Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Lagrange Method
Interpolation
COMPLETE SOLUTION SET

1. Given n+1 data pairs, a unique polynomial of degree ______________ passes through the $n+1$ data points.
   (A) $n+1$
   (B) n
   (C) n or less
   (D) $n+1$ or less

Solution
The correct answer is (C).

A unique polynomial of degree n or less passes through $n+1$ data points. Assume two polynomials $P_n(x)$ and $Q_n(x)$ go through $n+1$ data points,

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

Then

$$R_n(x) = P_n(x) - Q_n(x)$$

Since $P_n(x)$ and $Q_n(x)$ pass through all the $n+1$ data points,

$$P_n(x_i) = Q_n(x_i), \; i = 0, \ldots, n$$

Hence

$$R_n(x_i) = P_n(x_i) - Q_n(x_i) = 0, \; i = 0, \ldots, n$$

The n^{th} order polynomial $R_n(x)$ has $n+1$ zeros. A polynomial of order n can have $n+1$ zeros only if it is identical to a zero polynomial, that is,

$$R_n(x) \equiv 0$$

Hence

$$P_n(x) \equiv Q_n(x)$$

How can one show that if a second order polynomial has three zeros, then it is zero everywhere?
If $R_2(x) = a_0 + a_1 x + a_2 x^2$, then if it has three zeros at x_1, x_2, and x_3, then

$$R_2(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 = 0$$

$$R_2(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 = 0$$

$$R_2(x_3) = a_0 + a_1 x_3 + a_2 x_3^2 = 0$$

Which in matrix form gives

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The final solution $a_1 = a_2 = a_3 = 0$ exists if the coefficient matrix is invertible. The determinant of the coefficient matrix can be found symbolically with the forward elimination steps of naïve Gauss elimination to give

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = x_2 x_3^2 - x_2^2 x_3 - x_1 x_3^2 + x_1^2 x_3 + x_1 x_2^2 - x_1^2 x_2$$

$$= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$$

Since

$$x_1 \neq x_2 \neq x_3$$

the determinant is non-zero. Hence, the coefficient matrix is invertible. Therefore, $a_1 = a_2 = a_3 = 0$ is the only solution, that is, $R_2(x) \equiv 0$.

2. Given the two points $[a, f(a)], [b, f(b)]$, the linear Lagrange polynomial $f_1(x)$ that passes through these two points is given by

(A) $f_1(x) = \dfrac{x-b}{a-b} f(a) + \dfrac{x-a}{a-b} f(b)$

(B) $f_1(x) = \dfrac{x}{b-a} f(a) + \dfrac{x}{b-a} f(b)$

(C) $f_1(x) = f(a) + \dfrac{f(b) - f(a)}{b-a}(b-a)$

(D) $f_1(x) = \dfrac{x-b}{a-b} f(a) + \dfrac{x-a}{b-a} f(b)$

Solution

The correct answer is (D).

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$\begin{aligned} f_1(x) &= \sum_{i=0}^{1} L_i(x) f(x_i) \\ &= L_0(x) f(x_0) + L_1(x) f(x_1) \\ &= L_0(x) f(a) + L_1(x) f(b) \end{aligned}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$\begin{aligned} L_0(x) &= \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{x - x_j}{x_0 - x_j} \\ &= \frac{x - x_1}{x_0 - x_1} \\ &= \frac{x-b}{a-b} \end{aligned}$$

$$\begin{aligned} L_1(x) &= \prod_{\substack{j=0 \\ j \neq 1}}^{1} \frac{x - x_j}{x_1 - x_j} \\ &= \frac{x - x_0}{x_1 - x_0} \\ &= \frac{x-a}{b-a} \end{aligned}$$

$$\begin{aligned} f_1(x) &= L_0(x) f(a) + L_1(x) f(b) \\ &= \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b) \end{aligned}$$

3. The Lagrange polynomial that passes through the 3 data points is given by

x	15	18	22
y	24	37	25

$$f_2(x) = L_0(x)(24) + L_1(x)(37) + L_2(x)(25)$$

The value of $L_1(x)$ at $x = 16$ is

(A) –0.071430
(B) 0.50000
(C) 0.57143
(D) 4.3333

Solution

The correct answer is (B).

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^{2} \frac{x - x_j}{x_1 - x_j}$$

$$= \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$= \left(\frac{16 - 15}{18 - 15}\right)\left(\frac{16 - 22}{18 - 22}\right)$$

$$= \left(\frac{1}{3}\right)\left(\frac{-6}{-4}\right)$$

$$= 0.50000$$

4. The following data of the velocity of a body is given as a function of time.

Time (s)	10	15	18	22	24
Velocity (m/s)	22	24	37	25	123

A quadratic Lagrange interpolant is found using three data points, $t = 15$, 18 and 22. From this information, at what of the times given in seconds is the velocity of the body 26 m/s during the time interval of $t = 15$ to $t = 22$ seconds.

(A) 20.173
(B) 21.858
(C) 21.667
(D) 22.020

Solution

The correct answer is (B).

$$v_n(t) = \sum_{i=0}^{n} L_i(t) v(t_i)$$

where

$$t_0 = 15, \ v(t_0) = 24$$

$$t_1 = 18, \ v(t_1) = 37$$

$$t_2 = 22, \ v(t_2) = 25$$

gives

$$L_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{t - t_j}{t_i - t_j}$$

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{2} \frac{t - t_j}{t_0 - t_j}$$

$$= \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{2} \frac{t - t_j}{t_1 - t_j}$$

$$= \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j \neq 2}}^{2} \frac{t - t_j}{t_2 - t_j}$$

$$= \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)$$

$$v_2(t) = \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)v(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)v(t_1) + \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)v(t_2)$$

$$= \left(\frac{t-18}{15-18}\right)\left(\frac{t-22}{15-22}\right)v(15) + \left(\frac{t-15}{18-15}\right)\left(\frac{t-22}{18-22}\right)v(18) + \left(\frac{t-15}{22-15}\right)\left(\frac{t-18}{22-18}\right)v(22)$$

$$26 = \left(\frac{t-18}{15-18}\right)\left(\frac{t-22}{15-22}\right)\times 24 + \left(\frac{t-15}{18-15}\right)\left(\frac{t-22}{18-22}\right)\times 37 + \left(\frac{t-15}{22-15}\right)\left(\frac{t-18}{22-18}\right)\times 25$$

$$26 = \left(\frac{t-18}{-3}\right)\left(\frac{t-22}{-7}\right)\times 24 + \left(\frac{t-15}{3}\right)\left(\frac{t-22}{-4}\right)\times 37 + \left(\frac{t-15}{7}\right)\left(\frac{t-18}{4}\right)\times 25$$

$$26 = \left(\frac{t^2-40t+396}{21}\right)\times 24 + \left(\frac{t^2-37t+330}{-12}\right)\times 37 + \left(\frac{t^2-33t+270}{28}\right)\times 25$$

$$26 = \left(1.1429t^2 - 45.714t + 452.57\right) + \left(-3.0833t^2 + 114.08t - 1017.5\right)$$
$$+ \left(0.89286t^2 - 29.464t + 241.07\right)$$

$$26 = -1.0476t^2 + 38.905t - 323.86$$

$$0 = -1.0476t^2 + 38.905t - 349.86$$

$$t = \frac{-38.905 \pm \sqrt{(38.905)^2 - 4\times -1.0476 \times -349.86}}{2\times -1.0476}$$

$$= 21.858 \quad \text{and} \quad 15.278$$

5. The path that a robot is following on a *x*-*y* plane is found by interpolating four data points as

x	2	4.5	5.5	7
y	7.5	7.5	6	5

$$y(x) = 0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000$$

The length of the path from $x = 2$ to $x = 7$ is

(A) $\sqrt{(7.5-7.5)^2 + (4.5-2)^2} + \sqrt{(6-7.5)^2 + (5.5-4.5)^2} + \sqrt{(5-6)^2 + (7-5.5)^2}$

(B) $\int_2^7 \sqrt{1+(0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000)^2}\,dx$

(C) $\int_2^7 \sqrt{1+(0.45714x^2 - 4.5142x + 9.6048)^2}\,dx$

(D) $\int_2^7 (0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000)dx$

Solution

The correct answer is (C).

The length *S* of the curve $y(x)$ from *a* to *b* is given by

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

where

$$a = 2$$

$$b = 7$$

giving

$$S = \int_2^7 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$y(x) = 0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000$$

$$\frac{dy}{dx} = 0.45714x^2 - 4.5142x + 9.6048$$

Thus,

$$S = \int_2^7 \sqrt{1+\left(0.45714x^2 - 4.5142x + 9.6048\right)^2}\,dx$$

6. The following data of the velocity of a body is given as a function of time.

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

If you were going to use quadratic interpolation to find the value of the velocity at $t = 14.9$ seconds, what three data points of time would you choose for interpolation?

(A) 0, 15, 18
(B) 15, 18, 22
(C) 0, 15, 22
(D) 0, 18, 24

Solution

The correct answer is (A).

We need to choose the three points closest to $t = 14.9\,\text{s}$ that also bracket $t = 14.9\,\text{s}$. Although the data points in choice (B) are closest to 14.9, they do not bracket it. This would be performing extrapolation, not interpolation. Choices (C) and (D) both bracket $t = 14.9\,\text{s}$ but they are not the closest three data points.

Time (s)	Velocity (m/s)	How far is $t = 14.9\,\text{s}$
0	22	$\lvert 14.9 - 0 \rvert = 14.9$
15	24	$\lvert 14.9 - 15 \rvert = 0.1$
18	37	$\lvert 14.9 - 18 \rvert = 3.1$
22	25	$\lvert 14.9 - 22 \rvert = 7.1$
24	123	$\lvert 14.9 - 24 \rvert = 9.1$