# CS 228 : Logic in Computer Science

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# First-Order Logic : Semantics

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  - The structure $\mathcal{A}$ is finite if $A$ (or $u(\mathcal{A})$) is finite

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  - $\mathcal{G} = (V = \{1, 2, 3, 4\}, E^{\mathcal{G}} = \{(1, 2), (2, 3), (3, 4), (1, 1)\})$. We could just as well draw the graph for convenience.

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  - The structure with $u(\mathcal{W}) = \{0, 1, 2, \dots, 8\}$, ${Q_a}^{\mathcal{W}} = \{0, 1, 4, 6, 8\}$, ${Q_b}^{\mathcal{W}} = \{2, 3, 5, 7\}$,
  - $<^{\mathcal{W}} = \{(0, 1), (0, 2), \dots, (7, 8)\}$, $S^{\mathcal{W}} = \{(0, 1), (1, 2), \dots, (7, 8)\}$ uniquely defines the word $W = aabbababa$.
  - For convenience, we can just write the word instead of the structure.

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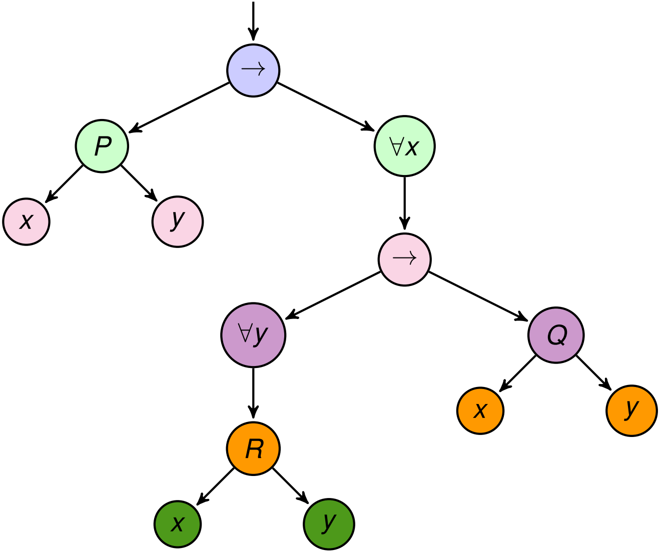
- Given $\varphi$, denote by $\varphi(x_1, \ldots, x_n)$, that $x_1, \ldots, x_n$ are the free variables of $\varphi$, also *free*$(\varphi)$
- A sentence is a formula $\varphi$ none of whose variables are free

## $P(x, y) \to \forall x((\forall y R(x, y)) \to Q(x, y))$



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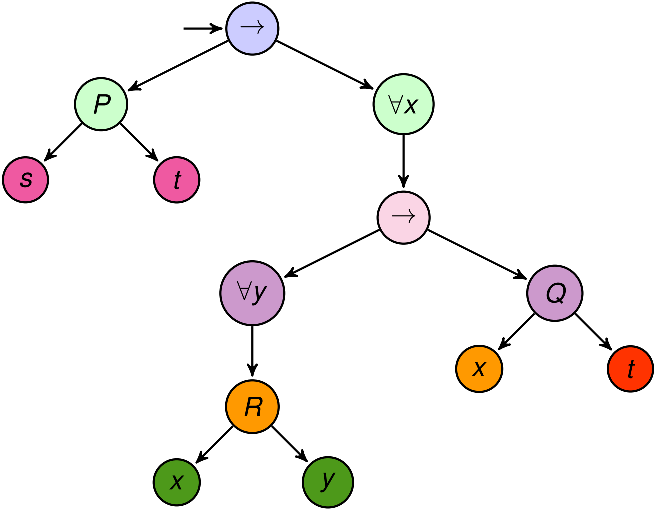
$\varphi(s, t) = P(s, t) \to \forall x((\forall y R(x, y)) \to Q(x, t))$



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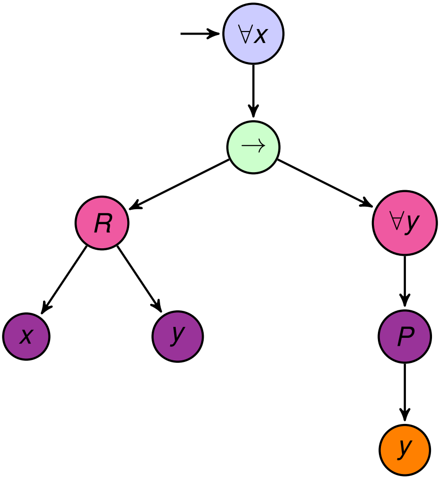

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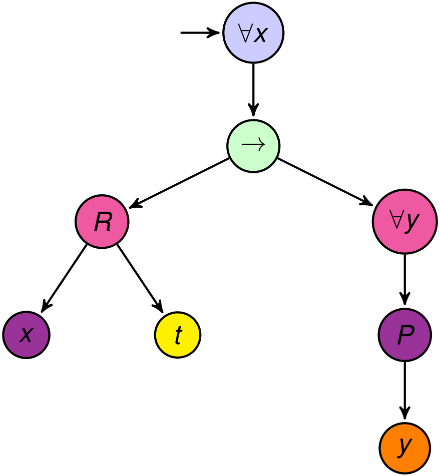


$\varphi(t) = \forall x(R(x,t) \to \forall y P(y))$

# Assignments on $\tau$-structures

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For a $\tau$-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a function $\alpha : \mathcal{V} \to u(\mathcal{A})$ that assigns every variable $x \in \mathcal{V}$ a value $\alpha(x) \in u(\mathcal{A})$. If $t$ is a constant symbol $c$, then $\alpha(t)$ is $c^{\mathcal{A}}$

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## Binding on a Variable

For an assignment $\alpha$ over $\mathcal{A}$, $\alpha[x \mapsto a]$ is the assignment

$$\alpha[x \mapsto a](y) = \begin{cases} \alpha(y), y \neq x, \\ a, y = x \end{cases}$$

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- $\mathcal{A} \models_\alpha (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

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- $\mathcal{A} \models_\alpha (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$
- $\mathcal{A} \models_\alpha (\exists x)\varphi$ iff there is some $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

Last two cases, $\alpha$ has no effect on the value of $x$. Thus, assignments matter only to free variables.