SC 639 Assignment 1

Due date: 02-09-2024 (Monday), 0900 hours

Instructions:

- All "Graded questions" carry equal weightage.
- Handwritten and typeset, both forms of submissions are acceptable.
- Answers should be legible. Excessive cutting, scratching, overwriting shall be penalized.
- Every answer to a particular question should be written on a different fresh page; for example do not start writing a solution for Q2 right after the end of Q1. Start from a new page.
- If a question have several parts, do not write solution of individual parts on different pages. Solution to all parts should be written one after another.
- If submitting typset in LaTeX, proof read your answer scripts before submitting. Any cribs related to typos will not be considered.
- The questions under "Graded questions" will be checked by the TAs. "Practice questions" are for your benefit and their solutions can be discussed in tutorials. Please don't submit solutions to any of the Practice questions for grading.
- Please submit your assignments at the SysCon office (the first room on the right as you enter the SysCon building) before 9:00 am.

Graded Questions:

1. Let $V = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_5 \end{bmatrix} : 0 < a_i \in \mathbb{R} \right\}$. If $v = \begin{bmatrix} a_1 \\ \vdots \\ a_5 \end{bmatrix}$ and $w = \begin{bmatrix} b_1 \\ \vdots \\ b_5 \end{bmatrix}$ belong to V and if $c \in \mathbb{R}$, set $v + w = \begin{bmatrix} a_1 b_1 \\ \vdots \\ a_5 b_5 \end{bmatrix}$ and $cv = \begin{bmatrix} a_1^c \\ \vdots \\ a_5^c \end{bmatrix}$. Do these operations turn V into a vector space over $\mathbb{R}$?

2. Let V be a vector space over a field F and let v and w be distinct vectors in V. Set $U = \{(1-t)v + tw : t \in F\}$. Show that there exists a vector $y \in V$ such that $\{u + y : u \in U\}$ is a subspace of V.

3. Find the matrix representation of the linear transformation $\alpha : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$\alpha : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} a+b+c \\ b+c \end{pmatrix}$$

with respect to bases $\left\{ \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \right\}$ of $\mathbb{R}^3$ and $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ of $\mathbb{R}^2$.

Practice Questions:

1. Let $1 < t \in \mathbb{R}$ and let $F = \{a \in \mathbb{R} : a < 1\}$. Define operations $\bigoplus$ and $\bigodot$ on F as follows:

   (a) $a \bigoplus b = a + b - ab \quad \text{for all} \quad a, b \in F;$

   (b) $a \bigodot b = 1 - t^{\log_t(1-a)\log_t(1-b)} \quad \text{for all} \quad a, b \in F$

2. Let $\mathbb{Q}(\sqrt{2})$ be the set of all real numbers of the form $\alpha + \beta\sqrt{2}$, where α and β are rational.

   (a) Is $\mathbb{Q}(\sqrt{2})$ a field? Prove.

   (b) What if α and β are required to be integers? Prove.

3. Consider the vector space $\mathcal{P}$ and the subsets V of $\mathcal{P}$ consisting of those vectors (polynomials) x for which

   (a) x has degree 3,

   (b) $x(t) = x(1-t)$ for all t

   For which of the cases is V a vector space? Prove.

4. Prove that if U_1 and U_2 are subspaces of a finite dimensional vector space U , then

$$dim(U_1 + U_2) \leq dim(U_1) + dim(U_2)$$

5. Let $S = \{1, x, x^2\}$ and $T = \{1 - x, 1 + x, 1 + x^2\}$. Show that the set S is linearly independent and also show that linear span of S and T is identical i.e. span$\{S\}$ = span$\{T\}$.

6. (a) Let V_1 and V_2 be distinct subspaces. Show that $V_1 \cap V_2$ is a subspace.

   (b) Extend the proof of part (a) to n distinct subspaces. That is, if $V_1, V_2, \cdots V_n$ are distinct subspaces, then show that $V_1 \cap V_2 \cap \cdots V_n$ is a subspace.

7. Let T be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1).$$

(a) If B is the standard ordered basis for $\mathbb{R}^3$ and $B^{'}$ is the standard ordered basis for $\mathbb{R}^2$, what is the matrix of T relative to the pair $B, B^{'}$?

(b) If $B = \{\alpha_1, \alpha_2, \alpha_3\}$ and $B^{'} = \{\beta_1, \beta_2\}$, where,

$$\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1), \alpha_3 = (1, 0, 0), \beta_1 = (0, 1), \beta_2 = (1, 0)$$

what is the matrix of T relative to the pair $B, B^{'}$?

8. Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

Is T invertible? If so, find a rule for T^{-1} like the one which defines T.

9. Is the subset $\left\{ \begin{bmatrix} 1+i \\ 3+8i \\ 5+7i \end{bmatrix}, \begin{bmatrix} 1-5i \\ 5 \\ 2+i \end{bmatrix}, \begin{bmatrix} 1+i \\ 3+2i \\ 4-i \end{bmatrix} \right\}$ of $\mathbb{C}^3$ linearly independent when we consider $\mathbb{C}^3$ as a vector space over $\mathbb{C}$? Is it linearly independent when we consider $\mathbb{C}^3$ as a vector space over $\mathbb{R}$?

10. (a) For the vector space $\mathcal{P}_2$ (polynominals of degree less than or equal to 2), find the dual basis of $\{1 + x, 1 + 2x, x^2\}$.

(b) Let $V \subset \mathcal{P}_2$. For $p \in V$, three linear functionals are defined as follows:

$$f_1(p) = \int_0^1 p(x)dx$$
$$f_2(p) = \int_0^2 p(x)dx$$
$$f_3(p) = \int_0^{-1} p(x)dx$$

If f_1, f_2, f_3 is a dual basis of V^*, then find the basis for V for which V^* is a dual.