

# 1 (Excursion) vector components as linear combinations of vectors

- Every vector can be written as a combination of scalars and unit vectors

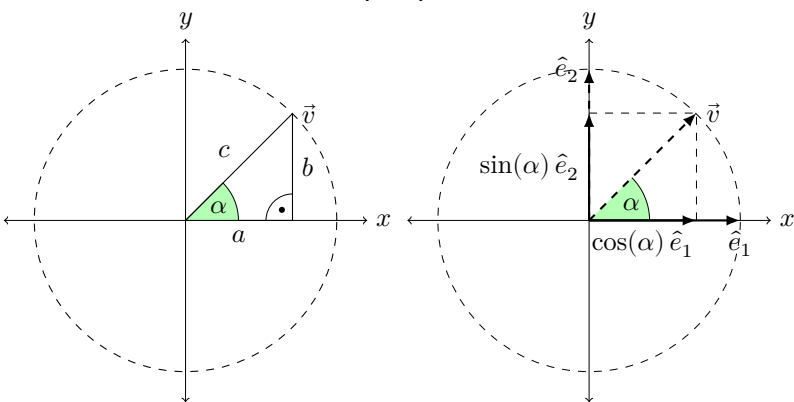
$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  are canonical base vectors in  $R^2$

For  $R^n$ :  $\hat{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$   $i$ th component being 1 all other 0

- Generally:  $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n x_i \cdot e_i$

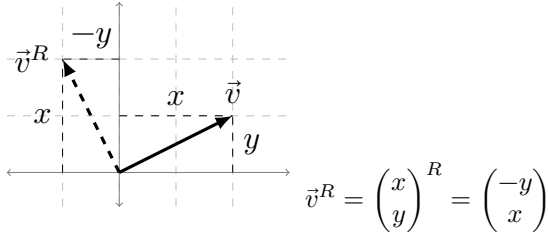
- Points on the unit circle (c=1):



- $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} = \cos(\alpha) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\alpha) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos(\alpha) \cdot \hat{e}_1 + \sin(\alpha) \cdot \hat{e}_2$

# 2 (Excursion) Orthogonal complement of a vector (90° Rotation CCW)

- Set up orthogonal vector to obtain up-vector for new coordinate frame



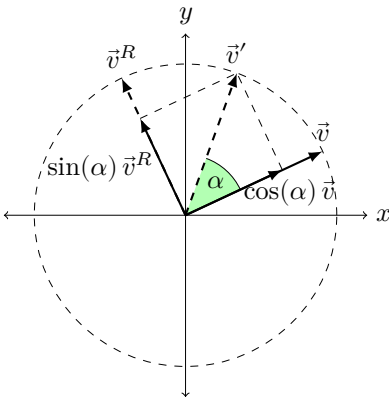
- Derive  $\vec{v}^R$ :

$$\begin{aligned} \vec{v}^R &= \begin{pmatrix} x \\ y \end{pmatrix}^R = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}^R = \begin{pmatrix} \cos(\alpha + 90^\circ) \\ \sin(\alpha + 90^\circ) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha) \cos(90^\circ) - \sin(\alpha) \sin(90^\circ) \\ \cos(\alpha) \sin(90^\circ) + \sin(\alpha) \cos(90^\circ) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha) \cdot 0 - \sin(\alpha) \cdot 1 \\ \cos(\alpha) \cdot 1 + \sin(\alpha) \cdot 0 \end{pmatrix} = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \end{aligned}$$

- $\vec{v}^R$  is perpendicular to  $\vec{v}$ :  $\begin{pmatrix} x \\ y \end{pmatrix} \bullet \begin{pmatrix} -y \\ x \end{pmatrix} = \underline{-xy + yx = 0}$

# 3 From 2D vector arithmetic to 2D rotation matrix

- Apply cos and sin to new coordinate frame with axes  $\vec{v}$  and  $\vec{v}^R$  to obtain rotated vector  $\vec{v}'$



- $\vec{v}' = \cos(\alpha) \vec{v} + \sin(\alpha) \vec{v}^R$

- rearrange to matrix arithmetic to apply transformation to arbitrary vector

$$\begin{aligned} v' &= \cos(\alpha) \begin{pmatrix} x \\ y \end{pmatrix} + \sin(\alpha) \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} \cos(\alpha) x \\ \cos(\alpha) y \end{pmatrix} + \begin{pmatrix} -\sin(\alpha) y \\ \sin(\alpha) x \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha) x - \sin(\alpha) y \\ \sin(\alpha) x + \cos(\alpha) y \end{pmatrix} = \underline{\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}} \end{aligned}$$

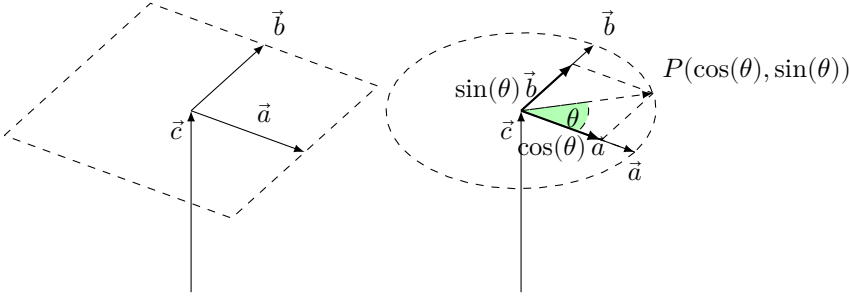
# 4 3D Angle-Axis Rotation(Rodrigues' rotation formula)

- Set up explicit plane equation placed orthogonal to rotation axis(normal) and rotate vector along plane.

- Points (x,y) on plane P ( $R^3$ ):

$$P(x, y) = \vec{c} + x \vec{a} + y \vec{b} \text{ with } \vec{a} \bullet \vec{b} = 0$$

$$\text{Circle on plane: } P(\cos(\theta), \sin(\theta)) = \vec{c} + \cos(\theta) \vec{a} + \sin(\theta) \vec{b}$$

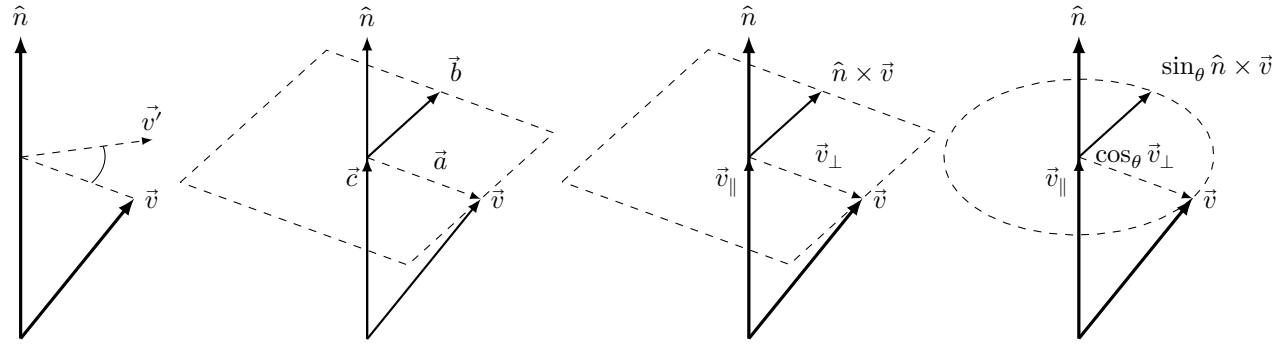


- With  $\hat{n}$  rotation axis and normal to the plane of rotation

(the rotation axis vector  $\hat{n}$  needs to be normalized, because of the projection operation)

- $\vec{v}$  vector to be rotated around axis vector  $\hat{n}$

- Construct plane of rotation with rejection(projection) of  $\vec{v}$  onto  $\hat{n}$  and cross product of  $\hat{n}$  with  $\vec{v}$



- $\vec{v} = \vec{v}_\perp + \vec{v}_\parallel$
- $\vec{v}_\parallel$  part of  $\vec{v}$  parallel to  $\hat{n}$  (Projection of  $\vec{v}$  onto  $\hat{n}$ )  
 $\vec{c} \rightarrow \vec{v}_\parallel = (\hat{n} \bullet \vec{v}) \hat{n}$
- $\vec{v}_\perp$  part of  $\vec{v}$  perpendicular to  $\hat{n}$   
 $\vec{a} \rightarrow \vec{v}_\perp = \vec{v} - \vec{v}_\parallel = \vec{v} - (\hat{n} \bullet \vec{v}) \hat{n}$
- vector in plane perpendicular to  $\vec{v}$  and  $\hat{n}$   
 $\vec{b} \rightarrow \hat{n} \times \vec{v} = \hat{n} \times \vec{v}_\perp$

- $\vec{v}' = \vec{v}_\parallel + \cos_\theta \vec{v}_\perp + \sin_\theta \hat{n} \times \vec{v}$

## 4.1 Derivation of Rodrigues Rotation Matrix:

- Express vector arithmetic terms with matrix arithmetic equivalents:

(Generally you can convert any vector arithmetic terms by transforming the base vectors of a space or the column vectors of a matrix. In case of the projection of a vector we can rearrange the operation to get it directly)

- $\mathbf{K}$  is skew symmetric matrix which can used calculate the cross product of  $\hat{n}$  with  $\vec{v}$ :

$$\mathbf{K} \vec{v} = \hat{n} \times \vec{v} = [\hat{n}]_\times \vec{v}$$

$$\mathbf{K} = [\hat{n}]_\times = \begin{bmatrix} \hat{n} \times \hat{e}_1 & \hat{n} \times \hat{e}_2 & \hat{n} \times \hat{e}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} n_y \cdot 0 - n_z \cdot 0 \\ n_z \cdot 1 - n_x \cdot 0 \\ n_x \cdot 0 - n_y \cdot 1 \end{pmatrix} & \begin{pmatrix} n_y \cdot 0 - n_z \cdot 1 \\ n_z \cdot 0 - n_x \cdot 0 \\ n_x \cdot 1 - n_y \cdot 0 \end{pmatrix} & \begin{pmatrix} n_y \cdot 1 - n_z \cdot 0 \\ n_z \cdot 0 - n_x \cdot 1 \\ n_x \cdot 0 - n_y \cdot 0 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$

- $\mathbf{P}$  is the matrix which multiplied with projects a vector  $\vec{v}$  onto normal  $\hat{n}$ :

$$\mathbf{P} \vec{v} = (\vec{v} \bullet \hat{n}) \hat{n} = (\hat{n} \bullet \vec{v}) \hat{n} = (\hat{n}^T \vec{v}) \hat{n} = \hat{n} \hat{n}^T \vec{v}$$

$$\mathbf{P} = \hat{n} \hat{n}^T = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \begin{bmatrix} n_x & n_y & n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{bmatrix}$$

- rearrange vector arithmetic rotation formula to matrix arithmetic (and factorize  $\vec{v}$ )

$$\begin{aligned} \vec{v}' &= \vec{v}_\parallel + \cos(\theta) \vec{v}_\perp + \sin(\theta) \hat{n} \times \vec{v} & \text{def. } \vec{v}_\parallel, \vec{v}_\perp \\ &= (\vec{v} \bullet \hat{n}) \hat{n} + \cos(\theta) (\vec{v} - (\vec{v} \bullet \hat{n}) \hat{n}) + \sin(\theta) \hat{n} \times \vec{v} & (\vec{v} \bullet \hat{n}) \hat{n} = (\hat{n}^T \vec{v}) \hat{n} = \hat{n} \hat{n}^T \vec{v} \\ &= \hat{n} \hat{n}^T \vec{v} + \cos(\theta) (\vec{v} - \hat{n} \hat{n}^T \vec{v}) + \sin(\theta) [\hat{n}]_\times \vec{v} & \mathbf{P} = \hat{n} \hat{n}^T, \mathbf{K} = [\hat{n}]_\times, \mathbf{I} \vec{v} = \vec{v} \\ &= \mathbf{P} \vec{v} + \cos(\theta) (\mathbf{I} \vec{v} - \mathbf{P} \vec{v}) + \sin(\theta) \mathbf{K} \vec{v} & \text{factorize } \vec{v}: \mathbf{A} \vec{v} + \mathbf{B} \vec{v} = [\mathbf{A} + \mathbf{B}] \vec{v} \\ &= [\mathbf{P} + \cos(\theta) (\mathbf{I} - \mathbf{P}) + \sin(\theta) \mathbf{K}] \vec{v} \\ &= \mathbf{R}(\hat{n}, \theta) \vec{v} \end{aligned}$$

- Matrix for rotating arbitrary vector around axis  $\hat{n}$  with angle  $\theta$ :

$$\mathbf{R}(\hat{n}, \theta) = \mathbf{P} + \cos(\theta) (\mathbf{I} - \mathbf{P}) + \sin(\theta) \mathbf{K}$$

$$\mathbf{R}(\hat{n}, \theta) = \begin{bmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{bmatrix} + \cos(\theta) \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{bmatrix} \right) + \sin(\theta) \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$