Scattering I

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1 Rayleigh scattering from the resonant oscillator

From Lesson 7, we have

$$\sigma(\omega) = \frac{8\pi e^4}{3m^2c^4}\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\alpha^2\omega^2}. \tag{1}$$

Equation for a forced classical (or equivalent quantum) oscillator, remember.

This has several useful limits beyond heuristic derivation of the Lorentz line profile. For example, for free electrons both ω_0 and α are zero. Hence

$$\sigma_T = \frac{8\pi e^4}{3m^2c^4} \tag{2}$$

Scattering by a free electron is *Thomson scattering.* If we plug in the numbers we get $\sigma_T = 6.65\times10^{-25}$ cm^2/molecule.

Now again assume $\alpha = 0$ but do $\omega << \omega_0$ to get

$$\sigma_R(\omega) = \sigma_T\frac{\omega^4}{(\omega_0^2-\omega^2)^2} \approx \sigma_T\frac{\omega^4}{\omega_0^4} \propto \lambda^{-4}. \tag{3}$$

Rayleigh scattering. Tells us there is a 4th power λ dependence for a scatterer far from its resonant frequency. Here the resonant frequencies are electronic (i.e. Lyman series in UV/EUV for hydrogen). The formula says nothing about the scattering phase function, though, and the magnitude does not account for quantum effects.

2 Rayleigh scattering from Hertzian EM theory

A dipole moment is $\mathbf{p} = q\mathbf{x}$. The dipole induced by an EM field is $\mathbf{p}_0 = \alpha\mathbf{E}_{ext}$, where α is the polarizability constant. α has units of volume and is a tensor in general. Here it is a scalar.

Hertz (1889) solution (can be derived from classical EM theory) for scattered E-field far from the dipole

$$\mathbf{E} = \frac{1}{c^2}\frac{1}{r}\frac{\partial^2\mathbf{p}}{\partial t^2}sin\theta_0. \tag{4}$$

θ_0 is the angle between the scattered dipole moment and the observation direction.

p is the scattered dipole moment in the $E-$field direction

$$\mathbf{p} = \mathbf{p}_0e^{-ik(r-ct)} = \alpha\mathbf{E}_{ext}e^{-ik(r-ct)}. \tag{5}$$

Hence

$$\frac{\partial^2 \mathbf{p}}{\partial t^2} = -k^2 c^2 \alpha \mathbf{E}_{ext} e^{-ik(r-ct)} \tag{6}$$

and

$$\mathbf{E} = -\frac{k^2}{r} \alpha \mathbf{E}_{ext} e^{-ik(r-ct)} sin\theta_0. \tag{7}$$

where θ_0 is the angle between the scattered dipole moment and the observer.

We can define *parallel* (to solar beam) and *perpendicular* components

$$E_\perp = -\frac{k^2}{r} \alpha E_{\perp,ext} e^{-ik(r-ct)} sin\theta_1 \tag{8}$$

$$E_\parallel = -\frac{k^2}{r} \alpha E_{\parallel,ext} e^{-ik(r-ct)} sin\theta_2. \tag{9}$$

$\theta_1 = \pi/2$ always because the scattered dipole moment is normal to the scattering plane. Also $\theta_2 = \pi/2 - \Theta$. This means that

$$E_\perp = -\frac{k^2}{r} \alpha E_{\perp,ext} e^{-ik(r-ct)} \tag{10}$$

$$E_\parallel = -\frac{k^2}{r} \alpha E_{\parallel,ext} e^{-ik(r-ct)} cos\Theta \tag{11}$$

Now remember

$$\mathcal{I} = \frac{c|E|^2}{8\pi} \tag{12}$$

hence

$$\mathcal{I}_\perp = \frac{k^4}{r^2} \alpha^2 \mathcal{I}_{\perp,ext} \tag{13}$$

$$\mathcal{I}_\parallel = \frac{k^4}{r^2} \alpha^2 \mathcal{I}_{\parallel,ext} cos^2\Theta \tag{14}$$

Now note for unpolarized light (sunlight) $\mathcal{I}_{\perp,ext} = \mathcal{I}_{\parallel,ext} = \mathcal{I}_{ext}/2$. Hence adding we get

$$\mathcal{I} = \frac{\mathcal{I}_{ext}}{r^2} \alpha^2 k^4 \frac{1 + cos^2\Theta}{2} \tag{15}$$

$$\frac{\mathcal{I}}{\mathcal{I}_{ext}} = \frac{8\pi^4}{r^2} \alpha^2 \lambda^{-4} (1 + cos^2\Theta). \tag{16}$$

This is the Rayleigh scattering formula.

Or we can use defn for phase function *with normalization* to write

$$\mathcal{P}(cos\Theta) = \frac{3}{4}(1 + cos^2\Theta) \tag{17}$$

Remember

$$\int_0^{2\pi} \int_0^{\pi} \mathcal{P}(cos\Theta) sin\Theta d\Theta d\phi = 4\pi. \tag{18}$$

and

$$\int_0^{\pi} (1 + cos^2\Theta) sin\Theta d\Theta = \frac{8}{3} \tag{19}$$

hence

$$\frac{\mathcal{I}}{\mathcal{I}_{ext}} = \frac{128\pi^5}{3r^2}\alpha^2\lambda^{-4}\frac{\mathcal{P}(cos\Theta)}{4\pi}. \tag{20}$$

The total scattered Rayleigh power [W] is

$$P_R = \int_\Omega \mathcal{I}\Delta\Omega r^2 d\Omega \tag{21}$$

where the integral is over solid angle. $\mathcal{I}\Delta\Omega$ is a flux [W/m^2]

$$P_R = \frac{128\pi^5\alpha^2}{3\lambda^4}\mathcal{I}_{ext}\Delta\Omega\int_\Omega \frac{\mathcal{P}(cos\Theta)}{4\pi}d\Omega \tag{22}$$

but

$$\int_\Omega \frac{\mathcal{P}(cos\Theta)}{4\pi}d\Omega \equiv 1 \tag{23}$$

so

$$P_R = \frac{128\pi^5\alpha^2}{3\lambda^4}\mathcal{F}_{ext} \tag{24}$$

with $\mathcal{F}_{ext} \equiv \mathcal{I}_{ext}\Delta\Omega$ the incoming flux [W/m^2]. Finally Rayleigh cross-section

$$\sigma_R \equiv \frac{\mathcal{F}_R}{\mathcal{F}_{ext}} = \frac{128\pi^5\alpha^2}{3\lambda^4}. \tag{25}$$

Compare this to what we had before from the oscillator argument.

Now α is given by the *Lorentz-Lorenz* formula

$$\alpha = \frac{3}{4\pi n_s}\frac{m^2-1}{m^2+2} \tag{26}$$

See Liou or Feynman Lectures vol. 2 for a good derivation of this.

n_s is the number of scatterers (here, molecules) per unit volume. m is refractive index. In practice for gases $m \approx 1$, so

$$\alpha \approx \frac{1}{4\pi n_s}(m^2-1) \tag{27}$$

so finally rewrite

$$\sigma_R = \frac{8\pi^3(m_r^2-1)^2}{3n_s^2\lambda^4}f(\delta). \tag{28}$$

and we've added an anisotropy factor $f(\delta)$.