$y'' + 2y' + y = 2x \quad y'(0) = 0 \quad y(0) = 0$

$y'' + y = x^2 \quad y(0) = 0 \quad y'(0) = 0$

$y'' + 2y' + y = 2x \quad y'(0) = 0 \quad y(0) = 0$

Green's Method

$x^2 + 2x + 1 = 0$

$(x+1)^2$ repeated root $r = -1$

$y_h = (C_1 + C_2 t)e^{-t}$

$G(t-s) = (t-s)e^{-(t-s)}$

$y_p(t) = \int_0^x G(t-s) f(s) ds$

$y_p(t) = \int_0^t (t-s)e^{-(t-s)} 2s \, ds$

$2t - 4$

$y(t) = (C_1 + C_2 t)e^{-t} + 2t - 4$

Undetermined Coefficients

$y''+2y'+2=2x$

$x^2+2x+1=0$

$r=-1$

$y_h=(C_1+C_2t)e^{-t}$

$y_p=at+b$

$y'_p=a \quad y''=0$

$0+2a+at+b=2t$

$at+(2a+b)=2t$

$a=2 \qquad 2a+b=0$

$4+b=0$

$b=-4$

$at+b=2t-4$

$(C_1+C_2t)e^{-t}+2t-4$

Variation of Parameters

$y''+2y+1=2x$

root (same as before): $r=-1$

$y_1=e^{-t} \qquad y_2=te^{-t}$

$W=y_1y_2'-y_1'y_2=e^{-t}((1-t)e^{-t})-(-e^{-t})(te^{-t})=e^{-2t}$

$$v_1' = -\frac{(te^{-t})(2t)}{e^{-2t}} = -2t^2e^t \qquad v_2' = \frac{(e^{-t})(2t)}{e^{-2t}} = 2te^t$$

$$v_1 \quad \int -2t^2e^t\,dt = -2e^t(t^2-2t+2)$$

$$v_2 \int 2te^t\,dt = 2e^t(t-1)$$

$$v_1y_1 + v_2y_2 = (-2e^t(t^2-2t+2))e^{-t} + (2e^t(t-1))(te^{-t})$$

$$-2(t^2-2t+2)+2(t^2-t) = t^2-4$$

$$((C_1+C_2t)e^{-t}+t^2-4$$

All 3 solutions are the same

Solving for C's

$y(0)=0 \to C_1-4=0 \qquad C_1=4$

$y'=(C_2-C_1-C_2t)e^{-t}+2$

$y'(0)=0 \to C_2-C_1+2=0 \qquad C_2=2$

$y(t)=(4+2t)e^{-t}+2t-4$

$y'' + y = x^2 \quad y(0)=0 \quad y'(0)=0$

Green's Method

$f(x)=x^2$

$$\begin{cases} x^2: a=1 \\ x^1: b=0 \\ x^0: 2a+c=0 \end{cases} \qquad 2a+(ax^2+bx+c)=x^2$$

$a=1$
$b=0$
$c=-2$

$y_p = x^2-2 \qquad C_1\cos(x)+C_2\sin(x)+x^2-2$

$C_1\cos(0)+C_2\sin(0)+0-2=0$

$C_1-2=0 \quad C_1=2$

$y'(x)= -C_1\sin x + C_2\cos x + 2x$

$y'(0)= -C_1(0)+C_2(1)+0 = C_2 = 0$

$2\cos x + x^2-2$

$W = Y_1Y_2' - Y_1'Y_2 = -1$

$$G(t,r)\begin{cases} 0 & t<r \\ \frac{\sin(t-r)}{1} & t\geq r \end{cases}$$

$$y(t)=\int_0^t G(t,r)f(r)\,dr = \int_0^t \sin(t-r)r^2\,dr$$

$\sin(t-r)=\sin(t)\cos(r)-\cos(t)\sin(r)$

$$y(t)=\sin t\int_0^t r^2\cos r\,dr - \cos(t)\int_0^t r^2\sin r\,dr$$

$y(t)=\sin t(t^2\sin t+2t\cos t-2\sin t)-\cos t(-t^2\cos t+2t\sin t+2\cos t)$

$y(t)=t^2-2$

$y(t) = t^2 - 2 + 2\cos t$

Undetermined Coefficients

$x^2 + 1 = 0$
$x^2 = -1$
$x = \pm i$
$C_1\cos x + C_2\sin x$

$y_p = ax^2 + bx + c$

$y_p' = 2ax + b \quad y_p'' = 2a$
$y'' = 2a$
$y_p'' + y = 2a + (ax^2 + bx + c) = x^2$

$x^2: a = 1 \qquad a = 1$
$x^1: b = 0 \qquad b = 0$
$c: 2a + c = 0 \qquad c = -2$

$y_p = x^2 - 2$

$y_h + y_p = C_1\cos(x) + C_2\sin x + x^2 - 2$

Variable parameters

homogeneous part
$C_1\cos x + C_2\sin x$

$y_p = v_1(x)\cos x + v_2\sin(x)$

$v_1' = -\dfrac{y_2 f(x)}{W}, \quad v_2 = \dfrac{y_1 f(x)}{W}$

$W = y_1 y_2' - y_1' y_2 = 1$

$f(x) = x^2$

$u_1' = -\sin x \cdot x^2 \quad u_2' = \cos x \cdot x^2$

$u_1 = -\int x^2 \sin x \, dx \quad u_2 = \int x^2 \cos x \, dx$

$u_1 = x^2 \cos x - 2x \sin x - 2\cos x \quad u_2 = x^2 \sin x + 2x \cos x - 2\sin x$

$y_p \ (x^2\cos x - 2x\sin x - 2\cos x)(\cos x) + (x^3 \sin x + 2x\cos x - 2\sin x)(\sin x)$

$y_p = x^2 - 2$

$y = C_1 \cos x + C_2 \sin x + x^2 - 2$

All 3 have the same solution!

$y(0) = C_1 - 2 \quad C_1 = 2$

$y'(0) = 0 = C_2$

$2\cos x + x^2 - 2$