CS 228 : Logic in Computer Science

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Check Satisfiability

Let $\psi(z) = \exists x[Q_a(x) \wedge \forall y[(y \leqslant x \wedge Q_b(y)) \rightarrow (z < x \wedge y < z \wedge Q_c(z))]]$ over the signature τ having the relational symbols $<$, Q_a, Q_b, Q_c and unary function S. Does $\psi(z)$ evaluate to true under some word structure?

Check Satisfiability

Let $\zeta = P(0) \wedge \forall x(P(x) \rightarrow P(S(x))) \wedge \exists x \neg P(x)$ over a signature τ containing the constant 0, unary function S and unary relation P.

Is ζ satisfiable?

Normal Forms in FOL

Recap : Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \bot$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha(t_1) = \alpha(t_2)$
- $\mathcal{A} \models_\alpha R(t_1, \ldots, t_k)$ iff $(\alpha(t_1), \ldots, \alpha(t_k)) \in R^{\mathcal{A}}$
- $\mathcal{A} \models_\alpha (\varphi \to \psi)$ iff $\mathcal{A} \not\models_\alpha \varphi$ or $\mathcal{A} \models_\alpha \psi$
- $\mathcal{A} \models_\alpha (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$
- $\mathcal{A} \models_\alpha (\exists x)\varphi$ iff there is some $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

Last two cases, α has no effect on the value of x. Thus, assignments matter only to free variables.

Equivalences

Let F, G be arbitrary FOL formulae.

1. $\neg \forall x F \equiv \exists x \neg F$
2. $\neg \exists x F \equiv \forall x \neg F$

$$
\begin{aligned}
\mathcal{A} \models_\alpha \neg \forall x F \text{ iff } & \mathcal{A} \not\models_\alpha \forall x F \\
\text{iff } & \mathcal{A} \not\models_{\alpha[x \mapsto a]} F \text{ for some } a \in U^{\mathcal{A}} \\
\text{iff } & \mathcal{A} \models_{\alpha[x \mapsto a]} \neg F \text{ for some } a \in U^{\mathcal{A}} \\
\text{iff } & \mathcal{A} \models_\alpha \exists x \neg F
\end{aligned}
$$

Equivalences

If x does not occur free in G then

1. $(\forall x F \wedge G) \equiv \forall x(F \wedge G)$
2. $(\forall x F \vee G) \equiv \forall x(F \vee G)$
3. $(\exists x F \wedge G) \equiv \exists x(F \wedge G)$
4. $(\exists x F \vee G) \equiv \exists x(F \vee G)$

$$
\begin{aligned}
\mathcal{A} \models_\alpha \forall x F \wedge G \text{ iff } & \mathcal{A} \models_\alpha \forall x F \text{ and } \mathcal{A} \models_\alpha G \\
\text{iff } & \text{for all } a \in U^{\mathcal{A}},\ \mathcal{A} \models_{\alpha[x \mapsto a]} F \text{ and } \mathcal{A} \models_\alpha G \\
\text{iff } & \text{for all } a \in U^{\mathcal{A}},\ \mathcal{A} \models_{\alpha[x \mapsto a]} F \text{ and } \mathcal{A} \models_{\alpha[x \mapsto a]} G \\
\text{iff } & \text{for all } a \in U^{\mathcal{A}},\ \mathcal{A} \models_{\alpha[x \mapsto a]} (F \wedge G) \\
\text{iff } & \mathcal{A} \models \forall x(F \wedge G)
\end{aligned}
$$

Equivalences

Let F, G be arbitrary FOL formulae.

1. $(\forall x F \wedge \forall x G) \equiv \forall x (F \wedge G)$
2. $(\exists x F \vee \exists x G) \equiv \exists x (F \vee G)$

1. $\forall x \forall y F \equiv \forall y \forall x F$
2. $\exists x \exists y F \equiv \exists y \exists x F$

Recap : Terms

Given a signature τ, the set of τ-terms are defined inductively as follows.

- Each variable is a term
- Each constant symbol is a term
- If $t_1, \ldots, t_k$ are terms and f is a k-ary function, then $f(t_1, \ldots, t_k)$ is a term
- Ground Terms : Terms without variables. For instance $f(c_1, \ldots, c_k)$ for constants $c_1, \ldots, c_k$.

Translation Lemma

Translation Lemma

If t is a term and F is a formula such that no variable in t occurs bound in F, then $\mathcal{A} \models_\alpha F[t/x]$ iff $\mathcal{A} \models_{\alpha[x \mapsto \alpha(t)]} F$.

$F[t/x]$ denotes substituting t for x in F, where x is free in F

- What if t contains a variable bound in F?
- Results in *Variable Capture*

Translation Lemma Proof : Optional

Proof by Induction on formulae.

- Base case. Atomic formulae $P(t_1, \ldots, t_k)$.
- $\mathcal{A} \models_\alpha P(t_1, \ldots, t_k)[t/x]$ iff $\mathcal{A} \models_\alpha P(t_1[t/x], \ldots, t_k[t/x])$.
- Show that $\mathcal{A} \models_{\alpha[x \mapsto \alpha(t)]} P(t_1, \ldots, t_k)$.
  - Base Cases within : $t_i = c$, $t_i = y$ for $y \neq x$, $t_i = x$ for each t_i.
  - Case $t_i = f(s_1, \ldots, s_j)$ for a function f.
  - $f(s_1, \ldots, s_j)[t/x] = f(s_1[t/x], \ldots, s_j[t/x])$
- $\mathcal{A} \models_\alpha P(t_1[t/x], \ldots, t_k[t/x])$ iff ($\alpha(t_1[t/x]), \ldots, \alpha(t_k[t/x])$) $\in P^{\mathcal{A}}$
- iff ($\alpha([x \mapsto \alpha(t)](t_1), \ldots, \alpha([x \mapsto \alpha(t)](t_k)$) $\in P^{\mathcal{A}}$
- iff $\mathcal{A} \models_{\alpha[x \mapsto \alpha(t)]} P(t_1, \ldots, t_k)$
- Cases for formulae with propositional connectives is routine.
- Case with quantifier, $\forall y F[t/x]$, $\exists y F[t/x]$ where $y \neq x$.

Renaming

$\int_0^\infty f(s)ds$ has the same value as $\int_0^\infty f(t)dt$

Renaming Lemma

Let $F = Qx[G]$ be a formula with $Q \in \{\exists, \forall\}$. Let y be a variable which does not appear in G. Then $\mathcal{A} \models_\alpha F$ iff $\mathcal{A} \models_\alpha Qy(G[y/x])$.

Assume $Q = \forall$.
$\mathcal{A} \models_\alpha \forall y G[y/x]$ iff $\mathcal{A} \models_{\alpha[y \mapsto a]} G[y/x]$ for all $a \in U^{\mathcal{A}}$

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$$\begin{aligned} \mathcal{A} \models_\alpha \forall y G[y/x] &\text{ iff } \mathcal{A} \models_{\alpha[y \mapsto a]} G[y/x] \text{ for all } a \in U^{\mathcal{A}} \\ &\text{ iff } \mathcal{A} \models_{\alpha[y \mapsto a, x \mapsto \alpha[y \mapsto a](y)]} G \text{ for all } a \in U^{\mathcal{A}} \end{aligned}$$

(Translation Lemma)

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$$