Signals & Systems

Week 5

Spring 2018

05.03.2018

DT Sinusoids :

CT $x(t) = \cos(\omega_0 t)$ or $\xrightarrow{\text{complex exponential sinusoids}}$ $e^{jk\omega_0 t}$

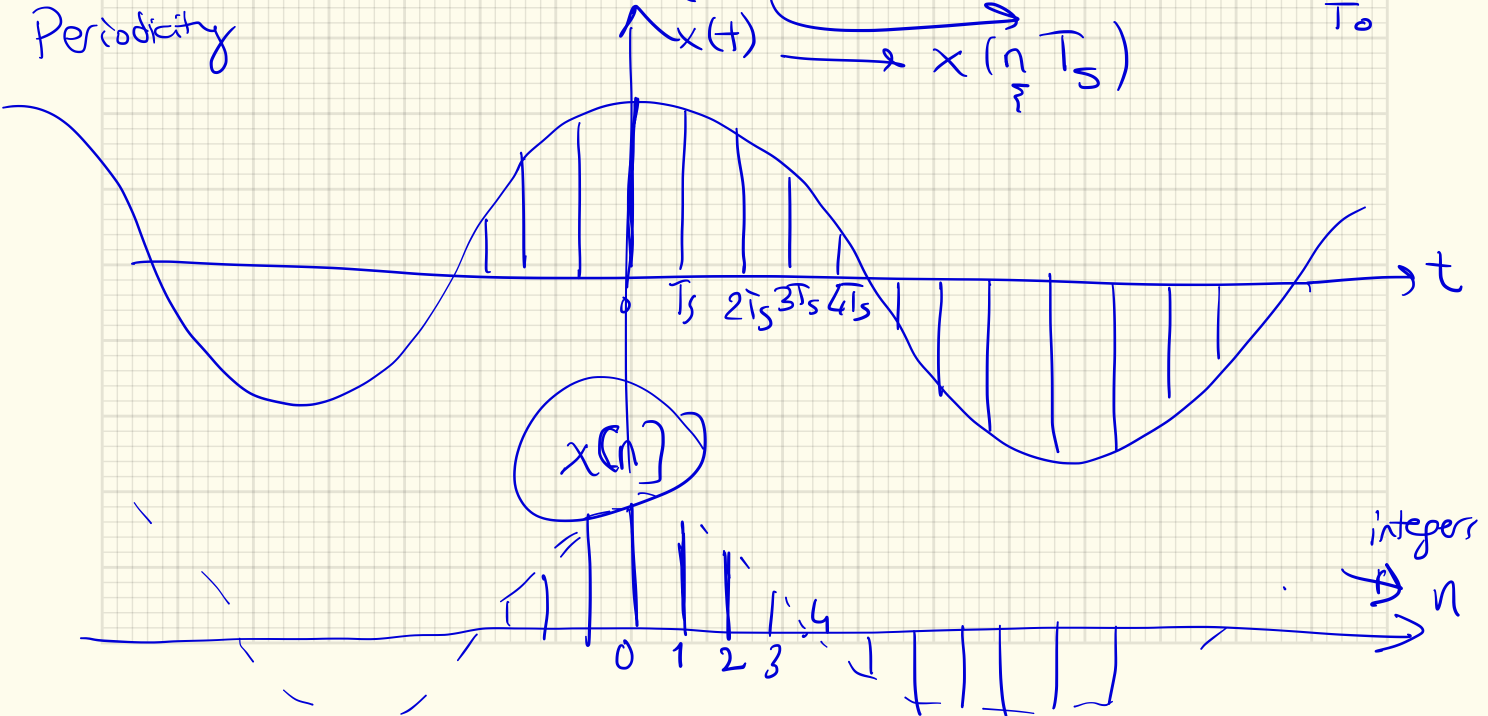
CT world.

$x(nT_s)$

DT $x[n] = \cos(\hat{\omega}_0 n)$

$\omega_0 = \frac{2\pi}{T_0}$

Periodicity



$\cos((\hat{\omega}+2\pi k)n)$ or $e^{j(\hat{\omega}+2\pi)n} = e^{j\hat{\omega}n}$

$\hat{\omega} = \frac{2\pi}{N}$

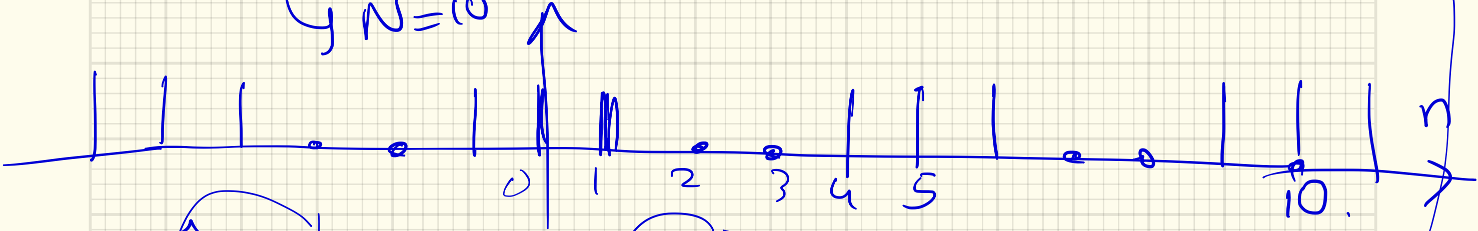
Periodicity of DT sinusoids

$$e^{j\hat{\omega}_0(n+N)} = e^{j\hat{\omega}_0 n} \cdot e^{j\hat{\omega}_0 N}$$

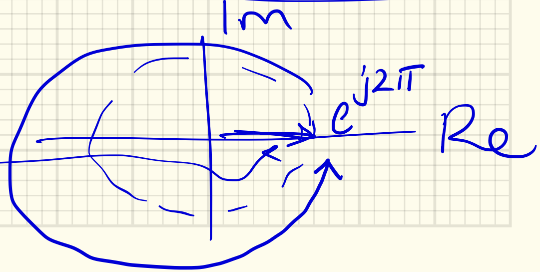
$$e^{j\hat{\omega}_0 N} = 1$$

$x[n+N] = x[n]$ w/ period N

↳ $N=10$



$$e^{j\hat{\omega}_0 N} = 1 = e^{j2\pi k}$$



$$\hat{\omega}_0 N = 2\pi k$$

$$\frac{\hat{\omega}_0}{2\pi} = \frac{k}{N} \Rightarrow$$

$\rightarrow \left(\frac{\hat{\omega}_0}{2\pi}\right)$ should be a rational # = ratio of 2 integers for DT sinusoid to be periodic.

Ex: $x_1[n] = \cos\left(\frac{2\pi}{12}n\right)$, $x_2(n) = \cos\left(\frac{8\pi}{31}n\right)$; $x_3(n) = \cos\left(\frac{n}{6}\right)$

$N = \left(\frac{2\pi}{\hat{\omega}_0}\right)k$ $\rightarrow$ See the Figure 1.25 (Openheim-Willsky)

DT Harmonically Related Complex Exponentials

$\Phi_k[n] = e^{jk\hat{\omega}_0 n}$ Recall $\Longleftrightarrow$ CT $\boxed{\Phi_k(t) = e^{jk\omega_0 t}}$

$\hat{\omega}_0 = \frac{2\pi}{N}$; N fund. period. $\rightarrow$ samples (integer)

$\omega = \frac{2\pi}{T_0}$ insert $t = nT_s$

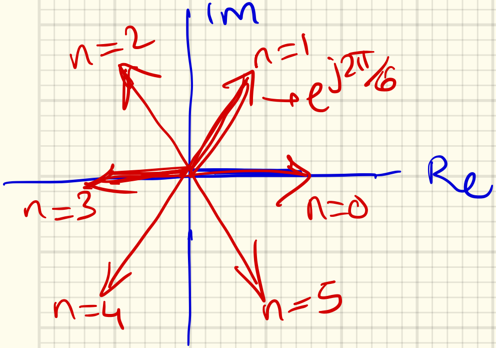
T_0: fund. period in sec

$\boxed{\omega_0 T_s = \hat{\omega}_0}$ normalized freq.

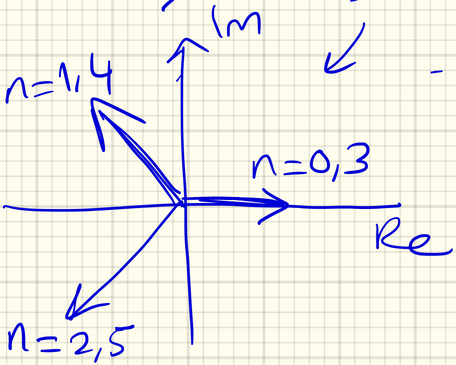
$\Phi_k(n) = e^{jk\left(\frac{2\pi}{N}\right)n}$, $k = 0, \ldots$

$\hookrightarrow$ we'll have N different signals ex: $\Phi_k(n) = e^{j\left(\frac{2\pi}{6}\right)kn}$; $N=6$ over 1 period $n=0,1,2,\dots,5$

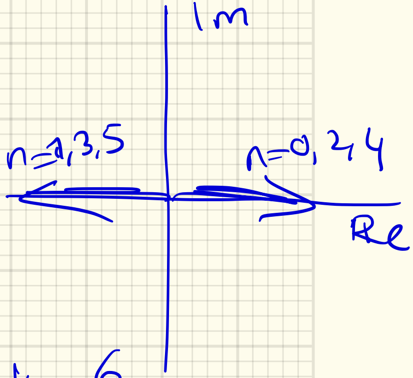
$k=1 \rightarrow \Phi_1[n]=e^{j\frac{2\pi}{6}n}$



$k=2$; $\Phi_2[n]$



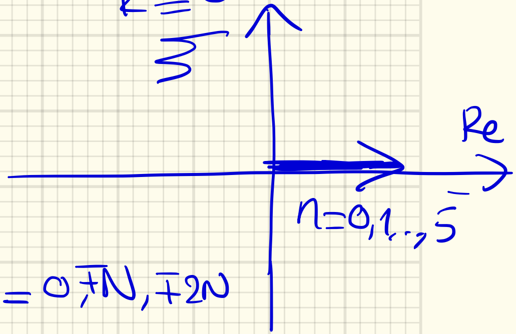
$k=3$ $\Phi_3[n]$



$k=4$
exercise
fill

$k=5$

$k=6$
$\equiv$



$$\sum_{n=0}^{N-1} \Phi_k[n] = \begin{cases} N, & k=0,\mp N,\mp 2N \\ 0, & k \text{ otherwise} \end{cases}$$

FS Representation DT Periodic Signals:

Consider a linear combination of N harmonically related complex exponential

Given a_k:

$$x[n] = \sum_{k=0}^{N-1} a_k \underbrace{e^{j\left(\frac{2\pi}{N}\right)kn}}_{\Phi_k[n]} ;$$

where $x(n)$ is periodic w/ fund. period N.

$$x[n+N] = \sum a_k e^{j\left(\frac{2\pi}{N}\right)k(n+N)} = \checkmark$$

show it is periodic.

— To determine a_k from $x(n)$ (similar to CT derivation, we did before)

multiply both sides by:

$x(n)\, e^{-j\frac{2\pi}{N}mn}$ $\Rightarrow$ sum from $n=0$ to $N-1$

$$\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}mn} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn} \cdot e^{-j\frac{2\pi}{N}mn}$$

$\Rightarrow$

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn} = \sum_{k=0}^{N-1} a_k \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

from orthogonality & zero-integral property in CT (DT counterpart)

$$= \begin{cases} N, & k=m \\ 0, & k\neq m \end{cases}$$

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}\cancel{m}kn} = N \cdot a_{\cancel{m}k} = a_k$$

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{+j\frac{2\pi}{N}kn} \quad \text{DTFS} \quad a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

Fourier Synthesis Equation

F.S. Analysis Equation

a_k are periodic w/ N : $a_{k+N} = a_k$.

ex: $x[n] = \sin(\hat{\omega}_0 n) = \sin\left(\frac{2\pi}{N} n\right) \rightarrow \boxed{N=5}$

$$x[n] = \underbrace{\frac{1}{2j}}_{a_1} e^{j\frac{2\pi}{5}n} \underbrace{- \frac{1}{2j}}_{a_{-1}} e^{-j\frac{2\pi}{5}n} \qquad \boxed{x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn}}$$

$a_{-1} = a_4 = a_{-1+N \to 5}$

→ Note: $a_k = a_{k+N}$: there are only N distinct harmonic components.

$|a_k|^2 \rightarrow$ power

$\longrightarrow \hat{\omega}$; unique in a 2π interval, $\quad -\pi \leq \hat{\omega} \leq \pi$

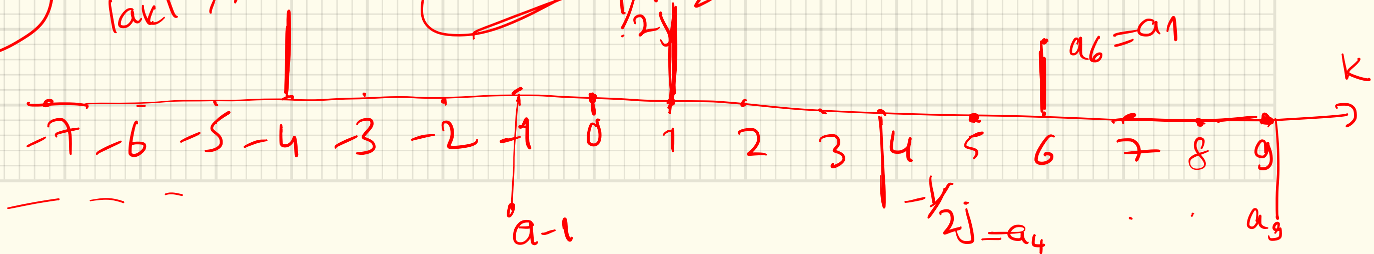
$0 \leq \hat{\omega} \leq 2\pi$

$a_k \rightarrow$ complex
Re, Im,
$|a_k|$, $\measuredangle a_k$

a_k plot

$\frac{1}{2j} = \frac{1}{2} e^{-j\pi/2} = a_1$

$N=5$



Now

find a_k for $N=5$

$$\to x(n) = \sin\left(3\left(\frac{2\pi}{N}\right)n\right) = \frac{1}{2j}\, e^{j\left(\frac{2\pi}{N}\right)3n} - \frac{1}{2j}\, e^{-j\left(\frac{2\pi}{N}\right)3n}$$

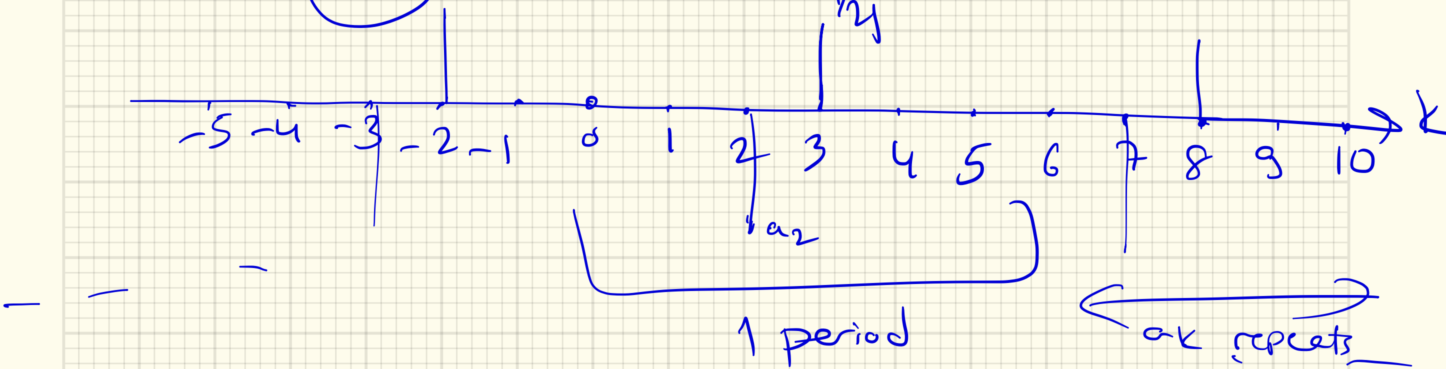
a_3 $\qquad$ a_{-3}

$$\sum a_k\, e^{j\left(\frac{2\pi}{N}\right)kn}$$

$$a_{-3} = a_2$$

freq. spectrum

a_k are periodic



Ex: $x[n] = 1 + \sin\left(\frac{2\pi}{N} n\right) + 3\cos\left(\frac{2\pi}{N} n\right) + \cos\left(\frac{4\pi}{N} n + \frac{\pi}{2}\right)$

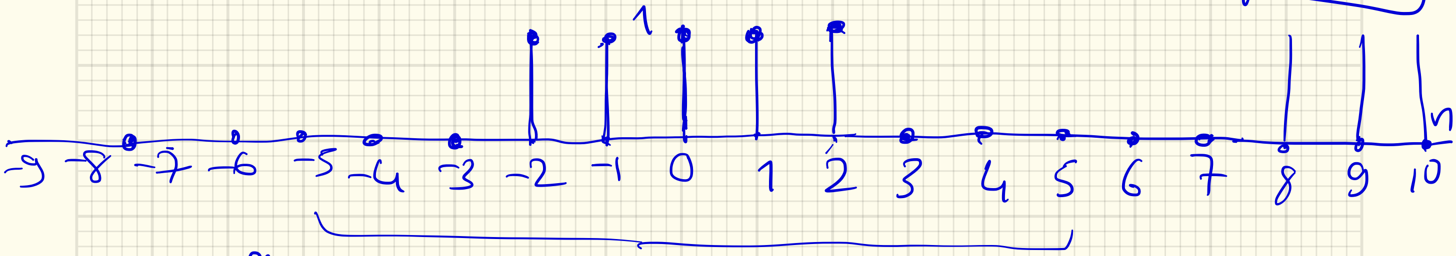
F.S. coef. of $x[n]$, Freq. spectrum of $x[n]$:

plot

exercise → Let $N = 10$ → find period.

Ex: Rectangular Pulse seq. w/ period $N = 10$

in a period $x[n] = \begin{cases} 1, & -2 \le n \le 2 \\ 0, & o/w \end{cases}$ → $\boxed{L = 2}$



Q: F.S. coeff for $x[n]$? Plot a_k.

Recall

$$a_k = \frac{1}{N} \sum_{n=-L}^{L} \underbrace{1}_{x(n)} \cdot e^{-j\left(\frac{2\pi}{N}\right)kn} \quad , \quad a_0 = \frac{2L+1}{N} \quad (k=0)$$

$\sum_{n=0}^{N-1}$ just sum in period.

change index from n to $m = n+L$

$$a_k = \frac{1}{N} \sum_{m=0}^{2L} e^{-j\frac{2\pi}{N}k(m-L)} = \frac{1}{N} e^{j\frac{2\pi}{N}kL} \sum_{m=0}^{2L} e^{-j\frac{2\pi}{N}km}$$

Use the geom. sum

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha} \quad (\alpha \neq 1)$$

$$a_k = \frac{1}{N} e^{+j\frac{2\pi}{N}kL} \left[\frac{1 - e^{-j\frac{2\pi}{N}k(2L+1)}}{1 - e^{-j\frac{2\pi}{N}k}} \right]$$

$$= \frac{1}{N} e^{j\frac{2\pi}{N}kL} \left(\frac{e^{-j\frac{2\pi}{N}k\left(\frac{2L+1}{2}\right)} \left(e^{+j\frac{2\pi k}{N}\left(\frac{2L+1}{2}\right)} - e^{-j\frac{2\pi k}{N}\left(\frac{2L+1}{2}\right)} \right)}{e^{-j\frac{2\pi}{N}\frac{k}{2}} \left(e^{j\frac{2\pi}{N}\frac{k}{2}} - e^{-j\frac{2\pi}{N}k/2} \right)} \right)$$

$$a_k = \frac{1}{N} \frac{\sin\left(\frac{2\pi}{N}k(L+1/2)\right)}{\sin\left(\frac{2\pi k}{N} \cdot \frac{1}{2}\right)}$$

Using

$$(1 - e^{-j\theta}) = e^{-j\frac{\theta}{2}} \underbrace{\left(e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}} \right)}_{2j \cdot \sin(\theta/2)}$$

$$a_k = \begin{cases} \dfrac{2L+1}{N}, & k = 0, \mp N, \mp 2N, \dots \\[2ex] \dfrac{1}{N}\,\dfrac{\sin\left(\frac{2\pi}{N}k\left(L+\frac{1}{2}\right)\right)}{\sin\left(\frac{2\pi}{N}k\frac{1}{2}\right)}, & \text{o/w} \end{cases}$$

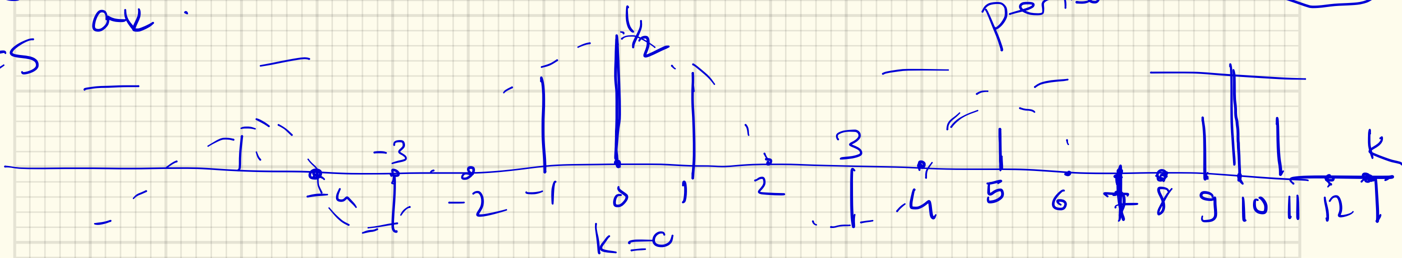
envelope is sinc fn. we'll see later

periodic

$L=2$
$N=10$

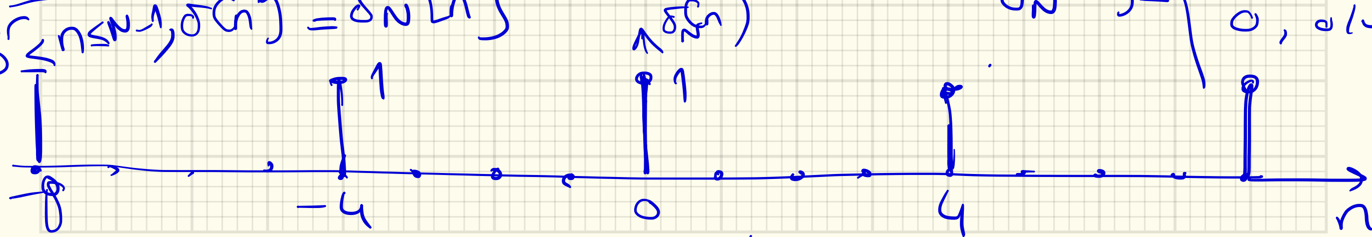
DTFS

a_k :



Exercise: Periodic impulse train: $\delta_N[n] \triangleq \begin{cases} 1, & n = mN \\ 0, & \text{o/w} \end{cases}$ (m: integer)

for $0 \le n \le N-1$, $\delta[n] = \delta_N[n]$



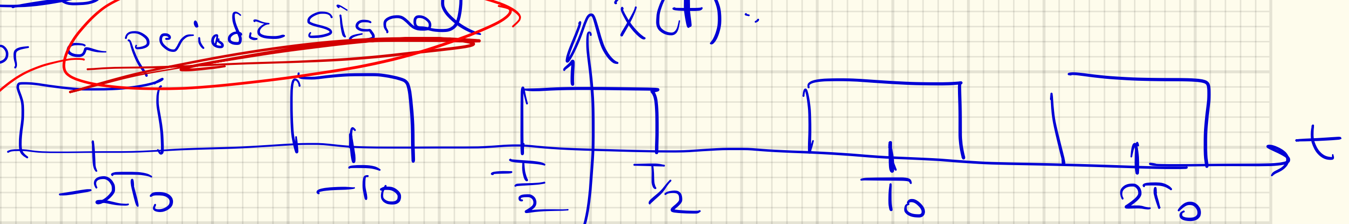
ex. $N=4$

F.S. coef for periodic impulse train. a_k ?

exercise → QUIZ next time.

Chap 11 1st sub-section: Want to develop a general defn of frequency spectrum of any signal $x(t)$? → non-periodic

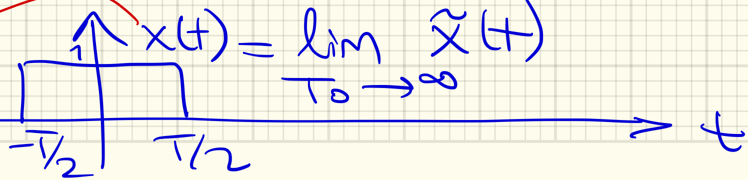
For a periodic signal



$x(t) = \lim_{T_0 \to \infty} \tilde{x}(t)$

non-periodic signal.

$T_0 \to \infty$



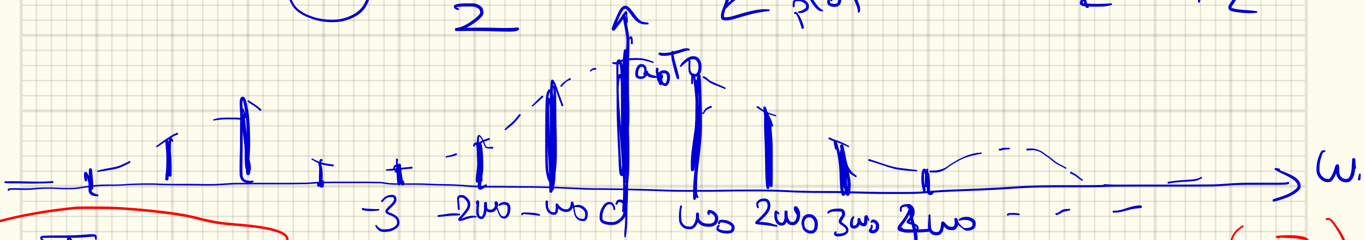
F.S. expansion $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $\qquad \omega_0 = \frac{2\pi}{T_0}$ → fund. period

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt \underset{\text{for the given } \tilde{x}(t)}{=} \frac{1}{T_0}\int_{-T/2}^{T/2} 1 \cdot e^{-jk\omega_0 t}\, dt$$

exercise: derive this!

$$a_k = \frac{\sin(\omega_0 k T/2)}{T_0\, k \frac{\omega_0}{2}} \Rightarrow a_k T_0 = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

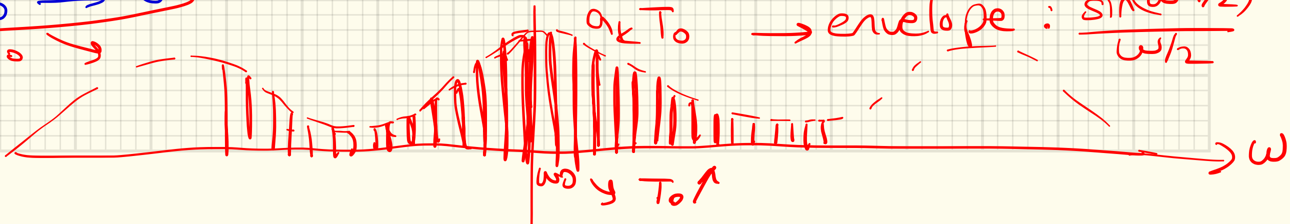
plot



as $T_0 \to \infty$

$\omega_0 \to$

$a_k T_0 \to$ envelope : $\frac{\sin(\omega T/2)}{\omega/2}$



$$\lim_{T_0 \to \infty} \tilde{x}(t) = x(t) \quad ; \quad \omega_0 \to d\omega, \quad k\omega_0 \to \omega$$

$$a_k T_0 = \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jk\omega_0 t} \, dt$$

envelope

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt}$$

freq. spectrum.

CT Fourier Transform (FT) of a non-periodic signal $x(t)$.

Note

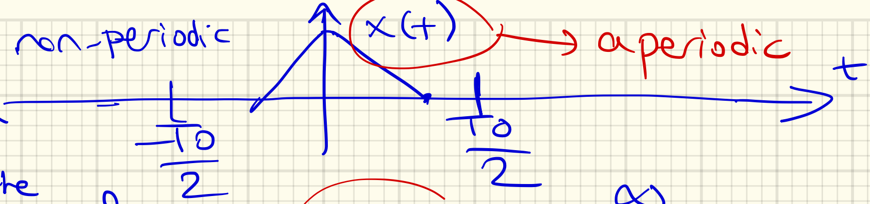
(i) $x(t)$ is not periodic $\rightarrow$ it has a F.T. not a F.S.

(ii) $X(j\omega)$ freq. spectrum representation of the signal

↳ "frequency content".

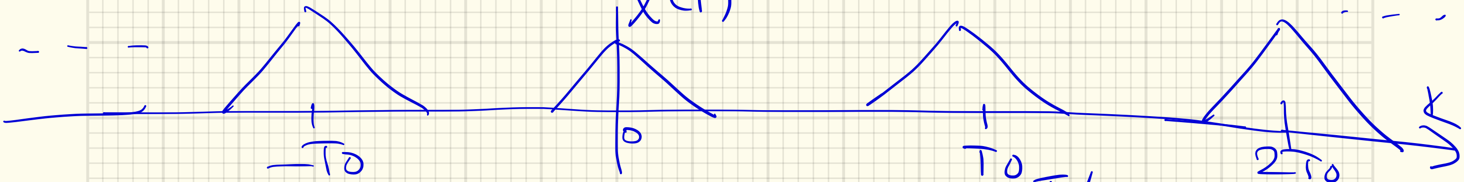
* Now, we derive inverse F.T. (Fourier Transform)

Take the non-periodic signal



create the periodic signal

$$\text{Let } \tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t - kT_0)$$



Find F.S. coef of $\tilde{x}(t)$: $a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$

$$a_k = \frac{1}{T_0}\int_{\cancel{-T_0/2}\ -\infty}^{\cancel{T_0/2}\ \infty} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0}\underbrace{\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt}_{X(jk\omega_0)}$$

$$\Rightarrow \boxed{a_k = \frac{1}{T_0} X(jk\omega_0)}$$

we tied F.S. coeff of a periodic signal w/ F.T. of the corresp. signal in 1 period & zero else.

* Can we obtain $x(t)$ from $X(j\omega)$?

non-periodic $x(t)$ time signal $\longrightarrow$ $X(j\omega)$ F.T.

$\longleftarrow$

periodic $\tilde{x}(t) = \sum a_k e^{jk\omega_0 t}$ insert $a_k = \frac{1}{T_0} X(jk\omega_0)$

$$= \sum \frac{1}{T_0} X(jk\omega_0) e^{jk\omega_0 t}$$

$T_0 = \frac{2\pi}{\omega_0}$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \cdot \omega_0$$

as $T_0 \to \infty$ $\tilde{x}(t) \to$ $\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega}$ *

$\omega_0 \to d\omega$

$k\omega_0 \to \omega$

Inverse Fourier transform / Synthesis Integral.

F.S. synthesis $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Fourier Transform Pairs

$$x(t) \xrightleftharpoons[\text{Inverse F.T.}]{\text{F.T. xform}} X(j\omega)$$

Note: F.T exists when $\int_{-\infty}^{\infty} |x(t)| < \infty$

$$|X(j\omega)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right| \leq \int |x(t) e^{-j\omega t}| dt$$

$$\leq \int |x(t)| \underbrace{|e^{-j\omega t}|}_{=1} dt$$

$$\leq \int |x(t)| dt$$

$$\leq \infty$$