1.7: INTRODUCTION TO PROOFS

Basic Definitions Needed for Proofs In The Remainder of This Section:

(1) Given an integer n.

- n is ***even*** if there exists an integer k such that
$$n = 2k.$$
- n is ***odd*** if there exists an integer k such that
$$n = 2k+1.$$

(2) A number x is ***rational*** if it can be written in the form $x = m/n$, where m and n are integers (with $n \neq 0$). x is ***irrational*** if it is not a rational number.

I. Direct Proof of $P \rightarrow Q$

Suppose P.

$\vdots$

Therefore Q.

Thus, $P \rightarrow Q$. ∎

Example 1: Prove that if n is odd, then $n + 1$ is even.

Your strategy for developing a direct proof of an implication should involve the following steps:

1. Determine precisely the hypothesis (P) and the conclusion (Q).
2. If necessary, replace P (and/or Q) with a more usable, but still equivalent, proposition.
3. Develop a chain of statements, each deducible from its predecesors or other known results, that leads from P to Q.

Example 2: Suppose n is an integer. Prove that n is even if n^2 is even.

Recall that $(P \to Q) \equiv (\neg Q \to \neg P)$.

Proof by Contrapositive of $P \to Q$

Suppose $\neg Q$.
$\vdots$
Therefore $\neg P$.
Thus, $P \to Q$. ■

Example 3: Prove that $\sqrt{2}$ is irrational.

Proof by Contradiction of *Q*	**Proof by Contradiction of $P \to Q$**
Suppose $\neg Q$. ⋮ Therefore R. ⋮ Therefore $\neg R$. Hence, $R \wedge \neg R$, a contradiction. Thus, Q. ■	Suppose P and $\neg Q$. ⋮ Therefore $\neg P$. Hence, $P \wedge \neg P$, a contradiction. Thus, Q. ■

Example 4: Let m and n be positive real numbers. Prove if $m \geq n$, then $m^0 \geq n^0$.

Trivial proof: $P \to Q$ is automatically true whenever Q is always true.

Example 5: Let m and n be positive real numbers. Prove if $mn < 0$, then $m < 0$ or $n < 0$.

Vacuous proof: $P \to Q$ is automatically true if P is always false.

SECTION 1.7 – INTRODUCTION TO PROOFS

METHODS OF PROVING THEOREMS

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