QUESTION 1 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

a) **(13 pts)** Find the equation of the plane that passes through the point $\mathbf{P(1,3,-2)}$ and contains the line of intersection of the planes $\mathbf{x+y+z=5}$ and $\mathbf{3x-y=4}$.

b) **(12 pts)** Let $\mathbf{f(x,y)}$ be given by: $\mathbf{f(x,y)} = \begin{cases} \dfrac{x^2y^2}{x^4+y^4} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

Is $\mathbf{f(x,y)}$ continuous at the point $(0,0)$? Explain your answer.

a) We need two more points of the plane other than P.

Take two arbitrary points on the line of intersection of $x+y+z=5$ and $3x-y=4$

P_0: If $x=0$ then $y=-4$, hence $0-4+z=5$ gives $z=9$

$\Rightarrow P_0(0,-4,9)$

P_1: if $x=1$ then $y=-1$ Thus $1-1+z=5$ gives $z=5$

$\Rightarrow P_1(1,-1,5)$

The normal vector $\vec{n}$ of the desired plane is $\vec{n} = \overrightarrow{PP_0} \times \overrightarrow{PP_1}$

$$\Rightarrow \vec{n} = \begin{vmatrix} i & j & k \\ -1 & -7 & 11 \\ 0 & -4 & 7 \end{vmatrix} = -5i + 7j + 4k$$

Thus the plane is $-5(x-1)+7(y-3+4(z+2)=0$

b) Since $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist f is not continuous at $(0,0)$

Approaching $(0,0)$ along $y=mx$ for $m \in \mathbb{R}$ we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^4 \cdot m^2}{x^4(1+m^4)} = \frac{m^2}{m^4+1}$$ which depends on m

QUESTION 2 The blanks below will be filled by students. (Except the score)

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a) **(13 pts)** Find the length of the curve $\vec{r}(t) = (t\cos t)\,\vec{i} + (t\sin t)\,\vec{j} + (\frac{2\sqrt{2}}{3}t^{\frac{3}{2}})\,\vec{k}$ for $0 \le t \le \pi$.

b) **(12 pts)** Find the directional derivative of $f(x,y,z) = 2x + 2y - \sin(xyz)$ at the point $P(1,1,0)$ in the direction of $\vec{v} = \vec{i} + \vec{j} - 3\vec{k}$.

a) $\vec{v} = \frac{d\vec{r}}{dt} = (\cos t - t\sin t)\, i + (\sin t + t\cos t)\, j + \sqrt{2t}\, k$

$\Rightarrow |\vec{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t} = \sqrt{t^2+2t+1}$

$= t+1$ when $t \geq 0$

So length $= \int_0^\pi |\vec{v}|\, dt = \int_0^\pi t+1\, dt = \left(\frac{t^2}{2} + t\right)\Big|_0^\pi = \frac{\pi^2}{2} + \pi$

b) f is a differentiable function on $\mathbb{R}^3$, (being the sum of a polynomial and a trigonometric function which are also differentiable) therefore $D_u f(P) = \nabla f(P) \cdot u$ for any unit vector u.

$\nabla f = (2 - yz\cos(xyz))\, i + (2 - xz\cos(xyz))\, j - (xy\cos xyz)\, k$

$\Rightarrow \nabla f(P) = 2i + 2j - k$

The unit vector $\vec{u}$ in the direction of $\vec{v}$ is

$\vec{u} = \frac{1}{|\vec{v}|} \cdot \vec{v} = \frac{1}{\sqrt{11}} \cdot (i + j - 3k)$

Thus $D_u f(P) = \nabla f(P) \cdot \vec{u} = \frac{1}{\sqrt{11}}(2 + 2 + 3) = \frac{7}{\sqrt{11}}$

QUESTION 3 The blanks below will be filled by students. (Except the score)

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a) **(13 pts)** Find all the points of the hyperbola $\mathbf{xy = 1}$ that are closest to the origin, using the method of Lagrange multipliers..

b) **(12 pts)** Find and classify all the critical points of $\mathbf{f(x, y) = 3x - x^3 - 3xy^2}$.

a) Let $f(x,y) = x^2+y^2$, which calculates the square of the distance of a point (x,y) to $(0,0)$. The question asks those points which minimize f subject to the costraint $xy-1=0=g(x,y)$

$\nabla f = \langle 2x, 2y \rangle$, $\nabla g = \langle y, x \rangle$

solve $\begin{cases} \nabla f = \lambda \cdot \nabla g \\ g = 0 \end{cases}$ $\Rightarrow$ $2x = \lambda y$ and $2y = \lambda x$

$\Rightarrow \lambda = \dfrac{2x}{y} = \dfrac{2y}{x}$ or $x^2 = y^2$
$x = \mp y$

Since $xy = 1$ we have $x = y$. Thus $x^2 = 1$ gives $x = \mp 1$

Hence the closest points are $(1,1)$ and $(-1,-1)$

b) $\nabla f = \langle 3 - 3x^2 - 3y^2, -6xy \rangle = \vec{0}$ implies

$x^2+y^2 = 1$ and $-6xy = 0$
$\Downarrow$
$x = 0$ or $y = 0$.

Thus $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$ are crit. pts.

$f_{xx} = -6x$ $f_{xy} = -6y$ $f_{yy} = -6x$ gives $\Delta = f_{xx}f_{yy} - f_{xy}^2$
$= 36x^2 - 36y^2 = 36(x^2-y^2)$

$\Delta(0,1) = \Delta(0,-1) = -36 < 0 \Rightarrow f$ has saddle pts. at $(0,1), (0,-1)$
$\Delta(1,0) = 36$ and $f_{xx}(1,0) = -6 \Rightarrow f$ has a loc. max at $(1,0)$
$\Delta(-1,0) = 36$ and $f_{xx}(-1,0) = 6 \Rightarrow f$ has a loc. min. at $(-1,0)$

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QUESTION 4 The blanks below will be filled by students. (Except the score)

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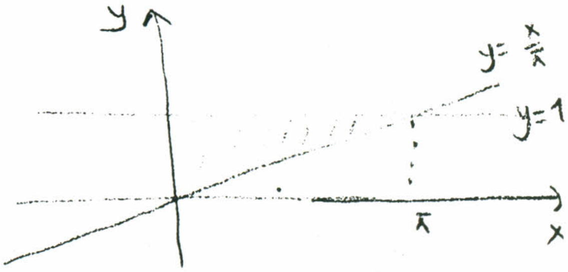
a) **(12 pts)** Evaluate the integral $\displaystyle\int_0^\pi \int_{\frac{x}{\pi}}^1 \mathbf{y^4 \sin(xy^2)\ dy\ dx}$.

b) **(13 pts)** Let **I** be the integral in cylindirical coordinates given by $\mathbf{I} = \displaystyle\int_0^\pi \int_0^1 \int_0^{\sqrt{4-r^2}} \mathbf{dz\ dr\ d\theta}$.
Express **I** in terms of spherical coordinates. Do not calculate the integral.

a) $\frac{x}{\pi} \le y \le 1$
$0 \le x \le \pi$ is the region

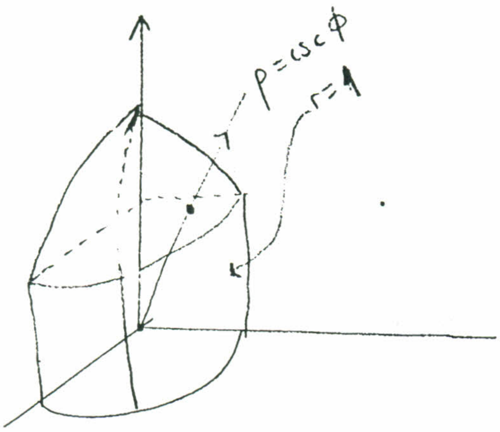


Changing the order we get

$$\int_0^\pi \int_{\frac{x}{\pi}}^1 y^4 \sin(xy^2)\, dy\, dx = \int_0^1 \int_0^{y\pi} y^4 \sin(xy^2)\, dx\, dy = \int_0^1 y^4 \cdot \frac{1}{y^2} \left[-\cos(xy^2)\right]\Big|_0^{y\pi} dy$$

$$= \int_0^1 y^2 \left(1 - \cos(y^3\pi)\right) dy = \left(\frac{y^3}{3} - \frac{1}{3\pi}\sin(y^3\pi)\right)\Bigg|_0^1 = \frac{1}{3}$$

b) $z = \sqrt{4-r^2}$ and $r = \rho \sin\phi$
$\Rightarrow x^2+y^2+z^2 = 4$
$\Rightarrow \rho = 2$



$$\int_0^\pi \int_0^1 \int_0^{\sqrt{4-r^2}} dz\, dr\, d\theta = \int_0^\pi \int_0^1 \int_0^{\sqrt{4-r^2}} \frac{1}{r} \cdot r\, dz\, dr\, d\theta$$

$$= \int_0^\pi \int_0^{\pi/6} \int_0^2 \frac{1}{\rho\sin\phi}\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$+ \int_0^\pi \int_{\pi/6}^{\pi/2} \int_0^{\csc\phi} \frac{1}{\rho\sin\phi}\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$