CHAPTER 2

Probability

In Chapter-1, we have discussed on the descriptive statistics, that is, methods for organising and summarising numerical data. Another important aspect is to present the fundamentals of inferential statistics, that is, methods of drawing conclusions about a population based on information from a sample of the population. Because inferential statistics involves utilising information from part of a population (a sample) to infer conclusions about the entire population, we can never be certain that our conclusions (inferences) are correct or true, that is, uncertainty is inherent in inferential statistics. Therefore, we need to be familiar with uncertainty before we can understand, develop, and apply the methods of inferential statistics.

The science of uncertainty is called *probability theory*. Probability theory enables us to evaluate and control the likelihood that a statistical inference is correct. More generally, probability theory provides the mathematical basis for inferential statistics.

This chapter introduces the basic concepts and definitions on probability, events (simple and compound), Venn diagram, tree diagram, approaches to probability (classical, relative frequency concept of probability, subjective probability, marginal probability and conditional probability. Special multiplication rule and Baye's formula are also briefly presented.

2.1 EXPERIMENT, OUTCOME AND SAMPLE SPACE

The tossing of a coin or the rolling of a die constitutes an *experiment*. In probability and statistics, the term *experiment* is used in a very wide sense and refers to any procedure that yields a collection of outcomes. The knowledge of all possible outcomes when a coin is tossed, or a die is rolled, is important. This is always the case for determining the probabilities. Measuring the length of a bolt, weighing the contents of a box of materials, and measuring the breaking strength of a metal component are all examples of experiments.

An *experiment* is a process that, when performed, results in one and only one of many observations. These observations are known as the *outcomes* of the experiment. The collection of all outcomes for an experiment is called a *sample space*. For tossing a coin, we can use the set {Heads, Tails} as the sample space. For rolling a six-sided die, we can use the set {1, 2, 3, 4, 5, 6}. These sample spaces are finite. Some experiments have sample spaces with an infinite number of outcomes. The elements of a sample space are called *sample points*. The *sample space* for an experiment can be described by drawing either a *Venn diagram* or a *tree diagram*. A *Venn diagram* is a picture that depicts all the possible outcomes for an experiment. In a *tree diagram*, a branch of the tree represents each outcome.

2.2 SIMPLE AND COMPOSITE EVENTS

An *event* is a collection of one or more of the outcomes of an experiment. An event may be simple event or a compound event as shown in Fig. 2.1. A simple event is also called an *elementary event*, and a composite event is also called a *compound event*. An event that includes one and only one of the final outcomes for an experiment is called a *simple event* and is generally denoted by E_i. A compound event is a collection of more than one outcome of an experiment. The probability of a composite event is the sum of the probabilities of the simple events of which it is composed. When all the sample points or simple events of an experiment are equally likely and so have equal probabilities, then the probability of a composite event is easily found. For instance, if there are k sample points and a composite event A contains r of them, then $P(A) = \frac{r}{k}$.

The probability that a simple event E_1 will occur is denoted by $P\ (E_1)$, and the probability that a compound event A will occur is denoted by $P(A)$.

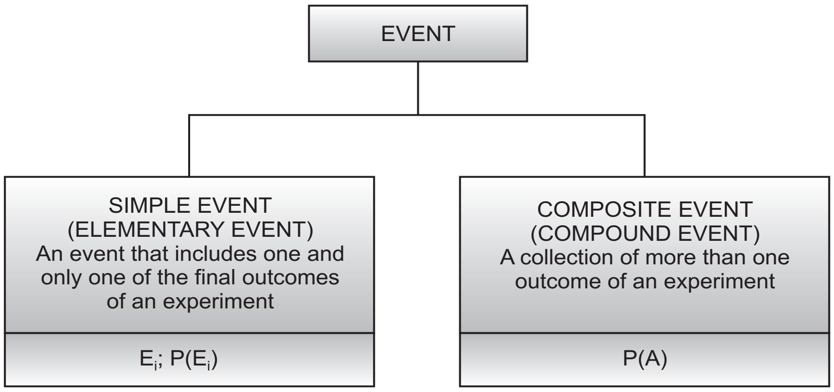


Fig. 2.1: Classification of an event

An *impossible event* is one that has no outcomes in it and consequently, cannot occur. On the other hand, a *sure event* is one that has all the outcomes of the sample space in it and will, therefore, definitely occur when the experiment is performed. Thus, the sample space constitutes a sure event.

Probability Definition: Let us assume that the sample space S has N outcomes e_1, e_2, e_N so that there are N simple events $\{e_1\}$, $\{e_2\}$, $\{e_N\}$.

The *probability of a simple event* $\{e\}$ is a number denoted by $P[\{e\}]$ and satisfies the following conditions:

1. $P\ [\{e\}]$ is always between 0 and 1 that is $0 \leq P\ [\{e\}] \leq 1$.
2. The sum of the probabilities of all the simple events is 1; that is, $P\ [\{e_1\}] + P\ [\{e_2\}] + \cdots + P\ [\{e_N\}] = 1$

The *probability of an event A*, denoted by $P(A)$, is defined as the sum of the probabilities assigned to the simple events that comprise the event A. The impossible event has probability 0 and the sure event has probability 1.

Example E2.1

Find the probability of getting *(a)* exactly two tails (event A), *(b)* at least two tails (event B) in tossing 3 balanced fair coins.

SOLUTION:

(a) The event A of exactly two tails comprises of the sample points (*TTH, THT, HTT*).

Hence, $$P(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

(b) The event B of at least two tails comprises of the sample points (*TTT*), (*TTH*), (*THT*) and (*HTT*).

Hence, $$P(B) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Example E2.2

Find the probability that the sum of the numbers shown in the two faces, when two dice are thrown *(a)* is 8 and *(b)* 10.

SOLUTION:

(a) The event A that the sum of the numbers shown on the two faces is eight consists of the sample points:

(1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) and (7, 1)

There are 7 sample points. Therefore, the required probability is $\frac{7}{36}$.

(b) The event that the sum is 10 consists of the following sample points:

(4, 6), (5, 5) and (6, 4)

Hence, the required probability is $\frac{3}{36} = \frac{1}{12}$.

2.3 AXIOMS OF PROBABILITY

The subject of probability is based on three rules, known as axioms. They are:

1. Let S be a sample space. Then $P(S) = 1$.
2. For any event A, $0 \le P(A) \le 1$.
3. If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

   More generally, if $A_1, A_2, A_3, \ldots$ are mutually exclusive events then $P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

2.4 FINITE PROBABILITY SPACES

Consider a sample space S and the class C of all events. We assume here that if S is finite, then all subsets of S are events. S then become a probability space by assigning probabilities to the events in C so that they satisfy the probability axioms.

If S is a finite sample space with n elements and suppose the physical characteristics of the experiment suggest that the various outcomes of the experiment be assigned equal probabilities. Then S becomes a probability space, called a *finite equiprobable space*, if each point in S is assigned the probability $1/n$ and if each event A containing r points is assigned the probability r/n.

Hence, $$P(A) = \frac{\text{number of elemens in } A}{\text{number of elements in } S} = \frac{n(A)}{n(S)} \tag{2.1}$$

or $$P(A) = \frac{\text{number of ways that the event } A \text{ can occur}}{\text{number of ways that the sample sapce } S \text{ can occur}} \tag{2.2}$$

Let $S = \{a_1, a_2, \ldots, a_n\}$ be a finite sample space. Then, a *finite probability space* or *finite probability model* is obtained by assigning each point a_i in S a real number p_i, called the probability of a_i, satisfying the following properties:

1. Each p_i is nonnegative, $p_i \geq 0$.
2. The sum of the $p_i = 1$, that is, $p_1 + p_2 + \cdots + p_n = 1$.

The *probability* $P(A)$ of an event A is defined to be the sum of the probabilities of the points in A.

2.5 INFINITE PROBABILITY SPACES

There are two cases: S is countably infinite and S is uncountable. A finite or countably infinite probability space S is said to be *discrete*. An uncountable space S which consists of a continuum of points is said to be *continuous*.

Case 1: Countably Infinite Sample Spaces

Suppose S is a countably infinite sample space: $S = \{a_1, a_2, \ldots\}$. Then, a probability space is obtained by assigning $a_i \in S$ a real number p_i, called its *probability*, such that

1. Each p_i is nonnegative, or $p_i \geq 0$.
2. The sum of the p_i is equal to 1

   or $p_1 + p_2 + \cdots = \sum_{i=1}^{\infty} p_i = 1$.

The probability $P(A)$ of an event A is then equals the sum of the probabilities of its points.

Case 2: Uncountable Spaces

The probability of an event A is given by the ratio of $m(A)$ to $m(S)$:

$$P(A) = \frac{\text{length of } A}{\text{length of } S}$$

or $$P(A) = \frac{\text{area of } A}{\text{area of } S} \tag{2.3}$$

or $$P(A) = \frac{\text{volume of } A}{\text{volume of } S}$$

where the uncountable sample spaces S are those with some finite geometrical measurement such as length, area, or volume and in which a point is selected at random. Such a probability space is said to be *uniform.*

2.6 PROPERTIES OF PROBABILITY

The basic properties of probabilities are:

1. The probability of an event always lies in the range of 0 and 1. That is

   $0 \le P(E_1) \le 1$ Simple event

   $0 \le P(A) \le 1$ Compound event

2. The sum of the probabilities of all simple events or final outcomes for an experiment, denoted by $\Sigma P(E_i)$, is always equal to 1.

   Hence, $$\Sigma P(E_i) = P(E_1) + P(E_2) + \cdots = 1 \qquad (2.4)$$

3. The probability of an event that cannot occur is 0. (An event that cannot occur is called an *impossible event*).
4. The probability of an event that must occur is 1. (An event that must occur is called a *certain event*).

2.7 VENN DIAGRAM

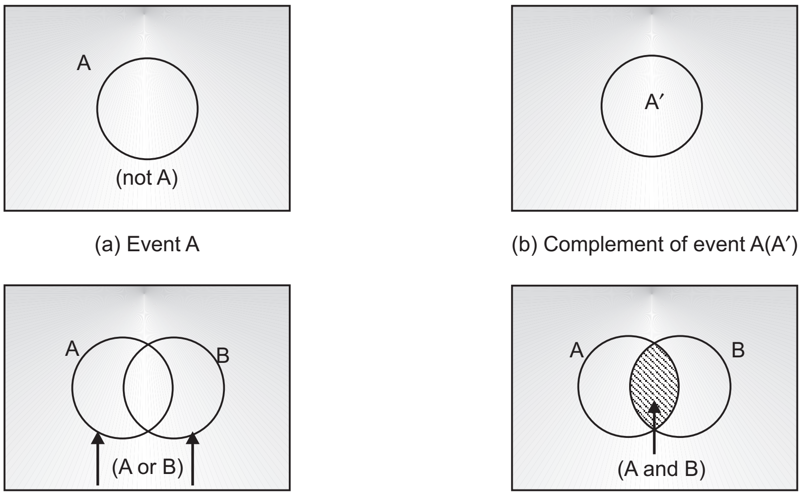
Graphical display of events are helpful for explaining and understanding probability. Venn diagrams, named after English logician John Venn (1834–1923) are considered one of the excellent ways to visually display events and relationships among events. The sample space is shown as a rectangle and the various events are drawn as circles (or other geometric shapes) inside the rectangle. Venn diagram for one event is shown in Fig. 2.2.

Each event A has a corresponding event defined by the condition that "A does not occur". That event is called the *complement* of A and is denoted (not A) or (A') as shown in Fig. 2.2(b). Event (not A) consists of all outcomes not in A, as shown in the Venn diagram in Fig. 2.2(a). With any two events, say, A and B, we can associate two new events. One new event is defined by the condition that "both event A and event B occur" and is denoted (A & B) or (A and B). Event (A and B) consists of all outcomes common to both event A and event B, as shown in Fig. 2.2(d). The other new event associated with A and B is defined by the condition "either event A or event B or both occur" or equivalently, that, "at least one of events A and B occur". That event is denoted (A or B) and consists of all outcomes in either event A or event B or both, as shown in Fig. 2.2(c).

Relationship among Events

(not A):	The event "A does not occur".
(A and B) or (A & B):	The event "both A and B occur".
(A or B):	The event "either A or B or both occur".

Note here that event "both A and B occurs" is the same as the event "both B and A occur", event (A and B) or (A & B) is the same as event (B and A) or (B & A). Similarly, event (A or B) is the same as event (B or A).



(a) Event A

(b) Complement of event A(A′)

(c) Union of events A and B (A∪B)
Event A or B

(d) Intersection of events A and B (A∩B)
Event (A and B)

Fig. 2.2: Venn diagram

2.8 PROBABILITY TREE OR TREE DIAGRAM

In a *(finite) stochastic process*, in which a finite sequence of experiments where each experiment has a finite number of outcomes with given probabilities, the most convenient way to describe such a process is by means of a *labeled tree diagram*.

Various events are drawn as circles (or other geometric shape such as a rectangle, a square, or a circle) that depicts all the possible outcomes for an experiment. For instance, the shaded regions of the four Venn diagrams of Fig. 2.2 represents respectively, event A, the complement of event A, the union of events A and B, and the intersection of events A and B.

If A and B are any two subsets of a sample space S, their union $A \cup B$ is the subset of S that contains all the elements that are either in A, in B, or in both; their intersection $A \cap B$ is the subset. S that contains all the elements that are in both A and B; on the complement A' of A is the subset of S that contains all the elements of S that are not in A.

Figure 2.3(*a*) indicates that events A and B are *mutually exclusive*, that is, the two sets have no elements in common (or the two events cannot both occur). This is written $A \cap B = \phi$ denotes the *empty set*, which has no elements at all. Figure 2.3(*b*) shows that A is contained in B, we write this as $A \subset B$.

(a) Events A and B mutually exclusive

(b) Event A contains in Event B

Fig. 2.3: Venn diagram sharing special relationships among events A and B

Example E2.3

A bin contains a certain number of manufactured mechanical components, a few of which are defective. Two components are selected at random from this bin and inspected to determine, if they are non-defective or defective. How many total outcomes are possible? Draw a tree diagram for this experiment. Show all the outcomes in a Venn diagram.

Solution:

Let G = the selected component is good

D = the selected component is defective

The four outcomes for this experiment are: GG, GD, DG and DD.

The Venn and tree diagrams are shown in Figs. E2.3(*a*) and (*b*).

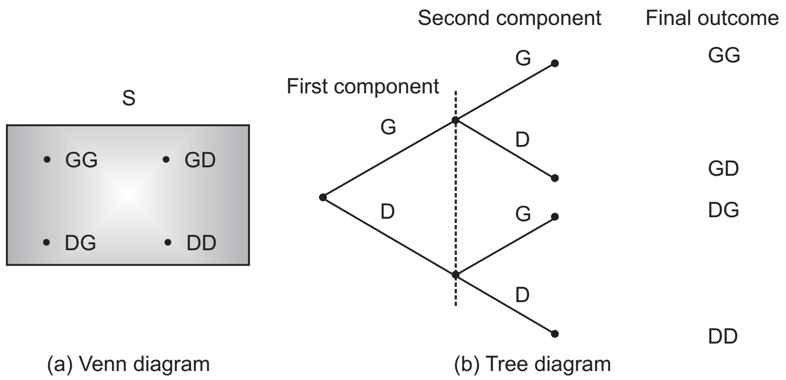


Fig. E2.3: Venn and tree diagrams

Example E2.4

In Fig. E2.4, H is the event that an employee has health insurance and D is the event that the employee has disability insurance.

(*a*) express in words what events are represented by regions 1, 2, 3 and 4

(*b*) what events are represented by regions 1 and 2 together

(*c*) regions 2 and 4 together

(*d*) regions 1, 2 and 3 together

(*e*) regions 2, 3 and 4 together.

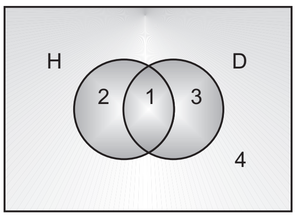


Fig. E2.4: Venn diagram

SOLUTION:

(a) 1 : The employee has health insurance and disability insurance
2 : The employee has health insurance but no disability insurance
3 : The employee has disability insurance but no health insurance
4 : The employee has neither health insurance nor disability insurance

(b) The employee has health insurance.

(c) The employee does not have health insurance.

(d) The employee has either health or disability insurance, but not both.

(e) The employee does not have both kinds of insurance.

Example E2.5

A small box contains a few red and a few blue balls. If two balls are randomly drawn and the colours of these balls are observed, how many total outcomes are possible? Draw a tree diagram for this experiment. Show all the outcomes in a Venn diagram.

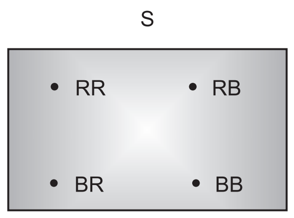
SOLUTION:

Let R = a red ball is selected

B = a blue ball is selected

The experiment has 4 outcomes: RR, RB, BR and BB.

The Venn diagram and tree diagram are shown in Figs. E2.5(*a*) and (*b*).



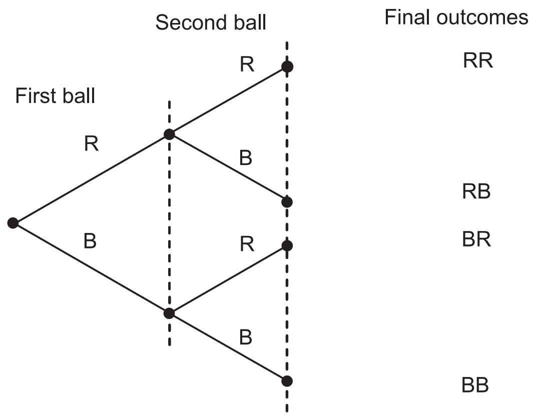


Fig. E2.5

Example E2.6

Draw a tree diagram for 3 tosses of a balanced (fair) coin. List all outcomes for this experiment in a sample space S.

SOLUTION:

Let H = a toss results in head

T = a toss results in tail.

Hence, the sample space is written as $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

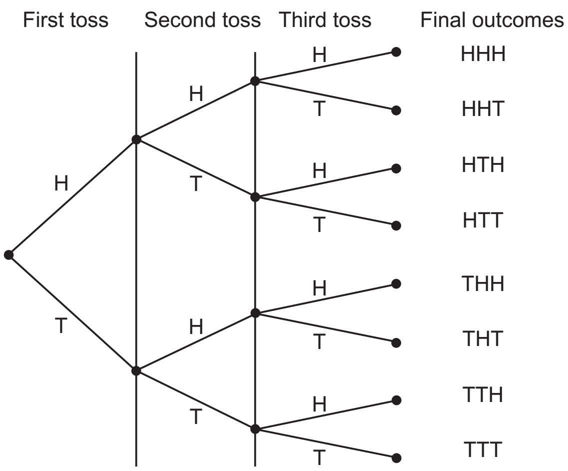


Fig. E2.6: Tree diagram

Example E2.7

Refer to Example E2.5. List all the outcomes included in each of the following events. Indicate which are simple and which are compound events.

(a) both balls are different colours

(b) at least one ball is red

(c) not more than one ball is blue

(d) the first ball is blue and the second is red.

SOLUTION:

(a) $\{RB, BR\}$; a compound event

(b) $\{RR, RB, BR\}$; a compound event

(c) $\{RR, RB, BR\}$; a compound event

(d) $\{BR\}$; a simple event.

2.9 APPROACHES TO PROBABILITY

There are three conceptual approaches to probability. They are

1. classical probability,
2. the relative frequency concept of probability and
3. the subjective probability concept.

2.9.1 Classical Probability

Outcomes that have the same probability of occurrence are called *equally likely outcomes*.

Two or more outcomes or events that have the same probability of occurrence are said to be *equally likely outcomes* or *events*. If E is an event, then $P(E)$ stands for the probability that event E occurs.

$$P(E_i) = \frac{1}{\text{total number of outcomes for the experiment}} \tag{2.5}$$

$$P(A) = \frac{\text{number of outcomes favourable to } A}{\text{total number of outcomes for the experiment}} \tag{2.6}$$

2.9.2 Relative Frequency Concept of Probability

If an experiment is repeated n times and an event A is observed f times, then, according to the relative frequency concept of probability

$$P(A) = \frac{f}{n}$$

The relative frequencies are not probabilities but approximate probabilities.

The *law of large numbers* states that if an experiment is repeated again and again, the probability of an event obtained from the relative frequency approaches the actual or theoretical probability.

2.9.3 Subjective Probability

Subjective probability is assigned arbitrarily. It is the probability assigned to an event based on subjective judgement, experience, information, and belief.

2.9.4 Marginal Probability

Marginal probability is the probability of a single event without consideration of any other event. Marginal probability is also known as *simple probability*.

2.9.5 Conditional Probability

It is the probability that an event will occur given that another event has already occurred.

If A and B are two events, then the conditional probability of A and B is written as $P(A|B)$ and read as "the probability of A given that B has already occurred".

The conditional probability of an event is the probability that the event occurs under the assumption that another event has occurred. The probability that event B occurs given that event A has occur is called a *conditional probability*. It is denoted by the symbol $P(B/A)$, which is read as "the probability of B given A". A is called the given event.

Joint and Marginal Probabilities

Data obtained by observing values of one variable of a population are known as *univariate data*. Data obtained by observing two variables of a population are called *bivariate data*, and a frequency distribution for bivariate data is called a *contingency table* or *two-way table*.

Example E2.8

A small bin contains 50 rubber balls. Of them, 26 are red and 24 are green. If one ball is randomly selected out of this bin, what is the probability that this ball is

(a) red

(b) green

Solution:

(a) $P(\text{ball selected is red}) = \dfrac{26}{50} = 0.52$

(b) $P(\text{ball selected is green}) = \dfrac{24}{50} = 0.48$

Example E2.9

A multiple-choice question in a test contains five answers. If a student chooses one answer based on "pure guess", what is the probability that this student answer is

(a) correct

(b) wrong

Does these probabilities add up to 1? If yes, why?

Solution:

(a) $P(\text{student's answer is correct}) = \dfrac{1}{5} = 0.2$

(b) $P(\text{student's answer is wrong}) = \dfrac{4}{5} = 0.8$

Yes, these probabilities add up to 1 because this experiment has two and only two outcomes, and according to the second property of probability, the sum of their probabilities must be equal to 1.

Example E2.10

A sample of 1000 families showed that 35 of them own no automobiles, 207 own one automobile each, 377 own two automobiles each, 264 own 3 automobiles each and 117 own 4 or more automobiles each. Write a frequency distribution for this problem. If one family is chosen randomly from these 1000 families, find the probability that this family owns

(a) two automobiles

(b) four or more automobiles.

Solution:

Refer to Table E2.10.

(a) $P(\text{family selected owns 2 automobiles}) = 0.377$

(b) $P(\text{family selected owns 4 or more automobiles}) = 0.117$

Table E2.10

Automobiles owned	Frequency	Relative frequency
0	35	0.035
1	207	0.207
2	377	0.377
3	264	0.264
4 or more	117	0.117
Total	1000	1.000

2.10 MUTUALLY EXCLUSIVE EVENTS

Events that cannot occur together are known as *mutually exclusive events*. Such events do not have any common outcomes. If two or more events are mutually exclusive, then at most one of them will occur every time we repeat the experiment. Hence, the occurrence of one event excludes the occurrence of the other event or events. For example, consider tossing a balanced coin twice. This experiment has 4 outcomes: *HH*, *HT*, *TH* and *TT*. These outcomes are mutually exclusive since one and only one of them will occur when we toss this coin twice.

Two or more events are *mutually exclusive events* if no two of them have outcomes in common. The Venn diagram shown in Fig. 2.4 show the difference between two events that are mutually exclusive and two events that are not mutually exclusive. Similarly Fig. 2.5 shows three mutually exclusive events and two cases of three events that are not mutually exclusive. Two events are said to be *independent*, if the occurrence of one does not affect the probability of the occurrence of the other. Thus, if A and B are *independent events*, then either $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$. If the occurrence of one event affects the probability of the occurrence of the other event, then the two events are said to be dependent events.

Hence, two events will be dependent if either $P(A \mid B) \neq P(A)$ or $P(B \mid A) \neq P(B)$. Two events are either *mutually exclusive* or *independent*. That is, mutually exclusive, events are always dependent, and dependent and independent events are never exclusive. Similarly, dependent events may or may not be mutually exclusive.

Two mutually exclusive events that are taken together include all the outcomes for an experiment are called *complementary event*. Thus, the complement of event A, denoted by $\bar{A}$ is the event that includes all the outcomes for an experiment that are not in A. It is clear that

$$P(A) + P(\bar{A}) = 1 \tag{2.7}$$

Also $$P(A) = 1 - P(\bar{A}) \text{ and } P(\bar{A}) = 1 - P(A) \tag{2.8}$$

The *intersection* of two events is given by the outcomes that are common to both events. If A and B are two events defined in a sample space, then, the intersection of A and B represent the collection of all outcomes that are common to both A and B and is denoted by A and B or AB or BA.

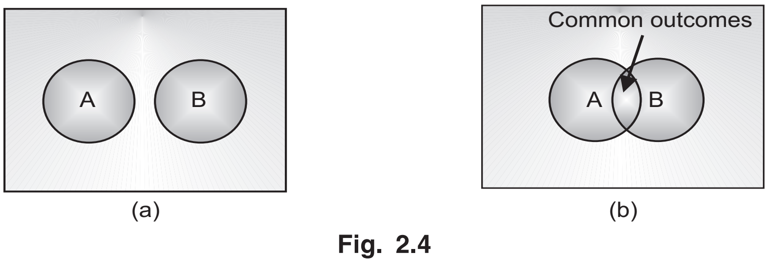


Fig. 2.4

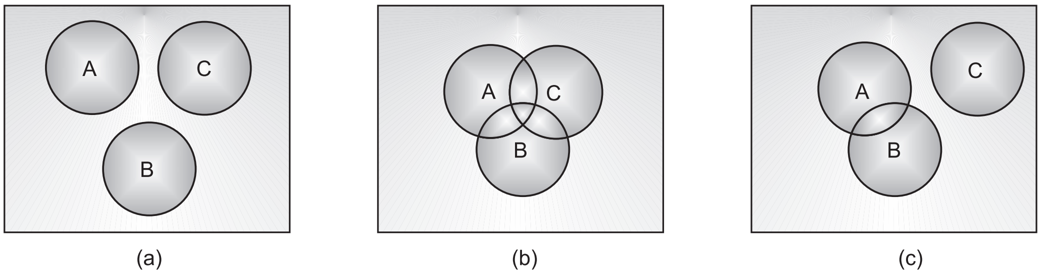


Fig. 2.5

2.11 INDEPENDENT AND DEPENDENT EVENTS

Two events are said to be *independent* if the occurrence of one does not affect the probability of the occurrence of the other. In other words, A and B are *independent events* if either

$$P(A \mid B) = P(A) \text{ or } P(B \mid A) = P(B) \tag{2.9}$$

It can be shown that if one of these two conditions is true, then the second will also be true, and if one is not true then the second will also not be true.

If the occurrence of one event affects the probability of the occurrence of the other event, then the two events are said to be *dependent events*.

Using the probability notation, the two events will be *dependent* if either

$$P(A \mid B) \neq P(A) \tag{2.10}$$

or

$$P(B \mid A) \neq P(B) \tag{2.11}$$

2.12 COMPLEMENTARY EVENTS

The complement event A denoted by $\bar{A}$ and read as "A bar" or "A complement" is the event that includes all the outcomes for an experiment that are not in A.

Events A and $\bar{A}$ are complements of each other. The Venn diagram in Fig. 2.6 shows the complementary events A and $\bar{A}$.

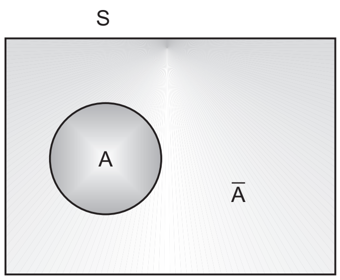


Fig. 2.6: Venn diagram of two complementary events

Since two complementary events are taken together, they include all the outcomes for an experiment and because the sum of the probabilities of all outcomes is 1, it is clear that

$$P(A) + P(\bar{A}) = 1 \tag{2.12}$$

Also $$P(A) = 1 - P(\bar{A}) \tag{2.13}$$

and $$P(\bar{A}) = 1 - P(A) \tag{2.14}$$

Example E2.11

A group of 2000 randomly selected adults were asked if they are in favour or against building a nuclear power plant in the city. The following table gives the results of this survey.

	In favour	Against	Total
Male	500	400	900
Female	650	550	1100
Total	1150	950	2000

(a) If one person is selected at random from these 2000 adults, find the probability that this person is
- *(i)* in favour of the plant
- *(ii)* against the plant
- *(iii)* in favour of the plant given the person is a female
- *(iv)* a male given the person is against the plant

(b) Are the events "male" and "in favour" mutually exclusive? What about the events "in favour" and "against"? Why or why not?

(c) Are the events "female" and "male" independent? Why or why not?

SOLUTION:

(a) *(i)* $P(\text{in favour}) = \dfrac{1150}{2000} = 0.575$

(ii) $P(\text{against}) = \dfrac{950}{2000} = 0.475$

(iii) $P(\text{in favour}|\text{female}) = \dfrac{650}{1100} = 0.59091$

(iv) $P(\text{male}|\text{against}) = \dfrac{400}{950} = 0.42105$

(b) The events "male" and "in favour" are not mutually exclusive because they can occur together. The events "in favour" and "against" are mutually exclusive because they cannot occur together.

(c) $P(\text{female}) = \dfrac{1100}{2000} = 0.55$

$P(\text{female}|\text{in favour}) = \dfrac{650}{1150} = 0.56522$

Since these two probabilities are not equal, the events "female" and "in favour" are not independent.

Example E2.12

There are a total of 170 practicing physicians in a big city. Of them, 60 are female and 30 are pediatricians. Of the 60 females, 10 are pediatricians. Are the events "female" and "pediatrician" independent? Are they mutually exclusive? Explain why or why not.

Solution:

$$P(\text{pediatrician}) = \frac{30}{170} = 0.17647$$

$$P(\text{pediatrician}|\text{female}) = \frac{10}{60} = 0.16667$$

Since these two probabilities are not equal, the events "female" and "pediatrician" are not independent. The events are not mutually exclusive because they can occur together.

Example E2.13

A company hired 50 new graduates last month. Of these 25 are male and 15 are business majors. Of the 25 males, 15 are business majors. Are the events "male" and "business major" independent? Are they mutually exclusive? Explain why or why not.

Solution:

$$P(\text{business major}) = \frac{15}{50} = 0.30$$

$$P(\text{business major}|\text{male}) = \frac{15}{25} = 0.60$$

Since these two probabilities are not equal, the events "male" and "business major" are not independent. The events are not mutually exclusive because they can occur together.

Example E2.14

Let A be the event that a number less than 3 is obtained if a die is rolled once. What is the probability of A? What is the complementary event of A, and what is its probability?

Solution:

Event A will occur if either 1-spot or a 2-spot is obtained on the die. Hence,

$$P(A) = \frac{2}{6} = 0.3334$$

The complementary event of A is that either a 3-spot or 4-spot or a 5-spot, or a 6-spot is obtained on the die. Hence,

$$P(\bar{A}) = 1 - 0.3334 = 0.6666$$

2.13 INTERSECTION OF EVENTS AND MULTIPLICATION RULE

In this section, we introduce the intersection of two events and the application of multiplication rule to find the probability of the intersection of events.

2.13.1 Intersection of Events

Let A and B be two events defined in a sample space. The *intersection* of A and B represents the collection of all outcomes that are common to both A and B and is denoted by A and B. The intersection of events A and B is also denoted by either $A \cap B$ or AB. Figure 2.7 shows the intersection of events A and B. The shaded area in Fig. 2.7 gives the intersection of events A and B.

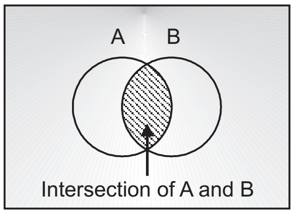


Fig. 2.7: Intersection of events A and B

2.13.2 Multiplication Rule

The probability that events A and B happen together is called the *joint probability* of A and B and is written as $P(A \text{ and } B)$. The probability of the intersection of two events is obtained by multiplying the marginal probability of one event by the conditional probability of second event. The rule is called the *multiplication rule*. Hence,

$$P(A \text{ and } B) = P(A)\, P(B|A) \tag{2.15}$$

The joint probability of events A and B can also be denoted by $P(A \cap B)$ or $P(AB)$.

Hence, for any two events, their joint probability is equal to the probability that one of the events occurs times the conditional probability of the other given event.

Conditional Probability

If A and B are two events, then,

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} \tag{2.16}$$

and $$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \tag{2.17}$$

given that $P(A) \neq 0$ and $P(B) \neq 0$.

Hence, for any two events A and B, the conditional probability that one event occurs given that the other event has occurred equals the joint probability of the two events divided by the probability of the given event.

Multiplication Rule for Independent Events

The probability of the intersection of two independent events A and B is

$$P(A \text{ and } B) = P(A)\, P(B) \tag{2.18}$$

The Special Multiplication Rule

The general multiplication rule for any two events A and B is given by Eq. (2.16)

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

If A and B are independent events, then

$$P(B \mid A) = P(B) \tag{2.19}$$

Hence, for the special case of independent events, we replace the term $P(B \mid A)$ in the general multiplication rule by the term $P(B)$.

If A and B are independent events, then

$$P(A \text{ and } B) = P(A) \cdot P(B) \tag{2.20}$$

and conversely, if

$$P(A \text{ and } B) = P(A)\, P(B)$$

then A and B are independent events. Hence, two events are independent if and only if their joint probability equals the product of their marginal probabilities.

Similarly, if events A, B, C,… are independent, then

$$P(A \text{ and } B \text{ and } C \text{ and } \ldots) = P(A) \cdot P(B) \cdot P(C) \ldots \tag{2.21}$$

Mutually exclusive events are those that cannot occur simultaneously. *Independent events* are those for which the occurrence of some does not affect the probabilities of the others occurring. Two or more (non-impossible) events cannot be both mutually exclusive and independent.

Total Probability Rule

Events $A_1, A_2, A_3, \ldots, A_k$ are said to be *exhaustive*, if one or more of them must occur. Suppose events $A_1, A_2, A_3, \ldots, A_k$ are mutually exclusive and exhaustive; that is exactly one of the events must occur. Then for any event B

$$P(B) = \sum_{j=1}^{k} P(A_j) \cdot P(B \mid A_j) \tag{2.22}$$

Joint Probability of Mutually Exclusive Events

The joint probability of two mutually exclusive events is always 0. If A and B are two mutually exclusive events, then

$$P(A \text{ and } B) = 0 \tag{2.23}$$

Example E2.15

In a group of 10 adults, 4 have type A personality and 6 have a type B personality. If two persons are selected at random from this group of 10, what is the probability that the first of them has a type A personality and the second has a type B personality? Draw a tree diagram for this experiment.

Solution:

Let C = first person selected has a type A personality

D = first person selected has a type B personality

E = second person selected has a type A personality

F = second person selected has a type B personality

Figure E2.15 shows the tree diagram for the experiment.

The probability that the first person has a type A personality and the second has a type B personality is given by

$$P(C \text{ and } F) = P(C)\,P(F|C) = \left(\frac{4}{10}\right)\left(\frac{6}{9}\right) = 0.2667.$$

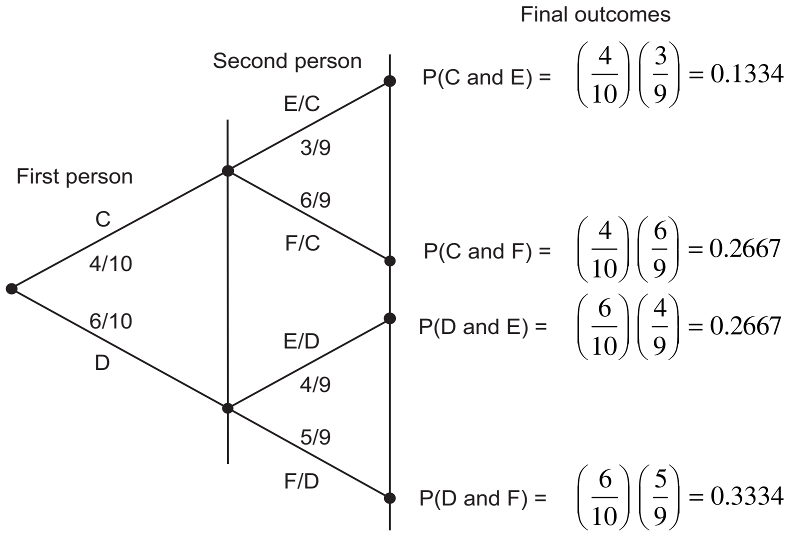


Fig. E2.15: Tree diagram

Example E2.16

Ten per cent of all items sold by a mail order company are returned by customers for a refund. Find the probability that of two items sold during a given hour by this company.

(*a*) both will be returned for a refund

(*b*) neither will be returned for a refund

Draw a tree diagram for this experiment.

Solution:

Let A = first item is returned

B = first item is not returned

C = second item is returned

D = second item is not returned

(*a*) $P(A \text{ and } C) = P(A)\,P(C) = (0.10)(0.10) = 0.01$

(*b*) $P(B \text{ and } D) = P(B)\,P(D) = (0.90)(0.90) = 0.81$

The tree diagram for this equipment is shown in Fig. E2.16.

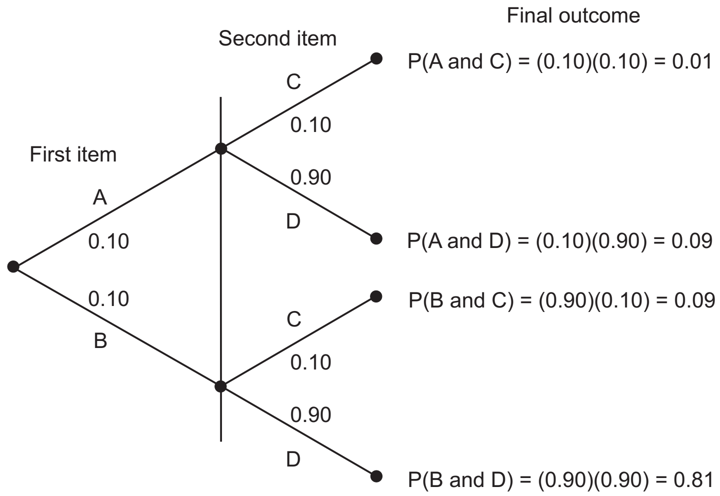


Fig. E2.16: Tree diagram

Example E2.17

The probability that a farmer is in debt is 0.80. What is the probability that three randomly selected farmers are all in debt? Assume independence of events.

Solution:

Let D_1 = the first farmer is in debt

D_2 = the second farmer is in debt

D_3 = the third farmer is in debt

Hence, $P(D_1 \text{ and } D_2 \text{ and } D_3) = P(D_1)P(D_2)P(D_3) = (0.80)(0.80)(0.80) = 0.512$

Example E2.18

According to some private estimates, the probability that a randomly selected student from the population of all students enrolled in all institutions of higher education in a particular state is a female is 0.60, and the probability that this student is a female and a part-time student is 0.25. What is the probability that this student is part-time given that the student is a female?

Solution:

Let F = student selected is a female

PT = student selected is a part-time

It is given that

$P(F) = 0.60$ and $P(PT \text{ and } F) = 0.25$

Hence, $P(PT \mid F) = \dfrac{P(PT \text{ and } F)}{P(F)} = \dfrac{0.25}{0.60} = 0.41667$

2.14 UNION OF EVENTS AND THE ADDITION RULE

In this section, we will discuss the union of events and the addition rule to compute the probability of the union of events.

2.14.1 Union of Events

The *union of two events* A and B includes all outcomes that are either in A or in B or in both A and B. Let A and B be two events defined in a sample space. The *union of events* A and B is the collection of all outcomes that belong either to A or to B or to both A and B and is denoted by A or B.

The union of events A and B is also denoted by "$A \cup B$".

2.14.2 Addition Rule

The method used to calculate the probability of the union of events is called the addition rule.

The probability of the union of two events A and B is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

where $P(A)$ and $P(B)$ are the marginal probabilities of A and B respectively and $P(A$ and $B)$ is the joint probability of A and B.

Addition Rule for Mutually Exclusive Events

The probability of the union of two mutually exclusive events A and B is

$$P(A \text{ or } B) = P(A) + P(B) \tag{2.24}$$

Special Addition Rule

If event A and event B are mutually exclusive, then

$$P(A \text{ or } B\ ) = P(A) + P(B) \tag{2.25}$$

Generalizing, if events A, B, C, … are mutually exclusive, then

$$P(A \text{ or } B \text{ or } C \text{ or } \ldots) = P\,(A) + P\,(B) + P(C) + \cdots \tag{2.26}$$

Therefore, for mutually exclusive events, the probability that one or another of the events occurs equals to the sum of the individual probabilities.

The General Addition Rule

For events that are not mutually exclusive, the general addition rule is applied. Referring to Fig. 2.8, we see that

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \tag{2.27}$$

For any two events, the probability that one or the other occurs equals the sum of the individual probabilities less the probability that both occur.

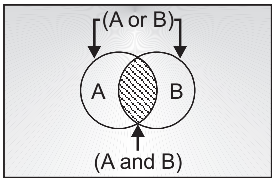


Fig. 2.8: Non-mutually exclusive events

Example E2.19

All 500 employees of a company were asked whether they are smokers or non-smokers of cigarettes and whether or not they are college graduates. Based on this information, the following two-way classification table was prepared:

	College graduate	Not a college graduate	Total
Smoker	55	100	155
Non-smoker	150	195	345
Total	205	295	500

Suppose one employee is selected at random from this company. Find the following probabilities:

(a) P(college graduate or smoker)

(b) P(smoker or not a college graduate)

(c) P(smoker or non-smoker)

Solution:

Let S = smoker

N = non-smoker

C = college graduate

D = not a college graduate

(a) $P(C \text{ or } S) = P(C) + P(S) - P(C \text{ and } S) = \dfrac{205}{500} + \dfrac{155}{500} - \dfrac{55}{500} = 0.60$

(b) $P(S \text{ or } D) = P(S) + P(D) - P(S \text{ and } D) = \dfrac{155}{500} + \dfrac{295}{500} - \dfrac{100}{500} = \dfrac{350}{500} = 0.7$

(c) Since S and N are mutually exclusive events

$$P(S \text{ or } N) = P(S) + P(N) = \frac{155}{500} + \frac{345}{500} = 1.0$$

Example E2.20

The probability that a family owns a desktop computer is 0.78 that it owns a DVD player is 0.70, and that it owns both the computer and a DVD is 0.60. Find the probability that a randomly selected family owns a computer or a DVD player.

Solution:

Let C = family selected owns a computer

D = family selected owns a DVD player

Then, $P(C \text{ or } D) = P(C) + P(D) - P(C \text{ and } D) = 0.78 + 0.70 - 0.60 = 0.88$

Example E2.21

The probability that an open-heart surgery is successful is 0.85. What is the probability that in two randomly selected open-heart surgeries at least one will be successful? Draw a tree diagram for this experiment.

SOLUTION:

Let A = first open-heart surgery is successful

B = first open-heart surgery is not successful

C = second open-heart surgery is successful

D = second open-heart surgery is not successful

P(at least one open-heart surgery is successful) = $P(A \text{ and } C) + P(A \text{ and } D) + P(B \text{ and } C)$

$= (0.85)(0.85) + (0.85)(0.15) + (0.15)(0.85) = 0.7225 + 0.1275 + 0.1275 = 0.9775$

The tree diagram is shown in Fig. E2.21.

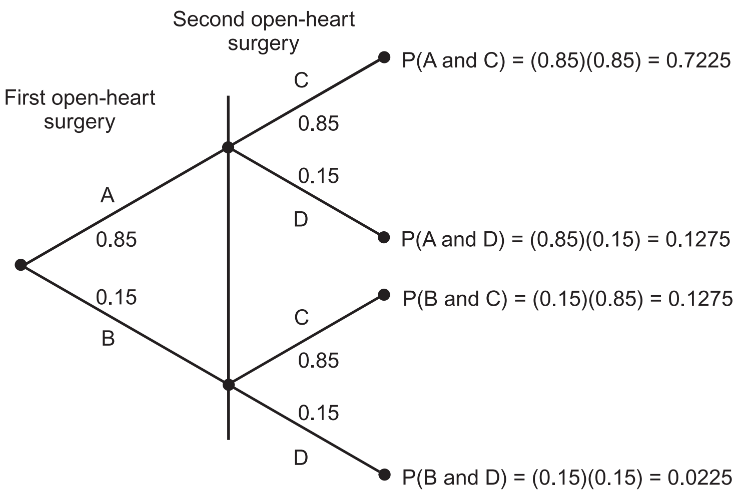


Fig. E2.21

Example E2.22

(a) In a class of 35 students, 11 are seniors, 8 are juniors, 9 are sophomores, and 7 are freshmen. If one student is selected at random from this class, what is the probability that this student is

(i) a junior

(ii) a freshman

(b) If two students are selected at random from this class of 35 students, find the probability that the first student selected is a junior and the second is a sophomore.

SOLUTION:

(a) *(i)* $P(\text{student selected is a junior}) = \dfrac{8}{35} = 0.22857$

(ii) $P(\text{student selected is a freshman}) = \dfrac{7}{35} = 0.2$

(b) Let J_1 = first student selected is a junior

S_2 = second student selected is a sophomore

$$P(J_1 \text{ and } S_2) = P(J_1)\, P(S_2|J_1) = \left(\frac{8}{35}\right)\left(\frac{9}{35}\right) = 0.05878$$

Example E2.23

A manufacturing company makes automobile steering wheels. The manufacturing system involves two independent processing machines so that each steering wheel passes through these two processes. The probability that the first processing machine is not working properly at any time is 0.09, and the probability that the second machine is not working properly at any time is 0.065. Find the probability that both machines will not be working properly at any given time.

Solution:

P(both machines not working properly)

= P(first machine is not working properly) P(second machine not working properly)

= (0.09)(0.065) = 0.00585

2.15 BAYE'S FORMULA

Baye's formula is quite useful in modifying a probability estimate as additional information becomes available. We have seen that a conditional probability is one in which the probability of the event depends upon whether the other event has occurred.

$$P(A/B) = \frac{P(AB)}{P(B)} \tag{2.28}$$

or

$$P(B/A) = \frac{P(AB)}{P(A)} \tag{2.29}$$

From Eqs. (2.28) and (2.29), we have

$$P(AB) = P(A/B)\, P(B) = P(B/A)\, P(A) \tag{2.30}$$

since $AB = BA$

and

$$P(A/B) = \frac{P(A/B)\, P(A)}{P(B)} \tag{2.31}$$

Since $P(A) + P(\bar{A}) = 1$, it follows that event B occurs jointly with either A or $\bar{A}$.

$$P(B) = P(AB) + P(\bar{A}B)$$

and from Eq. (2.30)

$$P(B) = P(A)\, P(B/A) + P(\bar{A})\, P(B/\bar{A}) \tag{2.32}$$

Substituting Eq. (2.32) into Eq. (2.31), we get

$$P(A/B) = \frac{P(A)\, P(B/A)}{P(A)\, P(B/A) + P(\bar{A})\, P(B/\bar{A})} \tag{2.33}$$

If event A has more than two available alternatives, we can write Eq. (2.33) as

$$P(A/B) = \frac{P(A_i)\, P(B/A_i)}{\sum_i P(A_i)\, P(B/A_i)} \tag{2.34}$$

Example E2.24

Find the joint probability of A and B for the following:

(*a*) $P(B) = 0.60$ and $P(A/B) = 0.70$

(*b*) $P(A) = 0.2$ and $P(B/A) = 0.15$

Solution:

(*a*) $P(A \text{ and } B) = P(B \text{ and } A) = P(B)\, P(A/B) = (0.60)(0.70) = 0.42$

(*b*) $P(A \text{ and } B) = P(A)\, P(B/A) = (0.2)(0.15) = 0.03$

Example E2.25

A certain manufacturing company has 456 employees. Based on the information obtained by asking all the employees whether they are smokers or non-smokers and whether or not they are married, the following two-way classification table of data was obtained.

Smoker/Non-smoker	Married	Not married	Total
Smoker	37	72	109
Non-smoker	121	226	347
Total	158	298	456

(*a*) If one employee is selected at random from this company, find the probabilities of the following:

(*i*) P(Married and non-smoker)

(*ii*) P(Smoker and not married)

(*b*) Draw a tree diagram and compute all the joint probabilities for the given data.

Solution:

(*a*) Let M = Married

NM = Not married

S = Smoker

NS = Non-smoker

(*i*) $P(M \text{ and } NS) = P(M)\, P(NS/M) = \left(\frac{158}{456}\right)\left(\frac{121}{158}\right) = 0.2653$

(*ii*) $P(S \text{ and } NM) = P(S)\, P(NM/S) = \left(\frac{109}{456}\right)\left(\frac{72}{109}\right) = 0.1579$

(*b*) The tree diagram is shown in Fig. E2.25.

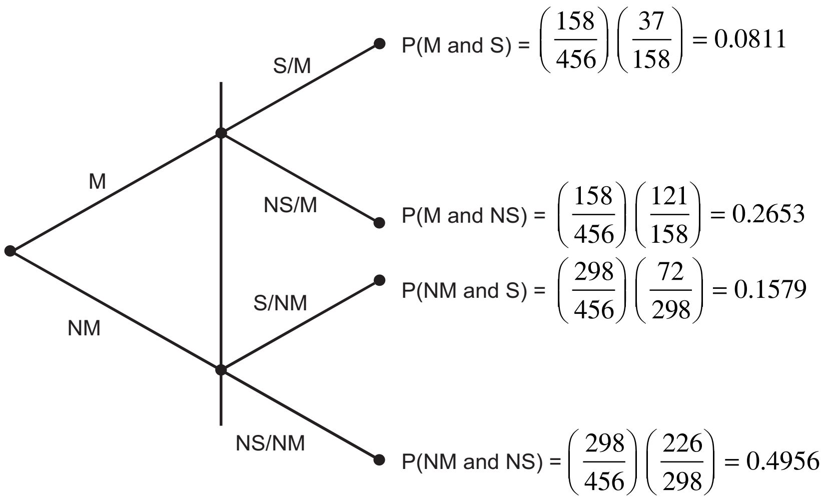


Fig. E2.25: Tree diagram

Example E2.26

Refer to the data of Example E2.25, if one employee is selected at random from the company, calculate the following probabilities.

(*a*) P(married or smoker)

(*b*) P(smoker or not married)

(*c*) P(smoker or non-smoker)

SOLUTION:

Referring to the solution of Example E2.25, we have

(*a*) $P(M \text{ or } S) = P(M) + P(S) - P(M \text{ and } S) = \left(\frac{158}{456}\right) + \left(\frac{109}{456}\right) - \left(\frac{37}{456}\right) = 0.5044$

(*b*) $P(S \text{ or } NM) = P(S) + P(NM) - P(S \text{ and } NM) = \left(\frac{109}{456}\right) + \left(\frac{298}{456}\right) - \left(\frac{72}{456}\right) = 0.7346$

(*c*) Since S and NS are mutually exclusive events,

$$P(S \text{ or } NS) = P(S) + P(NS) = \left(\frac{109}{456}\right) + \left(\frac{347}{456}\right) = 1.0$$

Example E2.27

The probability that a manufacturer is in a debt is 0.7. What is the probability that three randomly selected manufacturers are all in debt? Assume independence of events.

SOLUTION:

Let M_1 = first manufacturer in debt

M_2 = second manufacturer in debt

M_3 = third manufacturer in debt

Then, $P(M_1 \text{ and } M_2 \text{ and } M_3) = P(M_1)\, P(M_2)\, P(M_3) = (0.7)(0.7)(0.7) = 0.343$

Example E2.28

Find $P(A \text{ or } B)$ for the following:

(*a*) $P(A) = 0.17$, $P(B) = 0.5$ and $P(A \text{ and } B) = 0.12$

(*b*) $P(A) = 0.8$, $P(B) = 0.7$ and $P(A \text{ and } B) = 0.6$.

SOLUTION:

(*a*) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 0.17 + 0.5 - 0.12 = 0.55$

(*b*) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 0.8 + 0.7 - 0.6 = 0.9$

Example E2.29

The average acceptability of three automobile transmissions from three suppliers is given below:

Supplier	Relative probability of supply	Average acceptability
1	0.56	0.97
2	0.32	0.89
3	0.12	0.77

(*a*) Find the probability that any transmission received will perform acceptably

(*b*) Find the probability that a particular transmission delivered by supplier 2 when it is known to have performed acceptably.

SOLUTION:

(*a*) $P(A) = (0.56)(0.97) + (0.32)(0.89) + (0.12)(0.77) = 0.5432 + 0.2848 + 0.0924 = 0.9204$

(*b*) $$P(T_2 / A) = \frac{P(T_2)P(A/T_2)}{\Sigma P(T_1)P(A/T_1)} = \frac{(0.32)(0.89)}{(0.56)(0.97)+(0.32)(0.89)+(0.12)(0.77)} = \frac{0.2848}{0.9204} = 0.3094$$

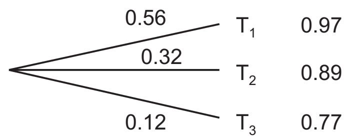


Fig. E2.29

2.16 ADDITIONAL EXAMPLES AND SOLUTIONS

Example E2.30

Following is a percentage distribution for the number of years of school completed by adult employees, 25 years old or over in a manufacturing company.

Years completed	Percentage	Event
0–4	2	A
5–7	4	B
8	5	C
9–11	11	D
12	40	E
13–15	20	F
16 or more	18	G

Determine the probability that an employee randomly selected

(a) has completed 8 years of school or less

(b) has completed 12 years of school or less

(c) has completed 13 years of school or more

(d) interpret each of your answers in *(a)–(c)* in terms of percentages.

SOLUTION:

(a) $P(\text{the employee has completed 8 years of school or less}) = P(A) + P(B) + P(C)$
$= 0.02 + 0.04 + 0.05 = 0.11$

(b) $P(\text{the employee has completed 12 years of school or less}) = P(A) + P(B) + P(C) + P(D) + P(E)$
$= 0.02 + 0.04 + 0.05 + 0.11 + 0.40 = 0.62$

(c) $P(\text{the employee has completed 13 years of school or more}) = P(F) + P(G)$
$= 0.20 + 0.18 = 0.38$

(d) The interpretation of each of the results above in terms of percentage is as follows:

(i) 11% of the adults aged 25 years old and over have completed 8 years of school or less

(ii) 62% of the adults aged 25 years old and over have completed 12 years of school or less

(iii) 38% of the adults aged 25 years old and over have completed 13 years of school or more.

Example E2.31

In a particular corporation, 56% of the employees are white, 95% are male, and 53% are white males. For a randomly selected employee, let

W = Event the employee selected is white

M = Event the employee selected male

(a) Find $P(W)$, $P(M)$ and $P(W \text{ and } M)$

(b) Determine $P(W \text{ or } M)$ and express the answer in percentages

(c) Find the probability that a randomly selected employee is female.

SOLUTION:

(a) $P(W) = 0.56$
$P(M) = 0.95$
$P(W \text{ or } M) = 0.53$

(b) $P(W \text{ or } M) = 0.56 + 0.95 - 0.53 = 0.98$
98% of the employees are either white or male.

(c) $P(F) = 1 - 0.95 = 0.05$

Example E2.32

The number of two door cars and four door vehicles in use by the employees of a company by age are as shown in the following Table E2.32.

Table E2.32

Age (years)		Type		Total
		2 Door T	4 Door F	
Under 6	A_1	25	45	70
6–8	A_2	15	5	20
9–11	A_3	20	10	30
12 and over	A_4	13	7	20
Total		73	67	140

For a randomly selected vehicle (2 Door or 4 Door), determine the probability that the vehicle selected

(a) is under 6 years old
(b) is under 6 years old, given that it is a 2 Door
(c) is a 2 Door
(d) is a 2 Door; given that it is under 6 years old
(e) interpret your answers in *(a)* to *(c)* in terms of percentages.

Solution:

The probability that the vehicle selected is

(a) Under 6 years old is 70/140 = 0.5
(b) Under 6 years old, given that it is a 2 Door car is 25/73 = 0.343
(c) A 2 door car is 73/140 = 0.521
(d) A 2 door car; given that it is under 6 years old is 25/70 = 0.357
(e) *(i)* 50% of all vehicles are under 6 years old
(ii) 34.3% of all 2 door cars are under 6 years old
(iii) 52.1% of all vehicles are 2 door
(iv) 35.7% of all vehicles under 6 years old are 2 door cars.

Example E2.33

Table E2.33 presents a joint probability distribution for engineers and scientists by highest degree obtained in a particular *R&D* corporation.

Table E2.33

Highest degree	Type		
	Engineer, T_1	Scientist, T_2	$P(D_1)$
Bachelors, D_1	0.35	0.29	0.64
Masters, D_2	0.10	0.2	0.30
Doctorate, D_3	0.02	0.025	0.045
Other, D_4	0.01	0.005	0.015
$P(T_1)$	0.48	0.52	1.0

Determine the probability that the person selected

(a) is an engineer
(b) has a doctorate
(c) is an engineer with a doctorate
(d) is an engineer given the person has a doctorate
(e) has a doctorate, given the person is an engineer
(f) interpret your answers in *(a)*–*(e)* in terms of percentages.

SOLUTION:

The probability that the person selected

(a) is an engineer is 0.48.
(b) has a doctorate is 0.045.
(c) is an engineer with a doctorate is 0.02.
(d) is an engineer given the person has a doctorate is 0.02/0.045 = 0.444.
(e) has a doctorate, given that the person is an engineer is 0.02/0.48 = 0.0417.
(f) *(i)* 48% of all engineers and scientists are engineers.
(ii) 4.5% of all engineers and scientists have doctorates.
(iii) 2% of all engineers and scientists are engineers with a doctorate.
(iv) 44.4% of all engineers and scientists with doctorates are engineers.
(v) 4.17% of all engineers have doctorates.

Example E2.34

A frequency distribution for the class level of students in "Introduction to JAVA programming" course is as given in Table E2.34.

Table E2.34

Class	Frequency
Freshman	5
Sophomore	14
Junior	11
Senior	6
Total	36

Two students are randomly selected without replacement. Find the probability that

(a) the first student selected is a junior and the second a senior
(b) both students selected are sophomores
(c) draw a tree diagram for the solution of all the probabilities
(d) what is the probability that one of the students selected is a freshman and the other a sophomore?

SOLUTION:

Let F_i = Freshman on selection i, where $i = 1, 2$
S_i = Sophomore on selection i, where $i = 1, 2$

J_i = Junior on selection i, where i = 1, 2

H_i = Senior on selection i, where i = 1, 2

(a) $P(J_1 \text{ and } H_2) = (11/36)/(6/35) = 0.0524$

(b) $P(S_1 \text{ and } S_2) = (14/36)/(13/35) = 0.1444$

(c) The tree diagram is shown in Fig. E2.34.

(d) The probability that one of the students selected is a freshman and the other student selected is a sophomore is:

$$P(F_1 \text{ and } S_2) + P(S_1 \text{ and } F_2) = 0.0555 + 0.0555 = 0.111$$

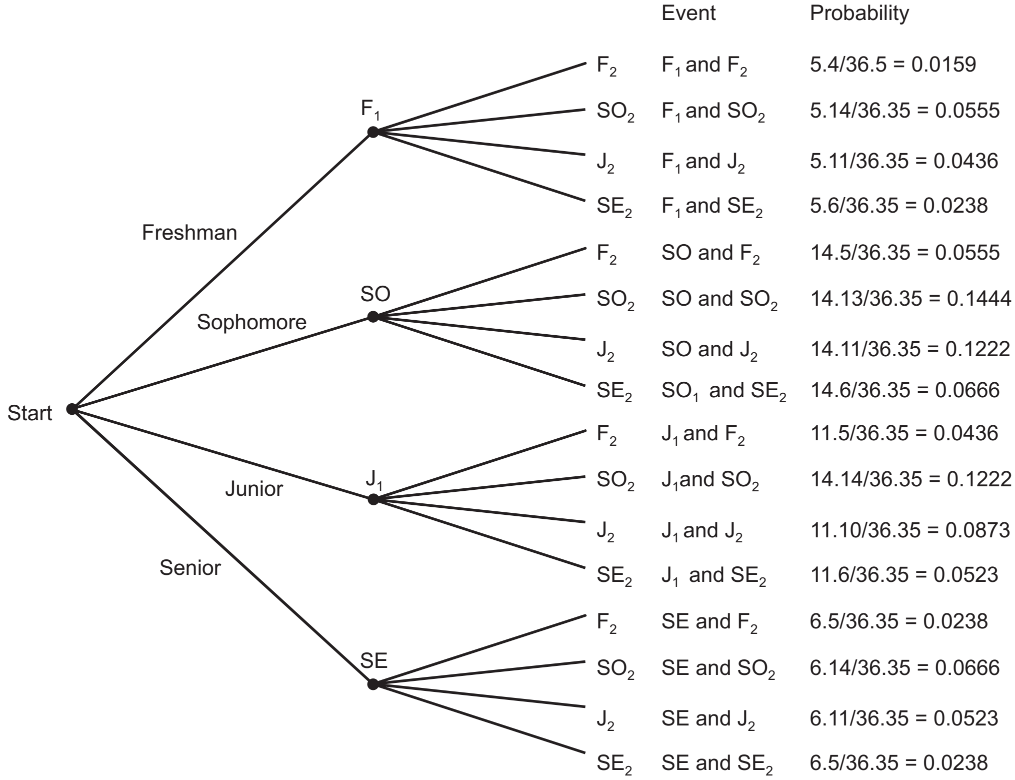


Fig. E2.34: Tree diagram

Example E2.35

Refer to Example E2.33

(a) determine $P(T_2)$, $P(D_3)$ and $P(T_2 \text{ and } D_3)$

(b) are T_2 and D_3 independent events? Why?

Solution:

(a) $P(T_2) = 0.52$

$P(D_3) = 0.045$

$P(T_2 \text{ and } D_3) = 0.025$

(b) The special multiplication rule states that two events T_2 and D_3 are independent if,

$P(T_2 \text{ and } D_3) = P(T_2) \cdot P(D_3)$

Using the data in *(a)*

$P(T_2) \cdot P(D_3) = (0.52)(0.045) = 0.0234$

Since this product does not equal $P(T_2 \text{ and } D_3) = 0.025$, the events T_2 and D_3 are not independent.

Example E2.36

The two-way classification of data of 2000 randomly selected employees of a manufacturing company from a city based on gender and commuting time from residence to work place is given in Table E2.36.

Table E2.36

Gender	Commuting time from residence to work place			Total
	Less than 30 min	30 min to 1 hour	More than 1 hour	
Men	525	450	225	1200
Women	410	250	140	800
Total	935	700	365	2000

(a) If one employee is selected at random from these 2000 employees, find the probability that this employee

(i) commutes from more than 1 hour

(ii) commute for less than 30 minutes

(iii) is a man commuting for 30 min to 1 hour

(iv) is a woman commuting for more than 1 hour.

(b) Determine if the following events are mutually exclusive

(i) 'men' and 'commuting for more than 1 hour'

(ii) 'less than 30 min' 'more than 1 hour'

(iii) 'woman' and 'commutes for 30 minutes to 1 hour'. Are these events independent?

Solution:

(a) *(i)* $P(\text{commutes for more than 1 hour}) = \dfrac{365}{2000} = 0.1825$

(ii) $P(\text{commutes for less than 30 min.}) = \dfrac{935}{2000} = 0.4675$

(iii) $P(\text{commutes for 30 min to 1 hour}) = \dfrac{450}{700} = 0.6428$

(iv) $P(\text{commutes for more than 1 hour/woman}) = \dfrac{140}{800} = 0.175$

(b) *(i)* The term 'man' and 'commutes for more than 1 hour' are not mutually exclusive. The two events can occur together.

(ii) The events 'less than 30 minutes' and 'more than 1 hour' are mutually exclusive because they cannot occur together.

(*iii*) $P(\text{woman}) = \dfrac{800}{2000} = 0.4$

$$P(\text{woman/commutes for 30 min to 1 hour}) = \frac{250}{700} = 0.3571$$

Since these two probabilities are not equal, the events 'woman' and 'commute for 30 minutes to 1 hour' are not independent.

2.17 SUMMARY

The term probability is used to describe the uncertainty *probability*, which measures the likelihood that an event will occur, is an important part of statistics. Probability theory is used extensively in analysing decision-making situations that involve risk or incomplete information.

PROBLEMS

P2.1 In Fig. P2.1, E, T, and N are the events that an automobile brought to a service garage needs an engine oil-change, tune-up, or new tires. Express in words the events represented by

(*a*) region 1
(*b*) region 3
(*c*) region 7
(*d*) regions 1 and 4 together
(*e*) regions 2 and 5 together
(*f*) regions 3, 5, 6, and 8 together.

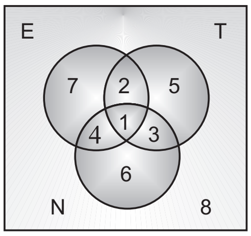


Fig. P2.1: Venn diagram

P2.2 In Problem 2.1, list the region or combinations of regions representing the events that an automobile brought to the garage needs

(*a*) tune-up, but neither engine oil-change nor new tires
(*b*) an engine oil-change and tune-up
(*c*) tune-up or new tires, but not an engine oil-change
(*d*) new tires.

P2.3 An electronic security system may be used in any one of the three modes, in any two of the three modes, or in all three modes: manual (M), semi-automatic (S) and automatic (A). In tests on 200 such systems, the following uses were found.

Number	Use
30	$M \cap S \cap A$
40	$M \cap S$
20	A-only
40	$S \cap A$
All remaining	M-only

(a) define *M*, *S*, and *A* as sets of systems used. Draw a Venn diagram, and show the number of systems in each use

(b) how many systems are in *M*? In *S*? In $(M \cup S)$?

P2.4 An age distribution of 100 employees at a manufacturing company are given as follows:

Age (years)	Number of employees
Under 40	1
40–49	14
50–59	41
60–69	27
70 and over	17
Total	100

For an employee selected at random, let

A : event the employee is under 40

B : event the employees is in his or here 40's

C : event the employee is in his or her 50's

S : event the employee is under 60

(a) use the tables and the *f/n* rule to find $P(S)$

(b) express event *S* in terms of events *A*, *B* and *C*

(c) determine $P(A)$, $P(B)$ and $P(C)$

(d) complete $P(S)$ using special addition rule and write the answers from parts (*b*) and (*c*). Also compare the answer in part (*a*).

P2.5 Refer to Problem P2.4. Find the probability that a randomly selected employee is

(a) 40 years old or under

(b) under 60 years old.

P2.6 In a particular company, 50% of the adult employees are female, 10% are divorced, and 5% are divorced females. For an employee selected at random, let

F = event the person is female

D = event the person is divorced.

(*a*) find $P(F)$, $P(D)$ and $P(F$ and $D)$

(*b*) determine $P(F$ or $D)$ and interpret your answer in terms of percentages

(*c*) find the probability that a randomly selected adult is male.

P2.7 A fair coin is tossed five times. Find the probability of getting at least one tail.

P2.8 Suppose we randomly select two students from a typical co-ed class and observe whether the student selected each time is a male or a female. Write all the outcomes for this experiment. Draw the Venn and tree diagrams for the experiment.

P2.9 In a group of freshman engineering class, some are in favour of mechanical engineering and others are against it. Two students are selected at random from this class and asked whether they are in favour of or against mechanical engineering. How many distinct outcomes are possible? Draw a Venn diagram and a tree diagram for this experiment. List all the outcomes included in each of the following events and state whether they are simple or compound events.

(*a*) both students are in favour of mechanical engineering

(*b*) at most one student is against mechanical engineering

(*c*) exactly one student in favour of mechanical engineering.

P2.10 Determine the conditional probability P(in favour/male) for the response data in Table P2.10 for the 100 employees. They were asked whether they are in favour or against paying higher salaries to CEOs of corporations.

Table P2.10

	In favour	Against	Total
Male	20	40	60
Female	4	36	40
Total	24	76	100

Draw the tree diagram to illustrate the computations.

P2.11 Refer to Problem P2.10. Find the conditional probability that a randomly selected employee is a female given that this employee is in favour of paying high salaries to the CEOs. Draw the tree diagram for this computation.

P2.12 (*a*) Determine the probability of getting 3 tails in tossing 3 coins.

(*b*) Two dice are thrown. Draw the sample space. Determine the probability of getting 6 in both the dice.

(*c*) Determine the probability of getting *(i)* exactly two tails (event A) and *(ii)* at least two tails (event B) in tossing 3 coins.

P2.13 (*a*) Determine the probability that the sum of the numbers shown on the two faces, when two dice are thrown

(*i*) is 7

(*ii*) is 10

(*b*) A die is thrown. Determine the probability of getting either an even number or a number greater than 4 or both.

P2.14 Table P2.14 gives the contingency table for age and rank of employees in a particular company. If an employee is selected at random, (*a*) find the unconditional probability that the employee selected is in his/her 50's

(*b*) find the unconditional probability that the employee selected is in his or her 50's given that Grade 1 employee is selected

(*c*) interpret the probabilities found in parts *(a)* and *(b)* in terms of percentages.

Table P2.14

			Rank				
			R_1	R_2	R_3	R_4	Total
Age	Under 30	A_1	5	8	57	7	77
	30–39	A_2	50	170	160	20	400
	40–49	A_3	150	120	60	6	336
	50–59	A_4	150	60	30	5	245
	60 & over	A_5	70	20	5	0	95
	Total		425	378	312	77	1192

P2.15 Refer to Problem P2.14. Compute the conditional probability $P(A_4/R_3)$ using unconditional probabilities.

P2.16 Data on the marital status of employees in a company are as given in the table below. The table provides a joint probability distribution for the marital status of adults by sex. The term "Single" is used for "Never Married".

Table P2.16

			Marital status				
			Single M_1	Married M_2	Widowed M_3	Divorced M_4	$P(S_1)$
Sex	Male	S_1	0.10	0.30	0.01	0.04	0.45
	Female	S_2	0.15	0.35	0.03	0.02	0.55
	$P(M_1)$		0.25	0.65	0.04	0.06	1.00

If an employee is selected at random,

(*a*) find the probability that the employee selected is divorced, given that the employee selected is male, given that the employee selected is divorced.

(*b*) use the conditional probability rule for *(a)*.

P2.17 In the research and development department of a company, the number of male and female employees are shown in the frequency distribution as given below in a tabular form.

Sex	Frequency
Male	30
Female	20
Total	50

Two employees selected at random from this department. The first employee obtained is not returned to the department for possible reselection; that is, the sampling is without replacement.

(*a*) determine the probability that the first employee obtained is female and the second is male

(*b*) draw the tree diagram.

P2.18 A bin contains 20 machine parts, 4 of which are defective. If 2 machine parts are selected at random (without replacement) from this box, what is the probability that both parts are defective? Draw the tree diagram.

P2.19 Table P2.19 gives the classification of all employees of a particular department in a large corporation by gender and college degree.

Table P2.19

	College graduate (G)	Not a college graduate (N)	Total
Male (M)	8	20	28
Female (F)	3	9	12
Total	11	29	40

If one of these employees is selected at random for membership on the planning committee, determine the probability that this employee is a female and a college graduate. Draw the tree diagram.

P2.20 A balanced coin is tossed five times. Determine the probability of getting at least one head.

P2.21 The probability that a patient is allergic to penicillin is 0.10. If the drug is administered to three individual patients, determine the probability that

(a) all three of them are allergic to it

(b) at least one of them is not allergic to it.

(c) draw the tree diagram.

P2.22 Draw the Venn and tree diagram for selecting three times two persons (a man or a woman) from the members of a club.

P2.23 The search committee of the school of engineering that plans to hire one new faculty member has prepared a final of five candidates, all of whom are equally qualified. Out of these five candidates two of them are woman. If the committee decides to select at random one person out of these five candidates, find the probability that this person will be a woman or a man? Also find the sum of these two probabilities and comment the result.

P2.24 In a random sample of 1000 companies surveyed showed that 35 of them have no design engineer, 210 have one design engineer each, 379 have two design engineers, 270 have three design engineers and 106 have four or more design engineers.

(a) prepare a frequency distribution table and calculate the relative frequencies for all categories

(b) if one company is selected randomly from these 1000 companies, find the probability that this company has to design engineers

(c) find the probability that the company selected has four or more design engineers.

P2.25 Of the 25 new graduates hired by a company, 14 are female and 9 are mechanical engineering majors. Of the 14 females, 6 are mechanical engineering majors.

(a) are these events "female" and "mechanical engineering major" independent?

(b) are these events "mutually exclusive"?

P2.26 An individual is picked at random from a group of 56 athletes. Suppose 26 of the athletes are female and 6 of them are swimmers. Also, there are 10 swimmers among the males.

(*a*) find the probability that she is a swimmer if that the individual picked is a female

(*b*) given that the individual picked is a swimmer, find the probability that the person is a male.

P2.27 (*a*) The probability that the stock market goes up on Monday is 0.7. Given that it goes up on Monday, the probability that it goes up on Tuesday is 0.3. Find the probability that the market goes up on both days.

(*b*) The probability that a student passes the midterm exam is 0.7. Given that a student fails the midterm exam, the probability that the student passes the final exam is 0.8. Find the probability that John (a student in that class) fails the midterm and passes the final.

(*c*) Past records show that the probability that the Chairman of the board attends a meeting is 0.7, that the President of the company attends a meeting is 0.8, and that they both attend a meeting is 0.6. Can we say that the Chairman and the President act independently regarding their attendance at the meeting of the board of directors?

P2.28 Two students are selected at random from a class without replacement whose frequency distribution of males and females is given in Table P2.28.

Table P2.28

Gender	Frequency
Male	17
Female	23
Total	40

(*a*) find the probability that the first student selected is female and the second is male

(*b*) draw a tree diagram.

P2.29 Given that A and B are two independent events, find their joint probability for the following data:

(*a*) $P(A) = 0.25$ and $P(B) = 0.78$

(*b*) $P(A) = 0.55$ and $P(B) = 0.23$

P2.30 The following data gives a two-way classification of all faculty members of a university based on gender and tenure.

	Tenured	Non-tenured
Male	75	48
Female	34	19

(*a*) if one of these faculty members is selected at random, find the following probabilities:

(*i*) P(male and non-tenured)

(*ii*) P(tenured and female)

(*b*) find P(tenured and non-tenured).

P2.31 In a group of 12 employees of a company, 5 are graduates and 7 are non-graduates. If 2 persons are selected at random from this group, find the probability that the first of them is a graduate and the second is a non-graduate. Draw a tree diagram for the solution.

P2.32 The probability that an employee at a university is a female is 0.3. The probability that an employee is a female and married is 0.2. Find the conditional probability that a randomly selected female employee from this university is a married person.

P2.33 For the data given in Problem P2.30, if one of these faculty members is selected at random, find the following probabilities.

(*a*) *P*(female or non-tenured)

(*b*) *P*(tenured or male).

P2.34 The probability that a new product introduced to the market is successful is 0.78. What is the probability that in two randomly selected new products introduced, at least one will be successful? Draw a tree diagram for the solution.

P2.35 Consider the following events for one roll of a die

A = an odd number is observed (1, 3, 5)

B = an even number is observed (2, 4, 6)

C = a number less than 5 is observed (1, 2, 3, 4)

(*a*) Are events A and B mutually exclusive?

(*b*) Are events B and C mutually exclusive?

P2.36 An experimental submarine is estimated to have a probability of 0.90 for a successful first flight. In case of failure there is a 0.03 probability of a catastrophic explosion, in which case the abort system cannot be used. The abort system has a reliability of 0.97. Calculate the probability of every possible outcome of the initial flight.

P2.37 With voltage fluctuations present, the probability of an electromechanical system going down is 75%; the probability of it going down during no voltage fluctuations condition is only 8%. The probability of voltage fluctuations occurring is 20%. What is the probability that voltage fluctuations are present given that the electro mechanical system is down?

P2.38 Out of a collection of 100 gaskets the following are defective as indicated:

Defective type	Number of gaskets
Type I defect	10
Type II defect	8
Type III defect	3
Type I and II defect	5
Type I, II and III defect	4
No defect	70

(*a*) if a gasket picked out the collection has a type II defect, what is the probability that it also has a type I defect?

(*b*) if the gasket picked has a type I defect, what is the probability that it also has a type II defect?

P2.39 Three persons: A, B and C are working independently to solve a mathematical puzzle. Based on their previous performance, the probabilities that they will succeed are 0.1, 0.2 and 0.3 respectively. What is the probability that the puzzle will be solved?

P2.40 A system consists of 25 different components, of which one; component A, has been identified as the critical component. The probability that the critical component is non-defective is 0.97. If the critical component is non-defective, then the system succeeds with at probability of 0.99. If not, then the probability of the system success drops to 0.93. What is the probability of system success?

P2.41 An electronic test set up has 97% probability of correctly identifying a defective component and a 3% probability of identifying a non-defective component as defective. A batch of 100 components of which 3 are known as defective is subjected to this electronic testing. If the test identifies a component as defective, what is the probability that it is truly defective?

P2.42 A mechanical system assembly plant purchases a certain component from three independent vendors: *A*, *B* and *C*, who supply 20%, 30% and 50% respectively of the total number of components needed by the assembly plant. On an average, the percentage of defective components supplied by vendors *A*, *B* and *C* are 5%, 3% and 2% respectively.

(a) if a component is selected at random, find the probability that it is defective

(b) if the selected component is defective, what is the probability that it was supplied by vendor *C*? by vendor *A*? by vendor *B*?

P2.43 A bowl contains five white balls, two red balls, and three green balls. What is the probability of getting either a white ball or a red ball in one draw from the bowl?

P2.44 One bowl contains five white balls, two red balls, and three green balls. Another bowl contains three yellow balls and seven black balls. Determine the probability of getting a red ball from the first bowl and a yellow ball from the second bowl in one draw from each bowl.

P2.45 The probability that a student is in favour of engineering ethics course is 0.60 and a student is against is 0.40. Two students are randomly selected, and it is observed whether they favour or oppose engineering ethics course.

(a) draw a tree diagram for this experiment

(b) determine the probability that at least one of the two students favours engineering ethics course.

P2.46 Customers who buy a certain make of an automobile can order an engine in any of three sizes. Of all automobiles sold, 50% have the smallest engine, 30% have the medium-size one, and 20% have the largest. Of automobiles with the smallest engine, 15% fail the emissions test within two years of purchase, while 10% of those with the medium size and 15% of those with the largest engine fail. Determine the probability that a randomly chosen automobile will fail on emissions test within two years.

P2.47 A company manufactures aluminium soft drink cans. It was found that the probability that a can has a flaw on its side is 0.03, the probability that a can has a flaw on the top is 0.02, and the probability that a can has a flaw on both sides and the top is 0.01.

(a) what is the probability that a randomly chosen can has a flaw?

(b) what is the probability that it has no flaw?

(c) what is the probability that a can will have a flaw on the side, given that it has a flaw on top?

(d) what is the probability that a can will have a flaw on the top, given that it has a flaw on the side?

P2.48 *(a)* Two coins are tossed. Determine the conditional probability of getting two heads (event *B*) given that at least one coin shows a head (event *A*)

(b) A box contains 5 red and 4 blue balls. Two balls are drawn one-by-one without replacement. Given that the first ball drawn is red, find the probability that both the balls drawn will be blue.

P2.49 Of the gas turbine engine blades manufactured by a certain process, 15% are defective. Five turbine blades are chosen at random. Assume they function independently. Find the probability that they all work.

P2.50 Refer Problem P2.49. What is the probability that at least one of the gas turbine engine blade manufactured works?

P2.51 In a certain city, 45% of the people consider themselves conservatives (C), 30% Consider themselves to be Liberals (L) and 25% consider themselves to be Independents (I). During the particular election, 50% of the Conservatives voted, 40% of the Liberals voted, and 60% of the Independents voted. If a randomly selected person voted, determine the probability that the voter is

(a) Conservative

(b) Liberal

(c) Independent.

P2.52 In a certain company employees, 5% of the men and 1% of the women are heavier than 100 kg. Furthermore, 70% of the employees are women. Suppose a randomly selected employee is heavier than 100 kg. Find the probability that the employee is a woman.

P2.53 In a certain engineering college hostel, 20% are freshmen of whom 10% own a car, 30% are sophomores of whom 20% own a car, 20% are juniors of whom 30% own a car, and 20% are seniors of whom 40% own a car. A student is randomly selected from the hostel.

(a) find the probability that the student owns a car

(b) if the student owns a car, find the probability that the student is a junior.

P2.54 At a certain grocery store, eggs come in cartons that hold a dozen eggs. The store experience indicates that 78% of the cartons have no broken eggs, 19% have one broken egg, 2% have two broken eggs, and 1% have three broken eggs, and that the percentage of cartons with four or more eggs broken eggs is negligible. An egg selected at random from a carton is found to be broken. What is the probability that this egg is the only broken one in the carton?

P2.55 The probability that a college student being male is 1/3 and that being female is 2/3. The probability that a male student completes the probability and statistics course successfully is 8/10 and that the female student does it is 4/5. A student is selected at random is found to have completed the course. Determine the probability that the student is a

(a) male

(b) female.

P2.56 In a car dealership, assume 55% of the cars are made in US and 15% of these are compact; 25% of the cars are made in Europe and 40% of these are compact; and finally, 20% are made in Japan and 60% of these are compact. If a car is picked at random from this dealership:

(a) find the probability that it is a compact

(b) draw a tree diagram for part *(a)*

(c) given that the car is a compact, find the probability that it is a European.

P2.57 In a survey of the evaluation of preliminary product designs by the customers, 90% of highly successful products received good reviews, 60% moderately successful products received good reviews, and 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful, and 25% have been poor products.

(a) find the probability that a product attains a good review

(b) if the new design attains a good review, what is the probability that it will be a highly successful product?

(*c*) if a product does not attain a good review, what is the probability that it will be a highly successful product?

P2.58 A cat is in a room and each of the four walls of the room has a door through which the cat could try to escape. However, there is a trap at each of the doors D_1, D_2, D_3 and D_4 and they work with probabilities 0.3, 0.2, 0.4 and 0.5 respectively. If the cat picks a door at random:

(*a*) find the probability that the cat will escape

(*b*) draw the tree diagram for the solution in part *(a)*

(*c*) given that the cat escapes, find the probability that the cat chose door D_3 to escape.

P2.59 According to a particular survey, 8% of the population has a lung disease. Of those having lung disease, 90% are smokers; of those not having lung disease, 25% are smokers. Find the probability that a randomly selected smoker has lung disease.

P2.60 Three different machines m_1, m_2 and m_3 were used for producing a large batch of similar manufactured items. Suppose that 30% of the items were produced by machine m_1, 30% by machine m_2, and 40% by machine m_3. Also, 1% of the items produced by machine m_1 are defective, 2% of the items produced by machine m_2 are defective, and 3% of the items produced by machine m_3 are defective. If one item is selected at random from the entire batch, and it is found to be defective. Determine the probability that this item was produced machine m_2.

REVIEW QUESTIONS

1. Define the following terms:
   (*a*) experiment
   (*b*) outcome
   (*c*) sample space
   (*d*) simple event
   (*e*) compound event.
2. Describe the properties of probability.
3. Describe an impossible event and a sure event. What is the probability of the occurrence of each of these two events?
4. Describe the three approaches to probability.
5. Explain the difference between the marginal and conditional probability of events.
6. What is meant by two mutually exclusive events?
7. Explain the meaning of independent and dependent events.
8. What is the complement of an event? What is the sum of the probabilities of two complementary events?
9. Explain the meaning of the following terms:
   (*a*) intersection of two events
   (*b*) the joint probability of two or more events
   (*c*) multiplication rule of probability
   (*d*) joint probability of two mutually exclusive events.
10. Explain the meaning of the following:
    (*a*) union of two events
    (*b*) addition rule of probability
    (*c*) classical probability rule
    (*d*) equally likely outcomes
    (*e*) sample point.

STATE TRUE OR FALSE

1. The probability of an event is always between 0 and 1, inclusive. (True/False)
2. The probability of an event that cannot occur is 1. (True/False)
3. The probability of an event that must occur is 1. (True/False)
4. An event that cannot occur is called a certain event. (True/False)
5. An event that must occur is called a certain event. (True/False)
6. An experiment is an action whose outcome cannot be predicted with certainty. (True/False)
7. An event is some specified result that may or may not occur when an experiment is performed. (True/False)
8. The experiment has a finite number of possible outcomes, all equally likely. (True/False)
9. The probability of an event equals the ratio of the number of ways that the event can occur to the total number possible outcomes. (True/False)
10. If a member is selected at random from a finite population, probabilities are identical to percentages (relative frequencies). (True/False)
11. The probability of an event is the proportion of times it occurs in a small number of repetitions of the experiment. (True/False)
12. Sample space is the collection of all possible outcomes for an experiment. (True/False)
13. An event is a collection of outcomes for the experiment, that is, any subset of the sample space. (True/False)
14. Two or more events are mutually exclusive events if no three of them have outcomes in common. (True/False)
15. If event A and event B are mutually exclusive, so are events A, B and C for every event C. (True/False)
16. If event A and event B are not mutually exclusive, neither are events A, B and C for every event C. (True/False)
17. If E is an event, then $P(E)$ represents the probability that event E occurs. (True/False)
18. If event A and event B are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$. (True/False)
19. For any event E, $P(E) = 1 - P(\text{not } E)$. (True/False)
20. If A and B are any two events, then $P(A \text{ or } B) = P(A) + P(B) - P(A \,\&\, B)$. (True/False)
21. The probability that event B occurs given that event A occurs is called a conditional probability. (True/False)
22. If A and B are any two events with $P(A) > 0$, then $P(B|A) = P(A \,\&\, B)/P(B)$. (True/False)
23. If A and B are any two events, then $P(A \,\&\, B) = P(A)/P(B|A)$. (True/False)
24. Event B is said to be independent of event A if $P(B|A) = P(B)$. (True/False)
25. If A and B are independent events, then $P(A \,\&\, B) = P(A) \cdot P(B)$. (True/False)
26. Two or more events are said to be mutually exclusive if at most one of them can occur when the experiment is performed, that is, if no two of them have outcomes in common. (True/False)

27. For any two events, the probability that one or the other of the events occurs equals the sum of the two individual probabilities. (True/False)
28. For any event, the probability that it occurs equals 1 minus the probability that it does not occur. (True/False)
29. Data obtained by observing values of one variable of a population are called univariate data. (True/False)
30. Data obtained by observing values of two variables of a population are called bivariate data. (True/False)
31. The joint probability equals the product of the marginal probabilities. (True/False)

ANSWERS TO STATE TRUE OR FALSE

1. **True**	2. **False**	3. **True**	4. **False**	5. **True**	6. **True**	7. **True**	8. **True**	9. **True**	10. **True**
11. **False**	12. **True**	13. **True**	14. **False**	15. **False**	16. **True**	17. **True**	18. **True**	19. **True**	20. **True**
21. **True**	22. **False**	23. **False**	24. **True**	25. **True**	26. **True**	27. **False**	28. **True**	29. **True**	30. **True**
31. **True**									