CS 228: Logic for CS

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Languages, Machines and Logic

A language $L \subseteq \Sigma^*$ is called regular iff there exists some DFA/NFA A such that $L = L(A)$.

A language $L \subseteq \Sigma^*$ is called FO-definable iff there exists a FO sentence φ such that $L = L(\varphi)$.

For a sentence φ, $L(\varphi) = \{w \in \Sigma^* \mid w \models \varphi\}$

Agenda

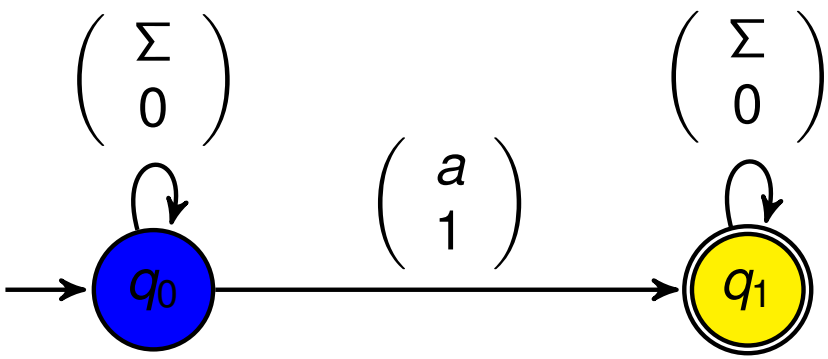
- FO-definable $\Rightarrow$ regular
- Given an FO formula φ, construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$
- If $L(A_\varphi) = \emptyset$, then φ is unsatisfiable
- If $L(A_\varphi) \neq \emptyset$, then φ is satisfiable

FO to Regular Languages

- Every FO sentence φ over words can be converted into a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.
- Start with atomic formulae, construct DFA for each of them.
- Conjunctions, disjunctions, negation of formulae easily handled via union, intersection and complementation of of respective DFA
- Handling quantifiers?

Atomic Formulae to DFA

- $Q_a(x)$: All words which have an a. Need to fix a position for x, where a holds.
- *baab* satisfies $Q_a(x)$ with assignment $x = 1$ or $x = 2$.
- Think of this as $\begin{matrix} baab \\ 0010 \end{matrix}$ or $\begin{matrix} baab \\ 0100 \end{matrix}$
- The first row is over Σ, and the second row captures a possible assignment to x
- Think of an extended alphabet $\Sigma' = \Sigma \times \{0, 1\}$, and construct an automaton over Σ'.
- Deterministic, not complete.

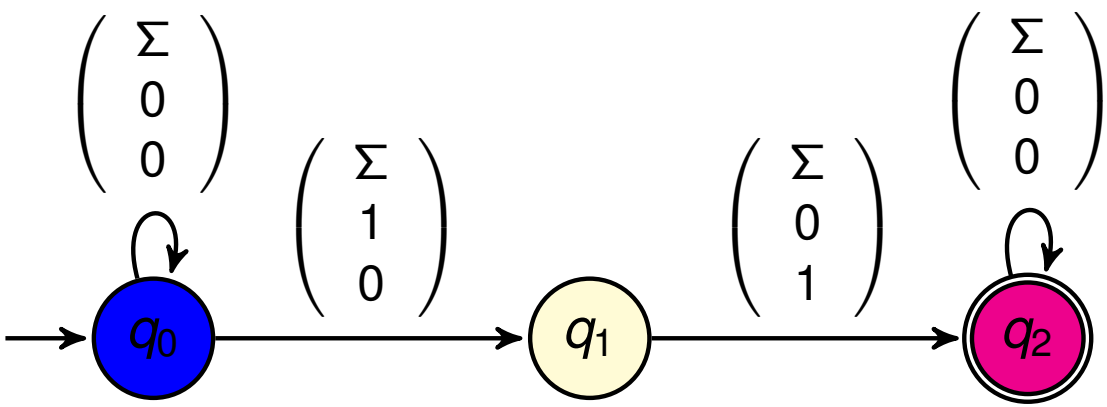


Atomic Formulae to DFA : $S(x, y)$

- *bab* satisfies $S(x, y)$ with assignment $x = 0$ or $y = 1$ or $x = 1, y = 2$.
- Think of this as

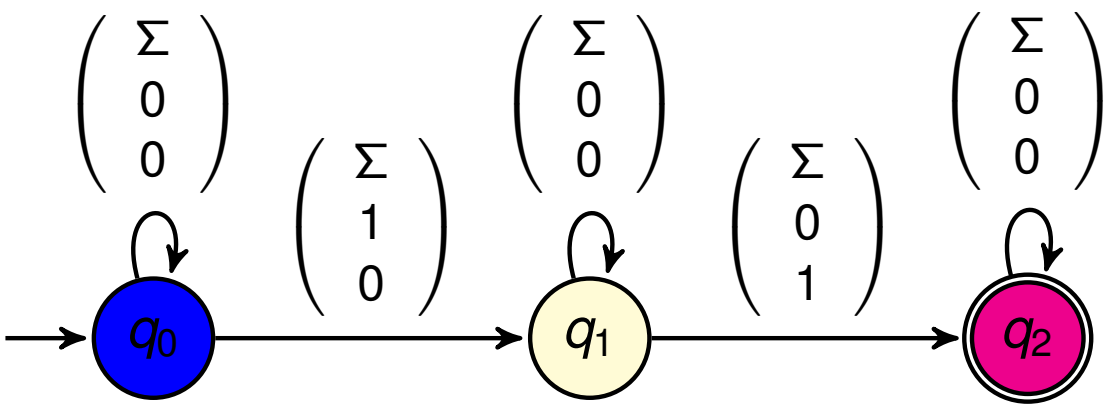
	bab		*bab*
	100	or	010
	010		001

- The first row is over Σ, and the second, third rows capture a possible assignment to x, y
- Think of an extended alphabet $\Sigma' = \Sigma \times \{0, 1\}^2$, and construct an automaton over Σ'.
- Deterministic, not complete.



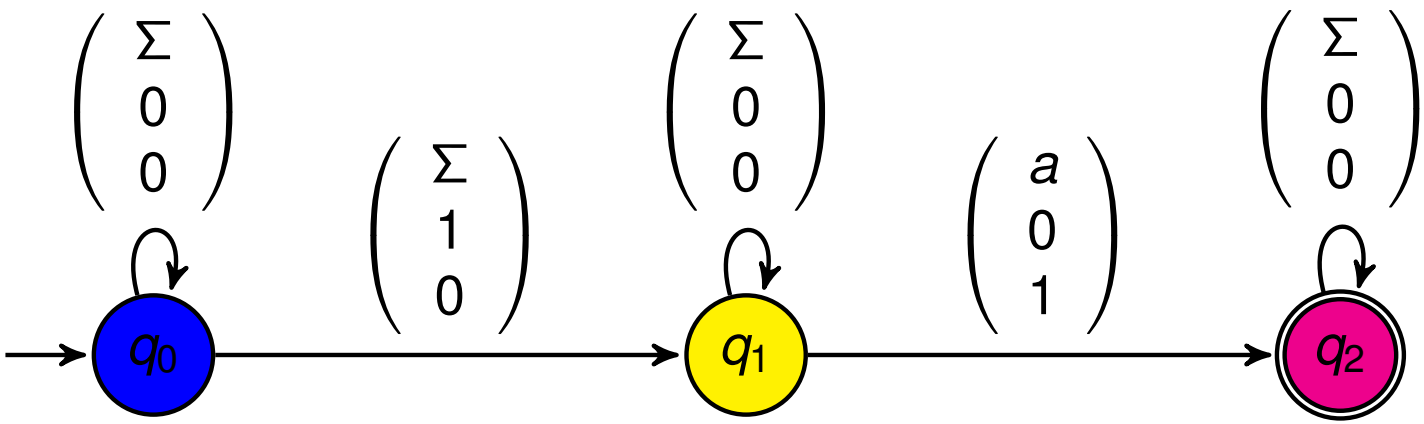
Atomic Formulae to DFA : $x < y$

- *bab* satisfies $x < y$ with assignment $x = 0$ or $y = 1$ or $x = 1, y = 2$ or $x = 0, y = 2$.



Simple Formulae to DFA

- $x < y \wedge Q_a(y)$
- $\Sigma' = \Sigma \times \{0,1\} \times \{0,1\}$
- Obtain intersection of DFA for $x < y$ and $Q_a(y)$



Formulae to DFA

- Given $\varphi(x_1, \ldots, x_n)$, a FO formula over Σ, consider the extended alphabet
$$\Sigma' = \Sigma \times \{0,1\}^n$$
- Assign values to x_i at every position as seen in the cases of atomic formulae
- Keep in mind that every x_i can be assigned 1 at a unique position