MATHEMATICS 102E — MIDTERM — 04.04.2015

QUESTION 1 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
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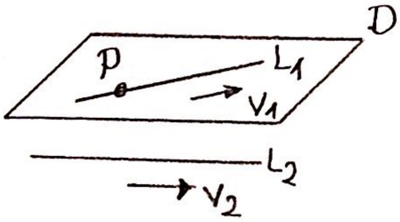
For the solution of this question please use only the front face and if necessary the back face of this page.

[10p] a) Find an equation of the plane containing the line $\mathbf{L_1}: \mathbf{x} = \mathbf{t}+3,\ \mathbf{y} = 2\mathbf{t}-1,\ \mathbf{z} = -\mathbf{t}+1$, $-\infty < \mathbf{t} < \infty$, and parallel to the line $\mathbf{L_2}: \mathbf{x} = 2\mathbf{s}+1,\ \mathbf{y} = \mathbf{s}-2,\ \mathbf{z} = -\mathbf{s}+1,\ -\infty < \mathbf{s} < \infty$.

[8p] b) Find the length of the curve $\vec{\mathbf{r}}(\mathbf{t}) = (1+\mathbf{t})\vec{\mathbf{i}} - \mathbf{t}^2\vec{\mathbf{j}} + (1+\frac{2}{3}\mathbf{t}^3)\vec{\mathbf{k}}$ from $\mathbf{t} = 0$ to $\mathbf{t} = 1$.

[7p] c) Find the direction of most rapid increase and decrease for $\mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{y}\tan^{-1}\mathbf{x} + \cos(\mathbf{xy})$ at the point $(-1, 0)$.

a)



$P(3,-1,1)$, $V_1 = i+2j-k$, $V_2 = 2i+j-k$

P is on L_1 and on D.

$V_1 \parallel L_1$, $V_2 \parallel L_2 \Rightarrow V_1, V_2 \parallel D$

$V_1 \times V_2 \perp V_1$ and $V_2 \Rightarrow V_1 \times V_2 \perp D$

$$V_1 \times V_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = -i-j-3k$$

D is the plane through P with the normal $n = V_1 \times V_2$.

$-(x-3)-(y+1)-3(z-1) = 0 \Rightarrow x+y+3z = 5$

b) $V = \frac{dr}{dt} = i - 2tj + 2t^2k$, $|V| = (1+4t^2+4t^4)^{1/2} = 1+2t^2$

$$L = \int_0^1 |V|\,dt = t + \frac{2}{3}t^3 \Big|_0^1 = 1 + \frac{2}{3} = \frac{5}{3}$$

c) $\nabla f = \left(\frac{y}{1+x^2} - y\sin(xy)\right)i + \left(\tan^{-1}x - x\sin(xy)\right)j$

$\nabla f\Big|_{(-1,0)} = 0i - \frac{\pi}{4}j \Rightarrow |\nabla f|\Big|_{(-1,0)} = \frac{\pi}{4}$

Most rapid increase is in the direction $\frac{\nabla f}{|\nabla f|}\Big|_{(-1,0)} = -j$

Most rapid decrease is in the direction $-\frac{\nabla f}{|\nabla f|}\Big|_{(-1,0)} = j$

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[10p] a) Let $f(x,y)=\begin{cases}\dfrac{x^3y}{x^5+2y^3}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

Determine if f is continuous at $(0,0)$. Explain your answer.

[15p] b) Let $f(x,y,z)=cx+\ln(x^2+y^2)+\cos(cz)$ where c is a constant. Find the value of c if the tangent plane at the point $P(1,-1,0)$ passes through the origin.

a) If f is continuous at the origin, then $\lim\limits_{(x,y)\to(0,0)} f(x,y)=f(0,0)$

$y=mx^2 \Rightarrow \lim\limits_{x\to 0}\dfrac{x^3mx^2}{x^5+2m^3x^6}=\lim\limits_{x\to 0}\dfrac{m}{1+2m^3x}=m$

The value of limit depends on the value of m Therefore, the limit of f at the origin does not exist Thus, f is not continuous at (0,0)

b) $\nabla f=\left(c+\dfrac{2x}{x^2+y^2}\right)i+\dfrac{2y}{x^2+y^2}j-c\sin(cz)k$

$\nabla f\Big|_P=(c+1)i-j+0k$

Tangent plane at P is the plane through P with the normal $\nabla f\Big|_P$.

Tang. pl: $(c+1)(x-1)-(y+1)=0$ $(*)$

Since the plane passes through the origin, (0,0,0) must satisfy the equation $(*)$

$(c+1)(-1)-1=0 \Rightarrow c=-2$

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Let $f(x,y) = x^2y - 2xy + y^2 - 3y$.

[10p] a) Find and classify the critical points of f.

[15p] b) Find the absolute extreme values of f on the region $D : \{(x,y) \mid x \le 0, y \ge -1, y - 2x \le 3\}$.

a) $f_x = 2xy - 2y = 2y(x-1) = 0 \Rightarrow y=0$ or $x=1$

$f_y = x^2 - 2x + 2y - 3 \Rightarrow y=0: x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1)=0$

$\Rightarrow x=3$ or $x=-1$

$\Rightarrow x=1 \Rightarrow 1-2+2y-3=0 \Rightarrow 2y=4 \Rightarrow y=2$

Critical points: $A(-1,0)$, $B(3,0)$, $C(1,2)$

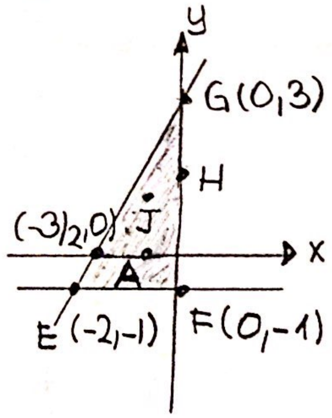
$f_{xx} = 2y$, $f_{xy} = 2x-2$, $f_{yy} = 2$

$\Delta = f_{xx}f_{yy} - f_{xy}^2 = 4y - 4(x-1)^2$

$\Delta\big|_A = -16 < 0 \Rightarrow A$: saddle point, $\Delta\big|_B = -16 < 0 \Rightarrow B$: saddle point

$\Delta\big|_C = 8 > 0$, $f_{yy} > 0 \Rightarrow C$: Local minimum point

b)



$[EF]: y=-1$, $f(x,-1) = -x^2+2x+1+3 = -x^2+2x+4$

$\frac{d}{dx} f(x,-1) = -2x+2=0 \Rightarrow x=1 \Rightarrow (1,-1) \notin D$

$[GF]: x=0$, $f(0,y) = y^2-3y$

$\frac{d}{dy} f(0,y) = 2y-3=0 \Rightarrow y=3/2 \Rightarrow H(0,3/2)$

$[GE]: y-2x=3$, $f(x,3+2x) = (3+2x)(x^2 \cancel{-2x} + \cancel{3} + \cancel{2x} \cancel{-3}) = (3+2x)x^2$

$\frac{d}{dx} f(x,3+2x) = 6x+6x^2=0 \Rightarrow x=0, x=-1 \Rightarrow G(0,3), J(-1,1)$

$f(E)=-4$, $f(F)=4$, $f(G)=0$, $f(A)=0$, $f(H)=-9/4$, $f(J)=1$

Absolute maximum value is 4 whereas absolute minimum value is -4

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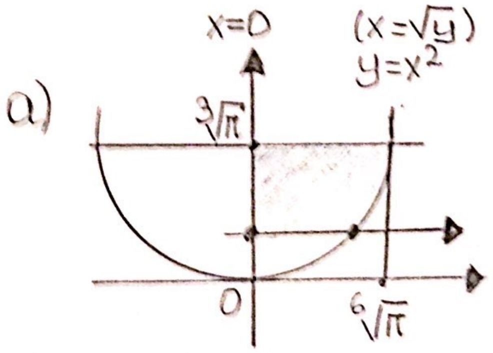
[10p] a) Evaluate

$$\int_0^{\sqrt[6]{\pi}} \int_{x^2}^{\sqrt[3]{\pi}} x^3 \sin(y^3)\,dy\,dx \,.$$

[15p] b) Let D be the region in space bounded on the top by the sphere $x^2+y^2+z^2=2$ and on the bottom by the paraboloid $z=x^2+y^2$. Sketch the region D. Express the volume of D in the given coordinates and orders of integration. Do not evaluate the integrals.

i) $dzdydx$ ii) $dzdrd\theta$ iii) $d\rho d\phi d\theta$

a)



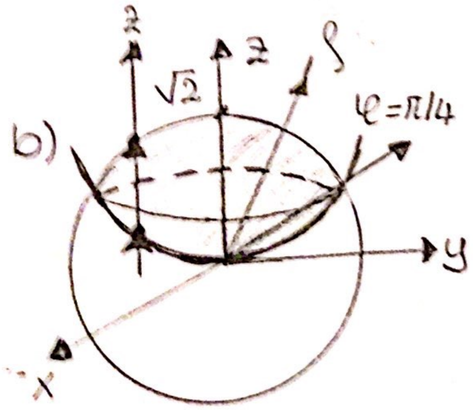
$0 \le x \le \sqrt[6]{\pi}$, $x^2 \le y \le \sqrt[3]{\pi}$

$$\int_0^{\sqrt[3]{\pi}} \int_0^{\sqrt{y}} x^3 dx \cdot \sin y^3\, dy = \int_0^{\sqrt[3]{\pi}} \frac{x^4}{4}\Big|_0^{\sqrt{y}} \sin y^3\, dy$$

$$= \frac{1}{4}\int_0^{\sqrt[3]{\pi}} y^2 \sin y^3\, dy = -\frac{1}{12}\cos y^3\Big|_0^{\sqrt[3]{\pi}}$$

$$= -\frac{1}{12}(\cos\pi - \cos 0) = \frac{1}{6}$$

b)

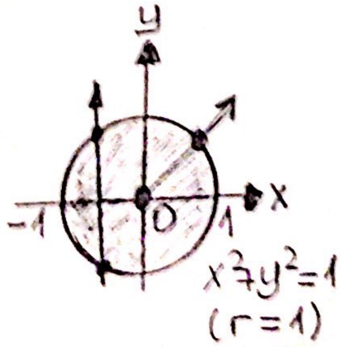


$x^2+y^2+z^2=2 \Rightarrow r^2+z^2=2 \Rightarrow z=\sqrt{2-r^2}$, $\rho=\sqrt{2}$

$z=x^2+y^2 \Rightarrow z=r^2$, $\rho\cos\varphi=\rho^2\sin^2\varphi \Rightarrow \rho=\frac{\cos\varphi}{\sin^2\varphi}$

$$\left.\begin{matrix} z=x^2+y^2 \\ x^2+y^2+z^2=2 \end{matrix}\right\} \begin{matrix} z^2+z-2=0 \Rightarrow (z+2)(z-1)=0 \\ \Rightarrow z=1=\sqrt{2}\cos\varphi \Rightarrow \varphi=\pi/4 \end{matrix}$$

$x^2+y^2=1$



i) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{2-(x^2+y^2)}} dz\,dy\,dx$$

ii) $$\int_0^{2\pi}\int_0^{1}\int_{r^2}^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta$$

iii) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta + \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{(\cos\varphi)/(\sin^2\varphi)} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$