Discrete Math I
Exam II (2/9/12) – Page 1

Name: ______________________________

Instructions: Provide all steps necessary to solve the problem. Simplify your answer as much as possible. Additionally, clearly indicate the value or expression that is your final answer! Please put your name on BOTH pages. Calculators are allowed. An integer n is ***even*** if there exists an integer k such that $n = 2k$ and n is ***odd*** if there exists an integer k such that $n = 2k+1$. A real number x is ***rational*** if it can be written in the form $x = m/n$, where m and n are integers (with $n \neq 0$) and x is ***irrational*** if it is not a rational number. If $f : A \rightarrow B$, $S \subseteq A$, and $T \subseteq B$, then $f(S) = \{f(x) \in B \mid x \in S\}$ and $f^{-1}(T) = \{x \in A \mid f(x) \in T\}$, f is 1-1 $\Leftrightarrow \forall\, x,y \in A$ [$(f(x) = f(y)) \rightarrow (x = y)$], and f is onto $\Leftrightarrow \forall y \in B\ \exists x \in A\ [y = f(x)]$. For $m \in \mathbf{Z} - \{0\}$ and $n \in \mathbf{Z}$, $m \mid n \Leftrightarrow \exists\, k \in \mathbf{Z}\ (n = km)$. For $m \in \mathbf{Z}^+$, $\mathbf{Z}_m = \{n \in \mathbf{N} \mid 0 \leq n < m\}$. Given $a,b \in \mathbf{Z}$ and $m \in \mathbf{Z}^+$, $a \equiv b \pmod{m} \Leftrightarrow m \mid (a - b)$.

1. TRUE or FALSE: The sum of two irrational numbers is irrational.

2. If $A = \{0,4,5\}$, what is $|\mathcal{P}(A)|$?

3. Let $A = [0,3]$ and $B = [2,7)$ be intervals of the real line. Determine each of the following.

(*a*) $A \cup B$

(*b*) $A \cap B$

(*c*) $A - B$

4. Let $U = \{n \in \mathbf{Z}^+ \mid n \leq 8\}$ be the universal set. Suppose $A \subseteq U$ and $B \subseteq U$ satisfy the following conditions: $A \cap \overline{B} = \{1,5,7\}$, $B - A = \{2,3,6\}$, and $|A \cup B| = 8$. Determine each of the following.

$A =$

$B =$

$\overline{A \cup B} =$

5. *Short answer.*

(*a*) TRUE or FALSE: If $f: A \to B$ with $x \in A$, $y,z \in B$, then

$$(x,y),(x,z) \in f \to (y = z).$$

(*b*) TRUE or FALSE: $\lfloor x \rfloor = \lceil x \rceil - 1$ for all $x \in \mathbf{R} - \mathbf{Z}$.

(*c*) (Circle all that apply) The function $f: \mathbf{Z} \to \mathbf{Z}$ is defined by $f(x) = x + 1$ is

ONE-TO-ONE ONTO NEITHER.

(*d*) (Circle all that apply) The function $f: \mathbf{Z} \to \mathbf{Z}$ is defined by $f(x) = x^3$ is

ONE-TO-ONE ONTO NEITHER.

(*e*) (Circle all that apply) The function $f: \mathbf{Z} \to \mathbf{Z}$ is defined by $f(x) = x^2 - 4$ is

ONE-TO-ONE ONTO NEITHER.

(*f*) (Circle all that apply) The function $f: \mathbf{Z} \to \mathbf{Z}$ is defined by $f(x) = \lfloor x/3 \rfloor$ is

ONE-TO-ONE ONTO NEITHER.

(*g*) If $f: \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = 4x+3$, then f is both one-to-one and onto (you don't need to prove this). This implies that f is invertible. Determine $f^{-1}(9)$.

(*h*) Let $f: \mathbf{R} \to \mathbf{Z}$ and $g: \mathbf{Z} \to \mathbf{R}$ be functions defined by $f(x) = \lceil x/4 \rceil$ and $g(n) = 2^n$.

(*i*) What is the domain of $f \circ g$?

(*ii*) Calculate $(g \circ f)(-6)$.

(*i*) Evaluate $\sum_{k \in \mathbf{Z}_5} (k!)$. [*Note:* Your answer should be a single number.]

6. Let $g: \mathbf{R} \to \mathbf{R}$ be the function defined as $g(x) = \lfloor 2x \rfloor$. Determine each of the following.

(*a*) $g(\mathbf{R})$

(*b*) $g(S)$, where $S = [2,4]$

(*c*) $g^{-1}(\{5\})$

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7. Let $A = \{a,b,c\}$ and $B = \{x,y\}$.

(***a***) How many functions are there from A to B?

(***b***) How many functions from A to B are one-to-one?

(***c***) How many functions from A to B are onto?

(***d***) How many partial functions are there from A to B?

8. *Short answer.*

(***a***) Evaluate 299 **mod** 7.

(***b***) Evaluate 50! **mod** 31.

(***c***) Determine all values of $q \in \mathbf{Z}$ and $r \in \mathbf{Z}_9$ such that $53 = 9q + r$.

(***d***) Find the prime factorization of the five digit integer 23,569.

(***e***) TRUE or FALSE: $45 \equiv 21 \pmod 8$.

(***f***) Find the smallest positive integer a such that $a + 3 \equiv 3a \pmod{11}$.

(***g***) TRUE or FALSE: If p and q are primes (>2), then $pq + 1$ is never prime.

9. *Briefly* explain in words the difference between $a \mid b$ and $\dfrac{b}{a}$.

10. Give a *mathematical proof* (not a valid proof) of the following using the definitions: If $a,b,c,d \in \mathbf{Z}$ with $a|c$ and $b|d$, then $ab|cd$.

Extra Credit: Clearly sketch the line(s) and/or the region(s) in the plane defined by the equation

$$\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 1.$$

(If drawing regions, use solid lines for borders that are included in the region and dashed lines for borders that are not included in the region.)

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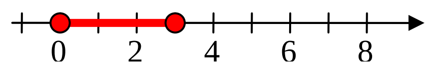
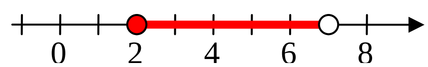
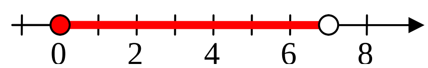
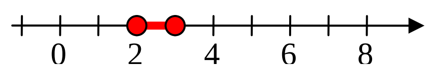
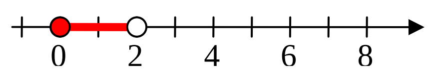
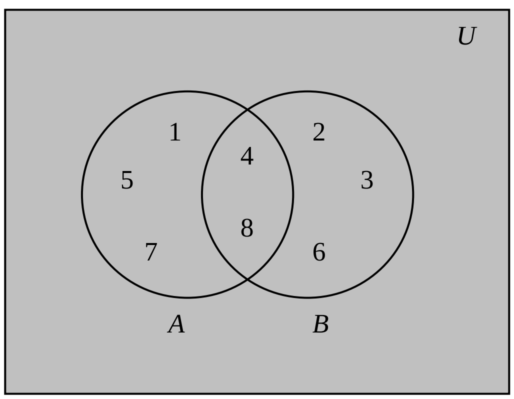
1. FALSE. Both π and $-\pi$ are irrational, but their sum, 0, is rational.

2. $|\mathcal{P}(A)| = 2^{|A|} = 2^3 = 8$

3. Note that these sets are intervals and do not represent discrete sets.

Problem	Interval	Picture
	$A = [0,3]$	
	$B = [2,7)$	
(*a*)	$A\cup B = [0,7)$	
(*b*)	$A\cap B = [2,3]$	
(*c*)	$A - B = [0,2)$	

4. Since $A\cap\overline{B} = A - B$, then $A = \{1,4,5,7,8\}$, $B = \{2,3,4,6,8\}$, and $\overline{A\cup B} = \varnothing$.



5.

(*a*) TRUE. This was the original definition of a function in terms of ordered pairs. The definition of a 1-1 function is slightly different.

(*b*) TRUE. This would be false for all $x\in\mathbf{R}$, but since we threw out the values for which the statement is false (i.e., all integers), we are left with a true statement for all $x\in\mathbf{R}-\mathbf{Z}$.

(*c*) f is 1-1 since $f(x) = f(y) \Rightarrow x + 1 = y + 1 \Rightarrow x = y$. f is onto since for any $x\in\mathbf{Z}$, there exists a number $y = x - 1\in\mathbf{Z}$ such that $f(y) = f(x-1) = (x-1) + 1 = x$. Thus, the function is ONE-TO-ONE and ONTO.

(*d*) f is 1-1 since $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$. f is not onto since there does not exist a number $y\in\mathbf{Z}$ such that $f(y) = 2$. Thus, the function is ONE-TO-ONE.

(*e*) f is not 1-1 since $f(1) = f(-1)$. f is not onto since there does not exist a number $y\in\mathbf{Z}$ such that $f(y) = -5$. Thus, the function is NEITHER.

(*f*) f is not 1-1 since $f(1) = f(2)$. f is onto since for any $x\in\mathbf{Z}$, there exists a number $y = 3x\in\mathbf{Z}$ such that $f(y) = f(3x) = \lfloor {}^{3x}/_{3} \rfloor = \lfloor x \rfloor = x$. Thus, the function is ONTO.

(*g*) Let $a = f^{-1}(9)$. This implies that $(9,a) \in f^{-1}$. By the definition of the inverse function, we have $(a,9) \in f$. In functional notation, this means that $f(a) = 9$. Since $f(x) = 4x+3$, then we need to find a such that $9 = 4a + 3$. Thus, $a = f^{-1}(9) = {}^{3}/_{2}$.

(*h*)

(*i*) Since $f \circ g : \mathbf{Z} \to \mathbf{Z}$, then the domain is **Z**.

(*ii*) $(g \circ f)(-6) = g(f(-6)) = g(\lceil -6/4 \rceil) = g(-1) = 2^{-1}$

(*i*) Since $\mathbf{Z}_5 = \{0,1,2,3,4\}$, then

$$\sum_{k \in \mathbf{Z}_5} (k!) = \sum_{k=0}^{4} (k!) = 0! + 1! + 2! + 3! + 4! = 1 + 1 + 2 + 6 + 24 = 34 .$$

6.

(*a*) **Z**

(*b*) $\{4,5,6,7,8\}$

(*c*) $[{}^{5}/_{2}, 3)$

7. **(*a*)** $2^3 = 8$; **(*b*)** 0; **(*c*)** 6; **(*d*)** $3^3 = 27$

8.

(*a*) 299 **mod** 7 = 5

(*b*) Since $50! = 50 \cdot 49 \cdot \cdots \cdot 32 \cdot 31 \cdot 30 \cdot \cdots \cdot 3 \cdot 2 \cdot 1$, then $31 \mid 50!$. This implies that 50! **mod** 31 = 0.

(*c*) By the Division Algorithm, these values exist and are unique: $q = \lfloor 53/9 \rfloor = 5$ and $r = 53 - 9 \cdot 5 = 8$.

(*d*) $7^2 \cdot 13 \cdot 37$

(*e*) TRUE

(*f*) $a = 7$

(*g*) TRUE. Any prime numbers greater than 2 is odd. The product of two odd numbers is still odd. Adding one to an odd number gives an even number, which is never prime as long as it is greater than 2.

9. Assume a and b are integers with a nonzero. The main difference between these expressions is

- $a \mid b$ is a proposition, it is either true or false, and
- $\dfrac{b}{a}$ is a rational number.

10. We need to show that $ab \mid cd$.

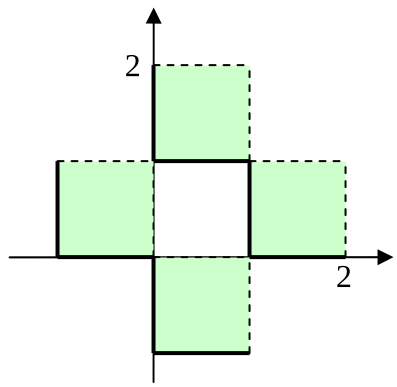
By definition, this means we need to find an integer k such that $ab \cdot k = cd$.
Since $a \mid c$, then there is an integer i such that $ai = c$.
Since $b \mid d$, then there is an integer j such that $bj = d$.
This implies that $cd = (ai)(bj) = ab(ij)$.
ij is an integer since an integer times integer is still an integer.
So, let $k = ij$.
Thus, ab is a divisor of cd and so $ab \mid cd$.

Comments (from the instructor) written on many of the proofs:

- Do not use fractions <u>at all</u>!
- Use complete sentences!
- The definition of divides is NOT "$m \mid n \Leftrightarrow \frac{n}{m} \in \mathbf{Z}$".
- $a \mid c$ is not a rational number, it is a proposition.
- Do not use the symbol ≡ for an equal sign.
- Contradiction is a bad idea for this type of proof.
- Examples are ignored in proofs.

Extra Credit: It would have been more efficient just to ask you to sketch the set

$$\{(x,y)\in\mathbf{R}\times\mathbf{R} \mid \lfloor x\rfloor^2 + \lfloor y\rfloor^2 = 1\}.$$



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P	**Answer/Solution**	*A %*	*M %*	**O**
1	FALSE	*18*	*10*	**1.5**
2	8	*81*	*100*	**3**
3a	[0,7)	*97*	*100*	**3**
3b	[2,3]	*96*	*100*	**3**
3c	[0,2)	*95*	*100*	**2**
4a	$A = \{1,4,5,7,8\}$	*93*	*100*	**2**
4b	$B = \{2,3,4,6,8\}$	*92*	*100*	**2**
4c	$\overline{A \cup B} = \varnothing$	*91*	*100*	**2**
5a	TRUE	*50*	*10*	**1.5**
5b	TRUE	*81*	*100*	**2**
5c	ONE-TO-ONE and ONTO	*97*	*100*	**2**
5d	ONE-TO-ONE	*81*	*100*	**2**
5e	NEITHER	*97*	*100*	**2**
5f	ONTO	*82*	*100*	**2**
5g	$a = f^{-1}(9) = {}^{3}/_{2}$	*95*	*100*	**3**
5h1	$\mathbf{Z}$	*78*	*100*	**2**
5h2	2^{-1}	*92*	*100*	**2**
5i	34	*75*	*90*	**4**
6a	$\mathbf{Z}$	*78*	*100*	**2**
6b	$\{4,5,6,7,8\}$	*80*	*80*	**3**
6c	$[{}^{5}/_{2}, 3)$	*68*	*90*	**3**
Overall		*78*	*80*	**50**

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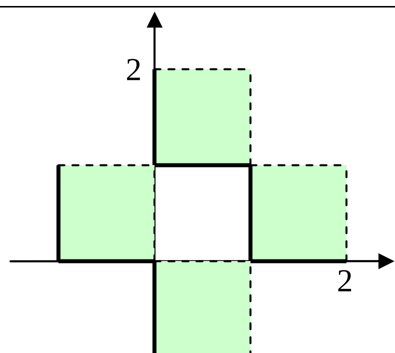
P	**Answer/Solution**	*A %*	*M %*	**O**
7a	$2^3 = 8$	*56*	*40*	**3**
7b	0	*68*	*100*	**1.5**
7c	6	*60*	*50*	**1.5**
8a	5	*95*	*100*	**2**
8b	0	*57*	*25*	**2**
8c	$q = 5$ and $r = 8$ is the only solution	*89*	*100*	**4**
8d	$7^2 \cdot 13 \cdot 37$	*82*	*100*	**3**
8e	TRUE	*70*	*100*	**2**
8f	$a = 7$	*79*	*100*	**3**
8g	TRUE	*100*	*100*	**2**
9	$a \mid b$ is a proposition, it is either true or false, and $^b/_a$ is a rational number.	*81*	*80*	**6**
10	Since $a \mid c$ and $b \mid d$, then there exists $i, j \in \mathbf{Z}$ such that $ai = c$ and $bj = d$. This implies that $cd = (ai)(bj) = ab(ij)$. $ij \in \mathbf{Z}$ since an integer times integer is still an integer. Thus $ab \mid cd$.	*63*	*60*	**10**
Overall		*74*	*75*	**40**
ec				**1.5**