Discrete Mathematics

Trees

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Topics

Tree

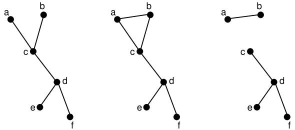
Definition

tree: a connected graph that contains no cycle

- *forest*: a graph where the connected components are trees

Tree Examples

Example

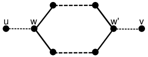


Tree Theorems

Theorem

In a tree, there is one and only one path between any two distinct nodes.

- there is at least one path because the tree is connected
- if there were more than one path, they would form a cycle



Tree Theorems

Theorem

let $T = (V, E)$ *be a tree:*

$$|E| = |V| - 1$$

- proof method: induction on the number of edges

Tree Theorems

Proof: Base step.

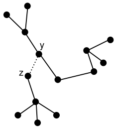
- $|E| = 0 \Rightarrow |V| = 1$
- $|E| = 1 \Rightarrow |V| = 2$
- $|E| = 2 \Rightarrow |V| = 3$

- assume that $|E| = |V| - 1$ for $|E| \leq k$

Tree Theorems

Proof: Induction step.

- $|E| = k+1$



- let's remove the edge (y,z):
  $T_1 = (V_1, E_1)$, $T_2 = (V_2, E_2)$

$$\begin{aligned} |V| &= |V_1| + |V_2| \\ &= |E_1| + 1 + |E_2| + 1 \\ &= (|E_1| + |E_2| + 1) + 1 \\ &= |E| + 1 \end{aligned}$$

□

Tree Theorems

Theorem

In a tree, there are at least two nodes with degree 1.

Proof.

- $2|E| = \sum_{v \in V} d_v$
- assume that there is only 1 node with degree 1:
  $\Rightarrow 2|E| \geq 2(|V|-1)+1$
  $\Rightarrow 2|E| \geq 2|V|-1$
  $\Rightarrow |E| \geq |V| - \frac{1}{2} > |V| - 1$

□

Tree Theorems

Theorem

T is a tree (T is connected and contains no cycle).
$\Leftrightarrow$
There is one and only one path between any two distinct nodes in T.
$\Leftrightarrow$
T is connected, but if any edge is removed it will no longer be connected.
$\Leftrightarrow$
T contains no cycle, but if an edge is added between any pair of nodes one and only one cycle will be formed.

Tree Theorems

Theorem

T is a tree (T is connected and contains no cycle).
$\Leftrightarrow$
T is connected and $|E| = |V| - 1$.
$\Leftrightarrow$
T contains no cycle and $|E| = |V| - 1$.

Rooted Tree

- a hierarchy is defined between nodes
- hierarchy creates a natural direction on edges: in and out degrees
- in-degree 0: root (only 1 such node)
- out-degree 0: leaf
- not a leaf: internal node

Node Level

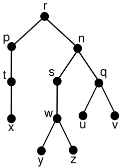
Definition

level of a node: distance from the root

- *parent*: adjacent node in the next upper level (only 1 such node)
- *child*: adjacent node in the next lower level
- *sibling*: node which has the same parent

Rooted Tree Example

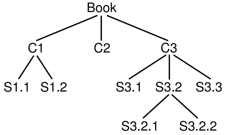
Example



- root: r
- leaves: $x\ y\ z\ u\ v$
- internal nodes: $r\ p\ n\ t\ s\ q\ w$
- parent of y: w
  children of w: y and z
- y and z are siblings

Rooted Tree Example

Example



Book
- C1
  - S1.1
  - S1.2
- C2
- C3
  - S3.1
  - S3.2
    - S3.2.1
    - S3.2.2
  - S3.3

Ordered Rooted Tree

- sibling nodes are ordered from left to right
- universal address system
  - assign the address 0 to the root
  - assign the positive integers $1, 2, 3, \ldots$ to the nodes at level 1, from left to right
  - let v be an internal node with address a, assign the addresses $a.1, a.2, a.3, \ldots$ to the children of v from left to right

Lexicographic Order

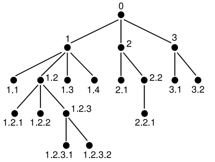
Definition

Let b and c be two addresses.
b comes before c if one of the following holds:

1. $b = a_1 a_2 \ldots a_m x_1 \ldots$
   $c = a_1 a_2 \ldots a_m x_2 \ldots$
   x_1 comes before x_2
2. $b = a_1 a_2 \ldots a_m$
   $c = a_1 a_2 \ldots a_m a_{m+1} \ldots$

Lexicographic Order Example

Example



- 0 - 1 - 1.1 - 1.2
  - 1.2.1 - 1.2.2 - 1.2.3
  - 1.2.3.1 - 1.2.3.2
  - 1.3 - 1.4 - 2
  - 2.1 - 2.2 - 2.2.1
  - 3 - 3.1 - 3.2

Binary Trees

Definition

$T = (V, E)$ is a binary tree:
$\forall v \in V \;\; d_v{}^o \in \{0, 1, 2\}$

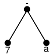
- $T = (V, E)$ is a *complete* binary tree:
  $\forall v \in V \;\; d_v{}^o \in \{0, 2\}$

Expression Tree

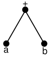
- a binary operation can be represented as a binary tree
  - operator as the root, operands as the children
- every mathematical expression can be represented as a tree
  - operators at internal nodes, variables and values at the leaves

Expression Tree Examples

Example $(7 - a)$

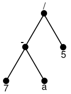


Example $(a + b)$

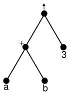


Expression Tree Examples

Example $((7 - a)/5)$

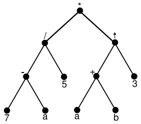


Example $((a + b) \uparrow 3)$



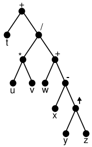
Expression Tree Examples

Example $(((7 - a)/5) * ((a + b) \uparrow 3))$



Expression Tree Examples

Example $(t + (u * v)/(w + x - y \uparrow z))$

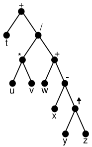


Expression Tree Traversals

1. inorder traversal:
   traverse left subtree, visit root, traverse right subtree
2. preorder traversal:
   visit root, traverse left subtree, traverse right subtree
3. postorder traversal:
   traverse left subtree, traverse right subtree, visit root
   - *reverse Polish notation*

Inorder Traversal Example

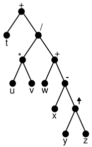
Example



$t + u * v / w + x - y \uparrow z$

Preorder Traversal Example

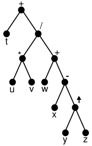
Example



$+ \ t \ / \ * \ u \ v \ + \ w \ - \ x \ \uparrow \ y \ z$

Postorder Traversal Example

Example



$t\ u\ v\ *\ w\ x\ y\ z\ \uparrow\ -\ +\ /\ +$

Expression Tree Evaluation

- inorder traversal requires parantheses for precedence
- preorder and postorder traversals do not require parantheses

Postorder Evaluation Example

Example ($t\ u\ v\ *\ w\ x\ y\ z\ \uparrow\ -\ +\ /\ +$)

$4\ 2\ 3\ *\ 1\ 9\ 2\ 3\ \uparrow\ -\ +\ /\ +$

4	2	3	$*$			
4	6	1	9	2	3	$\uparrow$
4	6	1	9	8	$-$	
4	6	1	1	$+$		
4	6	2	$/$			
4	3	$+$				
7						

Regular Trees

Definition

$T = (V, E)$ is an m-ary tree: $\forall v \in V\ d_v{}^o \leq m$

- $T = (V, E)$ is a complete m-ary tree:
  $\forall v \in V\ d_v{}^o \in \{0, m\}$

Regular Tree Theorem

Theorem

Let $T = (V, E)$ be a complete m-ary tree.

- *n: number of nodes*
- *l: number of leaves*
- *i: number of internal nodes*

Then:

- $n = m \cdot i + 1$
- $l = n - i = m \cdot i + 1 - i = (m - 1) \cdot i + 1$

$$i = \frac{l - 1}{m - 1}$$

Regular Tree Examples

Example

- how many matches are played in a tennis tournament with 27 players?

- every player is a leaf: $l = 27$
- every match is an internal node: $m = 2$
- number of matches: $i = \frac{l-1}{m-1} = \frac{27-1}{2-1} = 26$

Regular Tree Examples

Example

- how many extension cords with 4 outlets are required to connect 25 computers to a wall socket?

- every computer is a leaf: $l = 25$
- every extension cord is an internal node: $m = 4$
- number of cords: $i = \frac{l-1}{m-1} = \frac{25-1}{4-1} = 8$

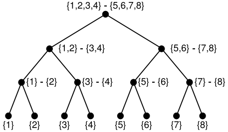
Decision Trees

Example

- one of 8 coins is counterfeit (it's heavier)
- find the counterfeit coin using a beam balance

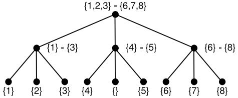
Decision Trees

Example (in 3 weighings)



Decision Trees

Example (in 2 weighings)



Spanning Tree

- T is a spanning tree of G:
  T is a subgraph of G such that T is a tree and contains all the nodes of G

Definition

MST is a minimum spanning tree of G:
MST is a spanning tree of G such that the total weight of the edges in MST is minimal

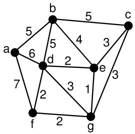
Kruskal's Algorithm

Kruskal's algorithm

1. $i \leftarrow 1$, $e_1 \in E$, $wt(e_1)$ is minimal
2. for $1 \leq i \leq n-2$:
   the selected edges are $e_1, e_2, \ldots, e_i$
   select a new edge e_{i+1} from the remaining edges such that:
   - $wt(e_{i+1})$ is minimal
   - $e_1, e_2, \ldots, e_i, e_{i+1}$ contains no cycle
3. $i \leftarrow i+1$
   - $i = n-1 \Rightarrow$ the subgraph G containing the edges $e_1, e_2, \ldots, e_{n-1}$ is a minimum spanning tree
   - $i < n-1 \Rightarrow$ go to step 2

Kruskal's Algorithm Example

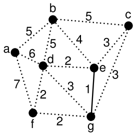
Example (initialization)



- $i \leftarrow 1$
- minimum weight: 1
  (e, g)
- $T = \{(e, g)\}$

Kruskal's Algorithm Example

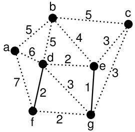
Example ($1 < 6$)



- minimum weight: 2
  $(d, e), (d, f), (f, g)$
- $T = \{(e, g), (d, f)\}$
- $i \leftarrow 2$

Kruskal's Algorithm Example

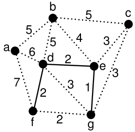
Example ($2 < 6$)



- minimum weight: 2
  $(d, e), (f, g)$
- $T = \{(e, g), (d, f), (d, e)\}$
- $i \leftarrow 3$

Kruskal's Algorithm Example

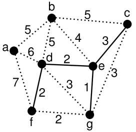
Example ($3 < 6$)



- minimum weight: 2
  (f, g) forms a cycle
- minimum weight: 3
  $(c, e), (c, g), (d, g)$
  (d, g) forms a cycle
- $T = \{(e, g), (d, f), (d, e), (c, e)\}$
- $i \leftarrow 4$

Kruskal's Algorithm Example

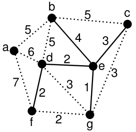
Example ($4 < 6$)



- $T = \{$
  $(e, g), (d, f), (d, e),$
  $(c, e), (b, e)$
  $\}$
- $i \leftarrow 5$

Kruskal's Algorithm Example

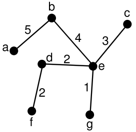
Example ($5 < 6$)



- $T = \{$
  $(e, g), (d, f), (d, e),$
  $(c, e), (b, e), (a, b)$
  $\}$
- $i \leftarrow 6$

Kruskal's Algorithm Example

Example ($6 \not< 6$)



- total weight: 17

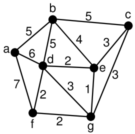
Prim's Algorithm

Prim's algorithm

1. $i \leftarrow 1$, $v_1 \in V$, $P = \{v_1\}$, $N = V - \{v_1\}$, $T = \emptyset$
2. for $1 \leq i \leq n - 1$:
   $P = \{v_1, v_2, \ldots, v_i\}$, $T = \{e_1, e_2, \ldots, e_{i-1}\}$, $N = V - P$
   select a node $v_{i+1} \in N$ such that for a node $x \in P$
   $e = (x, v_{i+1}) \notin T$, $wt(e)$ is minimal
   $P \leftarrow P + \{v_{i+1}\}$, $N \leftarrow N - \{v_{i+1}\}$, $T \leftarrow T + \{e\}$
3. $i \leftarrow i + 1$
   - $i = n \Rightarrow$: the subgraph G containing the edges $e_1, e_2, \ldots, e_{n-1}$ is a minimum spanning tree
   - $i < n \Rightarrow$ go to step 2

Prim's Algorithm Example

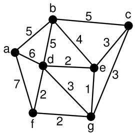
Example (initialization)



- $i \leftarrow 1$
- $P = \{a\}$
- $N = \{b, c, d, e, f, g\}$
- $T = \emptyset$

Prim's Algorithm Example

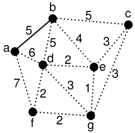
Example ($1 < 7$)



- $T = \{(a, b)\}$
- $P = \{a, b\}$
- $N = \{c, d, e, f, g\}$
- $i \leftarrow 2$

Prim's Algorithm Example

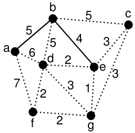
Example ($2 < 7$)



- $T = \{(a, b), (b, e)\}$
- $P = \{a, b, e\}$
- $N = \{c, d, f, g\}$
- $i \leftarrow 3$

Prim's Algorithm Example

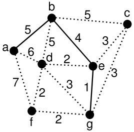
Example ($3 < 7$)



- $T = \{(a, b), (b, e), (e, g)\}$
- $P = \{a, b, e, g\}$
- $N = \{c, d, f\}$
- $i \leftarrow 4$

Prim's Algorithm Example

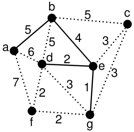
Example ($4 < 7$)



- $T = \{(a, b), (b, e), (e, g), (d, e)\}$
- $P = \{a, b, e, g, d\}$
- $N = \{c, f\}$
- $i \leftarrow 5$

Prim's Algorithm Example

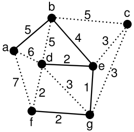
Example ($5 < 7$)



- $T = \{$
  $(a, b), (b, e), (e, g),$
  $(d, e), (f, g)$
  $\}$
- $P = \{a, b, e, g, d, f\}$
- $N = \{c\}$
- $i \leftarrow 6$

Prim's Algorithm Example

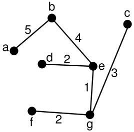
Example ($6 < 7$)



- $T = \{$
  $(a, b), (b, e), (e, g),$
  $(d, e), (f, g), (c, g)$
  $\}$
- $P = \{a, b, e, g, d, f, c\}$
- $N = \emptyset$
- $i \leftarrow 7$

Prim's Algorithm Example

Example ($7 \not< 7$)



- total weight: 17

References

Required Reading: Grimaldi

- Chapter 12: Trees
  - 12.1. Definitions and Examples
  - 12.2. Rooted Trees
- Chapter 13: Optimization and Matching
  - 13.2. Minimal Spanning Trees:
    The Algorithms of Kruskal and Prim