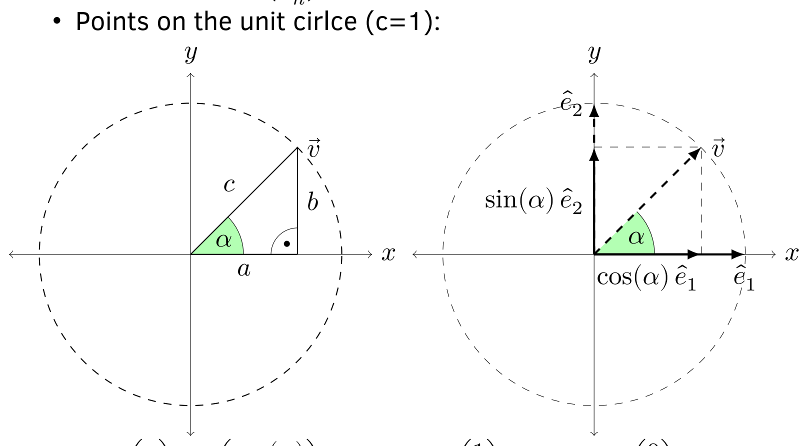
1 (Excursion) vector components as linear combinations of vectors

- Every vector can be written as a combination of scalars and unit vectors
- $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are canonical base vectors in R^2

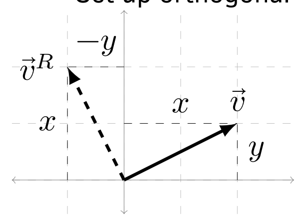
  For R^n: $\hat{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ith component being 1 all other 0
- Generally: $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i \cdot e_i$
- Points on the unit cirlce (c=1):



- $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} = \cos(\alpha) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\alpha) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos(\alpha) \cdot \hat{e}_1 + \sin(\alpha) \cdot \hat{e}_2$

2 (Excursion) Orthogonal complement of a vector ($90°$ Rotation CCW)

- Set up orthogonal vector to obtain up-vector for new coordinate frame



$$\vec{v}^R = \begin{pmatrix} x \\ y \end{pmatrix}^R = \begin{pmatrix} -y \\ x \end{pmatrix}$$

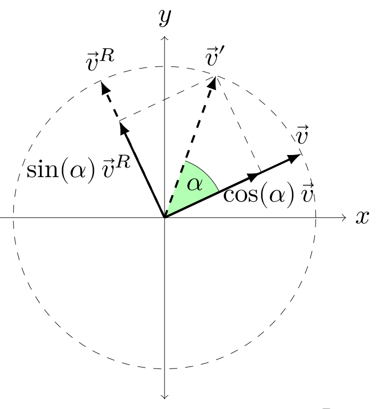
- Derive $\vec{v}^R$:

$$\begin{aligned} \vec{v}^R &= \begin{pmatrix} x \\ y \end{pmatrix}^R = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}^R = \begin{pmatrix} \cos(\alpha + 90°) \\ \sin(\alpha + 90°) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha)\cos(90°) - \sin(\alpha)\sin(90°) \\ \cos(\alpha)\sin(90°) + \sin(\alpha)\cos(90°) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha) \cdot 0 - \sin(\alpha) \cdot 1 \\ \cos(\alpha) \cdot 1 + \sin(\alpha) \cdot 0 \end{pmatrix} = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix} = \underline{\begin{pmatrix} -y \\ x \end{pmatrix}} \end{aligned}$$

- $\vec{v}^R$ is perpendicular to $\vec{v}$: $\begin{pmatrix} x \\ y \end{pmatrix} \bullet \begin{pmatrix} -y \\ x \end{pmatrix} = \underline{-xy + yx = 0}$

3 From 2D vector arithmetic to 2D rotation matrix

- Apply $\cos$ and $\sin$ to new coordinate frame with axes $\vec{v}$ and $\vec{v}^R$ to obtain rotated vector $\vec{v}'$



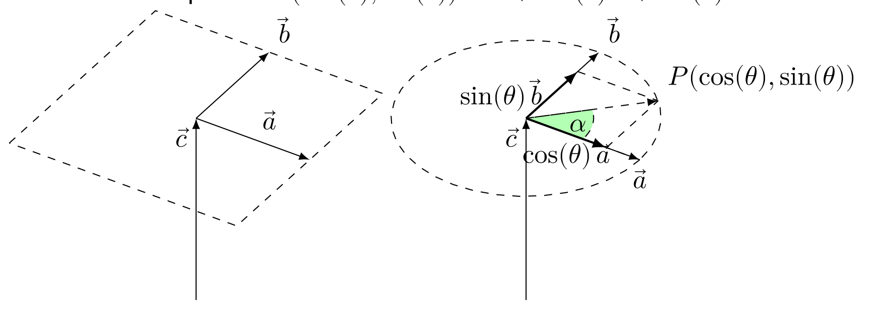
- $\vec{v}' = \cos(\alpha)\,\vec{v} + \sin(\alpha)\,\vec{v}^R$
- rearrange to matrix arithmetic to apply transformation to arbitrary vector

$$\begin{aligned} v' &= \cos(\alpha) \begin{pmatrix} x \\ y \end{pmatrix} + \sin(\alpha) \begin{pmatrix} -y \\ x \end{pmatrix} &&= \begin{pmatrix} \cos(\alpha)\,x \\ \cos(\alpha)\,y \end{pmatrix} + \begin{pmatrix} -\sin(\alpha)\,y \\ \sin(\alpha)\,x \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha)\,x - \sin(\alpha)\,y \\ \sin(\alpha)\,x + \cos(\alpha)\,y \end{pmatrix} &&= \underline{\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

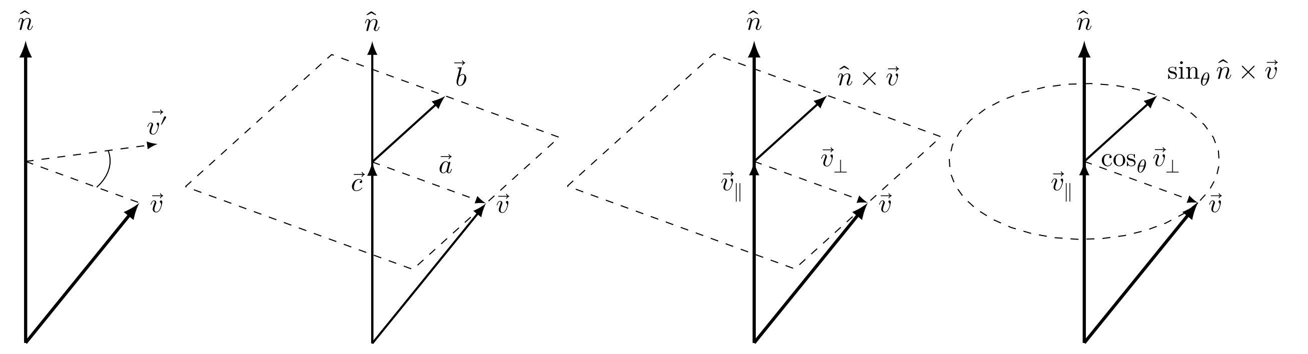
4 3D Angle-Axis Rotation(Rodrigues' rotation formula)

- Set up explicit plane equation placed orthogonal to rotation axis(normal) and rotate vector along plane.

- Points (x,y) on plane P (R^3):
  $P(x, y) = \vec{c} + x\,\vec{a} + y\,\vec{b}$ with $\vec{a} \bullet \vec{b} = 0$
  Circle on plane: $P(\cos(\theta), \sin(\theta)) = \vec{c} + \cos(\theta)\,\vec{a} + \sin(\theta)\,\vec{b}$



- With $\hat{n}$ rotation axis and normal to the plane of rotation
  (the rotation axis vector $\hat{n}$ needs to be normalized, because of the projection operation)
- $\vec{v}$ vector to be rotated around axis vector $\hat{n}$
- Construct plane of rotation with rejection(projection) of $\vec{v}$ onto $\hat{n}$ and cross product of $\hat{n}$ with $\hat{v}$



- $\vec{v} = \vec{v_\perp} + \vec{v_\parallel}$
- $\vec{v}_\parallel$ part of $\vec{v}$ parallel to $\hat{n}$ (Projection of $\vec{v}$ onto $\hat{n}$)
  $\vec{c} \to \vec{v_\parallel} = (\hat{n} \bullet \vec{v})\hat{n}$
- $\vec{v}_\perp$ part of $\vec{v}$ perpendicular to $\hat{n}$
  $\vec{a} \to \vec{v_\perp} = \vec{v} - \vec{v_\parallel} = \vec{v} - (\hat{n} \bullet \vec{v})\hat{n}$
- vector in plane perpendicular to $\vec{v}$ and $\hat{n}$
  $\vec{b} \to \hat{n} \times \vec{v} = \hat{n} \times \vec{v}_\perp$

- $\vec{v'} = \vec{v}_\parallel + \cos_\theta\,\vec{v}_\perp + \sin_\theta\,\hat{n} \times \vec{v}$

4.1 Derivation of Rodrigues Rotation Matrix:

- Express vector arithmetic terms with matrix arthimic equivalents:
  (Generally you can convert any vector arithmetic terms by transforming the base vectors of a space or the column vectors of a matrix. In case of the projection of a vector we can rearrange the operation to get it directly)

- $\mathbf{K}$ is skew symmetric matrix which can used calculate the cross product of $\hat{n}$ with $\vec{v}$:
  $\mathbf{K}\,\vec{v} = \hat{n} \times \vec{v} = [\hat{n}]_\times\,\vec{v}$

$$\mathbf{K} = [\hat{n}]_\times = \begin{bmatrix} \hat{n} \times \hat{e}_1 & \hat{n} \times \hat{e}_2 & \hat{n} \times \hat{e}_3 \end{bmatrix}$$

$$= \left[\begin{pmatrix} n_y \cdot 0 - n_z \cdot 0 \\ n_z \cdot 1 - n_x \cdot 0 \\ n_x \cdot 0 - n_y \cdot 1 \end{pmatrix} \begin{pmatrix} n_y \cdot 0 - n_z \cdot 1 \\ n_z \cdot 0 - n_x \cdot 0 \\ n_x \cdot 1 - n_y \cdot 0 \end{pmatrix} \begin{pmatrix} n_y \cdot 1 - n_z \cdot 0 \\ n_z \cdot 0 - n_x \cdot 1 \\ n_x \cdot 0 - n_y \cdot 0 \end{pmatrix} \right] = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$

- $\mathbf{P}$ is the matrix which multiplied with projects a vector $\vec{v}$ onto normal $\hat{n}$:
  $\mathbf{P}\,\vec{v} = (\vec{v} \bullet \hat{n})\,\hat{n} = (\hat{n} \bullet \vec{v})\hat{n} = (\hat{n}^T\vec{v})\hat{n} = \hat{n}\hat{n}^T\vec{v}$

$$\begin{array}{ccc} & & \begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \\ \mathbf{P} = \hat{n}\hat{n}^\mathsf{T} = & \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} & \begin{bmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{bmatrix} \end{array}$$

- rearrange vector arithmetic rotation formula to matrix arithmetic (and factorize $\vec{v}$)

$$\begin{aligned} \vec{v'} &= \vec{v}_\parallel + \cos(\theta)\,\vec{v}_\perp + \sin(\theta)\,\hat{n} \times \vec{v} && \mid \text{ def. } \vec{v}_\parallel, \vec{v}_\perp \\ &= (\vec{v} \bullet \hat{n})\hat{n} + \cos(\theta)\,(\vec{v} - (\vec{v} \bullet \hat{n})\hat{n}) + \sin(\theta)\,\hat{n} \times \vec{v} && \mid (\vec{n} \bullet \hat{n})\hat{n} = (\hat{n}^T\hat{v})\hat{n} = \hat{n}\hat{n}^T\vec{v} \\ &= \hat{n}\hat{n}^T\vec{v} + \cos(\theta)\,(\vec{v} - \hat{n}\hat{n}^T\vec{v}) + \sin(\theta)\,[\hat{n}]_\times\vec{v} && \mid \mathbf{P} = \hat{n}\hat{n}^T, \mathbf{K} = [\hat{n}]_\times, \mathbf{I}\,\vec{v} = \vec{v} \\ &= \mathbf{P}\,\vec{v} + \cos(\theta)\,(\mathbf{I}\,\vec{v} - \mathbf{P}\,\vec{v}) + \sin(\theta)\,\mathbf{K}\,\vec{v} && \mid \text{ factorize } \vec{v} : \mathbf{A}\vec{v} + \mathbf{B}\vec{v} = [\mathbf{A} + \mathbf{B}]\vec{v} \\ &= [\,\mathbf{P} + \cos(\theta)\,(\mathbf{I} - \mathbf{P}) + \sin(\theta)\,\mathbf{K}\,]\,\vec{v} \\ &= \mathbf{R}(\hat{n}, \theta)\,\vec{v} \end{aligned}$$

- Matrix for rotating arbitrary vector around axis $\hat{n}$ with angle θ:
  $\mathbf{R}(\hat{n}, \theta) = \mathbf{P} + \cos(\theta)\,(\mathbf{I} - \mathbf{P}) + \sin(\theta)\,\mathbf{K}$

$$\mathbf{R}(\hat{n}, \theta) = \mathbf{P} + \cos(\theta) \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{bmatrix} \right) + \sin(\theta) \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$