

INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

Department of Mathematics

SI 427 (Probability I)

Tutorial Sheet-VI

All random objects are in (Ω, \mathcal{F}, P) unless told otherwise.

1. Let X be a random variable with distribution function F and $a \in \mathbb{R}$. Define $Y = \max\{X, a\}$. Find the distribution function of Y in terms of F .
2. Let X be a random variable with distribution function F . Find the distribution function of the following random variables in terms of F .
(i) X^2 (ii) $|X|$ (iii) e^X .
3. Let X be an exponential(1) random variable and denote the distribution function of X by F . Find the distribution of $F \circ X$.
4. Let X be a continuous random variable with pdf f . Let X is symmetric¹ and X^2 is exponential with parameter λ . Find f .
5. Let X be a random variable with distribution function F and $Y = aX + b$. Find the distribution function of Y in terms of F .
6. Let X be Uniform $(-1, 1)$ random variable and $Y = X^2$. Find the pdf of Y .
7. Let X be uniform $(0, 1)$ random variable. Find $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g \circ X$ is uniform $(-1, 1)$.
8. Let X be a continuous random variable with distribution function F and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\varphi(x) = \begin{cases} x - 1 & \text{if } x < -1 \\ 0 & \text{if } -1 \leq x \leq 1 \\ x + 1 & \text{if } x > 1. \end{cases}$$

Find the distribution of $\varphi \circ X$ in terms of F .

¹A random variable X is said to be symmetric if X and $-X$ have the same distribution.

9. Are closed sets in \mathbb{R} are Borel sets? Justify your answer.
10. Is $(-1, 1) \setminus \{0\}$ a Borel set? Justify your answer.
11. Is $(-2, 1) \setminus \mathbb{Q}$ a Borel set? Here \mathbb{Q} denote the set of all rational numbers. Justify your answer.
12. Let D denote the set of all discontinuities of a distribution function? Is D a Borel set? Justify your answer.
13. Is $(\frac{1}{2}, 1] \cup [2, 3)$ a Borel set? Justify your answer.
14. Let $B = \{x \in [0, 1] | x \text{ doesn't contain } 2 \text{ in its decimal expansion}\}$. Is B a Borel set? Justify your answer.
15. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function? Is $f^{-1}([0, 1])$ a Borel set? Justify your answer.
16. Let $\mathcal{J} = \{[a, b) | a, b \in \mathbb{R}, a \leq b\}$. Find $\sigma(\mathcal{J})$.
17. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ -1 & \text{if } 0 \leq x < \frac{1}{2} \\ (x-2)^2 & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Is $f^{-1}(B)$ a Borel set, where B is given in question 11?

18. Let B be a Borel set. Is $2B := \{2x | x \in B\}$ a Borel set? Justify your answer.
19. Show that the closed region bounded by a triangle is a Borel set in \mathbb{R}^2 .
20. Let $B = \{(x, y) \in \mathbb{R}^2 | 1 < x^2 + y^2 < 2\}$ Is B a Borel set in \mathbb{R}^2 ? Justify your answer.
21. Let B_1 and B_2 are Borel sets in \mathbb{R} . Show that $B_1 \times B_2$ is a Borel set in \mathbb{R}^2 .
22. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Show that $B = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, y \leq f(x)\}$ is a Borel set in \mathbb{R}^2 .
23. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions such that $f \leq g$ and $D = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, f(x) \leq y \leq g(x)\}$. Show that D is a Borel set in \mathbb{R}^2 .