SOLUTIONS

MAT 102E – 102 MIDTERM EXAM 21 NOVEMBER 2015

QUESTION 1 The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

[8 pts] a) Find an equation of the plane that passes through the point $\mathbf{P(2,0,3)}$ and contains the line $\mathbf{L: x = 1+2t\ ,\ y = -1+t\ ,\ z = 2+3t}$, $-\infty < \mathbf{t} < \infty$.

[8 pts] b) Find the equation of **the tangent plane** and the parametric equations for **the normal line** to the surface $\mathbf{z+1 = x\ e^y \cos z}$ at the point $(\mathbf{1,0,0})$.

[9 pts] c) If the directional derivative of $\mathbf{f(x,y,z) = z^c \tan^{-1}(x+y)}$ at the point $\mathbf{P(0,0,4)}$ in the direction of $\vec{\mathbf{u}} = \vec{\mathbf{i}} + \vec{\mathbf{j}}$ is $\mathbf{2}$, find the real number $\mathbf{c}$.

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a) $L: x=1+2t,\ y=-1+t,\ z=2+3t,\ -\infty<t<\infty$

$\Rightarrow Q(1,-1,2)$ is a point on L, so is on the plane. Then,

$\vec{V} = \vec{QP} = i+j+k$ is parallel to the plane.

The direction vector $\vec{w} = 2i+j+3k$ is also parallel to the plane.

$$\Rightarrow \vec{V}\times\vec{W} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 2i-j-k$$ is a normal vector to the plane.

So, the plane through $P(2,0,3)$ with normal $\vec{n} = 2i-j-k$ is

$2(x-2)-1(y-0)-1(z-3)=0 \Rightarrow \underline{2x-y-z=1}$

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b) $f(x,y,z) = xe^y\cos z - z - 1 = 0$

$\vec{\nabla}f = (e^y\cos z)i + (xe^y\cos z)j - (1+xe^y\sin z)k$

$\vec{\nabla}f\big|_{(1,0,0)} = i+j-k$ is normal to the tangent plane at $(1,0,0)$.

• <u>Tangent plane at</u> $(1,0,0)$: $1(x-1)+1(y-0)-1(z-0)=0$

$\Rightarrow \underline{x+y-z=1}$

• <u>Normal line at</u> $(1,0,0)$: $\vec{\nabla}f\big|_{(1,0,0)}$ is the direction vector of the normal line at $(1,0,0)$. So, the parametric equations are

$x=1+t,\ y=t,\ z=-t,\ -\infty<t<\infty.$

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1- c) $\vec{\nabla}f = \left[z^c \cdot \frac{1}{1+(x+y)^2}\right]i + \left[z^c \frac{1}{1+(x+y)^2}\right]j + \left[cz^{c-1}\tan^{-1}(x+y)\right]k$

$\vec{\nabla}f\Big|_P = 4^c i + 4^c j$

$\vec{v} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{2}}(i+j) = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$

$(D_v f)_P = (4^c i + 4^c j)\left(\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j\right) = 2 \cdot \frac{4^c}{\sqrt{2}} = \sqrt{2} \cdot 2^{2c} = 2$

$\Rightarrow 2^{2c} = \sqrt{2} = 2^{1/2} \Rightarrow \underline{c = \frac{1}{4}}$

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QUESTION 2 The blanks below will be filled by students. (Except the score)

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[12 pts] a) Let $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.

Is $f(x,y)$ continuous at the point $(0,0)$? Explain your answer.

[13 pts] b) Let $\vec{r}(t) = \frac{\sqrt{3}}{2} t^2 \,\vec{i} + (\sin t - t\cos t)\vec{j} + (\cos t + t\sin t)\vec{k}$ be given.

i. Find the unit tangent vector $\vec{T}$ of the curve $\vec{r}(t)$ where $t > 0$.

ii. Find the length of the curve $\vec{r}(t)$ from $t = 0$ to $t = \pi$.

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a) First, we determine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists or not:

- Along x-axis ($y=0$): $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim\limits_{x\to 0} \frac{\overbrace{\sin 0}^{0}}{x^2} = 0$
- Along $y=x$: $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = \lim\limits_{x\to 0} \frac{\sin x^2}{2x^2} = \frac{1}{2}$

$\not\equiv$ $\Big\}$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ DOES NOT EXIST.

So, f is not continuous at (0,0).

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b) (i) $\frac{d\vec{r}}{dt} = \frac{\sqrt{3}}{2} \cdot 2t\, i + (\cancel{\cos t} - \cancel{\cos t} + t\sin t) j + (\cancel{-\sin t} + \cancel{\sin t} + t\cos t) k$

$\frac{d\vec{r}}{dt} = \vec{v}(t) = \sqrt{3}t\, i + (t\sin t) j + (t\cos t) k$

$\Rightarrow |\vec{v}(t)| = \sqrt{3t^2 + \underbrace{t^2\sin^2 t + t^2\cos^2 t}_{t^2}} = \sqrt{4t^2} = 2|t| = 2t$ since $t>0$

$\vec{T} = \frac{\vec{v}(t)}{|v(t)|} = \frac{1}{2t}[\sqrt{3}t\, i + t\sin t\, j + t\cos t\, k] \overset{t>0}{=} \frac{\sqrt{3}}{2} i + \frac{\sin t}{2} j + \frac{\cos t}{2} k$

(ii) $L = \int_0^{\pi} |\vec{v}(t)| dt = \int_0^{\pi} 2|t| dt \underset{t>0}{=} \int_0^{\pi} 2t\, dt = t^2 \Big|_0^{\pi} = \underline{\underline{\pi^2}}$

(3)

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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[13 pts] a) **Use the method of Lagrange Multipliers** to find the maximum and minimum values of $\mathbf{f(x,y,z) = 8x - 4z}$ subject to the constraint $\mathbf{x^2 + 10y^2 + z^2 = 5}$.

[12 pts] b) Find and classify the critical points of the function $\mathbf{f(x,y,\ )} = \mathbf{x^3 + 3xy^2 - 3x^2 - 3y^2 + 5}$.

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a) $f(x,y,z) = 8x - 4z \Rightarrow \vec{\nabla} f = f_x i + f_y j + f_z k = 8i - 4k$

$g(x,y,z) = x^2 + 10y^2 + z^2 - 5 \Rightarrow \vec{\nabla} g = g_x i + g_y j + g_z k$

$\Rightarrow \vec{\nabla} g = 2x i + 20y j + 2z k$

$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow 8i - 4k = \lambda(2xi + 20yj + 2zk)$

1. $8 = 2\lambda x \longrightarrow \boxed{x = \frac{4}{\lambda}}$
2. $0 = 20\lambda y \longrightarrow 20\lambda y = 0 \Rightarrow \lambda = 0$ or $y = 0$. But $\lambda \neq 0$ from 1 & 3 $\Rightarrow \boxed{y = 0}$
3. $-4 = 2\lambda z \longrightarrow \boxed{z = \frac{-2}{\lambda}}$
4. $x^2 + 10y^2 + z^2 = 5$

$x^2 + 10y^2 + z^2 = \left(\frac{4}{\lambda}\right)^2 + 10 \cdot 0^2 + \left(\frac{-2}{\lambda}\right)^2 = 5 \Rightarrow \frac{16+4}{\lambda^2} = 5 \Rightarrow \underline{\lambda = \pm 2}$

$\lambda = 2 \Rightarrow x = 2, z = -1 \Rightarrow P_1(2,0,-1)$

$\lambda = -2 \Rightarrow x = -2, z = 1 \Rightarrow P_2(-2,0,1)$

- <u>The maximum</u> of f subject to $x^2 + 10y^2 + z^2 = 5$ is $f(2,0,-1) = \underline{20}$
- <u>The minimum</u> of f subject to $x^2 + 10y^2 + z^2 = 5$ is $f(-2,0,1) = \underline{-20}$

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(3-b) $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 5$

$f_x = 3x^2 + 3y^2 - 6x = 0$

$f_y = 6xy - 6y = 0 \Rightarrow 6y(x-1) = 0 \Rightarrow \underline{x=1}$ OR $\underline{y=0}$

$\underline{x=1} \Rightarrow 3.1^2 + 3y^2 - 6.1 = 0 \Rightarrow 3y^2 = 3 \Rightarrow y = \pm 1$

$\underline{y=0} \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow \underline{x=0}$ OR $\underline{x=2}$

The critical points : (1,1), (1,−1), (0,0), (2,0)

$D = f_{xx} \cdot f_{yy} - f_{xy}^2$

$f_{xx} = 6x - 6 \quad f_{yy} = 6x - 6 \quad f_{xy} = 6y \Rightarrow D = (6x-6)^2 - 36y^2$

Point	D	fxx	Type
(1,1)	$-36 < 0$	no need	SADDLE POINT
(1,−1)	$-36 < 0$	no need	SADDLE POINT
(0,0)	$36 > 0$	$-6 < 0$	LOCAL MAXIMUM
(2,0)	$36 > 0$	$6 > 0$	LOCAL MINIMUM

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QUESTION 4 The blanks below will be filled by students. (Except the score)

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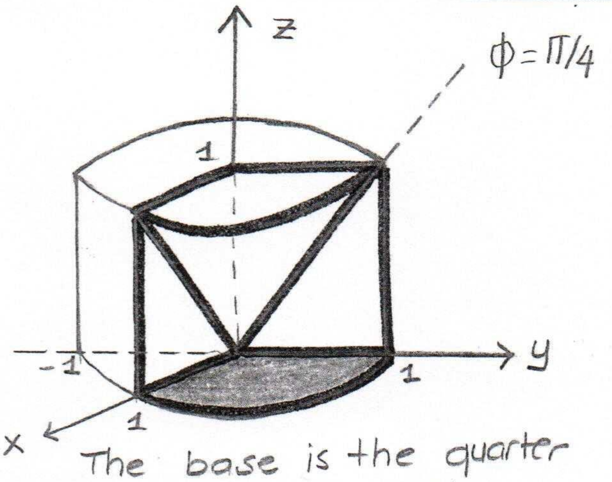
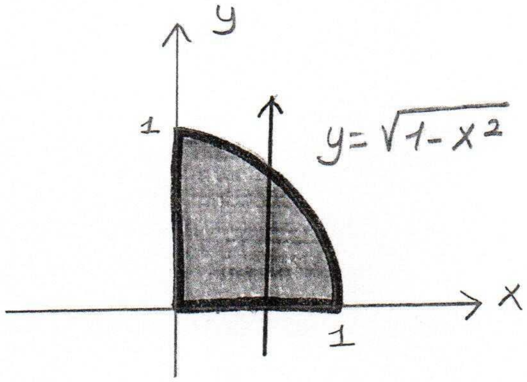
[15pts] a) Write an iterated triple integral for the integral of $\mathbf{f(x,y,z) = 6+4y}$ over the region **in the first octant**, bounded below by the **xy-plane**, above by the cone $\mathbf{z} = \sqrt{\mathbf{x^2+y^2}}$, laterally by the cylinder $\mathbf{x^2+y^2=1}$ and the coordinate planes, in

(i) rectangular coordinates,
(ii) cylindrical coordinates,
(iii) spherical coordinates.

Evaluate the integral in cylindrical coordinates.

[10pts] b) Evaluate $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{\mathbf{x\,e^{2y}}}{\mathbf{4-y}}\,\mathbf{dy\,dx}$.





The base is the quarter of the unit circle.

(i) RECTANGULAR : $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} (6+4y)\,dz\,dy\,dx$

(ii) CYLINDRICAL : $\displaystyle\int_0^{\pi/2} \int_0^1 \int_0^r (6+4r\sin\theta)\,r\,dz\,dr\,d\theta$

(iii) SPHERICAL : $\displaystyle\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{\csc\phi} (6+4\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$x^2+y^2=1 \Rightarrow \rho = \csc\phi$

4-a)

iv. $$\int_0^{\pi/2}\int_0^1\int_0^r \underbrace{(6+4r\sin\theta)r}_{6r+4r^2\sin\theta}\,dz\,dr\,d\theta=\int_0^{\pi/2}\int_0^1 (6rz+4r^2z\sin\theta)\Big|_0^r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\int_0^1(6r^2+4r^3\sin\theta)\,dr\,d\theta=\int_0^{\pi/2}(2r^3+r^4\sin\theta)\Big|_0^1\,d\theta$$

$$=\int_0^{\pi/2}(2+\sin\theta)\,d\theta=2\theta-\cos\theta\Big|_0^{\pi/2}=\pi-(-1)=\underline{\pi+1}$$

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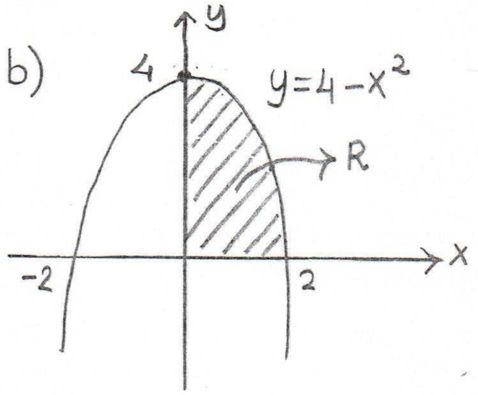
b)



change the order!

$$\int_0^2\int_0^{4-x^2}\frac{xe^{2y}}{4-y}\,dy\,dx=\int_0^4\int_0^{\sqrt{4-y}}\frac{xe^{2y}}{4-y}\,dx\,dy$$

$$=\int_0^4\frac{e^{2y}}{4-y}\cdot\frac{x^2}{2}\Big|_0^{\sqrt{4-y}}\,dy=\int_0^4\frac{e^{2y}}{\cancel{4-y}}\cdot\frac{\cancel{4-y}}{2}\,dy$$

$$=\int_0^4\frac{e^{2y}}{2}\,dy=\frac{e^{2y}}{4}\Big|_0^4=\underline{\frac{e^8-1}{4}}$$