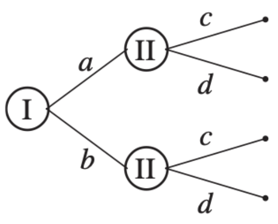
CS 405/6001: Game Theory and Algorithmic Mechanism Design

Problem Set 2

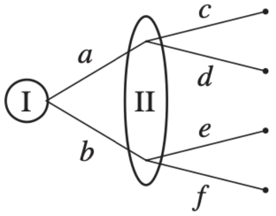
Course Homepage

[Week 5]

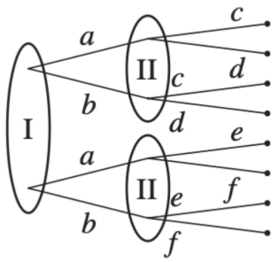
1. Each one of the following figures cannot depict a game in extensive form. For each one, explain why.



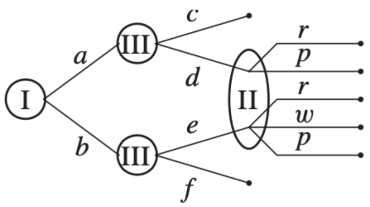
Part A



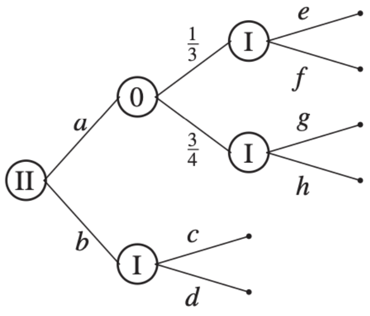
Part B



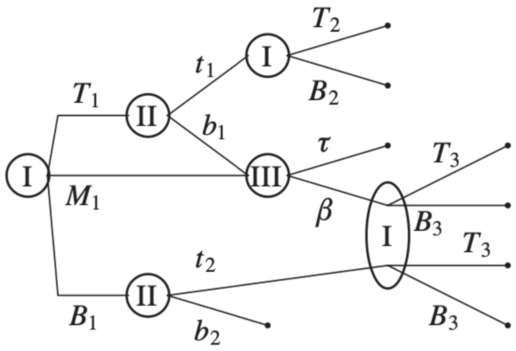
Part C



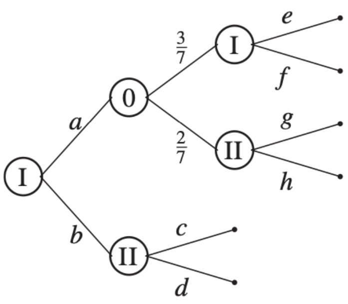
Part D



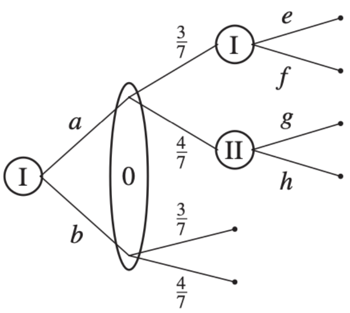
Part E



Part F

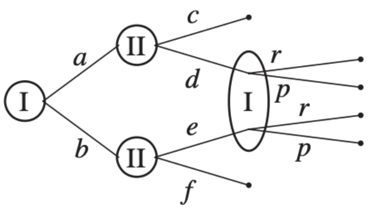


Part G

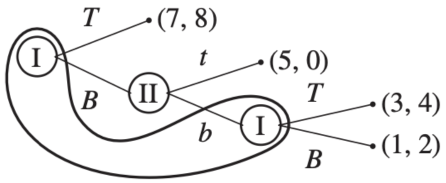


Part H

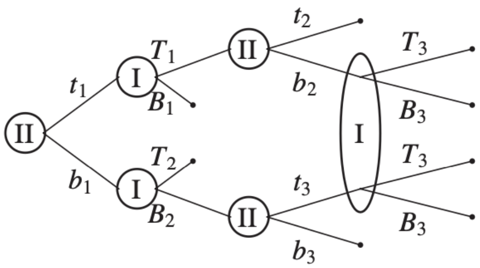
2. In each of the following games, Player I has an information set containing more than one vertex. What exactly has Player I "forgotten" (or could "forget") during the play of each game?



Game A

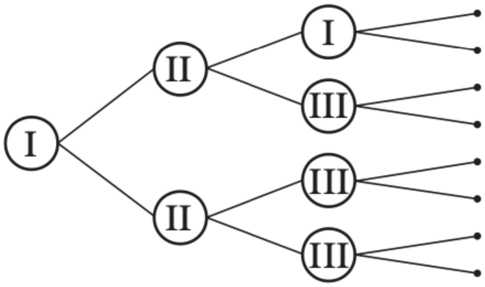


Game B



Game C

3. Sketch the information sets in the following game tree in each of the situations described in this exercise.

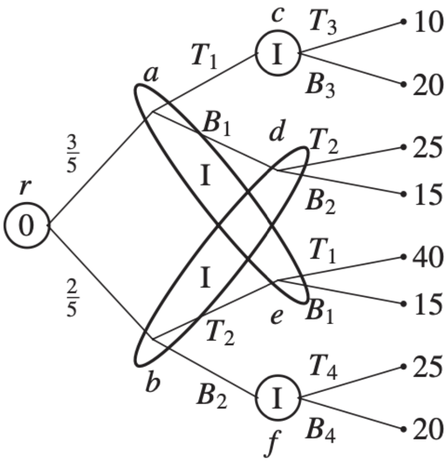


(a) Player II does not know what Player I selected, while Player III knows what Player I selected, but if Player I moved down, Player III does not know what Player II selected.

(b) Player II does not know what Player I selected, and Player III does not know the selections of either Player I or Player II.

(c) At every one of his decision points, Player I cannot remember whether or not he has previously made any moves.

4. (a) What does Player I know, and what does he not know, at each information set in the following game:



(b) How many strategies has Player I got?

(c) The outcome of the game is the payment to Player I. What do you recommend Player I should play in this game?

5. Consider the following game. Player I has the opening move, in which he chooses an action in the set $\{L, R\}$. A lottery is then conducted, with either λ or ρ selected, both with probability $\frac{1}{2}$. Finally, Player II chooses either l or r. The outcomes of the game are not specified. Depict the game tree associated with the extensive-form game in each of the following situations:

(a) Player II, at his turn, knows Player I's choice, but does not know the outcome of the lottery.

(b) Player II, at his turn, knows the outcome of the lottery, but does not know Player I's choice.

(c) Player II, at his turn, knows the outcome of the lottery only if Player I has selected L.

(d) Player II, at his turn, knows Player I's choice if the outcome of the lottery is λ, but does not know Player I's choice if the outcome of the lottery is ρ.

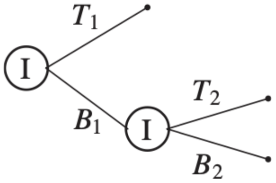
(e) Player II, at his turn, does not know Player I's choice, and also does not know the outcome of the lottery.

6. (a) Let i be a player with perfect recall in an extensive-form game and let σ_i be a mixed strategy of player i. Suppose that there is a strategy vector σ_{-i} of the other players such that $\rho(x; \sigma_i, \sigma_{-i}) > 0$ for each leaf x in the game tree. Prove that there exists a unique behavior strategy b_i equivalent to σ_i.

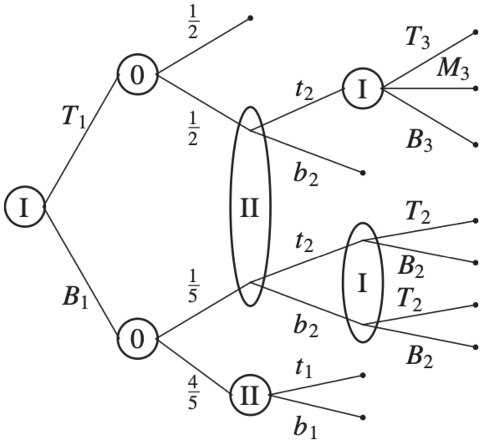
   (b) Give an example of an extensive-form game in which player i has perfect recall and there is a mixed strategy σ_i with more than one behavior strategy equivalent to it.

7. Find a behavior strategy equivalent to the given mixed strategies in each of the following games.

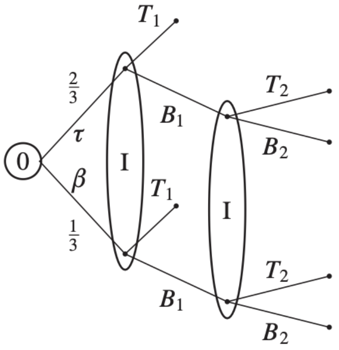
   (a) $s_I = [\frac{1}{2}(B_1, B_2), \frac{1}{2}(T_1, T_2)]$, in the game



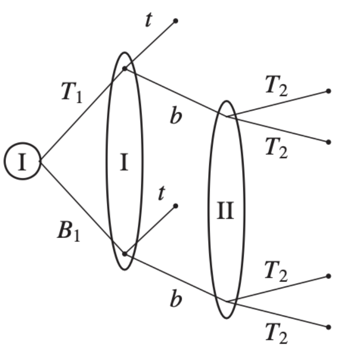
   (b) $s_I = [\frac{3}{7}(B_1B_2M_3), \frac{1}{7}(B_1T_2B_3), \frac{2}{7}(T_1B_2M_3), \frac{1}{7}(T_1T_2T_3)]$ and $s_{II} = [\frac{3}{7}(b_1b_2), \frac{1}{7}(b_1t_2), \frac{1}{7}(t_1b_2), \frac{2}{7}(t_1t_2)]$, in the game



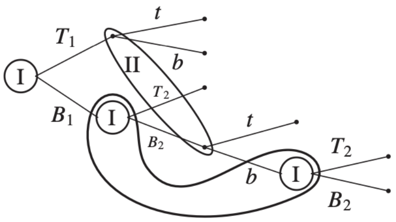
8. In each of the games in the following diagrams, identify which players have perfect recall. In each case in which there is a player with imperfect recall, indicate what the player may forget during a play of the game, and in what way the condition in definition of perfect recall fails to obtain.



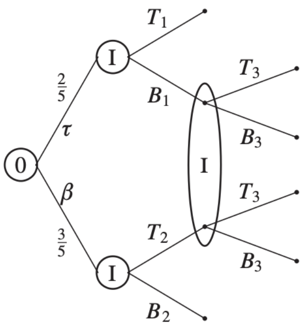
Game A



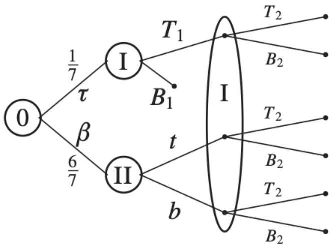
Game B



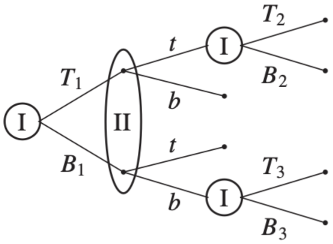
Game C



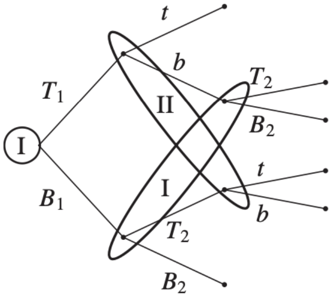
Game D



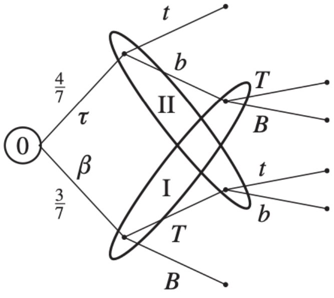
Game E



Game F



Game G

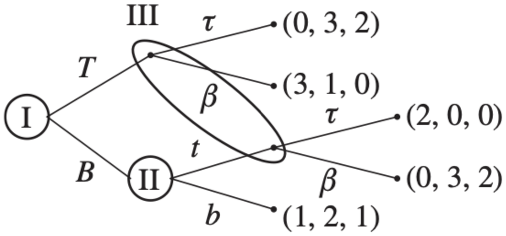


Game H

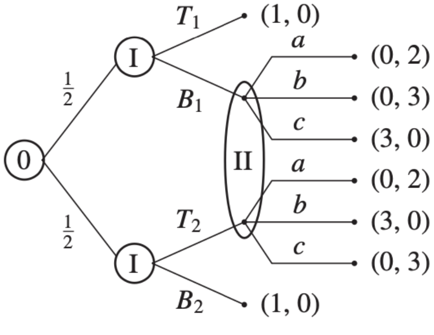
[Week 6]

9. Caesar is at a cafe, trying to choose what to drink with breakfast: beer or orange juice. Brutus, sitting at a nearby table, is pondering whether or not to challenge Caesar to a duel after breakfast. Brutus does not know whether Caesar is brave or cowardly, and he will only dare to challenge Caesar if Caesar is cowardly. If he fights a cowardly opponent, he receives one unit of utility, and he receives the same single unit of utility if he avoids fighting a brave opponent. In contrast, he loses one unit of utility if he fights a brave opponent, and similarly loses one unit of utility if he dishonors himself by failing to fight a cowardly opponent. Brutus ascribes probability 0.9 to Caesar being brave, and probability 0.1 to Caesar being a coward. Caesar has no interest in fighting Brutus: he loses 2 units of utility if he fights Brutus, but loses nothing if there is no fight. Caesar knows whether he is brave or cowardly. He can use the drink he orders for breakfast to signal his type, because it is commonly known that brave types receive one unit of utility if they drink beer (and receive nothing if they drink orange juice), while cowards receive one unit of utility if they drink orange juice (and receive nothing if they drink beer). Assume that Caesar's utility is additive; for example, he receives three units of utility if he is brave, drinks beer, and avoids fighting Brutus. Answer the following questions:
   (a) Describe this situation as an extensive-form game, where the root of the game tree is a chance move that determines whether Caesar is brave (with probability 0.9) or cowardly (with probability 0.1).
   (b) Find all the Nash equilibria of the game.
   (c) Find all the sequential equilibria of the game.

10. Find all the sequential equilibria of the following game.



11. Consider the following extensive-form game.



(a) Prove that in this game at every Nash equilibrium Player I plays (T_1, B_2).

(b) List all the Nash equilibria of the game.

(c) Which of these Nash equilibria can be completed to a sequential equilibrium, and for each such sequential equilibrium, what is the corresponding belief of Player II at his information sets? Justify your answer.

12. Two or three players are about to play a game: with probability $\frac{1}{2}$ the game involves Players 1 and 2 and with probability $\frac{1}{2}$ the game involves Players 1, 2, and 3. Players 2 and 3 know which game is being played. In contrast, Player 1, who participates in the game under all conditions, does not know whether he is playing against Player 2 alone, or against both Players 2 and 3. If the game involves Players 1 and 2 the game is given by the following matrix, where Player 1 chooses the row, and Player 2 chooses the column:

	L	R
T	0,0	2,1
B	2,1	0,0

with Player 3 receiving no payoff. If the game involves all three players, the game is given by the following two matrices, where Player 1 chooses the row, Player 2 chooses the column, and Player 3 chooses the matrix:

W

	L	R
T	1,2,4	0,0,0
B	0,0,0	2,1,3

E

	L	R
T	2,1,3	0,0,0
B	0,0,0	1,2,4

(a) What are the states of nature in this game?

(b) How many pure strategies does each player have in this game?

(c) Depict this game as a game with incomplete information.

(d) Describe the game in extensive form.

(e) Find two Bayesian equilibria in pure strategies.

(f) Find an additional Bayesian equilibrium by identifying a strategy vector in which all the players of all types are indifferent between their two possible actions.

13. Let $L > M > 0$ be two positive real numbers. Two players play a game in which the payoff function is one of the following two, depending on the value of the state of nature s, which may be 1 or 2:

		Player II	
		A	B
Player I	A	M, M	$1, -L$
	B	$-L, 0$	$0, 0$

The state game for $s = 1$

		Player II	
		A	B
Player I	A	$0, 0$	$0, -L$
	B	$-L, 1$	M, M

The state game for $s = 2$

The probability that the state of nature is $s = 2$ is $p < \frac{1}{2}$. Player I knows the true state of nature, and Player II does not know it. The players would clearly prefer to coordinate their actions and play (A, A) if the state of nature is $s = 1$ and (B, B) if the state is $s = 2$, which requires that both of them know what the true state is. Suppose the players are on opposite sides of the globe, and the sole method of communication available to them is e-mail. Due to possible technical communications disruptions, there is a probability of $\epsilon > 0$ that any e-mail message will fail to arrive at its destination. In order to transfer information regarding the state of nature from Player I to Player II, the two players have constructed an automated system that sends e-mail from Player I to Player II if the state of nature is $s = 2$, and does not send any e-mail if the state is $s = 1$. To ensure that Player I knows that Player II received the message, the system also sends an automated confirmation of receipt of the message (by e-mail, of course) from Player II to Player I the instant Player I's message arrives at Player II's e-mail inbox. To ensure that Player II knows that Player I received the confirmation message, the system also sends an automated confirmation of receipt of the confirmation message from Player I to Player II the instant Player II's confirmation arrives at Player I's e-mail inbox. The system then proceeds to send an automated confirmation of the receipt of the confirmation of the receipt of the confirmation, and so forth. If any of these e-mail messages fail to arrive at their destinations, the automated system stops sending new messages. After communication between the players is completed, each player is called upon to choose an action, A or B.

Answer the following questions:

(a) Depict the situation as a game with incomplete information, in which each type of each player is indexed by the number of e-mail messages he has received.

(b) Prove that the unique Bayesian equilibrium where Player I plays A when $s = 1$ is for both players to play A under all conditions.

(c) How would you play if you received 100 e-mail confirmation messages? Explain your answer.

14. This exercise illustrates that in situations in which a seller has more information than a buyer, transactions might not be possible. Consider a used car market in which a fraction q of the cars ($0 \leq q \leq 1$) are in good condition and $1 - q$ are in bad condition (lemons). The seller (Player 2) knows the quality of the car he is offering to sell while the buyer (Player 1) does not know the quality of the car that he is being offered to buy. Each used car is offered for sale at the price of \$$p$ (in units of thousands of dollars). The payoffs to the seller and the buyer, depending on whether or not the transaction is completed, are described in the following tables:

	Sell	Don't Sell
Buy	$6-p, p$	$0, 5$
Don't Buy	$0, 5$	$0, 5$

State game if car in good condition

	Sell	Don't Sell
Buy	$4-p, p$	$0, 0$
Don't Buy	$0, 0$	$0, 0$

State game if car in bad condition

Depict this situation as a Harsanyi game with incomplete information, and for each pair of parameters p and q, find all the Bayesian equilibria.