1.3 - PROPOSITIONAL EQUIVALENCES

Example 1: Determine the truth values of

(*a*) $p \leftrightarrow p$

(*b*) $\neg (p \vee q) \leftrightarrow \neg p \wedge \neg q$.

Definition:

(*i*) A ***tautology*** is a compound proposition that is always true.

(*ii*) A ***contradiction*** is a compound proposition that is always false.

(*iii*) Any compound proposition not satisfying **(*i*)** or **(*ii*)** is called a ***contingency***.

Let P and Q be compound propositions. If $P \leftrightarrow Q$ is a tautology, then P and Q are ***logically equivalent***, written $P \equiv Q$. (This is sometimes written as $P \Leftrightarrow Q$.)

Notes:

- If P and Q are logically equivalent, then P and Q have the exact same truth values.

- $P \equiv Q$ is not a proposition! As such, the symbol '$\equiv$' should not appear in any proposition.

Logical Equivalences involving $\neg$, $\wedge$ and $\vee$:

Let p, q and r be propositions.

Double negation law: $\neg(\neg p) \equiv p$

Identity laws: $p \vee \mathbf{F} \equiv p$, $\quad p \wedge \mathbf{T} \equiv p$

Domination laws: $p \vee \mathbf{T} \equiv \mathbf{T}$, $\quad p \wedge \mathbf{F} \equiv \mathbf{F}$

Negation laws: $p \vee \neg p \equiv \mathbf{T}$, $\quad p \wedge \neg p \equiv \mathbf{F}$

Idempotent laws: $p \vee p \equiv p$, $\quad p \wedge p \equiv p$

Commutative laws: $p \vee q \equiv q \vee p$, $\quad p \wedge q \equiv q \wedge p$

Associative laws: $p \vee (q \vee r) \equiv (p \vee q) \vee r$,

$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$,

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption laws: $p \vee (p \wedge q) \equiv p$, $\quad p \wedge (p \vee q) \equiv p$

DeMorgan's laws: $\neg(p \vee q) \equiv \neg p \wedge \neg q$,

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Example 2: Show that $p \to q$ is logically equivalent to its contrapositive.

Logical Equivalences involving $\to$

1. $p \to q \equiv \neg p \vee q$
2. $p \to q \equiv \neg q \to \neg p$
3. $p \vee q \equiv \neg p \to q$
4. $p \wedge q \equiv \neg(p \to \neg q)$
5. $\neg(p \to q) \equiv p \wedge \neg q$
6. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
7. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
8. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
9. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical Equivalences involving $\leftrightarrow$

10. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
11. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
12. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
13. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Example 3: Simplify the proposition

$$[a \wedge (a \rightarrow b)] \rightarrow b$$

using the laws of logical equivalences. Be sure to cite each law whenever used.

Example 4: Simplify the proposition

$$[\neg p \wedge (\neg p \rightarrow (q \rightarrow r))] \rightarrow (q \rightarrow r)$$

using the laws of logical equivalences.

NORMAL FORMS

A collection of logical operators is called ***functionally complete*** if every compound proposition is logically equivalent to a compound proposition involving only these logical expressions.

Example 5: Show that $\{\neg, \wedge, \vee\}$ form a functionally complete collection of logical operators.

A logical expression is in

- ***disjunctive normal form*** (DNF) if it is written as a disjunction of the the conjunctions of the variables or their negations.
- ***conjunctive normal form*** (CNF) if it is written as a conjunction of the the disjunctions of the variables or their negations.

SECTION 1.3 – PROPOSITIONAL EQUIVALENCES

Example 1: Determine the truth values of **(*a*)** $p \leftrightarrow p$ and **(*b*)** $\neg (p \vee q) \leftrightarrow \neg p \wedge \neg q$.

Definition:

- **(*i*)** A ***tautology*** is a compound proposition that is always true.
- **(*ii*)** A ***contradiction*** is a compound proposition that is always false.
- **(*iii*)** Any compound proposition not satisfying **(*i*)** or **(*ii*)** is called a ***contingency***.

Let P and Q be compound propositions. If $P \leftrightarrow Q$ is a tautology, then P and Q are ***logically equivalent***, written $P \equiv Q$. (This is sometimes written as $P \Leftrightarrow Q$.)

Notes:

- If P and Q are logically equivalent, then P and Q have the exact same truth values.
- $P \equiv Q$ is not a proposition! As such, the symbol '≡' should not appear in any proposition.

Logical Equivalences involving $\neg$, $\wedge$ and $\vee$: Let p, q and r be propositions.

Double negation law: $\neg(\neg p) \equiv p$

Identity laws:	$p \vee \mathbf{F} \equiv p$	$p \wedge \mathbf{T} \equiv p$
Domination laws:	$p \vee \mathbf{T} \equiv \mathbf{T}$	$p \wedge \mathbf{F} \equiv \mathbf{F}$
Negation laws:	$p \vee \neg p \equiv \mathbf{T}$	$p \wedge \neg p \equiv \mathbf{F}$
Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Associative laws:	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
DeMorgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Example 2: Show that $p \rightarrow q$ is logically equivalent to its contrapositive.

Logical Equivalences involving $\rightarrow$

1. $p \rightarrow q \equiv \neg p \vee q$
2. $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (*Contrapositive*)
3. $p \vee q \equiv \neg p \rightarrow q$
4. $p \wedge q \equiv \neg(p \rightarrow \neg q)$
5. $\neg(p \rightarrow q) \equiv p \wedge \neg q$
6. $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
7. $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
8. $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
9. $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences involving $\leftrightarrow$

10. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
11. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
12. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
13. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Example 3: Simplify the proposition $[a \wedge (a \to b)] \to \neg b$ using the laws of logical equivalences. Be sure to cite each law whenever used.

Example 4: Simplify the proposition $[\neg p \wedge (\neg p \to (q \to r))] \to (q \to r)$ using the laws of logical equivalences.

NORMAL FORMS

A collection of logical operators is called ***functionally complete*** if every compound proposition is logically equivalent to a compound proposition involving only these logical expressions.

Example 5: Show that $\{\neg, \wedge, \vee\}$ form a functionally complete collection of logical operators.

A logical expression is in

- ***disjunctive normal form*** (DNF) if it is written as a disjunction of the the conjunctions of the variables or their negations.
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LOGICAL EQUIVALENCES

Double negation law: $\neg(\neg p) \equiv p$

Identity laws: $p \vee \mathbf{F} \equiv p$, $p \wedge \mathbf{T} \equiv p$

Domination laws: $p \vee \mathbf{T} \equiv \mathbf{T}$, $p \wedge \mathbf{F} \equiv \mathbf{F}$

Negation laws: $p \vee \neg p \equiv \mathbf{T}$, $p \wedge \neg p \equiv \mathbf{F}$

Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$

Commutative laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$

Associative laws: $p \vee (q \vee r) \equiv (p \vee q) \vee r$, $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption laws: $p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$

DeMorgan's laws: $\neg(p \vee q) \equiv \neg p \wedge \neg q$, $\neg(p \wedge q) \equiv \neg p \vee \neg q$

1. $p \to q \equiv \neg p \vee q$
2. $p \to q \equiv \neg q \to \neg p$
3. $p \vee q \equiv \neg p \to q$
4. $p \wedge q \equiv \neg(p \to \neg q)$
5. $\neg(p \to q) \equiv p \wedge \neg q$
6. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
7. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
8. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
9. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$
10. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
11. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
12. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
13. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$