CS 228 : Logic in Computer Science

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Consider the formula
$\varphi = \forall x.Q_a(x) \vee [\forall x.(Q_a(x) \Rightarrow \exists y.(Q_b(y) \wedge x < y))]$.

1. The word *aaa* is a model for φ : True
2. The word *b* is a model for φ : True
3. The word *ab* is a model for φ : True
4. The word *aba* is a model for φ : False
5. The word *bab* is a model for φ : True
6. The word *abab* is a model for φ : True
7. The word *baaaaa* is a model for φ : False
8. The word *bbb* is a model for φ, but *bb* is not : False
9. The word *abb* is not a model for φ, but *bba* is : False
10. Every word over a, b is a model for φ : False

Satisfaction, Validity

- Given a FO formula $\varphi(x_1, \ldots, x_n)$ over a signature τ, is it satisfiable/valid?
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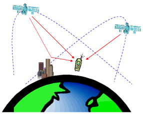
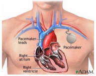
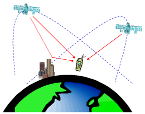




System —satisfy?→ specification

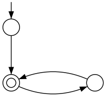
good/bad properties

Verification through Model Checking



System $\xrightarrow{\text{satisfy?}}$ specification

System Model $\xrightarrow[\models ?]{\text{model-checking}}$ Spec Model

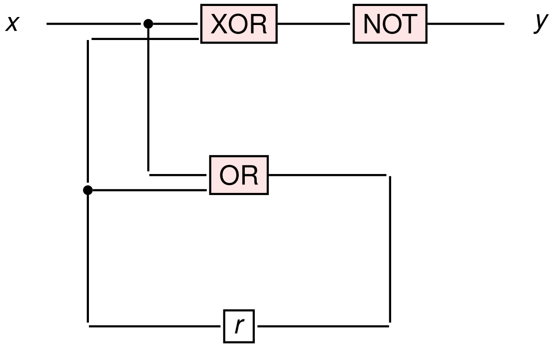


Logic formula ϕ

Model Checking

- Abstract the given system = code/circuit as a finite state transition system, G
- Behaviours of the system = sequence of actions taken by G (these are words, and the actions are the symbols of the alphabet)
- Write the property of interest in a chosen logic as formula φ
- Check $G \models \varphi$

Sequential Circuits



- Input variable x, output variable y, register r
- Output $\neg(x \bigoplus r)$ and register evaluates to $x \vee r$

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Initially, assume $r = 0$

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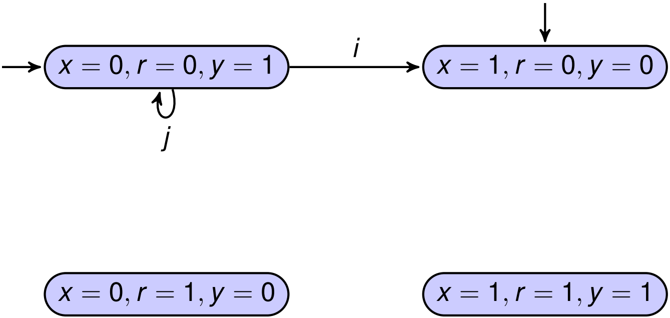

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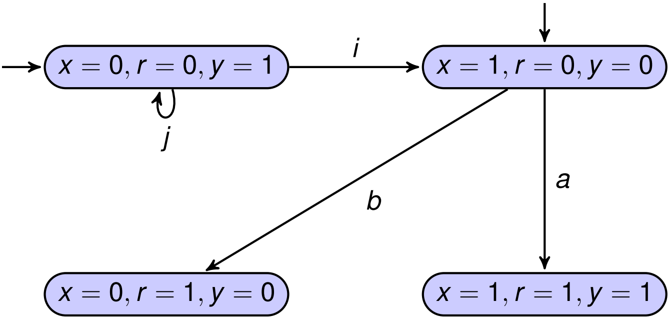

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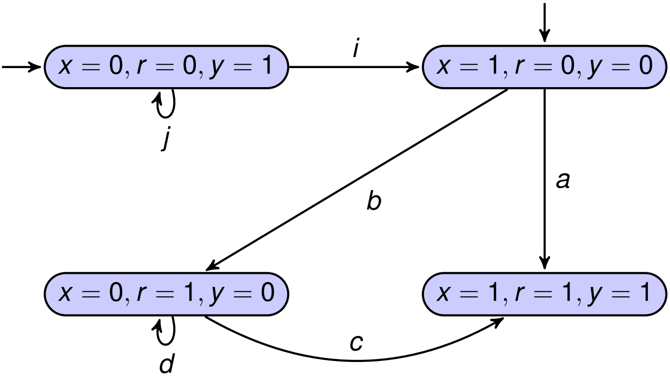

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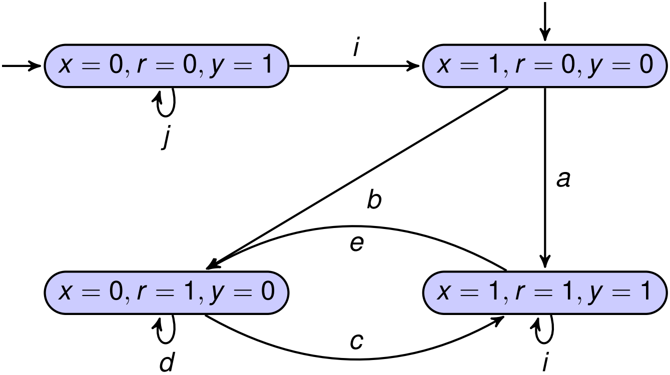

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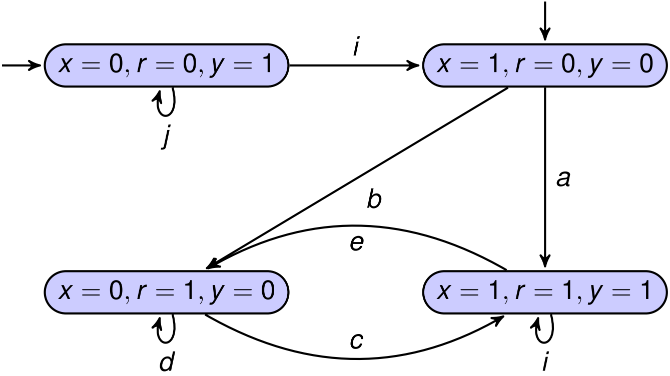

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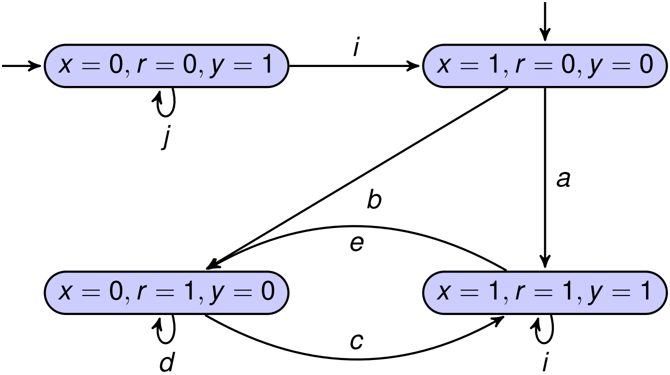

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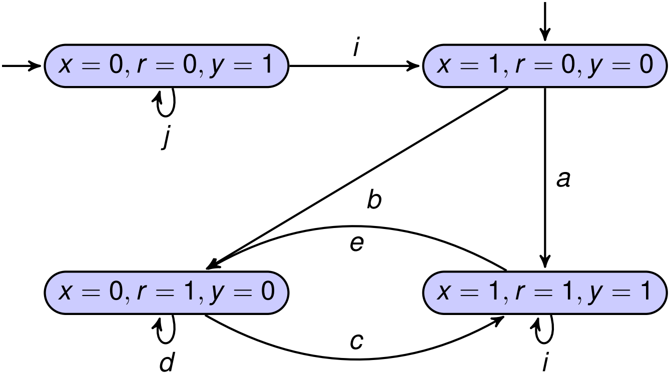

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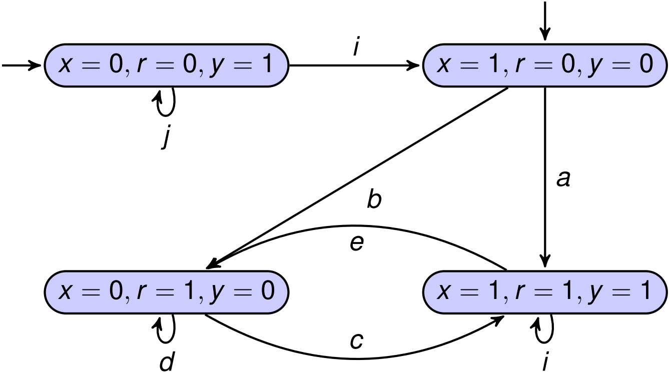

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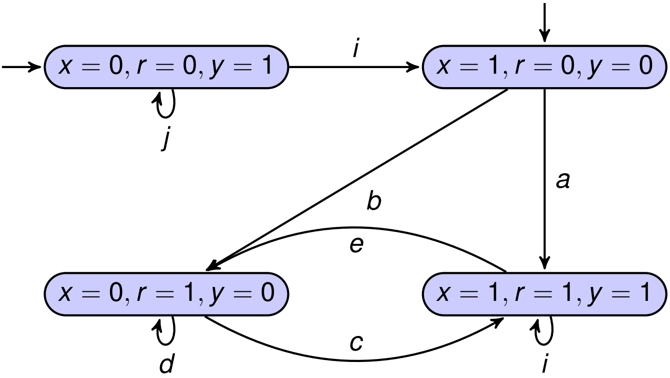

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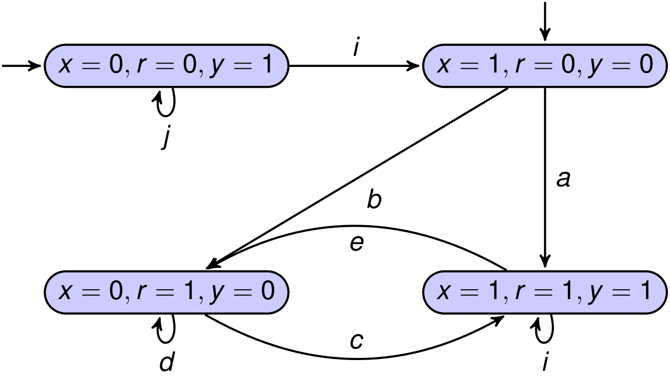

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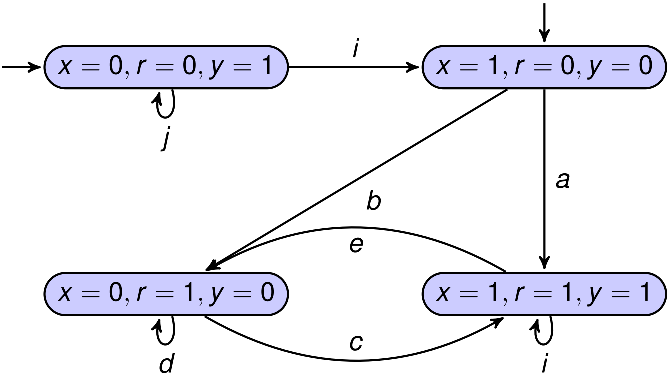

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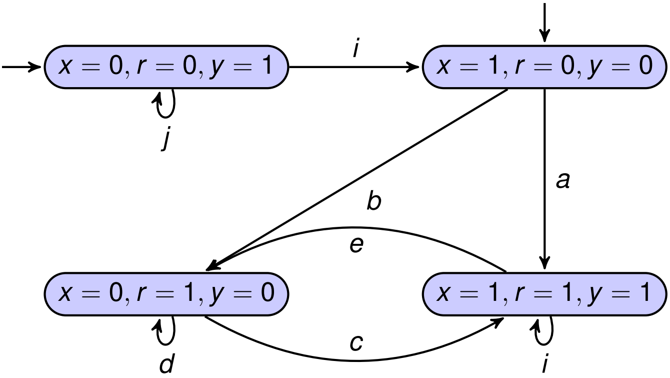

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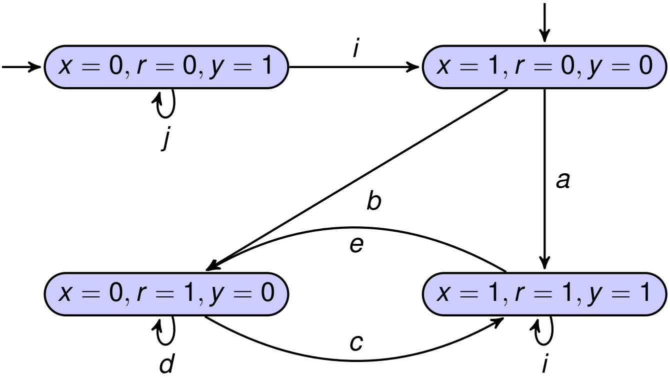

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- Property : Every i is followed by an a or b :
  $\forall x (Q_i(x) \Rightarrow \exists y (x < y \wedge [Q_a(y) \vee Q_b(y)]))$

First-Order Logic over Words

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- Given φ, write an algorithm to check $L(\varphi) = \emptyset$?

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  - Given a FO formula φ over words, is $L(\varphi)$ non-empty?

A Primer for Words

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  - $Pref(aaba) = \{\epsilon, a, aa, aab, aaba\}$
  - Proper prefixes = $\{a, aa, aab\}$
  - ϵ, $aaba$ improper prefixes

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- $\overline{A} = \{x \in \Sigma^* \mid x \notin A\}$
  - For $\Sigma = \{a\}$ and $A = (aa)^*, \overline{A} = \{a, a^3, a^5, \ldots\}$

Operation on Sets

Given a finite alphabet Σ, denote by $A, B, C, \ldots$ subsets of Σ^*

- Subsets of Σ^* are called languages
- $A \cup B = \{x \in \Sigma^* \mid x \in A \text{ or } x \in B\}$
  - $A = a^*$, $B = \{b, bb\}$, $A \cup B = a^* \cup \{b, bb\}$
- $A \cap B = \{x \in \Sigma^* \mid x \in A \text{ and } x \in B\}$
  - $A = (ab)^*$, $B = \{abab, \epsilon, bb\}$, $A \cap B = \{\epsilon, abab\}$
- $\overline{A} = \{x \in \Sigma^* \mid x \notin A\}$
  - For $\Sigma = \{a\}$ and $A = (aa)^*$, $\overline{A} = \{a, a^3, a^5, \ldots\}$
- $AB = \{xy \mid x \in A, y \in B\}$
  - $A = \{a, ba\}$, $B = \{\epsilon, aa, bb\}$
  - $AB = \{a, a^3, abb, ba, ba^3, babb\}$
  - $BA = \{a, ba, a^3, aaba, bba, bbba\}$

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- Concatenation does not distribute over interesection
  - $A = \{a, ab\}, B = \{b\}, C = \{\epsilon\}$
  - $A(B \cap C) \neq AB \cap AC$

FO for Languages

Formalize in FO

Write FO formulae φ_i such that $L(\varphi_i) = L_i$ for $i = 1, \ldots, 5$.

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- L_5 = Words in which whenever an a occurs, it is followed eventually by a b, and no c occurs in between the a and the b
  $aabbabab, aabbcbccaab \in L_5$, $aacaab \notin L_5$.

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- Given φ, can we easily convert φ into some other mechanism M, which we know how to deal with?