+\mu)}}$$

$$\frac{x_1+x_3}{2x_2} = \cos\mu$$

$$\boxed{\mu = \cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$$

$, T = \frac{2\pi}{\mu}$

To prove that, $T = \frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$

Here we get $\mu = \cos^{-1} \frac{x_1+x_3}{2x_2}$

$$\therefore T = \frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}$$

Hence proved.

A particle is executing an SHM

6m/ 10m

$$\frac{d^2x}{dt^2} = -\mu^2 x$$

CAT 2 SEM 2.

last sum either

At $t=0$, $x=a$ & $v=0$

Find the time taken to go from the position $x = \frac{a}{2}$ to $x = a$.

Also prove that the time taken is $\frac{1}{6}$ th of the period.

Ans :

$$\frac{d^2x}{dt^2} = -\mu x$$

$$\frac{d^2x}{dt^2} + \mu x = 0 \qquad \frac{d}{dt} = D$$

$$D^2x + \mu x = 0$$

$$(D^2 + \mu)x = 0$$

AE : $m^2 + \mu^2 = 0$

$m^2 = -\mu^2$

$m = \pm \mu i$

↳ Nature of roots: complex & distinct ③

$\alpha = 0 \qquad \beta = \mu$

↳ CF: $e^{0t} [C_1 \cos \mu t + C_2 \sin \mu t]$

$= C_1 \cos \mu t + C_2 \sin \mu t$

↳ complete solution

$$x = C_1 \cos \mu t + C_2 \sin \mu t \quad \text{---}(*)$$

Differentiate eqn (*) to find velocity 'v':

$$v = \frac{dx}{dt} = -\mu C_1 \sin \mu t + \mu C_2 \cos \mu t \quad \text{---}(**)$$

Apply I.C in (*)

1) At $t = 0 \quad x = a$

2) At $t = 0 \quad v = 0$

$\Rightarrow \quad a = C_1 \cos 0 + C_2 \sin 0$

$a = C_1$

Apply I.C in (**)

$0 = -\mu C_1 \sin 0 + \mu C_2 \cos 0$

$\mu C_2 = 0$

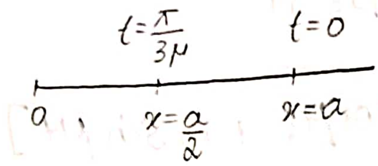
$C_2 = 0$

Sub C_1 & C_2 in (*)

$x = a \cos \mu t + 0$

$$x = a \cos \mu t \quad \text{---}(I)$$

$x = \frac{a}{2}$ to $x = a$



Put $x = a$ in eqn (I) $\quad x = a\cos\mu t$ —(I)

$$a = a\cos\mu t$$

$$1 = \cos(\mu t)$$

$$\cos^{-1}(1) = \mu t$$

$$0 = \mu t$$

$$t = \frac{0}{\mu}$$

$$t = 0$$

Put $x = \frac{a}{2}$ in eqn (II)

$$\frac{a}{2} = a\cos\mu t$$

$$\frac{1}{2} = \cos(\mu t)$$

$$\cos^{-1}\left(\frac{1}{2}\right) = \mu t$$

$$\frac{\pi}{3} = \mu t$$

$$t = \frac{\pi}{3\mu}$$

Time taken of the particle to move from the position

$x = \frac{a}{2}$ to $x = a$ is

$$t = \frac{\pi}{3\mu} - 0$$

$$t = \frac{\pi}{3\mu}$$

To prove: $\frac{1}{6}$ th of Period = t

To prove: $\frac{1}{6}T = \frac{\pi}{3\mu}$

$$\frac{1}{6}\frac{2\pi}{\mu} = \frac{\pi}{3\mu}$$

$\therefore \frac{1}{6}$ th of period $= t = \frac{\pi}{3\mu}$

Hence proved

A particle is moving on a straight line and its distance from fixed point O on it is x. And its velocity at time t and distance x is b. A relation connecting b & x is $4v^2 = 25 - x^2$. P.T the motion is the SHM and find the period & amplitude of the motion

$$4v^2 = 25 - x^2$$

Differentiating this eqn with respect to 't'.

$$8v\frac{dv}{dt} = -2x\cdot\left(\frac{dx}{dt}\right)$$

$$8v\frac{dv}{dt} = -2x\cdot v$$

$$4\frac{dv}{dt} = -x$$

$$4\left[\frac{d}{dt}\left(\frac{dx}{dt}\right)\right] = -x$$

$$4\frac{d^2x}{dt^2} = -x$$

$$\frac{d^2x}{dt^2} = -\frac{x}{4}$$

$$\frac{d^2x}{dt^2} + \frac{x}{4} = 0 \quad \text{—(1)}$$

compare eqn (1) with G.F

$$\frac{d^2x}{dt^2} + \mu^2 x = 0. \quad \text{—(2)} \quad (G.F)$$

$$\Rightarrow \mu^2 = \frac{1}{4}$$

$$\therefore \mu = \frac{1}{2}$$

At $t=0$, $V=0$

$$4V^2 = 25 - x^2$$

$$0 = 25 - x^2$$

$$x^2 = 25$$

$$x = \pm 5$$

$$x = \pm a$$

$$\therefore a = 5$$

(Amplitude can't be -ve)

Amplitude $a = 5$

Period $T = \dfrac{2\pi}{\mu}$

$$= \frac{2\pi}{\frac{1}{2}}$$

$$T = 4\pi$$

9.2.22

UNIT 5

LAPLACE TRANSFORM

↳ Let $f(t)$ be a function of t which is defined for $t>0$

↳ Then, the laplace transform of $f(t)$ is denoted by the symbol $L[f(t)]$ or $F(s)$

↳ And is defined by

$$L[f(t)] \text{ (or) } L(s) = \int_0^\infty e^{-st} f(t)\,dt$$

APP: To solve d.e in classical

Function of Exponential Order

↳ The function $f(t)$ is said to be of exponential order, if $\lim_{t\to\infty} e^{-st} \cdot f(t) = 0$

Sufficient condition for the existance of Laplace transform

2m

↳ $f(t)$ should be continous (or) Piecewise continuous (zoom:)

in the given closed interval $[a.b]$

↳ $f(t)$ should be of exponential order 2.

Laplace transforms of some basic / elementary functions.

2m 1. $L[K] = \frac{K}{S}$, $S>0$, where K is constant

$* \ L[1] = \frac{1}{S}$

Solution:

To find: $L[K]$

b'cause it time dom

W.K.T, $L[f(t)] = L(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\Rightarrow L[k] = \int_0^{\infty} e^{-st} \cdot k \, dt \qquad [f(t) = k]$$

$$= k \int_0^{\infty} e^{-st} \, dt$$

→ Integrate
↳ which we are going to integrate w.r.t 't'

$$= k \cdot \left[\frac{e^{-st}}{-s} \right]_0^{\infty}$$

$$= -\frac{k}{s} \left[e^{-st} \right]_0^{\infty}$$

$$= -\frac{k}{s} \left[e^{-\infty} - e^{0} \right] \qquad \left[\because e^{-\infty} = 0, \; e^{0} = 1 \right]$$

$$= -\frac{k}{s} [0 - 1]$$

$e^{\infty} \Rightarrow$ not defined

$$= -\frac{k}{s} (-1)$$

$$L[K] = \frac{K}{s}$$

$\Rightarrow L[1] = \frac{1}{s}$ (replace k by 1)

Thus, $L[K] = \frac{k}{s}$, $s > 0$ & k is constant

Hence Proved.

---

2. Find $L[e^{at}]$

To find: $L[e^{at}]$

W.K.T., $L[f(t)] = L(s) = \int_0^{\infty} e^{-st} f(t) \, dt$

$$\Rightarrow L[e^{at}] = \int_0^{\infty} e^{-st} \cdot e^{at} \, dt$$

$$\Rightarrow \quad = \int_0^{\infty} e^{-st+at} \, dt$$

$$= \int_0^{\infty} e^{-t(s-a)} \, dt$$

$$L[e^{at}] = \int_0^\infty e^{-t(s-a)}\, dt$$

$$= \left. \frac{e^{-t(s-a)}}{-(s-a)} \right|_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-t(s-a)} \right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-\infty} - e^{0} \right]$$

$$= \frac{-1}{s-a}\left[0-1 \right]$$

Thus, $L[e^{at}] = \frac{1}{s-a}$; $s > a$

---

3. Find $L[e^{-at}]$

Sol:

To find : $L[e^{-at}]$

W.K.T , $L[f(t)] = L(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$\Rightarrow L[e^{-at}] = \int_0^\infty e^{-st} \cdot e^{-at}\, dt$$

$$\Rightarrow \quad = \int_0^\infty e^{-st-at}\, dt$$

$$= \int_0^\infty e^{-t(s+a)}\, dt$$

$$= \left. \frac{e^{-t(s+a)}}{-(s+a)} \right|_0^\infty$$

$$= \frac{-1}{s+a}\left[e^{-t(s+a)} \right]_0^\infty$$

$$L[e^{-at}] = \frac{-1}{s+a}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{s+a}[0-1]$$

Thus, $L[e^{-at}] = \frac{1}{s+a}, \quad s>a$

---

4. find $L[\sin at]$

* if we use bernoulli rule the sum won't end
* ∴ we use direct formula

Solution:

To find: $L[\sin at]$

W.K.T $L[f(t)] = L[s] = \int_0^\infty e^{-st} f(t)\,dt$

$$\Rightarrow L[\sin at] = \int_0^\infty e^{-st} \cdot \sin at\,dt$$

W.K.T.,

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

Here $a=-s$; $b=a$; $x=t$

$$L[\sin at] = \left\{\frac{e^{-st}}{(-s)^2+a^2}\left[-s\sin at - a\cos at\right]\right\}_0^\infty$$

$$= \left[\frac{e^{-s}}{s^2+a^2}(-s\sin at)\right]_0^\infty - \left[(a\cos at)\left(\frac{e^{-s}}{s^2+a^2}\right)\right]_0^\infty$$

$$L[\sin at] = \left\{ \frac{e^{-st}}{s^2+a^2} \left[-s\sin at - a\cos at \right] \right\}_0^{\infty}$$

$$= \frac{e^{-s(\infty)}}{s^2+a2} \left(-s\sin a(\infty) - a\cos a(\infty) \right)$$

$$- \frac{e^{-s(0)}}{s^2+a2} \left(-\sin a(0) - a\cos a(0) \right)$$

$$= \underbrace{\frac{e^{-\infty}}{s^2+a^2}}_{\Rightarrow 0} \left(-s\sin a\infty - a\cos a\infty \right) \Rightarrow 0$$

$$- \frac{e^{0} \Rightarrow 1}{s^2+a2} (0-a)$$

$$= \frac{a}{s^2+a2}$$

Thus $L[\sin at] = \dfrac{a}{s^2+a2}$

---

5. Find $L[\cos at]$ Laplace form of $\cos at$

solution :

W.K.T $L[f(t)] = L(s) = \int_0^{\infty} e^{-st} f(t)\, dt$ 6. 13.2.22

$$L[\cos at] = \int_0^{\infty} e^{-st} \cos at\, dt$$

W.K.T

$$\int e^{ax} \cos bx\, dx$$

$$= \frac{e^{ax}}{a^2+b^2} \left[a\cos bx + b\sin bx \right]$$

here, $a=-s$, $b=a$, $x=t$

$$L[\cos at] = \left\{ \frac{e^{-st}}{s^2+a2} \left[-s\cos at + a\sin at \right] \right\}_0^{\infty}$$

$$= \frac{e}{s^2+a^2}\left[-s\cos a(\infty) + a\sin a(\infty)\right]$$

$$- \frac{e^{-s(0)}}{s^2+a^2}\left[-s\cos a(0) + a\sin a(0)\right]$$

$$= \frac{e^{-\infty \Rightarrow 0}}{s^2+a^2}\left[-s\cos a\infty + a\sin a\infty\right]$$

$$- \frac{e^{0 \Rightarrow 1}}{s^2+a^2}\left[-s\cos \overset{\Rightarrow 1}{0} + a\sin \overset{\Rightarrow 0}{0}\right]$$

$$= 0 - \frac{1}{s^2+a^2}\left[-s+0\right]$$

$$= \frac{s}{s^2+a^2}$$

Thus, $L[\cos at] = \dfrac{s}{s^2+a^2}$

---

6. 13.2.22 $L[\sinh at]$

soln :

W.K.T $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$L[\sin(h at)] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right]$$

$$L[\sinh at] = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

Thus,

$$L[\sinh at] = \frac{a}{s^2-a^2}, \quad s>|a|$$

---

6. $L[\cosh at]$

W.K.T $\cosh x = \dfrac{e^{x}+e^{-x}}{2}$

$$L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left\{L(e^{at}) + L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s-\cancel{a}+s+\cancel{a}}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2-a^2}\right]$$

$$= \frac{s}{s^2-a^2}$$

Thus, $L[\cosh at] = \dfrac{s}{s^2-a^2}, \quad s>|a|$

---

7. Find $L[t^n]$ $\quad * \; L[t] = \dfrac{1}{s^2} \quad (n=1)$

(X) repeatedly used function

W.K.T. $L[f(t)] = L[s] = \displaystyle\int_0^\infty e^{-st} f(t)\,dt$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$$

Put $u = st \Rightarrow \dfrac{u}{s} = t$

$\frac{du}{dt} = s$ $\quad t: 0 \to \infty$

$\frac{du}{s} = dt$ $\quad u: 0 \to \infty$ $\quad u = st,\ u = s(0),\ u = 0$ $\quad u = st,\ u = s(\infty),\ u = \infty$

$$\Rightarrow L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \cdot \frac{u^n}{s^n} \cdot \frac{du}{s}$$

$$= \int_0^\infty \frac{e^{-u} u^n}{s^{n+1}} du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n du$$

W.K.T $\int_0^\infty e^{-x} x^n dx = \Gamma(n+1) = n!$

$$\Rightarrow L[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0, \quad n = 0, 1, 2, \ldots$$

Basic Functions

1.	$L[K]$	$\frac{K}{s}, \ s > 0$		
2.	$L[1]$	$\frac{1}{s}, \ s > 0$		
3.	$L[e^{at}]$	$\frac{1}{s-a}, \ s > a$		
4.	$L[e^{-at}]$	$\frac{1}{s+a}, \ s > a$		
5.	$L[\sin at]$	$\frac{a}{s^2+a^2}$		
6.	$L[\cos at]$	$\frac{s}{s^2+a^2}$		
7.	$L[\sinh at]$	$\frac{a}{s^2-a^2}, \ s > \lvert a \rvert$		
8.	$L[\cosh at]$	$\frac{s}{s^2-a^2}, \ s > \lvert a \rvert$		
9.	$L[t^n]$	$\frac{n!}{s^{n+1}}, \ s > 0, \ n = 0, 1, 2, \ldots$	10. $L[t]$	$= \frac{1}{s^2}$ $(n=1)$

Problems based on basic functions.

1. Find $L[t^3]$

Soln: W.K.T $L[t^n] = \frac{n!}{s^{n+1}}$

$$L[t^3] = \frac{3!}{s^{3+1}} = \frac{3\times2\times1}{s^4}$$

$$L[t^3] = \frac{6}{s^4}\ ;\ s>0$$

2. Find $L[e^{3t}]$

W.K.T $L[e^{at}] = \frac{1}{s-a}$

$$L[e^{3t}] = \frac{1}{s-3}\ ,\ s\neq3$$

3. Find. $L[\frac{e^{at}}{a}]$

W.K.T $L[e^{at}] = \frac{1}{s-a}$

$$L[\frac{e^{at}}{a}] = L[\frac{1}{a}\cdot e^{at}]$$

$$= \frac{1}{a}\cdot L[e^{at}]$$

$\frac{1}{a} \to$ constant

$$L[\frac{e^{at}}{a}] = \frac{1}{a}\cdot\frac{1}{s-a}\ ,\ s\neq a$$

4. Find $L[2e^{(-3t)}]$

W.K.T $L[e^{at}] = \frac{1}{s+a}$

$$L[2e^{-3t}] = 2L[e^{-3t}]$$

$$= 2 \cdot \frac{1}{s+3}$$

$$L[2e^{-3t}] = \frac{2}{s+3}, \quad s \neq -3$$

---

5. $L[\sin 2t]$

W.K.T $L[\sin at] = \frac{s}{s^2+a^2}$

$$L[\sin 2t] = \frac{2}{s^2+4}, \quad s \neq -4$$

---

6. $L[\sin^2 2t]$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

W.K.T $\cos(2\theta) = 1 - 2\sin^2(\theta)$

$$\frac{1-\cos(2\theta)}{2} = \sin^2(\theta)$$

$$\sin^2 2t = \frac{1-\cos 2(2t)}{2}$$

$$= \frac{1-\cos 4t}{2}$$

$$L[\sin^2 2t] = L\left[\frac{1-\cos 4t}{2}\right]$$

$$= L\left[\frac{1}{2}\right] - L\left[\frac{\cos 4t}{2}\right]$$

$$= \frac{1}{2}L[1] - \frac{1}{2}L[\cos 4t]$$

$L[\cos at]$

$= \frac{s}{s^2+a^2}$

$(a=4)$

$$= \frac{1}{2s} - \frac{1}{2}\left[\frac{s}{s^2+4^2}\right]$$

$$L[\sin^2 2t] = \frac{1}{2s} - \frac{1}{2}\cdot\left(\frac{s}{s^2+16}\right), \quad s \neq 0, \; s \neq -16$$

W.K.T, $L[K] = \frac{K}{s}$ $K = 1$

$L[1] = \frac{1}{s}$

$L[\cos at] = \frac{s}{s^2+a^2}$

7. Find $L[\sin 5t \cdot \cos 2t]$

W.K.T $L[\sin at]$

Sol: $= \frac{a}{s^2+a^2}$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$
$\sin(a-b) = \sin a \cos b - \cos a \sin b$
(+)
$\sin(a+b) + \sin(a-b) = 2 \sin a \cos b$

$\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$

$\sin 5t \cos 2t = \frac{1}{2}[\sin 7t + \sin 3t]$

$L[\sin 5t \cdot \cos 2t]$

$= L\left[\frac{1}{2}(\sin 7t + \sin 3t)\right]$

$= \frac{1}{2} L[\sin 7t] + \frac{1}{2} L[\sin 3t]$

$= \frac{1}{2} \cdot \frac{7}{s^2+49} + \frac{1}{2} \cdot \frac{3}{s^2+9}$

$L[\sin 5t \cdot \cos 2t] = \frac{7}{2(s^2+49)} + \frac{3}{2(s^2+9)}$

8. $L[\cos 4t \cdot \cos 2t]$ $L[\cos 4t \sin 2t]$

$\cos(a+b) = \cos a \cos b - \cancel{\sin a \sin b}$
$\cos(a-b) = \cos a \cos b + \cancel{\sin a \sin b}$
(+)
$\cos(a+b) + \cos(a-b) = 2 \cos a \cos b$

$\frac{1}{2}[\cos(a+b) + \cos(a-b)] = \cos a \cos b$

W.K.T $L[\cos at] = \frac{s}{s^2+a^2}$

$L[\cos 4t \cos 2t] = L\left[\frac{1}{2}\left(\cos(4+2)t + \cos(4-2)t\right)\right]$

$= L\left[\frac{1}{2}(\cos 6t + \cos 2t)\right]$

$= \frac{1}{2} L[\cos 6t] + \frac{1}{2} L[\cos 2t]$

$= \frac{1}{2} \cdot \frac{s}{s^2+36} + \frac{1}{2} \cdot \frac{s}{s^2+4}$

$L[\cos 4t \cos 2t] = \frac{s}{2(s^2+36)} + \frac{s}{2(s^2+4)}$; $s \neq -36$, $s \neq -4$

14-2-23 Linear Property

1. If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$,
~~then~~ where a & b are constants:
then
$L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)]$
$L[a f(t) + b g(t)] = a F(s) + b G(s)$

2. First shifting Property

- If $L[f(t)] = F(s)$ then $L[\underline{e^{at}} f(t)] = F(s-a)$ (exponent)
- $L[e^{\mp at} f(t)] = F(s \pm a) = L[f(t)]/_{s \to s \pm a}$
$= F(s-a) = L[f(t)]/_{s \to s-a}$

3. Change of scale Property:

If $L[f(t)] = F(s)$ then
$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

$L[e^{-at} f(t)]$
$= F(s+a)$
$= L[f(t)]/_{s \to s+a}^{s \to s-a}$

Problems based on Properties.

1. Find $L[e^{-t}t^9]$

$e^{-at} f(t)$.

Soln : By first shifting Property, we have

$$L[e^{-at} f(t)] = L[f(t)] / s \to s+a$$

$$L[e^{-t} t^9] = L[t^9] / s \to s+1$$

$$= \frac{9!}{s^{10}} / s \to s+1$$

$$L[e^{-t}t^9] = \frac{9!}{(s+1)^{10}}$$

[Here, $f(t) = t^9$

$e^{-at} = e^{-t}$

$\therefore a = 1$

$\because L[t^n] = \frac{n!}{s^{n+1}}$

$L[t^9] = \frac{9!}{s^{9+1}}$]

3.

2. Find $L[e^{at} \cos bt]$

soln.

By first shifting property, we have,

If $L[f(t)] = F(s)$ then

$$L[e^{\pm at} f(t)] = L[f(t)] /_{s \to s \mp a}$$

$$L[e^{at} \cos bt] = L[\cos bt] / s \to s-a$$

Here $e^{at} = e^{at}$

$\therefore a = a$

$f(t) = \cos bt$

$L[\cos at] = \frac{s}{s^2+a^2}$ here $(a=b)$

$L[e^{at}\cos bt] = L[\cos bt] / s \to s-a$

$= \frac{s}{s^2+b^2} / s \to s-a$

$L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}$

---

3. Find $L[\cosh t . \sin 2t]$

~~cosa sinb~~ = W.K.T $\cosh x = \frac{e^x+e^{-x}}{2}$ $(x=t)$

$L[\cosh t . \sin 2t]$

$= L\left[\left(\frac{e^t+e^{-t}}{2}\right)\sin 2t\right]$

$= \frac{1}{2}L[e^t \sin 2t + e^{-t}\sin 2t]$

$= \frac{1}{2}L[e^t \sin 2t] + \frac{1}{2}L[e^{-t}\sin 2t]$

By first shifting property,

$L[e^{at}f(t)] = L[f(t)] / s \to s-a$

$L[e^{-at}f(t)] = L[f(t)] / s \to s+a$

$e^{at} = e^t$ $e^{-at} = e^{-t}$

$a = 1$ $a = 1$

$f(t) = \sin 2t$ $f(t) = \sin 2t$

$s \to s-a = s-1$ $s \to s+a = s+1$

$L[\cosh t \cdot \sin 2t]$

$$= \frac{1}{2} L[\sin 2t]\Big|_{s\to s-1} + \frac{1}{2} L[\sin 2t]\Big|_{s\to s+1}$$

W.K.T $L[\sin at] = \frac{a}{s^2+a^2}$ here $a=2$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{s^2+4}\right]\Big|_{s\to s-1} + \frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{s^2+4}\right]\Big|_{s\to s+1}$$

$$= \frac{1}{(s-1)^2+4} + \frac{1}{(s+1)^2+4}$$

$$L[\cosh t \cdot \sin 2t] = \frac{1}{(s-1)^2+4} + \frac{1}{(s+1)^2+4}$$

---

4. Find $L[e^{-3t} \cdot \sin^2 t]$ $\cos 2\theta = 1 - 2\sin^2\theta$

W.K.T $\sin^2 t = \frac{1-\cos 2t}{2}$

$L[e^{-3t} \sin^2 t]$

$$= L\left[e^{-3t} \cdot \left(\frac{1-\cos 2t}{2}\right)\right]$$

$$= \frac{1}{2} L[e^{-3t}] - \frac{1}{2} L[e^{-3t} \cos 2t]$$

W.K.T $L[e^{-at}] = \frac{1}{s+a}$ here $a=3$

$e^{-3t} = e^{-at}$

$a=3$

By first shifting property,

$$L[e^{-at} f(t)] = L[f(t)]\Big|_{s\to s+a}$$

$L[e^{-3t} \sin^2 t]$

$$= \frac{1}{2}\left(\frac{1}{s+3}\right) - \frac{1}{2} L[\cos 2t] \Big/ s \to s+3$$

W.K.T $L[\cos at] = \frac{s}{s^2+a^2}$ here $a=2$

$$= \frac{1}{2}\left[\frac{1}{(s+3)}\right] - \frac{1}{2}\left(\frac{s}{s^2+4}\right) \Big/ s \to s+3$$

$$= \frac{1}{2(s+3)} - \frac{1}{2}\left[\frac{s+3}{(s+3)^2+4}\right]$$

---

5. $L[8e^{8t} + \cosh 3t + \sin 5t]$

$$= 8 \cdot L[e^{8t}] + L[\cosh 3t]$$
$$+ L[\sin 5t]$$

$$= 8\left(\frac{1}{s-8}\right) + \frac{s}{s^2-9} + \frac{5}{s^2+25}$$

$$L[8e^{8t} + \cosh 3t + \sin 5t] = \frac{8}{s-8} + \frac{s}{s^2-9} + \frac{5}{s^2+25}$$

* $L[\sin^3 2t]$
* $L[\cos^3 3t]$
* $L[e^{-at} \cos b]$

---

6. $L[5 - 3t - 2e^{-t}]$

$$= \frac{5}{s} - \frac{3}{s^2} - \frac{2}{s+1}$$

---

7. $L[5e^{-7t} - \sinh 3t]$

$$= \frac{5}{s+7} - \frac{3}{s^2-9}$$

Derivatives of Laplace Transform

- If $L[f(t)] = F(s)$ then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$ * $f(t)$ t^n = poly.

$$= (-1)^n \frac{d^n}{ds^n}(L[f(t)])$$

Problems based on Derivatives of Laplace Transform.

1. Find the laplace transform of $t^2 e^{-3t}$

$$L[t^2 e^{-3t}] = (-1)^2 \frac{d^2}{ds^2} L[e^{-3t}]$$

$t^n \cdot f(t)$

$n=2$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}(s+3)^{-1}\right]$$

$$= \frac{d}{ds}\left[(-1)(s+3)^{-2}\right]$$

$$= (-1)(-2).(s+3)^{-3}$$

$$= \frac{2}{(s+3)^3}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\therefore L[t^2 e^{-3t}] = \frac{2}{(s+3)^3}$$

---

2. Find $L[t.\sin t]$

t^n $f(t)$

$n=1$

$$L[t\sin t] = (-1)\frac{d}{ds}L[\sin t]$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{d}{ds}(s^2+1)^{-1}$$

$$= -(-1)(s^2+1)^{-2}\cdot 2s$$

To make power as +ve.

$$L[t\sin t] = \frac{2s}{(s^2+1)^2}$$

---

3. $L[t^2\cos 2t]$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} L[f(t)]$$

$$L[\underset{t^n}{t^2}\ \underset{f(t)}{\cos 2t}] = (-1)^2 \frac{d^2}{ds^2} L[\cos 2t]$$

$n=2$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$$

$$\frac{vdu-udv}{v^2} = \frac{vu'-uv'}{v^2}$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{\overset{u}{s}}{\underset{v}{s^2+4}}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{\overset{v}{(s^2+4)}\overset{u'}{(1)} - \overset{u}{s}\overset{v'}{(2s)}}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-s^2+4}{(s^2+4)^2}\right]$$

$$= \frac{(s^2+4)^2(-2s) - (-s^2+4)(2s)(s^2+4)(2)}{(s^2+4)^4}$$

$$= \frac{\cancel{(s^2+4)}(2s)\left[-(s^2+4) - (-s^2+4)\cancel{(2s)}2\right]}{(s^2+4)^{\cancel{4}3}}$$

$$= \frac{2s}{(s^2+4)^3}\left[-s^2-4+2s^2-8\right]$$

$$= \frac{2s}{(s^2+4)^3}\left[s^2-12\right]$$

$$L[t^2\cos 2t] = \frac{2s^3-24s}{(s^2+4)^3}$$

2m

15.2.23 **Integrals of Laplace Transform** 2m

If $L[f(t)]=F(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$

Problems.
$$= \int_s^\infty L[f(t)]\,ds$$

1. Find $L\left[\frac{\sin at}{t}\right]$

Sol.

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\,ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= \cancel{a}\cdot\frac{1}{\cancel{a}}\tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$$

$$L\left[\frac{\sin at}{t}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

Here we used,

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

here $f(t) = \sin at$

W.K.T $L[\sin at] = \frac{a}{s^2+a^2}$

$\bullet \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

---

2 Find $L\left[\frac{1-\cos at}{t}\right]$

TO find: $L\left[\frac{1-\cos t}{t}\right]$

soln:

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^{\infty} L[1-\cos t]\, ds$$

$$= \int_s^{\infty} \left(L[1] - L[\cos t]\right) ds$$

$$= \int_s^{\infty} \left(\frac{1}{s} - \frac{s\times 2}{s^2+a^2\times 2}\right) ds$$

$$= \left[\log(s) - \frac{1}{2}\log(s^2+a^2)\right]\Big|_s^{\infty}$$

$$= \left[\log s - \log(s^2+1)^{1/2}\right]_s^{\infty}$$

when we sub ∞ = $\log\left(\frac{s}{\sqrt{s^2+1}}\right)\Big|_s^{\infty}$

$\frac{\infty}{\infty}$ = undefined

$\therefore$ cancel s in num.

$$= \log\left(\frac{s}{s\sqrt{1+1/s^2}}\right)\Big|_s^{\infty} \qquad \frac{1}{s^2} = \frac{1}{\infty} = 0$$

W.K.T., $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} L[f(t)]\, ds$

here $f(t) = 1-\cos t$

$\cos at = \cos t$

$\bullet\ L[\cos at] = \frac{s}{s^2+a^2}$ here $a=1$

$$L\left[\frac{1-\cos t}{t}\right] = \log\left(\frac{1}{\sqrt{1+1/s^2}}\right)\Big/_s^\infty$$

$$= \log 1 - \log\left(\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right)$$

$$= 0 - \log\left(\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right)$$

$$= -\log\left(\frac{1}{\sqrt{1+\frac{1}{s^2}}}\right)$$

$$L\left[\frac{1-\cos t}{t}\right] = \log\sqrt{1+\frac{1}{s^2}}$$

(*) 2m

UNIT STEP FUNCTION

Laplace transform of unit step function

* It is denoted by the symbol $U(t-a)$ and is defined by

(*) university Q

$$U(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

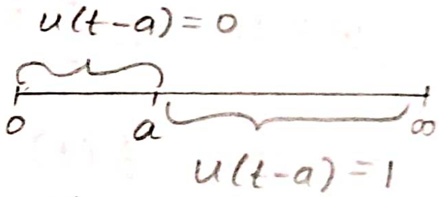
(*) 2m

Find the Laplace transform of unit step function.

To find: $L[U(t-a)]$

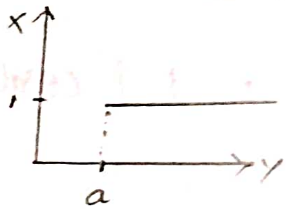
W.K.T., $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L[U(t-a)] = \int_0^\infty e^{-st}\, U(t-a)\,dt$$



Soln:

$L[u(t-a)]$



$$= \int_0^a e^{-st}\, U(t-a)\,dt + \int_a^\infty e^{-st}\, U(t-a)\,dt$$

$$L[U(t-a)] = \int_0^\infty e^{-st}\, U(t-a)\, dt$$

$$= \int_0^a e^{-st}\, \overset{\Rightarrow 0}{\underline{U(t-a)}}\, dt + \int_a^\infty e^{-st}\, \overset{\Rightarrow 1}{U(t-a)}\, dt$$

$$= \int_a^\infty e^{-st}\, U(t-a)\, dt$$

$$= \int_a^\infty e^{-st} (1)\, dt$$

$$= \left. \frac{e^{-st}}{-s} \right|_a^\infty$$

$$= -\frac{1}{s}\left[\overset{\Rightarrow 0}{e^{-\infty s}} - e^{-as} \right]$$

$$= \frac{e^{-as}}{s}$$

$$\boxed{L[U(t-a)] = \frac{e^{-as}}{s}}$$

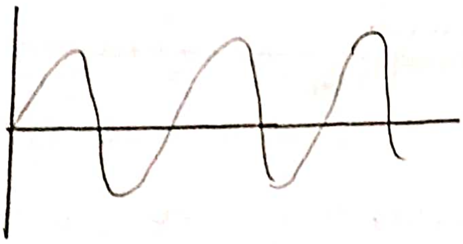
LAPLACE TRANSFORM OF PERIODIC FUNCTION

(1) **Periodic Function.** (General Definition)

A function $f(t)$ is said to have a period T or to be periodic if following condition is satisfied.

- $f(t+T) = f(t)$ for all t.
- where T is a positive constant.
- i.e $T>0$.
- The least value of T is known is known as period of the function $f(t)$.

Ex. • Sine t, $\cos t$ is a periodic function with period 2π.
- $\tan t$ is a periodic function with period π.

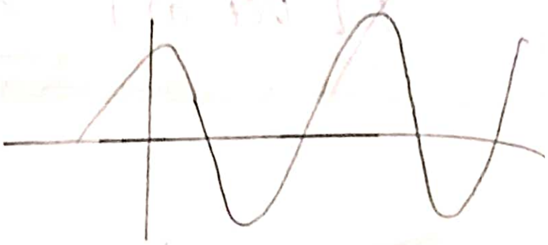


Sine wave

$0 \to \pi, 2\pi, 3\pi, 4\pi$

$T = \pi, 2\pi, 3\pi$

$\sin n\pi = 0 \ \forall n$



cos wave

$\cos n\pi = (-1)^n \ \forall n$

Laplace transform of Periodic Function

If $f(t)$ is a periodic function with period T, then the laplace transform of $f(t)$ is defined by

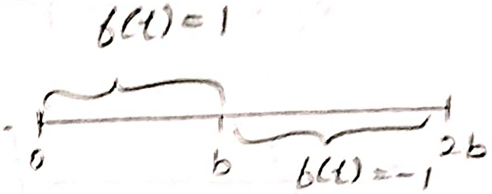
$$\boxed{L[f(t)] = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\, dt}$$

problem.

Find the laplace transform of

1. $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$ with

$f(t + \underset{T}{2b}) = f(t) \rightarrow$ condition



sol:

Here $T = 2b$.

$f(t)$ is continuous but discontinuous at '$\underline{b}$'

W.K.T

$$L[f(t)] = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \int_0^b e^{-st} f(t)\, dt + \int_b^{2b} e^{-st} f(t)\, dt \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \int_0^b e^{-st} \underline{(1)}\, dt + \int_b^{2b} e^{-st} \underline{(-1)}\, dt \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \left[\frac{e^{-st}}{-s} \right]_0^b - \left[\frac{e^{-st}}{-s} \right]_b^{2b} \right\}$$

limits are not same
∴ should not
combine.

$$L[f(t)] = \frac{1}{1-e^{-2bs}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^b - \left[\frac{e^{-st}}{-s}\right]_b^{2b}\right\}$$

$$= \frac{1}{1-e^{-2bs}}\left\{\frac{-1}{s}(e^{-bs} - e^0) + \frac{1}{s}(e^{-2bs} - e^{-bs})\right\}$$

$$= \frac{1}{s(1-e^{-2bs})}\left\{-e^{-bs} + 1 + e^{-2bs} - e^{-bs}\right\}$$

$$= \frac{1}{s(1-e^{-2bs})}\left\{\overset{-2ab}{-2e^{-bs}} + \overset{b^2}{e^{-2bs}} + \overset{a^2}{1}\right\}$$

$$= \frac{1}{s(1^2 - e^{-2bs})}\left[(1-e^{-bs})^2\right] \quad (a-b)^2 = a^2+b^2-2ab$$

$$= \frac{1}{s[1^2 - (e^{-bs})^2]} \cdot (1-e^{-bs})^2$$

$$= \frac{(1-e^{-bs})^{\cancel{2}}}{s\cancel{(1-e^{-bs})}(1+e^{-bs})}$$

$$= \frac{1-e^{-bs}}{s(1+e^{-bs})} \times \frac{e^{bs/2}}{e^{bs/2}} \qquad -1+\frac{1}{2} \quad \frac{-2+1}{2} = -\frac{1}{2}$$

$$= \frac{e^{bs/2} - e^{-bs/2}}{s(e^{bs/2} + e^{-bs/2})} \qquad \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x)$$

$$= \frac{1}{s}\tanh\left(\frac{bs}{2}\right)$$

---

2. Find the laplace transform of the Periodic function $f(t)$.

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 2-t, & 1 < t < 2 \end{cases}$$

with $f(\underset{T}{t+2}) = f(t)$

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left\{ \int_0^2 e^{-st} f(t)\, dt \right\}$$

$f(t) = t$

$f(t) = 2-t$

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st} f(t)\, dt + \int_1^2 e^{-st} f(t)\, dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st} (t)\, dt + \int_1^2 e^{-st} \overset{2-t}{\cancel{(t-2)}}\, dt \right\}$$

Using Bernoulli's Theorem,

$uv_1 - u'v_2$

I - Part	II - Part
$u = t$	$u = \overset{2-t}{\cancel{t-2}}$
$du = 1$	$du = -1$

common for both I & II ← $V = e^{-st}$

$$dv_1 = \frac{e^{-st}}{-s}$$

$$v_2 = \frac{e^{-st}}{s^2}$$

$L[f(t)]$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st}(t)\,dt + \int_1^2 e^{-st}(2-t)\,dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[t\left[\frac{e^{-st}}{-s}\right] - (1)\frac{e^{-st}}{s^2} \right]_0^1 \right.$$

$$\left. + \left[(2-t)\left(\frac{e^{-st}}{-s}\right) \overset{-(-1)}{+1} \left(\frac{e^{-st}}{s^2}\right) \right]_1^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[\left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\right) - \left(-\frac{e^0}{s^2}\right) \right] + \right.$$

$$\left. \left[\frac{e^{-2s}}{s^2} - \left(\frac{e^{-s}}{-s} + \frac{e^{-s}}{s^2} \right) \right] \right\}$$

$$= \frac{1}{1-e^{-2s}} \left[\cancel{-\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} + \cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{2e^{-s}}{s^2} + \frac{1}{s^2} + \left(\frac{e^{-s}}{s}\right)^2 \right]$$

$$= \frac{1}{s^2(1-e^{-2s})} \left[1 - 2e^{-s} + (e^{-s})^2 \right]$$

$(1)^2 - (e^{-s})^2$

$$= \frac{1}{s^2(1+e^{-s})\cancel{(1-e^{-s})}} (1-e^{-s})^{\cancel{2}}$$

$$\frac{e^x + e^{-x}}{e^x - e^{-x}} = \tan hx$$

$$= \frac{1-e^{-s}}{s^2(1+e^{-s})} = \frac{1}{s^2} \tan\left(h\,\frac{s}{2}\right)$$

Q.23. 3. Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & \pi<t<2\pi \end{cases}$$

with $f(t+2\pi) = f(t)$

Sol:

G.T, $f(t)$ is a periodic function with period 2π.

$\Rightarrow T = 2\pi$.

$$L[f(t)] = \frac{1}{1-e^{-Ts}} \left[\int_0^T e^{-st} f(t)\, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st} f(t)\, dt + \int_{\pi}^{2\pi} e^{-st} f(t)\, dt \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st} (\sin t)\, dt + \underbrace{\int_{\pi}^{2\pi} e^{-st} (0)\, dt}_{\to 0} \right\}$$

W.K.T $\int e^{ax} \sin bx = \dfrac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st} \sin t\, dt \right\} \quad \begin{matrix} a=-s \\ b=1 \end{matrix} \quad t=x$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{e^{-st}}{(-s)^2+(1)^2} \left[-s \sin t - \cos t\right] \right\}_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{e^{-\pi s}}{s^2+1} \left[-s \sin \pi - \cos \pi\right] - \frac{e^{0}}{s^2+1} \left[-s \sin 0 - \cos 0\right] \right\}$$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s}} \left\{ \frac{e^{-\pi s}}{s^2+1} \left[-s\overset{\to 0}{\sin\pi} - \overset{\to(-1)}{\cos\pi} \right] - \frac{\overset{\to 1}{e^{0}}}{s^2+1} \left[-s\overset{\to 0}{\sin 0} - \cos 0 \right] \right.$$

W.K.T., $\sin n\pi = 0 \;\forall n$

$\cos n\pi = (-1)^n \;\forall n$

$\cos\pi = (-1)^1 = -1$

$$L[f(t)] = \frac{1}{(1-e^{-2\pi s})(s^2+1)} \left[e^{-\pi s}(1) - 1(-1) \right]$$

$$= \frac{1}{s^2+1} \cdot \frac{1}{(1)^2-(e^{-\pi s})^2} \left[e^{-\pi s} + 1 \right]$$

$$= \frac{1}{s^2+1} \cdot \frac{\cancel{(e^{-\pi s}+1)}}{\cancel{(1+e^{-\pi s})}(1-e^{-\pi s})}$$

$$L[f(t)] = \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

INVERSE LAPLACE TRANSFORM

Inverse laplace Transform

If laplace transform of $f(t)$ is denoted by the symbol $L[f(t)]$ or $F(s)$ then the inverse laplace Transform is denoted by the symbol 'L^{-1}' i.e.,

$L^{-1}[F(s)] = f(t)$, then where

L^{-1} is called inverse laplace operator.

$L[f(t)] = F(s)$

IMPORTANT RESULTS

$f(t) = L^{-1}[F(s)]$

Inverse laplace transform of some elementary functions.

1. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

2. $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

3. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

4. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

5. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$ hyperbolic

6. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

7. $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

BASIC PROPERTIES OF INVERSE LAPLACE TRANSFORM.

2

1. LINEAR PROPERTY.

If $L^{-1}[F(s)] = f(t)$ &

$\Rightarrow$ if and only

$\Leftrightarrow$

$L^{-1}[G(s)] = g(t)$ then

$$L^{-1}[aF(s) + b\,G(s)] = a\,f(t) + b\,g(t)$$

2. FIRST SHIFTING PROPERTY

If $L^{-1}[F(s)] = f(t)$,

then $L^{-1}[F(s+a)] = e^{-at} f(t) = e^{-at} L^{-1}[F(s)]$

& $L^{-1}[F(s-a)] = e^{at} f(t) = e^{at} L^{-1}[F(s)]$

3. CHANGE OF SCALE PROPERTY

If $L^{-1}[F(s)] = f(t)$ then

$$L^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right)$$

Problems based of Basic Properties.

1. Find $L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$

$$\because L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$L(e^{at}) = \frac{1}{s-a}$$

$$\therefore L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$$

Find $L^{-1}\left[\frac{2s}{s^2-16}\right]$

2\. $L^{-1}\left[\frac{2s}{s^2-16}\right] = 2\, L^{-1}\left[\frac{s}{s^2-4^2}\right]$

$= 2 \cosh 4t$

$L[\cosh at] = \frac{s}{s^2-a^2}$

$\cosh at = L^{-1}\left(\frac{s}{s^2-a^2}\right)$

Find

3\. $L^{-1}\left[\frac{1}{s^2+25}\right]$

$L^{-1}\left[\frac{1}{s^2+25}\right] = \frac{\sin 5t}{5}$

4 Find $L^{-1}\left(\frac{s}{s^2+1}\right)$

$L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$

5\. Find $L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right]$

soln:

$\Rightarrow L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s+4}\right] + L^{-1}\left[\frac{1}{s^2+4}\right]$

$+ L^{-1}\left[\frac{s}{s^2-9}\right]$

$\Rightarrow t + e^{-4t} + \frac{\sin 2t}{2} + \cosh 3t$

6. Find $L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

In first shifting property

$$L\left[e^{-at} f(t)\right] = L\left[f(t)\right]\Big/_{s\to s+a}$$

$$\Rightarrow L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t} \cdot L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\Rightarrow \quad = e^{-t}\sin t.$$

$$L\left[e^{3t}\sin t\right] = L\left[\sin t\right]\Big/_{s\to s-3}$$

$$= \frac{1}{s^2+1}\Big/_{s\to s-3}$$

reverse process

$$= \frac{1}{(s-3)^2+1}$$

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7. $L^{-1}\left[\frac{s}{(s+2)^2+1}\right]$

$$= L^{-1}\left[\frac{s+2-2}{(s+2)^2+1}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}\right]$$

$$= e^{-2t}\, L^{-1}\left[\frac{s}{s^2+1}\right] - 2 e^{-2t}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-2t}\cos t - 2e^{-2t}\frac{\sin t}{1}$$

$$L^{-1}\left[\frac{s}{(s+2)^2+1}\right] = e^{-2t}\cos t - 2e^{-2t}\sin t$$

8. $L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right]$

$$= e^{3t} L^{-1}\left[\frac{s}{(s-3)^2+4}\right]$$

$$= e^{3t} L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= e^{3t} \cos 2t$$

[Using 1st shifting property]

9. Find $L^{-1}\left[\frac{s}{(s-b)^2+a^2}\right]$

$$= L^{-1}\left[\frac{s+b-b}{(s-b)^2+a^2}\right]$$

$$= L^{-1}\left[\frac{s-b}{(s-b)^2+a^2}\right] + L^{-1}\left[\frac{b}{(s-b)^2+a^2}\right]$$

$$= e^{bt} L^{-1}\left[\frac{s}{s^2+a^2}\right] + b e^{bt} L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= e^{bt} \cos at + b e^{bt} \frac{\sin at}{a}$$

$$= e^{bt}\left[\cos at + \frac{b \sin at}{a}\right]$$

?. $L^{-1}\left[\frac{1}{(s-4)^2} + \frac{5}{(s-2)^2+5^2} + \frac{s+3}{(s+3)^2+6^2}\right]$

$$= L^{-1}\left[\frac{1}{(s-4)^2}\right] + L^{-1}\left[\frac{5}{(s-2)^2+5^2}\right] + L^{-1}\left[\frac{s+3}{(s+3)^2+6^2}\right]$$

$$= e^{4t} L^{-1}\left[\frac{1}{s^2}\right] + e^{2t} L^{-1}\left[\frac{5}{s^2+5^2}\right] + e^{-3t} L^{-1}\left[\frac{s}{s^2+6^2}\right]$$

$$= e^{4t} t + e^{2t} \sin 5t + e^{-3t} \cos 6t$$

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There are two methods for finding inverse laplace transform.

1) Partial Fraction Method
2) Convolution Method.

Partial Fraction Method.

[for solving d.e, application of d.e]

TYPE 1

1. Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

10m

Soln: Using Partial Fraction method, we can split,

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \quad \text{---} (1)$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A(s+1)(s+2) + B(s+2)s + C(s+1)s}{s(s+1)(s+2)}$$

$$1 = A(s+1)(s+2) + B(s+2)s + C(s+1)s \quad \text{---} (*)$$

Put $s=0$ in eqn (*) ⇒ to find A

$1 = A(1)(2) + 0 + 0$

$\frac{1}{2} = A$

Put $s=-1$ in eqn (*) ⇒ to find B

$1 = 0 + B(-1+2)(-1) + 0$

$1 = -B$

$B = -1$

put $S = -2$ in eqn (*) $\Rightarrow$ to find c

$$1 = 0 + 0 - 2C(-2+1)$$

$$2C = 1$$

$$C = \frac{1}{2}$$

sub the values of A, B & C in eqn (1)

$$\frac{1}{S(S+1)(S+2)} = \frac{1}{2S} - \frac{1}{S+1} + \frac{1}{2(S+2)}$$

Taking inverse Laplace in both sides, we've

$$L^{-1}\left[\frac{1}{S(S+1)(S+2)}\right] = L^{-1}\left[\frac{1}{2S}\right] - L^{-1}\left[\frac{1}{S+1}\right] + L^{-1}\left[\frac{1}{(S+2)2}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{S}\right] + \left\{-L^{-1}\left[\frac{1}{S+1}\right]\right\} + \frac{1}{2}L^{-1}\left[\frac{1}{S+2}\right]$$

$L[1] = \frac{1}{k}$

$$= \frac{1}{2}(1) - e^{-t} + \frac{1}{2}e^{-2t}$$

$$L^{-1}\left[\frac{1}{S(S+1)(S+2)}\right] = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

⊗ TYPE 2

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2. Find $L^{-1}\left[\frac{1}{(S+1)^2(S+2)}\right]$ ↳ 3 factors

$(S+1)^2 = 0$
$S+1 = 0$
$S = -1$ (Twice)
↳ Repeated factor

Sol:

Consider $\frac{1}{(S+1)^2(S+2)}$

Using partial fraction, we've

$$\frac{1}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2} \quad \text{---(1)}$$

$$\frac{1}{(s+1)^2(s+2)} = \frac{A(s+1)(s+2) + B(s+2) + C(s+1)^2}{(s+1)^2(s+2)}$$

$$1 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \quad \text{---}(*)$$

Put $s=-1$ in $(*)$

for B & C there is no common factor, first find B.

$1 = 0 + B$

$\boxed{B=1}$

Put $s=-2$ in $(*)$

$1 = \cancel{A(-1)}^{0} + C(-2+1)^2$

$1 = \cancel{-A} + C$

$\boxed{C=1}$

Put $s=0$

$1 = 2A + 2B + C$

$1 = 2A + 2 + 1$

$1-3 = 2A$

$2A = -2$

$\boxed{A=-1}$

sub A, B & C values in (1)

$$\frac{1}{(s+1)^2(s+2)} = \frac{-1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}$$

taking inverse laplace in both sides.

$$L^{-1}\left[\frac{1}{(s+1)^2(s+2)}\right] = L^{-1}\left[\frac{-1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}\right]$$

$$L^{-1}\left[\frac{1}{(s+1)^2(s+2)}\right] = L^{-1}\left[\frac{-1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$+ L^{-1}\left[\frac{1}{s+2}\right]$$

$$\downarrow$$

$$= -e^{-t} + e^{-2t} + \underbrace{e^{-t}.L^{-1}\left[\frac{1}{(s\ \ )^2}\right]}_{\text{using first shifting prop}}$$

$$\cancel{= -e^{-t} + e^{-2t} + L^{-1}\left[\frac{s+1}{(s+1)^2}\right] - L^{-1}\left[\frac{s}{(s+1)^2}\right]}$$

$$= -e^{-t} + e^{-2t} \quad L^{-1}\ e^{-t}.t$$

$$= e^{-t}[t-1] + e^{-2t} \qquad L[t] = \frac{1}{s^2}$$

TYPE 3 – sq inside bracket

combination of TYPE II & III

3. Find $L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right] \Rightarrow$ 4 factors $\Rightarrow$ 4 constants

soln:

Consider $\frac{s}{(s+1)^2(s^2+1)}$

Using partial fraction, we have

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1} \quad \text{—(1)}$$

$$\frac{s}{\cancel{(s+1)^2(s^2+1)}} = \frac{A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2}{\cancel{(s+1)^2(s^2+1)}}$$

$$s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 \quad \text{—(*)}$$

Put $s=0$ in (*)

$0 = A + B + D \quad \Rightarrow \quad A + D = \frac{1}{2}$

Put $s=-1$ in (*)

$-1 = 0 + 2B + 0$

$\boxed{B = \frac{-1}{2}}$

$$S = A(S+1)(S^2+1) + B(S^2+1) + (CS+D)(S+1)^2 \quad (*)$$

Put $S = 1$ in $(*)$

$$1 = A(2)(2) + B(2) + (C+D)(2)^2$$

$$1 = 4A + 2B + 4C + 4D$$

$$1 = 4A + 2\left(-\frac{1}{2}\right) + 4C + 4D$$

$$2 = 4A + 4C + 4D$$

$$1 = 2A + 2C + 2D \quad ; \quad 0 = A + B + D \quad ; \quad A + D = \frac{1}{2} \quad -(2)$$

$$\cancel{1 = 2A + C + \frac{1}{2} - A} \qquad 0 = \frac{1}{2} - A$$

$$\cancel{\frac{1}{2} = A + C}$$

$$1 = 2(A + C + D)$$

$$A + C + D = \frac{1}{2} \quad -(3)$$

$$\underset{(-)}{A} \quad \underset{(-)}{+D} = \underset{(-)}{\frac{1}{2}} \quad -(2)$$

$$\boxed{C = 0} \qquad B = -\frac{1}{2}$$

Put $S = 2$ in eqn $(*)$

$$2 = A(3)(5) + B(5) + (2C+D)(9)$$

$$2 = 15A + 5B + 18C + 9D$$

$$2 = 15A + 5\left(-\frac{1}{2}\right) + 18(0) + 9D$$

$$2 = 15A - \frac{5}{2} + 9D$$

$$2 + \frac{5}{2} = 15A + 9D$$

$$3\frac{9}{2} = 3(5A + 3D)$$

$$5A + 3D = \frac{3}{2} \quad -(4)$$

solve eqns (2) & (4)

$(2) \times 3; \Rightarrow \quad 3A + 3D + \frac{3}{2}$

$(4) \Rightarrow \quad 5A + 3D = \frac{3}{2}$

$\quad (-) \quad (-) \quad (-)$

$2A = 0$

$A = 0$

Sub $A = 0$ in $(2) \Rightarrow A + D = \frac{1}{2}$

$\boxed{D = \frac{1}{2}}$

Sub the values of A, B, C & D in (1).

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+d}{s^2+1}$$

$$= \frac{0}{s+1} + \frac{-1}{2(s+1)^2} + \frac{1}{2(s^2+1)}$$

$$\frac{s}{(s+1)^2(s^2+1)} = \frac{-1}{2(s+1)^2} + \frac{1}{2(s^2+1)}$$

Taking inverse laplace in both sides

$$L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right] = \frac{-1}{2} L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{-1}{2} e^{-t} \cdot L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{2} \sin t$$

$$= \frac{-1}{2} e^{-t} \cdot t + \frac{1}{2} \sin t \qquad L[t] = \frac{1}{s^2}$$

$$L^{-1}\left[\frac{s}{(s+1)^2(s^2+1)}\right] = \frac{1}{2}\left[\sin t - te^{-t}\right]$$

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Applications of Laplace Transform

Steps to solve linear Differential equations:

1. Take laplace transform on both sides of the given equation.
2. Write the formulas for $L[y''(t)]$, $L[y'(t)]$, etc.
3. Apply the given initial condition
4. From the resulting expression find inverse laplace Transform to get the solution.

Formulas:

1. $L[y(t)] = L[y(t)]$
2. $L[y'(t)] = sL[y(t)] - y(0)$
3. $L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$

Problems.

10M 1. Solve $\frac{d^2y}{dt^2} + 4\frac{dt}{dy} - 5y = 5$, given that

$y = 0$; $\frac{dy}{dt} = 2$ at $t = 0$.

$y(\frac{0}{t}) = 0$

Soln:

$$y''(t) + 4y'(t) - 5y(t) = 5;$$

G.T $y(0) = 0$; $y'(0) = 2$

Taking Laplace Transform (LT) on both sides (b.s), we get,

$$L[y''(t) + 4y'(t) - 5y(t)] = L[5]$$

$$L[y''(t)] + 4L[y'(t)] - 5L[y(t)] = \frac{5}{s}$$

$$\Rightarrow s^2L[y(t)] - sy(0) - y'(0) + 4\{sL[y(t)] - y(0)\}$$
$$- 5L[y(t)] = \frac{5}{s}$$

Now, apply initial condition

$$\Rightarrow y(0) = 0$$
$$\Rightarrow y'(0) = 2$$

$$\Rightarrow s^2L[y(t)] - s(0) - 2 + 4sL[y(t)] - 4(0)$$
$$- 5L[y(t)] = \frac{5}{s}$$

$$\Rightarrow s^2L[y(t)] - 2 + 4sL[y(t) - 5L[y(t)] = \frac{5}{s}$$

$$\Rightarrow L[y(t)]\{s^2+4s-5\} = \frac{5}{s} + 2$$

$$\Rightarrow L[y(t)](s-1)(s+5) = \frac{5+2s}{s}$$

-5
$-1 | 5$
4

$$\Rightarrow L[y(t)] = \frac{5+2s}{s(s-1)(s+5)}$$

Taking inverse LT on b.s, ~~we~~ of the above result,

$$\Rightarrow y(t) = L^{-1}\left[\frac{5+2s}{s(s-1)(s+5)}\right]$$

consider $\left[\frac{5+2s}{s(s-1)(s+5)}\right]$

using partial fraction method (Part-I) (Type)
we can split the above terms as follows.

$$\frac{5+2s}{s(s+5)(s-1)} = \frac{A}{s} + \frac{B}{s+5} + \frac{C}{s-1} \quad \text{---} (1)$$

$$\frac{5+2s}{\cancel{s(s+5)(s-1)}} = \frac{A(s-1)(s+5) + B(s)(s-1) + C(s)(s+5)}{\cancel{s(s+5)(s-1)}}$$

$$5+2s = A(s-1)(s+5) + Bs(s-1) + Cs(s+5) \quad \text{---} (*)$$

Put $s=0$ in $(*)$

$5 = A(-5) + 0 + 0$

$A = -1$

Put $s=1$ in $(*)$

$7 = 0 + 0 + \overset{(1)(1+5)}{6C}$

$\frac{7}{6} = C$

Put $s=-5$ in $(*)$

$\cancel{-5} = B(-5)(-5-1)$

$1 = B(-6)$

$B = -\frac{1}{6}$

sub the values of A, B & C in (1)

$$\frac{5+2s}{s(s+5)(s-1)} = \frac{-1}{s} - \frac{1}{6(s+5)} + \frac{7}{6(s-1)}$$

Taking inverse L.T on b.s, we have

$$L^{-1}\left[\frac{5s}{s(s+5)(s-1)}\right] = L^{-1}\left[-\frac{1}{s} - \frac{1}{6(s+5)} + \frac{7}{6(s-1)}\right]$$

$$= -L^{-1}\left[\frac{1}{s}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{s+5}\right]$$

$$+ \frac{7}{6}L^{-1}\left[\frac{1}{s-1}\right.$$

$$L^{-1}\left[\frac{5s}{s(s+5)(s-1)}\right] = -1 - \frac{1}{6}e^{-5t} + \frac{7}{6}e^{t}$$

$$= \frac{1}{6}\left[7e^{t} - e^{-5t} - 6\right]$$

2\. Find $y''(t) - 3y'(t) + 2y(t) = e^{2t}$, $y(0) = -3$, $y'(0) = 5$ $\to (1)$

soln:

Take Laplace transform for the given (1) eqn on both sides

$$\Rightarrow L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L[e^{2t}]$$

$$\Rightarrow s^2L[y(t)] - sy(0) - y'(0) - 3\left[sL[y(t)] - y(0)\right] + 2L[y(t)] = \frac{1}{s-2}$$

By applying I.C, we get,

$-9-5=-14$

$$\Rightarrow s^2L[y(t)] + 3s - 5 - 3sL[y(t)] + 3(-3) + 2L[y(t)] = \frac{1}{s-2}$$

$$\Rightarrow L[y(t)]\{s^2 - 3s + 2\} = \frac{1}{s-2} - 3s + 14$$

$$\Rightarrow L[y(t)]\{(s-2)(s-1)\} = \frac{1 - 3s(s-2) + 14(s-2)}{s-2}$$

$$\begin{array}{c|c} & 2 \\ -2 & -1 \\ \hline & -3 \end{array}$$

$$\Rightarrow L[y(t)] = \frac{1 - 3s^2 + 6s + 14s - 28}{(s-2)(s-2)(s-1)}$$

$$\Rightarrow L[y(t)] = \frac{-3s^2 + 20s - 27}{(s-2)^2(s-1)}$$

$$\Rightarrow y(t) = L^{-1}\left[\frac{-3s^2 + 20s - 27}{(s-2)^2(s-1)}\right]$$

(Taking inverse LT in above eqn, we get).

Consider $\dfrac{-3s^2 + 20s - 27}{(s-2)^2(s-1)}$

$$\frac{-3s^2+20s-27}{(s-1)(s-2)^2} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s-1} \quad \text{—(1)}$$

$$\frac{-3s^2+20s-27}{(s-2)^2(s-1)} = \frac{A(s-2)(s-1) + B(s-1) + C(s-2)^2}{(s-2)^2(s-1)}$$

$$-3s^2+20s-27 = A(s-2)(s-1) + B(s-1) + C(s-2)^2 \quad \text{—}(*)$$

Put $s = +1$ in $(*)$

$$-3+20-27 = C$$

$$C = -10$$

Put $s = 2$ in $(*)$

$$-12+40-27 = B$$

$$B = 1$$

$$\begin{array}{r} 27 \\ 12 \\ \hline -39 \end{array}$$

Put $s = 0$ in $(*)$

$$-27 = 2A + (-B) + 4C$$

$$-27 = 2A - 1 - 40$$

$$-27 + 41 = 2A$$

$$\frac{14}{2} = A$$

$$A = 7$$

$$\begin{array}{r} 341 \\ -27 \\ \hline 14 \end{array}$$

Sub A, B, C in (1)

$$\frac{-3s^2+20s-27}{(s-2)^2(s-1)} = \frac{7}{s-2} + \frac{1}{(s-2)^2} - \frac{10}{s-1}$$

Taking inverse LT on b.s

$$L^{-1}\left[\frac{-3s^2+20s-27}{(s-2)^2(s-1)}\right] = L^{-1}\left[\frac{7}{s-2} + \frac{1}{(s-2)^2} - \frac{10}{s-1}\right]$$

$$= 7L^{-1}\left[\frac{1}{s-2}\right] + L^{-1}\left[\frac{1}{(s-2)^2}\right] - 10L^{-1}\left[\frac{1}{s-1}\right]$$

$$L^{-1}\left[\frac{-3s^2+20s-27}{(s-2)^2(s-1)}\right] = 7e^{2t} + e^{2t} L^{-1}\left[\frac{1}{s^2}\right] - 10e^{t}$$

$$= 7e^{2t} + te^{2t} - 10e^{t}$$

$$L^{-1}\left[\frac{-3s^2+20s-27}{(s-2)^2(s-1)}\right] = e^{2t}\left[7+t\right] - 10e^{t}$$

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CONVOLUTION

(x) 2m **Definition**

Convolution of two functions $f(t)$ and $g(t)$ is denoted by the symbol $f(t) * g(t)$ & is defined by,

$$f(t) * g(t) = \int_0^t f(u) \cdot g(t-u)\, du$$

(x) 2m **Convolution theorem**

~~The product~~. If $L[f(t)] = F[s]$ and $L[g(t)] = G[s]$ then

Laplace transform of product of 2 functions

i.e., $L[f(t) * g(t)] = L[f(t)] . L[g(t)]$

$$= F(S) \cdot G(S)$$

Note

1. $L^{-1}[f(t)] * L^{-1}[g(t)] = L^{-1}[F(s) . G(s)]$

Problem — if not mentioned, can use partial fraction method

1. Using convolution theorem, find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

Sol:

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{s+a} \cdot \frac{1}{s+b}\right]$$

W.K.T, $L^{-1}[f(t)] * L^{-1}[g(t)] = L^{-1}[F(s) . G(s)]$

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{s+a}\cdot\frac{1}{s+b}\right]$$

$$\cancel{= L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]}$$

$$= L^{-1}[e^{-at}] * L^{-1}[e^{-bt}]$$

W.K.T $f(t) * g(t) = \int_0^T f(u).\, g(t-u)\, du$

$$L^{-1}\left[\frac{1}{s+a}\cdot\frac{1}{s+b}\right]$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} \cdot e^{-b(t-u)}\, du$$

$$= \int_0^t e^{-au-bt+bu}\, du$$

$$= e^{-bt}\int_0^t e^{-au+bu}\, du$$

$$= e^{-bt}\int_0^t e^{-u(a-b)}\, du$$

$$= e^{-bt}\left[\frac{e^{-u(a-b)}}{-(a-b)}\right]_0^t$$

$$= \frac{-e^{-bt}}{(a-b)}\left[e^{-u(a-b)}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left[e^{-t(a-b)} - e^{0}\right]$$

$$= \frac{e^{-bt}\cdot e^{-at}\cdot e^{bt} - e^{-bt}}{b-a}$$

$$L^{-1}\left[\frac{1}{s+a}\cdot\frac{1}{s+b}\right] = \frac{e^{-bt+bt-at}-e^{-bt}}{b-a}$$

$$L^{-1}\left[\frac{1}{s+a}\cdot\frac{1}{s+b}\right] = \frac{e^{-at}-e^{-bt}}{b-a} \qquad \frac{e^{-2t}-e^{-t}}{2-1}$$

2. Using convolution theorem, Find

$$L^{-1}\left[\frac{1}{(s+2)(s+1)}\right]$$

Sol :

$$= L^{-1}\left[\frac{1}{s+2}\cdot\frac{1}{s+1}\right]$$

$$\Rightarrow L^{-1}\left[\frac{1}{(s+2)(s+1)}\right] = L^{-1}\left[\frac{1}{s+2}\cdot\frac{1}{s+1}\right]$$

$$= L^{-1}\left[\underbrace{e^{-2t}}_{f(t)}\right] * \left[\underbrace{e^{-t}}_{g(t)}\right]$$

W.K.T $f(t) * g(t) = \int_0^t f(u)\cdot g(t-u)\,du$

$$L^{-1}\left[\frac{1}{s+2}\cdot\frac{1}{s+1}\right] = e^{-2t} * e^{-t}$$

$$= \int_0^t e^{-2u}\cdot e^{-(t-u)}\,du$$

$$= \int_0^t e^{-2u-t+u}\,du$$

$$= \int_0^t e^{-u-t}\,du$$

$$= e^{-t}\int_0^t e^{-u}\,du$$

$$= e^{-t}\cdot\left[\frac{e^{-u}}{-1}\right]_0^t$$

$$= e^{-t}\left[-e^{-t}-(-e^{0})\right]$$

$$= e^{-t}\left[e^{-t}+1\right]$$

$$= e^{-t-t} + e^{-t}$$

$$\boxed{L^{-1}\left[\frac{1}{(s+2)(s+1)}\right] = e^{-2t} + e^{-t}}$$

$$= e^{(-2t)(-t)}$$

$$= e^{2t^2}$$