1.4 - PREDICATES AND QUANTIFIERS

Example 1: Let

$$p(x):\ x + 3 = 7.$$

Determine if the following is a proposition: "There exists an integer x such that $p(x)$."

- $p(x)$ is called a ***propositional function***.
- The set of values that x can possibly be in the propositional function is called the ***universe of discourse***.

Example 2: *Goldbach's conjecture.* Let

$q(n)$: "n can be written as the sum of two prime numbers"

The conjecture is $q(4) \wedge q(6) \wedge q(8) \wedge q(10) \wedge \ldots$. This can be written more compactly as

$$\text{"}\forall n \in \mathbb{Z}^+\ q(2n+2)\text{"}$$

Definition:

- The ***universal quantification*** of $p(x)$ is $\forall x\ p(x)$. It means "for all values x in the universe of discourse, $p(x)$ is true". The symbol $\forall$ is called the ***universal quantifier***.

- The ***existential quantification*** of $p(x)$ is $\exists x\ p(x)$. It means "there exists an element x in the universe of discourse such that $p(x)$ is true". The symbol $\exists$ is called the ***existential quantifier***.

Proposition	**Is true when…**	**Is false when…**
$\forall x\ p(x)$	$p(x)$ is true for every x	there exists one (or more) x such that $p(x)$ is false
$\exists x\ p(x)$	$p(x)$ is true for one (or more) x	$p(x)$ is false for every x

Example 3: $x + 3 = 7$ was determined not to be a proposition. Which quantifier makes it a proposition?

Example 4: Given the propositional function

$$F(n):\ 2^{2^n} + 1 \text{ is a prime number},$$

where the universe of discourse is the set of non-negative integers. Consider the following conjecture first written by Fermat in 1650.

Conjecture: $\forall n\ F(n)$.

The conjecture is false. Use your calculator to find the first ***counterexample***.

When a variable is given a quantifier, the variable is called ***bound***. Otherwise, the variable is ***free***.

Negation of Quantifiers:

$$\neg\forall x\ p(x) \equiv \exists x\ \neg p(x)$$

$$\neg\exists x\ p(x) \equiv \forall x\ \neg p(x)$$

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- $p(x)$ is called a ***propositional function***.
- The set of values that x can possibly be in the propositional function is called the ***universe of discourse*** (*a.k.a.* ***domain***).

Example 2: *Goldbach's conjecture.* Let

$$q(n):\ n \text{ can be written as the sum of two prime numbers.}$$

The conjecture is $q(4) \wedge q(6) \wedge q(8) \wedge q(10) \wedge \ldots$. This can be written more simply as: For all positive integers n, $q(2n+2)$.

QUANTIFIERS

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BINDING VARIABLES

When a variable is given a quantifier, the variable is called ***bound***. Otherwise, the variable is ***free***.

NEGATIONS and TRANSLATING FROM ENGLISH INTO LOGICAL EXPRESSIONS

$$\neg\forall x\, p(x) \equiv \exists x\ \neg p(x).$$

$$\neg\exists x\, p(x) \equiv \forall x\ \neg p(x).$$