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IIT Bombay
CS 405/6001: GT&AMD
Quiz 1, 2024-25-I
Date: August 28, 2024

Roll No : 12345 (e.g., 190040001) **Dept.:** CSE (e.g., CSE)

This is a sample solution. Your approach may be different and will be evaluated accordingly

CS 405/6001: Game Theory and Algorithmic Mechanism Design

Total: **10 + 8 + 12 = 30 marks,** *Duration:* **1 hour, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of 5 sheets of paper (10 pages, page 2 is blank). Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. One A4 assistance sheet (text **only on one side**) is allowed for the exam.
8. **After you are done with your exam or the exam duration is over, please DO NOT rush to the desk for submitting your paper**. Please remain seated until we collect all the papers, count them, and give a clearance to leave your seat.

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Problem 1 (10 points). A group of ten students are identified to perform Swachh Bharat Abhiyan in their hostel on the Independence Day. The task is to clean some common area of the hostel. If at least one student volunteers to participate in cleaning (i.e., picks action C), then the area will be clean and everyone gets a benefit of 10 units. However, every student who volunteers to clean, incurs a cost of 1 unit each and the other students who do not volunteer incur zero costs. If every student chooses not to clean (i.e., take action NC), the area remains unclean and everyone gets zero benefit but incurs zero cost as well.

(a) Represent this scenario as a *normal form game* (NFG) where the players are the students. Each player's utility is calculated as the benefit that player gets minus her own cost. Write the standard description of normal form games discussed in the class with a concise definition of the utility function for each player. **2 points.**

$N = \{1, 2, \ldots, 10\}$ The students

$S_i = \{C, NC\} \quad \forall i \in N$

$$u_i(s_i, s_{-i}) = \begin{cases} 9, & \text{if } s_i = C \quad \forall s_{-i} \\ 10, & \text{if } s_i = NC, \text{ and } \exists j \neq i \text{ s.t. } s_j = C \\ 0, & \text{if } s_i = NC \text{ and } \forall j \neq i, s_j = NC \end{cases}$$

(b) Find all the pure strategy Nash equilibria (PSNE) of this game with justification. **2 points.**

It is clear that all agents choosing NC is not a PSNE since each agent can improve their utility by choosing C.
Similarly, all agents choosing C is not a PSNE either since each can improve by choosing NC.
We can repeat this argument till exactly one player remains in C and all others choose NC.
Hence there are 10 PSNEs, in each of which exactly one player chooses C and others choose NC.

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(c) Find a symmetric mixed strategy Nash equilibrium of this game. A symmetric MSNE is an MSNE $(\sigma_1^*, \ldots, \sigma_n^*)$ such that $\sigma_i^* = \sigma_j^*$ for all $i, j \in N$. (Write the important steps of the derivation before arriving at the final answer) **3 points.**

None of the PSNEs are symmetric. Hence, the symmetric MSNE must have at least two strategies in its support. Say $\sigma_i^* = (p, 1-p)\ \forall i \in N$.

$\hat{C}$, $\hat{NC}$

Using the MSNE characterization theorem, we get

$$u_i(C, \sigma_{-i}^*) = u_i(NC, \sigma_{-i}^*)$$

$$\Rightarrow 9 = 10\left(1 - (1-p)^9\right) \Rightarrow (1-p)^9 = \frac{1}{10} \Rightarrow p = 1 - \frac{1}{10^{1/9}}$$

$$\sigma_i^* = \left(1 - \frac{1}{10^{1/9}}, \frac{1}{10^{1/9}}\right)$$ $\forall i \in N$ is the symmetric MSNE.

(d) What is the probability that, at the above symmetric MSNE, no one chooses to clean? (Explain your answer with a one-sentence justification) **1 point.**

This is the probability that everyone chooses NC

$$= \left(\frac{1}{10^{1/9}}\right)^{10} = \frac{1}{10^{10/9}}$$

(e) Now consider that there are n students instead of 10 in the group. Find a symmetric MSNE. **1 point.**

Using the same steps as in (c), we get

$$\sigma_i^* = \left(1 - \frac{1}{10^{1/(n-1)}}, \frac{1}{10^{1/(n-1)}}\right)$$

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(f) As n approaches infinity, what is the limit of the probability that no one volunteers to clean in this symmetric equilibrium? **1 point.**

using the same argument as in (e), the probability of nobody volunteering is $\frac{1}{10^{n/n-1}}$ (for n players) $\xrightarrow{n\to\infty} \frac{1}{10} = 0.1$

$= \frac{1}{10^{\frac{1}{1-1/n}}}$

(g) What implications can we draw from this analysis regarding volunteering in large groups? (Observation question, ungraded) **0 point.**

As the population increases, the probability that nobody volunteers is increasing. This is a phenomena where with a large population everyone thinks that at least one other person with clean the area and everyone will be benefitted. This is known as a "free-riding" behavior.

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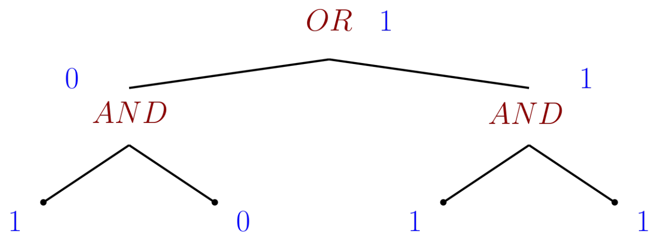
Problem 2 (8 points). OR-AND is a two-player game played on a full binary tree with a root, of depth n (e.g., see the figure below when $n = 2$). Players I and II take turns respectively to choose a leaf of the tree that has not previously been selected, and assigns it a logic value 1 and 0 respectively (i.e., Player I always puts 1 and II always puts 0).

After all the leaves have been assigned a value, a value for the entire tree is calculated (an example for $n = 2$ is shown in the figure). The first step involves calculating the value of the vertices at one level above the level of the leaves: the value of each such vertex is calculated using the logical **AND** function, operating on the values assigned to its children.



Next, a value is calculated for each vertex one level up, with that value calculated using the logical **OR** function, operating on the values previously calculated for their respective children. The values of all the vertices of the tree are calculated in this *bottom-up* manner, with the value of each vertex calculated using either the **AND** or **OR** functions, operating on values calculated for their respective children. Player I wins (i.e., Player II loses) if the value of the root vertex is 1, and loses (i.e., Player II wins) if the value of the root vertex is 0. Answer the following pairs of questions. Write only the conclusion in the first question and the explanation in the second question of that pair. Note that the explanation will not get any credit if the conclusion is incorrect.

(a) Which player has a winning strategy in a game played on a tree of depth two? Write "none" if no player has it. **2 points.**

Player II

(b) Explain your answer to the previous question in at most 4 sentences. **2 points.**

In order to win, player II needs to ensure that the value is zero at every AND node of this game tree. This is achieved by player II to simply follow whatever leaf node is picked by player I and picking the other sibling of that leaf node.

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(c) Which player has a winning strategy in a game played on a tree of depth $2k+1$, where k is any positive integer? Write "none" if no player has it. **2 points.**

Player I

(d) Explain your answer to the previous question in at most 5 sentences. **2 points.**

When the depth is "odd", 2k+1, the penultimate stage of the game tree has all OR nodes. Player I needs to ensure that every OR node gets a value of 1 to win. This can be achieved by player I to start at any random leaf in the first move and then picking the sibling node of what is picked by Player II.

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Problem 3 (12 points). Recall the definition of **correlated equilibrium** as discussed in the class. A *correlated equilibrium* (CE) is a correlated strategy π s.t.

$$\sum_{s_{-i}\in S_{-i}} \pi(s_i, s_{-i}) \cdot u_i(s_i, s_{-i}) \geqslant \sum_{s_{-i}\in S_{-i}} \pi(s_i, s_{-i}) \cdot u_i(s_i', s_{-i}),\ \forall s_i, s_i' \in S_i, \forall i \in N.$$

Answer the following questions that ask certain (im)possibilities. If your answer is **no**, i.e., it is impossible, or **cannot be determined**, argue in the most precise manner why. If your answer is **yes**, i.e., it is possible, provide an example of a normal form game with two players each having two strategies and utilities in $[0, 1]$ to show such a possibility. You should also provide an example if your answer is partly *yes* and partly *no* (see the options in the questions to understand this point).

Note: if the conclusion part of a question is incorrect, the explanation/example part will not get any credit.

(a) If s_i is a **strictly dominated** strategy of player i in some normal form game (NFG), then is it possible that there exists a CE π with $\pi(s_i, s_{-i}) > 0$ for some s_{-i}? (Yes/No) **1 point.**

No.

(b) Provide an explanation/example in support of your answer to the previous question. (See the second paragraph of Problem 3 for a guideline on how to write this answer) **3 points.**

Suppose the above conclusion is not true. Hence

$\exists\, s_{-i}$ s.t. $\pi(s_i, s_{-i}) > 0$

but since s_i is strictly dominated, $\exists\, t_i$ s.t.

$$u_i(s_i, \tilde{s}_{-i}) < u_i(t_i, \tilde{s}_{-i}), \ \forall \tilde{s}_{-i}$$

hence $\pi(s_i, s_{-i})\big(u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})\big) < 0$ for s_{-i}

and $$\sum_{\tilde{s}_{-i}} \pi(s_i, \tilde{s}_{-i})\left[u_i(s_i, \tilde{s}_{-i}) - u_i(t_i, \tilde{s}_{-i})\right]$$

$$\leq 0 + \underbrace{\pi(s_i, s_{-i})\big(u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})\big)}_{<0}$$

This contradicts that π is a CE.

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(c) Let s_i is a **weakly dominated** strategy of player i in some normal form game (NFG) (but not strictly dominated). Fix an s_{-i}. Is it possible that there exists a CE π with $\pi(s_i, s_{-i}) > 0$? (Select the option from below)

(i) No (for every s_{-i}).
(ii) Yes (for every s_{-i}).
(iii) Yes (for some s_{-i}) and No (for some s_{-i}).
(iv) Cannot be determined.

1 point.

(iii)

(d) Provide an explanation/example in support of your answer to the previous question. (See the second paragraph of Problem 3 for a guideline on how to write this answer) **3 points.**

Since s_i is weakly (but not strictly) dominated, $\exists\, t_i$ s.t.

$u_i(s_i, s_{-i}) \le u_i(t_i, s_{-i}) \quad \forall s_{-i}$

and $u_i(s_i, \tilde{s}_{-i}) < u_i(t_i, \tilde{s}_{-i})$ for some $\tilde{s}_{-i}$

for all s_{-i} s.t. $u_i(s_i, s_{-i}) = u_i(t_i, s_{-i})$, one can have $\pi(s_i, s_{-i}) > 0$ as it does not alter the inequality.

but for $\tilde{s}_{-i}$ s.t. $u_i(s_i, \tilde{s}_{-i}) < u_i(t_i, \tilde{s}_{-i})$, it must be true that $\pi(s_i, \tilde{s}_{-i}) = 0$. An example is where

	A	B
A	0, 0	1, 1
B	1, 1	1, 1

A is weakly dominated

But $\pi(A, B) = \frac{1}{3}$ is possible but $\pi(A, A) = 0$

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CS 405/6001: GT&AMD
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(e) If s_i is an **undominated** strategy (i.e., neither weakly or strictly dominated) of player i and also not a strategy in any pure strategy Nash equilibria in some normal form game (NFG), then is it possible that there exists a CE π with $\pi(s_i, s_{-i}) > 0$ for every s_{-i}? (Select the option from below)

(i) No.

(ii) Yes.

(iii) Cannot be determined.

1 point.

(ii)

(f) Provide an explanation/example in support of your answer to the previous question. (See the second paragraph of Problem 3 for a guideline on how to write this answer) **3 points.**

Example:

	A	B
A	1, 0	0, 1
B	0, 1	1, 0

$\pi(A,B) = \pi(B,A) = \pi(A,A) = \pi(B,B) = \frac{1}{4}$

is a CE.

[all strategies are undominated and none of them belong to any PSNE]

END OF QUESTION PAPER. GOOD LUCK!