QUESTION 1 The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

[12p] **a)** Find the length of the curve $\vec{r}(t) = \frac{1}{2}(\sin t - t\cos t)\,\vec{i} + \frac{1}{2}(\cos t + t\sin t)\,\vec{j}$, $\quad 1 \le t \le 3$ and find its unit tangent vector for $t = \frac{\pi}{3}$.

[13p] **b)** Find the distance from the point $P(1,1,2)$ to the plane passing through the points $A(2,2,-1)$, $B(0,3,1)$ and $C(-1,-1,-2)$.

a) $L = \int_a^b |\vec{v}|\,dt$

$\vec{v} = \frac{1}{2}(\cos t - \cos t + t\sin t)\,\vec{i} + \frac{1}{2}(-\sin t + \sin t + t\cos t)\,\vec{j}$

$\vec{v} = \frac{t}{2}\sin t\,\vec{i} + \frac{t}{2}\cos t\,\vec{j}$

$|\vec{v}| = \sqrt{\left(\frac{t}{2}\sin t\right)^2 + \left(\frac{t}{2}\cos t\right)^2} = \frac{|t|}{2} = \frac{t}{2} \quad (1 \leq t \leq 3)$

$L = \int_1^3 \frac{t}{2}\,dt = \frac{1}{2}\,\frac{t^2}{2}\Big|_1^3 = 2$ //

$\vec{T} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{t/2}\left(\frac{t}{2}\sin t\,\vec{i} + \frac{t}{2}\cos t\,\vec{j}\right) = \sin t\,\vec{i} + \cos t\,\vec{j}$

$t = \frac{\pi}{3} \Rightarrow \vec{T} = \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$

b) $d = \frac{|\vec{PS}\cdot\vec{n}|}{|\vec{n}|}$

P: the point from which the distance is to be calculated

S: a point on the plane

$\vec{n}$: normal vector of the plane

$\vec{BA} = (2-0)\vec{i} + (2-3)\vec{j} + (-1-1)\vec{k} = 2\vec{i} - \vec{j} - 2\vec{k}$

$\vec{BC} = (0-(-1))\vec{i} + (3-(-1))\vec{j} + (1-(-2))\vec{k} = \vec{i} + 4\vec{j} + 3\vec{k}$

$\vec{n} = \vec{BA} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -2 \\ 1 & 4 & 3 \end{vmatrix} = 5\vec{i} - 8\vec{j} + 9\vec{k}, \quad |\vec{n}| = \sqrt{170}$

Let $S = A(2,2,-1)$: $\vec{PS} = \vec{i} + \vec{j} - 3\vec{k} \Rightarrow d = \frac{|5-8-27|}{\sqrt{170}} = \frac{30}{\sqrt{170}}$ //

QUESTION 2 The blanks below will be filled by students. (Except the score)

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[12p] **a)** Investigate the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{2x^2}{x^2+2xy+y}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

at the point $(x,y) = (0,0)$.

[13p] **b)** Find the directional derivative of the function $f(x,y,z) = x + x\cos z - y\sin z + y$ at the point $(1,-2,\pi)$ in the direction of $\vec{v} = 2\vec{i} - 2\vec{j} - 2\vec{k}$ and at this point determine the direction in which the function increases most rapidly.

(a) • $f(0,0) = 1$ exists

• $f(x,y)\Big|_{x=0,\, y\neq 0} = \dfrac{2.0^2}{0^2+2.0.y+y} = 0$

so $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } x=0,\, y\neq 0}} f(x,y) = \lim\limits_{y\to 0} 0 = 0$

$f(x,y)\Big|_{y=0,\, x\neq 0} = \dfrac{2x^2}{x^2+2x\cdot 0+0} = 2$

so $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y=0,\, x\neq 0}} f(x,y) = \lim\limits_{x\to 0} 2 = 2$

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist by the two-path test.

f is not cont. at $(0,0)$.

(b) $\vec{v} = 2\vec{i} - 2\vec{j} - 2\vec{k}$, $|\vec{v}| = 2\sqrt{3} \rightarrow \vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{\sqrt{3}}(\vec{i} - \vec{j} - \vec{k})$

$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = (1+\cos z)\vec{i} + (-\sin z + 1)\vec{j} - (x\sin z + y\cos z)\vec{k}$

$\vec{\nabla} f\Big|_{(1,-2,\pi)} = 0.\vec{i} + \vec{j} - 2\vec{k}$

$(D_{\vec{u}} f)_{P_0} = \vec{\nabla} f\Big|_{P_0} \cdot \vec{u} = (0.\vec{i} + \vec{j} - 2\vec{k}) \cdot \left(\dfrac{1}{\sqrt{3}}(\vec{i} - \vec{j} - \vec{k})\right) = \dfrac{1}{\sqrt{3}}$ //

$\vec{\nabla} f\Big|_{(1,-2,\pi)} = \vec{j} - 2\vec{k}$ is the direction in which f increases most rapidly.

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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[10p] **a)** Find the equation of the tangent plane of the surface $\mathbf{z = e^{x+y} + 2xy - 2}$ at the point $\mathbf{(1, -1, -3)}$.

[15p] **b)** Using the method of Lagrange multipliers, find the maximum and minimum values of the function $\mathbf{f(x, y) = x^2 + y^2}$ subject to the constraint $\mathbf{x^2 + x + y^2 + 2y = 0}$.

(a) $g(x,y,z) = e^{x+y} + 2xy - z - 2 = 0$

$\vec{\nabla} g = (e^{x+y} + 2y)\vec{i} + (e^{x+y} + 2x)\vec{j} - 1.\vec{k}$

$\vec{n} = \vec{\nabla} g \Big|_{(1,-1,-3)} = -\vec{i} + 3\vec{j} - \vec{k}$

$-1.(x-1) + 3.(y-(-1)) - 1.(z-(-3)) = 0 \Rightarrow -x + 3y - z = -1$ //

(b) $g(x,y) = x^2 + x + y^2 + 2y = 0$

$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow 2x\vec{i} + 2y\vec{j} = \lambda[(2x+1)\vec{i} + (2y+2)\vec{j}]$

$$\left.\begin{aligned} 2x &= \lambda(2x+1) \\ 2y &= \lambda(2y+2) \\ x^2+x+y^2+2y &= 0 \end{aligned}\right\} \quad \begin{aligned} 2x(1-\lambda) &= \lambda \quad (1) \\ 2y(1-\lambda) &= 2\lambda \quad (2) \\ x^2+x+y^2+2y &= 0 \quad (3) \end{aligned}$$

(1) & (2) : $2y(1-\lambda) = 2.2x.(1-\lambda) \rightarrow 2(y-2x)(1-\lambda) = 0$

Two possibilities : If $\lambda = 1$, (1) & (2) gives $\lambda = 0$; so not possible.

$y = 2x$: (3) gives $x^2 + x + 4x^2 + 4x = 0 \rightarrow 5x(x+1) = 0$

$x = 0, \quad y = 0$: $f(0,0) = 0$ minimum value of f

$x = -1, \quad y = -2$: $f(-1,-2) = 5$ maximum value of f

QUESTION 4 The blanks below will be filled by students. (Except the score)

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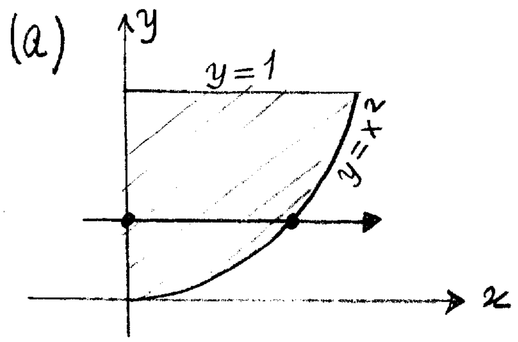
Lütfen bu soruyu bu kağıdın ön yüzünü ve gerekirse arka yüzünü kullanarak cevaplayınız.

[10p] **a)** Sketch the the region of integration of $\int_0^1 \int_{x^2}^1 \mathbf{4xe^{y^2}dydx}$ and evaluate the integral.

[15p] **b)** Using cylindrical coordinates, find the volume of the solid bounded from below by the cone $\mathbf{z = -\sqrt{x^2+y^2}}$, above by the sphere $\mathbf{x^2+y^2+z^2=1}$ and laterally (from the sides) by the cylinder $\mathbf{x^2+y^2=1}$.

(a)

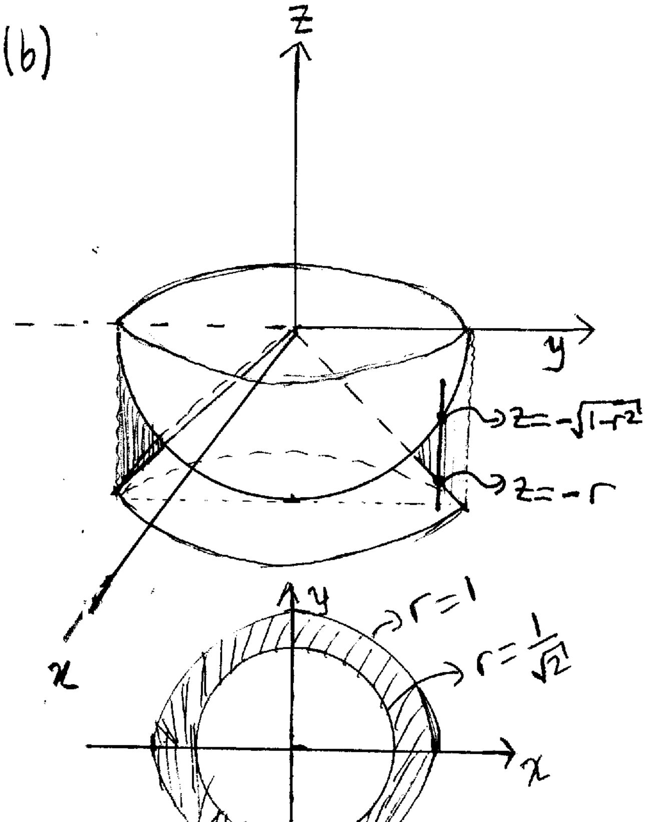


$$\int_{x=0}^{1}\int_{y=x^2}^{1} 4xe^{y^2}\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{x=\sqrt{y}} 4xe^{y^2}\,dx\,dy$$

$$= \int_{y=0}^{1}\left(2x^2e^{y^2}\Big|_{x=0}^{x=\sqrt{y}}\right)dy = \int_0^1 2ye^{y^2}\,dy$$

$$= e^{y^2}\Big|_0^1 = e-1$$

(b)



In Cylindrical coordinates:

$z=-\sqrt{x^2+y^2} \longrightarrow z=-r$

$x^2+y^2+z^2=1,\ z\leq 0 \longrightarrow z=-\sqrt{1-r^2}$

$x^2+y^2=1 \rightarrow r=1 \qquad -\sqrt{x^2+y^2} = -\sqrt{1-(x^2+y^2)} \rightarrow r=\frac{1}{\sqrt{2}}$

$$\text{Volume} = \int_{\theta=0}^{2\pi}\int_{r=1/\sqrt{2}}^{1}\int_{z=-r}^{-\sqrt{1-r^2}} dz\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_{1/\sqrt{2}}^{1}\left(-\sqrt{1-r^2}+r\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{3}(1-r^2)^{3/2}+\frac{r^3}{3}\right]_{1/\sqrt{2}}^{1} d\theta = \frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$