BLG456E
Robotics
2D spatial transforms

Lecture Contents:

Reference frames.
Transforming velocities.
Composing transforms.
Representing rotations.

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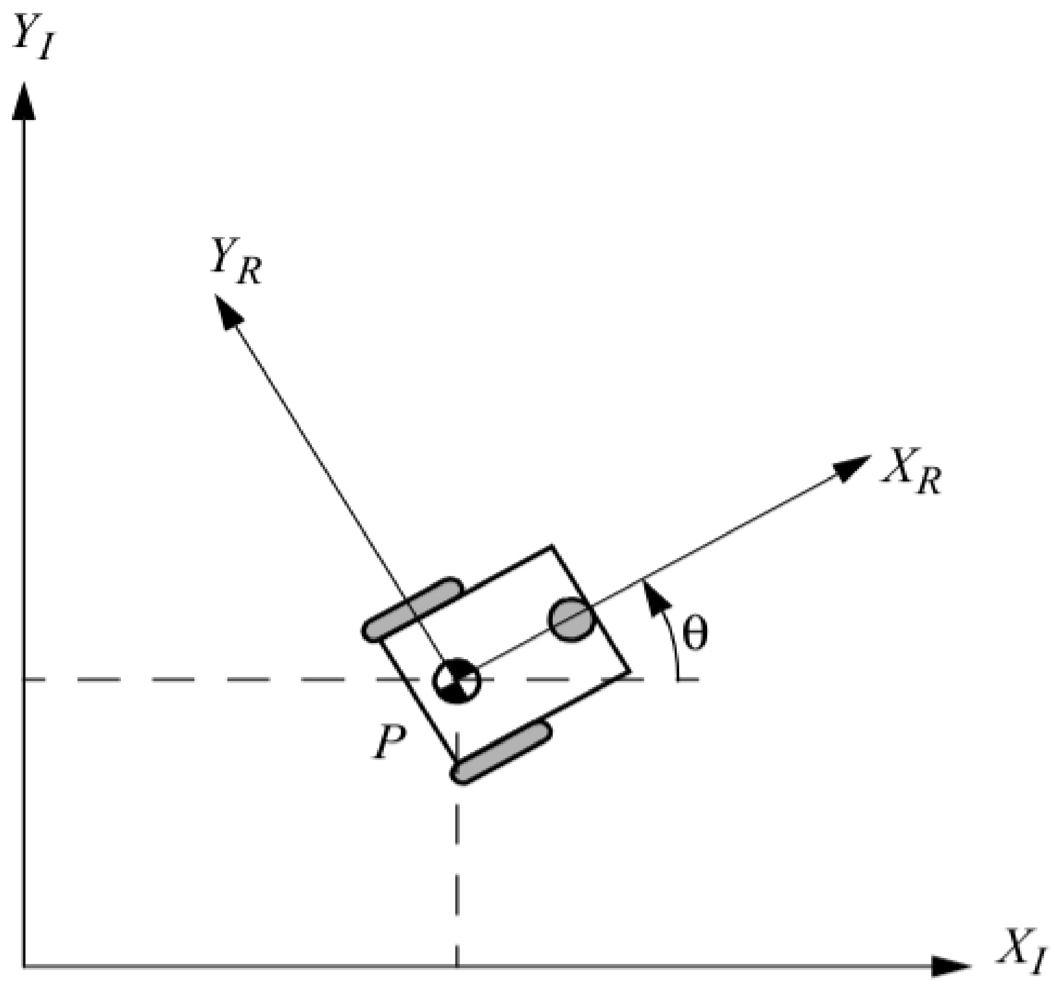
Reminder: reference frames



World reference frame:

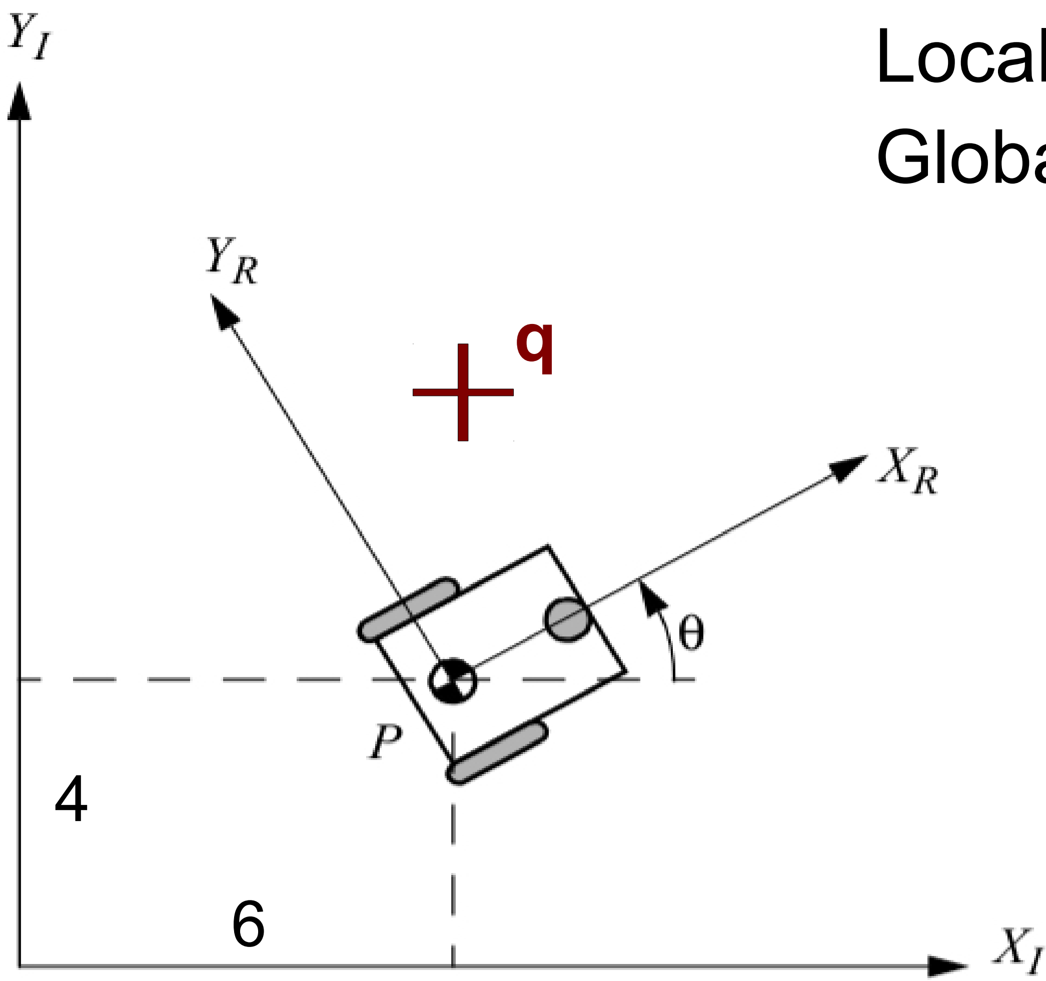
$x_I \quad y_I \quad \theta_I$

Robot reference frame:

$x_R \quad y_R \quad \theta_R$

Question: What is the relationship between θ_I and θ_R?

Points within reference frames



Local (to Robot): X_R, Y_R
Global (to the world): X_I, Y_I

Point q has two different addresses, $\mathbf{q}_I$ and $\mathbf{q}_R$.

$\mathbf{q}_R = [x_R, y_R] = [1,3]$

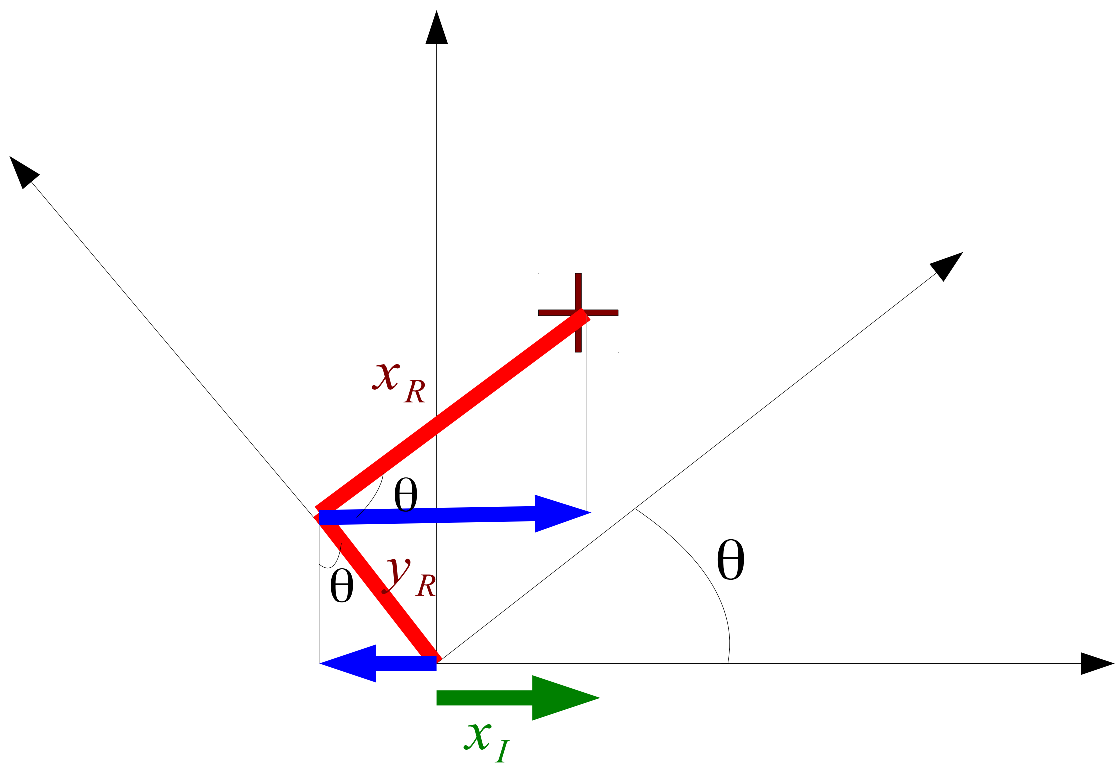
$\mathbf{q}_I = [x_I, y_I] = [5.3, 7.1]$

$(\theta = \pi/6, P = [6,4])$

From Siegwart & Nourbaksh.

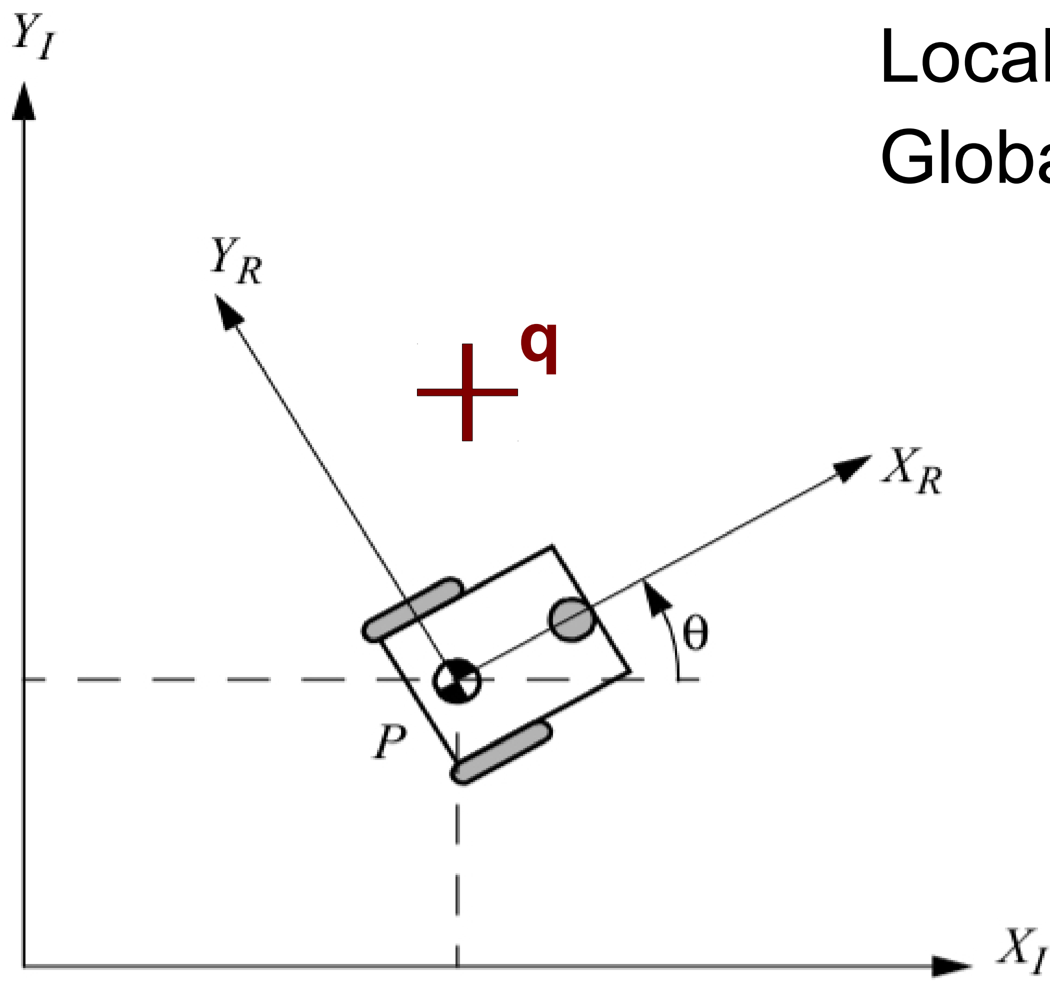
Calculate the X coordinate after rotation

$$x_I = \cos(\theta)\, x_R - \sin(\theta)\, x_R$$



Take-home exercises: Derive y_I in terms of x_R, y_R.
Derive y_R in terms of x_I, y_I.
Derive x_R in terms of x_I, y_I.

Transforming points between reference frames



Local (to Robot): X_R, Y_R

Global (to the world): X_I, Y_I

$$\mathbf{q}_I = r^{RI}(\theta)\mathbf{q}_R + \mathbf{P}$$

$$\begin{bmatrix} X_I \\ Y_I \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_R \\ Y_R \end{bmatrix} + \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

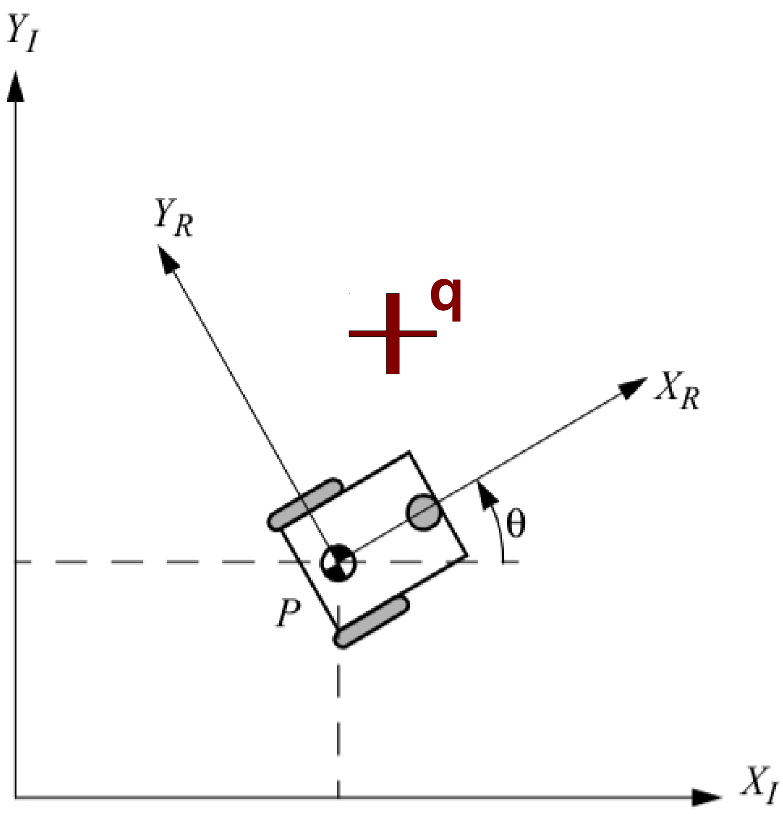
$$r^{IR}(\theta) = r^{RI}(\theta)^{-1} = r^{RI}(-\theta)$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

From Siegwart & Nourbaksh.

Transform point from local to global frame

$$\cos\frac{\pi}{6}=\sqrt{\frac{3}{4}}$$
$$\sin\frac{\pi}{6}=\frac{1}{2}$$



$\theta = \pi/6$

$P = [6,4]$

Let $\mathbf{q}_R = [1,3]$

(1 along X_R and 3 along Y_R)

Calculate $\mathbf{q}_I$!

(how far long along X_I and along Y_I)

$$\mathbf{q}_I = r^{RI}(\theta)\mathbf{q}_R + \mathbf{P}$$

$$r^{RI}(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

Translation, rotation: order important

$$r^{RI}(\theta)\boldsymbol{q_R}+\boldsymbol{P}$$

$$\neq$$

$$r^{RI}(\theta)(\boldsymbol{q_R}+\boldsymbol{P})$$

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}X_R\\ Y_R\end{bmatrix}+\begin{bmatrix}P_x\\ P_y\end{bmatrix}$$

$$=$$

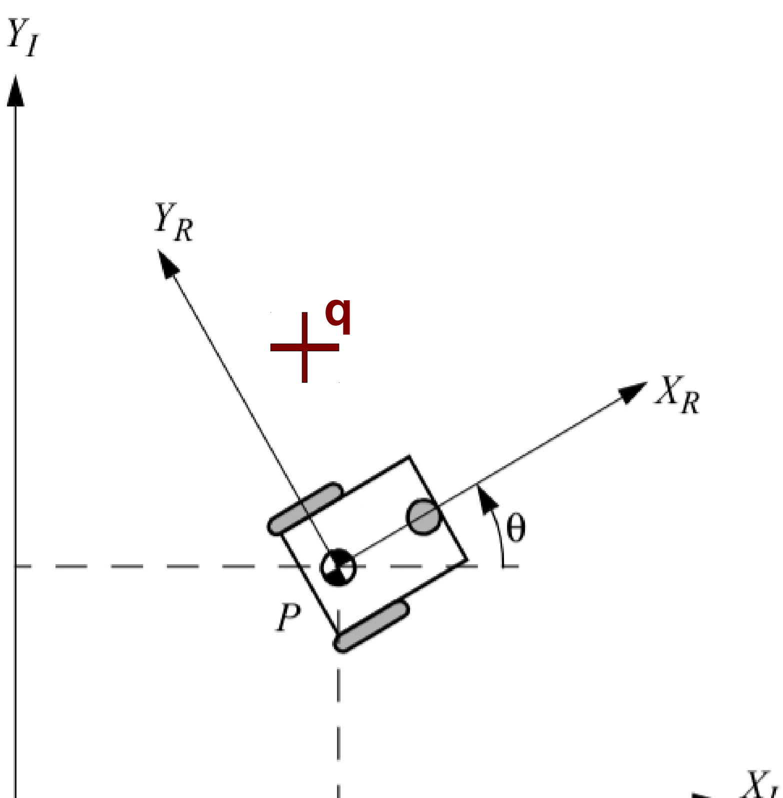
$$\begin{bmatrix}X_R\cos\theta-Y_R\sin\theta+P_x\\ X_R\sin\theta+X_R\cos\theta+P_y\end{bmatrix}$$

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}X_R+P_x\\ X_R+P_y\end{bmatrix}$$

$$=$$

???

Transform point from global to local frame



$\theta = \pi/6$

$P = [6,4]$

Let $\mathbf{q}_I = [5.3, 7.1]$

(1 along X_I and 3 along Y_I)

Calculate $\mathbf{q}_R$!

(how far along X_R and along Y_R)

$$\mathbf{q}_R = r^{IR}(\theta)(\mathbf{q}_I - \mathbf{P})$$

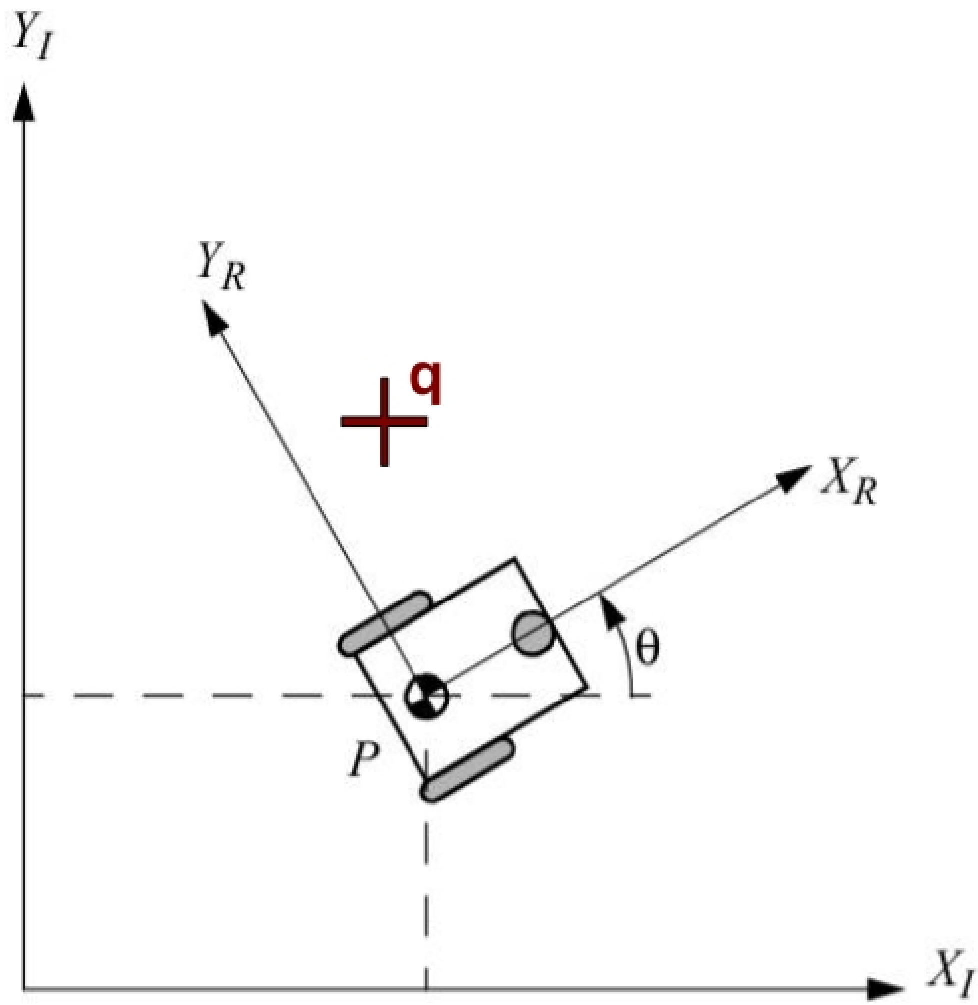
$$r^{RI}(\theta)^{-1} = r^{IR}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Coordinate transforms vs. object transforms

- Transforming a rigid object is mathematically equivalent to transforming the object's coordinate frame.
- The transform of the object is the inverse of the transform of the motion frame.

Review: transform a point

$$\cos\frac{\pi}{2}=0$$
$$\sin\frac{\pi}{2}=1$$



$$q_I = \begin{bmatrix} x_I \\ y_I \end{bmatrix} = R_{RI}(\theta)\, q_R + P$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_R \\ y_R \end{bmatrix} + \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

Exercise 1:
Rotate the point (3,3) by π/2.

Exercise 2:
Rotate the point (3,3) by π/2 and translate it by [2,2].

Coordinate frame transformations can apply to velocities too.

$$\boldsymbol{q}_I = r^{RI}(\theta)\,\boldsymbol{q}_R + P$$

$$\begin{bmatrix} x_I \\ y_I \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_R \\ y_R \end{bmatrix} + \begin{bmatrix} P_X \\ P_Y \end{bmatrix}$$

$$\dot{x} = \frac{dx}{dt}$$

$$\dot{y} = \frac{dy}{dt}$$

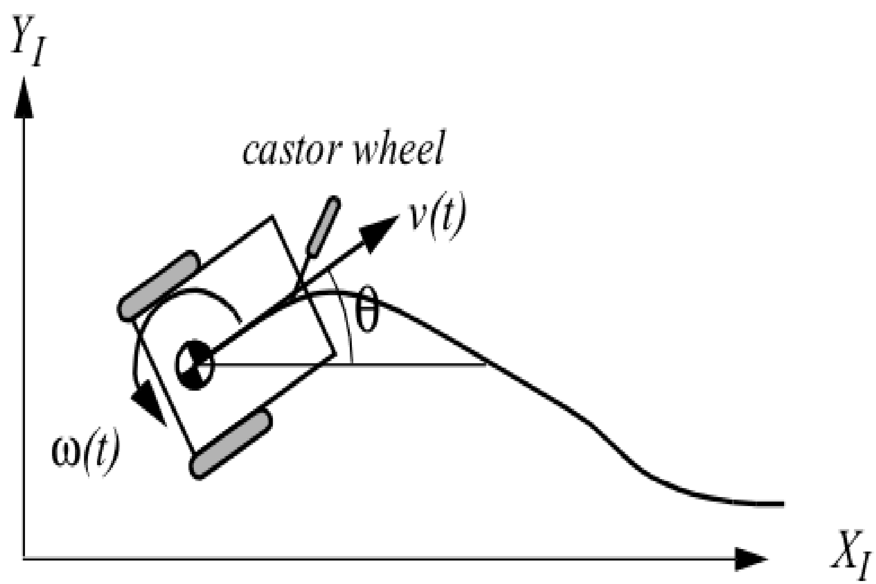
$$\dot{\boldsymbol{q}}_I = r^{RI}(\theta)\,\dot{\boldsymbol{q}}_R$$

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \end{bmatrix}$$

$$\dot{\boldsymbol{q}} = \frac{d\,\boldsymbol{q}}{dt}$$

Motion of a rigid body in two dimensions is captured by its linear and angular velocities.

(previously we saw velocities of points)



Transforming velocities between frames (2D):

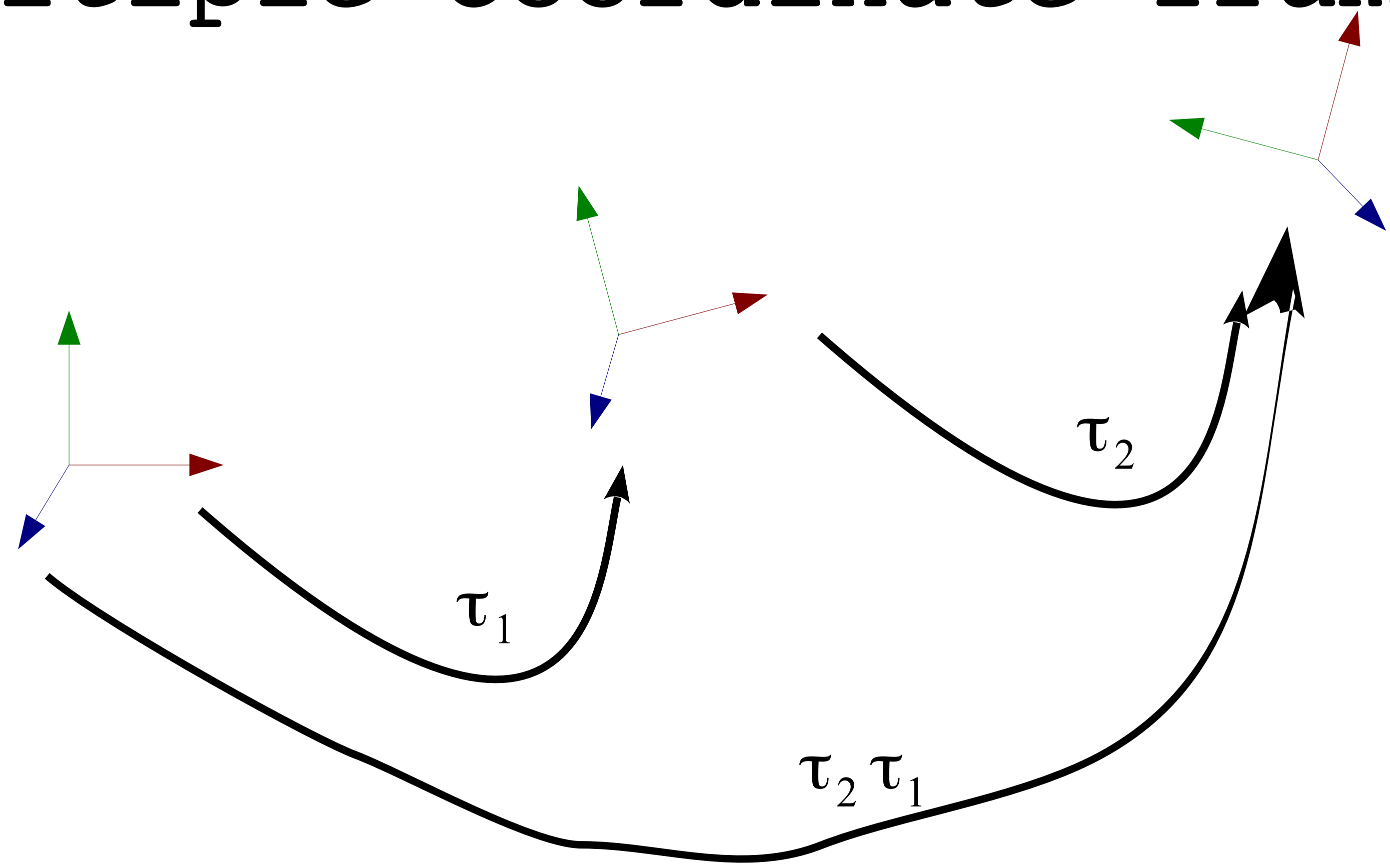
$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \end{bmatrix}$$

$$\omega_I = \omega_R$$

See Siegwart & Nourbaksh.

$$\left(\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \end{bmatrix} = v(t) \right)$$

Transformations between multiple coordinate frames



Composition of transformations can be represented by composition of transformation matrices.

This is easiest using homogeneous coordinates.

Introduction to homogeneous coordinates

Non-homogenous

$$q_{inh} = \begin{bmatrix} x_{inh} \\ y_{inh} \end{bmatrix} \xrightarrow{} q_{hom} = \begin{bmatrix} x_{inh} z_{hom} \\ y_{inh} z_{hom} \\ z_{hom} \end{bmatrix}$$

Homogeneous

choose a z_{hom}

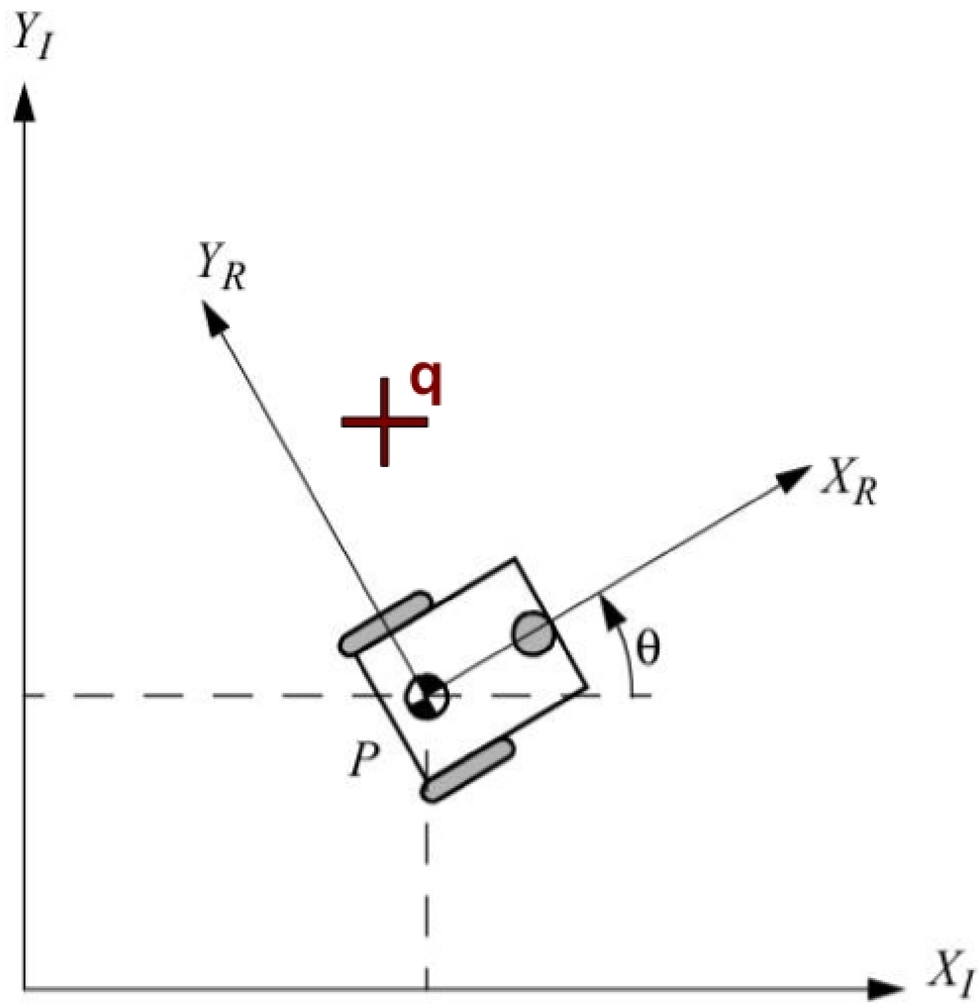
e.g. $z_{hom} = 1$

Represent 2D entities with 3 numbers!
Why?
Makes many calculations easier!
Can make the geometry easier too!

What to learn more? look up "projective geometry".

<u>**Exercise:**</u> (3 minutes) Convert the following vectors into homogeneous coordinates: $[5,4]^T$, $[0,0]^T$.

Transform a point with homogeneous coordinates



Homogeneous coordinates make writing transforms easier.

$$q_I = \begin{bmatrix} x_I \\ y_I \\ 1 \end{bmatrix} = T_{RI}(\theta, P)\, q_R$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & P_1 \\ \sin\theta & \cos\theta & P_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_R \\ y_R \\ 1 \end{bmatrix}$$

rotation matrix + translation vector
↓
translation matrix.

Converting from homogeneous coordinates

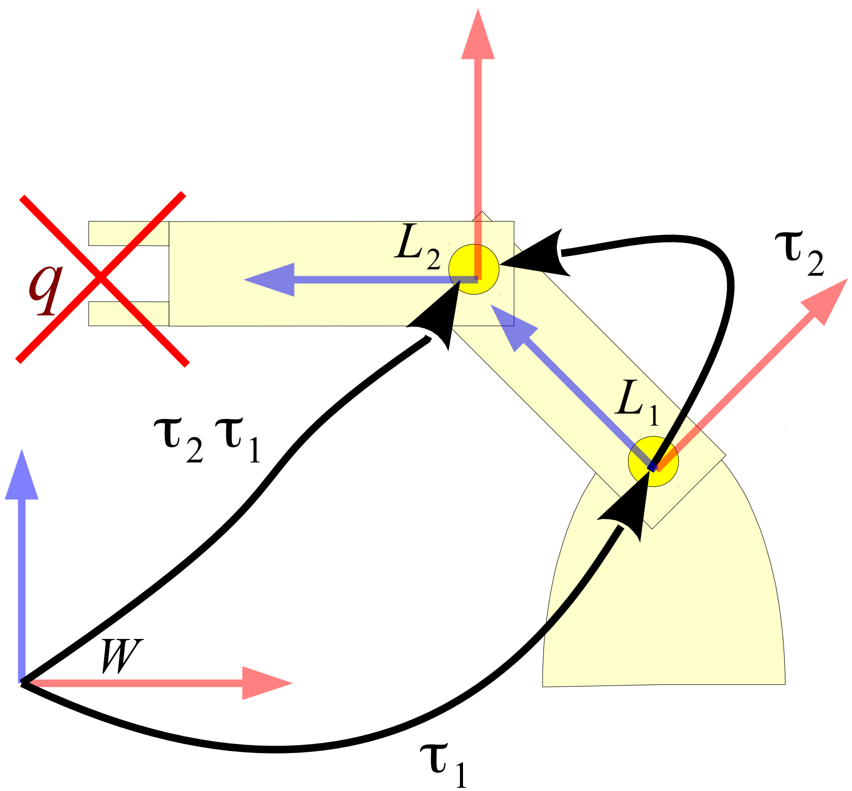
Homogenous **Non-homogeneous**

$$q_{hom} = \begin{bmatrix} x_{hom} \\ y_{hom} \\ z_{hom} \end{bmatrix} \xrightarrow{\text{divide by } z_{hom}} q_{inh} = \begin{bmatrix} x_{hom} / z_{hom} \\ y_{hom} / z_{hom} \end{bmatrix}$$

Exercise: (3 minutes) Convert the following homogeneous vectors into non-homogeneous coordinates: $[0,3,6]^T$, $[1,2,0]^T$.

$[1,2,0]^T$ is called a "point at infinity".
can only be represented in homogeneous coordinates.

Forward kinematics in a multiple link arm



$$\theta_1 = \pi/4 \quad P_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \; : \; \tau_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & P_{1x} \\ \sin\theta_1 & \cos\theta_1 & P_{1y} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta_2 = \pi/4 \quad P_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \; : \; \tau_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & P_{2x} \\ \sin\theta_2 & \cos\theta_2 & P_{2y} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tau_3 = \tau_2 \tau_1$$

Exercise: (on blackboard) Compose the matrices for the two transformations. [note: cos(π/4)=1/sqrt(2) , sin(π/3)=1/sqrt(2)]

Exercise: (3 minutes) If q_{L2} = (0,4) find q_W.

2D rotation representation

- Rotation angle.

$$\theta$$

- Rotation matrix.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3D rotation representation (1)

- Angle-axis:
  - Angle of rotation + axis of rotation.

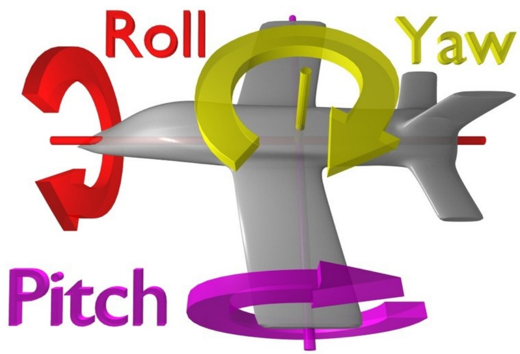
$$\left(\frac{\pi}{2},0,1,0\right)$$

- Rotation vector:
  - Length represents amount and direction represents axis (angle*axis).

$$\left(0,\frac{\pi}{2},0\right)$$

  - Used in Twist.

3D rotation representation (II)



- Euler angles (*intrinsic rotations*):
  - Rotate around axes moving with body:
    - Normally roll, pitch, yaw (X,Y Z axes).

      (Other directions possible.)

$$\left(0, \frac{\pi}{2}, 0\right)$$

- Fixed frame (*extrinsic rotations*).
  - Rotate around axes fixed in world frame.

$$\left(0, \frac{\pi}{2}, 0\right)$$

Exercise: Give a series of rotations in moving body frame that are different in fixed world frame

3D rotation representation (III)

- Rotation matrix.
  - The 3x3 matrix that, applied to a point, will rotate it.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

- Unit quaternion.
  - 4 numbers constrained to length 1, with special properties.

$$\left(0,\sqrt{\frac{1}{2}},0,\sqrt{\frac{1}{2}}\right)$$

Strengths of rotation representations

Type	Good	Bad
Euler angles.	Intuitive definition. Only 3 numbers.	Singularities. Order important. Composition not straightforward.
Fixed axis.	See Euler angles	
Angle-axis.	Intuitive definition. No order problem.	Composition difficult.
Rotation vector.	Intuitive definition. No order problem. Only 3 numbers.	Composition difficult.
Rotation matrix.	Composition easy. Directly applicable to rotate point.	9 numbers. Extra constraints.
Quaternion	Composition easy. Directly applicable to rotate point. No trigonometry	Not intuitive to non-mathematicians.

Rotation is a linear transform

Rotation is a linear transform.

$$R_\theta(x+y)=R_\theta(x)+R_\theta(y)$$
$$aR_\theta(x)=R_\theta(ax)$$

Rotation is orthogonal and orthonormal.

$$\|R_\theta(x)-R_\theta(y)\|=\|x-y\|,\ R_\theta(x)\cdot R_\theta(y)=x\cdot y$$

→ Rotation is invertible.

$$R_\theta^{-1}(x)=y \quad s.t. \quad y=R_\theta(x)$$

→ Rotation can be expressed by a matrix.

$$R_\theta\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=R_\theta\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Introduction to physical Degrees Of Freedom: poses and velocities

Dim	Quantity	DOF
2	Linear Displacement	2
2	Linear Velocity	2
2	Orientation	1
2	Rotational velocity	1
2	Pose	3
2	Rigid motion	3
2	Non-rigid pose/motion	?

Dim	Quantity	DOF
3	Linear Displacement	3
3	Linear Velocity	3
3	Orientation	3
3	Rotational velocity	3
3	Pose	6
3	Rigid motion	6
3	Non-rigid pose/motion	?

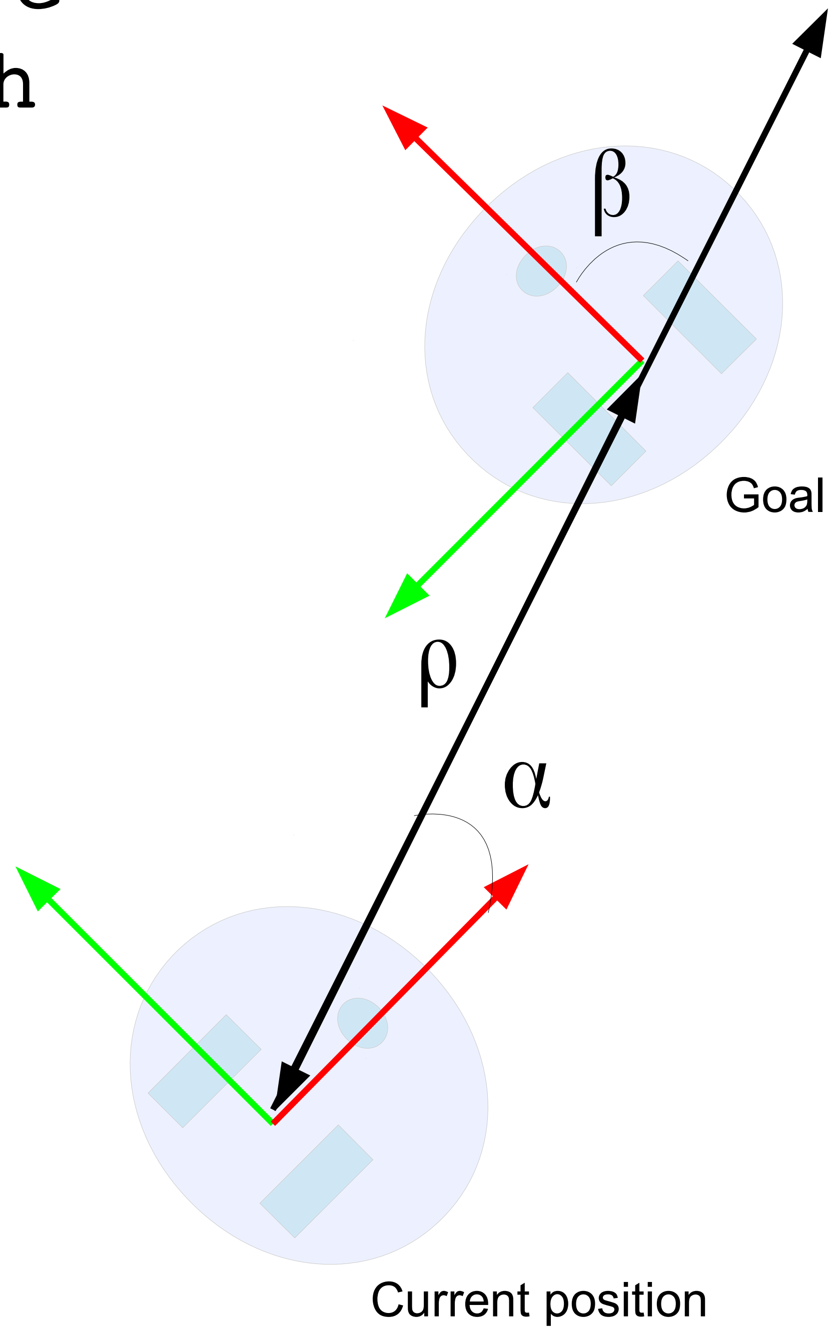
Constraints can reduce the degrees of freedom
(e.g. 3D body constrained to lie in plane, limited DOF robot arm).

Differential-drive pose control with goal-centred coordinates

A simple control law:

$$\dot{x}_r = k_\rho \rho$$

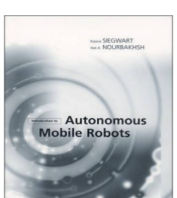
$$\dot{\theta}_r = k_\alpha \alpha + k_\beta \beta$$



<u>Question</u>: How to calculate

ρ, α, β?

Need $\Delta x, \Delta y, \Delta \theta$



See Siegwart & Nourbakhsh section 3.6 for more details.

$$\cos\frac{\pi}{2}=0$$
$$\sin\frac{\pi}{2}=1$$

Class exercises

Exercise (3 minutes):

Rotate the point (2,2) by π/2 and translate it by [1,5].

$$q_1=\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \quad q_2=\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

$$q_2=\begin{bmatrix} \cos\theta & -\sin\theta & P_1 \\ \sin\theta & \cos\theta & P_2 \\ 0 & 0 & 1 \end{bmatrix} q_1$$