

Signals & Systems

Week 6 Spring 2018

12.03.2018

CT F-S: $x(t)$ periodic; $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t}$

w/ $T_0 \rightarrow f_0 \rightarrow \omega_0$

$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j k \omega_0 t} dt$

DTFS: $x[n] = \sum_{k=0}^{N-1} a_k e^{+j \frac{2\pi}{N} k n}$

DT periodic w/ N
signal

$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} k n}$

for periodic signals

What about non-periodic signals?

CTFT: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

ω :

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{+j\omega t} d\omega$

ω :

Discrete-time Fourier Transform: DTFT:

$$\omega \rightarrow \boxed{\hat{\omega} = \omega T_s} \quad \begin{matrix} \text{sampling period in s.} \\ \text{normalized angular freq : Unit = radians} \end{matrix}$$

↑
rad/s

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\hat{\omega}n}$$

sampled signal

Exercise: Show that $X(e^{j\hat{\omega}})$ is periodic w/ 2π .

$$X(e^{j(\hat{\omega}+2\pi)}) = \sum x(n) e^{-j(\hat{\omega}+2\pi)n}$$
$$\sum x(n) e^{-j\hat{\omega}n} e^{-j2\pi n}$$
$$= 1$$

$$\hat{\omega} \in [0, 2\pi] \\ (-\pi, \pi)$$

n are integers

Inverse DTFT: $X[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\hat{\omega}}) e^{j\hat{\omega}n} d\hat{\omega}$

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\hat{\omega}n}$$

any 2π perio

Uniqueness of DTFT:

Q: Is DTFT a linear transform? Yes

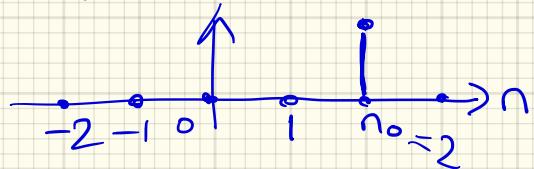
Let $x[n] = a x_1[n] + b x_2[n]$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} (a x_1(n) + b x_2(n)) e^{-j\hat{\omega}n}$$

show \downarrow $\sum_{n=-\infty}^{\infty} a x_1(n) e^{-j\hat{\omega}n} + b x_2(n) e^{-j\hat{\omega}n} = a X_1(e^{j\hat{\omega}}) + b X_2(e^{j\hat{\omega}})$



$$\text{Ex: } x(n) = \delta[n - n_0] \xrightarrow{\text{DTFT}} X(e^{j\hat{\omega}}) = ?$$



$$= \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\hat{\omega}n}$$

$$= e^{-j\hat{\omega}n_0}$$

~~↳~~ Uniqueness of DTFT

Show follows from linearity

Suppose 2 different signals $x_1(n), x_2(n)$ have the same DTFT. Let

$$x_3(n) = x_1(n) - x_2(n)$$

$$X_3(e^{j\hat{\omega}}) = X_1(e^{j\hat{\omega}}) - X_2(e^{j\hat{\omega}}) = 0 \xrightarrow[\text{from defn of DTFT, } x_3(n)]{} \text{has to be zero.}$$

$$\Rightarrow x_1(n) = x_2(n)$$

Existence of DTFT: A sufficient condn for $x(n)$ to have a DTFT $X(e^{j\hat{\omega}})$:

$$|X(e^{j\hat{\omega}})| = \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\hat{\omega}n} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n] e^{-j\hat{\omega}n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]| \underbrace{|e^{-j\hat{\omega}n}|}_{< 1}$$

$$\text{if } \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

DTFT
of $x(n)$
exists.

Discrete Fourier Transform : (DFT)

Discretize ω into
N values in $(0, 2\pi)$

$$\omega = \frac{2\pi}{N} k, k=0, 1, \dots, N-1$$

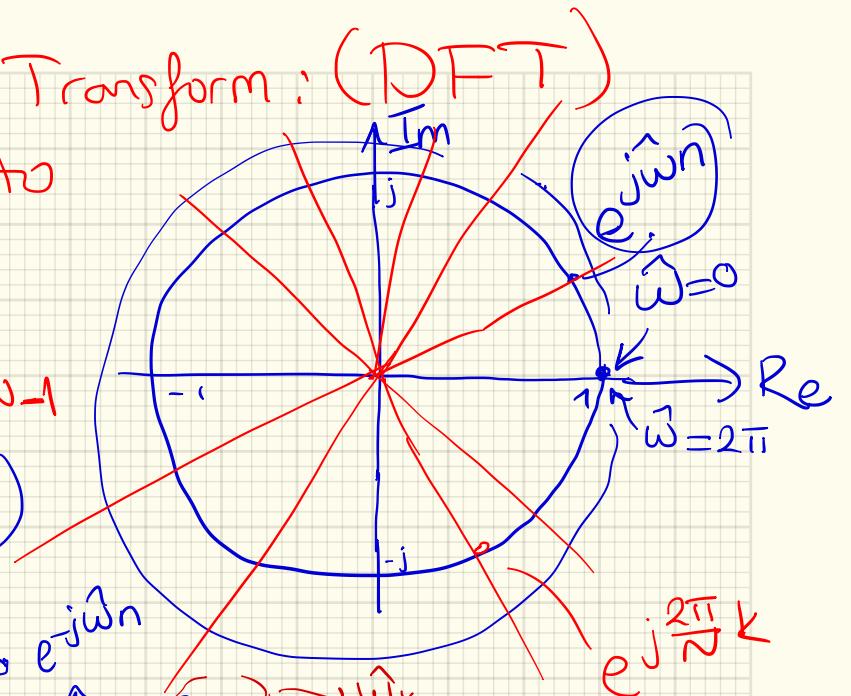
~~Recall~~
$$DTFT : X(e^{j\omega}) = \sum x(n) e^{-j\hat{\omega}n}$$

DFT :

$$X[k] = \sum_{n=-\infty}^{\infty} x(n) e^{-j \frac{2\pi}{N} k n} \quad k=0, 1, \dots, N-1$$

$\hat{\omega}_k = \frac{2\pi}{N} k$

range typically finite, $x(n)$ is finite duration length L



: N-pt DFT

$\Rightarrow X[k]$: both the freq. variable k , (k time variable
are discrete)
 n)

↳ numerically implementable

DFT: basis for Fast Fourier Transform (FFT)

$$\text{IDFT: } x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N} n}, \quad n=0, 1, \dots, N-1.$$

* We cannot compute for ∞ -length signals,
we operate on finite sections of a very long signal

Summary:

		Time	
		n discrete	t cont.
Frequency	k discrete	DFT, $X(k)$	Fourier Series $\sum a_k e^{j\omega k}$
	w cont	DTFT $X(e^{j\omega})$	CTFT $X(j\omega)$

Periodic signals

DFT
DTFT
CTFT

for
non-periodic
signals

(Later, we'll build a library of

$$\begin{array}{ccc} x(t) & \xrightarrow{\text{CTFT}} & X(j\omega) \\ x(n) & \xrightarrow{\text{DTFT}} & X(e^{j\omega}) \end{array}$$

Systems (Chap 5) → SP First

Also check DSP First

Maps an input signal to an output signal.

Systems

are used in SP

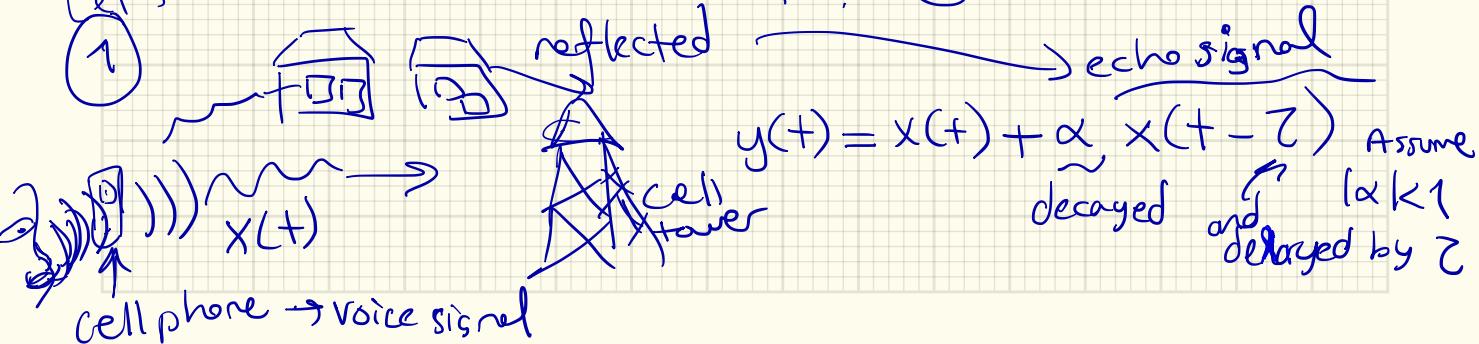
for 2 primary purposes:
Signal Processing

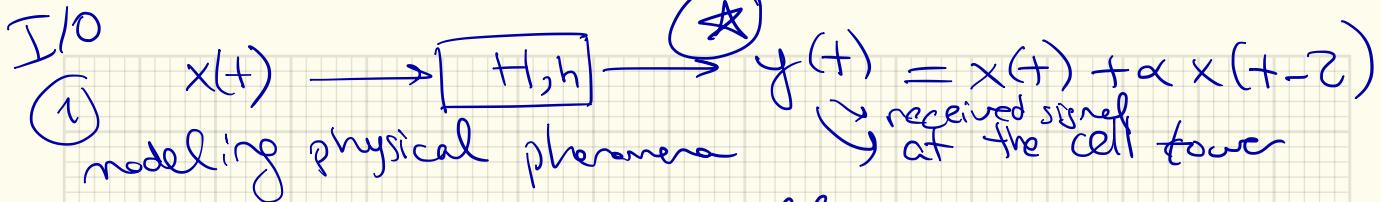
$$CT : X(t) \rightarrow \boxed{\quad} \rightarrow y(t)$$

$$DT : x[n] \rightarrow \boxed{\quad} \rightarrow y[n]$$

- ① Model the effects of a physical phenomenon
- ② Implement a desired effect on data or signals

Let's take a look at this example: ①





(2) Implement the desired effect: here, it is extracting the transmitted signal

$y(+)$ \rightarrow H_i \rightarrow $z(+)$ $= y(+)$ $- \alpha y(+ - 2)$

Goal: $z(+) \approx x(+) \quad$ I/O:

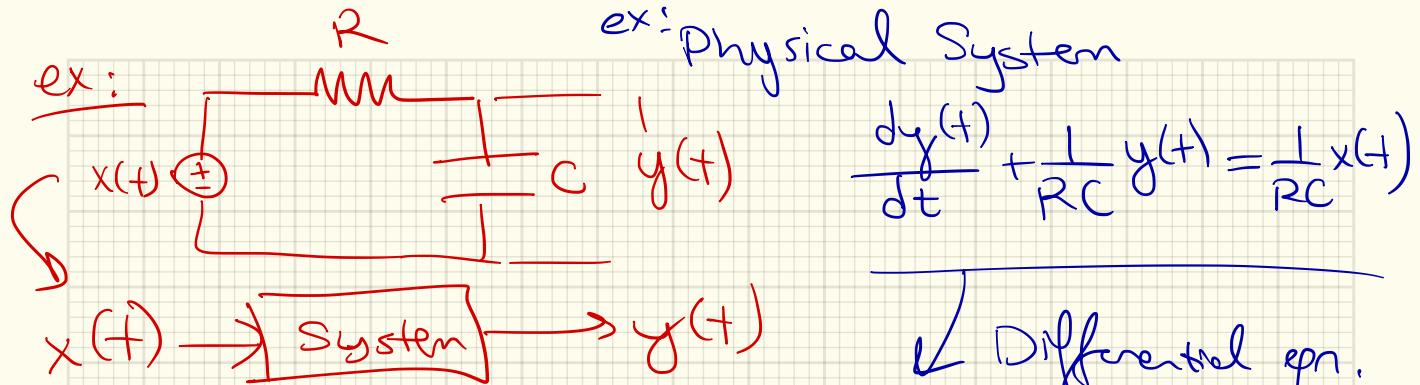
$+ \alpha^2 y(+ - 2z) - \alpha^3 y(+ - 3z)$

Insert $y(+)$ (eqn *)

$z(+) = x(+) - \alpha^4 x(+ - 4z)$

$\Rightarrow z(+) \approx x(+) \quad$ Recall $|\alpha| < 1$ $\alpha^4 \ll 1 \rightarrow 0$.

derive this eqn.



Difference eqn: Linear const coeff. D.E!

General form of

$$\sum_{k=0}^n a_k y[n-k] = \sum_{k=0}^m b_k x[n-k]$$

General const linear drift coeff. Eqn.

$$\sum_{k=0}^n a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^m b_k \frac{d^k x(t)}{dt^k}$$

RC Circuit: $a_0 = \frac{1}{RC}, a_1 = 1, b_0 = \frac{1}{RC}$

DT Systems

② Implementing a desired effect :

- ~~↳~~ i. Eliminate "noise" → Filters
ii. Extracting desired information
from a signal / eliminate the redundant info.

ex:



Define a rule or

I/O relation: $y[n] = \frac{1}{3} (x[n] + x[n-1] + x[n-2])$

3-pt averager

ex: $y[n] = (x[n])^2$; A squarer system

ex: $y[n] = \frac{1}{3} (x[n-1] + x[n] + x[n+1])$; 3 pt
last centered ← averager system

ex: 5-pt average RAF

$$y[n] = \frac{1}{5} (x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4])$$

In general: $(m+1)$ pt. Running Average Filter (RAF)

$$\downarrow \quad y[n] = \sum_{k=0}^m b_k x[n-k]$$

b_k : filter coefficients

Generalize to FIR filter: Finite Impulse Response

$$y[n] = \sum_{k=0}^m b_k x[n-k]$$

$$\begin{aligned} a_0 &= 1 \\ a_k &= 0 \quad k \neq 0 \end{aligned}$$



Recall this is a special case:

$$\sum_{k=-\infty}^{\infty} a_k y[n-k] = \sum_{k=-\infty}^{\infty} b_k x[n-k]$$

$$\text{I/O} \quad x(n) \xrightarrow{\text{FIR}} y(n)$$

ex. $y(n) = 3x(n) - x(n-1) + 2x(n-2) + x(n-3)$

Filter
coeff ?
 $b_k = \{3, -1, 2, 1\}$
 $b_0 \dots$

ex: Give $b_0 = 2$

$$\left. \begin{array}{l} b_1 = 0 \\ b_2 = 0 \\ b_3 = 0 \\ b_4 = -1 \end{array} \right\}$$

I/O ? FIR .

$$y(n) = 2x(n) - x(n-4)$$

Causal System: Causality important for real-time (online) systems.

Def: Output depends on current & past values of the input.

Q: Is this system causal? Yes

ex: $y(n) = \frac{1}{2}(x(n) + x(n+1))$ → Is this causal? No

$$y(0) = \frac{1}{2}(x(0) + x(1))$$
 a future point .

Linear System :

Time-Invariant System :

Causal ✓

Stability

Properties of systems ; we'll continue
w/ those next time.