SC 639 (Fall 2020) - Basics of Vector Spaces

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Outline

Field

A field is a set $\mathcal{S}$ (whose elements are called *scalars*) with two binary operations $+$ and $*$ that satisfy:

A. To every $\alpha,\beta \in \mathcal{S}$ there corresponds a scalar $\alpha + \beta$, called the *sum* of α and β. Further, given $\alpha,\beta,\gamma \in \mathcal{S}$

1. addition is associative, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$,
2. addition is commutative, $\alpha + \beta = \beta + \alpha$,
3. there exists a unique scalar 0 (called zero) such that $\alpha + 0 = \alpha$ for every scalar α, and
4. to every scalar α there corresponds a unique scalar $-\alpha$ such that $\alpha + (-\alpha) = 0$.

B. To every pair, α and β, of scalars there corresponds a scalar $\alpha * \beta$, called the *product* of α and β, in such a way that

1. multiplication is associative, $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$,
2. multiplication is commutative, $\alpha * \beta = \beta * \alpha$,
3. there exists a unique non-zero scalar 1 (called one) such that $\alpha * 1 = \alpha$ for every scalar α, and
4. to every non-zero scalar α there corresponds a unique scalar α^{-1} or $\frac{1}{\alpha}$ such that $\alpha * \alpha^{-1} = 1$.

Field

- Multiplication is distributive with respect to addition, $\alpha * (\beta + \gamma) = \alpha * \beta + \alpha * \gamma$

If addition and multiplication are defined within some set of objects (scalars) so that the conditions (A), (B), and (C) are satisfied, then that set (together with the given operations) is called a *field*.

Thus, for example, the set $\mathbb{Q}$ of all rational numbers (with the ordinary definitions of sum and product) is a field, and the same is true of the set $\mathbb{R}$ of all real numbers and the set $\mathcal{C}$ of all complex numbers

Definition of a Vector Space

Definition

A vector space is a set $\mathcal{V}$ of elements called vectors satisfying the following axioms:

a. To every pair, x and y, of vectors in $\mathcal{V}$ there corresponds a vector $x + y$, called the sum of x and y in such a way that:
   1. Addition is associative, $x + (y + z) = (x + y) + z$,
   2. Addition is commutative, $x + y = y + x$,
   3. there exists in $\mathcal{V}$ a unique vector 0 (called the origin) such that $x + 0 = x$ for every vector x, and
   4. to every vector x in $\mathcal{V}$ there corresponds a unique vector $-x$ such that $x + (-x) = 0$.

b. To every pair, α and x, where α is a scalar and x is a vector in $\mathcal{V}$, there corresponds a vector αx in $\mathcal{V}$, called the product of α and x, in such a way that
   1. multiplication by scalars is associative, $\alpha(\beta x) = (\alpha\beta)x$, and
   2. $1x = x$ for every vector x.

c.
   1. Multiplication by scalars is distributive with respect to vector addition, $\alpha(x + y) = \alpha x + \alpha y$, and
   2. multiplication by vectors is distributive with respect to scalar addition, $(\alpha + \beta)x = \alpha x + \beta x$.

Linear Dependence and Independence

Definition

A finite set $\{x_i\}$ of vectors is *linearly dependent* if there exists a corresponding set $\{\alpha_i\}$ of scalars, not all zero, such that

$$\sum_i \alpha_i x_i = 0$$

If on the other hand, $\sum_i \alpha_i x_i = 0$ implies that $\alpha_i = 0$ for each i, the set $\{x_i\}$ is *linear independent*.

We shall say, that whenever $x = \sum_i \alpha_i x_i$, that x is a linear combination of $\{x_i\}$.

Bases

Definition

A (linear) basis (or a coordinate system) in a vector space $\mathcal{V}$ is a set $\mathcal{X}$ of linearly independent vectors such that every vector in $\mathcal{V}$ is a linear combination of elements of $\mathcal{X}$. A vector space $\mathcal{V}$ is finite-dimensional if it has a finite basis.

Theorem

If $\mathcal{V}$ is a finite-dimensional vector space and if $\{y_i, \ldots, y_m\}$ is any set of linearly independent vectors in $\mathcal{V}$, then, unless the y's already form a basis, we can find vectors $y_{m+1}, \ldots, y_{m+p}$ so that the totality of the y's, that is, $\{y_1, \ldots, y_m, y_{m+1}, \ldots, y_{m+p}\}$,is a basis. In other words, every linearly independent set can be extended to a basis.

Example

The set $\{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,0,1)\}$ is a basis of $\mathbb{R}^n$,called the standard basis of $\mathbb{R}^n$.

Example

The set $\{(1,2),(3,5)\}$ is a basis of $\mathbb{R}^2$.

Dimension

Theorem

The number of elements in any basis of a finite-dimensional vector space V is the same as in any other basis.

Definition

The dimension of a finite-dimensional vector space $\mathcal{V}$ is the number of elements in a basis of $\mathcal{S}$.

Example

$\dim(\mathbb{R}^n) = n$ because the standard basis of $\mathbb{R}^n$ has n vectors.

Example

$\dim \mathcal{P}_m(\mathbb{R}) = m + 1$ because the basis $1, z, \ldots, z^m$ of $\mathcal{P}_m(\mathbb{R})$ has $m + 1$ vectors.

Subspaces

Definition

A non-empty subset $\mathcal{M}$ of a vector space $\mathcal{V}$ is a subspace if along with every pair, x and y, of vectors contained in $\mathcal{M}$, every linear combination $\alpha x + \beta y$ is also contained in $\mathcal{M}$.

- A subspace $\mathcal{M}$ in a vector space $\mathcal{V}$ is itself a vector space;

Example

$\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Example

The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

Linear Functionals

Definition

A linear functional on a vector space $\mathcal{V}$ is a scalar-valued function y defined for every vector x, with the property that (identically in the vectors x_1 and x_2 and the scalars α_1 and α_2)

$$y(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 y(x_1) + \alpha_2 y(x_2)$$

Theorem

If $\mathcal{V}$ is an n-dimensional vector space, if $\{x_1, \ldots, x_n\}$ is a basis in $\mathcal{V}$, and if $\{\alpha_i, \ldots, \alpha_n\}$ is any set of n scalars, then there is one and only one linear functional y on $\mathcal{V}$ such that $[x_i, y] = \alpha_i$ for $i = 1, \ldots, n$.

Examples

Example

Define $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$ by $\phi(x, y, z) = 4x - 5y + 2z$. Then ϕ is a linear functional on $\mathbb{R}^3$

Example

Let $V := \mathcal{C}[a, b]$ and $x \in [a, b]$.Then, $V \ni v \mapsto \phi(v) := v(x) \in \mathbb{R}$ is a linear functional.

Example

Let $V := \mathcal{C}[0, 1]$ .Then, $V \ni v \mapsto \phi(v) := \int_0^1 v(x)dx$ is a linear functional.

Dual Basis

Theorem

If $\mathcal{V}$ is an n-dimensional vector space and if $\mathcal{X} = \{x_1, \ldots, x_n\}$ is a basis in $\mathcal{V}$, then there is a uniquely determined basis $\mathcal{X}^{'}$ in $\mathcal{V}^{'}$ $\mathcal{X}^{'} = \{y_1, \ldots, y_n\}$ with the property that $[x_i, y_j] = \delta_{ij}$. Consequently the dual space of an n-dimensional space is n-dimensional. The basis $\mathcal{X}^{'}$ is called the dual basis of $\mathcal{X}$.

Example

If $\mathcal{V}$ is an n dimensional vector space, whose dual space is $\mathcal{V}'$ then what is the dual basis of the standard basis $e_1, \ldots, e_n$.

Solution: For $1 \leq j \leq n$, define ϕ_j to be the linear functional on $\mathcal{V}$ that selects the j^{th} coordinate of a vector in $\mathcal{V}$. In other words,

$$\phi_j(x_1, \ldots, x_n) = x_j$$

for $(x_1, \ldots, x_n) \in \mathcal{V}$. Clearly,

$$\phi_j(e_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

Thus, $\phi_1, \ldots, \phi_n$ is the dual basis of the standard basis $e_1, \ldots, e_n$

Example

Let $V = \mathbb{R}^2$ and basis be $b = \{(2,1),(3,1)\}$. The dual basis $b^* = \{-x + 3y, x - 2y\}$.

Linear Transformations

Definition

A linear transformation (or operator) A on a vector space $\mathcal{V}$ is a correspondence that assigns to every vector x in $\mathcal{V}$ a vector Ax in $\mathcal{V}$, in such a way that

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

identically in the vectors x and y and the scalars α and β.

Theorem

The set of all linear transformations on a vector space is itself a vector space.

Vector Space of Linear Transformations

Using the following observations, we see that the set of linear transformations on a vector space is itself a vector space

- If A and B are linear transformations, we define their *sum*, $S = A + B$, by the equation $Sx = Ax + Bx$ (for every x).
- We observe that the commutativity and associativity of addition in $\mathcal{V}$ imply immediately that the addition of linear transformations is commutative and associative.
- If we consider the sum of any linear transformation A and the linear transformation 0, we see that $A + 0 = A$.
- If, for each A, we denote by $-A$ the transformation defined by $(-A)x = -(Ax)$, then $A + (-A) = 0$. And that the transformation $-A$, so defined, is the only linear transformation B with the property that $A + B = 0$.

Continuing this way, we can verify all the axioms of a vector space.

Inverses

It may happen that the linear transformation A has one or both of the following two properties.

1. If $x_1 \neq x_2$, then $Ax_1 \neq Ax_2$.
2. To every vector y there corresponds (at least) one vector x such that $Ax = y$.

If A has both these properties, we shall say that A is invertible. If A is invertible, we define a linear transformation, called the inverse of A and denoted by A^{-1}.

Hence, for an invertible A, we have,

$$AA^{-1} = A^{-1}A = 1$$

Inverses

Properties of invertible linear transformations:

- If A, B, C are linear transformations, such that

$$AB = CA = 1,$$

  then A is invertible and $A^{-1} = B = C$

- A linear transformation A on a finite-dimensional vector space $\mathcal{V}$ is invertible if and only if $Ax = 0$ implies that $x = 0$, or, alternatively, if and only if every y in $\mathcal{V}$ can be written in the form $y = Ax$.

- If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. If A is invertible and a $\alpha \neq 0$, then αA is invertible and $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$. If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

Adjoint Transformation

- Let V be any vector space and V' be its dual space. A is a linear transformation V.
- For each fixed $y \in V'$, the function y' defined by $y'(x) = [Ax, y]$ is a linear functional on V.
- The adjoint $A' : V' \to V$ is defined by the following property.

$$[Ax, y] = [x, A'y] \tag{1}$$

Matrix representation of a Linear Transformation

Definition

Let $\mathcal{V}$ be an n-dimensional vector space, let $\mathcal{X} = \{x_1, \ldots, x_n\}$ be any basis of $\mathcal{V}$, and let A be a linear transformation on $\mathcal{V}$. Since every vector is a linear combination of the x_i; we have in particular

$$Ax_j = \sum_i \alpha_{ij} x_i$$

for $j = 1, \ldots, n$. The set (α_{ij}) of n^2 scalars, indexed with the double subscript i, j, is the matrix of A in the coordinate system $\mathcal{X}$. We shall generally denote it by $[A]$. A matrix (α_{ij}) is usually written in the form of a square array:

$$[A] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \ldots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \ldots & \alpha_{nn} \end{bmatrix}$$

This definition defines a matrix associated with a linear transformation.

Example

Example

Let $T : \mathcal{P}_2 \mapsto \mathbb{R}^3$ be a linear transformation with standard basis respectively. The transformation is given by, $T(x^2) = (1, 0, 0)$, $T(x) = (0, 1, 0)$, and $T(1) = (0, 0, 1)$. Then matrix representation is,

$$[T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

Let linear transformation $T(x, y) = (x + y, 2x - y, 3x + 5y)$ with standard basis of respective vector spaces. The matrix representation

$$[T] = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 5 \end{pmatrix}$$

An isomorphism - matrices and linear transformations

- Among the set of all matrices $(\alpha_{ij}), (\beta_{ij})$, etc., $i, j = 1, \ldots, n$ (not considered in relation to linear transformations), we define sum, scalar multiplication, product, o_{ij}, and e_{ij}, by

$$\begin{aligned}
(\alpha_{ij}) + (\beta_{ij}) &= (\alpha_{ij} + \beta_{ij}), \\
\alpha(\alpha_{ij}) &= (\alpha\alpha_{ij}), \\
(\alpha_{ij})(\beta_{ij}) &= (\sum_k \alpha_{ik}\beta_{kj}), \\
o_{ij} &= 0, \\
e_{ij} &= \delta_{ij}
\end{aligned}$$

- Then the correspondence (established by means of an arbitrary coordinate system $\mathcal{X} = \{x_1, \ldots, x_n\}$ of the n-dimensional vector space $\mathcal{V}$), between all linear transformations A on $\mathcal{V}$ and all matrices (α_{ij}), described by $Ax_j = \sum_i \alpha_{ij} x_i$ is an isomorphism. It is a one-to-one correspondence that preserves sum, scalar multiplication, product, 0 and 1.

Example

Example

Vector spaces $\mathbb{R}^3$ and $\mathcal{P}_2$ are isomorphic. A natural correspondence is $(a_0, a_1, a_2) \leftrightarrow a_0 + a_1x + a_2x^2$

Example

Let $\mathcal{M}$ be a set of all real 3×3 symmetric matrices, then $\mathcal{M}$ and $\mathbb{R}^6$ are isomorphic.

Example

The following map $f : \mathcal{M} \to \mathcal{P}_3$ given by,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c + (c + d)x + bx^2 + ax^3$$

Change of basis

- Let $\mathcal{V}$ be an n-dimensional vector space and let $\mathcal{X} = \{x_1, \dots, x_n\}$ and $\mathcal{Y} = \{y_1, \dots, y_n\}$ be two bases in $\mathcal{V}$.
- Let A be a linear transformation defined by $Ax_i = y_i, i = 1, \dots, n$. That is:

$$A(\sum_i \xi_i x_i) = \sum_i \xi_i y_i$$

- Let (α_{ij}) be the matrix of A in the basis $\mathcal{X}$, that is, $y_j = Ax_j = \sum_i \alpha_{ij} x_i$. We observe that A is invertible, since $\sum_i \xi_i y_i = 0$ implies that $\xi_1 = \xi_2 = \cdots = \xi_n = 0$

Based on the above data, we can arrive at the following relations between the two basis vectors $\mathcal{X}$ and $\mathcal{Y}$ of the vector space $\mathcal{V}$.

Change of Basis

1. If x is in $\mathcal{V}$, $x = \sum_i \xi_i x_i = \sum_i \eta_i y_i$, then the relation between its coordinates $(\xi_1, \ldots, \xi_n)$ with respect to $\mathcal{X}$ and its coordinates $(\eta_1, \ldots, \eta_n)$ with respect to $\mathcal{Y}$ will be:
   Since
   $$\begin{aligned}\sum_i \xi_i x_i = \sum_j \eta_j y_j &= \sum_j \eta_j A x_j \\ &= \sum_j \eta_j \sum_i \alpha_{ij} x_i \\ &= \sum_i (\sum_j \alpha_{ij} \eta_j) x_i\end{aligned}$$
   and so we have $\xi_i = \sum_j \alpha_{ij} \eta_j$
2. If $(\xi_1, \ldots, \xi_n)$ is an ordered set of n scalars, then the relation between the vectors $x = \sum_i \xi_i x_i$ and $y = \sum_i \xi_i y_i$ is:
   $$y = Ax$$

Thank You