QUESTION 1 The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

[**12pt**] **a)** Determine the convergence of the sequence $\{\mathbf{a_n}\}$ defined by

$$a_n = n \sin\left(\frac{1}{n}\right) \ln\left(\frac{n+2}{n}\right)^n \ (n \geq 1)$$

[**13pt**] **b)** Determine the convergence of the sequence $\{\mathbf{a_n}\}$ given by the recursion formula

$$\mathbf{a_1 = 2\ ,\ a_{n+1} = \frac{1+n}{1+2n}\, a_n\ (n \geq 1).}$$

by using the Monotonic Sequence Theorem.

SOLUTIONS:

a) $\lim\limits_{n\to\infty} n\sin\left(\frac{1}{n}\right) \ln\left(\frac{n+2}{n}\right)^n = \lim\limits_{n\to\infty} \overbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}^{1} \cdot \ln\underbrace{\left(1+\frac{2}{n}\right)^n}_{e^2} = 1 \cdot \ln e^2 = 2$

$\Rightarrow \{a_n\}$ converges.

b) $a_1 = 2,\quad a_{n+1} = \frac{1+n}{1+2n} a_n \Rightarrow \frac{a_{n+1}}{a_n} = \frac{1+n}{1+2n} < 1$

$\{a_n\}$ is decreasing and $a_n > 0$

Since $\{a_n\} \downarrow$ and $a_n > 0$, $\{a_n\}$ converges.

QUESTION 2 The blanks below will be filled by students. (Except the score)

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[6pt] a) Determine whether $\sum \mathbf{a_n}$ converges or diverges if $\sum(\mathbf{1} + \mathbf{a_n})$ converges?

[6pt] b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - \frac{1}{4}} = ?$

[13pt] c) Find the interval and radius of convergence of the series $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{(x-1)^n}{\ln n}$. For what values of **x** does the series converge absolutely or conditionally?

SOLUTIONS

a) $\sum(1+a_n)$: Converges. Thus $\lim_{n\to\infty} (1+a_n) = 0$

$\Rightarrow \lim_{n\to\infty} a_n = -1$

In this case, $\sum a_n$ diverges.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-\frac{1}{4}}$ $\quad a_n = \frac{1}{(n-\frac{1}{2})(n+\frac{1}{2})} = \frac{A}{n-\frac{1}{2}} + \frac{B}{n+\frac{1}{2}} = \frac{1}{n-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}}$

$S_n = a_1 + a_2 + \cdots + a_n = \left(\frac{1}{1/2} - \cancel{\frac{1}{3/2}}\right) + \left(\cancel{\frac{1}{3/2}} - \cancel{\frac{1}{5/2}}\right) + \left(\cancel{\frac{1}{5/2}} - \cancel{\frac{1}{7/2}}\right) + \cdots + \left(\cancel{\frac{1}{n-1/2}} - \frac{1}{n+1/2}\right)$

$S_n = 2 - \frac{1}{n+\frac{1}{2}} \Rightarrow \lim_{n\to\infty} S_n = 2 = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-\frac{1}{4}}$

c) $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{(-1)^n (x-1)^n} \right| = \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} |x-1|$

$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} |x-1| = |x-1| < 1 \rightarrow -1 < x-1 < 1 \rightarrow 0 < x < 2$ $\quad \boxed{R = \frac{2-0}{2} = 1}$

$\underline{x=0} \Rightarrow \displaystyle\sum_{n=3}^{\infty} (-1)^{2n} \frac{1}{\ln n} = \sum_{n=3}^{\infty} \frac{1}{\ln n}$. Let $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\ln n}{1/n} = \lim_{n\to\infty} \frac{n}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{1}{1/n} = \infty$ and $\sum \frac{1}{n}$ div.

$\Rightarrow \sum \frac{1}{\ln n}$ div.

$\underline{x=2} \Rightarrow \displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{1}{\ln n} \Rightarrow \sum_{n=3}^{\infty} \left|(-1)^n \frac{1}{\ln n}\right| = \sum_{n=3}^{\infty} \frac{1}{\ln n}$ diverges.

Leibnitz's Test $\quad u_n = \frac{1}{\ln n}$ $\quad$ i) $u_n = \frac{1}{\ln n} > 0 \; (n \geq 3)$ $\quad$ ii) $n+1 > n$, $\ln(n+1) > \ln n \Rightarrow u_{n+1} = \frac{1}{\ln(n+1)} < \frac{1}{\ln n} = u_n$

iii) $\lim_{n\to\infty} u_n = 0$

This series conv. conditionally at $x=2$

The interval of con. absolutely: $0 < x < 2$

The interval of convergence: $0 < x \leq 2$

QUESTION 3 The blanks below will be filled by students. (Except the score)

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Let $\mathbf{f(x) = \frac{1}{1-x}}$.

[5pt] a) Find a general formula for $\mathbf{f^{(n)}(x)}$.

[5pt] b) Find the Maclaurin series of $\mathbf{f(x)}$. (Do not show that $\mathbf{R_n(x) \to 0}$).

[5pt] c) Find the Maclaurin series of $\frac{1}{1+x}$ by using the series obtained in (b).

[5pt] d) Find the Maclaurin series of $\ln(1+x)$ by integrating term by term the series obtained in (c).

[5pt] e) By using the series obtained in (d), $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{-1}{2}\right)^{n-1} = ?$

SOLUTIONS:

a) $f(x) = (1-x)^{-1}$, $f'(x) = +(1-x)^{-2}$, $f''(x) = 2(1-x)^{-3}$, $f'''(x) = 2.3(1-x)^{-4}$ ...

$f^{(n)}(x) = n!\,(1-x)^{-(n+1)}$

b) $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$, $f^{(k)}(0) = k! \Rightarrow \sum_{k=0}^{\infty} \frac{k!}{k!} x^k = \sum_{k=0}^{\infty} x^k$

c) We must take $-x$ instead of x in (b).

Then $\sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k$

d) The Maclaurin series of $\ln(1+x) = \int \sum_{k=0}^{\infty} (-1)^k x^k$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} + c$$

$x = 0 \Rightarrow \ln 1 = c \Rightarrow c = 0$

$$\Rightarrow \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}}$$

e) $\sum_{n=1}^{\infty} \frac{1}{n}\left(-\frac{1}{2}\right)^{n-1}$, $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ $\quad n+1 \to n$, $x = \frac{1}{2}$

$\underbrace{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \cdot \left(\frac{1}{2}\right)^n 2}_{x = 1/2}$ $\qquad \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n} = \ln(1+x)$

$\longrightarrow 2 \cdot \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n} = \ln\left(1+\frac{1}{2}\right) \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}\left(\frac{1}{2}\right)^n = \frac{1}{2}\ln\frac{3}{2}$

QUESTION 4 The blanks below will be filled by students. (Except the score)

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[**10pt**] **a**) Find the length of the curve $\mathbf{r}(\mathbf{t}) = (\mathbf{e}^{\mathbf{t}}\cos\mathbf{t})\mathbf{i} + (\mathbf{e}^{\mathbf{t}}\sin\mathbf{t})\mathbf{j} + \mathbf{e}^{\mathbf{t}}\mathbf{k},\ -\mathbf{ln4} \le \mathbf{t} \le \mathbf{0}$.

[**15pt**] **b**) Find the point of intersection of the lines $\mathbf{x = 2s+2}$, $\mathbf{y = s+3}$, $\mathbf{z = 5s+6}$ and $\mathbf{x = t}$, $\mathbf{y = -t+2}$, $\mathbf{z = t+1}$. Then find the plane determined by these lines.

SOLUTIONS :

a) $\frac{d\vec r}{dt} = (e^t\cos t - e^t\sin t)\vec i + (e^t\sin t + e^t\cos t)\vec j + e^t\vec k = \vec v$

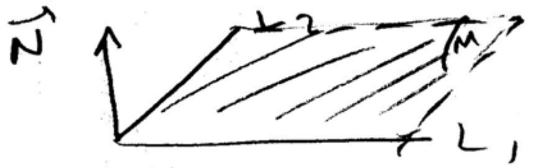
$$L = \int_{-\ln 4}^{0} |\vec v(t)|\,dt = \int_{-\ln 4}^{0} \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + e^{2t}}\,dt$$

$$= \int_{-\ln 4}^{0} \sqrt{e^{2t}(\cos^2 t - \cancel{2\cos t\sin t} + \sin^2 t + \sin^2 t + \cos^2 t + \cancel{2\sin t\cos t} + 1)}\,dt$$

$$= \sqrt3 \int_{-\ln 4}^{0} e^t\,dt = \sqrt3\, e^t \Big|_{-\ln 4}^{0} = \sqrt3\,(e^0 - e^{-\ln 4}) = \frac{3\sqrt3}{4}$$

b) $x = 2s+2$, $y = s+3$, $z = 5s+6$, $\vec v_1(2,1,5)$

$x = t$, $y = -t+2$, $z = t+1$, $\vec v_2(1,-1,1)$



$x = 2s+2 = t$
$y = s+3 = -t+2$
$z = 5s+6 = t+1$

$2s+2 = 5s+6$
$s = -1$

$2s+2 = t \Rightarrow t=0$

The point of intersection is
$x = 0,\ y = 2,\ z = 1$

The plane determined by these lines:

$$\vec N = \vec v_1 \times \vec v_2 = \begin{vmatrix} \vec i & \vec j & \vec k \\ 2 & 1 & 5 \\ 1 & -1 & 1 \end{vmatrix} = 6\vec i + 3\vec j - 3\vec k$$

$$6(x-0) + 3(y-2) - 3(z-1) = 0$$

$$\boxed{2x + y - z = 1}$$