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Radhey Theorem

Any polynomial $Q_n(x)$ of degree 'n' (≥ 2) which has atleast two real roots as $x = \pm 1$ can be expressed in terms of the derivatives of Legendre Polynomials as following :-

$$\boxed{Q_n(x) = (x-1)(x+1)\left[\sum_{i=1}^{n-1} \frac{\lambda_i}{i(i+1)} \, P_i'(x)\right]}$$

where
- λ_i is any arbitrary real number.
- $P_i'(x)$ denotes the derivative of the 'i th' degree Legendre Polynomial.
- $n \geqslant 2$

$$\boxed{Q_n(1) = 0 \quad \text{and} \quad Q_n(-1) = 0}$$

Proof :

Consider the Legendre differential equation :-

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0} \rightarrow (1)$$

which can be rewritten as :-

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0} \rightarrow (1.1)$$

integrating equation (1.1) w.r.t x, we get the following integro-differential equation :-

$$\boxed{(1-x^2)\frac{dy}{dx} + n(n+1)\int y\,dx + c = 0} \rightarrow (1.2)$$

c : constant that arises from indefinite integration rearranging (1.2) we get :-

$$(x^2-1)\frac{dy}{dx} - c = n(n+1)\int y\,dx$$

$$\boxed{\left(\frac{x^2-1}{n(n+1)}\right)\frac{dy}{dx} + c_n = \int y\,dx} \quad \left(c_n = \frac{-c}{n(n+1)}\right)$$

$\hookrightarrow$ (1.3)

Legendre Polynomials of degree 'n' are solutions of (1.3), therefore for a Legendre Polynomial $P_i(x)$ of degree 'i' we can write :-

$$\boxed{\frac{x^2-1}{i(i+1)}\frac{d}{dx}P_i(x) + c_i = \int P_i(x)\,dx}$$

multiplying the above eqn. by an arbitrary real number 'λ_i', we get :-

$$\boxed{(x^2-1)\frac{d}{dx}\left[\frac{\lambda_i}{i(i+1)}P_i(x)\right] + \lambda_i c_i = \int \lambda_i P_i(x)\,dx}$$

$$\boxed{(x^2-1)\frac{d}{dx}\left[\frac{\lambda_i}{i(i+1)}P_i(x)\right] + \lambda_i c_i = \int \lambda_i P_i(x)\,dx}$$

$\hookrightarrow i \neq 0,\ i \in \mathbb{N}$

summing over 'i', $i \in \mathbb{N}$, ranging from $i=1$ to $i=n-1$, ($\because n \geq 2$)

we get :-

$$\boxed{(x^2-1)\frac{d}{dx}\left[\sum_{i=1}^{n-1}\frac{\lambda_i}{i(i+1)}P_i(x)\right] + \sum_{i=1}^{n-1}\lambda_i c_i = \int \sum_{i=1}^{n-1}\lambda_i P_i(x)\,dx}$$

$(n \geq 2)$

$\hookrightarrow$ (1.5)

Adding $\int \lambda_0 P_0(x)\,dx$ on both sides of equation (1.5); where $P_0(x)$ is the zero degree Legendre Polynomial, 'λ_0' is another arbitrary constant :-

$$\boxed{(x^2-1)\frac{d}{dx}\left[\sum_{i=1}^{n-1}\frac{\lambda_i}{i(i+1)}P_i(x)\right] + \sum_{i=1}^{n-1}\lambda_i c_i + \int \lambda_0 P_0(x)\,dx = \int \sum_{i=0}^{n-1}\lambda_i P_i(x)\,dx}$$

$$\boxed{(x^2-1)\frac{d}{dx}\left[\sum_{i=1}^{n-1}\frac{\lambda_i}{i(i+1)}P_i(x)\right]+\sum_{i=1}^{n-1}\lambda_i c_i+\int\lambda_0 P_0(x)\,dx = \int\sum_{i=0}^{n-1}\lambda_i P_i(x)\,dx} \quad (1.6)$$

since we know that $P_0(x)=1$, we can further simplify the above equation:-

$$\boxed{(x^2-1)\frac{d}{dx}\left[\sum_{i=1}^{n-1}\frac{\lambda_i}{i(i+1)}P_i(x)\right]+\sum_{i=1}^{n-1}\lambda_i c_i+\lambda_0 x+\lambda_0 c_0 = \int\sum_{i=0}^{n-1}\lambda_i P_i(x)\,dx} \quad (1.7)$$

* using completeness of Legendre Polynomials, $\sum_{i=0}^{n-1}\lambda_i P_i(x)$ spans the set of all polynomials with degree $(n-1)$.

* Notation: $q_{n-1}(x)$ represents any random polynomial with degree $(n-1)$.

using the previous statements we can write :-

$$\boxed{g_{n-1}(x) = \sum_{i=0}^{n-1} \lambda_i P_i(x)} \longrightarrow (1.8)$$

using (1.8) in (1.7), we get :-

$$\boxed{(x^2-1)\frac{d}{dx}\left[\sum_{i=1}^{n-1} \frac{\lambda_i}{i(i+1)} P_i(x)\right] + \lambda_0 x + b = \int g_{n-1}(x)\,dx} \quad \left(b = \sum_{i=0}^{n-1} \lambda_i c_i\right)$$

$\hookrightarrow$ (1.9)

* $\int g_{n-1}(x)\,dx$ is nothing but a polynomial $\mathbb{P}_n(x)$ of degree 'n' $\left(\text{i.e., } \int g_{n-1}(x)\,dx = \mathbb{P}_n(x)\right)$

Therefore :-

$$\boxed{(x^2-1)\frac{d}{dx}\left[\sum_{i=1}^{n-1} \frac{\lambda_i}{i(i+1)} P_i(x)\right] = \mathbb{P}_n(x) - (\lambda_0 x + b) \qquad \boxed{n \geq 2}}$$

$\hookrightarrow$ (1.10)

say $$\boxed{P_n(x) - (\lambda_0 x + b) = Q_n(x) \; ; \; n \geq 2}$$

$\hookrightarrow$ (1.11)

hence
$$\boxed{Q_n(x) = (x-1)(x+1) \sum_{i=1}^{n-1} \frac{\lambda_i}{i(i+1)} P_i'(x)}$$

$\hookrightarrow$ (1.12)

where $P_i'(x) = \frac{d}{dx} P_i(x)$

From equation (1.12) we can easily say that :-
$$\boxed{Q_n(1) = 0 \quad \text{and} \quad Q_n(-1) = 0}$$

Therefore,

Any polynomial $Q_n(x)$ of degree 'n' (≥ 2) which has atleast two real roots as $x = \pm 1$ can be expressed in terms of the derivatives of Legendre Polynomials