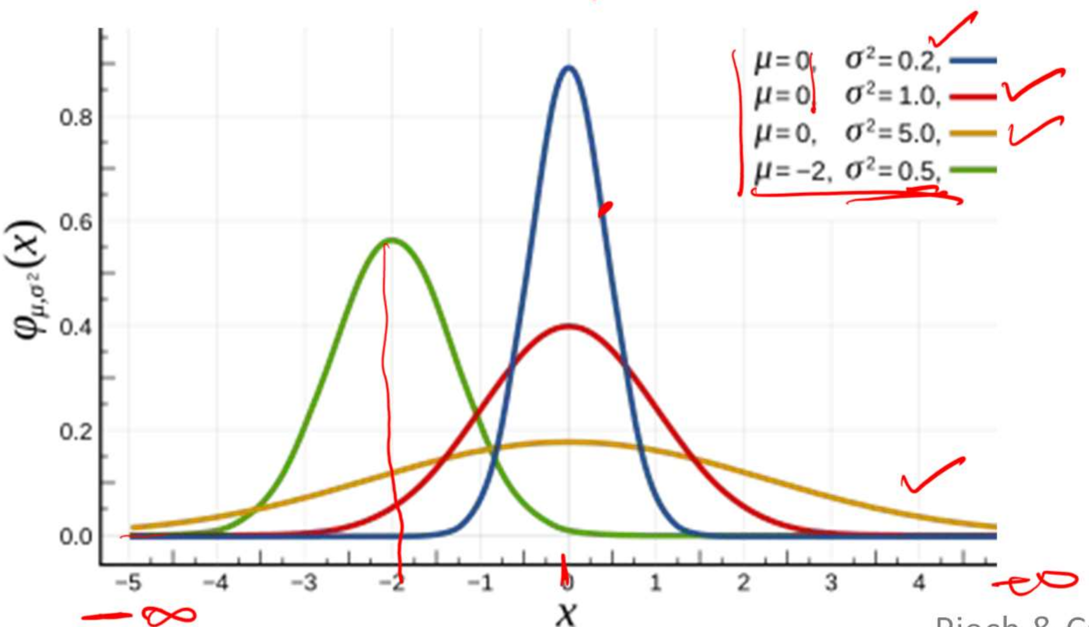
Normal (Gaussian) Random Variable

Support:
$(-\infty, \infty)$

mean variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$



density

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Support: $(-\infty, \infty)$

mean

variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

PDF:

$$f(X = x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Anatomy of a The Normal PDF

distance to the mean
(makes the PDF symmetric
around the mean)

$$f(X = x) = \underbrace{\frac{1}{\sigma\sqrt{2\pi}}}_{\text{a constant}} e^{\frac{-\overbrace{(x-\mu)^2}^{\text{distance to the mean}}}{2\sigma^2}}$$

a constant:
makes the integral
over all possible
outcomes sum to 1

...normalized by
the variance

Expected value of a normal distribution

verify that μ is the expected value of

$$x \sim N(\mu, \sigma^2)$$

$$E([x-\mu]) = E(x) - \mu$$

$$\int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \left. \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma\,(\sigma^2)} \right|_{-\infty}^{+\infty} = 0$$

$$E[(x-\mu)] = 0 \Rightarrow E(x) = \mu$$

Variance

$$E((X-\mu)^2) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/(2\sigma^2)}dx$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{-(y)^2/(2)}dy = \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(y)(ye^{-(y)^2/(2)})dy$$

u

$v\,du$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\left[\left(-ye^{-y^2/2}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -e^{-y^2/2}dy\right]$$

$$\int u\,dv = uv - \int v\,du$$

$$\int ye^{-y^2/2}dy = -e^{-y^2/2}$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy = \frac{\sigma^2}{\sqrt{2\pi}}\sqrt{2\pi} = \sigma^2$$

Properties

If $X \sim N(\mu, \sigma^2)$ and if $Y = aX + b$, then   a & b are scalars.

Top proves $Y \sim N(a\mu + b, a^2\sigma^2)$

Let F_Y be the cumulative density of Y

$F_Y = P(Y \le y)$ $f_Y = \frac{\partial}{\partial y} F(y)$

$F_X = P(X \le x)$ $f_X = \frac{\partial}{\partial x} F_X(x)$

Let $a > 0$

$$\boxed{P(Y \le y) = P(aX+b \le y) = P\left(X \le \frac{y-b}{a}\right)}$$

$$\Rightarrow \frac{\partial}{\partial y} F_Y(y) = \frac{\partial}{\partial y} F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y}\left[\frac{y-b}{a}\right]$$

$$= f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

$$f_X\left(\frac{y-b}{a}\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}}$$

Properties

- Median = mean (why?)
- Because of symmetry of the pdf about the mean

- Mode = mean – can be checked by setting the first derivative of the pdf to 0 and solving, and checking the sign of the second derivative.

$$f_x\left(\frac{y-b}{a}\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-(\mu a+b))^2}{2\sigma^2 \cdot a^2}}$$

$$f_y(y) = f_x\left(\frac{y-b}{a}\right)\cdot\frac{1}{a} = \frac{1}{\sqrt{2\pi}\sigma a} e^{-\frac{(y-(\mu a+b))^2}{2a^2\sigma^2}}$$

$$\Rightarrow Y \sim N(\mu a+b;\ a^2\sigma^2) \text{ if } a > 0$$

$a < 0$

$$F_y(y) = P(Y \le y) = P(aX+b \le y) = P\left(X \geq \frac{y-b}{a}\right)$$

$$= 1 - F_x\left(\frac{y-b}{a}\right)$$

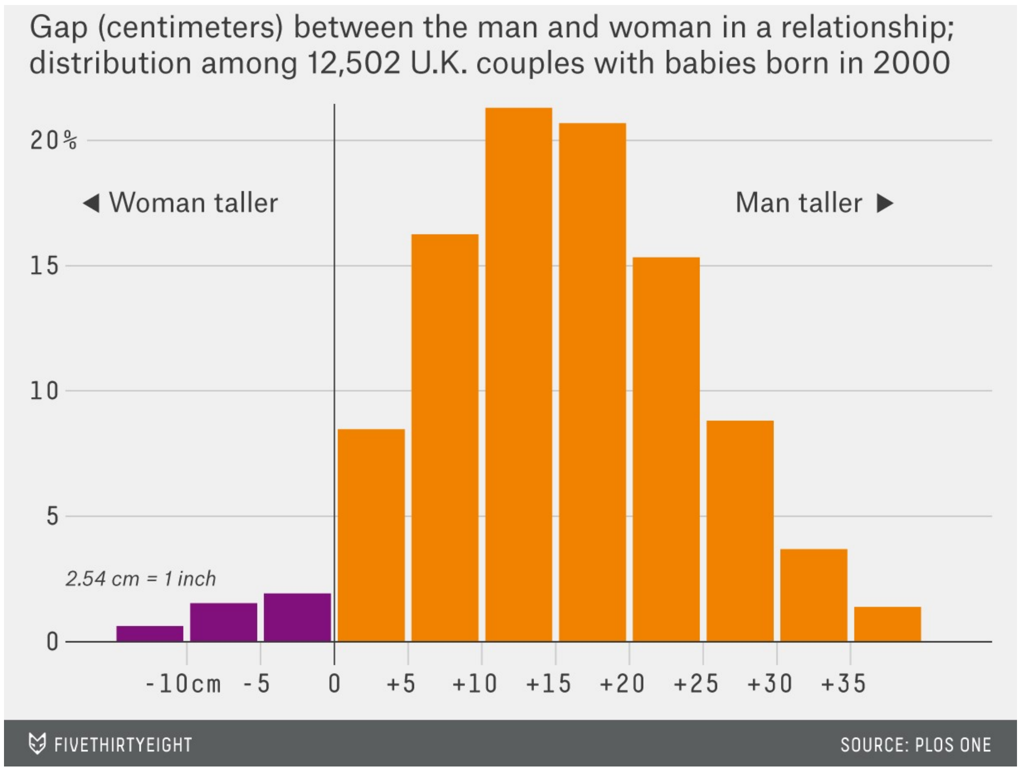
$$Y \sim N(a\mu+b;\ \sigma^2 a^2)$$

Carl Friedrich Gauss (1777-1855)

- German mathematician
- Sort-of invented the normal distribution
- Also astronomer, geologist, physicist
- Super influential in a lot of fields

Why the Normal?

- Common for natural phenomena: human height, weight, shoe sizes, etc.



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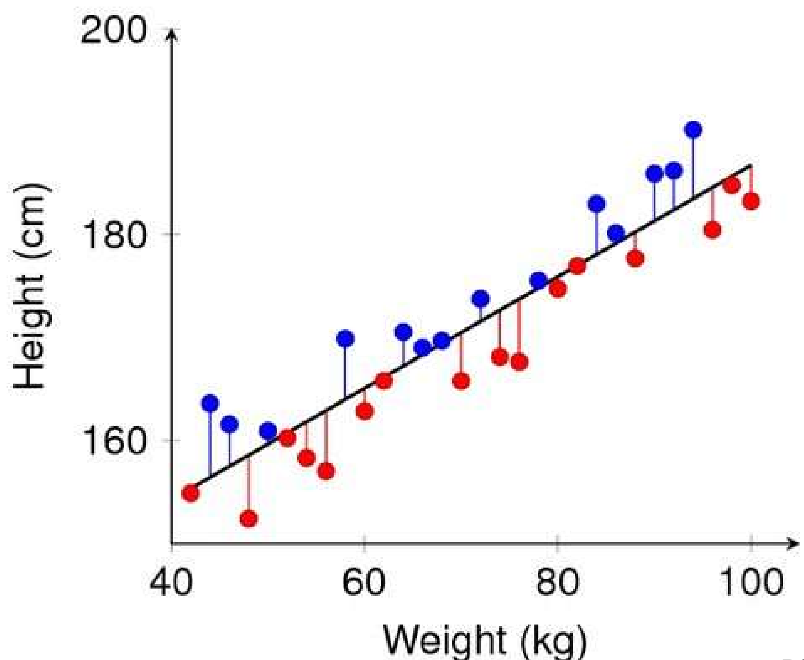

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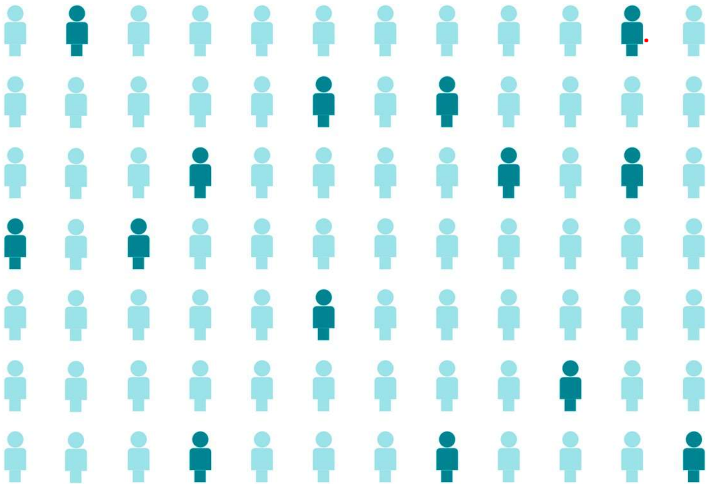
- Sample means are distributed normally – important for statistics



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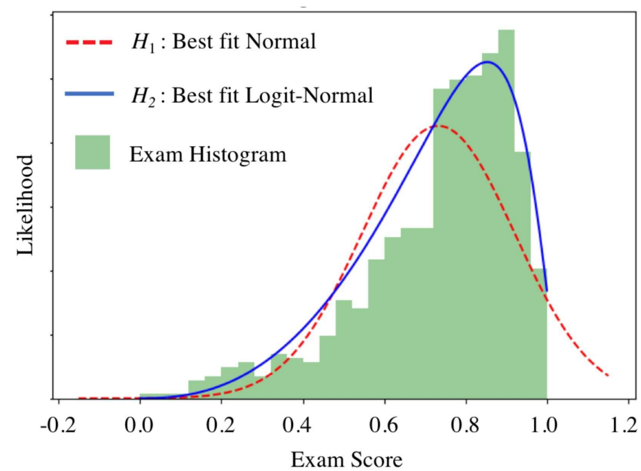
- Even things that aren't Normal might fit a normal-related distribution



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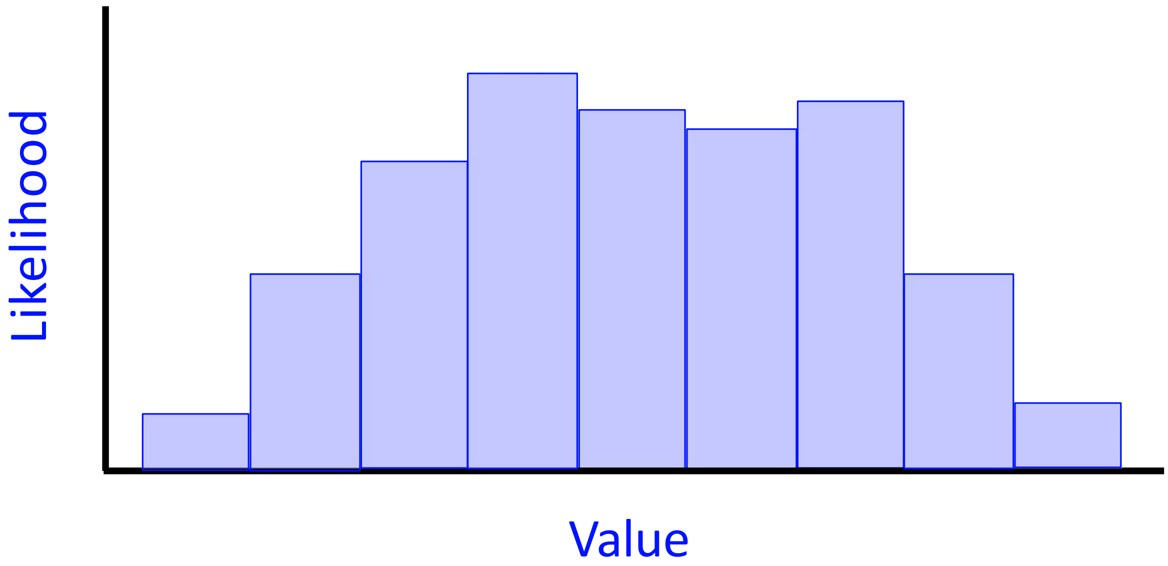
People also just assume things are normally distributed a lot.

- They can do this in part because the Normal is so common
- But there's a deeper reason to it…


Ockham's razor
Shaving your hypothesis since 14th Century
Ockham
AMAZING!

When We Fit Models To Data, We Try To Keep It Simple



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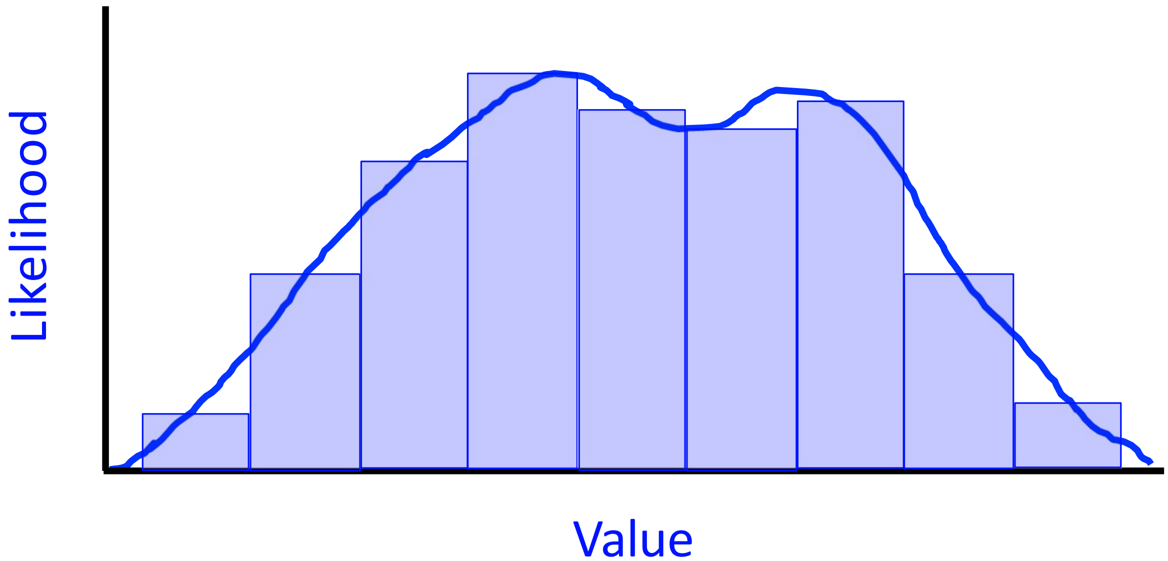

This curve fits the data well, but does it really represent the distribution?
Or is it "overfit", so that the curve captures too much of the noise?

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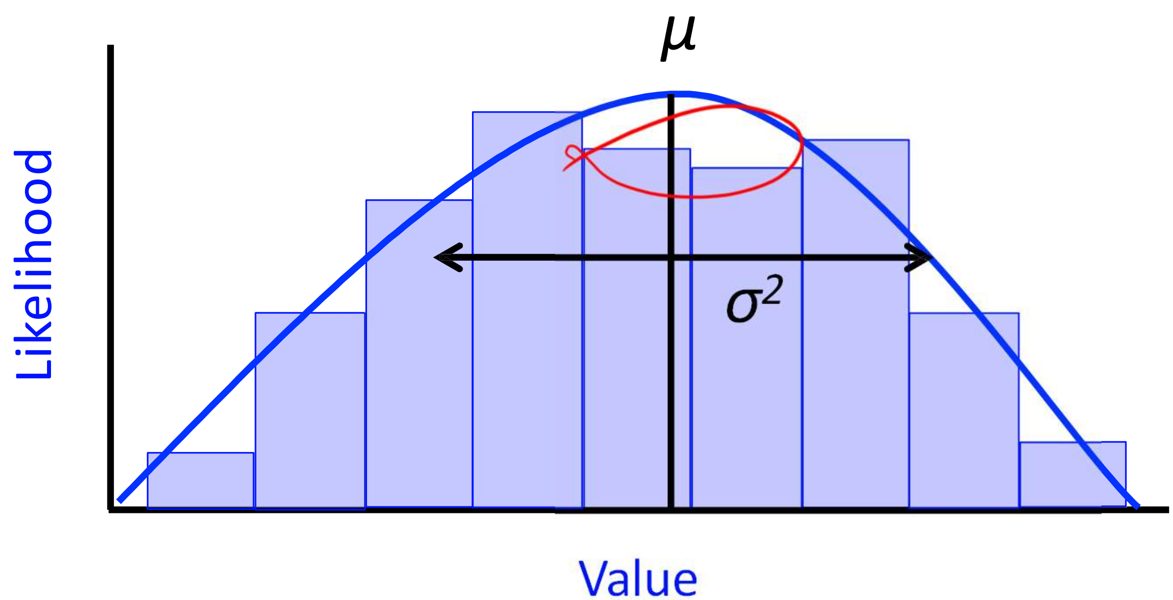
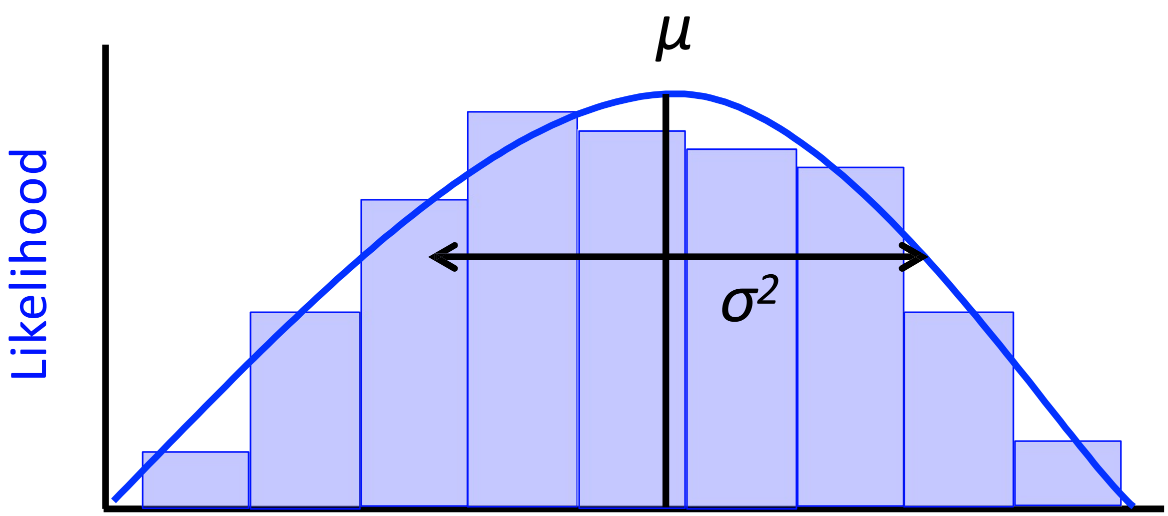


This curve fits the data about as well, but appears to overfit less.
We could say that this simpler distribution makes fewer assumptions.
The formal concept for this idea is **entropy**

When We Fit Models To Data, We Try To Keep It Simple



For a fixed mean and variance, the unique distribution that maximizes the entropy is the normal distribution.

https://medium.com/mathematical-musings/how-gaussian-distribution-maximizes-entropy-the-proof-7f7dcb2caf4d
https://statproofbook.github.io/P/norm-maxent.html