CS 228 : Logic in Computer Science

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Recap

Signatures, Formulae over signatures, Structure for a signature

Example of Satisfaction

- $\mathcal{G} = (\{1,2,3\}, E^{\mathcal{G}} = \{(1,2),(2,1),(2,3),(3,2)\})$
  - For any assignment α, $\mathcal{G} \models_{\alpha} \forall x \forall y (E(x,y) \rightarrow E(y,x))$ iff for every $a, b \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a, y \mapsto b]} (E(x,y) \rightarrow E(y,x))$

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- Formulae $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are equivalent denoted $\varphi \equiv \psi$ iff for every $\mathcal{A}$ and α, $\mathcal{A} \models_\alpha \varphi$ iff $\mathcal{A} \models_\alpha \psi$

Equisatisfiability

Let $\varphi_1(x) = \forall y R(x, y)$ and $\varphi_2 = \exists x \forall y R(x, y)$.

- It is clear that whenever $\mathcal{A} \models \varphi_2$, one can find an assignment α such that $\mathcal{A} \models_\alpha \varphi_1(x)$.
- Likewise, if $\mathcal{A} \models_\alpha \varphi_1(x)$, then $\mathcal{A} \models \varphi_2$.
- Thus, $\varphi_1(x), \varphi_2$ are equisatisfiable.

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \mathit{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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- Evaluate for all $a, b \in u(\mathcal{A})$, $R(a, \alpha) \to P(b)$
- $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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For a sentence φ, and any two assignments α_1 and α_2, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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No free variables!

Check Satisfiability

Let τ be a signature with a single unary relation P. Consider the structure $\mathcal{A} = (U_\mathcal{A} = \{0, 1\}, P^\mathcal{A} = \{1\})$.

Let $\varphi = \forall x_1 \forall x_2 \dots \forall x_n (P(x_1) \rightarrow (P(x_2) \rightarrow (P(x_3) \cdots \rightarrow (P(x_n) \rightarrow P(x_1))) \dots)))$.

Does $\mathcal{A} \models \varphi$?

Check Satisfiability

Let $\varphi(y) = \exists x(E(x, y) \wedge \neg(y = x) \wedge \forall z[E(z, y) \rightarrow z = x])$ over the signature τ containing a binary relation E. Is $\varphi(y)$ satisfiable under some graph structure?