Principal Component Analysis
- Review of Covariance & Linear Algebra

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Covariance

- The covariance of two random variables, X and Y, is the expected product of their deviations from their respective means

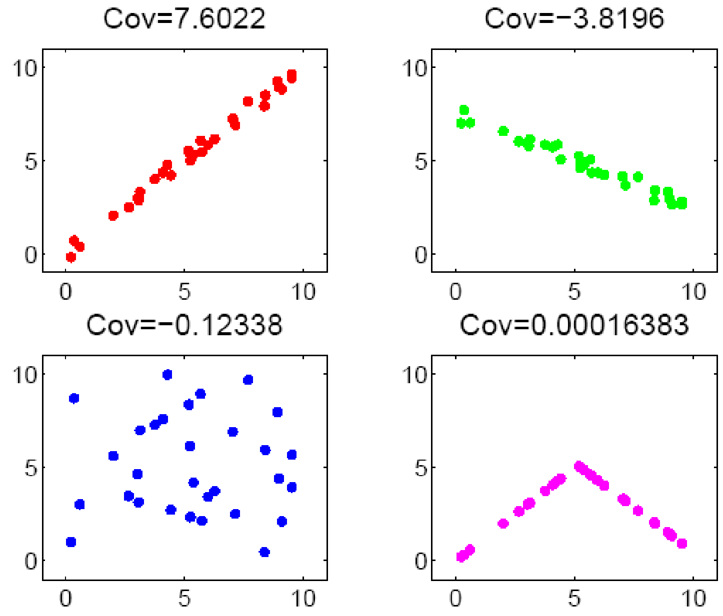
$$Cov(X,Y) = E\left(\left(X - E(X)\right)\left(Y - E(Y)\right)\right)$$
$$= \sum_x \sum_y \left(x - E(X)\right)\left(y - E(Y)\right) P_{X,Y}(x,y)$$

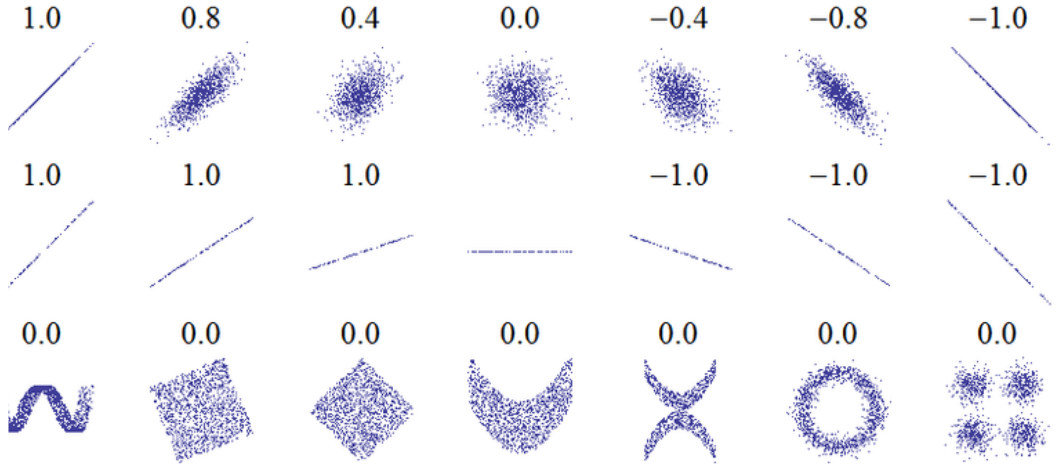
- Covariance measures the tendency of two random variables to "move together" in a certain way (linearly)
  - Covariance is >0 if, when X tends to be above its mean, Y also tends to be above its mean
  - Covariance is <0 if, when X tends to be above its mean, Y tends to be below its mean

Covariance and correlation

$$Covariance_{XY} = E[(X - E[X])(Y - E[Y])]$$

$$Correlation_{XY} = E[(X - E[X])(Y - E[Y])]/(\sigma_X \sigma_Y)$$





Wikipedia:Correlation

Linearly Independent

- Orthogonal vectors p,q:

$$p \cdot q = \sum_{i=1}^{n} p_i q_i = 0$$

- Linearly Dependent: One vector can be written as a linear combination of others:

$$a_1 p_1 + \cdots + a_k p_k = \vec{0}$$

$$p_i = -\frac{a_1}{a_i} p_1 - \cdots - \frac{a_k}{a_i} p_k = \sum_{j \neq i} \frac{-a_j}{a_i} p_j = \sum_{j \neq i} b_j p_j$$

- Linearly Independent vectors $p_1, p_2, \ldots, p_k$:

$$a_1 p_1 + \cdots + a_k p_k \neq \vec{0}$$

Eigen ~ of itself (German)

- x is an <u>eigenvector</u> of matrix *A*, if there exists a non-zero constant γ s.t.
  - $Ax = \gamma x$
  - γ is called an <u>eigenvalue</u> of A wrt x
- *A* may have more than one eigenvectors, each with its own eigenvalue.
- Eigenvectors corresponding to distinct eigenvalues are linearly independent
- Applications:
  - PCA, calculating a power of a matrix, finding solutions for a system of differential equations, and growth models

Inverse

- Matrix B is <u>inverse</u> of matrix A if:
  - $AB=I$
  - I is identity matrix
  - B is denoted as A^{-1}
- Some matrices do not have an inverse
  - E.g., when one row/column can be written as a linear combination of others.
- Every matrix A has a unique <u>pseudo-inverse</u> A^*, s.t.
  - $AA^*A=A$
  - $A^*AA^*=A^*$
  - $A^*A=(A^*A)^T$
  - $AA^*=(AA^*)^T$

Inverse Example

$A = \begin{bmatrix} 1 & 1 \\ 9 & 7 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -3.5 & 0.5 \\ 4.5 & -0.5 \end{bmatrix}$

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Pseudo-Inverse Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 4 & 7 \end{bmatrix}$$

A^-1 does not exist.

$$A^* = \begin{bmatrix} -0.3667 & -0.7333 & 0.8333 \\ 0.3333 & 0.6667 & -0.6667 \\ -0.0333 & -0.0667 & 0.1667 \end{bmatrix}$$

$$AA^* = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA^*A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 4 & 7 \end{bmatrix}$$

Basis Vectors

- Each data sample can be written as a linear combination of orthonormal (unit length and orthogonal) basis vectors.
- Example: The standard basis of the n-dimensional Euclidean space R^n
  - The set $\{e_1=\langle 1,0,0\rangle, e_2=\langle 0,1,0\rangle, e_3=\langle 0,0,1\rangle\}$ forms an orthonormal basis of R^n
  - $\langle x,y,z\rangle = xe_1 + ye_2 + ze_3$

Orthogonal matrix

- A square matrix whose rows (and columns) are orthonormal
  - $E^T = E^{-1}$
  - $E^T E = E E^T = I$