MATHEMATICS 102E MIDTERM March 30, 2013

QUESTION 1 The blanks below will be filled by students. (Except the score)

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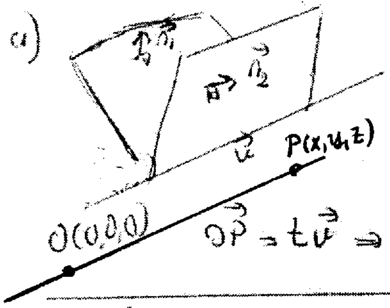
For the solution of this question please use only the front face and if necessary the back face of this page.

[10pt] **a)** Find parametric equations for the line through the origin that is parallel to the line of intersection of the planes $\mathbf{x+2y-z=5}$ and $\mathbf{2x-y+4z=2}$.

[5pt] **b)** Let $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$ be mutually orthogonal (perpendicular) unit vectors and let $\vec{\mathbf{D}} = 5\vec{\mathbf{A}} - 6\vec{\mathbf{B}} + 3\vec{\mathbf{C}}$. Find $|\vec{\mathbf{D}}|$, the length of $\vec{\mathbf{D}}$.

[10pt] **c)** For the curve $\vec{\mathbf{r}} = \cos t\,\vec{\mathbf{i}} + \sin t\,\vec{\mathbf{j}} + \ln\cos t\,\vec{\mathbf{k}}$ i) find the unit tangent vector $\vec{\mathbf{T}}$. ii) Find the length of the portion of the curve corresponding to the parameter interval $\mathbf{0 \le t \le \frac{\pi}{4}}$.

a)



line eqn.: $\vec{OP} = t\vec{v}$ where $\vec{v} = \vec{n}_1 \times \vec{n}_2$, $\vec{n}_1 = \vec{i}+2\vec{j}-\vec{k}$

$\vec{n}_2 = 2\vec{i}-\vec{j}+4\vec{k}$

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 2 & -1 & 4 \end{vmatrix} = (8-1)\vec{i} - (4+2)\vec{j} + (-1-4)\vec{k} = 7\vec{i} - 6\vec{j} - 5\vec{k}$$

$$\vec{OP} = t\vec{v} \Rightarrow (x-0)\vec{i} + (y-0)\vec{j} + (z-0)\vec{k} = (7\vec{i} - 6\vec{j} - 5\vec{k})t \Rightarrow \boxed{\begin{matrix} x = 7t \\ y = -6t \quad -\infty < t < \infty \\ z = -5t \end{matrix}}$$

b) $\vec{A} \perp \vec{B}$, $\vec{A} \perp \vec{C}$ and $\vec{B} \perp \vec{C}$, $\vec{D} = 5\vec{A} - 6\vec{B} + 3\vec{C}$

$$|\vec{D}|^2 = \vec{D}\cdot\vec{D} = (5\vec{A}-6\vec{B}+3\vec{C})\cdot(5\vec{A}-6\vec{B}+3\vec{C}) = 25\,\vec{A}\cdot\vec{A} - \underbrace{\cancel{30\,\vec{A}\cdot\vec{B}}}_{0} + \underbrace{\cancel{15\,\vec{A}\cdot\vec{C}}}_{0} - \underbrace{\cancel{30\,\vec{B}\cdot\vec{A}}}_{0} + 36\,\vec{B}\cdot\vec{B} - \underbrace{\cancel{18\,\vec{B}\cdot\vec{C}}}_{0} + \underbrace{\cancel{15\,\vec{C}\cdot\vec{A}}}_{0} - \underbrace{\cancel{18\,\vec{C}\cdot\vec{B}}}_{0} + 9\,\vec{C}\cdot\vec{C}$$

$$= 25\underbrace{|\vec{A}|^2}_{1} + 36\underbrace{|\vec{B}|^2}_{1} + 9\underbrace{|\vec{C}|^2}_{1} = 70$$

$$\Rightarrow |\vec{D}| = \sqrt{70}$$

c) i) $\vec{T} = \dfrac{\vec{v}}{|\vec{v}|}$, $\vec{v} = \dfrac{d\vec{r}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j} - \tan t\,\vec{k}$, $|\vec{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\tan t)^2}$

$= \sec t$, $0 \le t < \pi/4$.

$$\boxed{\vec{T} = \frac{-\sin t}{\sec t}\vec{i} + \frac{\cos t}{\sec t}\vec{j} - \frac{\tan t}{\sec t}\vec{k}}$$

ii) $$L = \int_0^{\pi/4} |\vec{v}|\,dt = \int_0^{\pi/4} \sec t\,dt = \ln|\sec t + \tan t|\Big|_0^{\pi/4} = \ln\left(\frac{2}{\sqrt{2}} + 1\right)$$

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[8pt] **a)** Let $\mathbf{f(x,y)} = \dfrac{\mathbf{(xy-y)\cos y}}{\mathbf{(x-1)^2+y^2}}$. Find the limit of $\mathbf{f}$ as $\mathbf{(x,y) \to (1,0)}$ or show that the limit does not exist.

[7pt] **b)** Find the plane tangent to the surface $\mathbf{x+y+z=e^{xyz}}$ at $\mathbf{(0,0,1)}$.

[10pt] **c)** Evaluate the integral $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \frac{\mathbf{y}}{\mathbf{x^5+1}}\mathbf{dxdy}$ by changing the order of integration.

a) $y = m(x-1)$, $m \in \mathbb{R}$, $x \to 1$, $y \to 0$

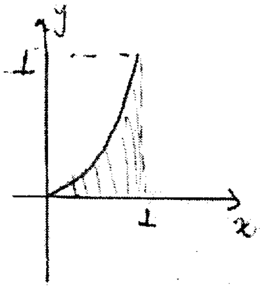
$$\lim_{(x,y)\to(1,0)} \frac{y(x-1)\cos y}{(x-1)^2+y^2} = \lim_{x\to 1} \frac{m(x-1)^2\cos(m(x-1))}{(x-1)^2+m^2(x-1)^2} = \frac{m}{m^2+1}$$

, for different values of m, we have different results $\Rightarrow$ no limit at $(1,0)$.

b) $\vec{P_0P} \cdot \vec{\nabla} f\big|_{P_0} = 0 \Rightarrow \dfrac{\partial f}{\partial x}\Big|_{P_0}(x-x_0) + \dfrac{\partial f}{\partial y}\Big|_{P_0}(y-y_0) + \dfrac{\partial f}{\partial z}\Big|_{P_0}(z-z_0) = 0$, $P_0(0,0,1)$, $f(x,y,z) = x+y+z-e^{xyz} = 0$

$\vec{\nabla} f = (1-yze^{xyz})\vec{i} + (1-xze^{xyz})\vec{j} + (1-xye^{xyz})\vec{k} \Rightarrow \vec{\nabla} f\big|_{(0,0,1)} = \vec{i}+\vec{j}+\vec{k}$

Tangent plane : $(x-0)+(y-0)+(z-1)=0 \Rightarrow \boxed{x+y+z=1}$

c) 

$$\int_{y=0}^{1}\int_{x=\sqrt{y}}^{1} \frac{y}{x^5+1}\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x^2} \frac{y}{x^5+1}\,dy\,dx = \int_0^1 \frac{1}{x^5+1}\left(\frac{y^2}{2}\Big|_{y=0}^{x^2}\right)dx$$

$$= \frac{1}{2}\int_0^1 \frac{x^4}{x^5+1}\,dx = \frac{1}{2.5}\ln|x^5+1|\Big|_0^1 = \frac{1}{10}\ln 2$$

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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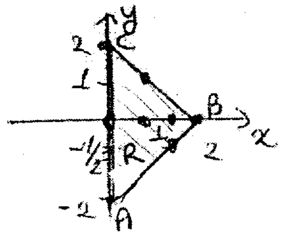
[17pt] **a)** Find the absolute extremum values of $\mathbf{f(x,y) = x^2 + y^2 - 2x}$ on the closed triangular region with vertices $(2,0)$, $(0,2)$ and $(0,-2)$.

[8pt] **b)** Find the extreme values of the function $\mathbf{f(x,y) = e^{xy}}$ on the curve $\mathbf{x^3 + y^3 = 16}$ by using the method of Lagrange multipliers.

a)



$f(x,y) = x^2+y^2-2x$

$f_x = 2x-2$, $f_y = 2y$

$f_{xx} = 2$, $f_{yy} = 2$

$f_{xy} = 0$

Interior points: $f_x = 0 \Rightarrow 2x-2=0 \Rightarrow x=1$

$f_y = 0 \Rightarrow 2y = 0 \Rightarrow y=0$

we have $(1,0) \in R$

This part is not necessary ← $\left\{ \Delta = f_{xx}f_{yy} - f_{xy}^2 \Big|_{(1,0)} = 2.2 - 0 = 4 > 0 \text{ and } f_{xx}\Big|_{(1,0)} = 2 > 0 \right.$

$f(1,0) = 1^2+0^2-2.1 = -1$ local minimum.

On the boundary of R:

AB: $y = x-2$, $0 \le x \le 2 \Rightarrow f(x,x-2) = x^2+(x-2)^2-2x \Rightarrow f_x = 2x+2(x-2)-2 = 4x-6 = 0$

$x = 3/2$, $y = -1/2$

$f(3/2,-1/2) = -1/2$, $f\Big|_{A(0,-2)} = 4$ and $f\Big|_{B(2,0)} = 0$

BC: $y = -x+2$, $0 \le x \le 2 \Rightarrow f(x,-x+2) = x^2+(2-x)^2-2x \Rightarrow f_x = 2x-2(2-x)-2 = 4x-6 = 0$

$x = \frac{3}{2}$, $y = \frac{1}{2}$

$f(\frac{3}{2},\frac{1}{2}) = -\frac{1}{2}$, $f\Big|_{C(0,2)} = 4$

CA: $x=0$, $-2 \le y \le 2 \Rightarrow f(0,y) = y^2 \Rightarrow f_y = 2y = 0$, $y=0, x=0$, $f\Big|_{(0,0)} = 0$

$\cdot f(1,0) = -1$, $\cdot f(3/2,-1/2) = -\frac{1}{2}$ $\cdot f(\frac{3}{2},\frac{1}{2}) = -\frac{1}{2}$, $f(0,0) = 0$, $f\Big|_A = 4$, $f\Big|_B = 0$, $f\Big|_C = 4$

f has the abs. min. of -1 at $(1,0)$ and the abs. max of 4 at A and C.

b) $f(x,y) = e^{xy}$

$g(x,y) = x^3+y^3-16 = 0$

$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow y e^{xy}\vec{i} + x e^{xy}\vec{j} = \lambda(3x^2\vec{i} + 3y^2\vec{j})$

$\Rightarrow (1)\ y e^{xy} = 3\lambda x^2$

$(2)\ x e^{xy} = 3\lambda y^2$

$\frac{(1)}{(2)} \Rightarrow \frac{y}{x} = \frac{x^2}{y^2} \Rightarrow y^3 - x^3 = 0 \Rightarrow \underbrace{(y-x)}_{=0}\underbrace{(y^2+xy+x^2)}_{=0} = 0$ or

i) $\Rightarrow y - x = 0 \Rightarrow x = y \overset{g(x,y)}{\Rightarrow} 2x^3 = 16 \Rightarrow x = 2, y = 2$ $\boxed{f(2,2) = e^4}$

ii) $y^2+xy+x^2 = 0 \Rightarrow x = y = 0 \overset{g(x,y)}{\Rightarrow} -16 = 0$ X

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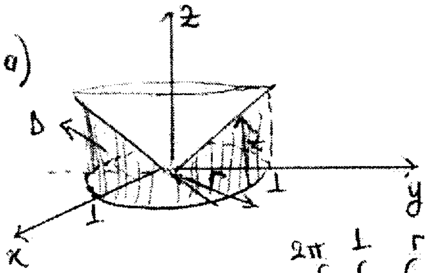
[10pt] a) Using the cylindrical coordinates, find the volume of the region bounded below by the xy-plane, above by the cone $z = \sqrt{x^2+y^2}$ and laterally by the cylinder $x^2+y^2=1$.

[15pt] b) Use the change of variables $u = x-2y$, $v = 3x-y$ to evaluate the integral $\int\int_R \frac{x-2y}{3x-y}dA$ where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$ and $3x-y=8$.

a)



In Cylindrical Coordinates

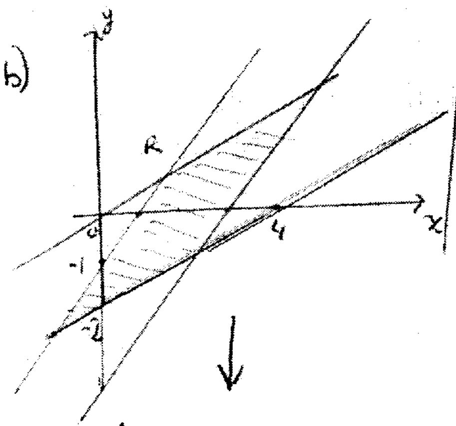
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

the cone : $z = \sqrt{x^2+y^2} \Rightarrow z = r$

$$V = \iiint_D dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{r} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(z\Big|_0^r\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta = \int_0^{2\pi}\frac{r^3}{3}\Big|_0^1 d\theta = \int_0^{2\pi}\frac{1}{3}\,d\theta$$

$$= \frac{1}{3}\theta\Big|_0^{2\pi} = \frac{2\pi}{3}$$

b)



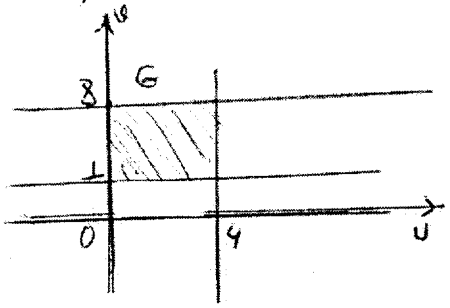
Boundries:

$x-2y=0 \Rightarrow u=0$
$x-2y=4 \Rightarrow u=4$
$3x-y=1 \Rightarrow v=1$
$3x-y=8 \Rightarrow v=8$

$f(x,y) = \frac{x-2y}{3x-y} \Rightarrow F(u,v) = \frac{u}{v}$

$J(u,v) = ?$ $x = \frac{2v-u}{5}$, $y = \frac{v-3u}{5}$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \frac{1}{5}$$



$$\iint_R f(x,y)\,dA = \iint_G F(u,v)\,|J(u,v)|\,du\,dv = \int_{v=1}^{8}\int_{u=0}^{4}\frac{u}{v}\cdot\frac{1}{5}\,du\,dv$$

$$= \frac{1}{5}\int_1^8 \frac{1}{v}\,\frac{u^2}{2}\Big|_0^4 dv = \frac{1}{10}\int_1^8 \frac{16}{v}\,dv = \frac{8}{5}\ln|v|\Big|_1^8$$

$$= \frac{8}{5}\ln 8$$