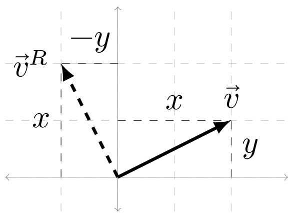
1 Orthogonal complement of a vector (90° Rotation CCW)



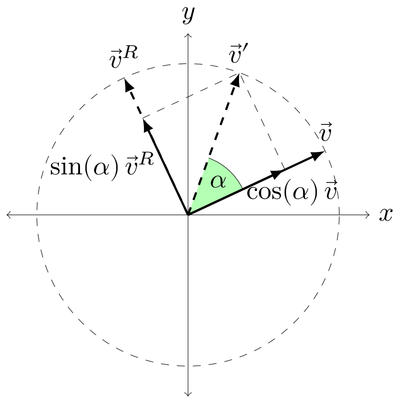
$$\vec{v}^R = \begin{pmatrix} x \\ y \end{pmatrix}^R = \begin{pmatrix} -y \\ x \end{pmatrix}$$

- $$\begin{aligned} \vec{v}^R &= \begin{pmatrix} x \\ y \end{pmatrix}^R = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}^R = \begin{pmatrix} \cos(\alpha + 90^\circ) \\ \sin(\alpha + 90^\circ) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha)\cos(90^\circ) - \sin(\alpha)\sin(90^\circ) \\ \cos(\alpha)\sin(90^\circ) + \sin(\alpha)\cos(90^\circ) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha)\cdot 0 - \sin(\alpha)\cdot 1 \\ \cos(\alpha)\cdot 1 + \sin(\alpha)\cdot 0 \end{pmatrix} = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix} = \underline{\begin{pmatrix} -y \\ x \end{pmatrix}} \end{aligned}$$
- $\vec{v}^R$ is perpendicular to $\vec{v}$: $\begin{pmatrix} x \\ y \end{pmatrix} \bullet \begin{pmatrix} -y \\ x \end{pmatrix} = \underline{-xy + yx = 0}$

2 vector components as linear combinations of vectors

- Every vector can be written as a combination of scalars and unit vectors
- $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Generally: $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i \cdot e_i$
- Points on the unit cirlce: $\vec{v} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = \cos(\theta) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\theta) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos(\theta) \cdot \hat{e}_1 + \sin(\theta) \cdot \hat{e}_2$

3 From 2D vector arithmetic to 2D rotation matrix

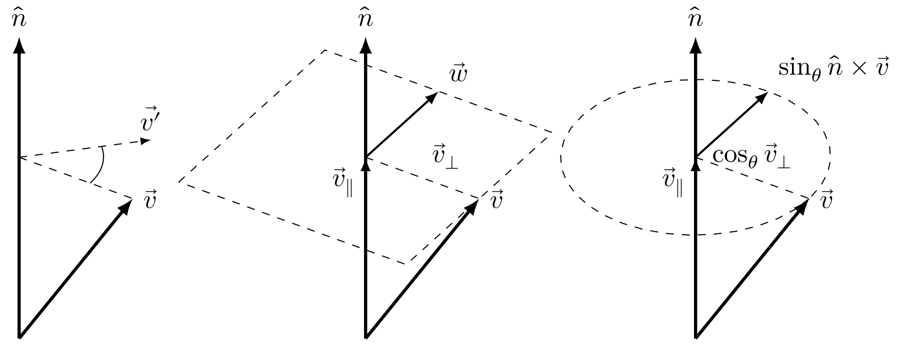


$\vec{v}' = \cos(\alpha)\,\vec{v} + \sin(\alpha)\,\vec{v}^R$

$$\begin{aligned} v' &= \cos(\alpha)\begin{pmatrix} x \\ y \end{pmatrix} + \sin(\alpha)\begin{pmatrix} -y \\ x \end{pmatrix} &&= \begin{pmatrix} \cos(\alpha)\,x \\ \cos(\alpha)\,y \end{pmatrix} + \begin{pmatrix} -\sin(\alpha)\,y \\ \sin(\alpha)\,x \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha)\,x - \sin(\alpha)\,y \\ \sin(\alpha)\,x + \cos(\alpha)\,y \end{pmatrix} &&= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

4 3D Angle-Axis Rotation(Rodrigues' rotation formula)

- Set up explicit plane equation placed orthogonal to rotation axis(normal) and rotate vector along plane.
- Points (x,y) on plane P (R^3):
  $P(x, y) = \vec{c} + x\,\vec{a} + y\,\vec{b}$ with $\vec{a} \bullet \vec{b} = 0$
  Circle on plane: $P(\cos(\theta), \sin(\theta)) = \vec{c} + \cos(\theta)\,\vec{a} + \sin(\theta)\,\vec{b}$



- $\vec{v}_\parallel = (\hat{n} \bullet \vec{v})\hat{n}$ (Proj. $\vec{v}$ onto $\hat{n}$)
- $\vec{v}_\perp = \vec{v} - \vec{v}_\parallel = \vec{v} - (\hat{n} \bullet \vec{v})\hat{n}$
- $\vec{v} = \vec{v}_\perp + \vec{v}_\parallel$
- $\vec{w} = \hat{n} \times \vec{v} = \hat{n} \times \vec{v}_\perp$
- $\vec{v}' = \vec{v}_\parallel + \cos(\theta)\,\vec{v}_\perp + \sin(\theta)\,\vec{w}$
- $\vec{v}' = \vec{v}_\parallel + \cos_\theta\,\vec{v}_\perp + \sin_\theta\,\hat{n} \times \vec{v}$

4.1 Derivation of Rodrigues Rotation Matrix:

- $$\begin{aligned} \vec{v}' &= \vec{v}_\parallel + \cos(\theta)\,\vec{v}_\perp + \sin(\theta)\,\vec{w} && \mid \text{ def. } \vec{v}_\parallel, \vec{v}_\perp, \vec{w} \\ &= (\vec{v} \bullet \hat{n})\hat{n} + \cos(\theta)\,(\vec{v} - (\vec{v} \bullet \hat{n})\hat{n}) + \sin(\theta)\,\hat{n} \times \vec{v} && \mid\ (\hat{n} \bullet \vec{v})\hat{n} = (\hat{n}^T\vec{v})\hat{n} = \hat{n}\hat{n}^T\vec{v} \\ &= \hat{n}\hat{n}^T\vec{v} + \cos(\theta)\,(\vec{v} - \hat{n}\hat{n}^T\vec{v}) + \sin(\theta)\,[\hat{n}]_\times\vec{v} && \mid\ \mathbf{P} = \hat{n}\hat{n}^T, \mathbf{K} = [\hat{n}]_\times, \mathbf{I}\,\vec{v} = \vec{v} \\ &= \mathbf{P}\,\vec{v} + \cos(\theta)\,(\mathbf{I}\,\vec{v} - \mathbf{P}\,\vec{v}) + \sin(\theta)\,\mathbf{K}\,\vec{v} && \mid \text{ factorize } \vec{v} : \mathbf{A}\vec{v} + \mathbf{B}\vec{v} = [\mathbf{A} + \mathbf{B}]\vec{v} \\ &= [\,\mathbf{P} + \cos(\theta)\,(\mathbf{I} - \mathbf{P}) + \sin(\theta)\,\mathbf{K}\,]\,\vec{v} \\ &= \mathbf{R}(\hat{n}, \theta)\,\vec{v} \end{aligned}$$
- $\mathbf{K} = [\hat{n}]_\times = \begin{bmatrix} \hat{n} \times \hat{e}_1 & \hat{n} \times \hat{e}_2 & \hat{n} \times \hat{e}_3 \end{bmatrix}$

$$= \left[\begin{pmatrix} n_y \cdot 0 - n_z \cdot 0 \\ n_z \cdot 1 - n_x \cdot 0 \\ n_x \cdot 0 - n_y \cdot 1 \end{pmatrix} \begin{pmatrix} n_y \cdot 0 - n_z \cdot 1 \\ n_z \cdot 0 - n_x \cdot 0 \\ n_x \cdot 1 - n_y \cdot 0 \end{pmatrix} \begin{pmatrix} n_y \cdot 1 - n_z \cdot 0 \\ n_z \cdot 0 - n_x \cdot 1 \\ n_x \cdot 0 - n_y \cdot 0 \end{pmatrix} \right] = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$

- $$\mathbf{P}_{\hat{n}} = \hat{n}\hat{n}^\mathsf{T} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \begin{array}{c} \begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \\ \begin{bmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{bmatrix} \end{array}$$
- Matrix for rotating arbitrary vector around axis $\hat{n}$ with angle θ:
  $\mathbf{R}(\hat{n}, \theta) = \mathbf{P} + \cos(\theta)\,(\mathbf{I} - \mathbf{P}) + \sin(\theta)\,\mathbf{K}$