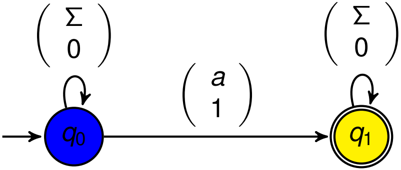
CS 228: Logic for CS

S. Krishna

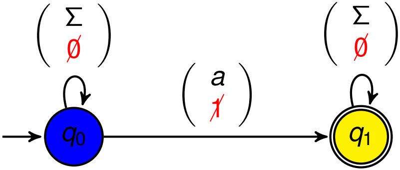
Quantifiers

- $Q_a(x)$



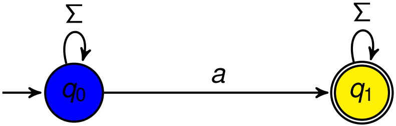
Handling Quantifiers

- $\exists x Q_a(x)$



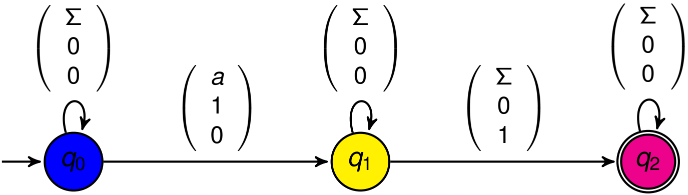
Handling Quantifiers

- $\exists x Q_a(x)$



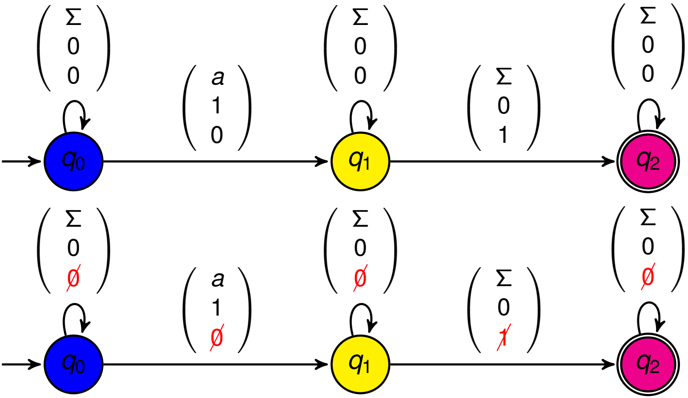
Handling Quantifiers

- $Q_a(x) \wedge \exists y(x < y)$



Handling Quantifiers

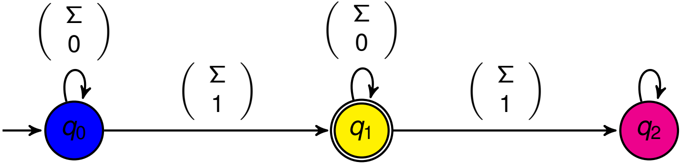
- $Q_a(x) \wedge \exists y(x < y)$



Handling Quantifiers: $\forall x (x \neq x)$

Handling Quantifiers: $\forall x(x \neq x)$

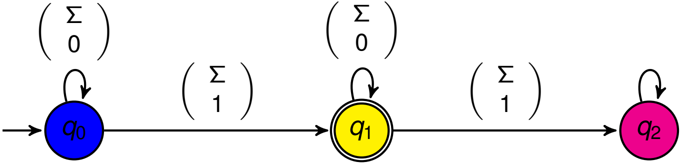
- $(x = x)$



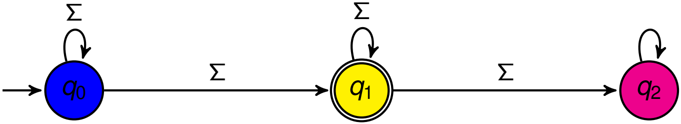
- $\exists x(x = x)$

Handling Quantifiers: $\forall x(x \neq x)$

- $(x = x)$



- $\exists x(x = x)$



- $\neg\exists x(x = x)$

Handling Quantifiers : Summary

- Let $L \subseteq (\Sigma \times \{0,1\}^n)^*$ be defined by $\varphi(x_1, \dots, x_n)$.
- Let $f : (\Sigma \times \{0,1\}^n)^* \to (\Sigma \times \{0,1\}^{n-1})^*$ be the projection $f(w, c_1, \dots, c_n) = (w, c_1, \dots, c_{n-1})$.
- Then $\exists x_n \varphi(x_1, \dots, x_{n-1})$ defines $f(L)$.

Handling Quantifiers : Done on Board

- $\exists x \forall y [x > y \vee \neg Q_a(x)] = \exists x [\neg \exists y [x \leqslant y \wedge Q_a(x)]]$
- Draw the automaton for $[x \leqslant y \wedge Q_a(x)]$
- Project out the y-row
- Determinize it, and complement it
- Fix the x-row : Intersect with $\begin{pmatrix} \Sigma \\ 0 \end{pmatrix}^* \begin{pmatrix} \Sigma \\ 1 \end{pmatrix} \begin{pmatrix} \Sigma \\ 0 \end{pmatrix}^*$
- Project the x-row

Points to Remember

- Given $\varphi(x_1, \ldots, x_n)$, construct automaton for atomic FO formulae over the extended alphabet $\Sigma \times \{0, 1\}^n$
- Intersect with the regular language where every x_i is assigned 1 exactly at one position
- Given a sentence $Q_{x_1} \ldots Q_{x_n}\varphi$, first construct the automaton for the formula $\varphi(x_1, \ldots, x_n)$
- Replace $\forall$ in terms of $\exists$

Points to Remember

- Given the automaton for $\varphi(x_1, \dots, x_n)$, the automaton for $\exists x_i \varphi(x_1, \dots, x_n)$ is obtained by projecting out the row of x_i
- This may result in an NFA
- Determinize it and complement it to get a DFA for $\neg\exists x_i \varphi(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$
- Intersect with the regular language where each of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ are assigned 1 exactly at one position

The Computational Effort

Given NFAs A_1, A_2 each with atmost n states,

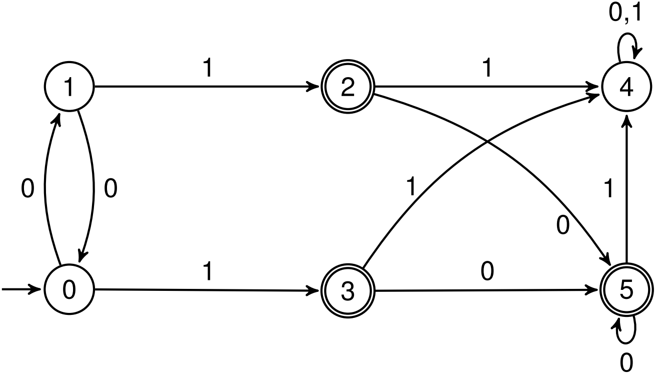
- The union has at most $2n$ states
- Intersection has at most n^2 states
- The complement has at most 2^n states
- The projection has at most n states

The Computational Effort

- $\psi = Q_1 \dots Q_n \varphi$. If $Q_i = \exists$ for all i, then size of A_ψ is same the size of A_φ.
- When $Q_1 = \exists, Q_2 = \forall, \dots$: each $\forall$ quantifier can create a 2^n blowup in automaton size
- Size of automaton is
$$2^{2^{2^{2^{2^{2^{2^{n}}}}}}}$$
where the tower height k is the quantifier alternation size.
- This number is indeed a lower bound!

The Automaton-Logic Connection

Given any FO sentence φ, one can construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.


0,1
1
2
4
0
0
1
1
1
0
1
0
0
3
5
0