MAT 102E MIDTERM EXAM 17 NOVEMBER 2012

QUESTION 1 The blanks below will be filled by students. (Except the score)

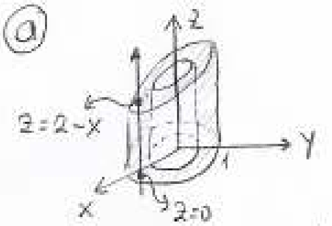
Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

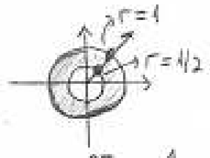
For the solution of this question please use only the front face and if necessary the back face of this page.

[16pt] a) D is the solid right cylinder whose base is the region between the circles $x^2+y^2=\frac{1}{4}$ and $x^2+y^2=1$, in the xy-plane and whose top lies in the plane $z=2-x$. Find the volume.

[10pt] b) Let $f(x,y)=\dfrac{(x+y+2)^2}{x^2+y^2-4x+8y+20}$. Determine if the limit of this function exists as $(x,y)\to(2,-4)$. Explain your answer.

(a)





$x^2+y^2=1/4 \Rightarrow r^2=1/4 \Rightarrow r=1/2$

$x^2+y^2=1 \Rightarrow r=1$

$$V=\int_{\theta=0}^{2\pi}\int_{r=1/2}^{1}\int_{z=0}^{2-r\cos\theta} dz\, r\, dr\, d\theta = \int_{\theta=0}^{2\pi}\int_{r=1/2}^{1}(2r-r^2\cos\theta)\,dr\,d\theta$$

$$=\int_{\theta=0}^{2\pi}\left(r^2-\frac{r^3}{3}\cos\theta\right)\Big|_{1/2}^{1} d\theta=\int_{\theta=0}^{2\pi}\left(\frac{3}{4}-\frac{7}{24}\cos\theta\right)d\theta=\frac{3}{4}\theta-\frac{7}{24}\sin\theta\Big|_{0}^{2\pi}=\frac{3}{4}2\pi=\frac{3\pi}{2}$$

(b) $\left(\frac{0}{0}\right) \Rightarrow \left.\begin{array}{l} x-2=r\cos\theta \\ y+4=r\sin\theta \end{array}\right\}$ $(x,y)\to(2,-4) \Rightarrow r\to 0$

$x+y+2=r(\cos\theta+\sin\theta)$, $x^2+y^2-4x+8y+20=(x-2)^2+(y+4)^2=r^2$

$$\lim_{r\to 0}\frac{\cancel{r^2}(\cos\theta+\sin\theta)^2}{\cancel{r^2}}=(\cos\theta+\sin\theta)^2$$

The value of limit depends on the value of θ

Therefore, the limit of f as $(x,y)\to(2,-4)$ does not exist

MAT 102E MIDTERM EXAM 17 NOVEMBER 2012

QUESTION 2 The blanks below will be filled by students. (Except the score)

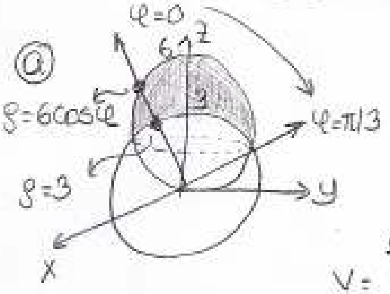
Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[10pt] a) Express the volume of the space region outside the surface $\mathbf{x^2+y^2+z^2=9}$ and inside the surface $\mathbf{x^2+y^2+(z-3)^2=9}$ in spherical coordinates. Do not evaluate the integral.

[12pt] b) Find the point on the sphere $\mathbf{x^2+y^2+z^2=4}$ farthest from the point $\mathbf{(1,-1,1)}$ by using the method of Lagrange multipliers.

ⓐ



$x^2+y^2+z^2=9 \Rightarrow \rho=3 \Rightarrow C(0,0,0),\ r=3$

$x^2+y^2+(z-3)^2=9 \Rightarrow \rho^2=6\rho\cos\varphi \Rightarrow \rho=6\cos\varphi$

$\Rightarrow C(0,0,3),\ r=3$

$3=6\cos\varphi \Rightarrow \cos\varphi=1/2 \Rightarrow \varphi=\pi/3$

$$V=\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi/3}\int_{\rho=3}^{6\cos\varphi}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

ⓑ $f(x,y,z)=(x-1)^2+(y+1)^2+(z-1)^2 \qquad g(x,y,z)=x^2+y^2+z^2-4=0$

$\nabla f=\lambda\nabla g \quad (\text{or } \nabla g=\lambda\nabla f)$

$$\left.\begin{array}{l}2(x-1)=2x\lambda\\ 2(y+1)=2y\lambda\\ 2(z-1)=2z\lambda\end{array}\right\} \left.\begin{array}{l}x-1=x\lambda\\ y+1=y\lambda\\ z-1=z\lambda\end{array}\right\} \left.\begin{array}{l}x=z=\dfrac{1}{1-\lambda}\\ y=-\dfrac{1}{1-\lambda}\end{array}\right\} x=z=-y$$

$\lambda\neq1$

$\lambda=1$ not possible

$g(x,-x,x)=3x^2=4 \Rightarrow x=z=\pm2/\sqrt{3} \Rightarrow y=\mp2/\sqrt{3}$

$P_0(2/\sqrt{3},-2/\sqrt{3},2/\sqrt{3}) \qquad P_1(-2/\sqrt{3},2/\sqrt{3},-2/\sqrt{3})$

$f(P_0)=3\left(\frac{2}{\sqrt{3}}-1\right)^2 < f(P_1)=3\left(\frac{2}{\sqrt{3}}+1\right)^2 \Rightarrow P_1$: farthest point

MAT 102E MIDTERM EXAM 17 NOVEMBER 2012

QUESTION 3 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[14pt] a) Show that the planes $\mathbf{x} - \mathbf{y} = \mathbf{3}$ and $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$ intersect and write the equation of the line of intersection.

[10pt] b) Find an equation of the plane that is tangent to the surface $\mathbf{x}^2 + 2\mathbf{y}^2 + 3\mathbf{z}^2 = \mathbf{6}$ at the point $(\mathbf{1}, -\mathbf{1}, \mathbf{1})$.

(a) $x-y=3 \Rightarrow$ normal of the plane : $n_1 = i - j$

$x+y+z=0 \Rightarrow$ " " " " : $n_2 = i+j+k$

$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -i-j+2k$

$n_1 \times n_2 \neq 0 \Rightarrow n_1, n_2$: not parallel $\Rightarrow$ planes intersect

$\left.\begin{array}{l} x-y=3 \\ z=0 \Rightarrow x+y=0 \end{array}\right\} \begin{array}{l} x=3/2 \\ y=-3/2 \end{array}$ $\Big\}$ $(3/2, -3/2, 0)$: a point on the line of intersection

$V = n_1 \times n_2 \,//\, L \Rightarrow$ $\left.\begin{array}{l} x=(3/2)-t \\ y=(-3/2)-t \\ z=2t \end{array}\right\}$ Line's equation $-\infty < t < \infty$

(b) $\nabla f = 2xi + 4yj + 6zk \Rightarrow \nabla f \big|_P = 2i - 4j + 6k$

Tangent plane : $2(x-1) - 4(y+1) + 6(z-1) = 0$

MAT 102E MIDTERM EXAM 17 NOVEMBER 2012

QUESTION 4 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[08pt] a) Find the length of the parametric curve $\mathbf{r(t) = a\cos t\ \bar{i} + a\sin t\ \bar{j} + bt\ \bar{k}}$ between the points $(\mathbf{a},\mathbf{0},\mathbf{0})$ and $(\mathbf{a},\mathbf{0},\mathbf{2\pi b})$.

[08pt] b) Let $\mathbf{f(x,y,z) = x^2y + xz}$. Find the derivative of $\mathbf{f}$ at the point $(\mathbf{1},\mathbf{1},\mathbf{1})$ and in the direction of $\bar{\mathbf{i}}+\bar{\mathbf{j}}+\bar{\mathbf{k}}$.

[12pt] c) Write the integral $\mathbf{I} = \int_0^1\int_{-1}^0\int_0^{y^2}$ dz dy dx as an iterated integral in the order **dx dy dz** and **dy dz dx**.

a) $\left.\begin{array}{l} x = a\cos t \\ y = a\sin t \\ z = bt \end{array}\right\}$ $\begin{array}{l} (a,0,0) \Rightarrow z=0 \Rightarrow t=0 \\ (a,0,2\pi b) \Rightarrow z = 2\pi b = bt \Rightarrow t = 2\pi \end{array}$

$v = \dfrac{dr}{dt} = -a\sin t\, i + a\cos t\, j + bk \Rightarrow |v| = \sqrt{a^2+b^2}$

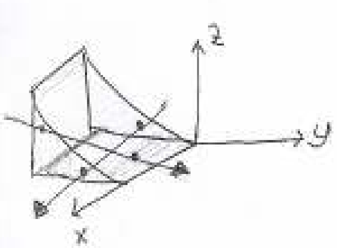
$L = \int_0^{2\pi} \sqrt{a^2+b^2}\, dt = 2\sqrt{a^2+b^2}\,\pi$

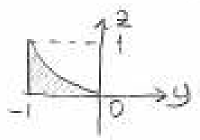
b) $u = i+j+k \Rightarrow |u| = \sqrt{3} \Rightarrow v = \dfrac{u}{|u|} = \dfrac{1}{\sqrt{3}}(i+j+k)$

$\nabla f = (2xy+z)\,i + x^2 j + xk \Rightarrow \nabla f\big|_P = 3i + j + k$

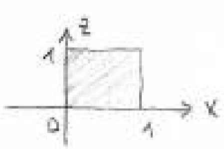
$(D_v f)_{P_0} = \nabla f\big|_{P_0} \cdot v = \dfrac{3+1+1}{\sqrt{3}} = \dfrac{5}{\sqrt{3}}$

c) $0 \le x \le 1,\ -1 \le y \le 0,\ 0 \le z \le y^2$





$I = \int_{z=0}^{1}\int_{y=-1}^{-\sqrt{z}}\int_{x=0}^{1} dx\,dy\,dz$



$I = \int_{x=0}^{1}\int_{z=0}^{1}\int_{y=-1}^{-\sqrt{z}} dy\,dz\,dx$