Discrete Math I – Practice Problems for Final Exam

The upcoming final on Tuesday, February 28 (2:45-4:45 pm in BRN-1110) will cover the material in sections on both midterms and sections 4.3, 4.4, 4.6, 5.1, 5.2, portions of 5.3, 6.1, 6.2, and 6.3 of your textbook. These practice problems almost exclusively focus on material since the last exam. For a review of the previous material, look at the previous practice exams.

Note that this practice exam is NOT "synchronized" with what you will see on exam day. I won't purposely present problems here and just give you the same problem with the numbers changed. That is, the following problems do not represent all of the possible types of problems that could appear on the exam. Problems chosen for the exam will be similar to homework problems, the quiz, examples done in class, and the basic problems from the workshops. Also note that the number of problems presented in this practice exam may not represent the actual length of the exam you see on the exam day.

IMPORTANT! First try these problems as if it were the real exam; work by yourself without the text or your notes. This is supposed to be a gauge on what you need to work on to prepare for the exam. Answering these problems as you might handle homework problems won't necessarily give you much of a clue on what you need to work on.

Instructions: Provide all steps necessary to solve the problem. Unless otherwise stated, your answer should be exact and reasonably simplified. Additionally, clearly indicate the value or expression that is your final answer! Calculators are not allowed.

1. Prove that if n^2 is even, then $n+3$ is odd. What kind of proof did you use?

2. Write the first 10 terms of the sequence that lists the odd positive integers in increasing order, listing each odd integer twice.

3. Find ALL pairs of integers m and n that satisfy the following conditions:

$$\gcd(m,n) = 12,\ \text{lcm}(m,n) = 360, \text{ and } 12 < m < n < 360.$$

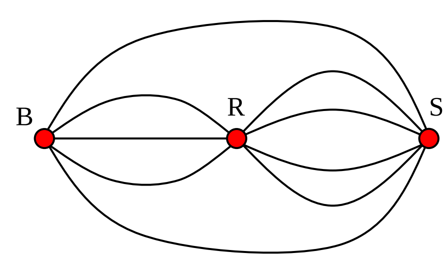
4. Find gcd(222,921). Determine $s, t \in \mathbf{Z}$ where $\gcd(222,921) = 222\,s + 921\,t$.

5. Encode the message "stop at noon" using the function $f(x) = (x + 8)$ **mod** 26.

6. Prove that $3 \mid f_{4n}$ for all $n \in \mathbf{Z}$, where f_m is the m-th Fibonacci number.

7. <u>Prove</u> that if A is a set of 20 distinct numbers chosen from the first 38 positive integers (that is, 1, 2, 3, and so on, up to 38), then there exists two numbers in A that are consecutive.

8. Three small towns, designated **B**, **R**, and **S**, are interconnected by a system of roads as given to the right.



(***a***) How many different ways can a person travel along the roads from **B** to **S** passing through each town at most once?

(***b***) How many different round trips can a person travel from **B** to **S** and back to **B**? As in the previous problem, throughout the round trip, the person can pass though **R** and **S** at most once.

9. A test contains 100 true/false questions. How many different ways can a student answer the questions on the test, if answers may be left blank?

10. Each locker in an airport is labeled with an uppercase letter followed by three digits. How many different labels for lockers are there?

11. Thirteen people on a softball team show up for a game.

(*a*) How many ways are there to choose 10 players to take the field?

(*b*) How many ways are there to assign to 10 positions by selecting players from the 13 people who show up?

(*c*) Of the 13 people who show up, 3 are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

12. Recall that a ***bit string*** is an ordered list of characters using only the digits 0 and 1.

(*a*) How many bit strings of length ten are there?

(*b*) How many bit strings of length ten have exactly three 1s?

(*c*) How many bit strings of length ten have exactly three 1s and none of these 1s are adjacent to each other?

Discrete Math I – Answers to Practice Problems for Final Exam

1. Prove the contrapositive: If $n + 3$ is even, then n^2 is odd.

Assume: $n + 3$ is even
Prove: n^2 is odd.

If n+3 is even, then there is an integer k such that $n + 3 = 2k$ which implies that $n = 2k - 3$.

So, $n^2 = (2k - 3)^2 = 4k^2 - 12k + 9 = 4k^2 - 12k + 8 + 1 = 2(2k^2 - 6k + 4) + 1$. Since $2k^2 - 6k + 4$ is an integer, then n^2 is of the form of an odd number.

Therefore, if $n + 3$ is even, then n^2 is odd.

2. $(1,1,3,3,5,5,7,7,9,9,\cdots\cdots)$.

3. There are three distinct solutions. Use $\gcd(m,n) = 2^2 \cdot 3^1 \cdot 5^0$ and $\operatorname{lcm}(m,n) = 2^3 \cdot 3^2 \cdot 5^1$.

Solution #1: $m = 2^2 \cdot 3^1 \cdot 5^1 = 60$ and $n = 2^3 \cdot 3^1 \cdot 5^0 = 72$.
Solution #2: $m = 2^2 \cdot 3^2 \cdot 5^0 = 36$ and $n = 2^3 \cdot 3^1 \cdot 5^1 = 120$.
Solution #3: $m = 2^3 \cdot 3^1 \cdot 5^0 = 24$ and $n = 2^2 \cdot 3^2 \cdot 5^1 = 180$.

4.

i	$\gcd(a_i,a_{i+1})$	a_i	q_i	s_i	t_i
0	gcd(921,222)	921	-	1	0
1	gcd(222,33)	222	4	0	1
2	gcd(33,24)	33	6	1	-4
3	gcd(24,9)	24	1	-6	25
4	gcd(9,6)	9	2	7	-29
5	gcd(6,3)	6	1	-20	83
6	gcd(3,0)	3	2	27	-112

Thus, gcd(921,222) = 3, s = 27, and t = -112.

5. No answer yet.

6. Use weak induction.
(*Basis Step*) For n=0, is it true that $3 \mid f_0 \Rightarrow 3 \mid 0$? Yes.
(*Inductive Step*) Assume that $3 \mid f_{4k}$ and prove that $3 \mid f_{4(k+1)}$. Repeatively applying the definition of the Fibonacci sequence to $f_{4(k+1)}$ yields

$$\begin{aligned} f_{4(k+1)} &= f_{4k+4} \\ &= f_{4k+3} + f_{4k+2} \\ &= (f_{4k+2} + f_{4k+1}) + f_{4k+2} = 2f_{4k+2} + f_{4k+1} \\ &= 2(f_{4k+1} + f_{4k}) + f_{4k+1} = 3f_{4k+1} + 2f_{4k}. \end{aligned}$$

Since $3 \mid 3$ and 3 divides any multiple of 3, then $3 \mid 3f_{4k+1}$. The inductive hypothesis is $3 \mid f_{4k}$ and so $3 \mid 2f_{4k}$. Thus, $3 \mid (3f_{4k+1} + 2f_{4k})$, implying that $3 \mid f_{4(k+1)}$.

7. Note that this problem asks to prove the statement is true no matter how the numbers were chose. If you pick the “worst-case scenario”, then you would be demonstrating that the statement is true for one case, but you still have lots of others you would still need to check.

Proof:

Use the Pigeonhole Principle.
Label 19 pigeonholes with {1,2}, {3,4}, {5,6}, …, {37,38}.
Place a chosen number into its corresponding pigeonhole.
There are 20 numbers and 19 pigeonholes, so, by the Pigeonhole Principle, there must be one pigeonhole with two numbers in it.
Those two numbers must be consecutive.

8. If you didn't guess already, these three towns are Buffalo, Rochester, and Syracuse.

(***a***) The person can take any of the $3 \cdot 4 = 12$ paths that pass through R or one of the 2 paths that do not pass through R. Thus, there are $12 + 2 = 14$ possibilities.

(***b***) Note that we may only pass through R not at all or just once.

Option #1: The person can start by taking any of the 12 paths that pass through R, but must take one of the two paths going back to B that do not pass through R.

Option #2: The person can take either of the two paths that do not pass through R and can take any of the 14 routs to return to B.

Thus, there are $12 \cdot 2 + 2 \cdot 14 = 52$ possibilities.

9.

The student has 3 choices for each question: true, false, and no answer. There are 100 questions, so by the product rule there are $3^{100} \approx 5.2 \times 10^{47}$ ways to answer the test.

10. By the product rule for counting there are $26 \cdot 101010 = 26000$ different labels for lockers.

11.

(***a***) $C(13,10) = 286$

(***b***) $P(13,10) = 1{,}037{,}836{,}800$

(***c***)

Solution 1: The long way to answer the problem is to count how many ways exactly 1 woman and 9 men could be on the team (which is $C(3,1)\ C(10,9) = 30$), how many ways exactly 2 women and 8 men could be on the team (which is $C(3,2)\ C(10,8) = 135$), and to count how many ways exactly 3 women and 7 men could be on the team (which is $C(3,3)\ C(10,7) = 120$); then add them all together. The answer is 285.

Solution 2: The short way to answer the problem is to use Union Rule (c) on pg 266 of your text. The total number of ways to choose any 10 to be on the team is 286 (from 2(a) above). The total number of ways to choose a team with 0 women and 10 men is $C(3,0)\ C(10,10) = 1$. The difference represents the number of ways the team can be chosen so that there is at least one woman on the team. The answer is $286 – 1 = 285$.

12.

(***a***) There are ten positions in the string where each position must be filled with one of two possibilities. Thus, there are $2^{10} = 1024$ ways.

(***b***) There are $C(10,3) = 120$ ways to designate places to choose the positions for the 1s in the string. Fill the remaining with 0s.

(***c***) Start with a string of seven 0s. There are eight places to put the 1s so that no two of the 1s are adjacent. We can put those three 1s into the eight places in $C(8,3) = 56$ different ways.