CS213/293 Data Structure and Algorithms 2024

Lecture 9: Pattern matching

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Compile date: 2024-09-13

Topic 9.1

Pattern matching problem

Pattern matching

Definition 9.1

In a pattern-matching problem, we need to find the position of all occurrences of a pattern string P in a string T.

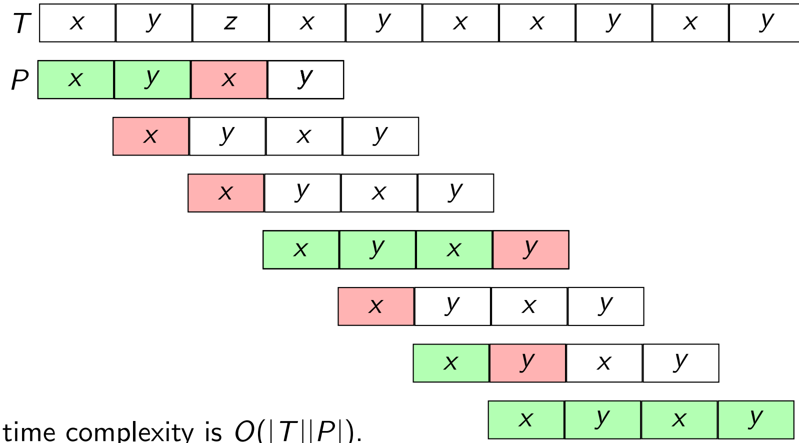
Usage:

- Text editor
- DNA sequencing

Example : Näive approach for pattern matching

Example 9.1

Consider the following text T and pattern P. We try to match the pattern in every position.



Running time complexity is $O(|T||P|)$.

Wasteful attempts of matching.

			i ↓									
T	x	y	z	x	y	x	x	y	x	y	p	x
P	x	y	x	y								
		x	y	x	y							
			x	y	x	y						

Should we have tried to match the pattern at the second and third positions?

No.

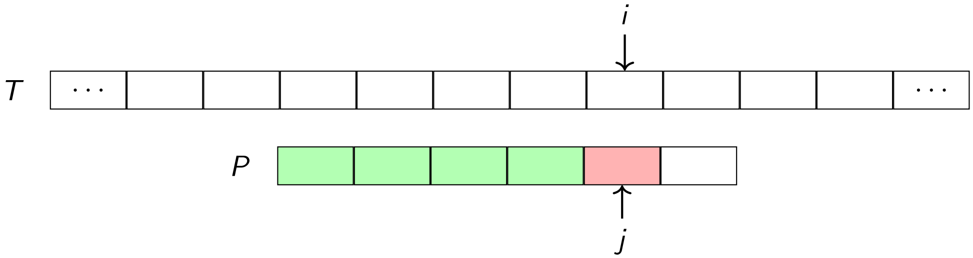
Commentary: In the drawing i is 2. However, we have named the position i to illustrate the argument using symbolic expressions.

Let us suppose we failed to match at position i of T and position 2 of P.

- We know that $T[i-1] = y$. Therefore, there is no matching starting at $i-1$. (Why?)
- We know that $T[i] \neq x$. Therefore, there is no matching starting at i. (Why?)

Shifting the pattern

Let us suppose at position i of T and j of P the matching fails.



Let us suppose we want to resume the search by only updating j.

If we assign j some value k, we are shifting the pattern forward by $j - k$.

Exercise 9.1

What is the meaning of $k = j - 1$, $k = 0$, or $k = -1$?

Side note: out-of-bounds access of P

If k takes value -1 or $|P|$, $P[k]$ is accessing the array out of bounds.

For consistency of the definitions, we will say $P[-1] = P[|P|] = Null$.

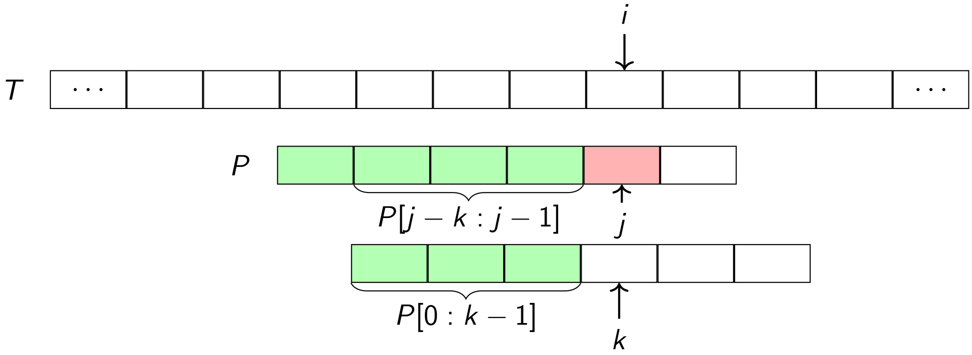
However, the algorithms will be carefully written and there will be no out-of-bound access in them.

Definition 9.2
Let $P[i:j]$ indicates the array containing elements $P[i], ...P[j]$.

Commentary: In a formal definition, we may overlook or simplify some implementation issues. This allows us to write clean mathematical definitions. However, the implementations need to be careful about the issues.

What is a good value of k?

We know $T[i-j : i-1] = P[0 : j-1]$ and $T[i] \neq P[j]$.



We must have $P[0 : k-1] = P[j-k : j-1]$ and $P[j] \neq P[k]$(Why?).

Exercise 9.2

Should we choose the largest k or smallest k?

The largest k implies the minimum shift

We choose the largest k such that

$$P[0:k-1] = P[j-k:j-1] \text{ and } P[j] \neq P[k].$$

k only depends on P and j. Since P is typically small, we pre-compute array h such that $h[j] = k$.

Example 9.2

P	x	y	x	y	
h	-1	0	-1	0	2

P	x	y	x	z	
h	-1	0	-1	1	0

We can compute h in $O(|P|)$ time. We will discuss this later.

Exercise 9.3

a. *Show that* $j > h(j) \geq -1$ *for each* $j \in [0..|P|)$

b. *Show that* $|P| > h(|P|) \geq 0$ *if* $|P| > 0$. *Is it true if* $|P| = 0$?

c. *If we drop condition* $P[j] \neq P[k]$, *what may go wrong?*

Commentary: Answer of b: Since $P[|P|] = null$, we are guaranteed that $P[|P|] \neq P[0]$. Since we have $P[0:-1] = P[j:j-1]$. $k=0$ will satisfy the condition for $P[|P|]$. Since we are looking the largest k, $P[|P|] \geq 0$.

Knuth–Morris–Pratt algorithm

Algorithm 9.1: KMP(string T,string P)

```
1  assume(|P| > 0);
2  i := 0; j := 0;  found := ∅;
3  h := KMPTable(P);
4  while i < |T| do
5      if P[j] = T[i] then
6          i := i + 1;   j := j + 1;
7          if j = |P| then
8              found.insert(i − j);
9              j := h[j];
10     else
11         j := h[j];
12         if j < 0 then
13             i := i + 1;  j := j + 1;
14 return found
```

Running time complexity:

- Since no. of increments of $i \leq |T|$, the line 6 and 13 will execute $\leq |T|$ times in total.
- How do we bound the number of iterations when the **else** branch does not increment i?
  1. The **else** branch reduces j because $h[j] < j$.
  2. Since every time at the loop head $j \geq 0$ (Why?), no. of reductions of $j \leq$ no. of increments of j.
  3. Since i and j are always incremented together, no. of reductions of $j \leq$ no. of increments of i.
  4. no. of reductions of $j \leq |T|$.
- $O(|T|)$ algorithm

Commentary: The step two is bounding the number of reductions over all iterations of the loop (needs some thinking). It is called amortized complexity. Note that the argument does not guarantee a constant bound over the number of consecutive reduction steps.

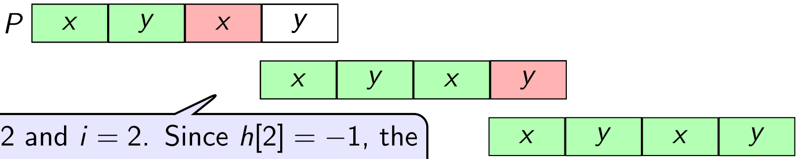
Example : KMP execution

Example 9.3

Consider the following text T and pattern P. Let us suppose, we have h.

P	x	y	x	y	
h	-1	0	-1	0	2

T	x	y	z	x	y	x	x	y	x	y	p	x



Shifting at $j = 2$ and $i = 2$. Since $h[2] = -1$, the pattern is shifted to 3, $j = 0$ and $i = 3$.

Topic 9.2

How to compute array h?

Recall: the definition of h

For a pattern P, $h[i]$ is the largest k such that

$$P[0:k-1] = P[i-k:i-1] \text{ and } P[i] \neq P[k].$$

We use KMP like algorithm again to compute h.

When we compute $h[i]$, we assume we have computed $h[i']$ for each $i' \in [0, i)$.

Self-matching: use KMP again for computing h

We run two indexes i and j on P such that $j < i$.

We assume that for each $k \in (j, i), \neg(P[0:k-1] = P[i-k:i-1] \wedge P[i] \neq P[k])$.

We will be computing $h[i]$. Let j be the current running match,i.e, $P[i-j:i-1] = P[0:j-1]$.

When we consider position i, we have two cases.

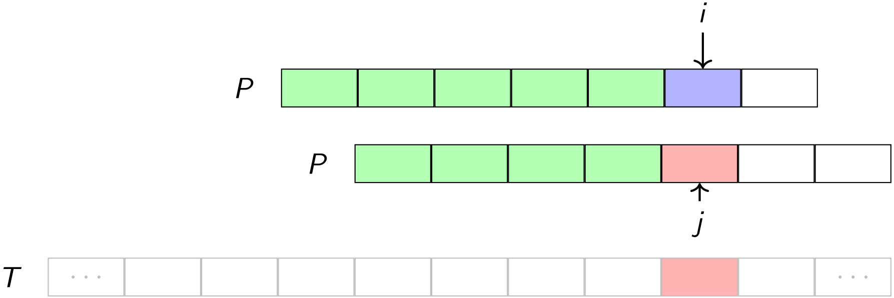
1. $P[i] \neq P[j]$
2. $P[i] = P[j]$

In both the cases, we need to update $h[i]$ and may update j.

We ensure that j is largest by updating j conservatively.

Commentary: Due to the assumption, no k larger j is a valid candidate or $h[i]$.

Case 1: $P[i] \neq P[j]$



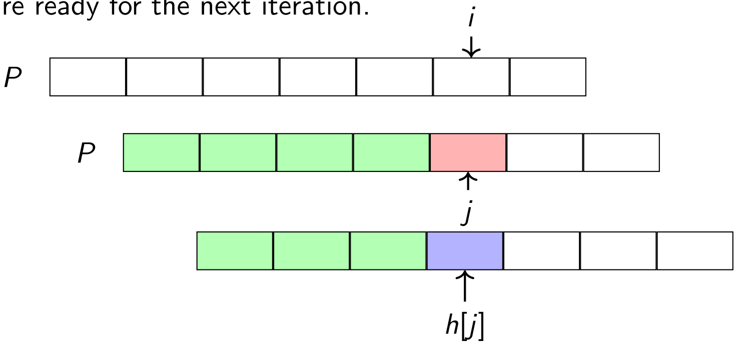
We assign $h[i] := j$, since j meets the requirements.

We have found the shift position for i.Now, we need to prepare for the next index $i+1$.

Now we need to move the pattern forward as little as possible.

Case 1 (continued): $P[i] \neq P[j]$

After the mismatch, we move the pattern forward as little as possible such that we have a match at position i and are ready for the next iteration.



We must have computed h for earlier indexes. We set $j := h[j]$.

We need to keep reducing j until $P[j] = P[i]$ or $j \leq 0$.

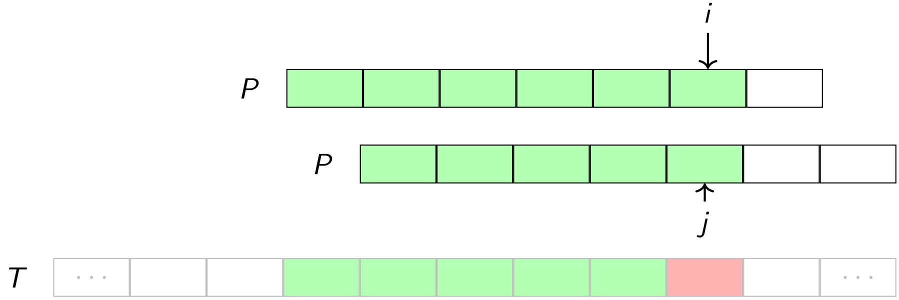
Exercise 9.4

a. Why the value of $h[j]$ be available?

b. Prove that $\forall k \in (h[j], j] : \neg(P[0:k-1] = P[i-k:i-1] \wedge P[i] \neq P[k])$. *(important point!)*

Case 2: $P[i] = P[j]$

Let us consider the case when matching continues. How should we assign $h[i]$?



We may use $h[i] := j$, but it does not meet the requirement $P[i] \neq P[j]$. (Why?)

Let us jump to $h[j]$, which will meet the requirements. (Why?) We assign $h[i] := h[j]$.

Computing h array

Algorithm 9.2: KMPTable(string P)

$i := 1; j := 0;\ \ h[0] := -1;$
while $i < |P|$ **do**
    if $P[j] \neq P[i]$ **then**
        $h[i] := j;$
        while $j \geq 0$ *and* $P[j] \neq P[i]$ **do**
            $j := h[j];$ // Prepare for the next iteration
    else
        $h[i] := h[j];$
    $i := i + 1;\ \ j := j + 1;$
$h[|P|] := j;$
return h

Commentary: Let $prop(i, k) = (P[0 : k-1] = P[i-k : i-1] \wedge P[i] \neq P[k])$. The loop invariant at the head of the outer loop is $P[i-j : i-1] = P[0 : j-1]$, $\forall k \in (j, i), \neg prop(i, k)$, and $\forall l < j\ \ prop(h[l], l) \wedge \forall k \in (h[l], l), \neg prop(l, k)$. We prove the correctness by proving the validity of the loop invariant.

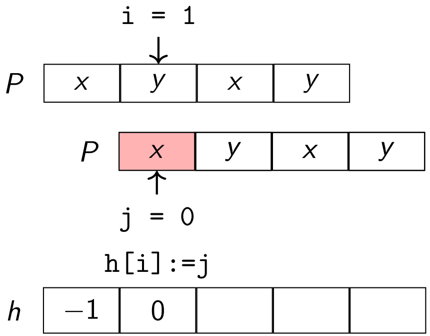
Exercise 9.5

Give proof of correctness of the algorithm.

Example: computing h

Example 9.4

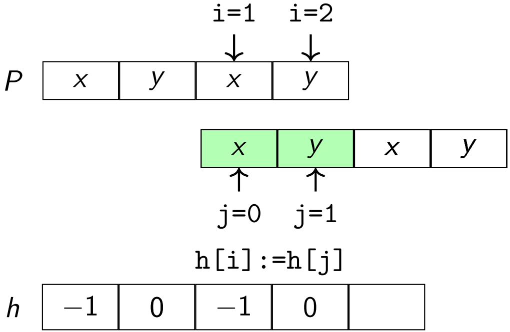
Consider the following pattern P and the first iteration of the outer loop, which is case 1.



We need to update `j := h[j]`. Therefore, `j = -1`.
Afterwards, we increment both j and i. Therefore, `i = 2;j = 0;`.

Example: computing h (cotinued) (2)

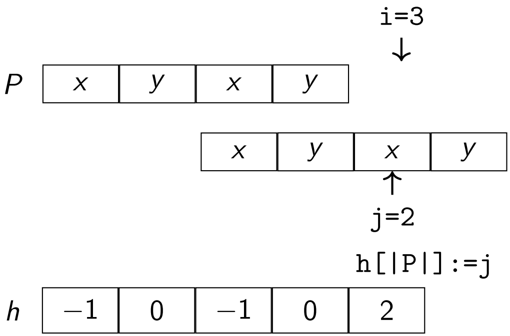
Let us consider the second and third iteration of the outer loop, which are case 2.



After the third iteration, the loop exits since $i \geq |P|$.

Example: computing h (cotinued) (3)

After the third iteration, the loop exists and we update $h[|P|]$.



Topic 9.3

Tutorial problems

Exercise: compute h

Exercise 9.6

Compute array h for pattern "babbaabba".

Exercise: version of KMPTABLE

Exercise 9.7

Is the following version of KMPTABLE *correct?*

Algorithm 9.3: KMPTABLEV2(string P)

$i := 1; j := 0;\ \ h[0] := -1;$
while $i < |P|$ **do**
    $h[i] := j;$
    while $j \geq 0$ *and* $P[j] \neq P[i]$ **do**
        $j := h[j];$ // Moving forward the pattern in minimum steps as in KMP
    $i := i + 1;\ \ j := j + 1;$
$h[|P|] := j;$
return h

Exercise: compute $h(i)$

Exercise 9.8

Suppose that there is a letter z in P of length n such that it occurs in only one place, say k, which is given in advance. Can we optimize the computation of h?

End of Lecture 9