Performance indicators in multiobjective optimization

We can evaluate the overall performance of our algorithm based on three widely used metrics that cover the convergence and diversity aspects of the retrieved solutions.

- *The convergence aspect* quantifies how close a set of non-dominated points is from the Pareto front in the objective space.
- *The diversity aspect* quantifies how well distributed the points are on the Pareto front approximation.

Final Generational Distance (GD) *(Convergence indicator)*

The GD indicator measures the closeness of the solutions to the true Pareto front.

$$GD = \frac{\sqrt{\sum_{i=1}^{n} d_i^2}}{n}$$

where $d_i = min\ ||f(x_i)\text{-}PF_{true}(x_j)||$ refers to the distance in objective space between individual x_i and the nearest member in the true Pareto front, and n is the number of individuals in the approximation front.

Interpretation: This metric, assuming PF_{true} is readily available, represents how ''far'' the approximation front is from the true Pareto front. Lower value of GD represents a better performance.

Inverted Generational Distance (IGD) *(Convergence & Diversity indicator)*

This metric measures both convergence and diversity. Let PFtrue is a set of uniformly distributed solutions in the true Pareto front. X is the set of nondominated solutions in the approximation front PF known as:

$$IGD = \frac{\sum_{v \in PF_{true}} d(v, X)}{|PF_{true}|}$$

d(v,X) denotes the minimum Euclidean distance between v and the points in X. To have a low value of IGD, the set X should be close to PF_{true} and cannot miss any part of the whole PF_{true}. ***The only difference between GD and IGD is that in the later we don't miss any part of the true Pareto Front set.***

Hypervolume metrics Hv *(Convergence & Diversity indicator)*

This metric measures the volume of the objective space between the obtained solutions set and a specified reference point in the objective space. For HV, a larger value is preferable. For example, an individual x_i in PF_{known} for a two-dimensional MOP defines a rectangle area, $a(x_i)$, bounded by an origin and $f(x_i)$. The union of such rectangle areas is referred to as Hypervolume of PF_{known},

$$H(PF_{known}) = \left\{ \bigcup_i a(x_i) \mid \forall x_i \in PF_{known} \right\}$$

The Hypervolume measure (Hv) is interesting because it captures both the proximity of the approximation solution set (F) to the true Pareto Front (PF) and the distribution of F over the objective space. Moreover, Hv is the only known indicator that reflects the Pareto dominance in the sense that, if an approximation set weakly dominates other, this fact will be reflected in the values of Hv.

Interpretation: Note that the hypervolume indicator is Pareto-dominance compliant, i.e., whenever a solution set $A \subseteq X$ is strictly better than a set $B \subseteq X$ with respect to the weak Pareto-dominance relation ($A >= B$) the hypervolume of A is also strictly better than the one for B ($H(A) > H(B)$). Therefore, a set $X* \subseteq X$ that maximizes the hypervolume indicator contains the Pareto front entirely.

This metric requires defining a reference point "r" and the true pareto front.
To do so, the true PF is assumed as the union of Pareto solution set of non-dominated solutions which are obtained from different algorithms (which can be NSGA-II RNSGA-II, NSGA-III, UNSGA-III, RNSGA-III because they are already implemented in the Framework Pymoo) by running them for a large number of cycles.
The evaluation of HV for a solution set requires an appropriate reference point r. For each problem instance, the nondominated sets from all the competing algorithms are collected to identify the **maximum value for each objective**, then "**r" is set 10% above each dimension's maximum value**. When assessing the metrics, both the obtained solution set and "r" are normalized based on the true PF

Maximum SPREAD *(Diversity indicator)*

It addresses the range of objective function values and takes into account the proximity to the true Pareto front, assuming available. This metric is applied to measure how well the PF_{true} is covered by the PF_{known}.

$$MS = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left[\frac{\min\left(PF^{\max}_{known,i}, PF^{\max}_{true,i}\right) - \max\left(PF^{\min}_{known,i}, PF^{\min}_{true,i}\right)}{PF^{\max}_{true,i} - PF^{\min}_{true,i}}\right]^2}$$

where $PF^{max}{}_{known;i}$ and $PF^{min}{}_{known;i}$ are the maximum and minimum of the ith objective in PF_{known}, respectively; and $PF^{max}{}_{true;i}$ and $PF^{min}{}_{true;i}$ are the maximum and minimum of the ith objective in PF_{true}, respectively. M denotes the number of objectives considered.

Interpretation: A higher value of MS reflects that a larger area of the PF_{true} is covered by the PF_{known}.