1. Grupa 1

$$
\begin{aligned}
f_1(n) &= n^{0.999999} \log n \\
f_2(n) &= 10000000n \\
f_3(n) &= 1.000001^n \\
f_4(n) &= n^2
\end{aligned}
$$

Twierdzę, że

$$f_1 \preccurlyeq f_2 \preccurlyeq f_4 \preccurlyeq f_3$$

Dowód. $f_1 \preccurlyeq f_2$

$$\lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} = \lim_{n\to\infty} \frac{n^{0.999999} \log n}{10000000n} = \lim_{n\to\infty} \frac{n*n^{-10^6} \log n}{10^7 n} = \lim_{n\to\infty} \frac{1}{10^7} \frac{\ln n}{n^{10^{-6}}} \overset{H}{=} \lim_{n\to\infty} \frac{1}{10} \frac{\frac{1}{n}}{n^{10^{-6}-1}} = \frac{1}{10} \lim_{n\to\infty} \frac{1}{n^{10^{-6}}} = 0$$

$f_2 \preccurlyeq f_4$

$$\lim_{n\to\infty} \frac{f_2(n)}{f_4(n)} = \lim_{n\to\infty} \frac{10^7 n}{n^2} \overset{H}{=} \lim_{n\to\infty} \frac{10^7}{2n} = 0$$

$f_4 \preccurlyeq f_3$

$$\lim_{n\to\infty} \frac{f_4(n)}{f_3(n)} = \lim_{n\to\infty} \frac{n^2}{1.000001^n} \overset{H}{=} \lim_{n\to\infty} \frac{2n}{1.000001^n \ln(1.000001)} \overset{H}{=} \lim_{n\to\infty} \frac{2}{1.000001^n \ln^2(1.000001)} = 0$$

□

2. Grupa 2

$$
\begin{aligned}
f_1(n) &= 2^{100n} \\
f_2(n) &= \binom{n}{2} \\
f_3(n) &= n\sqrt{n}
\end{aligned}
$$

Twierdzę, że

$$f_3 \preccurlyeq f_2 \preccurlyeq f_1$$

Dowód. $f_3 \preccurlyeq f_1$

$$\lim_{n\to\infty} \frac{f_3(n)}{f_2(n)} = \lim_{n\to\infty} \frac{n\sqrt{n}}{\binom{n}{2}} = \lim_{n\to\infty} \frac{n\sqrt{n}}{\frac{n!}{2!(n-2)!}} = \lim_{n\to\infty} \frac{n\sqrt{n}}{n(n-1)} \overset{H}{=} \lim_{n\to\infty} \frac{3\sqrt{n}}{2n-1} \overset{H}{=} \lim_{n\to\infty} \frac{3}{4\sqrt{n}} = 0$$

$f_2 \preccurlyeq f_1$

$$\lim_{n\to\infty} \frac{f_2(n)}{f_1(n)} = \lim_{n\to\infty} \frac{\binom{n}{2}}{2^{100n}} = \lim_{n\to\infty} \frac{\frac{n!}{2!(n-2)!}}{2^{100n}} = \lim_{n\to\infty} \frac{n(n-1)}{2^{100n+1}} \overset{H}{=} \lim_{n\to\infty} \frac{2n-1}{2^{100n+1} \ln 2 \cdot 100} \overset{H}{=} \lim_{n\to\infty} \frac{2}{2^{100n+1} \ln^2 2 \cdot 10^4} = 0$$

□

3. Grupa 3

$$
\begin{aligned}
f_1(n) &= n^{\sqrt{n}} \\
f_2(n) &= 2^n \\
f_3(n) &= 10^n 2^{n/2} \\
f_4(n) &= \sum_{i=1}^{n} (i+1)
\end{aligned}
$$

Twierdzę, że

$$f_4 \preccurlyeq f_1 \preccurlyeq f_2 \preccurlyeq f_3$$

Dowód. $f_4 \preccurlyeq f_1$

$\lim_{n\to\infty} \frac{f_4(n)}{f_1(n)} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(i+1)}{n^{\sqrt{n}}} = 0$, bo

$$0 \leqslant \frac{f_4(n)}{f_1(n)} = \frac{\frac{1}{2}n(n+1)}{n^{\sqrt{n}}} \leqslant \frac{n^2+n}{2n^3} \leqslant \frac{2n^2}{2n^3} = \frac{1}{n} \overset{n\to\infty}{\to} 0$$

i z twierdzenia o 3 ciągach

$f_1 \preccurlyeq f_2$

$\lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} = \lim_{n\to\infty} \frac{n^{\sqrt{n}}}{2^n} = \lim_{n\to\infty} \frac{\exp(\ln n\sqrt{n})}{\exp(n\ln 2)} = \exp\left(\lim_{n\to\infty} \ln n\sqrt{n} - n\ln 2\right) = \exp\left(\lim_{n\to\infty} n\left(\frac{\ln n}{\sqrt{n}\ln 2} - 1\right)\ln 2\right)$
$= 2\exp\left(\lim_{n\to\infty} n \cdot \lim_{n\to\infty}\left(\frac{\ln n}{\sqrt{n}\ln 2} - 1\right)\right) = 2\exp\left(\lim_{n\to\infty} n \cdot \left(\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}\ln 2} - \lim_{n\to\infty} 1\right)\right)$
$\overset{H}{=} 2\exp\left(\lim_{n\to\infty} n \cdot \left(\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{\ln 2}{2\sqrt{n}}} - \lim_{n\to\infty} 1\right)\right) = 2\exp\left(\lim_{n\to\infty} n \cdot \left(\lim_{n\to\infty} \frac{2\sqrt{n}}{n\ln 2} - \lim_{n\to\infty} 1\right)\right)$
$= 2\exp\left(\lim_{n\to\infty} n \cdot \left(\lim_{n\to\infty} \frac{2}{\sqrt{n}\ln 2} - \lim_{n\to\infty} 1\right)\right) = 2\exp\left(\infty \cdot (0-1)\right) = 2\exp\left(-\infty\right)) = 0$

$f_2 \preccurlyeq f_3$

$\lim_{n\to\infty} \frac{f_2(n)}{f_3(n)} = \lim_{n\to\infty} \frac{2^n}{10^n 2^{n/2}} = \lim_{n\to\infty} \frac{2^n}{10^n \sqrt{2}^n} = \lim_{n\to\infty} \left(\frac{2}{10\sqrt{2}}\right)^n = \lim_{n\to\infty} \left(\frac{1}{5\sqrt{2}}\right)^n = 0$

$\square$