Signals & Systems

Spring 2018

19.02.18

Recall:

Beat Notes: multiplication of sinusoids

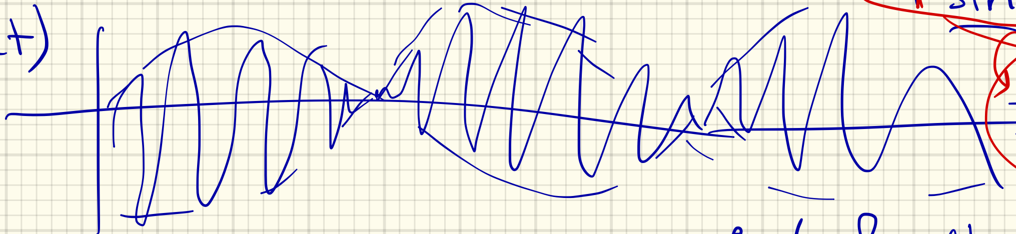
$$\cos(2\pi \underbrace{f_1}_{\text{high freq}} t) \cdot \cos(2\pi \underbrace{f_2}_{\text{low freq.}} t)$$

$\boxed{\text{Euler formula}}$

$$x(t) = \frac{1}{2}\left[\cos(2\pi(f_1+f_2)t) + \cos(2\pi(f_1-f_2)t) + \sin(2\pi 100 t)\right]$$

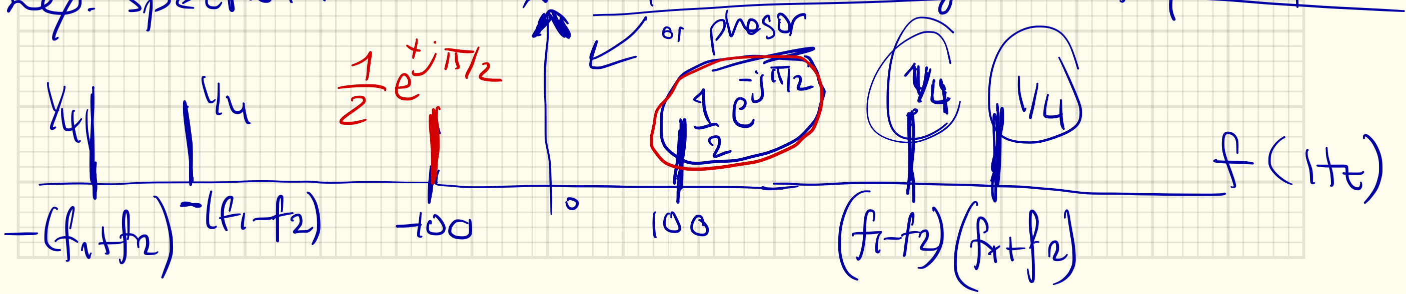
$$\sin(2\pi 100 t) = \frac{1}{2j}\left(e^{j2\pi 100 t} - e^{-j2\pi 100 t}\right)$$

$x(t)$



Freq. Spectrum → complex amplitude of each freq. component

or phasor



In general:

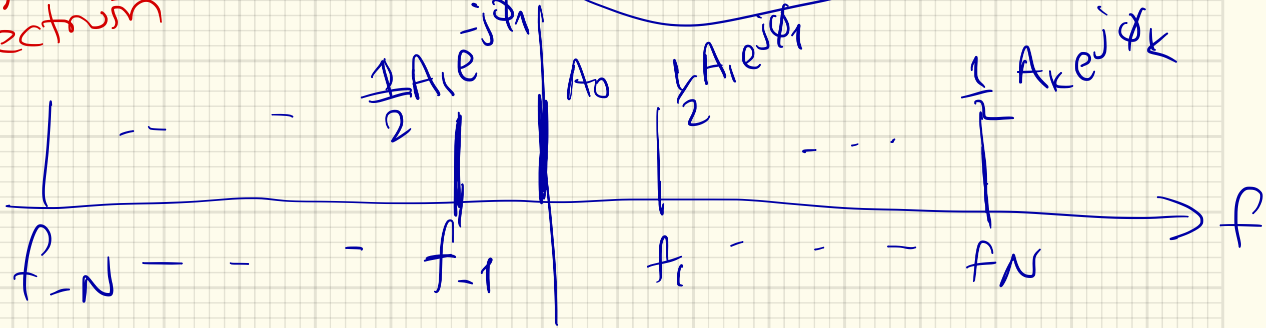
$$x(t) = \underbrace{A_0}_{a_0} + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

freq = 0 in the spectrum

for $f_k \longrightarrow \frac{1}{2} A_k e^{j\phi_k} \rightarrow a_k$ → phasors

for $f_{-k} \longrightarrow \frac{1}{2} A_k e^{-j\phi_k} \rightarrow a_{-k}$

frequency spectrum



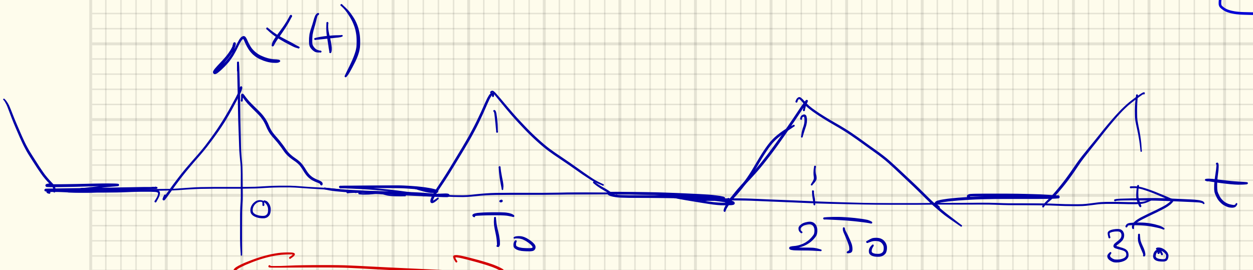
$x(t) = ?$ (f_k, a_k) pairs from the spectrum of $x(t)$.

Fundamental Frequency:

We have a sum of sinusoids signal:

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

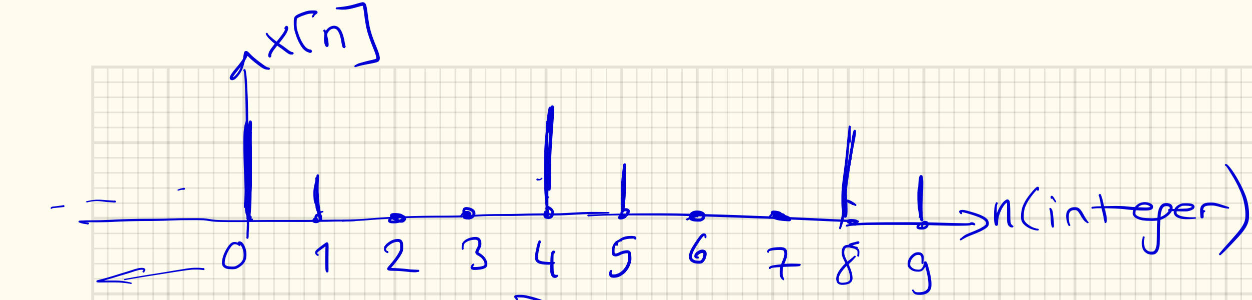
Q: If $x(t)$ is periodic; what is the fundamental freq?



kT_0 are all periods of $x(t)$

smallest T_0 = fundamental period

Fundamental Freq: $f_0 = \dfrac{1}{T_0}$ (Hz)



$N_0 = 4$: fundamental period $\longrightarrow$ $\frac{1}{N_0}$: fund. freq.

⇒ Answer to the Q: If we can find an f_0 such that

$$f_0 = \underbrace{gcd}_{\text{greatest common divisor}}(f_k) \quad \text{where } \left(\frac{f_k}{f_0}\right) \text{ are integers } \forall k,$$

then the signal $x(t)$ is periodic, f_0 is the fundamental frequency;

Def (Harmonic freq): The frequency $(k \cdot f_0)$; i.e. integer multiples of f_0 are called the harmonics of f_0.

Ex: $x(t) = \cos(2\pi \boxed{3} t) + \sin(2\pi \boxed{(4.5)} t)$

Is $x(t)$ periodic?

$f = 3\,\text{Hz}$, $f = 4.5\,\text{Hz}$.

$f_0 = 1.5\,\text{Hz}$: fundamental freq.

$\left(\frac{3}{1.5}\right) = 2$ → 2^{nd} & 3^{rd} harmonics exist in this signal.

$\left(\frac{4.5}{1.5}\right) = 3$

Fact: When we add sinusoids w/ frequencies that are harmonics of an f_0 (fund. freq), then we get a PERIODIC signal.

exercise: $x(t) = \cos(2\pi(5.5)t) + 3\sin(2\pi(7.5)t)$

Is this signal periodic?

$f_0 = 0.5\,\text{Hz}$ → Yes

Q: Which harmonics are there?

Q: Plot the spectrum.

Fourier Series Theory (FS)

Any periodic signal $x(t)$ w/ fundamental frequency f_0 can be written as sum of harmonically related sinusoids

Fourier Synthesis:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{j2\pi \boxed{k f_0} t}}_{\psi_k(t)}$$

$a_2 \rightarrow 2 f_0$

$a_{10} \rightarrow 10 f_0$.

More generally in maths

$$x(t) = \sum_{k=-\cancel{\infty}\,N}^{\cancel{\infty}\,N} a_k \psi_k(t)$$

$\hookrightarrow$ basis functions

FS theory:

$$\psi_k(t) = e^{j2\pi k f_0 t}$$

: f_0 : fund. freq. $\Leftrightarrow T_0 = \frac{1}{f_0}$

$k f_0$: harmonics

a_k : Fourier Series coefficients

* For real $x(t)$ (periodic) $\rightarrow$ property a_k's

$$\left(x(t)\right)^* = \left(\sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi k f_0 t}\right)^*$$

$$x(t) = \sum_{k=-\infty}^{\infty} (a_k)^* \cdot e^{-j2\pi k f_0 t}$$

$k' = -k$ change of vars

$$x(t) = \sum_{k'} a^*_{-k'}\, e^{j2\pi k' f_0 t}$$

$$=$$

$$a_k = a^*_{-k} \implies \boxed{a^*_k = a_{-k}}$$ for a real periodic signal.

Basis functions for F.S: $\boxed{v_k(t) = e^{j2\pi k f_0 t}}$ Basis func.

Property 1: Zero Integral: $\int_0^{T_0} v_k(t)\,dt = 0$

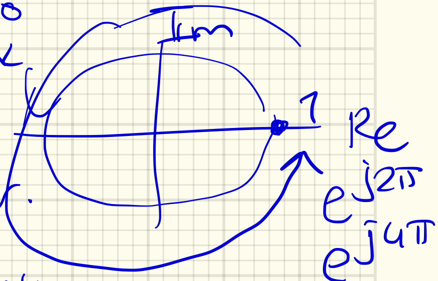
$k \neq 0$.

$$\int_0^{T_0} e^{j2\pi k f_0 t}\,dt = \frac{e^{j2\pi k f_0 t}}{j2\pi k f_0}\Bigg|_0^{T_0} = \dots \text{ fill this in}$$

$$= \frac{e^{j2\pi k} - 1}{j2\pi k f_0} = $$

$e^{j2\pi k}$

k integer.



$$e^{j2\pi k} = 1 \quad \forall k \text{ integer}$$

$k = 0$: $\int_0^{T_0} e^0\,dt = T_0$

$$\int_0^{T_0} v_k(t)\,dt = \begin{cases} 0, & \forall k,\ k \neq 0 \\ T_0, & k = 0 \end{cases}$$

② Orthogonality of harmonic complex exponentials ($v_k(t)$)

Recall $\langle \underline{v}, \underline{u} \rangle = 0 \Rightarrow \underline{u}, \underline{v}$ are orth.

$\underline{v}^T \underline{u} = 0 \Rightarrow$ for complex vectors $\underline{v}^H \underline{u}$

$(\underline{v}^T)^* \underline{u} = 0$

$\langle \underline{v_k(t)}, \underline{v_\ell(t)} \rangle = 0$

$$\int_0^{T_0} v_k(t)\, v_\ell^*(t)\, dt = 0 \quad \text{for } k \neq \ell.$$

$(v_\ell(t))^* = ?$

k, ℓ integers

$$\int_0^{T_0} e^{j2\pi k f_0 t}\, e^{-j2\pi \ell f_0 t}\, dt =$$

$$\int_0^{T_0} e^{j2\pi \underbrace{(k-\ell)}_{=m} f_0 t}\, dt = \int_0^{T_0} e^{j2\pi m f_0 t}\, dt = \begin{cases} 0, & m \neq 0 \;\; (k \neq \ell) \\ T_0, & m = 0 \;\; (k = \ell) \end{cases}$$

Use prop-1

Fourier synthesis: Given a_k, f_0:

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}}$$

$$a_k, f_0 \xrightarrow{\text{synthesis}} x(t)$$

Freq

$a_k, f_0 = ?$ $\xleftarrow{\text{analysis}}$ Given $x(t)$, periodic

~~Period~~ (fund) $\rightarrow f_0$.

How to find a_k?

Start w/

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

synthesis eqn

$* \; e^{-j2\pi l f_0 t}$

$$x(t) e^{-j2\pi l f_0 t} = \sum_k a_k e^{j2\pi k f_0 t} \cdot e^{-j2\pi l f_0 t}$$

take the integral of both sides $\longrightarrow$

$$\Rightarrow \int_0^{T_0} x(t)\, e^{-j2\pi l f_0 t}\, dt = \int_0^{T_0} \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi k f_0 t}\, e^{-j2\pi l f_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_0^{T_0} e^{j2\pi k f_0 t} \cdot e^{-j2\pi l f_0 t}\, dt}_{\text{Prop 2}} = \begin{cases} 0, & l \neq k \\ T_0, & l = k \end{cases}$$

only non-zero when $k = l$

$$\int_0^{T_0} x(t)\, e^{-j2\pi l f_0 t}\, dt = a_l \cdot T_0$$

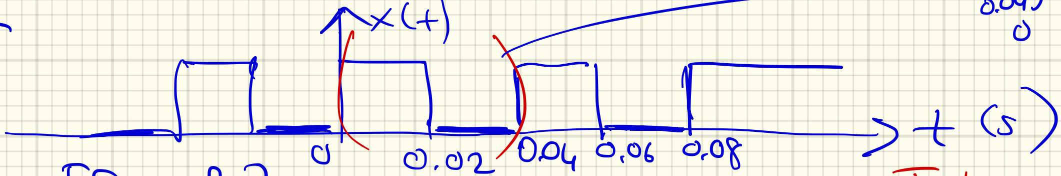
Fourier Analysis Integral

Fourier series coefficients

$$\boxed{a_k = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-j2\pi k f_0 t}\, dt}$$

Q: Given periodic $x(t) \rightarrow T_0 \rightarrow f_0 \rightarrow a_k$

Ex: $x(t)$ is the square wave: $\quad a_0 = \frac{1}{0.04}\int_0^{\cancel{0.04}\ 0.02} \cancel{e^{j0}}\ 1\, dt = \frac{1}{2}$



Q: Find a_k : FS coef?

$T_0 = 0.04$ s

$f_0 = 25$ Hz

$\int_{-T_0/2}^{T_0/2} \ldots = \int_{T_0}^{2T_0} \ldots$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi k f_0 t}\, dt$$

$$x(t) = \begin{cases} 1, & 0 \le t \le 0.02 \\ 0, & 0.02 \le t \le 0.04 \end{cases}$$

$$a_k = \frac{1}{0.04}\int_0^{\cancel{0.04}\ 0.02} 1 \cdot e^{-j2\pi k 25 t}\, dt = \ldots$$

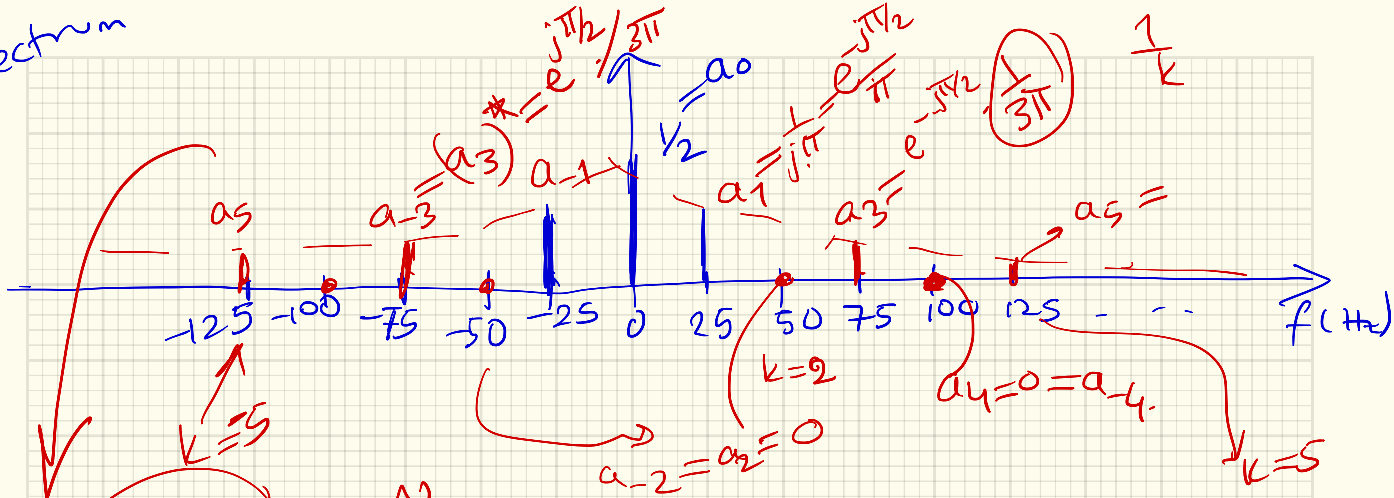
complete exercises

$$a_k = \frac{1}{j2\pi k}\left(1 - e^{-j\pi k}\right) \qquad \left(e^{-j\pi}\right)^k = (-1)^k = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd} \end{cases}$$

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{j\pi k}, & k \text{ odd} \\ 0, & k \text{ even}, \end{cases} \propto \frac{1}{k}$$

plot the spectrum

Spectrum



$$x_N(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

Approx. to $x(t)$

$N \nearrow$

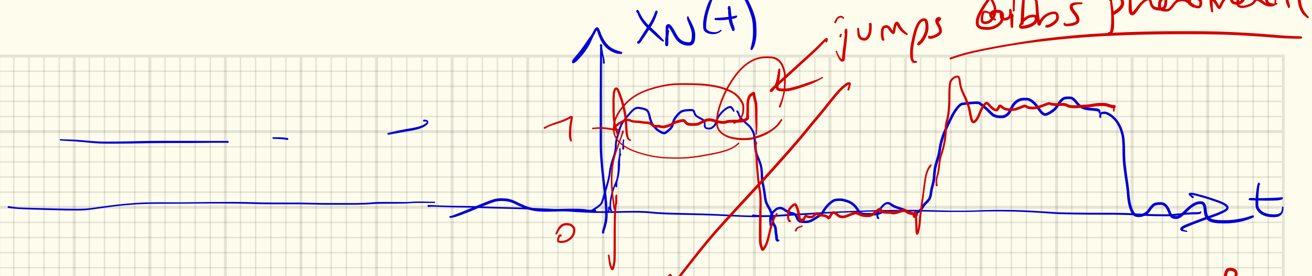
Error we make $\max e_N(t) = \max |x_N(t) - x(t)| \underset{N\to\infty}{\longrightarrow} 0$

for a continuous signal.

* For a discontinuous signal

$\max e_N(t) \underset{N\to\infty}{\not\longrightarrow} 0$

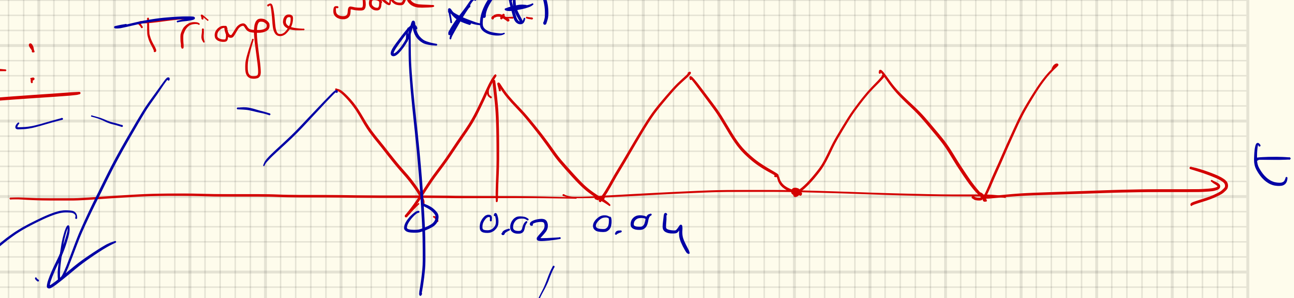
this does not hold.



Gibbs phenomenon

occurs at a discontinuity of a signal,

ex: Triangle wave



$$\max e_N(t) = |x(t) - x_N(t)| \xrightarrow[N \to \infty]{} 0 \quad \checkmark$$

F.S. coef $\quad a_k \propto \frac{1}{k^2}$

exercise $x(t) = \sin^3(3\pi t)$, find F.S. coefficients

plot the frequency spectrum

$f_0 = ?$

READING ASSIGNMENT

DO THIS !!!!

Read Chapter 3 from your textbook