IEEE Stochastic Workshop Solutions

$(x-2)(x-2)$
x^2-4x+4

1. $$f_X(x|b=0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(\frac{1}{2}x^2-2x+2)}$$

$$f_X(x|b=1) = \frac{1}{\sqrt{2\pi}} e^{-(\frac{1}{2}x^2+2x+2)}$$

ML Rule

$$\frac{\frac{1}{\sqrt{2\pi}} e^{-(\frac{1}{2}x^2-2x+2)}}{\frac{1}{\sqrt{2\pi}} e^{-(\frac{1}{2}x^2+2x+2)}} \underset{B_1}{\overset{B_0}{\gtrless}} 1$$

$$e^{(-\cancel{\frac{1}{2}x^2}+2x-\cancel{2})+(\cancel{\frac{1}{2}x^2}+2x+\cancel{2})} \underset{B_1}{\overset{B_0}{\gtrless}} 1$$

$$e^{4x} \underset{B_1}{\overset{B_0}{\gtrless}} 1$$

$$x \underset{B_1}{\overset{B_0}{\gtrless}} 0 \Longleftarrow 4x \underset{B_1}{\overset{B_0}{\gtrless}} \ln(1)$$

Decide B_0 if $x \geq 0$
otherwise decide B_1 $\therefore$ ML Rule

MAP Rule

$P(0)=0.3$, $P(1)=0.7$

$$e^{4x} \underset{B_1}{\overset{B_0}{\gtrless}} \frac{0.7}{0.3} \rightarrow x \underset{B_1}{\overset{B_0}{\gtrless}} \frac{\ln(2.333)}{4}$$

MAP Rule

Threshold

1. $x \underset{B_1}{\overset{B_0}{\gtrless}} 0.211824\ldots$

Decide B_0 if $x > .211824$, otherwise decide B_1

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2. $f_X(x|H_0) = \frac{1}{\sqrt{\frac{\pi}{2}}} e^{-2x^2}$

$f_Y(y|H_0) = \frac{1}{\sqrt{\frac{\pi}{2}}} e^{-2y^2}$

$f_{XY}(x,y|H_0) = \frac{1}{\sqrt{\frac{\pi}{2}}} \frac{1}{\sqrt{\frac{\pi}{2}}} e^{-2(x^2+y^2)}$

$f_{XY}(x,y|H_0) = \left(\frac{2}{\pi}\right) \cdot e^{-2(x^2+y^2)}$

$f_{XY}(x,y|H_0) = f_X(x|H_0) \cdot f_Y(y|H_0)$

b/c Independent

$f_{XY}(x,y|H_1) = \frac{1}{\pi} e^{-2(x^2-2pxy+y^2)}$

$P(H_1) = .40$

$P(H_0) = 1 - .40 = .60$

MAP Rule

$$\frac{\frac{1}{\pi} e^{-2(x^2-2pxy+y^2)}}{\frac{2}{\pi} \cdot e^{-2(x^2+y^2)}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{.60}{.40}$$

$$\frac{1}{2} e^{4pxy} \underset{H_0}{\overset{H_1}{\gtrless}} 1.5$$

2. $\frac{1}{2} e^{4\rho xy} \underset{H_0}{\overset{H_1}{\gtrless}} 3\frac{3}{8}$

a) $\sqrt{\frac{3}{4}}\, xy \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln(3)}{4}$

$\sqrt{\frac{3}{4}}\, xy \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln(3)}{4} \Bigg| = xy \underset{H_0}{\overset{H_1}{\gtrless}} 0.317142$

MAP Rule

If $xy > 0.317142$ decide H_1, otherwise decide H_0

2. b) ML Rule

$f_{xy}(xy) = f_x(x) \cdot f_y(y)$

Priors Don't Matter since ML, so ~~P(H₁) + P(H₀)~~

$\rho_{xy} = 0$ means SI

$f_{xy}(x,y|H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$

$f_{xy}(xy|H_0) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$

$f_{xy}(xy|H_1) = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}(x^2)} \cdot \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}y^2}$

$f_{xy}(xy|H_1) = \frac{1}{4\pi} e^{-\frac{1}{4}(x^2+y^2)}$

2. ML Rule

Likelihood Ratio. ML

$$\frac{\frac{1}{4\pi}e^{-\frac{1}{4}(x^2+y^2)}}{\frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\frac{1}{2}e^{-\frac{1}{4}(x^2+y^2)+\frac{1}{2}x^2+y^2} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \quad \Big| \quad e^{\frac{1}{4}(x^2+y^2)} \underset{H_0}{\overset{H_1}{\gtrless}} 2$$

$$(x^2+y^2) \underset{H_0}{\overset{H_1}{\gtrless}} 4\ln(2) \approx 2.77259$$

ML Rule

If $(x^2+y^2) > 2.77259$ decide H_1, otherwise decide H_0

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3. $$\hat{Y}_{MMSE} = E[Y|X] = E[Y] + \frac{Cov(X,Y)}{Var(X)}(X - E[X])$$

a) $$E[Y|X] = 0 + \frac{-.125}{1}(x-7) = -\frac{1}{8}x + 0.875$$

This is Linear So

b) $$\rho_{xy} = \frac{-.125}{\sqrt{1 \cdot \frac{1}{2}}} = -0.176777$$

$$MSE = \frac{1}{2}(1-(-0.176777)^2) = 0.484375$$

$$f_{XY}(xy) = \begin{cases} 2, & 0 \le x \le 1 \\ 0, & 0 \le y \le x \end{cases}$$

4.

a) $f_{Y|X}(y|x) = \dfrac{f_{XY}(xy)}{f_X(x)} = \dfrac{2}{2x} = \dfrac{1}{x}, \; 0 \le x \le 1$

$$f_X(x) = \int_0^x 2\,dy = 2x, \quad 0 \le x \le 1$$

$$E[Y|X=x] = \int_0^x y \cdot \frac{1}{x}\,dy = \frac{x}{2} = E[Y|X]$$

b) $E[\frac{x}{2}] = \frac{1}{2}E[x] = \frac{1}{2}\int_0^1 x \cdot 2x\,dx = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

c) Since $MMSE = E[Y|X] = \frac{x}{2}$

d) $MSE = E[(Y - E[Y|X])^2]$

$MSE = E[Y^2] - E[(E[Y|X])^2]$

$E[Y|X] = \dfrac{x}{2}$

$$E[X^2] = \int_0^1 x^2 \cdot 2x\,dx = \frac{1}{2}$$

so

$$E[(E[Y|X])^2] = \frac{1}{4}E[X^2] = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

$$E[Y^2] = \int_0^x y^2\,dy = \frac{x^3}{3}$$

$$= \int_0^1 2 \cdot \frac{x^3}{3}\,dx = \frac{1}{6}$$

$$MSE = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

5. $x = [13, 8, 6, 9]$ $\quad \bar{x} = \frac{13+8+6+9}{4} = 9$

$y = [20, 3, 11, 10]$ $\quad \bar{y} = \frac{20+3+11+10}{4} = 11$ } mean

$S = \begin{bmatrix} cov(x,x) & cov(x,y) \\ cov(y,x) & cov(y,y) \end{bmatrix}$ | $S = \begin{bmatrix} 8.667 & 14.667 \\ 14.667 & 48.667 \end{bmatrix}$

$cov(x,x) = \frac{1}{4-1}\left((13-9)^2 + (8-9)^2 + (6-9)^2 + (9-9)^2\right) = 8.6667$

$cov(x,y) = cov(y,x) = \frac{1}{4-1}\left((13-9)(20-11) + (8-9)(3-11) + (6-9)(11-11) + (9-9)(10-11)\right)$

$= 36 + 8 + 0 + 0 = \frac{44}{3}$ | $= 14.667$

$cov(y,y) = \frac{1}{4-1}\left((20-11)^2 + (3-11)^2 + (11-11)^2 + (10-11)^2\right) = \frac{146}{3} = 48.667$

det = ad − bc

$0 = \det(S - \lambda I)$

$0 = \det\left(\begin{bmatrix} 8.667 & 14.667 \\ 14.667 & 48.667 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{pmatrix} 8.667-\lambda & 14.667 \\ 14.667 & 48.667-\lambda \end{pmatrix}$

$0 = (8.667-\lambda)(48.667-\lambda) - (14.667)^2$

5. $0 = (8.667-\lambda)(48.667-\lambda)-(215.121)$

$\lambda_2 = 3.86537$

$\lambda_1 = 53.4686$

$(S-\lambda I) = \begin{bmatrix} 8.667-\lambda & 14.667 \\ 14.667 & 48.667-\lambda \end{bmatrix}$

$(S-\lambda_1 I)u = 0 \quad , u = (u_x, u_y)$

$(8.667-\lambda_1)u_x + 14.667u_y = 0$

$u_x = 0.327377u_y$

Temporary value to find second

$\tilde{u} = (u_x, u_y) = (0.3273, 1)$

$||\tilde{u}|| = \sqrt{0.3273^2+1^2} = 1.0522$

(First PCA unit vector) $= u_1 = \left(\frac{0.3273}{1.0522}, \frac{1}{1.0522}\right) = (0.311063, 0.95039) = u_1$

$(S-\lambda_2 I)u = 0, \; u = (u_x, u_x)$

$(14.667)u_x + (48.667-\lambda_2)u_y = 0$

$u_x = -3.05459u_y$

5. $\tilde{u} = (u_x, u_y) = (-3.05459, 1)$ temporary value

$||\hat{u}|| = \sqrt{-3.05459^2 + 1^2} = 3.21411$

$u_2 = \frac{\tilde{u}}{||\tilde{u}||} = \left(\frac{-3.05459}{3.21411}, \frac{1}{3.21411}\right)$

(second PCA unit vector) $u_2 = (-0.950368, 0.311128)$

$-u_2 = (0.950368, -0.311128)$

sign doesn't matter same axis

Mean centered Points

$(13-9, 20-11) = (4, 9)$, $(8-9, 3-11) = (-1, -8)$

$(6-9, 11-11) = (-3, 0)$, $(9-9, 10-11) = (0, -1)$