MATHEMATICS 102E MIDTERM EXAM 15 JULY 2014

QUESTION 1 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	**Score**
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[10 pts] a) Find parametric equations for the tangent line **L** of the curve $\vec{r}(t) = 2e^{(t-1)/2}\,\vec{i} - t^3\,\vec{j} + 3(t-1)\,\vec{k}$ at $t = 1$.

[10 pts] b) Find the equation of the plane **M** through the points $A(0,-2,-6)$, $B(-1,1,5)$, $C(2,3,-6)$.

[5 pts] c) Find the intersection point of the tangent line **L** and the plane **M**.

Solutions

a) $\frac{d\vec{r}}{dt} = e^{\frac{(t-1)}{2}}\vec{i} - 3t^2\vec{j} + 3\vec{k}$, $\quad \vec{v} = \left.\frac{d\vec{r}}{dt}\right|_{t=1} = \vec{i} - 3\vec{j} + 3\vec{k}$

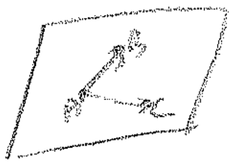
$\vec{r}(1) = 2\vec{i} - \vec{j}$

$P_0(2,-1,0)$ is a point on the curve. $\Big\}$ The line pass trough the point P_0 and parallel to $\vec{v}$;

$\overrightarrow{P_0P} = t\,\vec{v}$

$$L: \begin{cases} x = 2 + t \\ y = -1 - 3t \\ z = 3t \end{cases}, \quad -\infty < t < \infty$$

b)



$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

$\overrightarrow{AB} = -\vec{i} + 3\vec{j} + 11\vec{k}$

$\overrightarrow{AC} = 2\vec{i} + 5\vec{j}$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 3 & 11 \\ 2 & 5 & 0 \end{vmatrix} = -55\vec{i} + 22\vec{j} - 11\vec{k}$$

$A(0,-2,-6)$

$\vec{n} = -55\vec{i} + 22\vec{j} - 11\vec{k}$ $\Big\}$ $-55(x-0) + 22(y+2) - 11(z+6) = 0$

$M: \boxed{-55x + 22y - 11z = 22}$

c) $L: \; x = 2 + \lambda, \; y = -1 - 3\lambda, \; z = 3\lambda$

$M: -55x + 22y - 11z = 22$

$-55(2+\lambda) + 22(-1-3\lambda) - 11(3\lambda) = 22$

$-154\lambda = 154 \Rightarrow \boxed{\lambda = -1}$

$\therefore$ The intersection point:

$x = 2 - 1 = 1$

$y = -1 + 3 = 2$

$(1, 2, -3)$

MATHEMATICS 102E MIDTERM EXAM 15 JULY 2014

QUESTION 2 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	**Score**
Signature:	Electronic Post(e-mail) address:	Student Number:		

[15 pts] a) $\mathbf{f(x,y)}$ be a function given by

$$f(x,y)=\begin{cases}\dfrac{\mathbf{xy}}{(\mathbf{x^2+y^2})^2}\tan(\mathbf{x^2+y^2}) , & (\mathbf{x,y})\neq(\mathbf{0,0})\\[2ex] \dfrac{\mathbf{3}}{\mathbf{2}} , & (\mathbf{x,y})=(\mathbf{0,0})\end{cases}$$

Is $\mathbf{f(x,y)}$ continuous at the point $(\mathbf{0,0})$? Give reasons for your answer.

[10 pts] b) Calculate the partial derivative $\dfrac{\partial \mathbf{f}}{\partial \mathbf{y}}$ of the function $\mathbf{f(x,y) = 4+2x-3y-xy^2}$ at $(\mathbf{-2,1})$ by using limit definition.

Solution :

a) $f(0,0)=\frac{3}{2}$

b) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{\substack{r\to 0\\ \left(\substack{x=r\cos\theta\\ y=r\sin\theta}\right)}} \dfrac{\cancel{r^2}\sin\theta\cos\theta}{(r^2)^2}\tan r^2 = \lim\limits_{r\to 0} \sin\theta\cos\theta \cdot \dfrac{\tan r^2}{r^2}$

$= \sin\theta\cos\theta \cdot \underbrace{\lim\limits_{r\to 0}\dfrac{\tan r^2}{r^2}}_{\left(\frac{0}{0}\right)} = \sin\theta\cos\theta\cdot\lim\limits_{t\to 0}\dfrac{\tan t}{t}$

$= \sin\theta\cos\theta\cdot\lim\limits_{t\to 0}\underbrace{\dfrac{\sin t}{t}}_{1}\cdot\underbrace{\dfrac{1}{\cos t}}_{1}$

$=\sin\theta\cdot\cos\theta.$

limit depends on the value of θ. Since the limit does not exist at $(0,0)$, $f(x,y)$ is not continuous at $(0,0)$

b) $\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim\limits_{h\to 0}\dfrac{f(x_0, y_0+h)-f(x_0,y_0)}{h}$

$\left.\dfrac{\partial f}{\partial y}\right|_{(-2,1)} = \lim\limits_{h\to 0}\dfrac{f(-2,1+h)-f(-2,1)}{h} = \lim\limits_{h\to 0}\dfrac{-3(1+h)+2(1+h)^2-(-1)}{h} = \lim\limits_{h\to 0}\dfrac{2h^2+h}{h}$

$=\lim\limits_{h\to 0} 2h+1 = 1$

$\therefore \left.\dfrac{\partial f}{\partial y}\right|_{(-2,1)} = 1$ //

MATHEMATICS 102E MIDTERM EXAM 15 JULY 2014

QUESTION 3 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[15 pts] a) Find all the local maxima, local minima and saddle points of the function $f(x,y) = x^3 + 3xy + y^3$.

[10 pts] b) Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x,y,z) = x + 2y + 3z$ has its maximum and minimum values by using Lagrange multipliers methods.

Solution

a) $f_x = 3x^2+3y = 0$, $f_y = 3x+3y^2 = 0$ $\Big\}$ $3x+3(-x^2)^2 = 0$ $\quad x(1+x^3)=0$, $x=0,\ x=-1$

$\Rightarrow y=-x^2$, $y=0,\ y=-1$

$\therefore$ The critical points : $(0,0)$ and $(-1,-1)$

$f_{xx} = 6x$, $f_{xy} = 3$, $f_{yy} = 6y$

$D = f_{xx}f_{yy} - f_{xy}^2 = 6x \cdot 6y - 9 = 36xy - 9$

$D(0,0) = 36.0.0 - 9 = -9 < 0 \Rightarrow (0,0)$ is a saddle point.

$D(-1,-1) = 36(-1)(-1) - 9 = 27 > 0$, $f_{xx}\big|_{(-1,1)} = 6.(-1) = -6 < 0$ $\Big\}$ $\Rightarrow (-1,-1)$ is a local maximum.

b) $f(x,y,z) = x+2y+3z$ $\qquad \vec{\nabla} f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k} = \vec{i} + 2\vec{j} + 3\vec{k}$

$g(x,y,z) = x^2+y^2+z^2-25$ $\qquad \vec{\nabla} g = g_x\vec{i} + g_y\vec{j} + g_z\vec{k} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \vec{i} + 2\vec{j} + 3\vec{k} = \lambda(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$

$1 = 2\lambda x \Rightarrow x = \frac{1}{2\lambda}$
$2 = 2\lambda y \Rightarrow y = \frac{1}{\lambda}$
$3 = 2\lambda z \Rightarrow z = \frac{3}{2\lambda}$

$g(x,y,z) = 0$

$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 25$ $\qquad \frac{14}{4\lambda^2} = 25 \Rightarrow \lambda = \mp\frac{\sqrt{14}}{10}$

$\Rightarrow P\left(\frac{1}{2\lambda}, \frac{1}{\lambda}, \frac{3}{2\lambda}\right) = \mp\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right)$ //

QUESTION 4 The blanks below will be filled by students. (Except the score)

Surname:		Name:	Group Number:	List Number:	**Score**
Signature:	Electronic Post(e-mail) address:		Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[12 pts] a) Evaluate the integral $\int_0^1 \int_{-\sqrt{1-\mathbf{x}^2}}^{\sqrt{1-\mathbf{x}^2}} \frac{\mathbf{2}}{(\mathbf{1}+\mathbf{x}^2+\mathbf{y}^2)^2}\,\mathbf{dy\,dx}$ by changing to polar coordinates.

[13 pts] b) Let **D** be the space region cut from the solid cylinder $\mathbf{x}^2+\mathbf{y}^2 \leq \mathbf{1}$ by the sphere $\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2 = \mathbf{4}$. Write the triple integral that calculates the volume of **D** in terms of

i) Cartesian coordinates (Do not calculate the integral)

ii) Cylindrical coordinates (Do not calculate the integral)

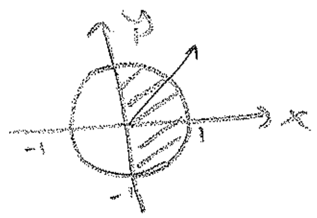
iii) Find the volume of **D** by using part-(ii) .

Solution

a)

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dx\,dy &= r\,dr\,d\theta \end{aligned}\right\} \qquad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \quad 0 \leq x \leq 1$$

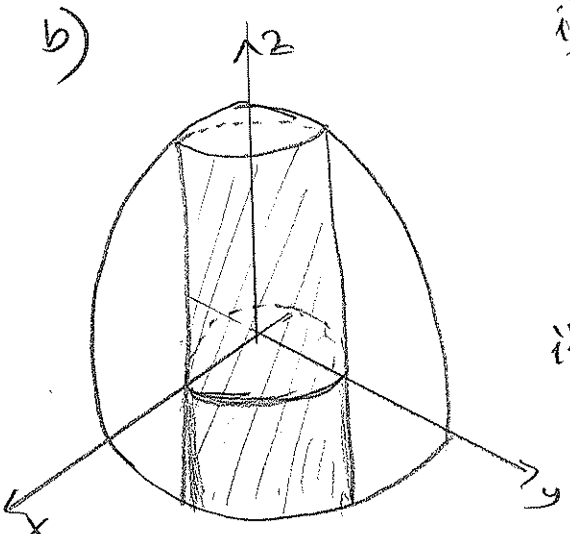
$$x^2+y^2=1 \Rightarrow r^2=1, \quad r=1$$



$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = \int_{-\pi/2}^{\pi/2} \int_0^1 \frac{2r\,dr\,d\theta}{(1+r^2)^2} \qquad \left(\begin{aligned} u &= 1+r^2 \\ du &= 2r\,dr \end{aligned}\right)$$

$$= \int_{-\pi/2}^{\pi/2} \int \frac{1}{u^2}\,du\,d\theta = \int_{-\pi/2}^{\pi/2} \left(-\frac{1}{1+r^2}\right)\Bigg|_{r=0}^{1} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta = \frac{1}{2}\theta\Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{2}$$

b)



i) $$V = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx.$$

ii) $$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\,r\,dr\,d\theta.$$

iii) $$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 z \Big|_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 2\sqrt{4-r^2}\, r\, dr\, d\theta$$

$$\begin{pmatrix} u = 4-r^2 \\ du = -2r\,dr \end{pmatrix}$$

$$= \int_0^{2\pi} \int \sqrt{u}\,(-du)\, d\theta = \int_0^{2\pi} -\frac{2}{3}(4-r^2)^{3/2} \Big|_{r=0}^{1} d\theta$$

$$= 2\pi \left[-\frac{2}{3} 3^{3/2} + \frac{2}{3} 4^{3/2} \right]$$

$$= \frac{4\pi}{3}\left(8 - 3\sqrt{3}\right) \quad //$$