CS 228 : Logic in Computer Science

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Satisfaction, Validity

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  - Satisfiable, if there exists a τ-structure $\mathcal{A}$ and an assignment α for $x_1, \ldots, x_n$ in $u(\mathcal{A})$ such that $\mathcal{A} \models_\alpha \varphi(x_1, \ldots, x_n)$

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  - Valid, if for any τ-structure $\mathcal{A}$ and any assignment α for $x_1, \ldots, x_n$ in $u(\mathcal{A})$, $\mathcal{A} \models_\alpha \varphi(x_1, \ldots, x_n)$
- Assume we fix the type of the structure $\mathcal{A}$, say words (why words?)

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- Given φ, write an algorithm to check $L(\varphi) = \emptyset$

First-Order Logic over Words

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- Satisfiability
  - Given a FO formula φ over words, is $L(\varphi)$ non-empty?

A Primer for Words

Alphabet

- An alphabet Σ is a finite set
  - $\Sigma = \{a, b, \ldots, z\}$
  - $\Sigma = \{+, \alpha, 100, B\}$
- Elements of Σ called letters or symbols
- A word or string over Σ is a finite sequence of symbols from Σ
- If $\Sigma = \{a, b\}$, then *abababa* is a word of length 7
- The length of a word w is denoted $|w|$
- There is a unique word of length 0 denoted ϵ, called the empty word
- $|\epsilon| = 0$

Notations for Words

- *aaaaa* denoted a^5
- $a^0 = \epsilon$
- $a^{n+1} = a^n.a = a.a^n$
- The set of all words over Σ is denoted Σ^*
  - $\{a, b\}^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$
  - $\{a\}^* = \{\epsilon, a, aa, aaa, \ldots\} = \{a^n \mid n \geqslant 0\}$
- By convention, $\{\}^* = \{\epsilon\}$

Notations for Words

- Σ is a finite set
- Σ^* is the infinite set of all finite words over alphabet Σ
- Each $w \in \Sigma^*$ is a finite word
  - $\{a, b\} = \{b, a\}$ but $ab \neq ba$
  - $\{a, a, b\} = \{a, b\}$ but $aab \neq ab$
  - $\emptyset$ is the set consisting of no words
  - $\{\epsilon\}$ is a set having the single word ϵ
  - ϵ is a word

Operations on Words

- Concatenation of words : $x.y = xy$
  - Concatenation is associative : $x.(yz) = (xy).z$
  - Concatenation not commutative in general $x.y \neq y.x$
  - ϵ is the identity for concatenation $\epsilon.x = x.\epsilon = x$
  - $|x.y| = |x| + |y|$
- x^n : catenating word x n times
  - $(aab)^5 = aabaabaabaabaab$
  - $(aab)^0 = \epsilon$
  - $(aab)^* = \{\epsilon, aab, aabaab, aabaabaab, \ldots\}$
  - $x^{n+1} = x^n x$

More Operations on Words

- For $a \in \Sigma$ and $x \in \Sigma^*$,

$$|x|_a = \textit{number of times the symbol a occurs in the word x}$$

  - $|aabbaa|_a = 4$, $|aabbaa|_b = 2$
  - $|\epsilon|_a = 0$
- Prefix of a word $w \in \Sigma^*$ is an initial subword of w

$$Pref(w) = \{x \in \Sigma^* \mid \exists y \in \Sigma^*, w = x.y\}$$

  - $Pref(aaba) = \{\epsilon, a, aa, aab, aaba\}$
  - Proper prefixes = $\{a, aa, aab\}$
  - ϵ, $aaba$ improper prefixes

Operation on Sets

Given a finite alphabet Σ, denote by $A, B, C, \ldots$ subsets of Σ^*

- Subsets of Σ^* are called languages
- $A \cup B = \{x \in \Sigma^* \mid x \in A \text{ or } x \in B\}$
  - $A = a^*, B = \{b, bb\}, A \cup B = a^* \cup \{b, bb\}$
- $A \cap B = \{x \in \Sigma^* \mid x \in A \text{ and } x \in B\}$
  - $A = (ab)^*, B = \{abab, \epsilon, bb\}, A \cap B = \{\epsilon, abab\}$
- $\overline{A} = \{x \in \Sigma^* \mid x \notin A\}$
  - For $\Sigma = \{a\}$ and $A = (aa)^*$, $\overline{A} = \{a, a^3, a^5, \ldots\}$
- $AB = \{xy \mid x \in A, y \in B\}$
  - $A = \{a, ba\}, B = \{\epsilon, aa, bb\}$
  - $AB = \{a, a^3, abb, ba, ba^3, babb\}$
  - $BA = \{a, ba, a^3, aaba, bba, bbba\}$

Operation on Sets

For a set $A \subseteq \Sigma^*$,

- $A^0 = \{\epsilon\}$
- $A^{n+1} = A.A^n$
  - $\{a, ab\}^2 = \{a, ab\}.\{a, ab\} = \{aa, aab, aba, abab\}$
  - $\{a, b\}^n = \{x \in \{a, b\}^* \mid |x| = n\}$
- $A^* = A^0 \cup A \cup A^2 \cup \cdots = \bigcup_{i \geqslant 0} A^i$
- $A^+ = AA^* = A \cup A^2 \cup \cdots = \bigcup_{i \geqslant 1} A^i$
- Union : Associative, commutative
- Concatenation : Associative, Non commutative
- $A \cup \emptyset = \emptyset \cup A = A$
- $\{\epsilon\}A = A\{\epsilon\} = A$
- $\emptyset A = A\emptyset = \emptyset$

Operation on Sets

- Union, Intersection distribute over union
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Concatenation distributes over union
  - $A(\cup_{i \in I} B_i) = \cup_{i \in I} AB_i$
  - $(\cup_{i \in I} B_i)A = \cup_{i \in I} B_i A$
- Concatenation does not distribute over interesection
  - $A = \{a, ab\}, B = \{b\}, C = \{\epsilon\}$
  - $A(B \cap C) \neq AB \cap AC$

FO for Languages

Formalize in FO

Write FO formulae φ_i such that $L(\varphi_i) = L_i$ for $i = 1, \ldots, 5$.

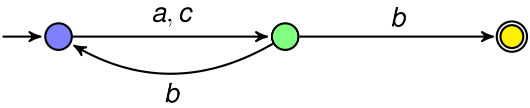
- L_1 = Words that have exactly one occurrence of the letter *c*
- L_2 = Words that begin with *a* and end with *b*
- L_3 = Words that have no two consecutive *a*'s
- L_4 = Words in which any *a* is followed immediately by a *b*
- L_5 = Words in which whenever an *a* occurs, it is followed eventually by a *b*, and no *c* occurs in between the *a* and the *b*
  aabbabab, *aabbcbccaab* $\in L_5$, *aacaab* $\notin L_5$.

Satisfiability of FO over Words

- Recall : Given an FO sentence φ over words, is $L(\varphi) = \emptyset$?
- Algorithm?

Core Idea

- Given FO formula φ over an alphabet Σ, construct an edge labeled graph G_φ : a graph whose edges are labeled by Σ.



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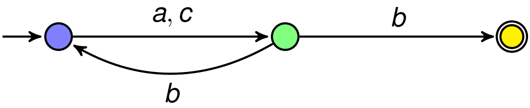

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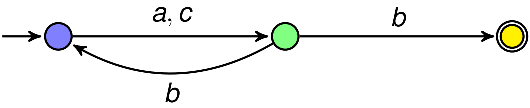

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  - There is a unique vertex called the start vertex (blue vertex)
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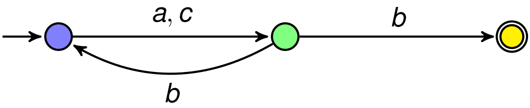

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- G_φ has some special kinds of vertices
  - There is a unique vertex called the start vertex (blue vertex)
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- Read off words on paths from the start vertex to any final vertex and call this set of words $L(G_\varphi)$
- Ensure that G_φ is constructed such that $L(\varphi) = L(G_\varphi)$.

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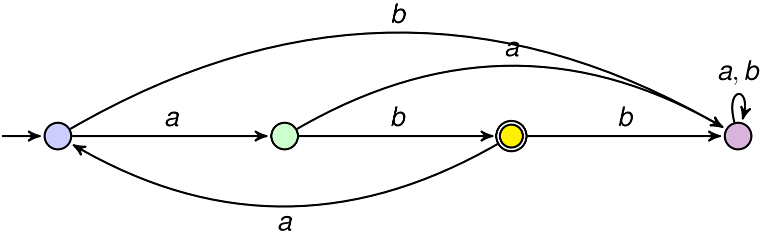
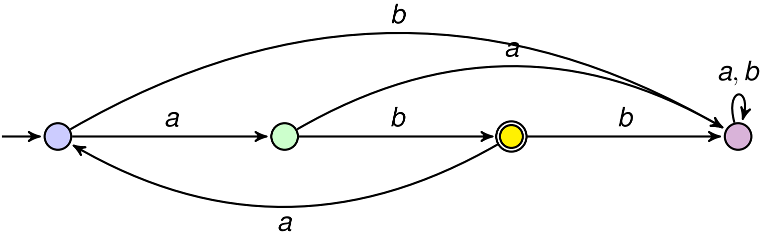
- If somehow we manage to construct G_φ correctly, then checking satisfiability of φ is same as checking the reachability of some good vertex from the start vertex of G_φ.
- How to construct G_φ?

A First Machine *A*



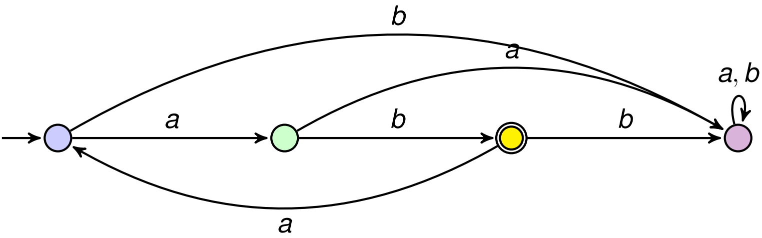
- Colored circles called states
- Arrows between circles called transitions
- Blue state called an initial state
- Doubly circled state called a final state

A First Machine A



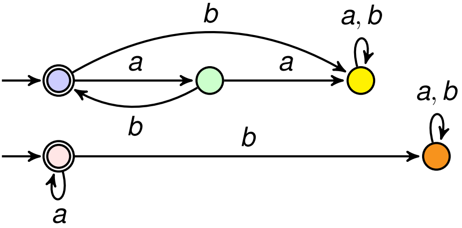
- A path from one state to another gives a word over $\Sigma = \{a, b, c\}$
- The machine accepts words along paths from an initial state to a final state
- The set of words accepted by the machine is called the language accepted by the machine

A First Machine A



- What is the language L accepted by this machine, $L(A)$?
- Write an FO formula φ such that $L(\varphi) = L(A)$

A Second and a Third Machine B, C



- What are $L(B), L(C)$?
- Give an FO formula φ such that $L(\varphi) = L(B) \cup L(C)$