CS 228 : Logic in Computer Science

S. Krishna

FOL : Notations

- Recall : A signature τ consists of a finite set of constants, relations and functions
- Recall : A formula φ is over some chosen signature τ
- Recall : A τ-structure (or a τ-model) consists of a universe (or domain), and gives meanings to the constants, functions and relations in τ.
- Recall notation : $\varphi(x_1, \ldots, x_n)$
- Recall Assignments, satisfaction

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- Recall Assignments, satisfaction
- Signature τ_E having a single binary relation E,
- Signature τ_W having binary relations $S, <$ as well as unary relations Q_a for finitely many symbols a.

Example of Satisfaction

$\mathcal{G} = (\{1, 2, 3\}, E^{\mathcal{G}} = \{(1, 2), (2, 1), (2, 3), (3, 2)\})$

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- $\mathcal{W} = abaaa$ or,
  $\mathcal{W} = (\{0, \ldots, 4\}, <^{\mathcal{W}}, S^{\mathcal{W}}, Q_a^{\mathcal{W}} = \{0, 2, 3, 4\}, Q_b^{\mathcal{W}} = \{1\})$

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  - There is no assignment α which satisfies
    $\exists x \exists y (Q_b(x) \wedge Q_b(y) \wedge x \neq y)$
  - Prove or disprove : $\mathcal{W} \models \exists x \forall y [Q_b(x) \wedge x < y \wedge Q_a(y)]$
  - Prove or disprove : $\mathcal{W} \models \exists x \forall y [Q_b(x) \wedge x < y \Rightarrow Q_a(y)]$

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- Formulae $\varphi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are equivalent denoted $\varphi \equiv \psi$ iff for every $\mathcal{A}$ and α, $\mathcal{A} \models_\alpha \varphi$ iff $\mathcal{A} \models_\alpha \psi$

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- $\psi(z) = \exists x[Q_a(x) \wedge \forall y[(y < x \wedge Q_b(y)) \rightarrow (z < x \wedge y < z \wedge Q_a(z))]]$. Does ψ evaluate to true under some word structure?

Consider the formula
$\varphi = \forall x.Q_a(x) \vee [\forall x.(Q_a(x) \Rightarrow \exists y.(Q_b(y) \wedge x < y))]$.

1. The word *aaa* is a model for φ
2. The word *b* is a model for φ
3. The word *ab* is a model for φ
4. The word *aba* is a model for φ
5. The word *bab* is a model for φ
6. The word *abab* is a model for φ
7. The word *baaaaa* is a model for φ
8. The word *bbb* is a model for φ, but *bb* is not
9. The word *abb* is not a model for φ, but *bba* is
10. Every word over a, b is a model for φ