



# Floating Point

**15-213/18-213/15-513: Introduction to Computer Systems**  
**4<sup>th</sup> Lecture, Sept. 7, 2017**

**Today's Instructor:**

Phil Gibbons

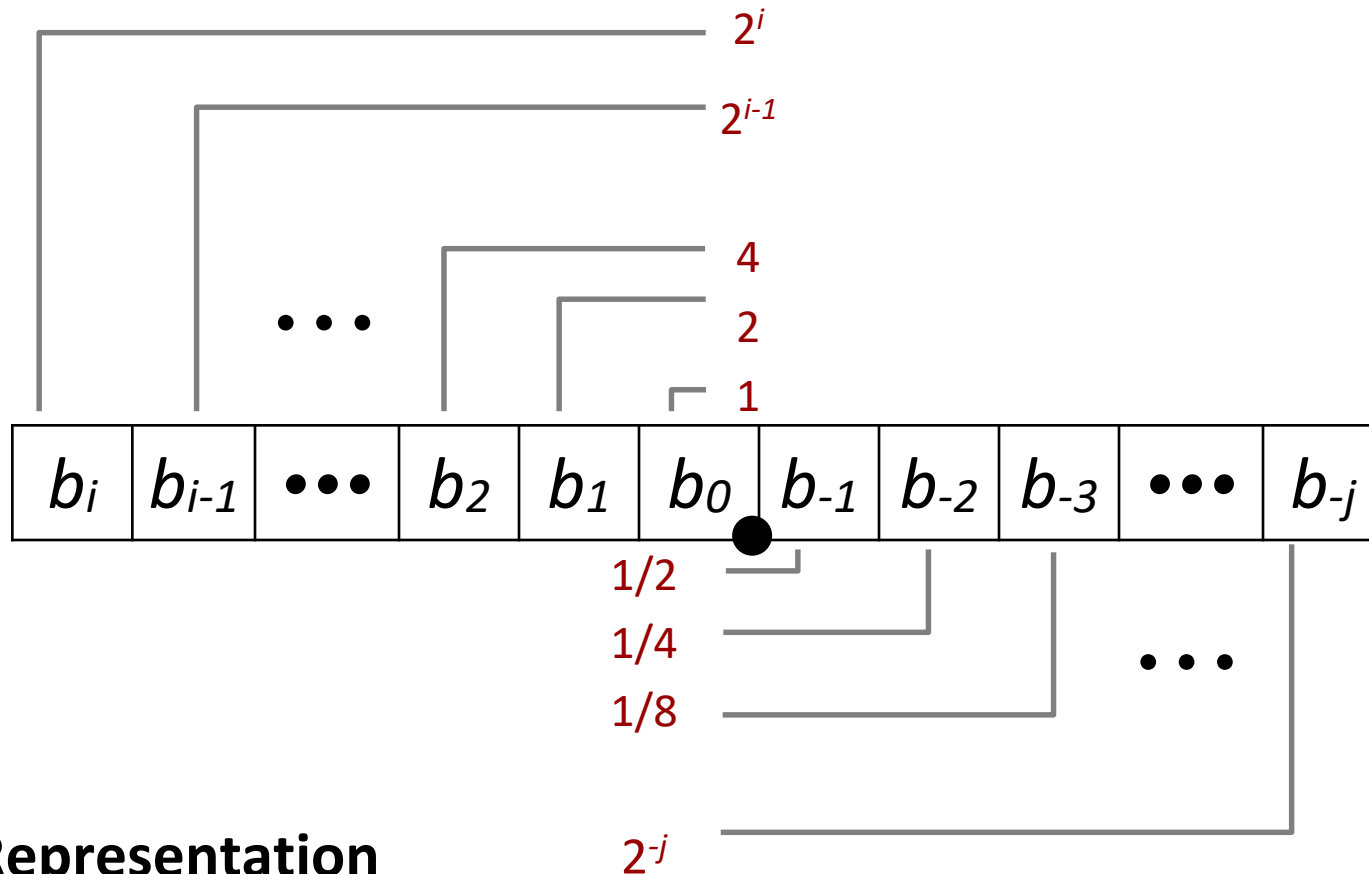
# Today: Floating Point

- **Background: Fractional binary numbers**
- **IEEE floating point standard: Definition**
- **Example and properties**
- **Rounding, addition, multiplication**
- **Floating point in C**
- **Summary**

# Fractional binary numbers

- What is  $1011.101_2$ ?

# Fractional Binary Numbers



## ■ Representation

- Bits to right of “binary point” represent fractional powers of 2
- Represents rational number:

$$\sum_{k=-j}^i b_k \times 2^k$$

# Fractional Binary Numbers: Examples

Value	Representation	
$5 \frac{3}{4} = 23/4$	$101.11_2$	$= 4 + 1 + 1/2 + 1/4$
$2 \frac{7}{8} = 23/8$	$10.111_2$	$= 2 + 1/2 + 1/4 + 1/8$
$1 \frac{7}{16} = 23/16$	$1.0111_2$	$= 1 + 1/4 + 1/8 + 1/16$

## Observations

- Divide by 2 by shifting right (unsigned)
- Multiply by 2 by shifting left
- Numbers of form  $0.111111..._2$  are just below 1.0
  - $1/2 + 1/4 + 1/8 + ... + 1/2^i + ... \rightarrow 1.0$
  - Use notation  $1.0 - \epsilon$

# Representable Numbers

## ■ Limitation #1

- Can only exactly represent numbers of the form  $x/2^k$ 
  - Other rational numbers have repeating bit representations
- Value                  Representation
  - $1/3$                    $0.0101010101 [01] \dots_2$
  - $1/5$                    $0.001100110011 [0011] \dots_2$
  - $1/10$                  $0.0001100110011 [0011] \dots_2$

## ■ Limitation #2

- Just one setting of binary point within the  $w$  bits
  - Limited range of numbers (very small values? very large?)

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# IEEE Floating Point

## ■ IEEE Standard 754

- Established in 1985 as uniform standard for floating point arithmetic
  - Before that, many idiosyncratic formats
- Supported by all major CPUs
- Some CPUs don't implement IEEE 754 in full  
e.g., early GPUs, Cell BE processor

## ■ Driven by numerical concerns

- Nice standards for rounding, overflow, underflow
- Hard to make fast in hardware
  - **Numerical analysts** predominated over **hardware designers** in defining standard

# Floating Point Representation

## ■ Numerical Form:

Example:

$$15213_{10} = (-1)^0 \times 1.1101101101101_2 \times 2^{13}$$

$$(-1)^s M 2^E$$

- **Sign bit**  $s$  determines whether number is negative or positive
- **Significand**  $M$  normally a fractional value in range  $[1.0, 2.0)$ .
- **Exponent**  $E$  weights value by power of two

## ■ Encoding

- MSB  $s$  is sign bit  $s$
- **exp** field encodes  $E$  (but is not equal to  $E$ )
- **frac** field encodes  $M$  (but is not equal to  $M$ )



# Precision options

- **Single precision: 32 bits**

≈ 7 decimal digits,  $10^{\pm 38}$



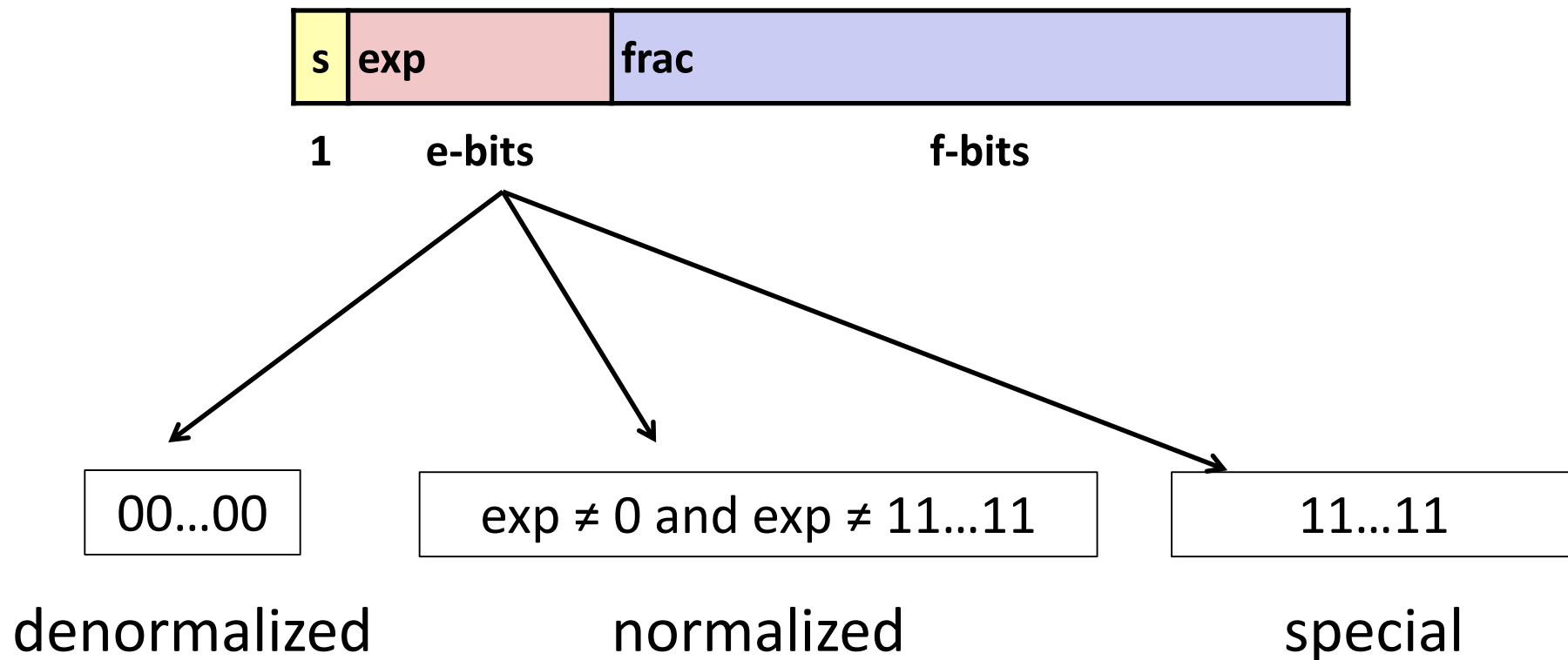
- **Double precision: 64 bits**

≈ 16 decimal digits,  $10^{\pm 308}$



- **Other formats: half precision, quad precision**

# Three “kinds” of floating point numbers



# “Normalized” Values

$$v = (-1)^s M 2^E$$

- When: **exp**  $\neq$  000...0 and **exp**  $\neq$  111...1
- Exponent coded as a *biased* value:  $E = \text{exp} - \text{Bias}$ 
  - *exp*: unsigned value of exp field
  - $\text{Bias} = 2^{k-1} - 1$ , where  $k$  is number of exponent bits
    - Single precision: 127 (**exp**: 1...254, E: -126...127)
    - Double precision: 1023 (**exp**: 1...2046, E: -1022...1023)
- Significand coded with implied leading 1:  $M = 1.\text{xxx}...\text{x}_2$ 
  - xxx...x: bits of frac field
  - Minimum when **frac**=000...0 ( $M = 1.0$ )
  - Maximum when **frac**=111...1 ( $M = 2.0 - \epsilon$ )
  - Get extra leading bit for “free”

# Normalized Encoding Example

$$v = (-1)^s M 2^E$$

$$E = \text{exp} - \text{Bias}$$

■ Value: float  $F = 15213.0;$

$$15213_{10} = 11101101101101_2$$

$$= 1.1101101101101_2 \times 2^{13}$$

■ Significand

$$M = 1.\underline{1101101101101}_2$$

$$\text{frac} = \underline{110110110110100000000000}_2$$

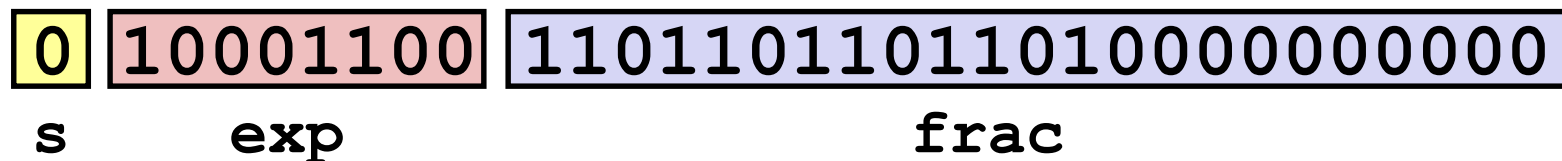
■ Exponent

$$E = 13$$

$$\text{Bias} = 127$$

$$\text{exp} = 140 = 10001100_2$$

■ Result:



# Denormalized Values

$$v = (-1)^s M 2^E$$

$$E = 1 - \text{Bias}$$

- **Condition:**  $\text{exp} = 000\dots 0$
- **Exponent value:**  $E = 1 - \text{Bias}$  (instead of  $\text{exp} - \text{Bias}$ ) (why?)
- **Significand coded with implied leading 0:**  $M = 0.\text{xxx}\dots\text{x}_2$ 
  - $\text{xxx}\dots\text{x}$ : bits of  $\text{frac}$
- **Cases**
  - $\text{exp} = 000\dots 0, \text{frac} = 000\dots 0$ 
    - Represents zero value
    - Note distinct values:  $+0$  and  $-0$  (why?)
  - $\text{exp} = 000\dots 0, \text{frac} \neq 000\dots 0$ 
    - Numbers closest to  $0.0$
    - Equispaced

# Special Values

- Condition:  $\text{exp} = 111\dots 1$
- Case:  $\text{exp} = 111\dots 1, \text{frac} = 000\dots 0$ 
  - Represents value  $\infty$  (infinity)
    - Operation that overflows
    - Both positive and negative
    - E.g.,  $1.0/0.0 = -1.0/-0.0 = +\infty$ ,  $1.0/-0.0 = -\infty$
- Case:  $\text{exp} = 111\dots 1, \text{frac} \neq 000\dots 0$ 
  - Not-a-Number (NaN)
    - Represents case when no numeric value can be determined
    - E.g.,  $\text{sqrt}(-1)$ ,  $\infty - \infty$ ,  $\infty \times 0$



# C float Decoding Example

float: 0xC0A00000

binary: \_\_\_\_\_



1

8-bits

23-bits

**E =**

**S =**

**M =**

**$v = (-1)^S M 2^E =$**

$$v = (-1)^S M 2^E$$

$$E = \text{exp} - \text{Bias}$$

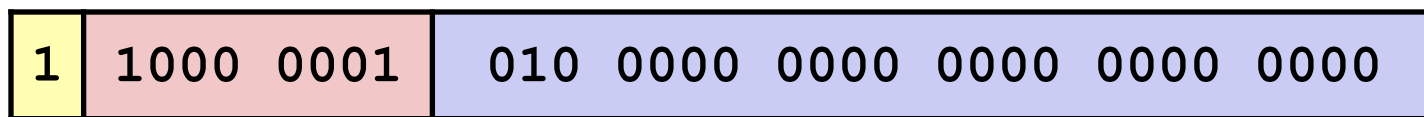
$$\text{Bias} = 2^{k-1} - 1 = 127$$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

# C float Decoding Example

float: 0xC0A00000

binary: 1100 0000 1010 0000 0000 0000 0000 0000



1

8-bits

23-bits

**E =**

**S =**

**M = 1.**

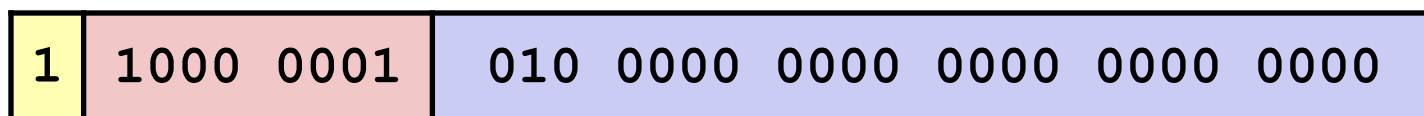
**$v = (-1)^S M 2^E =$**

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

# C float Decoding Example

float: 0xC0A00000

binary: 1100 0000 1010 0000 0000 0000 0000 0000



1

8-bits

23-bits

$$E = \text{exp} - \text{Bias} = 129 - 127 = 2 \text{ (decimal)}$$

$S = 1 \rightarrow$  negative number

$$M = 1.010 \ 0000 \ 0000 \ 0000 \ 0000 \ 0000$$

$$= 1 + 1/4 = 1.25$$

$$v = (-1)^S M 2^E = (-1)^1 * 1.25 * 2^2 = -5$$

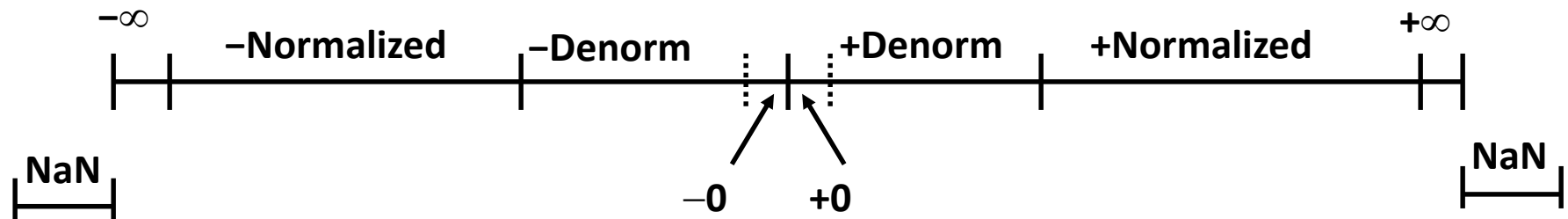
$$v = (-1)^S M 2^E$$

$$E = \text{exp} - \text{Bias}$$

$$\text{Bias} = 2^{k-1} - 1 = 127$$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

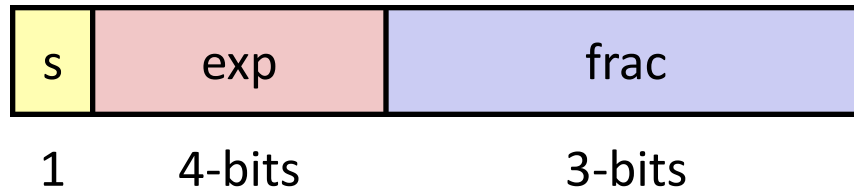
# Visualization: Floating Point Encodings



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# Tiny Floating Point Example



## ■ 8-bit Floating Point Representation

- the sign bit is in the most significant bit
- the next four bits are the **exp**, with a bias of 7
- the last three bits are the **frac**

## ■ Same general form as IEEE Format

- normalized, denormalized
- representation of 0, NaN, infinity

# Dynamic Range (s=0 only)

$$v = (-1)^s M 2^E$$

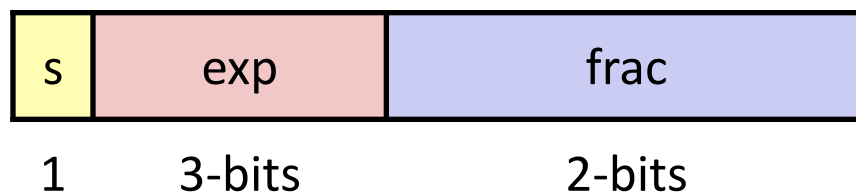
*norm:  $E = \text{exp} - \text{Bias}$*   
*denorm:  $E = 1 - \text{Bias}$*

	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	$1/8 * 1/64 = 1/512$	closest to zero
	0	0000	010	-6	$2/8 * 1/64 = 2/512$	$(-1)^0 (0 + 1/4) * 2^{-6}$
	...					
	0	0000	110	-6	$6/8 * 1/64 = 6/512$	
	0	0000	111	-6	$7/8 * 1/64 = 7/512$	largest denorm
Normalized numbers	0	0001	000	-6	$8/8 * 1/64 = 8/512$	smallest norm
	0	0001	001	-6	$9/8 * 1/64 = 9/512$	$(-1)^0 (1 + 1/8) * 2^{-6}$
	...					
	0	0110	110	-1	$14/8 * 1/2 = 14/16$	
	0	0110	111	-1	$15/8 * 1/2 = 15/16$	closest to 1 below
	0	0111	000	0	$8/8 * 1 = 1$	
	0	0111	001	0	$9/8 * 1 = 9/8$	closest to 1 above
	0	0111	010	0	$10/8 * 1 = 10/8$	
	...					
	0	1110	110	7	$14/8 * 128 = 224$	
	0	1110	111	7	$15/8 * 128 = 240$	largest norm
	0	1111	000	n/a	inf	

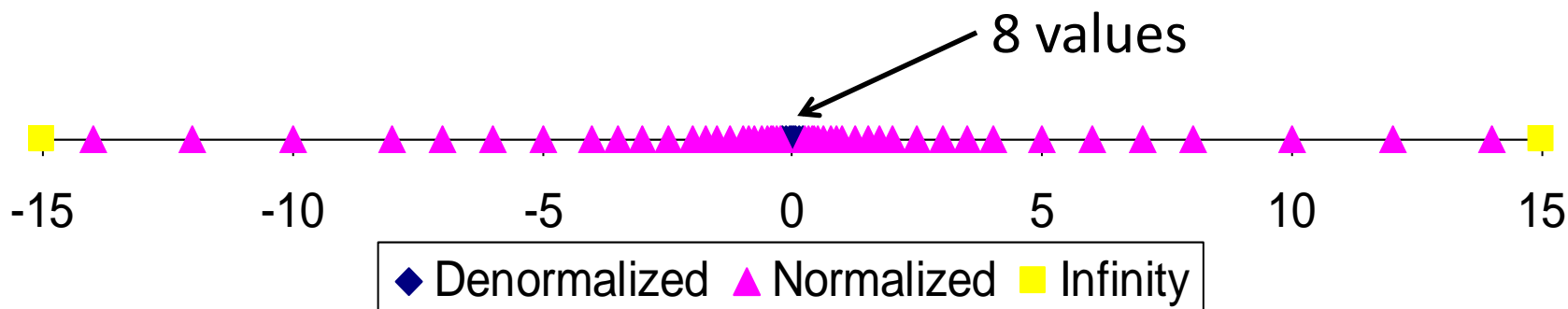
# Distribution of Values

## ■ 6-bit IEEE-like format

- $e = 3$  exponent bits
- $f = 2$  fraction bits
- Bias is  $2^{3-1}-1 = 3$



## ■ Notice how the distribution gets denser toward zero.

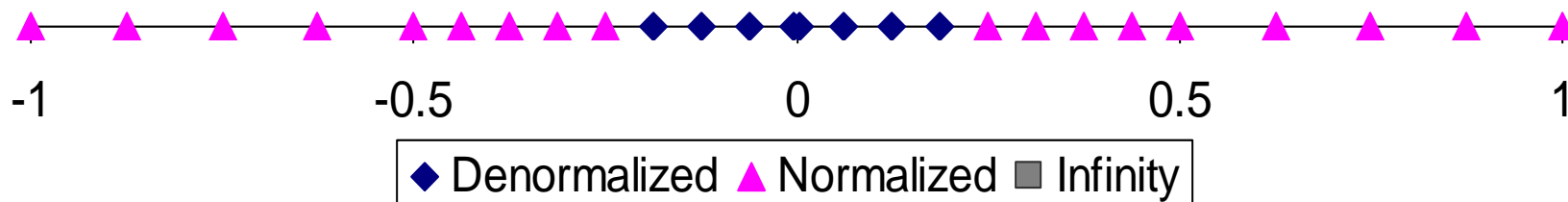
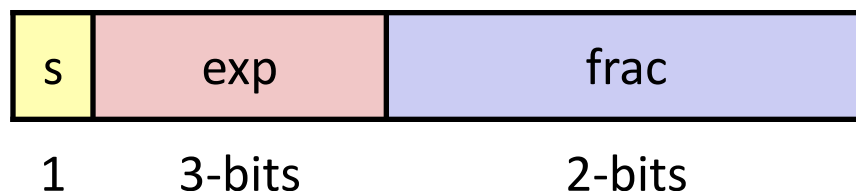




# Distribution of Values (close-up view)

## ■ 6-bit IEEE-like format

- $e = 3$  exponent bits
- $f = 2$  fraction bits
- Bias is 3



# Special Properties of the IEEE Encoding

## ■ FP Zero Same as Integer Zero

- All bits = 0

## ■ Can (Almost) Use Unsigned Integer Comparison

- Must first compare sign bits
- Must consider  $-0 = 0$
- NaNs problematic
  - Will be greater than any other values
  - What should comparison yield? The answer is complicated.
- Otherwise OK
  - Denorm vs. normalized
  - Normalized vs. infinity

# Quiz Time!

Check out:

<https://canvas.cmu.edu/courses/1221>

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# Floating Point Operations: Basic Idea

$$\blacksquare \mathbf{x} +_{\mathbf{f}} \mathbf{y} = \mathbf{Round}(\mathbf{x} + \mathbf{y})$$

$$\blacksquare \mathbf{x} \times_{\mathbf{f}} \mathbf{y} = \mathbf{Round}(\mathbf{x} \times \mathbf{y})$$

## ■ Basic idea

- First **compute exact result**
- Make it fit into desired precision
  - Possibly overflow if exponent too large
  - Possibly **round to fit into frac**

# Rounding

## ■ Rounding Modes (illustrate with \$ rounding)

	\$1.40	\$1.60	\$1.50	\$2.50	-\$1.50
■ Towards zero	\$1 ↓	\$1 ↓	\$1 ↓	\$2 ↓	-\$1 ↑
■ Round down ( $-\infty$ )	\$1 ↓	\$1 ↓	\$1 ↓	\$2 ↓	-\$2 ↓
■ Round up ( $+\infty$ )	\$2 ↑	\$2 ↑	\$2 ↑	\$3 ↑	-\$1 ↑
■ Nearest Even* (default)	\$1 ↓	\$2 ↑	\$2 ↑	\$2 ↓	-\$2 ↓

\*Round to nearest, but if half-way in-between then round to nearest even

# Closer Look at Round-To-Even

## ■ Default Rounding Mode

- Hard to get any other kind without dropping into assembly
- C99 has support for rounding mode management
- All others are statistically biased
  - Sum of set of positive numbers will consistently be over- or under-estimated

## ■ Applying to Other Decimal Places / Bit Positions

- When exactly halfway between two possible values
  - Round so that least significant digit is even
- E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

# Rounding Binary Numbers

## ■ Binary Fractional Numbers

- “Even” when least significant bit is 0
- “Half way” when bits to right of rounding position =  $100..._2$

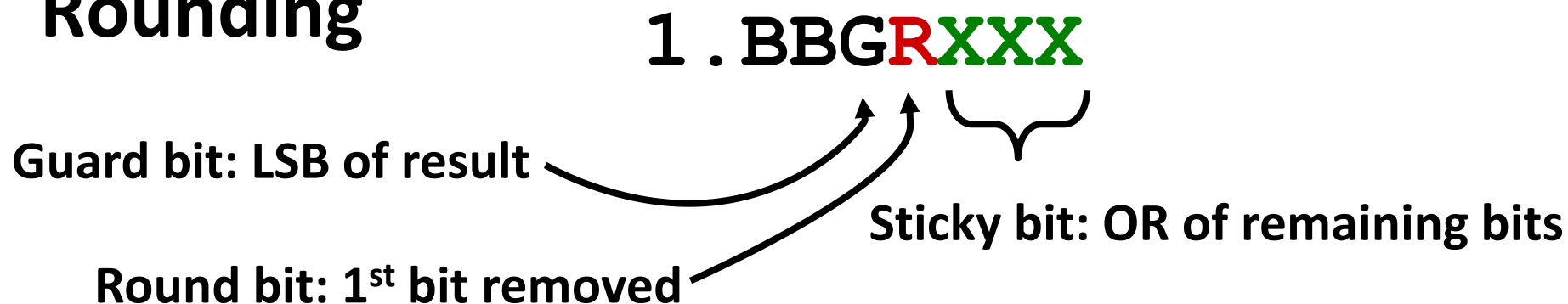
## ■ Examples

- Round to nearest  $1/4$  (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
$2 \frac{3}{32}$	$10.00\textcolor{red}{011}_2$	$10.00_2$	( $<1/2$ —down)	2
$2 \frac{3}{16}$	$10.00\textcolor{red}{110}_2$	$10.01_2$	( $>1/2$ —up)	$2 \frac{1}{4}$
$2 \frac{7}{8}$	$10.11\textcolor{red}{100}_2$	$11.00_2$	( $\textcolor{red}{1/2}$ —up)	3
$2 \frac{5}{8}$	$10.10\textcolor{red}{100}_2$	$10.10_2$	( $\textcolor{red}{1/2}$ —down)	$2 \frac{1}{2}$



# Rounding



## ■ Round up conditions

- Round = 1, Sticky = 1  $\rightarrow > 0.5$
- Guard = 1, Round = 1, Sticky = 0  $\rightarrow$  Round to even

<i>Value</i>	<i>Fraction</i>	<i>GRS</i>	<i>Incr?</i>	<i>Rounded</i>
128	1.000 <b>0</b> 000	0 <b>0</b> 0	N	1.000
15	1.101 <b>0</b> 000	1 <b>0</b> 0	N	1.101
17	1.000 <b>1</b> 000	0 <b>1</b> 0	N	1.000
19	1.001 <b>1</b> 000	1 <b>1</b> 0	Y	1.010
138	1.000 <b>1</b> 010	0 <b>1</b> 1	Y	1.001
63	1.111 <b>1</b> 100	1 <b>1</b> 1	Y	10.000

# FP Multiplication

■  $(-1)^{s1} M1 2^{E1} \times (-1)^{s2} M2 2^{E2}$

■ **Exact Result:**  $(-1)^s M 2^E$

- Sign  $s$ :  $s1 \wedge s2$
- Significand  $M$ :  $M1 \times M2$
- Exponent  $E$ :  $E1 + E2$

## ■ Fixing

- If  $M \geq 2$ , shift  $M$  right, increment  $E$
- If  $E$  out of range, overflow
- Round  $M$  to fit **frac** precision

## ■ Implementation

- Biggest chore is multiplying significands

$$\begin{aligned}
 \text{4 bit mantissa: } 1.010 \times 2^2 \times 1.110 \times 2^3 &= 1\mathbf{0}.0011 \times 2^5 \\
 &= 1.000\mathbf{11} \times 2^6 = 1.00\mathbf{1} \times 2^6
 \end{aligned}$$

# Floating Point Addition

$$\blacksquare (-1)^{s1} M1 2^{E1} + (-1)^{s2} M2 2^{E2}$$

- Assume  $E1 > E2$

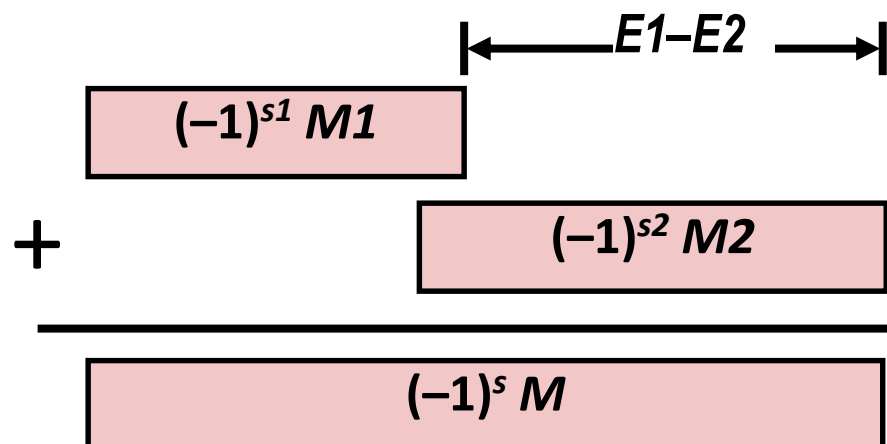
$$\blacksquare \text{Exact Result: } (-1)^s M 2^E$$

- Sign  $s$ , significand  $M$ :
  - Result of signed align & add
- Exponent  $E$ :  $E1$

## Fixing

- If  $M \geq 2$ , shift  $M$  right, increment  $E$
- if  $M < 1$ , shift  $M$  left  $k$  positions, decrement  $E$  by  $k$
- Overflow if  $E$  out of range
- Round  $M$  to fit **frac** precision

Get binary points lined up



$$1.010 * 2^2 + 1.110 * 2^3 = (0.1010 + 1.1100) * 2^3$$

$$= 1\textcolor{red}{0}.0110 * 2^3 = 1.001\textcolor{red}{10} * 2^4 = 1.010 * 2^4$$

# Mathematical Properties of FP Add

## ■ Compare to those of Abelian Group

- Closed under addition? *Yes*
  - But may generate infinity or NaN
- Commutative? *Yes*
- Associative? *No*
  - Overflow and inexactness of rounding
  - $(3.14 + 1e10) - 1e10 = 0$ ,  $3.14 + (1e10 - 1e10) = 3.14$
- 0 is additive identity? *Yes*
- Every element has additive inverse? *Almost*
  - Yes, except for infinities & NaNs

## ■ Monotonicity

- $a \geq b \Rightarrow a + c \geq b + c$  *Almost*
  - Except for infinities & NaNs

# Mathematical Properties of FP Mult

## ■ Compare to Commutative Ring

- Closed under multiplication? *Yes*
  - But may generate infinity or NaN
- Multiplication Commutative? *Yes*
- Multiplication is Associative? *No*
  - Possibility of overflow, inexactness of rounding
  - Ex:  $(1e20 * 1e20) * 1e-20 = \text{inf}$ ,  $1e20 * (1e20 * 1e-20) = 1e20$
- 1 is multiplicative identity? *Yes*
- Multiplication distributes over addition? *No*
  - Possibility of overflow, inexactness of rounding
  - $1e20 * (1e20 - 1e20) = 0.0$ ,  $1e20 * 1e20 - 1e20 * 1e20 = \text{NaN}$

## ■ Monotonicity

- $a \geq b \ \& \ c \geq 0 \Rightarrow a * c \geq b * c$  *Almost*
  - Except for infinities & NaNs

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- Summary

# Floating Point in C

## ■ C Guarantees Two Levels

- `float`      single precision
- `double`     double precision

## ■ Conversions/Casting

- Casting between `int`, `float`, and `double` changes bit representation
- `double/float → int`
  - Truncates fractional part
  - Like rounding toward zero
  - Not defined when out of range or NaN: Generally sets to TMin
- `int → double`
  - Exact conversion, as long as `int` has  $\leq 53$  bit word size
- `int → float`
  - Will round according to rounding mode

# Floating Point Puzzles

## ■ For each of the following C expressions, either:

- Argue that it is true for all argument values
- Explain why not true

```
int x = ...;
float f = ...;
double d = ...;
```

Assume neither  
**d** nor **f** is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (double)(float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0`  $\Rightarrow$  `((d*2) < 0.0)`
- `d > f`  $\Rightarrow$  `-f > -d`
- `d * d >= 0.0`
- `(d+f) - d == f`

✗

✓

✓

✗

✓

✗

✓

✓

✓

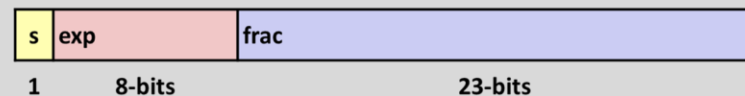
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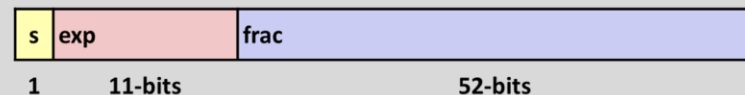
# Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form  $M \times 2^E$
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

Single precision: 32 bits



Double precision: 64 bits

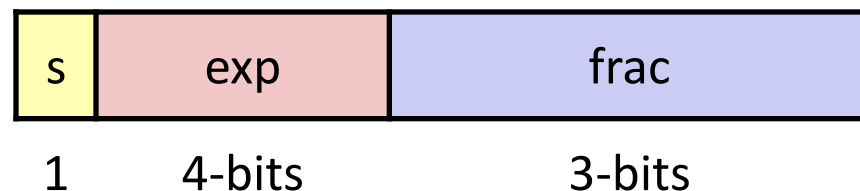


# Additional Slides

# Creating Floating Point Number

## ■ Steps

- Normalize to have leading 1
- Round to fit within fraction
- Postnormalize to deal with effects of rounding



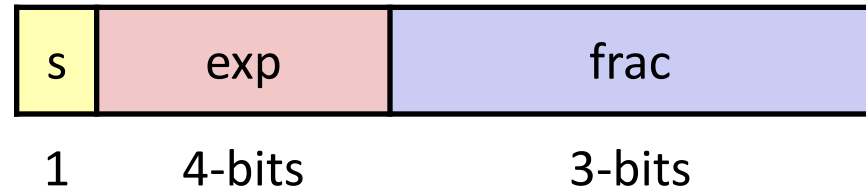
## ■ Case Study

- Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

128	10000000
15	00001101
33	00010001
35	00010011
138	10001010
63	00111111

# Normalize



## ■ Requirement

- Set binary point so that numbers of form 1.xxxxx
- Adjust all to have leading one
  - Decrement exponent as shift left

<i>Value</i>	<i>Binary</i>	<i>Fraction</i>	<i>Exponent</i>
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

# Postnormalize

## ■ Issue

- Rounding may have caused overflow
- Handle by shifting right once & incrementing exponent

<i>Value</i>	<i>Rounded</i>	<i>Exp</i>	<i>Adjusted</i>	<i>Numeric Result</i>
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

# Interesting Numbers

{single, double}

<i>Description</i>	<i>exp</i>	<i>frac</i>	<i>Numeric Value</i>
■ Zero	00...00	00...00	0.0
■ Smallest Pos. Denorm.	00...00	00...01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
<ul style="list-style-type: none"> <li>■ Single <math>\approx 1.4 \times 10^{-45}</math></li> <li>■ Double <math>\approx 4.9 \times 10^{-324}</math></li> </ul>			
■ Largest Denormalized	00...00	11...11	$(1.0 - \epsilon) \times 2^{-\{126,1022\}}$
<ul style="list-style-type: none"> <li>■ Single <math>\approx 1.18 \times 10^{-38}</math></li> <li>■ Double <math>\approx 2.2 \times 10^{-308}</math></li> </ul>			
■ Smallest Pos. Normalized	00...01	00...00	$1.0 \times 2^{-\{126,1022\}}$
<ul style="list-style-type: none"> <li>■ Just larger than largest denormalized</li> </ul>			
■ One	01...11	00...00	1.0
■ Largest Normalized	11...10	11...11	$(2.0 - \epsilon) \times 2^{\{127,1023\}}$
<ul style="list-style-type: none"> <li>■ Single <math>\approx 3.4 \times 10^{38}</math></li> <li>■ Double <math>\approx 1.8 \times 10^{308}</math></li> </ul>			