ODE intro

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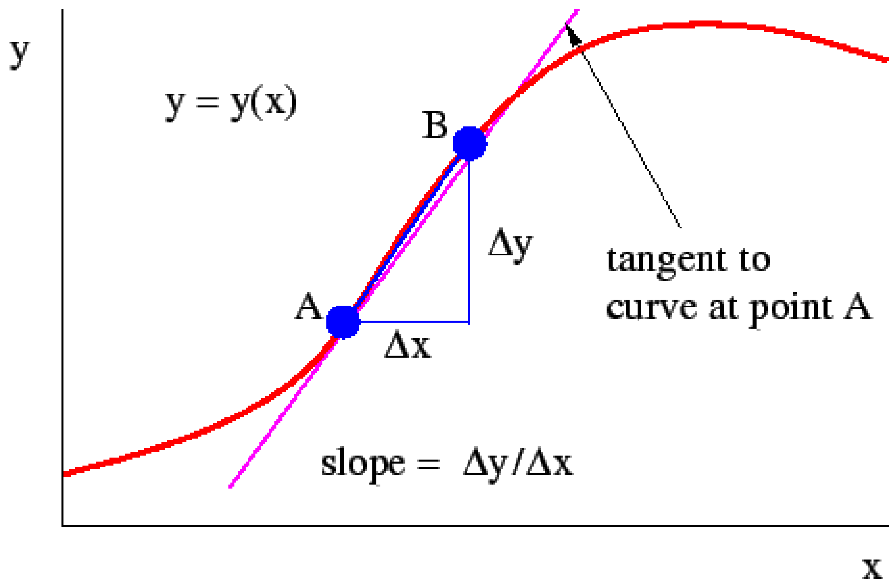
Some figures from:
Advanced Engineering Mathematics, Kreyszig
Wikipedia

Derivative: Definitions and Notations

- Derivative:
  - Amount of change in a variable as a function of another variable.
    - y'(x): How much does y change when x changes by 1 unit.
  - Geometric interpretation:
    - the slope of y(x)
    - tangent
- Notations:
  - y'
  - $\dot{y}$
  - $y'(x)$
  - $\frac{dy}{dx}$
  - $\frac{dy(x)}{dx}$
- Δy vs. dy
  - Δ: change
  - d: infinitely small amount of change

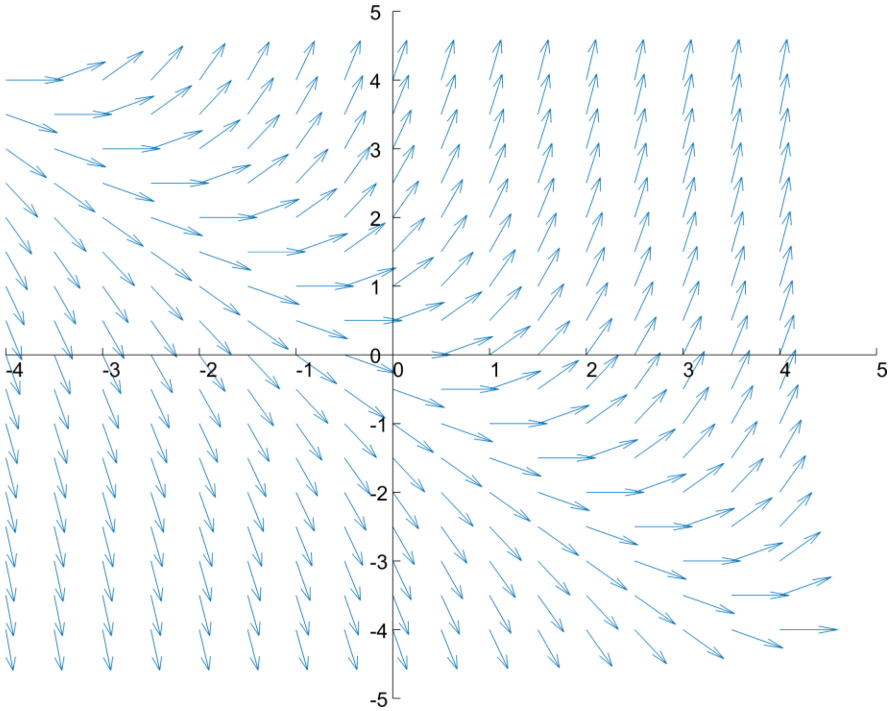


Number of dependent variables

- The independent variable in most problems we model is time (t)
- One dependent variable:
  - y(t), y′
- For one dependent variable, it is typical to use “x” instead of “t”:
  - y(x), y′
- Two dependent variables:
  - x(t), x′(t), x′
  - y(t), y′(t), y′
- **Watch out:** sometimes **f** is used to represent **y**, sometimes **f** is used to represent **y'**.

Direction/Slope/Vector Field

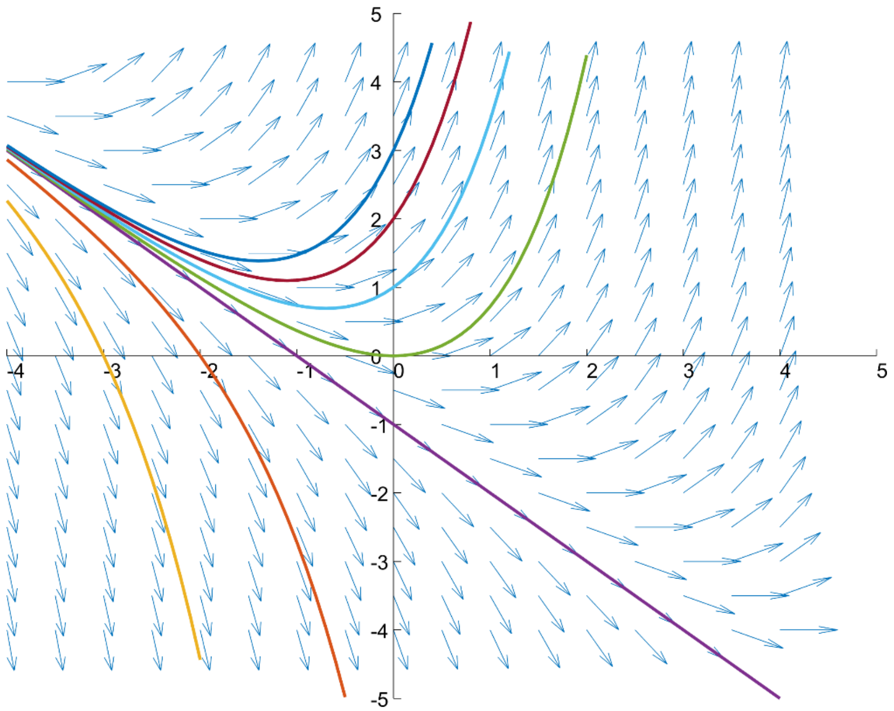
- Direction field shows the slope (the value of the derivative) at each point
- Shows behavior of the system at different conditions.



Direction field of y′=x+y

Integral Curves / Solution Curves

- Example solutions can be shown on the direction field as integral curves
- Integral curves can be created using:
  - The solution to the diffeq., if you know it.
  - Numerical simulations from example starting points.



Example integral curves

Phase Diagram

- When two dependent variables are dependent on a third independent variable, we typically factor out the independent variable and show the differential relationship of the two dependent variables.
  - Third variable is typically time.
- If dependent variables only depend on the third, but do not depend on each other, phase diagram is then useless. Just use two vector fields for such cases.

Phase Diagram: Pendulum

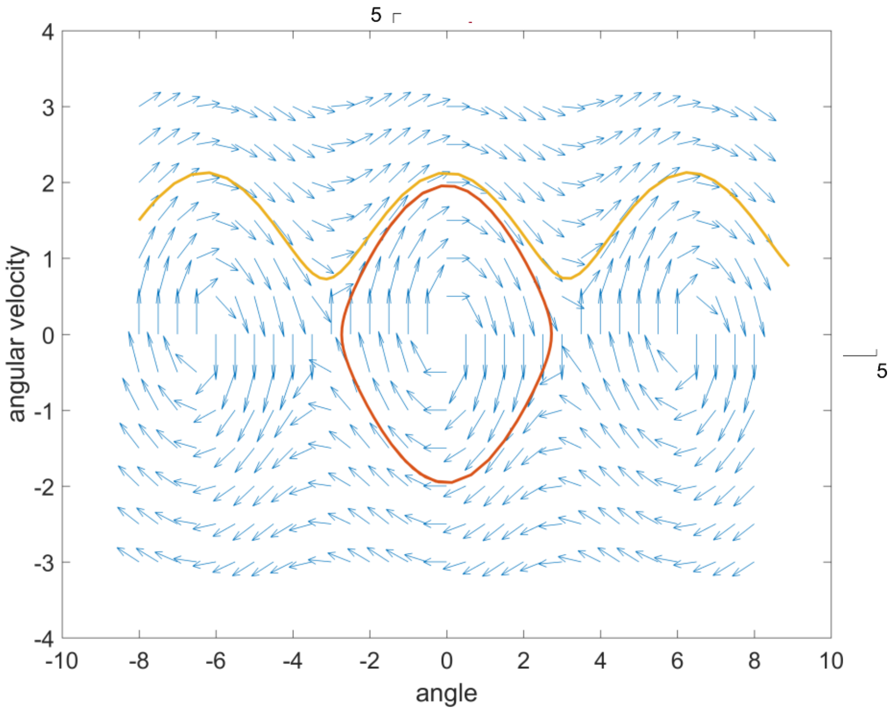
- Consider the angle x(t) and angular velocity y(t) of a swinging pendulum of length L.
- By definition of velocity:

$$\frac{dx}{dt} = y(t)$$

- Using the tangential component of the gravity on the pendulum:

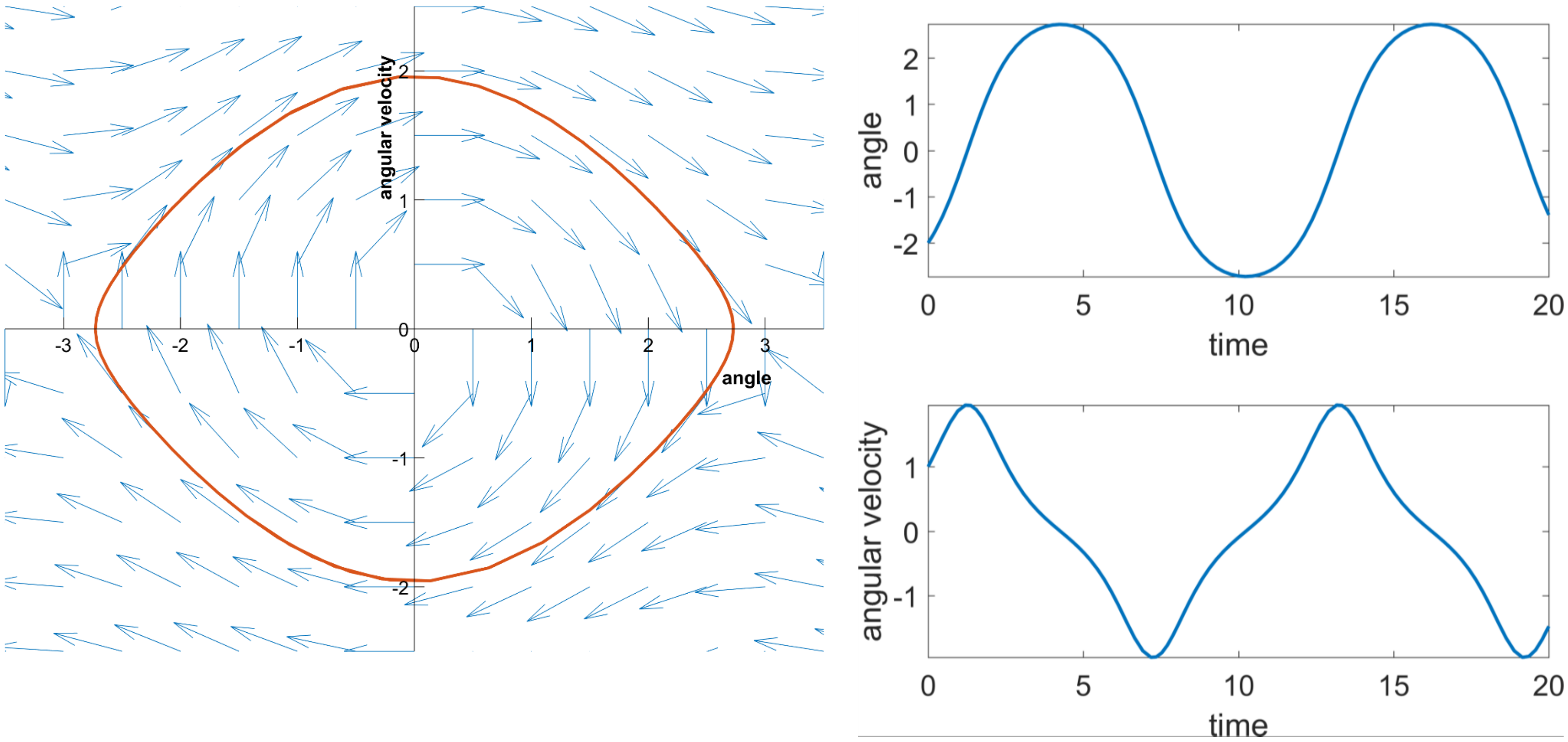
$$\frac{dy}{dt} = -\frac{g}{L}sin(x)$$

- Let L=g for simplicity.



phase diagram of pendulum

Phase Diagram: Pendulum



Matlab demo

- ode_graphicalanalysis.m
- phaseplot_animate.m

Euler's method

- Given an ODE and an initial value:

$$y'(x) = f(x, y)$$
$$y(x_0) = y_0$$

- Euler's method gives approximate solution values at equidistant x-values:

$$x_0, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \ \ldots$$

$$y_1 = y_0 \ + \ hf(x_0, y_0)$$
$$y_2 = y_1 \ + \ hf(x_1, y_1)$$
$$\ldots$$
$$y_n = y_{n-1} \ + \ hf(x_2, y_2)$$

Euler's Method

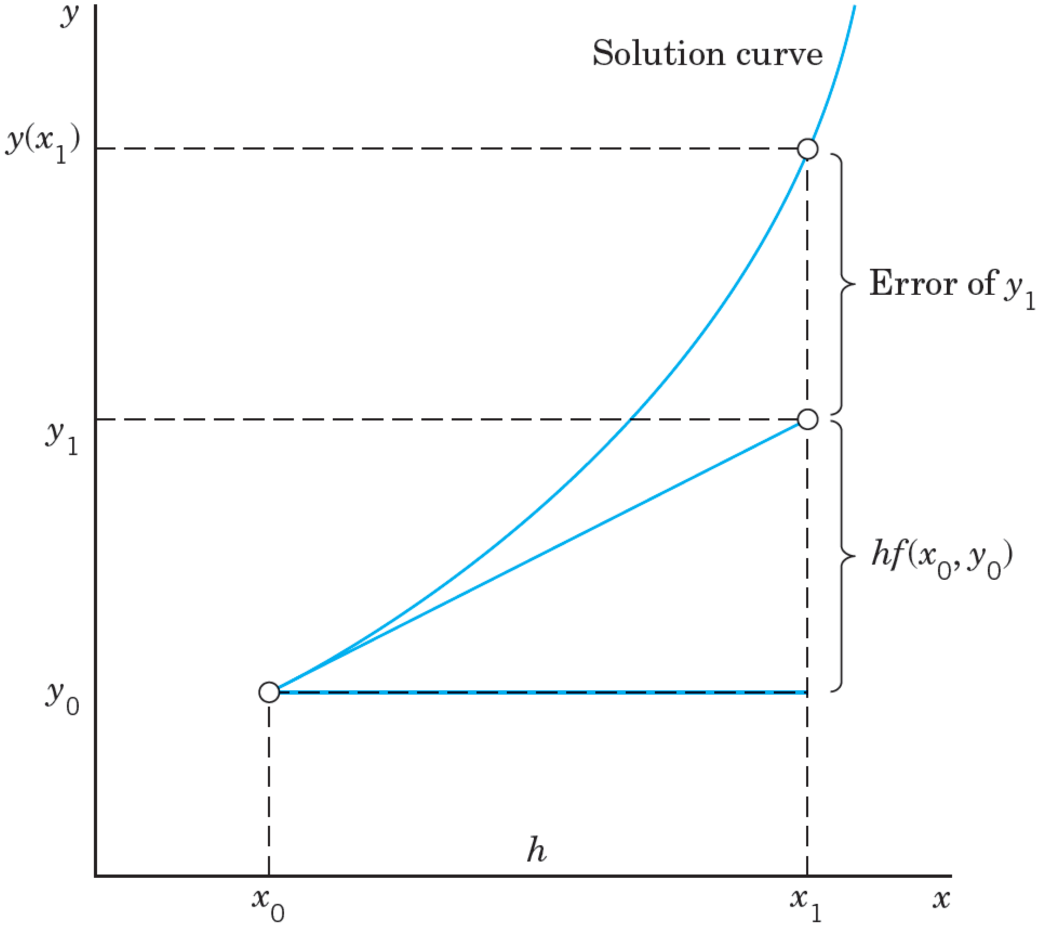


Fig. 8. First Euler step, showing a solution curve, its tangent at (x_0, y_0), step h and increment $hf(x_0, y_0)$ in the formula for y_1

Taylor Series Approximation

- Taylor Series approximation to a function f(x) around x=a is:

$$f(x) = \sum_{i=0..n\to\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- We use Taylor Series approximation when f(x) is impossible or difficult to calculate directly.
  - But we still require that the value of f(a), f'(a), f''(a), etc. are available.

- taylor_demo.m

Euler's Method

- Truncating the Taylor Series at n=1, 2, etc. gives us 1st order, 2nd order, etc. approximations.
- Euler's Method is essentially a 1st order Taylor Series approximation (repeated application).

$$f(x) = f(a) + f'(a)(x - a) + \boldsymbol{Error}$$

- Using step size h from a, x=a+h:

$$f(a + h) = f(a) + hf'(a) + \boldsymbol{Error}$$

Matlab demo

- eulermethod_demo.m