1. Grupa 1

$$\begin{aligned} f_1(n) &= n^{0.999999} \log n \\ f_2(n) &= 10000000n \\ f_3(n) &= 1.000001^n \\ f_4(n) &= n^2 \end{aligned}$$

Twierdzę, że

$$f_1 \preccurlyeq f_2 \preccurlyeq f_4 \preccurlyeq f_3$$

Dowód. $f_1 \preccurlyeq f_2$

$$\lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} = \lim_{n\to\infty} \frac{n^{0.999999} \log n}{10000000n} = \lim_{n\to\infty} \frac{n * n^{-10^6} \log n}{10^7 n} = \lim_{n\to\infty} \frac{1}{10^7} \frac{\ln n}{n^{10^{-6}}} \stackrel{H}{=} \lim_{n\to\infty} \frac{1}{10} \frac{\frac{1}{n}}{n^{10^{-6}-1}} = \frac{1}{10} \lim_{n\to\infty} \frac{1}{n^{10^{-6}}} = 0$$

$f_2 \preccurlyeq f_4$

$$\lim_{n\to\infty} \frac{f_2(n)}{f_4(n)} = \lim_{n\to\infty} \frac{10^7 n}{n^2} \stackrel{H}{=} \lim_{n\to\infty} \frac{10^7}{2n} = 0$$

$f_4 \preccurlyeq f_3$

$$\lim_{n\to\infty} \frac{f_4(n)}{f_3(n)} = \lim_{n\to\infty} \frac{n^2}{1.000001^n} \stackrel{H}{=} \lim_{n\to\infty} \frac{2n}{1.000001^n \ln(1.000001)} \stackrel{H}{=} \lim_{n\to\infty} \frac{2}{1.000001^n \ln^2(1.000001)} = 0$$

□

2. Grupa 2

$$\begin{aligned} f_1(n) &= 2^{100n} \\ f_2(n) &= \binom{n}{2} \\ f_3(n) &= n\sqrt{n} \end{aligned}$$

Twierdzę, że

$$f_3 \preccurlyeq f_2 \preccurlyeq f_1$$

Dowód. $f_3 \preccurlyeq f_1$

$$\lim_{n\to\infty} \frac{f_3(n)}{f_2(n)} = \lim_{n\to\infty} \frac{n\sqrt{n}}{\binom{n}{2}} = \lim_{n\to\infty} \frac{n\sqrt{n}}{\frac{n!}{2!(n-2)!}} = \lim_{n\to\infty} \frac{n\sqrt{n}}{n(n-1)} \stackrel{H}{=} \lim_{n\to\infty} \frac{3\sqrt{n}}{2n-1} \stackrel{H}{=} \lim_{n\to\infty} \frac{3}{4\sqrt{n}} = 0$$

$f_2 \preccurlyeq f_1$

$$\lim_{n\to\infty} \frac{f_2(n)}{f_1(n)} = \lim_{n\to\infty} \frac{\binom{n}{2}}{2^{100n}} = \lim_{n\to\infty} \frac{\frac{n!}{2!(n-2)!}}{2^{100n}} = \lim_{n\to\infty} \frac{n(n-1)}{2^{100n+1}} \stackrel{H}{=} \lim_{n\to\infty} \frac{2n-1}{2^{100n+1} \ln 2 \cdot 100} \stackrel{H}{=} \lim_{n\to\infty} \frac{2}{2^{100n+1} \ln^2 2 \cdot 10^4} = 0$$

□

3. Grupa 3

$$\begin{aligned} f_1(n) &= n^{\sqrt{n}} \\ f_2(n) &= 2^n \\ f_3(n) &= n^{10} 2^{n/2} \\ f_4(n) &= \sum_{i=1}^{n} (i+1) \end{aligned}$$

Twierdzę, że

$$f_4 \preccurlyeq f_1 \preccurlyeq f_3 \preccurlyeq f_2$$

Dowód. $f_4 \preccurlyeq f_1$

$\lim_{n\to\infty} \frac{f_4(n)}{f_1(n)} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(i+1)}{n^{\sqrt{n}}} = 0$, bo

$$0 \leqslant \frac{f_4(n)}{f_1(n)} = \frac{\frac{1}{2}n(n+1)}{n^{\sqrt{n}}} \leqslant \frac{n^2+n}{2n^3} \leqslant \frac{2n^2}{2n^3} = \frac{1}{n} \overset{n\to\infty}{\to} 0$$

i z twierdzenia o 3 ciągach

$f_1 \preccurlyeq f_3$
$\lim_{n\to\infty} \frac{f_1(n)}{f_3(n)} = \lim_{n\to\infty} \frac{n^{\sqrt{n}}}{n^{10}2^{n/2}} = \lim_{n\to\infty} \frac{n^{\sqrt{n}-10}}{\sqrt{2}^n} = 0$,bo

$$0 \leqslant \frac{f_1(n)}{f_3(n)} \leqslant \frac{n^{\sqrt{n}}}{\sqrt{2}^n} = \left(\frac{n}{\sqrt{2}^{\sqrt{n}}}\right)^{\sqrt{n}},$$

a ponieważ $\lim_{n\to\infty} \frac{n}{\sqrt{2}^{\sqrt{n}}} = 0$, to prawa strona dąży do 0 ($0^\infty = 0$, symbol oznaczony). Więc z twierdzenia o 3 ciągach szukana granica również dąży do 0.

$f_3 \preccurlyeq f_2$
$\lim_{n\to\infty} \frac{f_3(n)}{f_2(n)} = \lim_{n\to\infty} \frac{n^{10}2^{n/2}}{2^n} = \lim_{n\to\infty} \frac{n^{10}}{2^{n/2}}$,

korzystając teraz 10 razy z reguły de l'Hospitala otrzymujemy

$\lim_{n\to\infty} \frac{f_3(n)}{f_2(n)} \lim_{n\to\infty} \frac{10!}{\ln^{10}(2)2^{n/2}\left(\frac{1}{2}\right)^{10}} = 0$

□