

# NUMERICAL METHODS

## Week-13

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# Linear Programming (LP)

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# Linear Programming (LP)

- **What** is Optimization and LP?
- **Why** do we use LP?
- **How** do we solve LP?

# What is Optimization & LP?

- **Optimization** is the mathematical discipline which is concerned with finding the maxima and minima of functions, possibly subject to constraints.
- **A Linear Programming (LP) model** seeks to maximize or minimize **a linear function**, subject to a set of linear constraints.
- The linear model consists of the following components:
  - A set of decision variables.
  - An objective function.
  - A set of constraints.

# Why Linear Programming?

- The Importance of Linear Programming
  - Many real world problems lend themselves to linear programming modeling.
  - Many real world problems can be approximated by linear models.
  - There are well-known successful applications in:
    - Engineering
    - Manufacturing
    - Marketing
    - Finance (investment)
    - Advertising
    - Agriculture
    - Transportation
    - ...

# How to represent and Solve LP?

- First Primal Form

$$\begin{cases} \text{maximize: } c^T x \\ \text{constraints: } \begin{cases} Ax \leq b \\ x \geq 0 \end{cases} \end{cases}$$

$$\begin{cases} \text{maximize: } 2x_1 + 3x_2 \\ \text{constraints: } \begin{cases} 4x_1 + 5x_2 \leq 6 \\ 7x_1 + 8x_2 \leq 9 \\ 10x_1 + 11x_2 \leq 12 \\ x_1 \geq 0 \quad x_2 \geq 0 \end{cases} \end{cases}$$

# Problem Formulation

To Formulate the LP problem

- (i) Pick out important information
- (ii) Formulate constraints
- (iii) Formulate objective function

# Problem Formulation- Example-1

Suppose that a certain factory uses **two raw materials** to produce two products. Suppose also that the following are true:

- Each unit of the first product requires 5 units of the first raw material and 3 of the second.
- Each unit of the second product requires 3 units of the first raw material and 6 of the second.
- On hand are 15 units of the first raw material and 18 units of the second.
- The profits on sales of the products are 2 per unit for the first product and 3 per unit for the second product.

**How should the raw materials be used to realize a maximum profit?**

- To answer this question,  
variables  $x_1$  and  $x_2$  are introduced to represent  
the number of units of the two products to be  
manufactured.

**In terms of these variables, the profit is**

$$2x_1 + 3x_2$$

- The process uses up  $5x_1 + 3x_2$  units of the first raw material and  $3x_1 + 6x_2$  units of the second.

The limitations in the fact above are expressed by these inequalities:

$$\begin{cases} 5x_1 + 3x_2 \leq 15 \\ 3x_1 + 6x_2 \leq 18 \end{cases}$$



**The LP formulation for this problem is:**

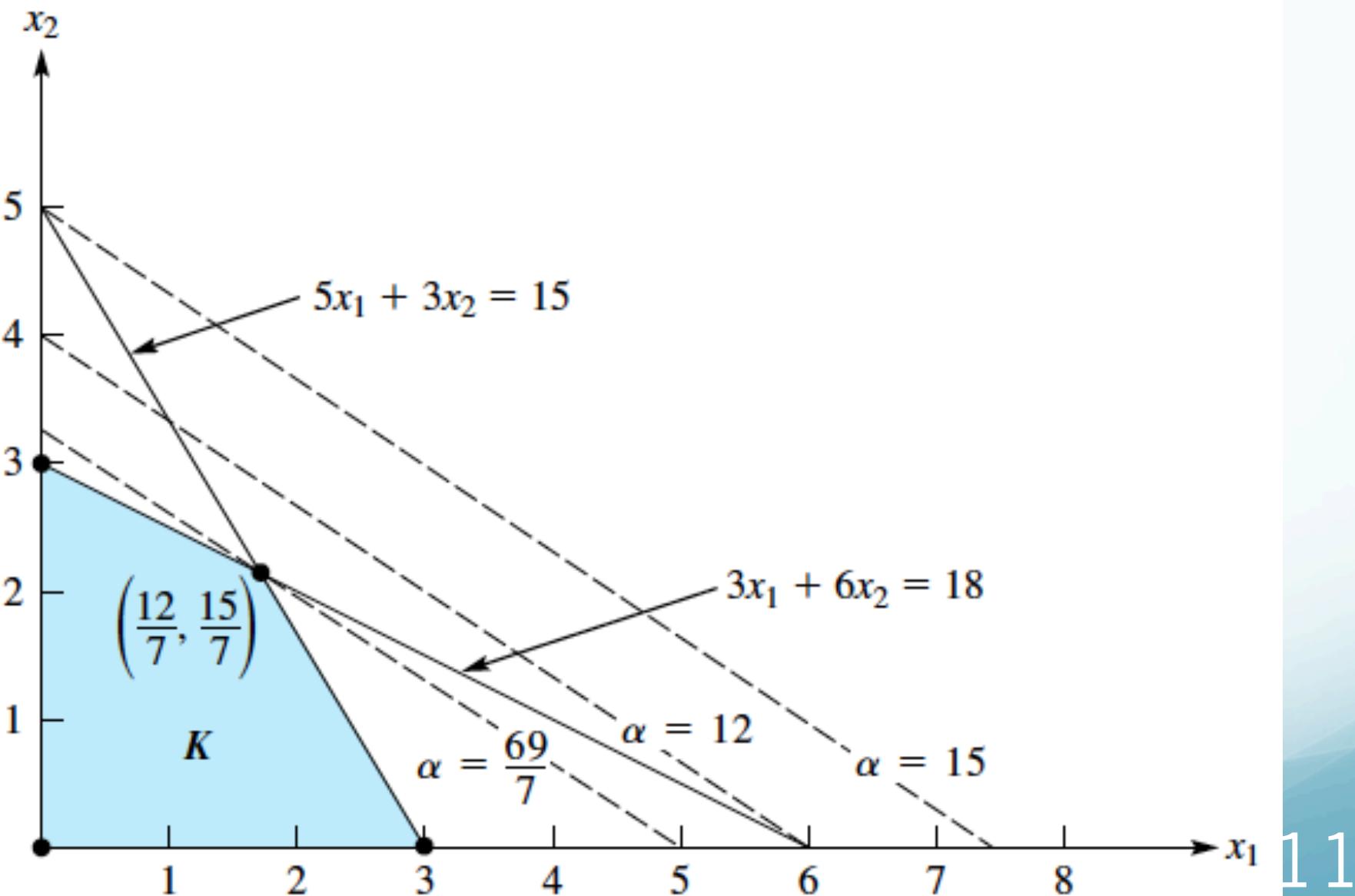
$$\begin{cases} \text{maximize: } 2x_1 + 3x_2 \\ \text{constraints: } \begin{cases} 5x_1 + 3x_2 \leq 15 \\ 3x_1 + 6x_2 \leq 18 \\ x_1 \geq 0 \quad x_2 \geq 0 \end{cases} \end{cases}$$

- More precisely, among all vectors  $x$  in the set

$$K = \{x: x \geq 0, 5x_1 + 3x_2 \leq 15, 3x_1 + 6x_2 \leq 18\}$$

we want the one that makes  $2x_1 + 3x_2$  as large as possible.

# Graphical Solution



## Example-2:

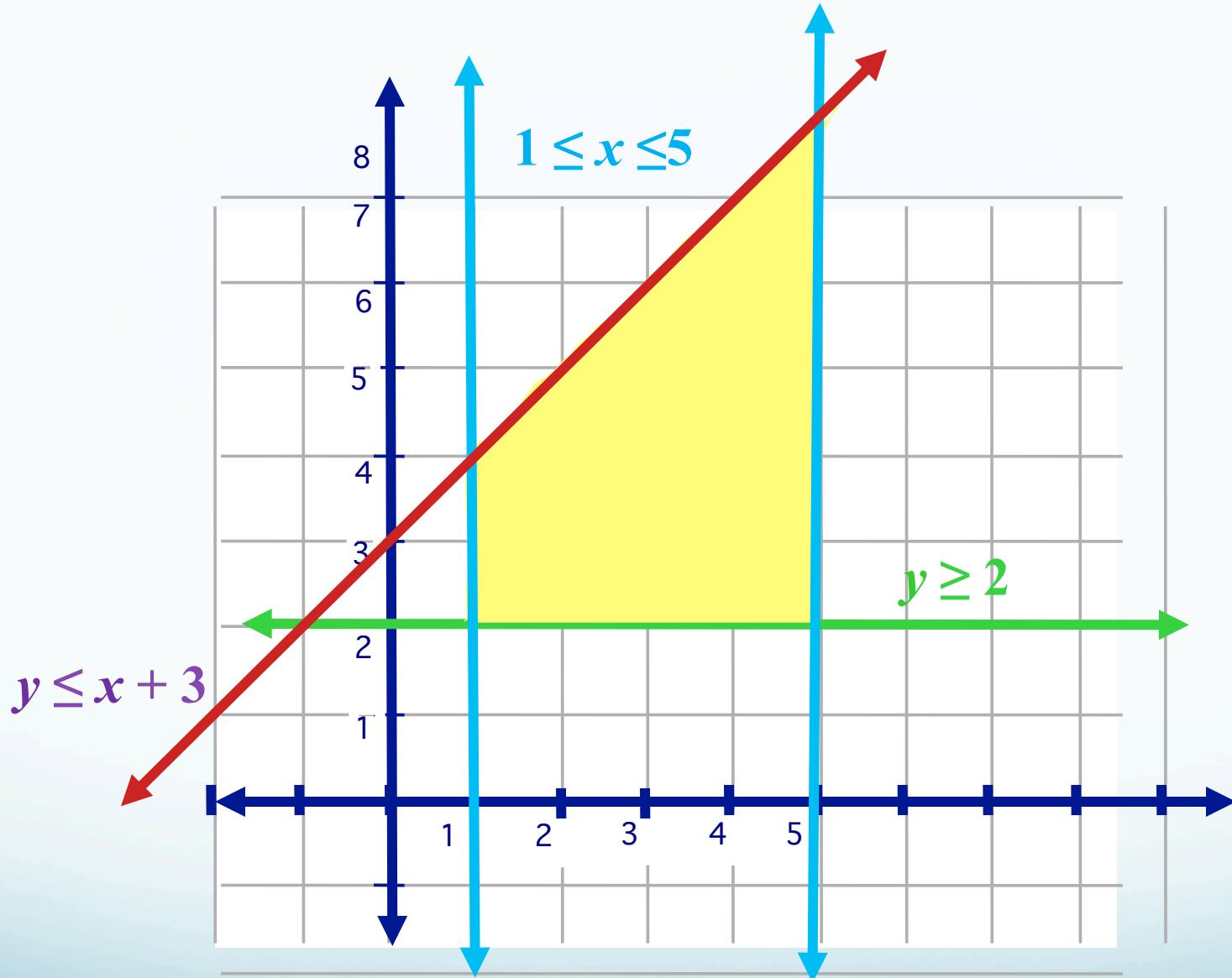
Find the minimum and maximum value of the function  $f(x, y) = 3x - 2y$ .

We are given the constraints:

- $y \geq 2$
- $1 \leq x \leq 5$
- $y \leq x + 3$

# In LP Perspective..

- Find the minimum and maximum values by graphing the inequalities and finding the vertices of the polygon formed.
- Substitute the vertices into the function and find the largest and smallest values.



- The vertices of the quadrilateral formed are:  
 $(1, 2) (1, 4) (5, 2) (5, 8)$
- Plug these points into the function  $f(x, y) = 3x - 2y$

$$f(x, y) = 3x - 2y$$

- $f(1, 2) = 3(1) - 2(2) = 3 - 4 = -1$
- $f(1, 4) = 3(1) - 2(4) = 3 - 8 = -5$
- $f(5, 2) = 3(5) - 2(2) = 15 - 4 = 11$
- $f(5, 8) = 3(5) - 2(8) = 15 - 16 = -1$

# Solution

- $f(1, 4) = -5$  minimum
- $f(5, 2) = 11$  maximum

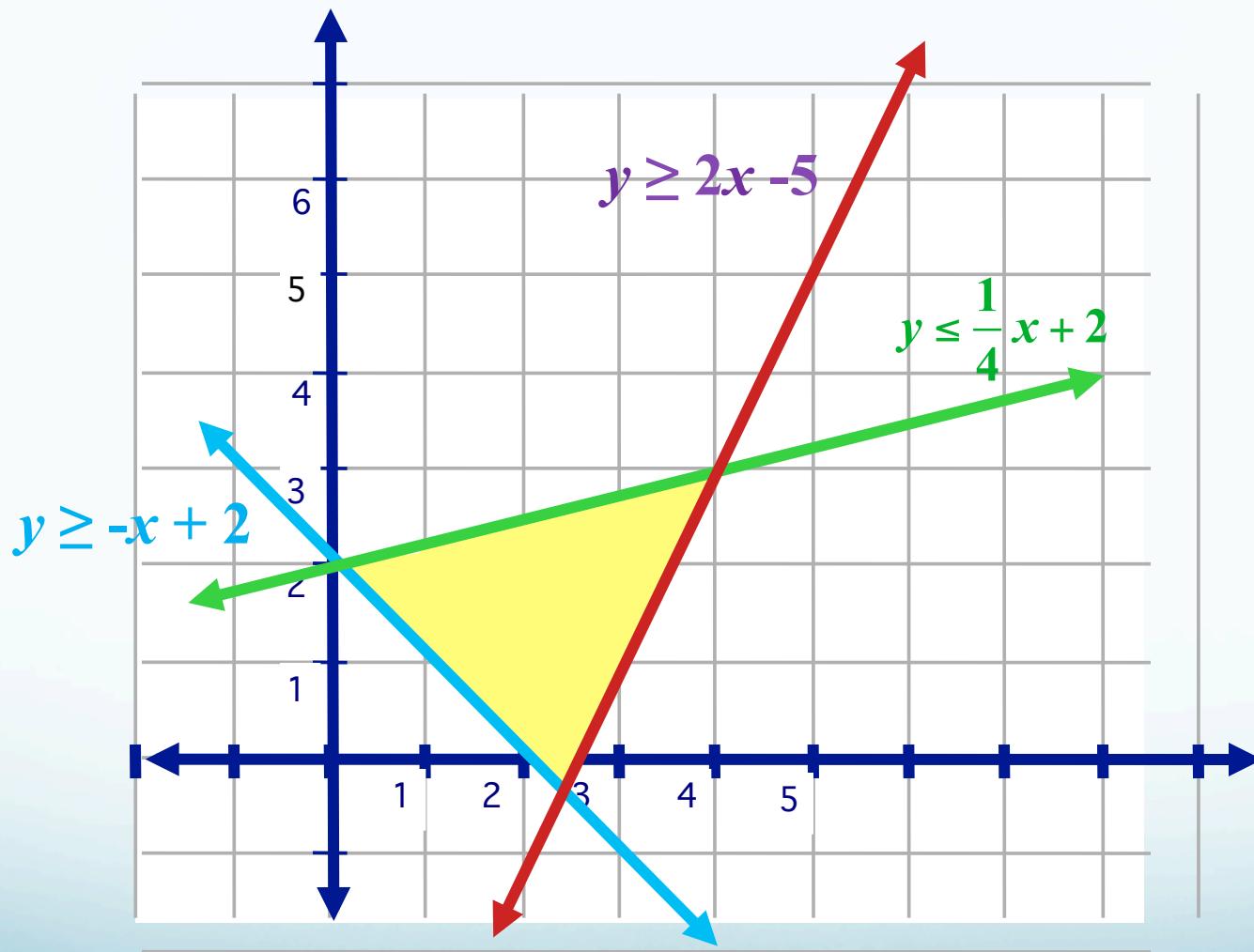
## Example 3:

Find the minimum and maximum value of the function

$$f(x, y) = 4x + 3y$$

We are given the constraints:

- $y \geq -\frac{1}{x} + 2$
- $y \leq \frac{1}{4}x + 2$
- $y \geq 2x - 5$



$$f(x, y) = 4x + 3y$$

- $f(0, 2) = 4(0) + 3(2) = 6$
- $f(4, 3) = 4(4) + 3(3) = 25$
- $f\left(\frac{7}{3}, -\frac{1}{3}\right) = 4\left(\frac{7}{3}\right) + 3\left(-\frac{1}{3}\right) = \frac{28}{3} - 1 = \frac{25}{3}$



- $f(0, 2) = 6$  minimum

- $f(4, 3) = 25$  maximum

# Problem Formulation- Example 4

- A small factory produces two types of toys: cars and diggers. In the manufacturing process two machines are used: the moulder and the colouriser. A digger needs 2 hours on the moulder and 1 hour on the colouriser. A car needs 1 hour on the moulder and 1 hour on the colouriser. The moulder can be operated for 16 hours a day and the colouriser for 9 hours a day. Each digger gives a profit of £16 and each car gives a profit of £14. The profit needs to be maximised.
- How do we formulate this problem?

# PICKING OUT IMPORTANT INFORMATION

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- The moulder can be operated for 16 hours a day and the colouriser for 9 hours a day.
- Using the decision variables

$d$  = number of diggers

$c$  = number of cars

make two constraints from this information.

# FORMING CONSTRAINT 1

## THE MOULDER

- A digger needs 2 hours on the moulder and 1 hour on the colouriser. A car needs 1 hour on the moulder and 1 hour on the colouriser.
- The moulder can be operated for 16 hours a day and the colouriser for 9 hours a day.

$$2d + c \leq 16$$

# FORMING CONSTRAINT 2

## THE COLOURISER

- A digger needs 2 hours on the moulder and 1 hour on the colouriser. A car needs 1 hour on the moulder and 1 hour on the colouriser.
- The moulder can be operated for 16 hours a day and the colouriser for 9 hours a day.

$$d + c \leq 9$$

# FORMING THE OBJECTIVE FUNCTION

- Each digger gives a profit of £16 and each car gives a profit of £14.

$$Z = 16d + 14c$$

- MAXIMIZE  $Z = 16d + 14c$   
subject to the constraints:
  - (i)  $2d + c \leq 16$
  - (ii)  $d + c \leq 9$
  - (iii)  $c \geq 0, d \geq 0$

- **VERY IMPORTANT**

**DON'T FORGET YOUR NON – NEGATIVITY  
CONSTRAINTS !**

# The Prototype Production- Example 5

- A Factory manufactures two chip models:
  - Chip A.
  - Chip B.
- Resources are limited to
  - 1000 units of special material.
  - 40 hours of production time per week.

# The Prototype Production- Example 5

- Marketing requirement
  - Total production cannot exceed 700 dozens.
  - Number of dozens of Chip A cannot exceed number of dozens of Chip B by more than 350.
- Technological input
  - A requires 2 units of material and 3 minutes of labor per dozen.
  - B requires 1 units of material and 4 minutes of labor per dozen.

- The current production plan calls for:
  - Producing as much as possible of the more profitable product, A (\$8 profit per dozen).
  - Use resources left over to produce B (\$5 profit per dozen), while remaining within the marketing guidelines.

# The Linear Programming Model

- Decisions variables:
  - $X_1$  = Weekly production level of A (in dozens)
  - $X_2$  = Weekly production level of B (in dozens).
- Objective Function:
  - Weekly profit, to be maximized

# The Linear Programming Model

Max  $8X_1 + 5X_2$  (Weekly profit)

subject to

$$2X_1 + 1X_2 \leq 1000 \quad (\text{Material})$$

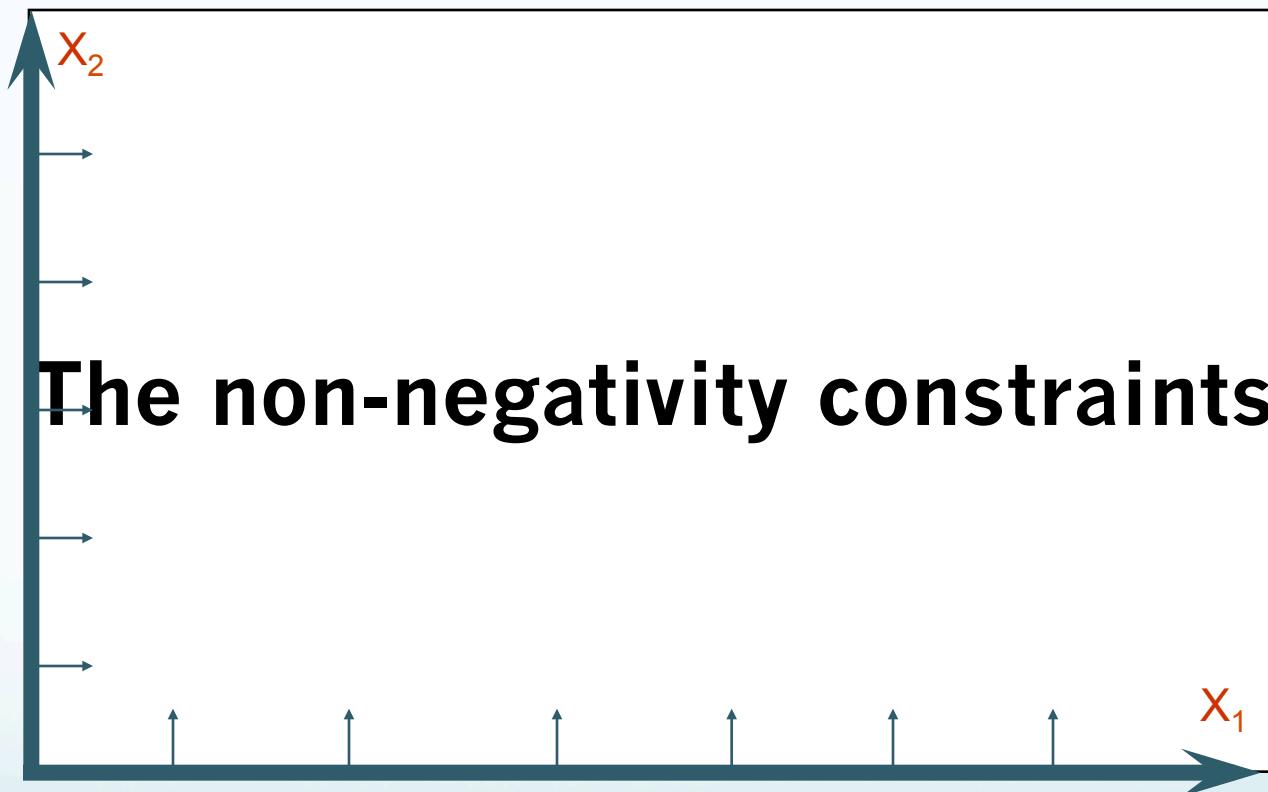
$$3X_1 + 4X_2 \leq 2400 \quad (\text{Production Time})$$

$$X_1 + X_2 \leq 700 \quad (\text{Total production})$$

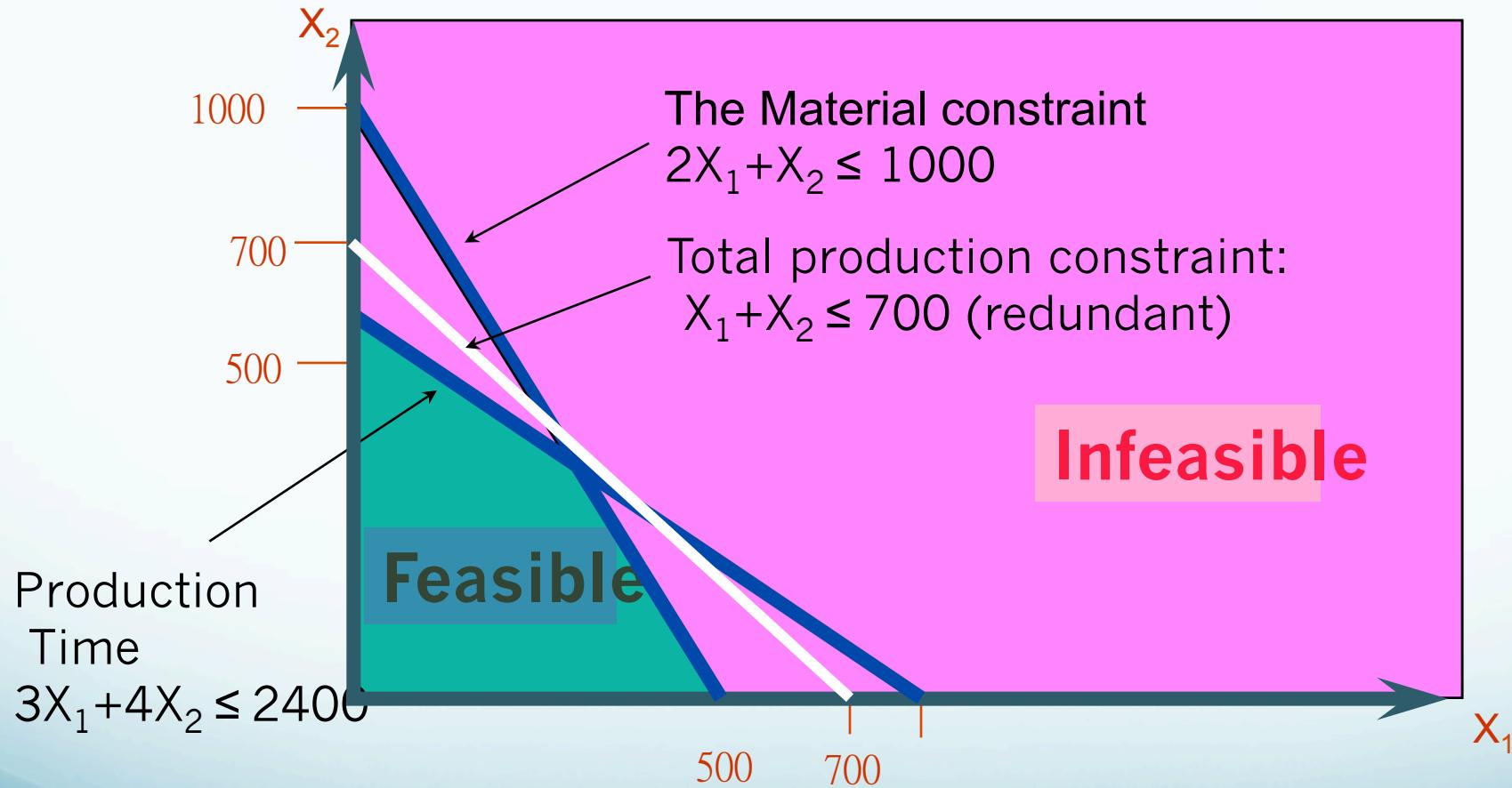
$$X_1 - X_2 \leq 350 \quad (\text{Mix})$$

$$X_j >= 0, \quad j = 1, 2 \quad (\text{Nonnegativity})$$

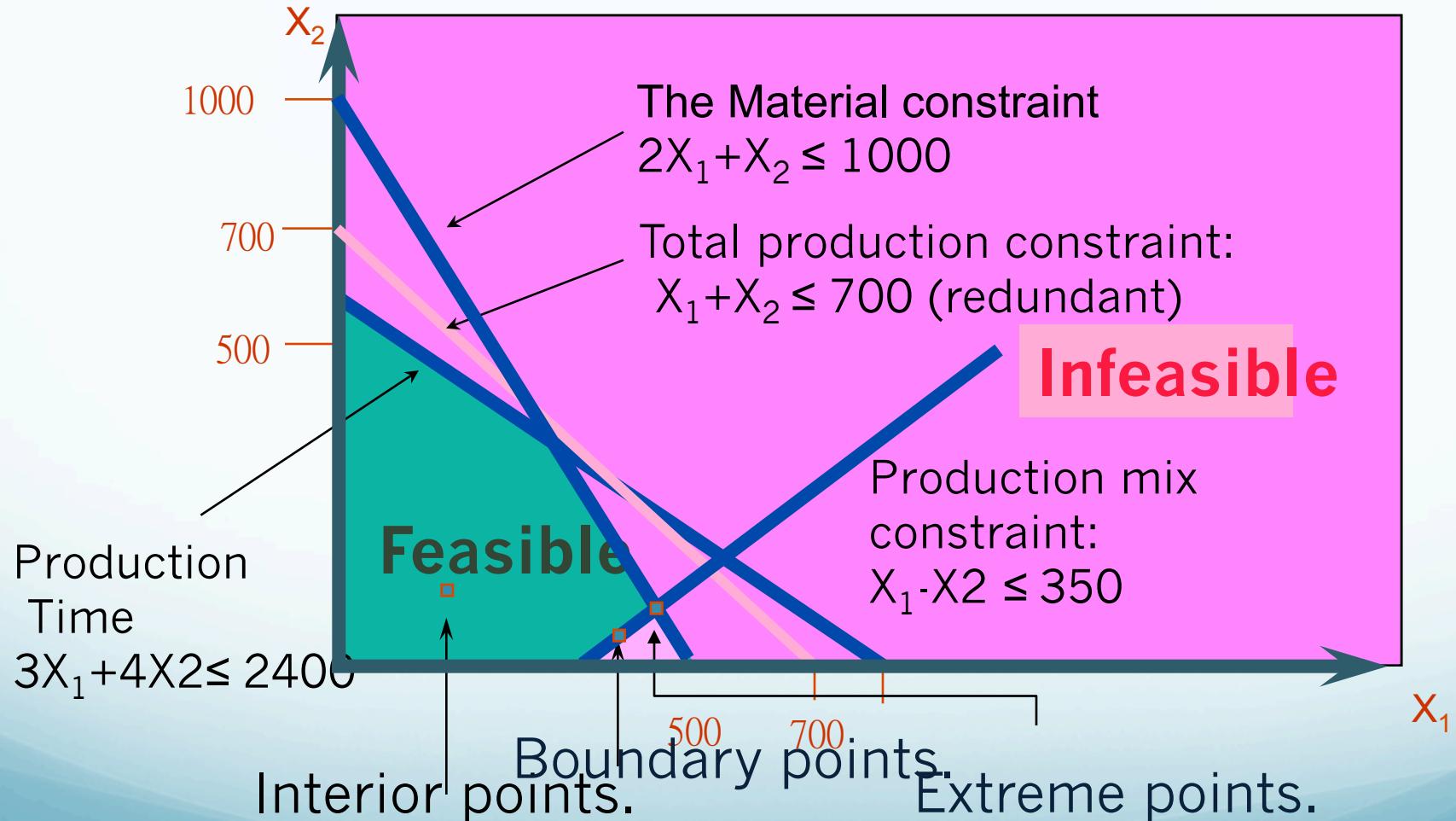
# Graphical Analysis – the Feasible Region



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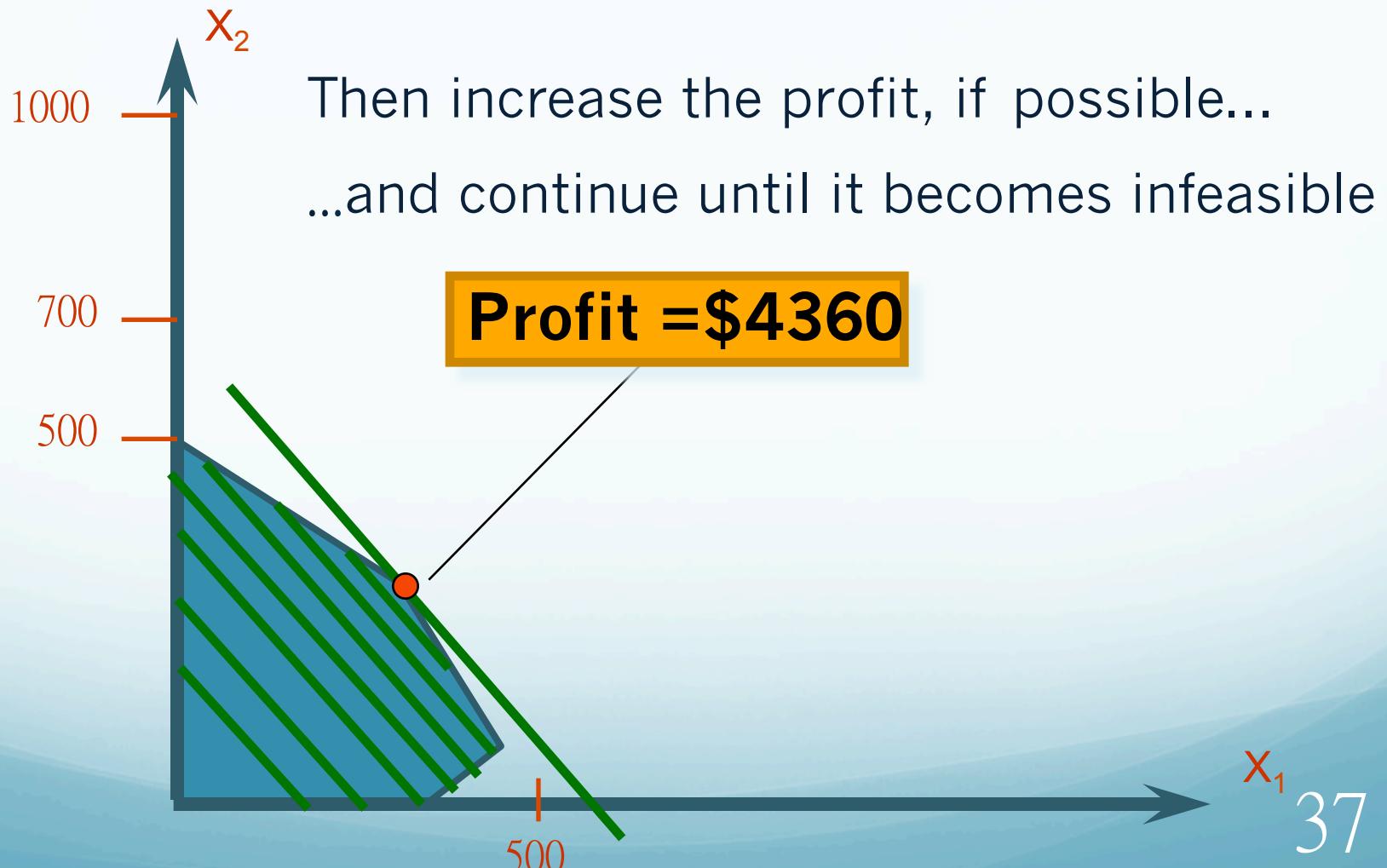
# Graphical Analysis – the Feasible Region



- There are three types of feasible points 36

# The search for an optimal solution

Start at some arbitrary profit, say profit = \$2,000...



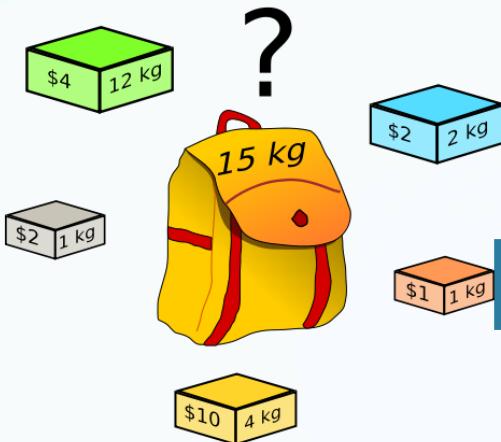
# Summary of the optimal solution

Chip A = 320 dozen

Chip B = 360 dozen

Profit = \$4360

- This solution utilizes all the materials and all the production hours.
- Total production is only 680 (not 700).
- Chip A production exceeds B's production by only 40 dozens.



# Knapsack Problem

- You have a knapsack that has capacity (weight)  $C$ .
- You have several items  $I_1, \dots, I_n$ .
- Each item  $I_j$  has a weight  $w_j$  and a benefit  $b_j$ .
- You want to place a certain number of copies of each item  $I_j$  in the knapsack so that:
  - The knapsack weight capacity is not exceeded and
  - The total benefit is maximal.

# Example

<b>Item</b>	<b>Weight</b>	<b>Benefit</b>
A	2	60
B	3	75
C	4	90

Capacity = 5

# Key question

- Suppose  $f(w)$  represents the *maximal possible benefit* of a knapsack with weight  $w$ .
- We want to find (in the example)  $f(5)$ .
- Is there anything we can say about  $f(w)$  for arbitrary  $w$ ?

# Key observation

- To fill a knapsack with items of weight  $w$ , we must have added items into the knapsack in some order.
- Suppose the last such item was  $I_j$  with weight  $w_i$  and benefit  $b_i$ .
- Consider the knapsack with weight  $(w - w_i)$ . Clearly, we chose to add  $I_j$  to this knapsack because of all items with weight  $w_i$  or less,  $I_j$  had the max benefit  $b_i$ .

# Key observation

- Thus,  $f(w) = \text{MAX} \{ b_j + f(w-w_j) \mid I_j \text{ is an item}\}.$
- This gives rise to an immediate recursive algorithm to determine how to fill a knapsack.

# $f(0), f(1)$

- $f(0) = 0$ . Why? The knapsack with capacity 0 can have nothing in it.
- $f(1) = 0$ . There is no item with weight 1.

$$f(2)$$

- $f(2) = 60$ . There is only one item with weight 60.
- **Choose A.**

# f(3)

- $f(3) = \text{MAX} \{ b_j + f(w \cdot w_j) \mid l_j \text{ is an item}\}.$   
 $= \text{MAX} \{ 60+f(3 \cdot 2), 75 + f(3 \cdot 3)\}$   
 $= \text{MAX} \{ 60 + 0, 75 + 0 \}$   
 $= 75.$

**Choose B.**

# f(4)

- $f(4) = \text{MAX} \{ b_j + f(w \cdot w_j) \mid l_j \text{ is an item}\}.$   
 $= \text{MAX} \{ 60 + f(4 \cdot 2), 75 + f(4 \cdot 3), 90 + f(4 \cdot 4)\}$   
 $= \text{MAX} \{ 60 + 60, 75 + f(1), 90 + f(0)\}$   
 $= \text{MAX} \{ 120, 75, 90\}$   
 $= 120.$

**Choose A.**

# f(5)

- $f(5) = \text{MAX} \{ b_j + f(w \cdot w_j) \mid l_j \text{ is an item}\}.$   
 $= \text{MAX} \{ 60 + f(5 \cdot 2), 75 + f(5 \cdot 3), 90 + f(5 \cdot 4)\}$   
 $= \text{MAX} \{ 60 + f(3), 75 + f(2), 90 + f(1)\}$   
 $= \text{MAX} \{ 60 + 75, 75 + 60, 90 + 0\}$   
 $= 135.$

**Choose A or B.**

# Result

- Optimal knapsack weight is 135.
- Two possible optimal solutions:
  - Choose A during computation of  $f(5)$ . Choose B in computation of  $f(3)$ .
  - Choose B during computation of  $f(5)$ . Choose A in computation of  $f(2)$ .
- Both solutions coincide. Take A and B.