Computability

Assignment 1 Kacper Korban

1. Every effectively calculable function on the positive integers can be computed using a Turing machine.

That means: Every function defined in a sane way (that can be computed using some list of steps) can be computed using a Turing machine (an abstract computer architecture).

2. a) Injective but not surjective:

$$f = (+1) \quad \text{i.e.} \quad f\,x = x + 1$$

$$\neg \exists_{x \in \mathbb{N}}\, f(x) = 0 \ \wedge\ \forall_{x,y \in \mathbb{N}}.\ f(x) = f(y) \Rightarrow x = y$$

b) Bijective:

$$f = id \quad \text{i.e.} \quad f\,x = x$$

$$\forall_{x \in \mathbb{N}}.\ f(x) = x \ \wedge\ \forall_{x,y \in \mathbb{N}}.\ f(x) = f(y) \Rightarrow x = y$$

3. YES. Let's define $f \in \mathbb{N} \times Bool \to \mathbb{N}$, such as

$$f(n, true) = 2 \cdot n + 1$$

$$f(n, false) = 2 \cdot n$$

Proof: Let's take any $x, y \in \mathbb{N} \times Bool$ such that $f(x) = f(y)$ then we have $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $2 \cdot x_1 + s(x_2) = 2 \cdot y_1 + s(y_2)$. where $s(x) = 1$ if $x = true$ else 0. Since $x_1, y_1 \in \mathbb{N}$ and $s(x_2), s(y_2) \in \{0, 1\}$ then we have:

$$2 \cdot x_1 = 2 \cdot y_1 \ \wedge\ s(x_2) = s(y_2)$$

$$x_1 = y_1 \wedge x_2 = y_2 \Rightarrow (x_1, x_2) = (y_1, y_2) \quad \square$$

4. No.

Proof: Let's assume that $\mathbb{N} \to Bool$ is countable.

We know that there is at least one such function, so we have a surjection $f \in \mathbb{N} \to (\mathbb{N} \to Bool)$.

We define $g \in \mathbb{N} \to Bool$ as $g\, n = \neg f\, n\, n$. (1)

Because f is a surjection, we know that $g \overset{(2)}{=} f\, i$ for some i.

So:

$$f\, i\, i \overset{(2)}{=} g\, i \overset{(1)}{=} \neg f\, i\, i \quad ↯$$

We get a contradiction.

By proof by contradiction, we have that $\mathbb{N} \to Bool$ is not countable.