Greedy Pebbling: Towards Proof Space Compression

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Abstract. Bruno

1 Introduction

Bruno

Propositional Resolution proofs usually are huge. So huge that when processing such proofs memory limits are reached and exceeded. However not the whole proof needs to be kept in memory. Proof nodes that are not used further on can be removed from memory. Information when to remove which node can be added as extra lines in the proof output. The maximum number of proof nodes that have to be kept in memory at once using deletion information is called the space measure. This measure is closely related to the Black Pebbling Game [7, 5]. A strategy for this game corresponds to a topological ordering of the proof nodes plus deletion information. Finding an optimal strategy for the Black Pebbling Game is PSPACE-complete [5] and therefore not a feasible approach to compress big proofs. This paper investigates heuristic approaches to the problem.

The motivation of this work is to compress propositional resolution proofs w.r.t. the space measure. However the results can easily be transferred to more general kinds of DAGs.

2 Propositional Resolution Calculus

A *literal* is a propositional variable or the negation of a propositional variable. The *complement* of a literal ℓ is denoted $\overline{\ell}$ (i.e. for any propositional variable p, $\overline{p} = \neg p$ and $\overline{\neg p} = p$). The set of all literals is denoted $\mathcal{L}$. A *clause* is a set of literals. $\bot$ denotes the *empty clause.*

Definition 1 (Proof). *A directed acyclic graph $\langle V, E, \Gamma \rangle$, where V is a set of nodes and E is a set of edges labeled by literals (i.e. $E \subset V \times \mathcal{L} \times V$ and $v_1 \xrightarrow{\ell} v_2$ denotes an edge from node v_1 to node v_2 labeled by ℓ), is a proof of a clause Γ iff it is inductively constructible according to the following cases:*

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1. *If Γ is a clause, $\widehat{\Gamma}$ denotes some proof $\langle \{v\}, \emptyset, \Gamma \rangle$, where v is a new node.*
2. *If ψ_L is a proof $\langle V_L, E_L, \Gamma_L \rangle$ and ψ_R is a proof $\langle V_R, E_R, \Gamma_R \rangle$ and ℓ is a literal such that $\overline{\ell} \in \Gamma_L$ and $\ell \in \Gamma_R$, then $\psi_L \odot_\ell \psi_R$ denotes a proof $\langle V, E, \Gamma \rangle$ s.t.*

$$V = V_L \cup V_R \cup \{v\}$$
$$E = E_L \cup E_R \cup \left\{ v \xrightarrow{\overline{\ell}} \rho(\psi_L), v \xrightarrow{\ell} \rho(\psi_R) \right\}$$
$$\Gamma = \left(\Gamma_L \setminus \{\overline{\ell}\}\right) \cup \left(\Gamma_R \setminus \{\ell\}\right)$$

where v is a new node and $\rho(\varphi)$ denotes the root node of φ. □

If $\psi = \varphi_L \odot_\ell \varphi_R$, then φ_L and φ_R are *direct subproofs* of ψ and ψ is a *child* of φ_L and φ_R. The transitive closure of the direct subproof relation is the *subproof* relation. A subproof which has no direct subproof is an *axiom* of the proof. V_ψ, E_ψ, A_ψ and Γ_ψ denote, respectively, the nodes, edges, axioms and proved clause (conclusion) of ψ. P_v^ψ denotes the premises and C_v^ψ the children of a node v in a proof ψ. When a proof is represented graphically, the root is drawn at the bottom and the axioms at the top.

3 Pebbling Game

Pebbling games were introduced in the 1970's to model programming language expressivity [9, 12] and compiler construction [11]. More recently, pebbling games have been used to investigate various questions in parallel complexity [2] and proof complexity [1, 4, 8]. They are used to get bounds for space and time requirements and tradeoffs between the two measures [3]. The *pebbling game* from definition 2 is a slight variation of the Black Pebbling Game presented in [6, 10]. *To pebble* a node is to put a pebble on it; *to unpebble* is to remove a pebble; a node is *pebbleable* if it is not pebbled but can be pebbled.

ToDo: briefly explain the difference...

Definition 2 (Pebbling Game). *The* Pebbling Game *is played by one player on a DAG $G = (V, E)$ with one distinguished node s. The goal of the game is to pebble s, respecting the following rules:*

1. *If all predecessors of a node v are pebbled, then v is pebbleable.*
2. *Nodes can be unpebbled at any time.*
3. *Each node can be pebbled only once.*

As a consequence of rule 1, pebbles can be put on nodes without predecessors at any time. A pebbling strategy *for G and node s is a sequence of moves in the pebbling game, where the last move pebbles s. The* pebble number of a pebbling strategy *is the maximum number of pebbles that are placed on nodes simultaneously, following the moves of the strategy. The* pebble number of a graph G and node s *is the minimum pebble number of all pebble strategies, for the pebbling game played on G and s.* □

The Black Pebbling Game defined in [6, 10] introduces another rule according to which a pebble can be moved from a predecessor to the node instead of using a fresh one. Including this rule results in strategies that use exactly one pebble less, as shown in [3]. Omitting rule 3 allows pebbling strategies with lower pebbling numbers ([11] has an example on page 1). However, this possibly has a cost of exponentially more moves in the game [3]. Deciding the question whether the pebbling number of a graph G and node s is smaller than k is PSPACE-complete in the absence of rule 3 [5] and NP-complete when rule 3 is included [11].

4 Pebbling and Proof Checking

The problem of checking the correctness of a proof while minimizing the memory consumption is analogous to the problem of playing the pebbling game on the proof while minimizing the number of pebbles. Checking the correctness of a node and storing it in memory corresponds to pebbling it. In order to check the correctness of a node v, its premises must have been checked before and must still be stored in memory (rule 1 of the pebbling game). A node that has already been checked can be removed from memory at any time (rule 2). The correctness of a node should be checked only once (rule 3).

Proof files generated by sat- and smt-solvers usually already list the nodes of a proof in a topological order. Even if this were not the case, it is simple though memory-consuming to generate a topological order by traversing the proof once.

Definition 3 (Topological Order). *A topological order of a proof ψ is a total order relation $<_T$ on V_ψ, such that for all $v \in V_\psi$, for all $p \in P^\psi_v, p <_T v$* □

A topological order $<_T$ can be represented by a sequence $(v_1, \dots, v_n)$ of proof nodes, by defining $<_T := \{(v_i, v_j) \mid 1 \le i < j \le n\}$. This sequence can be interpreted as a particular pebbling strategy that pebbles nodes according to the topological order and unpebbles a node v soon after all its last child is pebbled. This is formally defined below.

Definition 4 (Canonical Topological Pebbling Strategy). *The* canonical topological pebbling strategy *$S(\psi, \rho(\psi), <_T)$ for a DAG ψ and node $\rho(\psi)$ w.r.t. a topological order $<_T$ represented as a sequence $(v_1, \dots, v_n)$ is defined recursively:*

$$S(\psi, \rho(\psi), t) = \begin{cases} () & \text{, if } t = () \\ pebble(v) :: (U(v, \psi, t) ::: S(\psi, \rho(\psi), r) & \text{, if } t = v :: r \end{cases}$$

$$U(v, \psi, t) = (unpebble(v_p) \mid v_p \in P^\psi_v \text{ and } v_c \le_T v \text{ forall } v_c \in C^\psi_{v_p})$$

where :: is the cons *list constructor and ::: is the list concatenation operator.* □

Theorem 1. *$S(\psi, \rho(\psi), <_T)$ has the minimum pebbling number among all pebbling strategies that pebble nodes according to the topological order $<_T$.*

Proof. (Sketch) All the pebbling strategies respecting $<_T$ differ only w.r.t. their unpebbling moves. Consider the unpebbling of an arbitrary node v in the canonical strategy $S(\psi, \rho(\psi), <_T)$. Unpebbling it later could only possibly increase the pebble number. To reduce the pebble number, v would have to be unpebbled earlier than some preceding pebbling move. But, by definition of canonical strategy, the immediately preceding pebbling move pebbles the last child of v. Therefore, unpebbling v earlier would make it impossible for its last child to be pebbled later without violating the rules of the game. □

Theorem 1 shows that, in the version of the pebbling game considered here, the problem of finding a strategy with a low pebble number can be reduced to the problem of finding a topological order whose canonical strategy has a low pebble number. Unpebbling moves can be omitted, because the optimal moment to unpebble a node is immediately after its last child has been pebbled.

Assuming that nodes are approximately of the same size, the maximum memory needed to check a proof is proportional to the maximum number of nodes that have to be kept in memory while checking the proof according to a given topological order. Thus memory consumption can be estimated by the following measure:

Definition 5 (Space). *The* space $s(\psi, <_T)$ *of a proof* ψ *and a topological order* $<_T$ *is the pebbling number of the canonical topological pebbling strategy* $S(\psi, \rho(\psi), <_T)$. □

The problem of compressing the space of a proof ψ and a topological order $<_T$ is the problem of finding another topological order $<'_T$ such that $s(\psi, <'_T) < s(\psi, <_T)$. The following theorem shows that the number of possible topological orders is very large; hence, enumeration is not a feasible option when trying to find a good topological order.

Theorem 2. *There is a sequence of proofs* $(\psi_1, \ldots, \psi_m, \ldots)$ *such that* $\mathit{length}(\psi_m) \in O(m)$ *and* $|T(\psi_m)| \in \Omega(m!)$, *where* $T(\psi_m)$ *is the set of possible topological orders for* ψ_m.

Use v instead of s for nodes here

Proof. Let ψ_m be a perfect binary tree with m axioms. Clearly, $\mathit{length}(\psi_m) = 2m - 1$. Let $(s_1, \ldots, s_n)$ be a topological order for ψ_m. Let $A_\psi = \{s_{k_1}, \ldots, s_{k_m}\}$, then $(s_{k_1}, \ldots, s_{k_m}, s_{l_1}, \ldots, s_{l_{n-m}})$, where $(l_1, \ldots, l_{n-m}) = (1, \ldots, n) \setminus (k_1, \ldots, k_m)$, is a topological order as well. Likewise, $(s_{\pi(k_1)}, \ldots, s_{\pi(k_m)}, s_{l_1}, \ldots, s_{l_{n-m}})$ is a topological order, for every permutation π of $\{k_1, \ldots, k_m\}$. There are $m!$ such permutations, so the overall number of topological orders is at least factorial in m (and also in n).

5 Pebbling as a Satisfiability Problem

To find the pebble number of a proof, the question whether this proof can be pebbled using no more than k pebbles, can be encoded as a propositional satisfiability problem. Let ψ be a proof with nodes $v_1, \ldots, v_n$ and let $v_n = \rho(\psi)$.

Due to rule 3 of the pebbling game, the number of pebbling moves is exactly n. For every $x \in \{1, \dots, k\}$, every $j \in \{1, \dots, n\}$ and every $t \in \{1, \dots, n\}$ there is a propositional variable $p_{x,j,t}$, denoting if that pebble x is on node v_j at the time of the pebbling move t. The following constraints, combined conjunctively, are satisfiable *iff* there is a pebbling strategy for ψ, using at most k pebbles. If satisfiable, a pebbling strategy can be read off from any satisfying assignment.

1. The root is pebbled at the time of the last move

$$\bigvee_{x=1}^{k} p_{x,n,n}$$

2. At most one node is pebbled initially

$$\bigwedge_{x=1}^{k} \bigwedge_{j=1}^{n} \left(p_{x,j,1} \rightarrow \bigwedge_{y=1, y\neq x}^{k} \bigwedge_{i=1}^{n} p_{y,i,n} \right)$$

3. At least one axiom is pebbled initially

$$\bigvee_{x=1}^{k} \bigvee_{j \in A_\psi} p_{x,j,1}$$

4. A pebble can only be on one node

$$\bigwedge_{x=1}^{k} \bigwedge_{j=1}^{n} \bigwedge_{t=1}^{n} \left(p_{x,j,t} \rightarrow \bigwedge_{i=1, i\neq j}^{n} \neg p_{x,i,t} \right)$$

5. For pebbling a node, its premises have to be pebbled and only one node is pebbled each move

$$\bigwedge_{x=1}^{k} \bigwedge_{j=1}^{n} \bigwedge_{t=1}^{n} \left((\neg p_{x,j,t} \wedge p_{x,j,(t+1)}) \rightarrow \left(\bigwedge_{i \in P_j^\psi} \bigvee_{y=1, y\neq x}^{k} p_{y,i,t} \right) \wedge \left(\bigwedge_{i=1}^{n} \bigwedge_{y=1, y\neq x}^{k} \neg(\neg p_{y,i,t} \wedge p_{y,i,(t+1)}) \right) \right)$$

The sets A_ψ and P_j^ψ are interpreted as sets of indices of the respective nodes. This encoding is polynomial, both in n and k. However constraint 5 accounts to $O(n^3 * k^2)$ clauses. Even small resolution proofs have more than 1000 nodes and pebble numbers bigger than 100, which adds up to 10^{13} clauses for constraint 5 alone. Therefore, although theoretically possible to play the pebbling game or compress proof space via sat-solving, this is practically infeasible.

6 Greedy Pebbling Algorithms

Theorem 2 and the remarks in the end of section 5 indicate that obtaining an optimal topological order either by enumerating topological orders or by encoding the problem as a satisfiability problem is impractical. This section presents two greedy algorithms that aim at finding a better though not necessarily optimal topological order. They are both parameterized by the same heuristics described in Section 7, but differ from each other in the traversal direction in which the algorithms operate on proofs.

6.1 Top-Down Pebbling

Top-Down Pebbling (Algorithm 1) constructs a topological order of a proof ψ by traversing it from the axioms to the root node $\rho(\psi)$. This approach closely corresponds to how a human would play the pebbling game. A human would look at the nodes that are available for pebbling at a given state, choose one of them to pebble and remove pebbles if possible. Similarly the algorithm keeps track of pebblable nodes in a set P, initialized as A_ψ. When a node v is pebbled, it is removed from P and added to the sequence representing the topological order. The children of v that become pebbleable are added to P. When P becomes empty, all nodes have been pebbled once and a topological order has been found.

```
   Input: a proof $\psi$
   Output: a topological order $<_T$ of $\psi$ represented by a sequence of nodes
1  $S = ()$; // the empty sequence
2  $P = A_\psi$;
3  while $P$ is not empty do
4      choose $v \in P$ heuristically;
5      for each $c \in C_v^\psi$ do
6          if $\forall p \in P_c^\psi : p \in S$ then
7              $P = P \cup \{c\}$;
8      $P = P \setminus \{v\}$;
9      $S = S ::: (v)$; // ::: is the concatenation of sequences
10 return $S$;
```

Algorithm 1: `Top-Down Pebbling`

Example 1. It is easy to see how top-down pebbling may end up finding a sup-optimal pebbling strategy. Consider the graph shown in figure 3 and suppose that top-down pebbling has already pebbled the initial sequence of nodes $(1, 2, 3)$. For a greedy heuristic that only has information about pebbled nodes, their premises and children, all nodes marked with '4?' are considered equally worthy to pebble next. Suppose the node marked with '4' in the middle graph is chosen and pebbled next. Subsequently, pebbling '5' opens up the possibility to remove a pebble in the next move, which is done by pebbling '6'. After that only '7' and '8' are pebbleable. In this situation, it does not matter which is pebbled first. After pebbling '7' and '8', four pebbles are used, which is one more than what an optimal strategy needs.

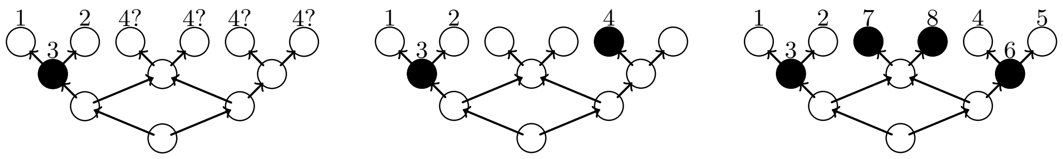

Fig. 1: Top-Down Pebbling

6.2 Bottom-Up Pebbling

Bottom-Up Pebbling (Algorithms 2 and 3) constructs a topological order of a proof ψ while traversing it from its root node $\rho(\psi)$ to its axioms. The algorithm constructs the order by visiting nodes and their premises recursively. At every node v the order in which the premises of v are visited is decided heuristically. After visiting the premises, n is added to the current sequence of nodes. Since axioms do not have any premises, there is no recursive call for axioms and these nodes are simply added to the sequence. The recursion is started by a call to visit the root. Since all proof nodes are ancestors of the root, the recursive calls will eventually visit all nodes once and a topological total order will be found.

```
   Input: a proof $\psi$
   Output: a topological order $<_T$ of $\psi$ represented by a sequence of nodes
1  $S = ()$; // the empty sequence
2  $V = \emptyset$;
3  return visit($\psi$,$\rho(\psi)$,$V$,$S$);
```

Algorithm 2: `Bottom-Up Pebbling`

```
   Input: a proof $\psi$
   Input: a node $v$
   Input: a set of visited nodes $V$
   Input: initial sequence of nodes $S$
   Output: a sequence of nodes
1  $V_1 = V \cup \{v\}$;
2  $D = P_v^\psi \setminus V$;
3  $S_1 = S$
4  while $D$ is not empty do
5      choose $p \in D$ heuristically;
6      $D = D \setminus p$;
7      $S_1 = S_1 ::: visit(\psi, p, V, S)$; // ::: is the concatenation of sequences
8  return $S_1 ::: (v)$;
```

Algorithm 3: `visit`

Example 2. Figure 2 shows part of an execution of Bottom-Up Pebbling on the same graph of Figure 3. Nodes chosen by the heuristic, during the bottom-up traversal, to be processed before the respective other premise are marked in gray. Similarly to the Top-down Pebbling scenario, nodes have been chosen in such a way that the initial pebbling sequence is $(1, 2, 3)$. However, the choice of where to go next is predefined by the gray nodes. Consider the gray child of node '3'. Since '3' has been completely processed and pebbled, the other premise of its gray child is visited next. The result is that node '7' is pebbled earlier and at no point more than 3 pebbles will be used for pebbling the root node. This is so independently of the heuristic choices.

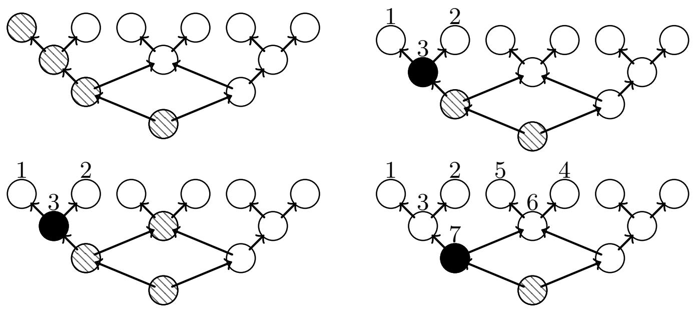

Fig. 2: Bottom-Up Pebbling

6.3 Remarks about Top-Down and Bottom-Up Pebbling

In principle every topological order of a given proof can be constructed using Top-down or Bottom-up Pebbling. Both algorithms traverse the proof only once and have linear run-time in the proof length (assuming that the heuristic choice requires constant time). Example 1 shows a situation where Top-Down Pebbling may pebble a node that is far away from the previously pebbled nodes. This results in a sub-optimal pebbling strategy. As discussed in Example 2, Bottom-Up Pebbling is more immune to this non-locality issue, because queuing up the processing of premises enforces local pebbling. This suggests that Bottom-Up is better than Top-Down, which is confirmed by the experiments in Section 8.

Theoretically, can we prove that BUP is always better than TDP if they both use the same heuristics? Experimentally, does it ever happen that TDP is better than BUP? We could make a scatter plot with compression ratios of TDP and BUP on the y and x axis to investigate this...

7 Heuristics

Pebbling heuristics for a proof ψ are defined by a function $h : V_\psi \to H$, where $(H, \prec)$ is a totally ordered set. Top-down-, as well as Bottom-Up-, Pebbling select one node v out of a set of nodes $N \subseteq V_\psi$ where $v = max_{n \in N} h(n)$ using the order $\prec$. For Top-down Pebbling, N is the set of pebbleable nodes and for Bottom-Up Pebbling, N is the set of premises of a node.

7.1 Number of Children Heuristic

This heuristic uses the number of children nodes of a node v the deciding characteristic, i.e. $h(v) = |C_v^{\psi}|$, $H = \mathbb{N}$ and $\prec$ is the natural smaller relation.
The motivation of this heuristic is that nodes with many children, will require many pebbles. Example 3 shows why it is a good idea to process hard subproofs first.

Example 3. Figure 3 shows a simple proof with two subproofs. The numbers in the subproofs denote the pebbling number, after it has been pebbled. The left, easy subproof needs 4 pebbles and the right, hard subproof requires 5 pebbles. After pebbling one of them, the pebble on its root node has to be kept there until the other is pebbled fully. This increases the pebbling number of the other subproof by one. So pebbling the easy subproof first increases the overall pebbling number to 6, while pebbling the hard one first leaves it at 5.
Note that this is a simplified situation with two independent subproofs, for which pebbling one does not influence the pebbling number of the other, which is not true if they share nodes.

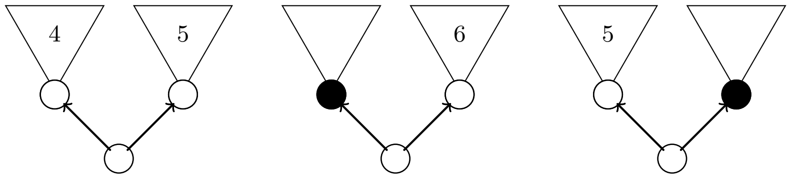

Fig. 3: Hard subproof first

7.2 Last Child Heuristic

In section 3 we explained the implicit unpebbling of a node v as soon as all of its children have been pebbled. More precisely, v can be unpebbled as soon as the its last child, w.r.t. a topological order $<_T$, is pebbled. The idea of this heuristic is to prefer nodes that are last children of other nodes.
Pebbling a node, which allows another one to be unpebbled, is always a good move. It does not increase the pebbling number, it might decrease the current number of pebbles used, if more than one premise can be unpebbled, and it possibly opens up new nodes for pebbling.
For determining the number of premises of which a node is the last children of, it first has to be traversed top-down once, using some topological order $<_T$. Before the traversal, $h(v)$ is set to 0 for every node v. During the traversal $h(v)$ is increased by 1, if v is the last child of the current node w.r.t. $<_T$. So for this heuristic, just like as the Number of Children Heuristic, $H = \mathbb{N}$ and $\prec$ is the natural smaller relation.

To some extent, this heuristic is a paradox, because pebbling the last child v of some node early, might results in v not being the last child anymore.
However some nodes are forced to be the last child of another node with more than one child by the structure of the proof, as shown in figure 4, where the bottom node is the last child of the top right node in every topological order.

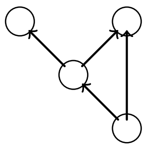

Fig. 4: Forced last child

7.3 Node Distance Heuristic

In section Section 6.3 the issue with non local pebbling was explained. The Node Distance Heuristic searches for pebbled nodes that are close to the decision node. It does this by calculating spheres with a limited radius around nodes. A sphere with radius r around the node v in the graph $G = (V, E)$ is defined in the following way:

$$K_r^G(v) =: \{p \in V \mid \text{ there are at most } r \text{ (undirected) edges between } p \text{ and } v\}$$

Using these spheres, the characteristic for this heuristic is defined as follows:

$$\begin{aligned} d(v) &:= -min\{r \mid K_r^G(v) \text{ contains a pebbled node}\} \\ s(v) &:= |K_{-d(v)}^G| \\ l(v) &:= max_{<_P} K_{-d(v)}^G \\ h(v) &:= (d(v), s(v), l(v)) \end{aligned}$$

where $<_P$ denotes the total order on the initial sequence of pebbled nodes P, i.e. nodes in spheres that were pebbled later are preferred. So $H = \mathbb{Z} \times \mathbb{N} \times P$ and the order used is the lexicographic order of two times the natural smaller relation and $<_P$. The spheres $K_r(v)$ can grow exponentially in r. Therefore the maximum sphere calculated has to be limited and if no pebbled node is found in any of the calculated spheres, another heuristic has to be used.

Copied from example: Pebbling '7' after '3' can be done with a heuristic that measures the number of edges between pebbled and pebbleable nodes. However for every d, one can easily construct a similar example in which '7' and '3' have $d + 1$ edges between them. Computing d-Spheres w.r.t. this measure can be exponential in d. Therefore only relatively small spheres can be used, in order not to slow down the whole process too much.

7.4 Decay Heuristics

Decay Heuristics denote a family of meta heuristics. The idea is to not only use the measure of a single node, but also to include the measures of its premises. Such a heuristic has four parameters: a heuristic function $h_u : V \rightarrow H$, a recursion depth $d \in \mathbb{N}$, a decay factor $\gamma \in \mathbb{R}^+ \cup \{0\}$ and a family of combining functions $com : H^n \rightarrow H$ for $n \in \mathbb{N}$.
The resulting heuristic function $h : V \rightarrow H$ is defined with the help of the recusive function $rec : (V \times \mathbb{N}) \rightarrow H$:

$$
\begin{aligned}
rec(v, 0) &:= h_u(v) \\
rec(v, k) &:= h_u(v) + com(rec(p_1, k-1), \ldots, rec(p_n, k-1)) * \gamma \qquad \text{where } P_v^\psi = \{p_1, \ldots, p_n\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } k \in \{1, \ldots, d\} \\
h(v) &:= rec(v, d)
\end{aligned}
$$

8 Experiments

All the `Pebbling Heuristics` mentioned in the last chapter, have been implemented in the functional programming language Scala[1] as part of the Skeptik library[2].

In order to evaluate the heuristics, experiments were run on two sets of test cases, consisting of proofs produced by the SMT-solver veriT[3] on unsatisfiable benchmarks from the SMT-Lib[4]. The details on the number of proofs per SMT category and their size in proof nodes are shown in Table 1. The proofs were translated into pure resolution proofs by considering every non-resolution inference as an axiom.

By default Skeptik stores proofs using a topological order that is found in a Bottom-up fashion, by visiting premises in the order they were parsed. The compression of space measures of heuristics is compared to this default topological order. Note that it is another Bottom-up heuristic, which is referred to as *uncompressed*.

The experiments were executed on the Vienna Scientific Cluster[5] VSC-2. Each algorithm was executed in a single core and had up to 16 GB of memory available. This amount of memory has been useful to compress the biggest proofs (with more than 10^6 nodes).

The overall results of the experiments are shown in Table 3 and **??**. The compression ratios are computed according to the formulas (1) and (2), in which ψ ranges over all the proofs in the corresponding set of test cases and $sp(\psi, heuristic)$ denotes the space measure of ψ w.r.t. the topological order computed by the respective heuristic.

[1] http://www.scala-lang.org/
[2] https://github.com/Paradoxika/Skeptik
[3] http://www.verit-solver.org/
[4] http://www.smtlib.org/
[5] http://vsc.ac.at/

Table 1: Proofs benchmarks and statistics

Benchmark Category	Set 1 Number of Proofs	Set 2 Number of Proofs
QF_UF	3936	199
QF_IDL	446	184
QF_LIA	587	450
QF_UFIDL	16	2
QF_UFLIA	123	78
QF_RDL	30	1
Sum	914	5138
Total number of proof nodes	4926946	441244454
Average number of proof nodes	85879	5391

Table 2: Total compression ratios

Heuristic	(1)	(2)
Children	-1 %	0,27 %
LastChild	0,7 %	7,8 %

$$\frac{\sum sp(\psi, uncompressed) - \sum sp(\psi, heuristic)}{\sum sp(\psi, uncompressed)} \tag{1}$$

$$\frac{\sum \frac{sp(\psi, uncompressed) - sp(\psi, heuristic)}{sp(\psi, uncompressed)}}{total\ number\ of\ proofs} \tag{2}$$

8.1 Test set 1

The following heuristics were evaluated using this test set:

Childen: the Bottom-Up version of the Number of Children Heuristic, see 7.1
LastChild: the Bottom-Up version of the Last Child Heuristic, see 7.2
ChildenTD: *experiments on the Top-down version of the Number of Children Heuristic are still running*
LastChildTD: *experiments on the Top-down version of the Last Child Heuristic are still running*

The results suggest, that smaller proofs benefit more from the heuristics. This claim is supported by Figure **??**, which compares the proof length in nodes to the achieved compression.

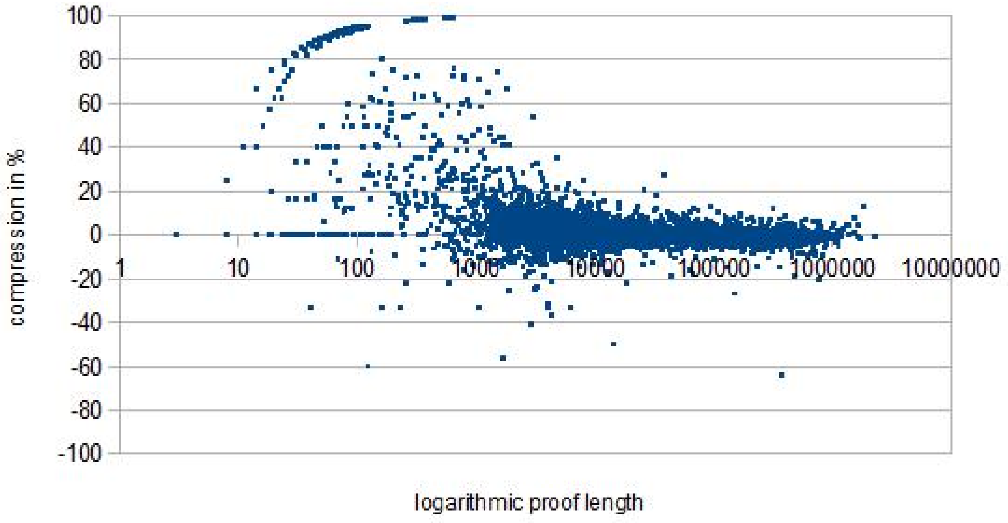

Fig. 5: Proof length compared to compression of Last Child Heuristic

8.2 Test set 2

The following heuristics were evaluated using this test set:

Childen: the Bottom-Up version of the Number of Children Heuristic (7.1)
LastChild: the Bottom-Up version of the Last Child Heuristic (7.2)
ChildenTD: the Top-down version of the Number of Children Heuristic
LastChildTD: the Top-down version of the Last Child Heuristic
Distance3: the Bottom-Up version of the Node Distance Heuristic with a maximum radius of 3
Distance3TD: the Top-down version of the Node Distance Heuristic with a maximum radius of 3
LCllmax: the Decay Heuristic, using **LastChild** as underlying heuristic, with $\gamma = 0.5$, $d = 1$ and $com = max(\dots)$
LChhmax: the Decay Heuristic, using **LastChild** as underlying heuristic, with $\gamma = 3$, $d = 7$ and $com = max(\dots)$
LCllavg: the Decay Heuristic, using **LastChild** as underlying heuristic, with $\gamma = 0.5$, $d = 1$ and $com = average(\dots)$
LChhavg: the Decay Heuristic, using **LastChild** as underlying heuristic, with $\gamma = 3$, $d = 7$ and $com = average(\dots)$

The results of this set imply, that Bottom-up heuristics produce significantly better topological orders than their Top-down pendants. Also just like **Test Set**

Table 3: Total compression ratios

Heuristic	Compression (1)	Compression (2)
Children	8,4 %	8,9 %
LastChild	25,1 %	36 %
ChildenTD	-15,8 %	0,7 %
LastChildTD	-16,7 %	5,1 %
Distance3	-0,7 %	-3,2 %
Distance3TD	-30,8 %	-37,6 %
LCllmax	25,4 %	36,6 %
LChhmax	26,6 %	37,4 %
LCllavg	25,5 %	36,7 %
LChhavg	25,7 %	34,4 %

1 the results presented in Table 3 show that small proofs benefit more from the heuristics.

We could evaluate the effect of RPI and LUV on the space of proofs

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9 Conclusions and Future Work

Bruno

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