

Exponential Random Variable

For any **Poisson Process**, the **Exponential** RV models *time until an event*:

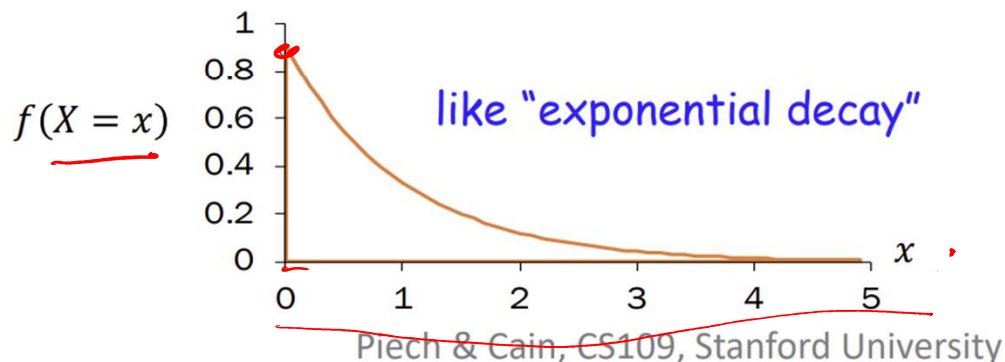
$$X \sim \text{Exp}(\lambda)$$

PDF:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

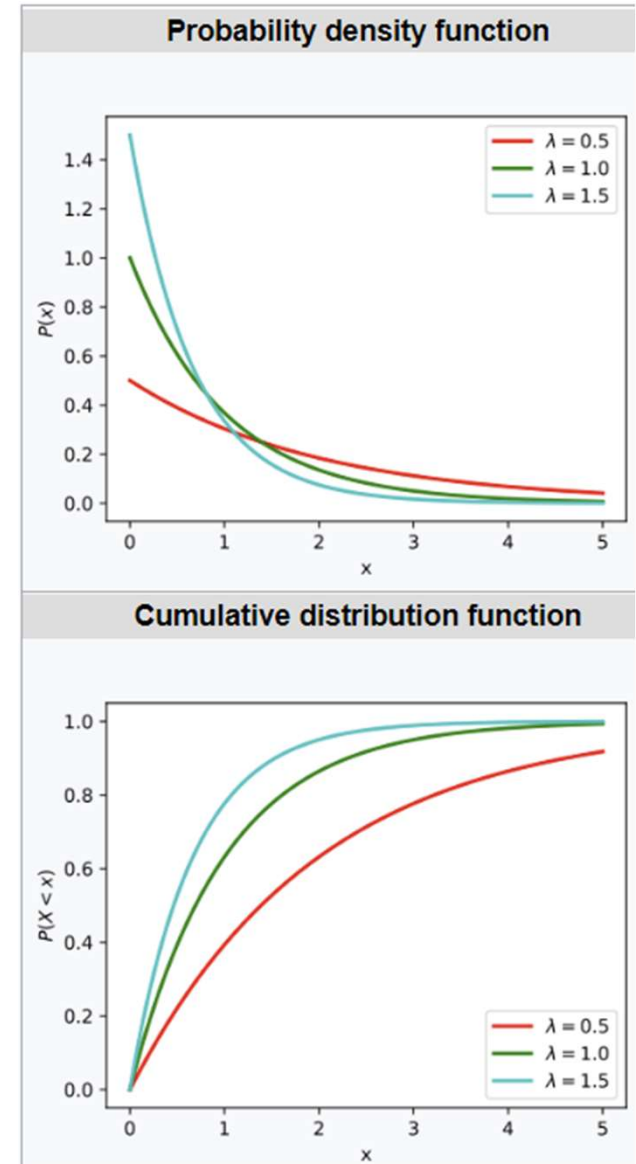
Examples:

- Time until next earthquake
- Time until a ping reaches a web server
- Time until a Uranium atom decays



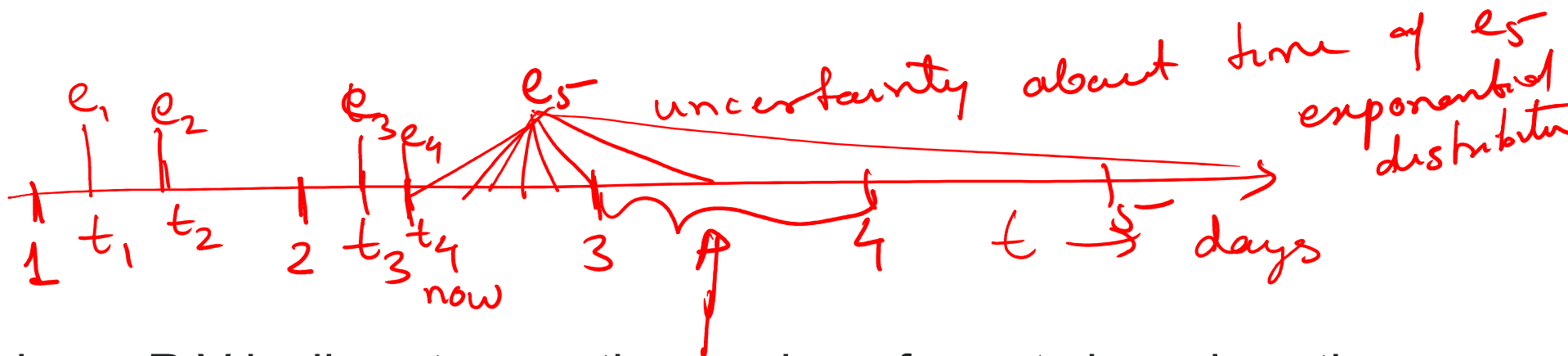
Cumulative Distribution function

$$\begin{aligned} F(x) &= P\{X \leq x\} \\ &= \int_0^x \lambda e^{-\lambda y} dy \\ &= 1 - e^{-\lambda x}, \quad x \geq 0 \end{aligned}$$



Relationship to Poisson distribution

- Both are applicable when events occur continuously and independently at a constant average rate λ



- Poisson R.V is discrete over the number of events in a given time
- Exponential R.V is continuous and is the distance between two events.

Moment Generating Function , Mean, Variance

$$\phi(t) = \underline{E[e^{tX}]} \quad X \sim \text{exp}(\lambda)$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx$$

$$= \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

Differentiation yields

$$\underline{\phi'(t)} = \frac{\lambda}{(\lambda - t)^2}$$

$$\underline{\phi''(t)} = \frac{2\lambda}{(\lambda - t)^3} \cdot \frac{2\lambda}{\lambda^3}$$

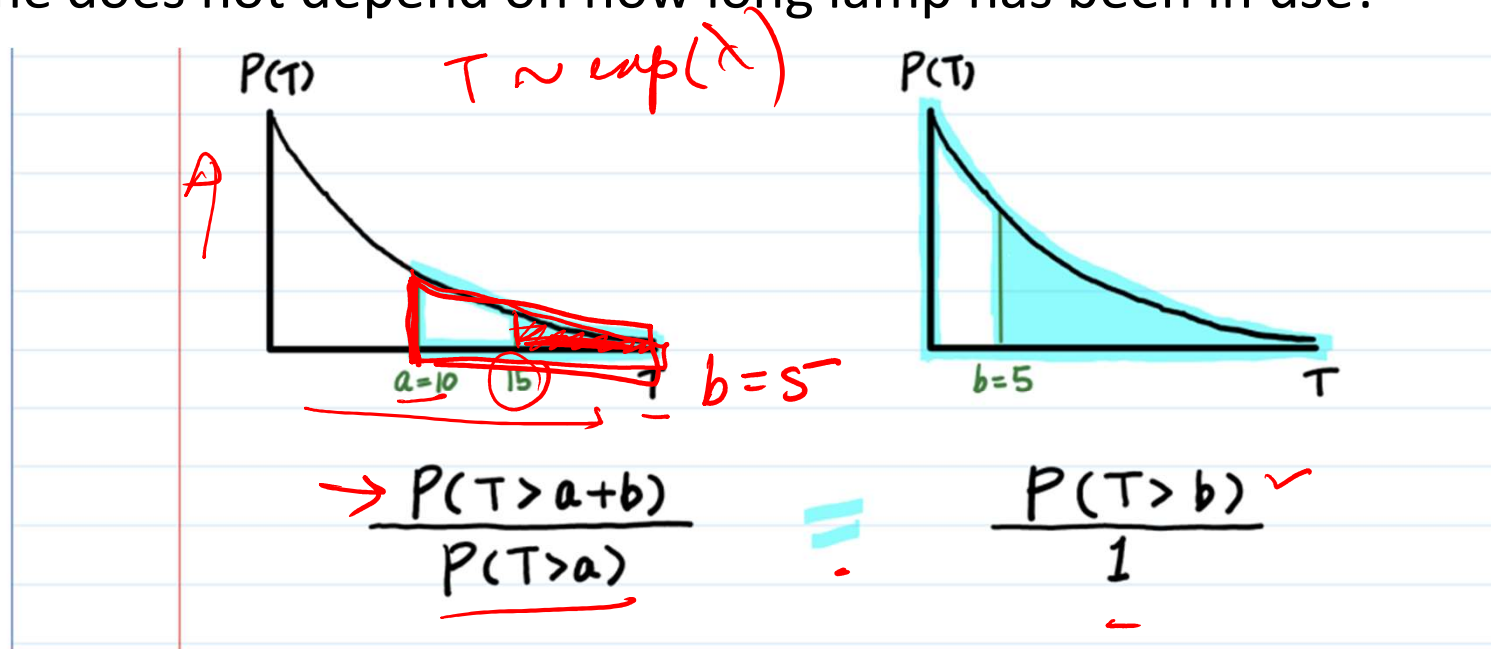
$$E[X] = \phi'(\underline{0}) = 1/\lambda$$

$$\begin{aligned} \underline{\text{Var}(X)} &= \underline{\phi''(0)} - (E[X])^2 \\ &= \underline{2/\lambda^2} - \underline{1/\lambda^2} \\ &= \underline{1/\lambda^2} \end{aligned}$$

Memoryless property of exponential distribution

$$P(X > \underline{s} + \underline{t} \mid X > \underline{s}) = P(X > \underline{t})$$

Example: lifetime T of a lamp if exponentially distributed, then remaining lifetime does not depend on how long lamp has been in use!



Proof of the memory-less property

$$P(A|B) = \frac{P(A, B)}{P(B)}$$

$$X \sim \exp(\lambda)$$

$$CDF(X) = \frac{1 - e^{-\lambda x}}{P(X \leq x)}$$

$$P(X \leq x)$$

$t > 0$

$$P(X > s+t | X > s) = \frac{P(X > s+t, X > s)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{1 - CDF_X(s+t)}{1 - CDF_X(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$= 1 - CDF_X(t)$$

$$= \underline{P(X > t)}$$

$\Rightarrow X$ is memoryless

Memoryless property is unique to exponential!

- If X is a continuous random variable where $P(X > s+t | X > s) = P(X > t)$ then $P(X)$ is an exponential distribution. [Proof not part of the syllabus]

Proof

Let F be the CDF of X , and let $G(x) = P(X > x) = 1 - F(x)$. The memoryless property says $G(s + t) = G(s)G(t)$, we want to show that only the exponential will satisfy this.

Try $s = t$, this gives us $G(2t) = G(t)^2, G(3t) = G(t)^3, \dots, G(kt) = G(t)^k$.

Similarly, from the above we see that $G(\frac{t}{2}) = G(t)^{\frac{1}{2}}, \dots, G(\frac{t}{k}) = G(t)^{\frac{1}{k}}$.

Combining the two, we get $G(\frac{m}{n}t) = G(t)^{\frac{m}{n}}$ where $\frac{m}{n}$ is a rational number.

Now, if we take the limit of rational numbers, we get real numbers. Thus, $G(xt) = G(t)^x$ for all real $x > 0$.

If we let $t = 1$, we see that $G(x) = G(1)^x$ and this looks like the exponential. Thus, $G(1)^x = e^{x \ln G(1)}$, and since $0 < G(1) \leq 1$, we can let $\ln G(1) = -\lambda$.

Therefore $e^{x \ln G(1)} = e^{-\lambda x}$ and only exponential can be memoryless.

Example

- Suppose the number of kms that a car can run before the battery wears down is exponentially distributed with average distance as 10000. If the person takes a 5000 km trip, what is the probability that the battery will not run down.

$$X \sim \exp(\lambda) \quad \lambda = \frac{1}{10000} \quad E(X) = \frac{1}{\lambda}$$

$$P(X > 5000) = e^{-\lambda 5000} = e^{-\frac{1}{2}}$$

Another interesting property of exponential distribution

Proposition 5.6.1. If X_1, X_2, \dots, X_n are independent exponential random variables having respective parameters $\lambda_1, \lambda_2, \dots, \lambda_n$, then $\min(X_1, X_2, \dots, X_n)$ is exponential with parameter $\sum_{i=1}^n \lambda_i$.

$$Y = \min(X_1, X_2, \dots, X_n) \quad X_i \sim \text{exp}(\lambda_i)$$

$$P(Y > b) = P(X_1 > b, X_2 > b, \dots, X_n > b)$$

$$P(\min(X_1, \dots, X_n) > b) = \prod_{i=1}^n P(X_i > b)$$

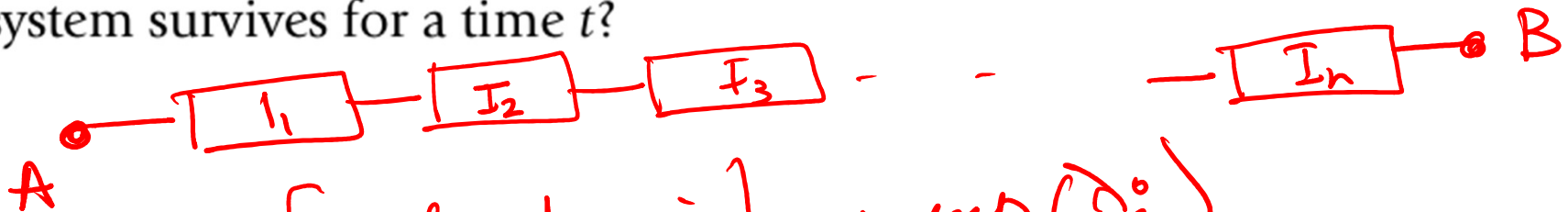
$$= \prod_{i=1}^n e^{-\lambda_i b} = e^{-b(\sum_{i=1}^n \lambda_i)}$$

$$\Rightarrow Y \sim \text{exp}\left(\sum_{i=1}^n \lambda_i\right)$$

X_1, \dots, X_n are independent

Example

Example 5.6.c. A series system is one that needs all of its components to function in order for the system itself to be functional. For an n -component series system in which the component lifetimes are independent exponential random variables with respective parameters $\lambda_1, \lambda_2, \dots, \lambda_n$, what is the probability that the system survives for a time t ?



$$[I_j \text{ functioning}] \sim \exp(\lambda_j)$$

Y = A to B connection is functioning

$$P(Y \geq t) = P(\min(I_1, I_2, \dots, I_n) > t) = e^{-t \left[\sum_{i=1}^n \lambda_i \right]}$$

Another fun property of exponential distribution

Maximum entropy distribution

Among all continuous probability distributions with support $[0, \infty)$ and mean μ , the exponential distribution with $\lambda = 1/\mu$ has the largest differential entropy. In other words, it is the maximum entropy probability distribution for a random variate X which is greater than or equal to zero and for which $E[X]$ is fixed.^[2]