In below table, CPU temperature measurements are shown. Calculate true error and relative true error.

Samples	Temperature (C)			
	Truth	Observed	True Error	Relative True Error
1	40,58	40,32	40,58-40,32=0,26	$0,26/40,58=\%0,6407$
2	40,65	40,77	40,65-40,77=-0,12	-0,12/40,65=-%0,2952
3	41,34	41,36	41,34-41,36=-0,02	-0,02/41,34=-%0,0484
4	41,91	42,20	41,91-42,20=-0,29	-0,29/41,91=-%0,6920

Derivative of a function f at a point x is defined by the equation;

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad f'(x)\approx\frac{f(x+h)-f(x)}{h}$$ for small h values.

- If $f(x)=x^3+3x^2$ and $h=0,6$

  Find approximate value of $f(3)$

  Find true value of $f'(3)$

  Find true error, relative true error.

$$f'(x)\approx\frac{f(x+h)-f(x)}{h} \qquad f'(x)\approx\frac{f(3+0,6)-f(3)}{0,6}\approx\frac{f(3,6)-f(3)}{0,6}$$

$f'(x)\approx\dfrac{(46,656+38,88)-(54)}{0,6}\approx\dfrac{31,536}{0,6}\approx 52,56$ **is approximate value**.

$f'(x)=3x^2+6x$ is derivative function of function $f(x)$

$f'(3)=27+18=45$ **is true value**.

True error = true value – approximate value = $45-52,56=-7,56$

Relative true error = true error / true value = $-7,56/45=-0,168(\%-16,8)$

- Compute approximate error and relative approximate error for $f'(3)$ using $h=0,6$ and $h=0,4$

Approximate error = present approximation – previous approximation

$f^{'}(x) \approx \dfrac{f(x+h)-f(x)}{h}$ $\quad f^{'}(x) \approx \dfrac{f(3+0,4)-f(3)}{0,4} \approx \dfrac{f(3,4)-f(3)}{0,4}$

$f^{'}(x) \approx \dfrac{(39,304+34,68)-(54)}{0,4} \approx \dfrac{73,984-54}{0,4} \approx \dfrac{19,984}{0,4} \approx 49,96$ **is present approximation**.

Approximate error = $49,96 - 52,56 - = 2,6$

Relative approximation error = approximation error / present approximation

$= \dfrac{-2,56}{49,96} =$ -0,051241(%5,1241)

Error measurement on floating point format.

Floating decimal point scientific form : $sign\ x\ mantisa\ x\ 10^{exponent}$

$\sigma\ x\ m\ x\ 10^{e} \qquad -2,5678x10^{2} \qquad \sigma=-1 \quad m=2,5678 \quad e=2$

$\left[(1)_2 < m < (10)_2\right]$ 1 is not stored as it is always given to be 1.

Let we have a floating point format ten bit word;

sn	se	e	e	e	e	m	m	m	m

* 1 bit for sign of number
* 1 bit for sign of exponent
* 4 bit for exponent
* 4 bit for mantissa

Display $(123,9631)_{10}$ floating point number for the given format and calculate the true error while representation if it occurs.

$(123)_{10} = (1111011)_2 \qquad (0*2^7)+(1*2^6)+(1*2^5)+(1*2^4)+(1*2^3)+(0*2^2)+(1*2^1)+(1*2^0)$

$(0,9631)_{10} = (?)_2$

	Result	Integer part	Fractional part
0,9631*2	1,9262	1	0,9262
0,9262*2	1,8524	1	0,8524
0,8524*2	1,7048	1	0,7048
0,7048*2	1,4096	1	0,4096

0,4096*2	0,8192	0	0,8192
0,8192*2	1,6384	1	0,6384
0,6384*2	1,2768	1	0,2768
0,2768*2	0,5536	0	0,5536
0,5536*2	1,1072	1	0,1072

1111011,111101101 = **1,111011111101101** * 2^6

sn	se	e	e	e	e	m	m	m	m
0	0	0	1	1	0	1	1	1	0

$1{,}1110 * 2^6 = (1111000)_2 = 120$

True error = 123,9631 – 120 = 3,9631

8 – bit mantissa example for the same question

sn	se	e	e	e	e	m	m	m	m	m	m	m	m
0	0	0	1	1	0	1	1	1	0	1	1	1	1

$1{,}11101111 * 2^6 = (1111011{,}11{,})_2 = 123{,}75$

True error = 123,9631 – 123,75 = 0,2131

Use Naive Gauss Elimination to solve the system of linear equations.

$x_1 + 2x_2 + x_3 = -8$

$2x_1 - 2x_2 - 3x_3 = 0$

$-x_1 + x_2 + 2x_3 = 3$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 3 \end{bmatrix}$$

Add -2 times row1 to row2

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

Add -1 times row1 to row3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ -5 \end{bmatrix}$$

Add (1/2) times row2 to row3

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

Now elimination stops.

$0.5x_3 = 3$

$x_3 = 6$

$-6x_2 - 5 * 6 = 16$

$x_2 = -\frac{23}{3}$

$x_1 + 2x_2 + x_3 = -8$

$x_1 = \frac{4}{3}$

Solve same linear system using LU decomposition.

Ax=B

Apply gauss elimination **only matrix A**.

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$ same operation as gauss elimination $U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 0.5 \end{bmatrix}$

Now find [L] matrix

$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ $l_{21} = \frac{a_{21}}{a_{11}}$, $l_{31} = \frac{a_{31}}{a_{11}}$, $l_{32} = \frac{a_{32}}{a_{22}}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & \mathbf{-0.5} & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 3 \end{bmatrix}$$

$Z = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \\ 3 \end{bmatrix}$$

$0.5x_3 = 3$

$x_3 = 6$

$-6x_2 - 5 * 6 = 16$

$x_2 = -\frac{23}{3}$

$x_1 + 2x_2 + x_3 = -8$

$x_1 = \frac{4}{3}$

Details of LU decomposition.

A=LU

LUx=B

Ux=L^{-1}B

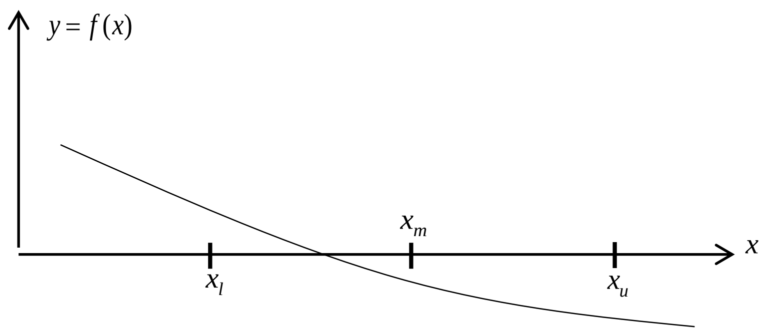
x=$U^{-1}L^{-1}$B

x=L^{-1}Z

Example of Bisection Method

A function $f(x)$ is defined as $f(x) = x^2 - 4x - 5$. For $y = f(x) = 0$ estimate a root of this function using Bisection method. Use $[0, 48]$ as initial estimation range points and use absolute relative approximate error notation for error of estimated root at each iteration.

Recall



$$x_m = \frac{x_l + x_u}{2}$$

$$|E| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| * 100$$

$f(x_l) * f(x_m) < 0$ ***the root lies between*** x_l ***and*** x_m

$f(x_l) * f(x_m) > 0$ ***the root lies between*** x_m ***and*** x_u

$f(x_l) * f(x_m) = 0$ x_m ***is root***

$$x_l = 0 \qquad f(x_l) = f(0) = 0 - 0 - 5 = -5$$
$$x_u = 48 \qquad f(x_u) = f(48) = 2304 - 192 - 5 = 2107 \qquad x_m = (0 + 48)/2 = 24$$
$$x_m = 24 \qquad f(x_m) = f(24) = 576 - 96 - 5 = 475$$
$$f(x_l) * f(x_m) < 0 \ \ the\ root\ lies\ between\ x_l\ and\ x_m\ so\ new\ estimation\ range\ [0, 24]$$

$$x_l = 0 \qquad f(x_l) = f(0) = 0 - 0 - 5 = -5$$
$$x_u = 24 \qquad f(x_u) = f(24) = 576 - 96 - 5 = 475 \qquad x_m = (0 + 24)/2 = 12$$
$$x_m = 12 \qquad f(x_m) = f(12) = 144 - 48 - 5 = 91$$
$$f(x_l) * f(x_m) < 0 \ \ the\ root\ lies\ between\ x_l\ and\ x_m\ so\ new\ estimation\ range\ [0, 12]$$

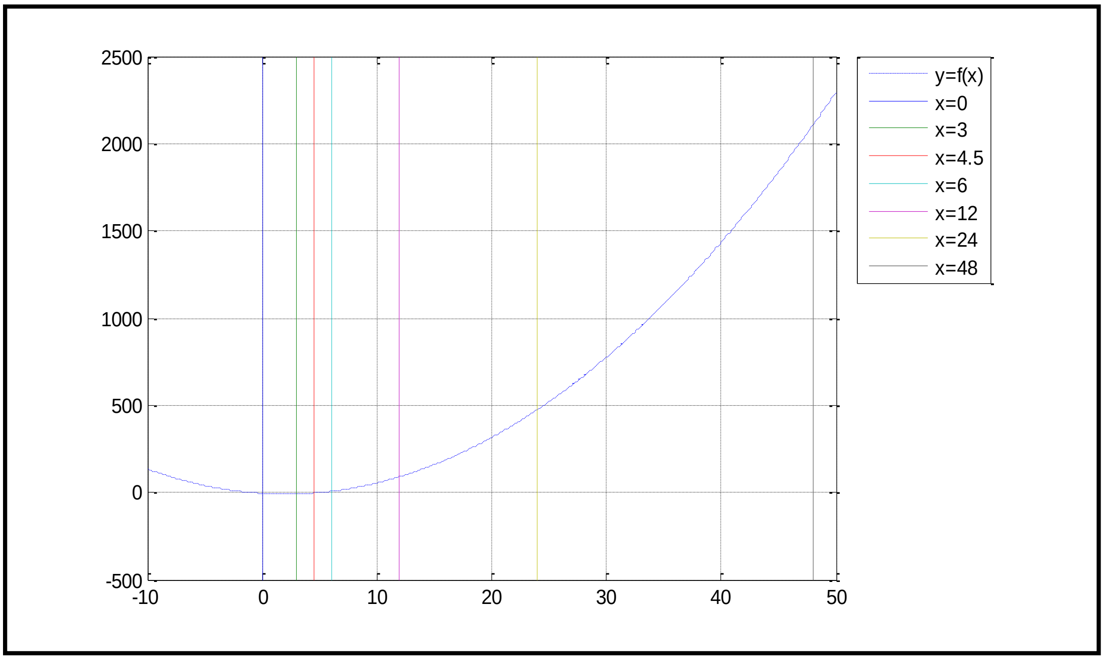
$$x_l = 0 \qquad f(x_l) = f(0) = 0 - 0 - 5 = -5$$
$$x_u = 12 \qquad f(x_u) = f(12) = 144 - 48 - 5 = 91 \qquad x_m = (0 + 12)/2 = 6$$
$$x_m = 6 \qquad f(x_m) = f(6) = 36 - 24 - 5 = 7$$
$$f(x_l) * f(x_m) < 0 \ \ the\ root\ lies\ between\ x_l\ and\ x_m\ so\ new\ estimation\ range\ [0, 6]$$

$$x_l = 0 \qquad f(x_l) = f(0) = 0 - 0 - 5 = -5$$
$$x_u = 6 \qquad f(x_u) = f(6) = 36 - 24 - 5 = 7 \qquad x_m = (0 + 6)/2 = 3$$
$$x_m = 3 \qquad f(x_m) = f(3) = 9 - 12 - 5 = -8$$
$$f(x_l) * f(x_m) > 0 \ \ the\ root\ lies\ between\ x_m\ and\ x_u\ so\ new\ estimation\ range\ [3, 6]$$
$$x_l = 3 \qquad f(x_l) = f(3) = 9 - 12 - 5 = -8$$
$$x_u = 6 \qquad f(x_u) = f(6) = 36 - 24 - 5 = 7 \qquad x_m = (3 + 6)/2 = 4{,}5$$
$$x_m = 4{,}5 \qquad f(x_m) = f(4{,}5) = 20{,}25 - 18 - 5 = -2{,}75$$
$$f(x_l) * f(x_m) > 0 \ \ the\ root\ lies\ between\ x_m\ and\ x_u\ so\ new\ estimation\ range\ [4.5\ ,\ 6]$$

Table - Estimation of root for initial range given at each iteration. Iteration will be continued until root is estimated with an acceptable error.

Iteration	x_l	x_u	x_m	$f(x_l)$	$f(x_u)$	$f(x_m)$	E
1	**0**	**48**	**24**	**-5**	**2107**	**475**	**-**
2	**0**	**24**	**12**	**-5**	**475**	**91**	**%100**
3	**0**	**12**	**6**	**-5**	**91**	**7**	**%100**
4	**0**	**6**	**3**	**-5**	**7**	**-8**	**%100**
5	**3**	**6**	**4,5**	**-8**	**7**	**-2,75**	**%33,33**

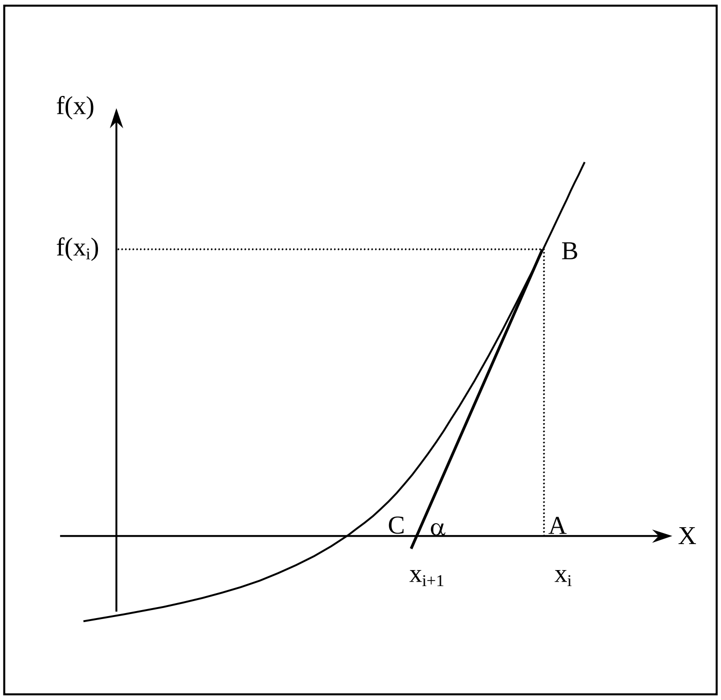
Figure - Bisection lines for given non-linear equation.



Example of Newton Raphson Method

A function $f(x)$ is defined as $f(x)=x^3-10x^2+100$. For $y=f(x)=0$ estimate a root of this function using Newton-Raphson method. Initial guess value of root x_0 is 15 and use absolute relative approximate error notation for error of estimated root.

Recall



$$\tan(\alpha)=\frac{AB}{AC}$$

$$f'(x_i)=\frac{f(x_i)}{x_i-x_{i+1}}$$

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

Gist of prediction of x_{i+1}

$$|E|=\left|\frac{x_{i+1}-x_i}{x_{i+1}}\right|*100$$

Function : $f(x) = x^3 - 10x^2 + 100$

Derivative of function : $f'(x) = 3x^2 - 20x$

$$x_1 = 15 - \frac{f(15)}{f'(15)} = 15 - \frac{3375 - 2250 + 100}{675 - 300} = 15 - \frac{1225}{375} = 15 - 3,26 = 11,74$$

$$x_2 = 11,74 - \frac{f(11,74)}{f'(11,74)} = 11,74 - \frac{1618,09 - 1378,27 + 100}{413,48 - 234,8} = 11,74 - \frac{339,82}{178,68} = 11,74 - 1,89 = 9,85$$

$$x_3 = 9,85 - \frac{f(9,85)}{f'(9,85)} = 9,85 - \frac{955,67 - 970,22 + 100}{291,06 - 197} = 9,85 - \frac{85,45}{94,06} = 9,85 - 1,89 = 8,94$$

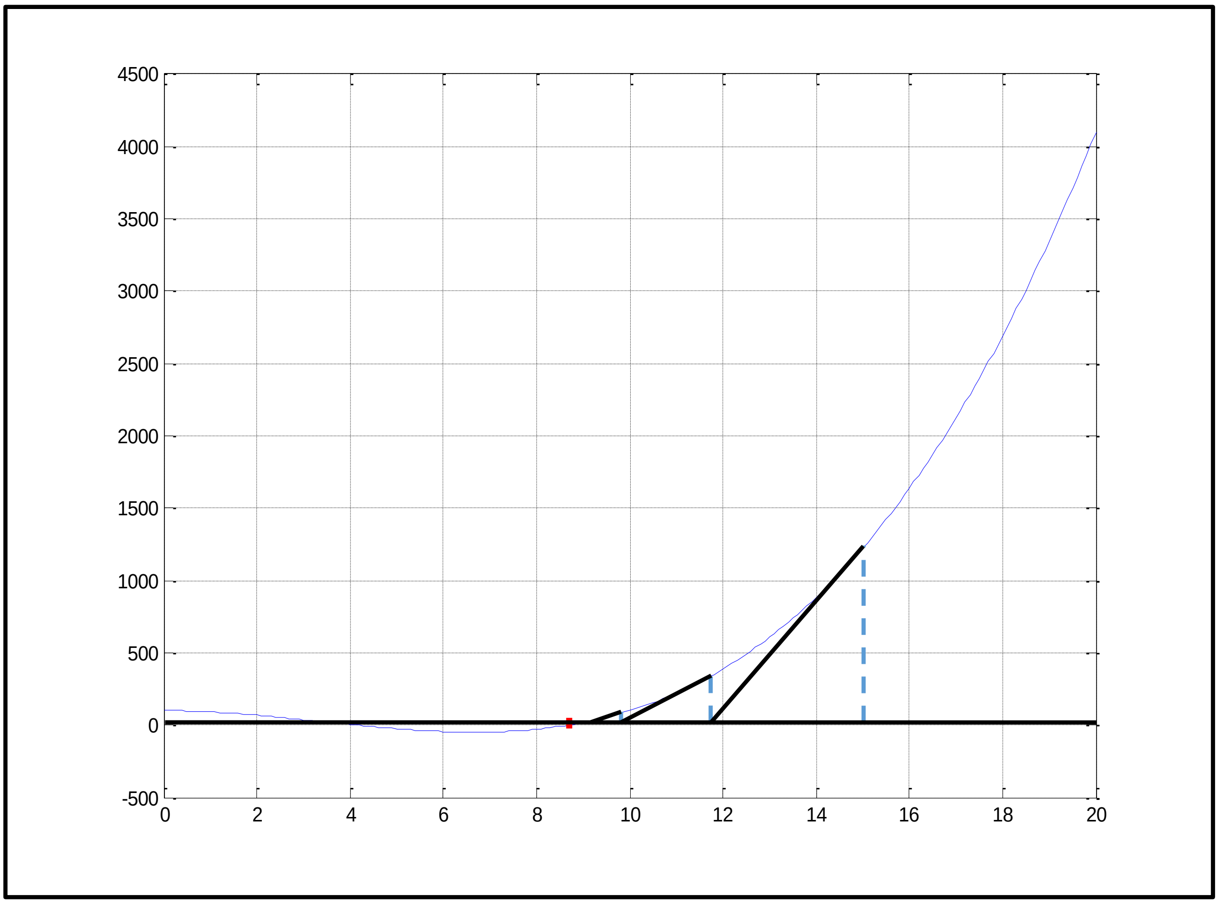
$$x_4 = 8,94 - \frac{f(8,94)}{f'(8,94)} = 8,94 - \frac{714,51 - 799,23 + 100}{239,77 - 178,8} = 8,94 - \frac{15,28}{60,97} = 8,94 - 0,25 = 8,69$$

$$x_5 = 8,69 - \frac{f(8,69)}{f'(8,69)} = 8,69 - \frac{656,23 - 755,16 + 100}{226,54 - 173,8} = 8,69 - \frac{1,07}{52,74} = 8,69 - 0,02 = 8,67$$

Table - Estimation of root for the function described in Q2 at each iteration. Iteration will be continued until root is estimated with an acceptable error.

Iteration	x	$f(x)$	$f'(x)$	E
0	$x_0 = 15$	**1225**	**375**	**-**
1	$x_1 = 11,74$	**339,82**	**178,68**	**%27,76**
2	$x_2 = 9,85$	**85,45**	**94,06**	**%19,18**
3	$x_3 = 8,94$	**15,28**	**60,97**	**%10,17**
4	$x_4 = 8,69$	**1,07**	**52,74**	**%2,87**
5	$x_5 = 8,67$	**-**	**-**	**%0,23**

Figure – Graphical analysisof Newton-Raphson method for given non-linear equation.


4500
4000
3500
3000
2500
2000
1500
1000
500
0
-500
0
2
4
6
8
10
12
14
16
18
20

Direct Method Example

General formula for an n th order polynomial can be written as
$f(x)=a_0+a_1x+a_2x^2+.........+a_nx^n$ for (n+1) data points. Data points :
$(x_0,y_0),(x_1,y_1),.........(x_n,y_n)$

Direct Method with linear interpolation
For given data table below model the linear interpolating polynomial and estimate the function value at given point.

x	5	12	6	9
f(x)	28	63	**?**	**?**

$x_1=5 \qquad f(5)=28 \qquad f(5)=a_0+a_15 \qquad 28=a_0+a_15 \qquad \boxed{a_0=3}$

$x_2=12 \qquad f(12)=63 \qquad f(12)=a_0+a_112 \qquad 63=a_0+a_112 \qquad \boxed{a_1=5}$

$f(x)=3+5x \qquad f(6)=3+5\cdot 6=33$

$f(9)=3+5\cdot 9=48$

Direct Method with quadratic interpolation
For given data table below model the quadratic interpolating polynomial and estimate the function value at given point.

x	5	12	20	17
f(x)	28	63	80	**?**

$x_1=5 \qquad f(5)=28 \qquad f(5)=a_0+a_15+a_225 \qquad 28=a_0+a_15+a_225$

$x_2=12 \qquad f(12)=63 \qquad f(12)=a_0+a_112+a_2144 \qquad 63=a_0+a_112+a_2144$

$x_2=20 \qquad f(20)=80 \qquad f(20)=a_0+a_120+a_2400 \qquad 80=a_0+a_120+a_2400$

$a_0=-8.5$

$a_1=8,25 \qquad \boxed{f(x)=-8,5+8,25x-0,19x^2}$

$a_2=-0,19 \qquad \boxed{f(17)=-8,5+8,25\cdot 17-0,19\cdot(17\cdot 17)=-8,25+140,25-54,91=77,09}$

Lagrangian Method Example

General form of Lagrangian interpolating polynomial;

$$f_n(x)=\sum_{i=0}^{n} L_i(x)\,f(x_i) \qquad \text{for (n+1) data points.} \qquad L_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j}$$

Data points : $(x_0,y_0),(x_1,y_1),\ldots\ldots\ldots(x_n,y_n)$

Lagrangian Method with quadratic interpolation

For given data table below model the quadratic interpolating polynomial and estimate the function value at given point.

x	5	9	18	13
f(x)	21	34	49	**?**

$$f_2(x)=L_0(x)\,f(x_0)+L_1(x)\,f(x_1)+L_2(x)\,f(x_2)$$

$$f_2(x)=L_0(x)\cdot 21+L_1(x)\cdot 34+L_2(x)\cdot 49$$

$$L_0(x)=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)=\left(\frac{x-9}{5-9}\right)\left(\frac{x-18}{5-18}\right)=\left(\frac{9-x}{4}\right)\left(\frac{18-x}{13}\right)$$

$$L_1(x)=\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)=\left(\frac{x-5}{9-5}\right)\left(\frac{x-18}{9-18}\right)=\left(\frac{x-5}{4}\right)\left(\frac{18-x}{9}\right)$$

$$L_2(x)=\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)=\left(\frac{x-5}{18-5}\right)\left(\frac{x-9}{18-9}\right)=\left(\frac{x-5}{13}\right)\left(\frac{x-9}{9}\right)$$

$$f_2(x)=\left(\frac{9-x}{4}\right)\left(\frac{18-x}{13}\right)\cdot 21+\left(\frac{x-5}{4}\right)\left(\frac{18-x}{9}\right)\cdot 34+\left(\frac{x-5}{13}\right)\left(\frac{x-9}{9}\right)\cdot 49$$

$$f_2(13)=(-1)\left(\frac{5}{13}\right)\cdot 21+(2)\left(\frac{5}{9}\right)\cdot 34+\left(\frac{8}{13}\right)\left(\frac{4}{9}\right)\cdot 49=-8{,}08+37{,}78+13{,}40=43{,}1$$

Newton's Method Example

General form of Newton's interpolating polynomials;
Given $(n+1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots, x_0]$$
$$b_n = f[x_n, x_{n-1}, \ldots, x_0]$$

and

$$f[x_n, x_{n-1}, \ldots, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

For given data table below model the quadratic interpolating polynomial and estimate the function value at given point.

x	5	9	18	13
f(x)	21	34	49	**?**

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f[x_0] = f(x_0) = f(5) = 21$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(9) - f(5)}{9 - 5} = \frac{34 - 21}{4} = \frac{13}{4} = 3,25$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\frac{f(18) - f(9)}{18 - 9} - \frac{f(9) - f(5)}{9 - 5}}{18 - 5} = \frac{\frac{49 - 34}{18 - 9} - \frac{34 - 21}{9 - 5}}{18 - 5} = \frac{\frac{15}{9} - \frac{13}{4}}{13} = \frac{60 - 117}{13 \cdot 36} = -\frac{57}{468} = -0,12$$

So,

$$f_2(x) = 21 + 3,25(x - 5) - 0,12(x - 5)(x - 9)$$
$$f_2(13) = 21 + 3,25(13 - 5) - 0,12(13 - 5)(13 - 9)$$
$$f_2(13) = 21 + 26 - 3,84$$
$$f_2(13) = 43,16$$

Linear regression with least square method

Minimize error; $\sum_{i=1}^{n}(y_i-(ax_i+b))^2$ n=number of data points

$$\frac{\partial_{err}}{\partial_a}=-2\sum_{i=1}^{n}x_i(y_i-ax_i-b)=0$$

$$\frac{\partial_{err}}{\partial_b}=-2\sum_{i=1}^{n}(y_i-ax_i-b)=0$$

Rewrite,

$$a\sum x_i^2+b\sum x_i=\sum(x_iy_i)$$

$$a\sum x_i+(b*n)=\sum y_i$$

Matrix form,

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}\cdot\begin{bmatrix} b \\ a \end{bmatrix}=\begin{bmatrix} \sum y_i \\ \sum(x_iy_i) \end{bmatrix}$$

x	y=f(x)
0	8,4121
1	7,4882
2	6,4038
3	7,0530
4	6,6072
5	5,3039
6	5,9597
7	5,4933
8	5,7356
9	5,9598

$$n=10$$

$$\sum x_i=45$$

$$\sum x_i^2=285$$

$$\sum y_i=64,4166$$

$$\sum(x_iy_i)=268,1374$$

$$\begin{bmatrix} 10 & 45 \\ 45 & 285 \end{bmatrix}\cdot\begin{bmatrix} b \\ a \end{bmatrix}=\begin{bmatrix} 64,4166 \\ 268,1374 \end{bmatrix}$$

$b=7,6275$, $a=-0,2635$

$$\mathsf{y}=\mathsf{ax}+\mathsf{b}=-0,2635\mathsf{x}+7,6273$$