Signals & Systems

Spring 2018

Week 7

15.03.2018

Systems in SP:

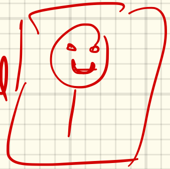
1) **Filtering**:

$$x[n] \rightarrow \boxed{\begin{matrix} h \\ H \end{matrix}} \rightarrow y[n]$$

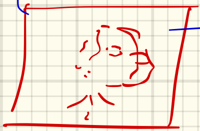
2)

$$x[n] \rightarrow \boxed{H} \rightarrow y[n] \xrightarrow{h} \boxed{H^{-1}} \rightarrow \tilde{x}[n]$$

Signal



Image



Blurred Image.

need system identification (estimating H) to undo effects of the unknown system

3) **Modeling physical effects**:

$$x[n] \rightarrow \boxed{H} \rightarrow y(n)$$

Recall General form of Linear Difference eqns

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

if $a_0 = 1$, $a_k = 0$, $k = 0$ $\Rightarrow y[n] = \sum_{k=0}^{M} b_k x[n-k]$

$$\Rightarrow y[n] = \sum \cancel{b_k}\, x(n-k) \longrightarrow \text{FIR (: Finite Impulse Response system)}$$

$$\boxed{y(n) = \sum_{k=0}^{M} h[k]\,x[n-k]}$$

convolution

Ex:

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M-1} b_k\, x(n-k)$$

$$y(n) - \frac{1}{2}y(n-1) = x(n) \Rightarrow y(n) = \frac{1}{2}y(n-1) + x(n)$$

↳ we'll do later.

IIR (Infinite Impulse Response) System.

Linearity of Systems : satisfies the property of superposition.

Sum of Inputs $\rightarrow \boxed{H} \rightarrow$ Sum of Outputs

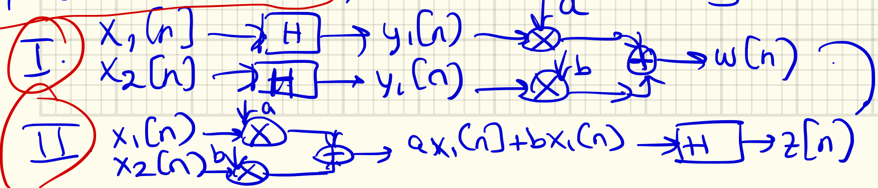
let $x_1[n] \rightarrow \boxed{H} \rightarrow y_1(n)$

$x_2(n) \rightarrow \boxed{H} \rightarrow y_2(n)$

for a linear system:

$a x_1(n) + b x_2(n) \rightarrow \boxed{H} \rightarrow a y_1(n) + b y_2(n)$

Testing for a linear system:



$= ?$

If $w(n) = z(n)$

System is linear.

Time Invariance (TI) of a System: System responds the same now as it does later.

$x(n) \longrightarrow \boxed{H} \longrightarrow y(n)$

$x[n-n_0] \longrightarrow \boxed{H} \longrightarrow y[n-n_0]$

Testing (TI):

I. $x(n) \longrightarrow \boxed{H} \longrightarrow y(n) \longrightarrow \boxed{\text{Delay } n_0} \longrightarrow y(n-n_0) = z(n)$

II. $x(n) \longrightarrow \boxed{\text{Delay } n_0} \longrightarrow x(n-n_0) \longrightarrow \boxed{H} \longrightarrow w(n)$ $\quad = ?$

n_0: integer

If $z(n) = w(n)$ then the system is time-invariant

Def: LTI (Linear & Time Invariant)

If A system is both linear & time invariant $\rightarrow$ LTI.

ex: $y(n) = (x(n))^2$: Is this an LTI? exercise:

Show that it is not linear but is time-invariant.

ex: $y[n] = x[-n]$: Is this LTI? Time-reversal System

Test TI:

I. $x(n) \longrightarrow \boxed{H} \longrightarrow y[n] = x[-n] \longrightarrow \boxed{\text{Delay } n_0} \longrightarrow x[-(n-n_0)] = x[-n+n_0]$

II. $x(n) \longrightarrow \boxed{\text{Delay } n_0} \longrightarrow y(n) = x[n-n_0] \longrightarrow \boxed{\text{System}} \longrightarrow w(n) = x[-n-n_0]$

$\neq$

Time-varying system (Not TI)

Test for linearity

I. $x_1(n) \rightarrow \boxed{H} \rightarrow y_1(n) = x_1(-n) \xrightarrow{\times a}$

$x_2(n) \rightarrow \boxed{H} \rightarrow y_2[n] = x_2(-n) \xrightarrow{\times b}$

$\oplus \rightarrow a x_1(-n) + b x_2(-n)$

II. $(a x_1(n) + b x_2(n)) \rightarrow \boxed{H} \rightarrow a x_1(-n) + b x_2(-n)$

$=$ ✓ Linear system.

Ex: $\boxed{y(n) = n\, x(n)}$: I/O relation : $q(n) \rightarrow \boxed{H} \rightarrow y(n) = n\, q[n]$

Test TI:

I. $x(n) \rightarrow \boxed{H} \rightarrow n\, x(n) = y(n) \rightarrow \boxed{\text{Delay } n_0} \rightarrow w(n) = y(n-n_0)$

$= (n-n_0)\, x(n-n_0)$

II. $x(n) \rightarrow \boxed{\text{Delay } n_0} \rightarrow \boxed{q(n)} = x(n-n_0) \xrightarrow{q(n)} \boxed{H} \rightarrow y_2(n) = n.\underbrace{q(n)}_{n.\,x(n-n_0)}$

$\neq$

Time-varying.

Test for linearity:

I. $x_1(n) \rightarrow \boxed{H} \rightarrow y_1(n) = n x_1(n) \xrightarrow{\times a}$

$x_2(n) \rightarrow \boxed{H} \rightarrow y_2(n) = n x_2(n) \xrightarrow{\times b}$

$\oplus \rightarrow a n x_1(n) + b n x_2(n)$

II. $\underbrace{(a x_1(n) + b x_2(n))}_{x_3(n)} \rightarrow \boxed{H} \rightarrow n.x_3(n) = n.(a x_1(n) + b x_2(n))$

$=$

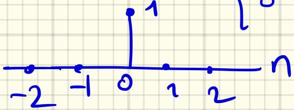
Linear System.

LTI Systems:

$x(n) \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

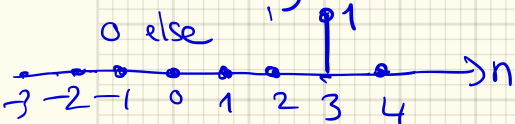
have a simple I/O relationship. Impulse Response tells us all we need to know to compute the output of the system.

Recall $\delta(n) = \begin{cases} 1, & n=0 \\ 0, & o/w \end{cases}$

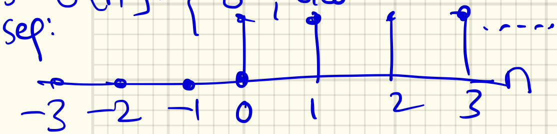


$\delta[n-3]$

0 else



step sep: $u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & o/w \end{cases}$



Impulse Response

Set $x[n] = \delta[n]$ $\longrightarrow \boxed{H} \longrightarrow$ $y(n) = h[n] \triangleq$ Impulse Response fn. (sep.)

Input = Impulse sep

$h(n)$

We know $h[n]$

When $x(n)$ is the input

$x(n) \longrightarrow \boxed{H} \longrightarrow y[n] = h[n] * x(n)$

LTI System

convolution operator

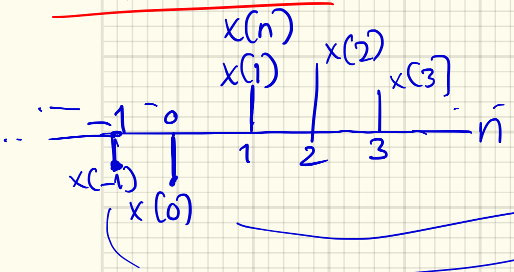
$$y[n] \triangleq \sum_{k=-\infty}^{\infty} \underbrace{h[k]}_{b_k} \, x[n-k]$$

let $\ell = n-k$

change of variables

$$y(n) = \sum_{\ell=-\infty}^{\infty} x[\ell] \, h[n-\ell]$$

LTI System: $h(n) * x(n) = \sum h[k]\,x(n-k)$



$$x[n] = \ldots + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + \ldots$$

$$x(n) = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

LTI System:

$x(n) \rightarrow \boxed{T} \rightarrow y[n] = T\{x(n)\} = \underbrace{T\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\}}_{\text{linearity}}$

$$= \sum_{k=-\infty}^{\infty} \underbrace{T\{x(k)\,\delta(n-k)\}}$$

$$= \sum x[k]\,\underbrace{T\{\delta[n-k]\}}_{\text{time invariance}}$$

impulse response

$\delta[n] \rightarrow \boxed{T} \rightarrow h[n]$

$\delta[n-k] \rightarrow \boxed{T} \rightarrow h[n-k]$

due to Time Inv.

$$y(n) = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

We showed: for an LTI system w/ impulse response $h(n)$, the I/O relation is given $\longrightarrow$ $y(n) = x(n) * h(n)$ convolution.

by the convolution btw input and $h(n)$.

Properties of Convolution Operator $*$

Show these / Prove as exercise

(1) Commutative : $x_1(n) * x_2(n) = x_2(n) * x_1(n)$

(2) Associative : $(x_1[n] * x_2(n]) * x_3(n) = x_1[n] * (x_2[n] * x_3[n])$

(3) Distributive over Addition : $h[n] * (x_1[n] + x_2(n]) = h[n] * x_1(n) + h[n] * x_2[n]$

(4) Identity Element : $\delta[n]$: unit impulse seq.

$$x(n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k] \underbrace{\delta[n-k]}_{= \begin{cases} 1, & k=n \\ 0, & o/w \end{cases}} = x(n]$$

(5) $\delta(n-n_0] * x(n] = x(n-n_0)$

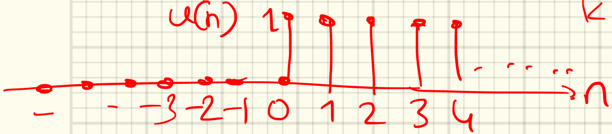
$$\delta[n-1] * x(n] = \sum_{k=-\infty}^{\infty} x(k) \, \delta[(n-1)-k] = x[n-1]$$

Recall

$x(n) \rightarrow \boxed{\text{Delay System } n_0} \rightarrow y(n) = x[n-n_0]$: Compute the impulse response ?

Set $x(n) = \delta[n] \rightarrow \boxed{\text{"}} \rightarrow h(n) = \delta[n-n_0]$: of the delay system.

STEP sequence (fn.): $u(n) = \sum_{k=0}^{\infty} \delta[\underbrace{n-k}_{\triangleq \ell}] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

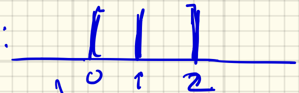


$\hookrightarrow u[n] = \sum_{\ell=-\infty}^{n} \delta[\ell]$

Step Response of a LTI System

$x(n) = u[n] \longrightarrow$ LTI $\longrightarrow s[n]$

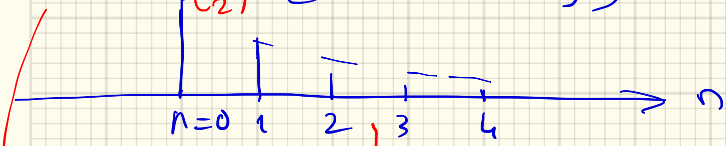
$x(n) = \underbrace{u[n] - u[n-3]}$:



$\rightarrow$ creates a window fn. of length 3.

ex: $x(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$ (exponential seq.)

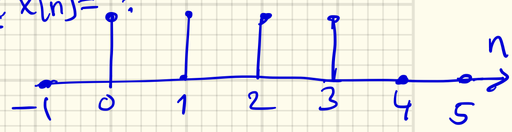
or $= \left(\frac{1}{2}\right)^n \cdot \left(u(n) - u[n-3]\right)$



pointwise multiplication.

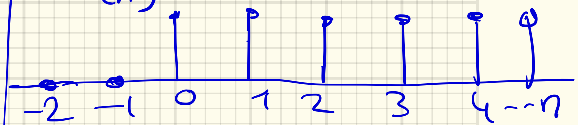
windowed signal

ex: $x[n] = ?$ $\delta(n) + \delta(n-1) + \delta[n-2] + \delta[n-3]$

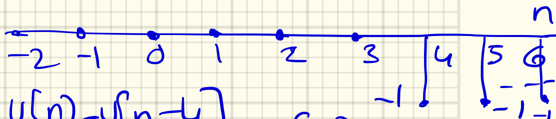


Use 2 step fn's to represent this signal.

$u[n]$



$-u[n-4]$



$u[n] - u[n-4] = x(n)$

Step Response of an LTI System

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \longrightarrow \boxed{LTI} \longrightarrow s(n)$$

$$\|$$

$$x(n) \longrightarrow \boxed{LTI} \longrightarrow y(n) = h(n) * x(n)$$

$h[n]$ ← impulse response : $\delta[n] \longrightarrow \boxed{LTI} \longrightarrow h(n)$

$$y(n) = h(n) * u(n) = h(n) * \left(\sum_{k=0}^{\infty} \delta[n-k] \right)$$

distributive over addition

$$s(n) = \sum_{k=0}^{\infty} \left(h(n) * \delta[n-k] \right)$$

$$s(n) = \sum_{k=0}^{\infty} h[n-k]$$; Given h (impulse resp) → going to $s(n)$ (step response)

ex: Given step response → $h(n) = ?$

Use $\delta[n] = u[n] - u(n-1]$

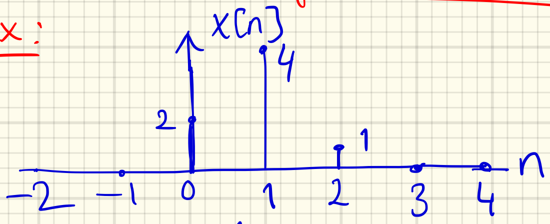
$$u[n] \longrightarrow \boxed{LTI} \longrightarrow s[n]$$

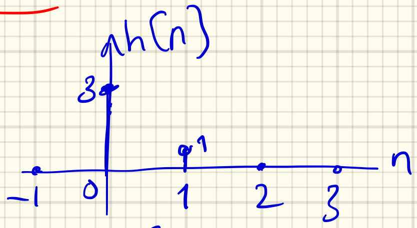
$$u[n-1) \longrightarrow \boxed{LTI} \longrightarrow s[n-1]$$

$$= \delta[n] \longrightarrow \boxed{LTI} \longrightarrow h(n) = s[n] - s(n-1]$$

Computation of Convolution: Given $x[n]$ & $h[n]$, Compute $y[n]=?$

Ex:





$$y[n] = \sum_{k=0}^{1} h[k]\,x[n-k]$$

1 way
n=0

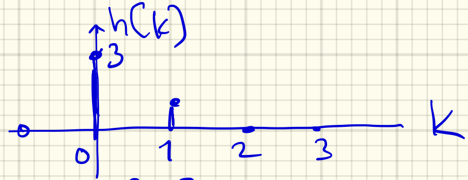
$$y[0] = \sum_{k=0}^{1} h[k]\,x[-k]$$

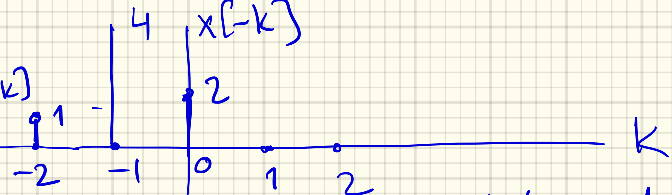
$$y[1] = \sum_{k=0}^{1} h[k]\,x[1-k]$$

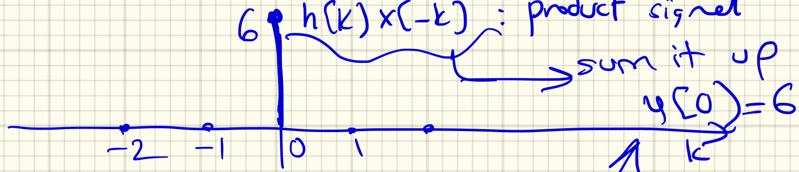
$$y[2] = \sum h[k]\,x[2-k]$$

$$y[3] = \sum h[k]\,x[3-k]$$

n=0
(1) use h(k) as it is,
(2) flip & shift x(k)

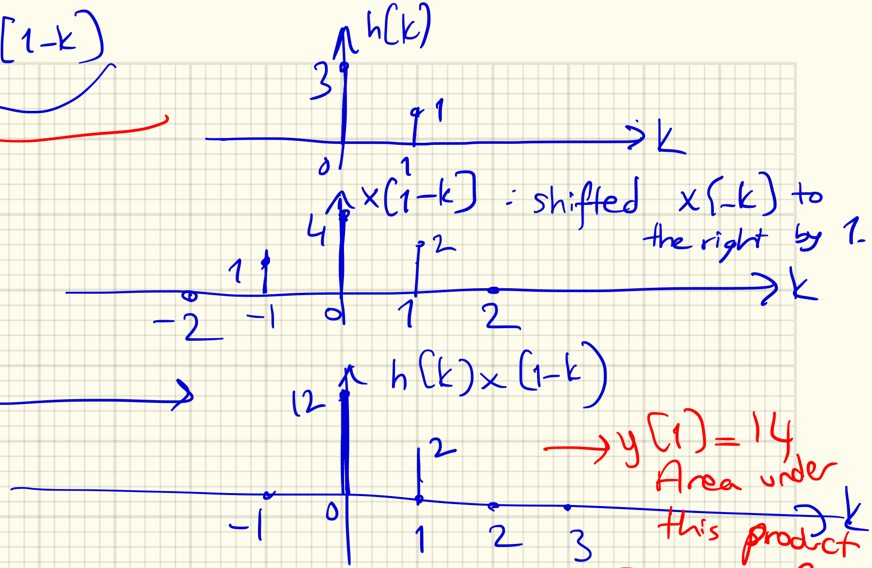






n=0

$n=1 \rightarrow y(1) = \sum_{k=0}^{1} h[k]\,x[1-k]$



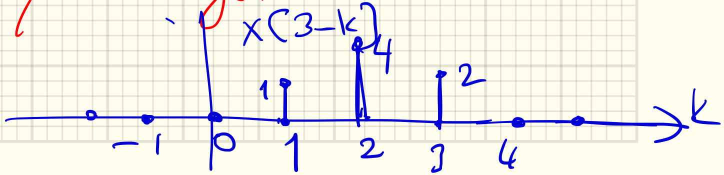
x[1-k] : shifted x[-k] to the right by 1.

$\rightarrow y[1] = 14$ Area under this product signal.

Exercise: calculate these at home: y(2), y(3) ← non-zero values

y[-1], y[4] ← check.

This is Graphical convoln method / works for CT convolution.



Another Way: $y(n) = \sum_{k=0}^{1} h[k]\,x[n-k] = h[0]\,x[n] + h[1]\,x[n-1]$

Table Method

n	0	1	2	3	4	5
$x(n)$	2	4	1	0	0	0
$h(n)$	3	1	0	0	0	0
$h(0)x(n)$	6	12	3	0	0	0
+ $h(1)x(n-1)$	0	2	4	1	0	0
$y(n)$	6	14	7	1	0	0

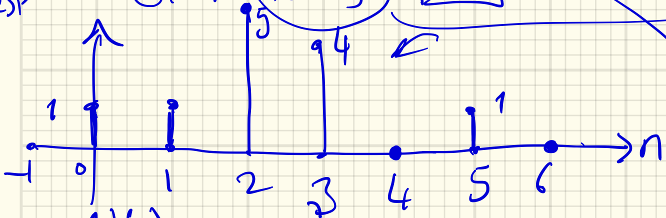
only valid for DT convol.

Ex: FIR Filter: $y[n] = x[n] - x[n-1]$: First order Difference filter

$h[n] = \delta[n] - \delta[n-1]$ impulse response

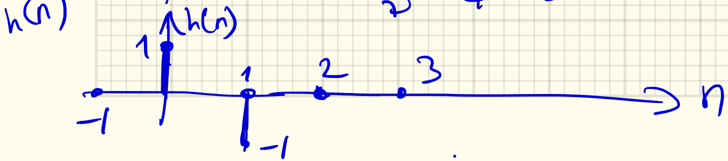
Given $x[n] \rightarrow \boxed{FIR} \rightarrow y(n) = ?$

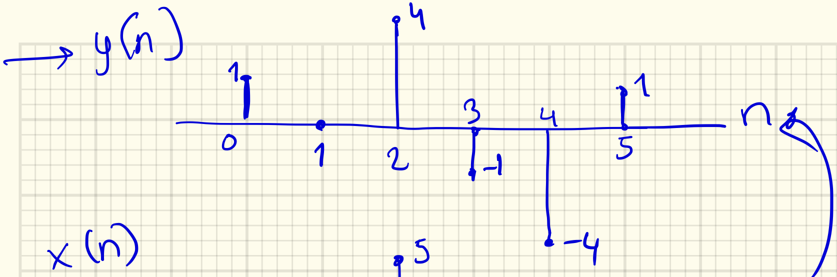
$x[n] = \delta[n] + \delta[n-1] + 5\delta[n-2] + 4\delta[n-3] + \delta[n-5]$



$y[n] = h(n) * x[n]$

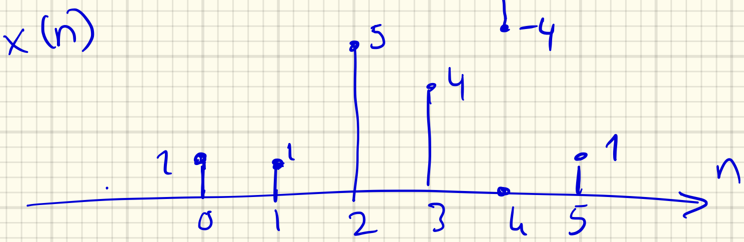
$= \delta[n] + \delta[n-1] + 5\delta[n-2] + 4\delta[n-3] + \delta[n-5]$
$- (\delta[n-1] + \delta[n-2] + 5\delta[n-3] + 4\delta[n-4] + \delta[n-6])$





$$h(n) = \delta(n) - \delta(n-1)$$

derivative $\equiv$ difference filter



smoothing filter.

e.g. if we had 5pt-RAF : exercise do this

$$y(n) = \sum_{k=0}^{4} \frac{1}{5} x(n-k)$$