DICTIONARY OF REAL NUMBERS

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Introduction

In this paper I have tried to prove set $\mathbb{R}$ to be countable or countably infinite. In order to prove this theorem some postulates have been proposed so that we have a systematic arrangement of real numbers inside an array which we would call the 'primary array', this is done so that all the real numbers are contained in this arrangement so that it becomes easier to list them chronologically. Further, I have substantiated this theorem by proving Georg Cantor's diagonalisation rule to be false because he had used the diagonalisation rule to prove set $\mathbb{R}$ to be uncountable. This one theorem has also led me to prove set $\mathbb{C}$, which contains all the complex numbers, to be countable or countably infinite.Hence this paper would eventually solve the continuum hypothesis which according to David Hilbert is one of the most important problems in mathematics.

THEOREMS

Theorem 1. *The set $\mathbb{R}$ which spans the entire real number field is a countable/countably infinite set.*

Proof. Let us consider a two dimensional array with equal number of columns and rows which has its entries as matrices rather than real numbers or complex numbers. In order to prove the above theorem we will have to define some postulates in regard to which the properties of the entries of such an array will be defined.

Postulates.

1. Each entry a_{ij}; where i,j are integers and $i, j \leq 0$, represents a matrix: m_{ij}.

   **In order to avoid any confusion lets call matrices m_{ij} to be 'mini-matrices' and the array as the 'primary array'.

2. Order of a particular mini-matrix m_{ij} belonging to this array would depend on the column number in which it is placed in the following manner:

$$order_{m_{ij}} = 2 * 10^j$$

hence the mini-matrix m_{ij} will have:

- rows=2;
- $columns = 10^j$

Now to put constraints on the entries of any mini-matrix m_{ij}, we propose the following postulates:

3. All the entries in the first row of a given matrix m_{ij} will be real numbers such that their greatest integer function equals 'i' i.e., the row number of the primary array to which the mini-matrix belongs.

4. As specified above, each entry in the mini-matrix would be a real number, expressed through the decimal representation and in its decimal representation it will have exactly 'j' decimal places.

5. (**rule for constructing the first row of the mini-matrix)

[5.1] The entries of the mini-matrix m_{ij} will be sorted in an increasing order from left to right in the first row s.t these components form an arithmetic progression with common difference 10^{-j} and first term as 'i'.

(**rule for constructing the second row of the mini-matrix)

[5.2] The entries of the second row will be the same as the first row in magnitude but opposite in sign i.e.,

$b_{2,k} = -b_{1,k}$,

where $b_{p,k}$ represents any entry inside the mini-matrix m_{ij}, $0 < p \leq 2$, $0 < k \leq j$ and p,k belong to $\mathbb{Z}$

Construction of the array

$$\begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} & m_{04} & m_{05} & \cdots \\ m_{10} & m_{11} & m_{12} & m_{13} & m_{14} & m_{05} & \cdots \\ m_{20} & m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & \cdots \\ \cdots & \cdots & \cdots & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & & & \\ \vdots & \vdots & \vdots & \vdots & & & \end{bmatrix}$$

Since our main objective is to prove that the set $\mathbb{R}$ is countable hence we must start talking in the context that whether we can list all the entries of this 'primary array' or not. It can be easily proved that the number of entries contained in this 'primary array' is countably infinite. The procedure is as follows:

Say, assuming $i + j = k$, where k too is an integer.On varying k from 0 to n (here 'n' has been used just to indicate the absence of any upper-bound, i.e., the primary array has infinite entries) we will be able to cover all the mini-matrices present in this given primary array. It can be illustrated in the following way:

$k = 0$ corresponds to mini-matrices m_{00}

$k = 1$ corresponds to mini-matrices m_{01} , m_{10}

$k = 2$ corresponds to mini-matrices m_{02} ,m_{11}, m_{20}

$k = 3$ corresponds to mini-matrices m_{03} ,m_{12}, m_{21},m_{30}

$\vdots$

$\vdots$

$\vdots$

through this mechanism will be able to list all the mini-matrices present in this array. Hence now we have proved that the mini-matrices in this 'primary array' are countably infinite.

Now, in order to link the countably infinite nature of these mini-matrices with that of real numbers we must re centre our attention to the postulates defined above. The relation established between the entries of a particular mini-matrix and the position of that mini-matrix in the array by the postulates eventually helps us to prove set $\mathbb{R}$ to be countably infinite.

following the postulates, we get:

for k=0;

for mini-matrix m_{00}*:*

- number of rows=2;(from **postulate 2**)

- number of columns=10^j=10^1=10;(from **postulate 2**)

- number of decimal places each real entry in the matrix has=j=0;(from **postulate 4**)

- since an arithmetic progression is formed in the first row, its:(from **postulate 5.1**)
  first term= i=0;
  common difference =10^{-1}=0.1;

- greatest integer function for all the real numbers present in the first row of the given matrix=i=0;(from **postulate 3**)

$$m_{00} = \begin{bmatrix} 0 \\ -0 \end{bmatrix}$$

for k=1;

for mini-matrix m_{01}*:*

- number of rows=2;

- number of columns=10^j=10^1=10;

- number of decimal places each real entry in the matrix has=j=1;

- since an arithmetic progression is formed in the first row, its:
  first term= i=0;
  common difference =10^{-1}=0.1;

- greatest integer function for all the real numbers present in the first row of the given matrix=i=0;

therefore on construction the matrix looks like this:

$$m_{01} = \begin{bmatrix} 0.0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ -0.0 & -0.1 & -0.2 & -0.3 & -0.4 & -0.5 & -0.6 & -0.7 & -0.8 & -0.9 \end{bmatrix}$$

for mini-matrix m_{10}*:*

- number of rows=2;

- number of columns=10^j=10^0=1;

- number of decimal places each real entry in the matrix has=j=0;

- since an arithmetic progression is formed in the first row, its:
  first term= i=1;
  common difference =10^{-j}=1;

- greatest integer function for all the real numbers present in the first row of the given matrix=i=1;

$$m_{10} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for k=2;

for mini-matrix m_{02}:

Similarly,
constructing it using the postulates, we get:

$$m_{02} = \begin{bmatrix} 0.00 & 0.01 & 0.02 & 0.03 & \dots & 0.99 \\ -0.00 & -0.01 & -0.02 & -0.03 & \dots & -0.99 \end{bmatrix}$$

for mini-matrix m_{11}:

Similarly,
constructing it using the postulates, we get:

$$m_{11} = \begin{bmatrix} 1.0 & 1.1 & 1.2 & 1.3 & 1.4 & 1.5 & 1.6 & 1.7 & 1.8 & 1.9 \\ -1.0 & -1.1 & -1.2 & -1.3 & -1.4 & -1.5 & -1.6 & -1.7 & -1.8 & -1.9 \end{bmatrix}$$

for mini-matrix m_{20}:

Similarly,
constructing it using the postulates, we get:

$$m_{20} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

for k=3:

⋮

⋮

⋮

**In order to avoid any confusion lets call matrices m_{ij} to be mini-matrices.

The best thing about having mini-matrices as the entries of the primary array is that, even if any mini-matrix has infinite entries, they still are listable. Taking analogy with the primary array,which itself is a matrix with infinite entries, but still we were able to list all the entries by following a particular algorithm i.e., by varying the sum of column number and row number through integers from '0' to 'n'. Here, 'n' has been used just to indicate the absence of any upper-bound.

Since we had proved these mini-matrices of the primary array to be countably infinite, hence these mini-matrices can be listed in one to one correspondence with natural numbers. Since we were able to list these mini-matrices hence while doing so we have indirectly proved that real numbers too are listable as on listing the entries of all the mini-matrices we will be able to span the entire real number field.
Recall the properties of any mini-matrix m_{ij}:

$b_{2,k} = -b_{1,k}$,

where $b_{p,k}$ represents any entry inside the mini-matrix m_{ij}, $1 \leq p \leq 2$, $1 \leq k \leq 10^j$ and p,k belong to $\mathbb{Z}$.

Therefore,now, we can easily make a list which spans all the real numbers because we can individually count/list the entries in each mini-matrix by varying the sum $p + k$ through integers as 2 ≤p+k≤ $2 + 10^j$.

Now in order to show that the postulates applied on the mini-matrices actually help in spanning the entire real number field, we take any arbitrary number from the real number line and try to locate its position in this new systematic arrangement of real numbers.

For example, we have taken a real number which approximates π to fifty(50) decimal places.
3.14159265358979323846264338327950288419716939937510
But, the question is can we locate it in some mini-matrix in the primary array which has been constructed? The answer is Yes, for that we first have to find the greatest integer function of this real number i.e.,

$$[3.14159265358979323846264338327950288419716939937510] = 3$$

hence now we have the value of 'i' and $i = 3$. And since the number of decimal places in this real number equal 50, therefore j=50.
Hence this real number is located in "$m_{3,50}$" mini-matrix.

Hence we can locate any arbitrary real number in this arrangement, given its greatest integer function and the number of decimal places. Say if 'x' is the floor function of the real number and it has 'y' decimal places then;
it will simply lie in the mini-matrix "$m_{x,y}$".

Therefore we have proved **Theorem 1** as we can locate any arbitrary real number in the prescribed array of mini-matrices, hence the set $\mathbb{R}$ is countably infinite as there does not exist any real number which escapes this systematic arrangement.Plus in the case of irrational numbers and non-terminating rational numbers, they will find themselves in the mini-matrices of the rightmost columns of the array because they have infinite decimal places. And in order to deal with such cases, we have not put any upper bound on the values of 'i' and 'j'.

Theorem 2. $\mathbb{N} \times \mathbb{N}$ *is countable.*

Proof:left for reader, as an exercise.

Corollary 2.1. *If a set X is countable then $X \times X$ is countable as well.*

Proof: left for reader, as an exercise.

Theorem 3. *The set $\mathbb{C}$ containing all the complex numbers, which spans the entire real number plane is a countable/countably finite set.*

Proof: Since we have proved set $\mathbb{R}$ to be countably finite, hence using *Corollary 2.1* we can say that $\mathbb{R} \times \mathbb{R}$ is countable as well. And since the basis of $\mathbb{C}$ is $\mathbb{R} \times \mathbb{R}$ hence the set $\mathbb{C}$ is countable as well.

Flaws in Cantor's diagonalization rule

Recap of Cantor's diagonal rule

Suppose that f : N $\to$ [0, 1] is any function. Make a table of values of f, where the 1st row contains the decimal expansion of f(1), the 2nd row contains the decimal expansion of f(2), . . . the nth row contains the

so that the table starts out like this:
n f(n)
1 - 0 .**1** 2 3 4 5 6 7 8 9 1 . . .
2 - 0 .2 **5** 1 8 9 7 3 9 3 3 . . .
3 - 0 .3 4 **7** 2 7 7 5 1 4 2 . . .
4 - 0 .4 0 7 **8** 0 6 7 8 1 1 . . .
5 - 0 .5 8 5 1 **6** 1 0 3 2 0 . . .
$\vdots$

The highlighted digits are 0.12798 . . . . Suppose that we add 1 to each of these digits, to get the number 0.26897 . . . . Now, this number can't be in the table. Why not? Because
- it differs from f(1) in its first digit;
- it differs from f(2) in its second digit;
- . . .
- it differs from f(n)

Therefore we have just found a number which is not contained in this list, hence we have failed proving function 'f' to be an onto function.Hence, there does not exist a bijection between $\mathbb{N}$ and [0,1].
This is Cantor's diagonal rule.

Now lets discuss the flaws in it:
The main flaw lies in the construction of the list, the list was incomplete at the first place itself, this can be proved in the following manner:
Say, if there are 'j' decimal places in the decimal expansion of a real number with floor function as '0', then we can have 10^j such real numbers because each decimal place has ten options(i.e., 0,1,2,...,9) and since there are j such decimal places, therefore '10^j' such combinations are possible. On listing them in the following manner:

$$0.a_{00}a_{01}a_{02}a_{03}a_{04}a_{05}....$$

$$0.a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}....$$

$$0.a_{20}a_{21}a_{22}a_{23}a_{24}a_{25}....$$

$\vdots$
$\vdots$
$\vdots$

If we cover the zeroes before the decimal point then we are left with an array which has columns equal to 'j'(i.e., number of decimal places) and rows equal '10^j'(i.e., total number of real numbers).And hence in no way can we draw a primary diagonal which has elements from all the rows in the array because number of rows and columns are unequal.

Consider the following example:
List all the real numbers with floor function as 0 and decimal places in its decimal expansion equal to two().
0.**0**0
0.0**1**
0.02
0.03
$\vdots$
0.99
when we cover zeroes before the decimal point then we are left with an array of order 10^j*j i.e., 10^2*2, and when we try to draw a diagonal we can at maximum cover elements from two rows only, i.e., the numbers which we have highlighted and hence by default the remaining 98 elements are left uncovered by the diagonal.
The highlighted digits are 0.01 , Suppose that we add 1 to each of these digits, to get the number 0.12, but now we see that this number is present in the list in the 13th row;
what if we had followed cantors diagonal rule:
then our list would appear like this:
0.**0**0
0.0**1**
because according to him the array formed by deleting the zeroes column before the decimal point should be a square matrix such that the diagonal covers every row in this matrix.
But now we can see how incorrect this arrangement is because it already has 98 elements missing.Hence it is no big deal if one proves that some elements are missing in this list, because the list has been constructed in a way that it will not be able to span all the elements with floor function equal to 0 and decimal places equal to 2(two).
Hence if we would have followed Cantors diagonal rule here then we would have concluded that the number 0.12 is absent in the list, because cantors diagonal is only able to cover the first two rows and the rest are ignored.

Therefore our only requirement at this point in time is to find an arrangement which at first claims to list all the real numbers in reality and then if we can find a systematic pattern of counting those real numbers such that this "complete" list is completely exhausted. I have tried to achieve the same objective through this paper by first proposing a list (in the form of an array) which spans the entire real number field and then finding a pattern present in the formation of the list so that we can count all the real numbers such that no real number is left uncounted in that list.
Therefore, now the continuum hypothesis stands resolved as we have proven the size of the two infinite sets i.e., $\mathbb{N}$ and $\mathbb{R}$ to be the same.