CS228 Logic for Computer Science 2023

Lecture 18: FOL - conjunctive normal form

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CNF normalization steps

We can convert any FOL sentence into a first-order logic conjunctive normal form(FOL CNF).

We will define FOL CNF by following the process of transformation.

The following transformations results in the CNF.

1. Rename apart : rename variables for each quantifier
2. Negation normal form : push negation inside
3. Prenex form : pull quantifiers to front
4. Skolemization: remove existential quantifiers (only satisfiability preserving)
5. CNF transformation: turn the quantifier-free part of the sentence into CNF
6. Syntactical removal of universal quantifiers: a CNF with free variables.

Topic 18.1

Step 1: rename apart

Name does not matter

Theorem 18.1

If $x, y \notin FV(F(z))$, then $\forall x.F(x)$ and $\forall y.F(y)$ are provably equivalent .

Proof.

1. $\{\forall x.F(x)\} \vdash \forall x.F(x)$ — Assumption
2. $\{\forall x.F(x)\} \vdash F(y)$ — $\forall$-Instantiation applied to 1
3. $\{\forall x.F(x)\} \vdash \forall y.F(y)$ — $\forall$-Intro applied to 2, since $y \notin FV(\forall x.F(x))$

We can run the proof in both the directions. □

Exercise 18.1

a. Prove: if $x, y \notin FV(F(z))$, then $\exists x.F(x)$ and $\exists y.F(y)$ are provably equivalent.

b. Give proof for renaming a quantified variable to a fresh name that is not on the top.

Step 1: rename apart

Definition 18.1

A formula F is renamed apart if no quantifier in F use a variable that is used by another quantifier or occurs as free variable in F.

Due to the previous theorem, we can assume that every quantifier has different variable. If not, we can rename quantified variables apart.

Example 18.1

Consider formula $\neg(\exists x.\forall y R(x,y) \Rightarrow \forall y.\exists x(R(x,y) \land P(x)))$. After renaming apart we obtain the following

$$\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.(R(w,z) \land P(w)))$$

Topic 18.2

Step 2: negation normal form

Relating $\forall$ and $\exists$

Theorem 18.2 *If we have $\Sigma \vdash \neg\exists x.\, \neg F(x)$, we can prove $\Sigma \vdash \forall x. F(x)$.*

Proof.

1. $\Sigma \vdash \neg\exists x.\, \neg F(x)$ — Premise
2. $\Sigma \cup \{\neg F(y)\} \vdash \neg F(y)$ — Assumption (choose fresh y (why?))
3. $\Sigma \cup \{\neg F(y)\} \vdash \exists x.\, \neg F(x)$ — $\exists$-Intro
4. $\Sigma \vdash F(y)$ — propositional rules applied to 1 and 3
5. $\Sigma \vdash \forall x. F(x)$ — $\forall$-Intro on 4

$\square$

Exercise 18.2

a. *Prove: if we have $\Sigma \vdash \neg\forall x. F(x)$, we can prove $\Sigma \vdash \exists x.\, \neg F(x)$.*

b. *Prove: if we have $\Sigma \vdash \neg\exists x.\, F(x)$, we can prove $\Sigma \vdash \forall x. \neg F(x)$.* *(Hint: replace $\neg F(.)$ by $F(.)$ in the above proof)*

Commentary: The reverse direction of the above equivalences are proven in the extra slides of this lecture.

Step 2: negation normal form(NNF)

Definition 18.2

A formula F is in negation normal form if all the negation symbols in the formula occur in form of atomic formulas.

Due to the previous theorems and the properties of propositional connectives, we can translate any formula in negation normal form.

Example: negation normal form

Exercise 18.3

We convert $\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.(R(w,z) \wedge P(w)))$ *into NNF as follows*

$$
\begin{aligned}
\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.(R(w,z) \wedge P(w))) &\equiv (\exists x.\forall y.R(x,y) \wedge \neg\forall z.\exists w.(R(w,z) \wedge P(w))) \\
&\equiv (\exists x.\forall y.R(x,y) \wedge \exists z.\neg\exists w.(R(w,z) \wedge P(w))) \\
&\equiv (\exists x.\forall y.R(x,y) \wedge \exists z.\forall w.\neg(R(w,z) \wedge P(w))) \\
&\equiv (\exists x.\forall y.R(x,y) \wedge \exists z.\forall w.(\neg R(w,z) \vee \neg P(w)))
\end{aligned}
$$

Topic 18.3

Step 3: prenex form

No occurrence; no issues

Theorem 18.3

Let x be a variable such that $x \notin FV(F)$. Then F, $\exists x.F$, and $\forall x.F$ are provably equivalent.

Proof.

We have already seen $\forall x.F$ to $\exists x.F$.

Proving from F to $\forall x.F$

1. $\Sigma \vdash F$ Premise
2. $\Sigma \vdash \forall x.\ F$ $\forall$-Intro applied to 1

Since x is not in F, we choose $y, z \notin FV(\Sigma \cup \{F\})$ and say $F(z)\{z \mapsto y\} = F$. $\forall$-Intro conditions are met.(why?)

Proving from $\exists x.F$ to F

1. $\Sigma \vdash \exists x.F$ Premise
2. $\Sigma \cup \{F\} \vdash F$ Assumption
3. $\Sigma \vdash F \Rightarrow F$ propositional rules applied to 2
4. $\Sigma \vdash \exists x.F \Rightarrow F$ $\exists$-Elim applied to 3
5. $\Sigma \vdash F$ propositional rules applied to 4 and 1

$\square$

Commentary: Please check if the side conditions of $\exists$-Elim are met in step 4 of the right proof. Why absence of x is important in the proof?

No occurrence; we can pull quantifiers to top

Theorem 18.4

If $x \notin FV(G)$, then $\exists x.F(x) \land \exists x.G$ and $\exists x.(F(x) \land G)$ are provably equivalent.

Proof.

Commentary: If x occurs in G, which step of the following proof does not work?

Reverse direction is trivial. Consider the forward direction.

1. $\Sigma \vdash \exists x.F(x) \land \exists x.G$ — Premise
2. $\Sigma \vdash \exists x.G$ — propositional rules applied to 1
3. $\Sigma \vdash G$ — previous theorem applied to 2
4. $\Sigma \cup \{F(x)\} \vdash F(x) \land G$ — propositional rules applied to 3
5. $\Sigma \cup \{F(x)\} \vdash \exists x.(F(x) \land G)$ — $\exists$-Intro applied to 4
6. $\Sigma \vdash F(x) \Rightarrow \exists x.(F(x) \land G)$ — $\Rightarrow$-Intro applied to 5
7. $\Sigma \vdash \exists x.F(x) \Rightarrow \exists x.(F(x) \land G)$ — $\exists$-Elim applied to 6
8. $\Sigma \vdash \exists x.(F(x) \land G)$ — propositional rules applied to 7 and 1 $\square$

Exercise 18.4

If $x \notin FV(G)$, then $\forall x.F(x) \lor \forall x.G$ and $\forall x.(F(x) \lor G)$ are provably equivalent.

Step 3: prenex form

Definition 18.3

A formula F is in prenex form if all the quantifiers of the formula occur as prefix of F. The quantifier-free suffix of F is called matrix of F.

Due to the previous theorems, we move quantifiers to the front.

Exercise 18.5

Show that the following equivalences hold.

- $\forall x.F \Rightarrow G \equiv \exists x.(F \Rightarrow G)$
- $\exists x.F \Rightarrow G \equiv \forall x.(F \Rightarrow G)$
- $F \Rightarrow \forall x.G \equiv \forall x.(F \Rightarrow G)$
- $F \Rightarrow \exists x.G \equiv \exists x.(F \Rightarrow G)$

Example: prenex form

Exercise 18.6

We convert $(\exists x.\forall y.R(x,y) \land \exists z.\forall w.(\neg R(w,z) \lor \neg P(w)))$ *into prenex form as follows*

- $(\exists x.\forall y.R(x,y) \land \exists z.\forall w.(\neg R(w,z) \lor \neg P(w)))$
- $\exists z.(\exists x.\forall y.R(x,y) \land \forall w.(\neg R(w,z) \lor \neg P(w)))$
- $\exists z.\forall w.(\exists x.\forall y.R(x,y) \land (\neg R(w,z) \lor \neg P(w)))$
- $\exists z.\forall w.\exists x.(\forall y.R(x,y) \land (\neg R(w,z) \lor \neg P(w)))$
- $\underbrace{\exists z.\forall w.\exists x.\forall y.}_{\textit{Quantifiers}}\underbrace{(R(x,y) \land (\neg R(w,z) \lor \neg P(w)))}_{\textit{body/matrix of the formula}}$

We move quantifier forward step by step.

In the standard definition of prenex, the body need not be in NNF. Our body is in NNF due to the order of steps we have followed.

Topic 18.4

Step 4: skolemization

Step 4: skolemization

Skolemization removes $\exists$ quantifiers from prenex sentences and only $\forall$ quantifiers are left.

Example 18.2

Let us suppose. We know "for every man there is a woman".

$$\forall m.\ \exists w.\ Relationship(m, w)$$

To satisfy the sentence, we need to find a woman for each man.

In other words, there is a function $f : Men \rightarrow Women$.

In terms of FOL, we may write

$$\forall m. Relationship(m, f(m))$$

The replacement of $\exists$ by a function is called skolemization and f is called skolem function.

Introduction of skolem function with free variables

Theorem 18.5

Let F be a $\mathbf{S}$-formula, $FV(F) = \{x, y_1, \dots, y_n\}$ and $f/n \in \mathbf{F}$ does not occur in F. For each model m', there is a model m such that

Commentary: m is not with any assignment, which means for any assignment.

$$m \models \exists x.F \Rightarrow F\{x \mapsto f(y_1, \dots, y_n)\}.$$

and m and m' only differ on the interpretation of f.

Proof.

Consider a model m'. We will construct m. Before, let us construct an interpretation $f' : D^n_{m'} \to D_{m'}$ of f as follows.

Why d exists?

$$f'(d_1, ..., d_n) \triangleq \begin{cases} d & \text{if } m', \{y_1 \mapsto d_1, ..., y_n \mapsto d_n\} \models \exists x.F, \\ & \text{Choose } d \in D_{m'} \text{ such that } m', \{y_1 \mapsto d_1, ..., y_n \mapsto d_n, x \mapsto d\} \models F \\ & \\ d & \text{otherwise choose any } d \in D_{m'} \end{cases}$$

Introduction of skolem function with free variables(contd.)

Proof(contd.)

Let us define $m \triangleq m'[f \mapsto f']$.

Since f does not occur in F, if $m, \nu \models \exists x.F$ then $m', \nu \models \exists x.F$.

Due to the construction of m,

$$m, \nu \models F\{x \mapsto f(y_1, \ldots, y_n)\}_{\text{(why?)}}.$$

$\square$

Exercise 18.7

Show there is m such that $m \models F\{x \mapsto f(y_1, \ldots, y_n)\} \Rightarrow \forall x.F$

Introduction of skolem functions under quantifiers

Theorem 18.6

Let $F(x)$ be a $(\mathbf{F}, \mathbf{R})$-formula with $FV(F) = \{x, y_1, \dots, y_n\}$ and $f/n \in \mathbf{F}$ such that f does not occur in $F(x)$.

$$\forall y_1, \dots, y_n. \exists x. F(x) \textit{ is sat} \qquad \textit{iff} \qquad \forall y_1, \dots, y_n.\ F(f(y_1, \dots, y_n)) \textit{ is sat}$$

Proof.

Forward direction:

Assume $m' \models \forall y_1, ...y_n. \exists x. F(x)$. Therefore, $m' \models \exists x. F(x)$(why?).

Due to the last theorem, there is m such that $m \models \exists x. F(x) \Rightarrow F(f(y_1, \dots, y_n))$.

Since m and m' only differ on f, $m \models \exists x. F(x)$.

Therefore, $m \models F(f(y_1, .., y_n))$. Therefore, $m \models \forall y_1, ..., y_n.\ F(f(y_1, .., y_n))$. ...

Introduction of skolem functions under quantifiers(contd.)

Proof.

Reverse direction

1. $\{\forall y_1, \dots, y_n.\ F(f(y_1, \dots, y_n))\} \vdash \forall y_1, \dots, y_n.\ F(f(y_1, \dots, y_n))$ Assumption
2. $\{\forall y_1, \dots, y_n.\ F(f(y_1, \dots, y_n))\} \vdash F(f(y_1, \dots, y_n))$ $\forall$-Elim
3. $\{\forall y_1, \dots, y_n.\ F(f(y_1, \dots, y_n))\} \vdash \exists x. F(x)$ $\exists$-Intro
4. $\{\forall y_1, \dots, y_n.\ F(f(y_1, \dots, y_n))\} \vdash \forall y_1, \dots, y_n.\ \exists x. F(x)$ $\forall$-Intro

$\square$

Skolemization of prenex sentence

Since the quantifiers are in prenex form, all $\exists$s can be removed using skolem functions.

Skolemization should be applied from out to inside, i.e.,

remove outermost $\exists$ first.

Example 18.3

Let us skolemize the following sentence

- $\exists z.\forall w.\exists x.\forall y.(R(x,y) \land (\neg R(w,z) \lor \neg P(w)))$
- *Since there are no universals before $\exists z$, we introduce a function $c/0$.*
  $\forall w.\exists x.\forall y.(R(x,y) \land (\neg R(w,c) \lor \neg P(w)))$
- *Since there is a universal $\forall w$ before $\exists x$, we introduce a function $f/1$.*
  $\forall w.\forall y.(R(f(w),y) \land (\neg R(w,c) \lor \neg P(w)))$

Topic 18.5

Step 5-6: FOL CNF

Step 5: convert body of the sentence to CNF

Consider skolemized prenex sentence $\forall x_1, \dots, x_n.\ F$.

Since F is quantifier-free, we can use propositional logic methods to convert F into CNF

$$\forall x_1, \dots, x_n.\ C_1 \wedge \dots \wedge C_k.$$

Example 18.4

In our running example, the body of the sentence was already in CNF

$$\forall w. \forall y. (R(f(w), y) \wedge (\neg R(w, c) \vee \neg P(w))).$$

Exercise 18.8

We may use Tseitin encoding to obtain CNF, which introduces fresh propositional predicates. Is there a quantifier over the propositional predicates? (Hint: there are no propositional variables in FOL and we cannot quantify over predicates.)

Step 6: drop of explicit mention of quantifiers

Consider skolemized prenex clauses $\forall x_1, \dots, x_n.\ C_1 \land \dots \land C_k$.

Since $\forall$ distributes over $\land$, we translate to

$$(\forall x_1, \dots, x_n.\ C_1) \land \dots \land (\forall x_1, \dots, x_n.\ C_k).$$

We may view the above sentence as conjunction of clauses

$$C_1 \land \dots \land C_k,$$

Since clauses have different quantifiers, even if two clauses share a variable name, they are referring to different variables.

without any explicit mention of quantifiers.

Since we started with sentences, we will assume that the free variables are universally quantified.

Example 18.5

We write the sentence as $R(f(w), y) \land (\neg R(w, c) \lor \neg P(w))$

Commentary: Observe that both the occurrences of w in $(\neg R(w, c) \lor \neg P(w))$ refer to same variable. However, the w in $R(f(w), y)$ is a different variable from the w in $(\neg R(w, c) \lor \neg P(w))$

Topic 18.6

Problems

Skolemization

Exercise 18.9

Demonstrate that skolemization does not produce equivalent formula.

Minimize skolem functions

Exercise 18.10

The order of quantifiers determines the number of parameters in the skolem functions. Give a greedy and efficient(linear) strategy for producing prenex formula such that the total number of parameters in skolem functions is minimal?

FOL CNF

Exercise 18.11

Convert the following formulas in FOL CNF

- $\exists z.(\exists x.Q(x,z) \lor \exists x.P(x)) \Rightarrow \neg(\neg\exists x.P(x) \land \forall x.\ \exists z.Q(z,x)\ )$
- $\neg\exists n.\ \forall w.\ (Gtn(w) \Rightarrow \exists x,y,z.\ (f(x,y,z) = w \land NonEmpty(y) \land \neg Gtn(f(x,y)) \land (\forall k.\ L(x,y,z,k))))$

Convert into CNF

Exercise 18.12

Consider the following formulas

$$\begin{aligned}\Sigma = \{ & \forall x, y, z.\ (z \in x \Leftrightarrow z \in y) \Rightarrow x \approx y, \\ & \forall x, y.\ (x \subseteq y \Leftrightarrow \forall z.\ (z \in x \Rightarrow z \in y)), \\ & \forall x, y, z.\ (z \in x - y \Leftrightarrow (z \in x \land z \notin y))\}.\end{aligned}$$

Convert the following formula into FOL CNF.

$$\bigwedge \Sigma \land \neg \forall x, y.\ (x \subseteq y \Rightarrow \exists z.(y - z \approx x))$$

Theorem prover

Exercise 18.13

Download EPROVER *a first order theorem prover from the following web page.*

`http://wwwlehre.dhbw-stuttgart.de/~sschulz/E/Usage.html`

Run the prover to prove the validity of the following sentence.

$$\forall x.\exists y.\forall z.\exists w.(R(x,y) \lor \neg R(w,z))$$

Report the proof generated by the prover. Explain the proof steps.

Topic 18.7

Extra slides: proofs for pulling negations out

Relating $\forall$ and $\exists$ (reverse direction)

Commentary: $c \neq c$ is proxy for false. Reflex rule allows us to have a representation of true without having true as a symbol. It may feel like cheating.

Theorem 18.7 *If we have $\Sigma \vdash \forall x.F(x)$, we can prove $\Sigma \vdash \neg\exists x.\neg F(x)$.*

Proof.

1. $\Sigma \vdash \forall x.F(x)$ — Premise
2. $\Sigma \cup \{\neg F(x)\} \vdash \forall x.F(x)$ — Monotonic applied to 1
3. $\Sigma \cup \{\neg F(x)\} \vdash F(x)$ — $\forall$-Elim applied to 2
4. $\Sigma \cup \{\neg F(x)\} \vdash \neg F(x) \wedge F(x)$ — propositional rules applied to 3
5. $\Sigma \vdash \neg F(x) \Rightarrow c \neq c$ — Contra applied to 4
6. $\Sigma \vdash \exists x.\neg F(x) \Rightarrow c \neq c$ — $\exists$-Elim applied to 5
7. $\Sigma \vdash c = c$ — Reflex
8. $\Sigma \vdash \neg\exists x.\neg F(x)$ — propositional rules applied to 6 and 7 $\square$

Exercise 18.14

Prove: if we have $\Sigma \vdash \forall x.\neg F(x)$, we can derive $\Sigma \vdash \neg\exists x.F(x)$. *Hint : replace $F(.)$ by $\neg F(.)$ in the above.*

End of Lecture 18