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BLG 202E – Numerical Methods
Assignment 3
Solutions

Solution 1

Solution 1.a

Linear interpolant through the points with abscissae 7 and 14 can be obtained like following:

(7,98); (14,101)

$$f_1(x) = \frac{(x-14)}{(7-14)}.98 + \frac{(x-7)}{(14-7)}.101$$

$$f_1(x) = 0.43x + 95.06$$

(0,100); (7,98); (14,101)

$$f_2(x) = \frac{(x-7)(x-14)}{(0-7)(0-14)}.100 + \frac{x(x-14)}{7(7-14)}.98 + \frac{x(x-7)}{14(14-7)}.101$$

$$f_2(x) = 0.05x^2 - 0.63x + 99.96$$

(7,98); (14,101); (21,50)

$$f_3(x) = \frac{(x-14)(x-21)}{(7-14)(7-21)}.98 + \frac{(x-7)(x-21)}{(14-7)(14-21)}.101 + \frac{(x-7)(x-14)}{(21-7)(21-14)}.50$$

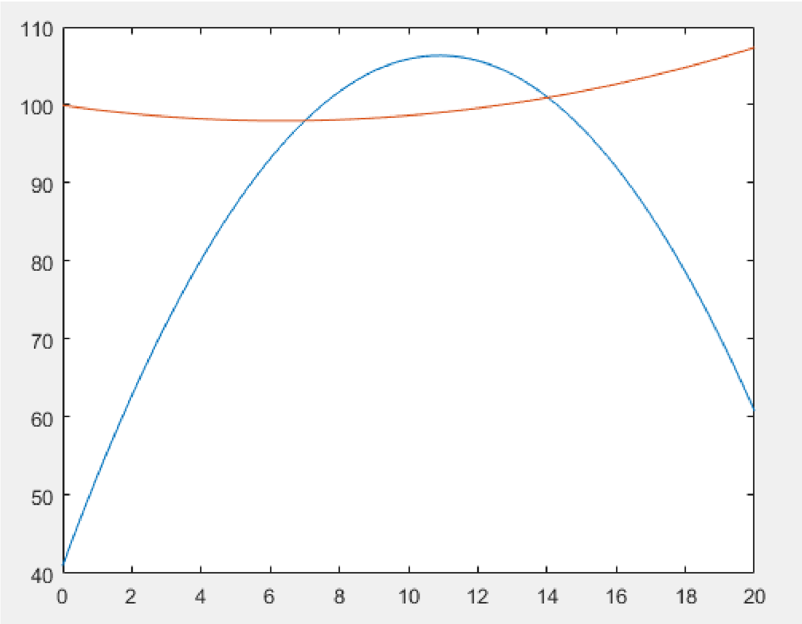
$$f_3(x) = -0.55x^2 + 11.99x + 41$$

$f_1(12) = 100.14,\quad f_2(12) = 99.6,\quad f_3(12) = 105.65$

$f_0(12)$ *and* $f_1(12)$ are the most accurate ones because 12 is more near to interval of 0-14 data points; not 21. So that to observe the most accurate function, interpolant data point must be selected properly.

Solution 1.b

```
x = linspace(0,20);
y1 = -0.55*x.^2+11.99*x+41
y2 = 0.05*x.^2-0.63*x+99.96
figure
plot(x,y1,x,y2)
```



The interval of 6 and 14, result is close each other so that more accurate. On the other hand, lower and upper bound is equal. These two quadratic interpolant just give an approximate result in short area.

Solution 2

Solution 2.a

To find the upper bound of interval we need to interpolating polynomial found in solution b.

$$\max_{0\le x\le 1} |e^x - p_2(x)|$$

$$\max_{0\le x\le 1} \left|e^x - 2\left(x-\frac{1}{2}\right)(x-1) + 4x(1-x)\sqrt{e} + 2x(x-\frac{1}{2})e\right|$$

$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\psi_n(x)$ is expression for the error in polynomial interpolation.

$$f(x) - p_2(x) = \frac{f'''(\xi)}{3!}\psi_2(x)$$

$$\psi_2(x) = \prod_{i=0}^{2}(x - x_i) = (x-x_0)(x-x_1)(x-x_2)$$

$$\max_{0\le x\le 1}|e^{x}-p_2(x)| \le \frac{1}{3!}\max_{0\le x\le 1}|f'''(x)| \max_{0\le x\le 1}|\psi_2(x)|$$

$$\max_{0\le x\le 1}|e^{x}-p_2(x)| \le \frac{1}{3!}\max_{0\le x\le 1}|e^{x}| \max_{0\le x\le 1}|(x-x_0)(x-x_1)(x-x_2)|$$

Maximum value of e^{x} can be obtained while $x=1$. So that $e^{1}=e$ is max value.

Maximum value of third degree polynomial can be found by derivation .

Solution 2.b

Using <u>Lagrangian Interpolation,</u> $p_2(x)$ can be found as following:

$$\boldsymbol{x_0=0},\ \ f(x_0)=e^{x_0}=e^{0}=\mathbf{1}$$

$$\boldsymbol{x_1=\frac{1}{2}},\ \ f(x_1)=e^{x_1}=e^{1/2}=\sqrt{\boldsymbol{e}}$$

$$\boldsymbol{x_2=1},\ \ f(x_2)=e^{x_2}=e^{1}=\boldsymbol{e}$$

$$L_0(x)=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)=\left(\frac{x-1/2}{0-1/2}\right)\left(\frac{x-1}{0-1}\right)=2(x-\frac{1}{2})(x-1)$$

$$L_1(x)=\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)=\left(\frac{x-0}{1/2-0}\right)\left(\frac{x-1}{1/2-1}\right)=4x(1-x)$$

$$L_2(x)=\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)=\left(\frac{x-0}{1-0}\right)\left(\frac{x-1/2}{1-1/2}\right)=2x(x-\frac{1}{2})$$

$$p_2(x)=L_0(x)f(x_0)+\ L_1(x)f(x_1)+L_2(x)f(x_2)$$

$$p_2(x)=2\left(x-\frac{1}{2}\right)(x-1)+4x(1-x)\sqrt{e}+2x(x-\frac{1}{2})e$$

$$p_2(x)=0.82x^{2}+0.89x+1$$

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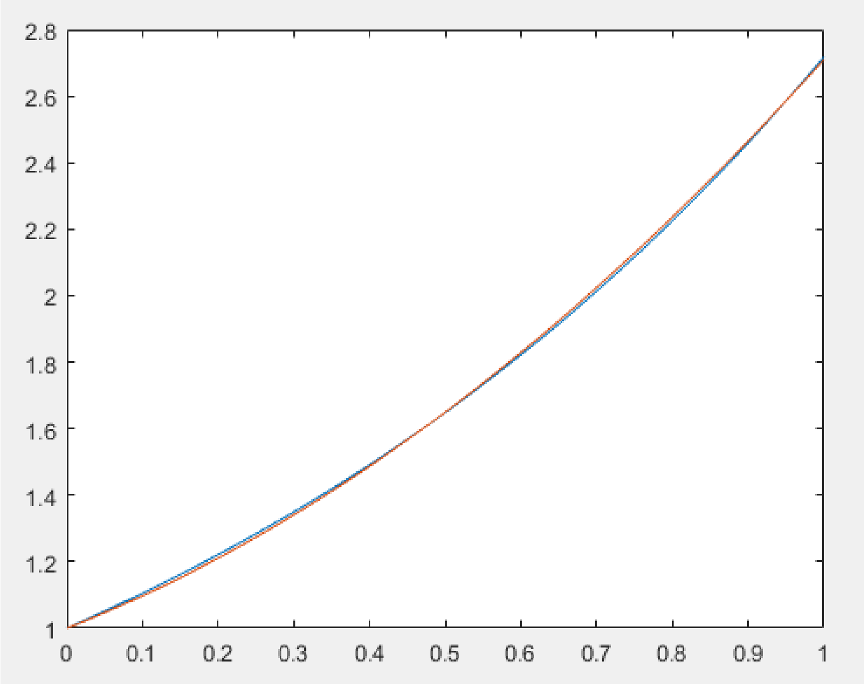
Solution 2.c

```
x = linspace(0,1);
y1 = 2.718182.^x;
y2 = 0.82*x.^2+0.89*x+1;
figure
plot(x,y1,x,y2)
```

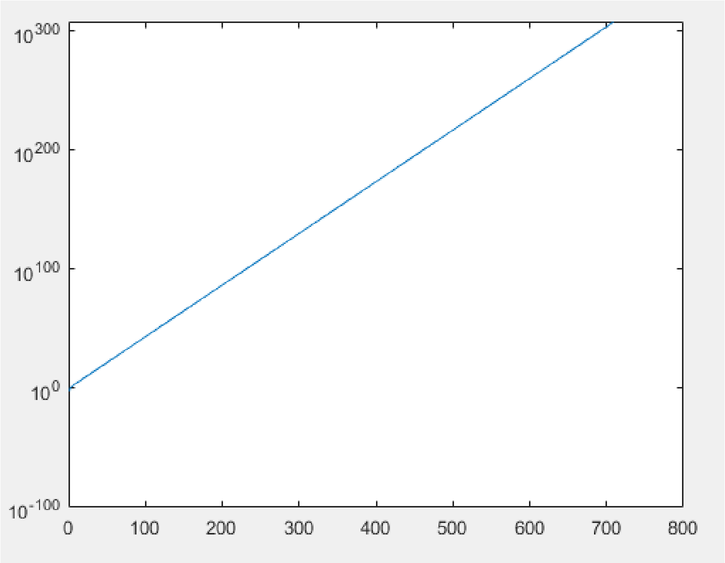


It can be seen easily that two function is same (first one is second derivative which is e^x, and second one is interpolating polynomial found by Lagrange Method).

Solution 2.d

```
x = 0:1000;
y = (2.718182.^x)-(0.82*x.^2+0.89*x+1);

figure
semilogy(x,y)
```



Solution 3

Solution 3.a

$f'''(x_0)$ can be derived by using Taylor approximation 5-pt formula. A centered formula (at x_0):

Expand about $x = x_0$ at $x = x_0 - h$, $x = x_0 + h$, $x = x_0 + 2h$, $x = x_0 - 2h$

$$f(x_0 + 2h) = f(x_0) + f'(x_0)2h + f''(x_0)2h^2 + f'''(x_0)\frac{4h^3}{3} + f^{(4)}(x_0)\frac{2h^4}{3} + f^{(5)}(\xi_1)\frac{4h^5}{15}$$

$$f(x_0 - 2h) = f(x_0) - f'(x_0)2h + f''(x_0)2h^2 - f'''(x_0)\frac{4h^3}{3} + f^{(4)}(x_0)\frac{2h^4}{3} - f^{(5)}(\xi_2)\frac{4h^5}{15}$$

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2} + f'''(x_0)\frac{h^3}{6} + f^{(4)}(x_0)\frac{h^4}{24} + f^{(5)}(\xi_3)\frac{h^5}{120}$$

$$f(x_0 - h) = f(x_0) - f'(x_0)h + f''(x_0)\frac{h^2}{2} - f'''(x_0)\frac{h^3}{6} + f^{(4)}(x_0)\frac{h^4}{24} - f^{(5)}(\xi_4)\frac{h^5}{120}$$

Then by adding and multiplying, it can be like following equation:

$$f(x_0 + 2h) - f(x_0 - 2h) - 2[f(x_0 + h) - f(x_0 - h)] = f'''(x_0)2h^3 + \xi\, h^5$$

Third derivative is determined as following:

$$f'''(x_0) = \frac{1}{2h^3}(f(x_0 + 2h) - f(x_0 - 2h) - 2[f(x_0 + h) - f(x_0 - h)]) + \boxed{\frac{h^2}{4} f^{(5)}(\xi)}$$

Truncation Error

$O(h^2)$

Solution 3.b

$$f'''(x_0) = \frac{1}{2h^3}(f(x_0 + 2h) - f(x_0 - 2h) - 2[f(x_0 + h) - f(x_0 - h)]) + \frac{h^2}{4} f^{(5)}(\xi)$$

As can bee seen for the first three values of n, when we reduce h by a factor of 10, the error goes down by afactor of 100, so the method is second order.

For example, apply the formula with one value of h and then h/2. It can be seen easily the error should be reduced by a factor of 4.

n	approx. to $f'''(x_0)$	absolute error
1.0000e-01	1.0025e+00	2.5025e-03
1.0000e-02	1.0000e+00	2.5000e-05
1.0000e-03	1.0000e+00	2.4927e-07
1.0000e-04	9.9998e-01	2.2122e-05
1.0000e-05	1.0547e+00	5.4712e-02
1.0000e-06	5.5511e+01	5.4511e+01
1.0000e-07	1.6653e+05	1.6653e+05
1.0000e-08	5.5511e+07	5.5511e+07
1.0000e-09	-1.1102e+11	1.1102e+11

```
syms x
syms h
f = inline(2.718182.^x, 'x');
fpp = inline(1/(2*h.^3 )*(f(2*h)-f(-2*h)-2*(f(h)- f(-h))),'h'); %% approximate derivation
k = [1:1:9]
h = 10.^-k
result = fpp(h)
err = abs(1-result) %% absolute error
plot(k,err)
```

For $h = e - 2 \ and \ e - 3$ gives the closest approximation to $e^0 = 1$

Solution 3.c

When h decreases, absolute error increase rapidly doe to round of error. The round off error is inversely proportional to h^2 as h approach to 0.

Solution 3.d

By using 7-pt central difference formula, it can be obtained $f'''(x_0)$ in fourth order.

$x = x_0$ at $x = x_0 - h$, $x = x_0 + h$, $x = x_0 + 2h$, $x = x_0 - 2h$, $x = x_0 - 3h$, $x = x_0 + 3h$

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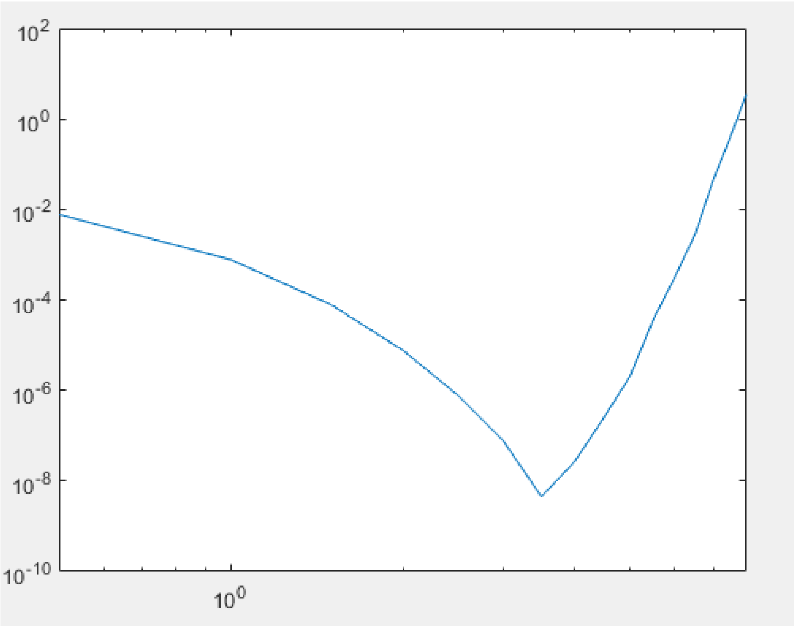
Solution 4

```
syms x
syms h
fpp = inline((sin(1.2+h)-2*sin(1.2)+sin(1.2-h))/h^2, 'h'); %% approximate derivation
true_value = -sin(1.2) %% absolute derivative of sin(x)
k = [0:.5:8]
h = 10.^-k
result = fpp(h)
err = abs(true_value-result) %% absolute error
loglog(k,err)
```



Increasing k values corresponding to decreasing h values, absolute error increase rapidly in $10^{-7.5}$ and maximum error is obtained in 10^{-8}. It arise from the floating point error of h. So that approximating the socond derivative gives closest values while $1.5 \leq k \leq 5$ which means absolute error is zero.