CS 228 : Logic in Computer Science

S. Krishna

Let $\varphi = \forall x(x \neq x)$. Lets consider only words as possible structures for satisfying φ.

1. φ is unsatisfiable
2. $\epsilon \in L(\varphi)$
3. $L(\varphi) = \{\epsilon\}$
4. $\{\epsilon\} \subset L(\varphi)$
5. $L(\neg\varphi) = \Sigma^*$
6. $\Sigma^+ \subseteq L(\neg\varphi)$
7. $\neg\varphi$ is the formula $\exists x(x \neq x)$
8. $\neg\varphi$ is the formula $\forall x(x \neq x)$
9. Flipping the $\forall$ quantifier in φ to $\exists$ results in the formula $\psi = \exists x(x \neq x)$ which is satisfied by words of length 1
10. Flipping the $\forall$ quantifier in φ to $\exists$ results in the formula $\psi = \exists x(x \neq x)$ which is unsatisfiable

Recap

- We focus on FO over words : the signature has $<, S, Q_a, Q_b, \ldots$. Remember you always have $=$ with you. Recall the terms structure, universe, and assignment.

- Formula φ is satisfiable iff $L(\varphi) \neq \emptyset$.

- Formula φ is valid iff $L(\varphi) = \Sigma^*$.

- Question : How to check satisfiability of FO over words?

Idea for SAT checking

▶ Given FO formula φ over an alphabet Σ, construct an edge labeled graph G_φ : a graph whose edges are labeled by Σ.
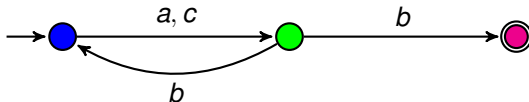

▶ Each path in the graph gives rise to a word over Σ, obtained by reading off the labels on the edges
▶ G_φ has some special kinds of vertices
   ▶ There is a unique vertex called the start vertex (blue vertex)
   ▶ There are some vertices called good vertices (magenta vertex)
▶ Read off words on paths from the start vertex to any final vertex and call this set of words $L(G_\varphi)$
▶ Ensure that G_φ is constructed such that $L(\varphi) = L(G_\varphi)$.

Idea for SAT checking

- Why does this help?
- We know how to check the existence of a path between 2 vertices in a graph easily (how?)
- If somehow we manage to construct G_φ correctly, then checking satisfiability of φ is same as checking the reachability of some good vertex from the start vertex of G_φ.
- How to construct G_φ?

A First Labeled Graph A
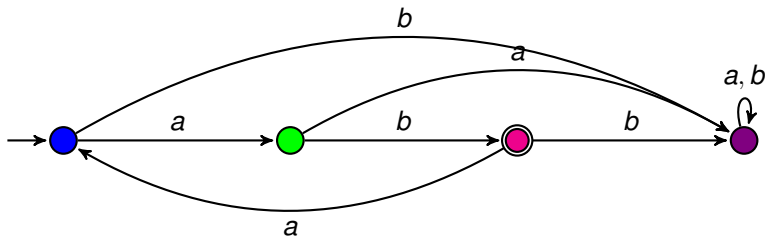
- Let us call the vertices of the graph states
- A path from one state to another gives a word over $\Sigma = \{a, b\}$
- The graph accepts words along paths from an initial state to a good state
- The set of words accepted by the graph is called the language of the graph
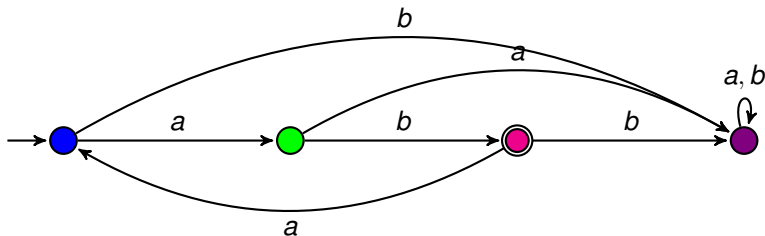
- What is the language *L* accepted by this graph, *L*(*A*)?
- Write an FO formula φ such that $L(\varphi) = L(A)$

A Second and a Third Graph B, C
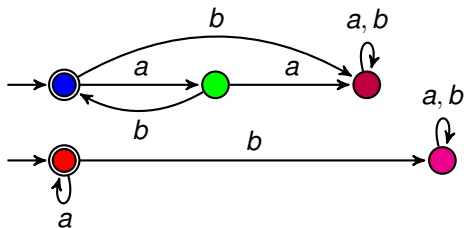
- What are $L(B), L(C)$?
- Give an FO formula φ such that $L(\varphi) = L(B) \cup L(C)$

$\neg\exists x(x = x) \vee \exists x(Q_a(x) \wedge first(x)) \wedge \exists y(Q_b(y) \wedge last(y)) \wedge$
$\forall x \forall y[(S(x, y) \wedge Q_a(x) \rightarrow Q_b(y)) \wedge (S(x, y) \wedge Q_b(x) \rightarrow Q_a(y))]$

$$\vee$$

$\forall x(Q_a(x))$

These graphs are called........

A deterministic finite state automaton (DFA) $A = (Q, \Sigma, \delta, q_0, F)$

- ▶ Q is a finite set of states
- ▶ Σ is a finite alphabet
- ▶ $\delta : Q \times \Sigma \to Q$ is the transition function
- ▶ $q_0 \in Q$ is the initial state
- ▶ $F \subseteq Q$ is the set of final states
- ▶ $L(A)$=all words leading from q_0 to some $f \in F$

Languages, Machines and Logic

A language $L \subseteq \Sigma^*$ is called regular iff there exists some DFA A such that $L = L(A)$.

A language $L \subseteq \Sigma^*$ is called FO-definable iff there exists an FO formula φ such that $L = L(\varphi)$.

Is it Regular? Is it FO-definable?

$\Sigma = \{a, b\}$. Consider the following languages $L \subseteq \Sigma^*$:

► Even length words