Discrete Math I
Exam I (1/12/12) – Page 1

Name: ______________________________

Instructions: Provide all steps necessary to solve the problem. Simplify your answer as much as possible. Additionally, clearly indicate the value or expression that is your final answer! Please put your name on both pages. Clearly distinguish answers of 'T' from answers of 'F'.

1. Determine the truth value of the following statement:

If $1 + 1 = 2$ or $1 + 1 = 3$, then $2 + 2 = 3$ and $2 + 2 = 4$.

2. What is the contrapositive of the statement "If $0 = 1$, then I am rich"?

3. Let p and q be propositions. Exactly one of the following compound propositions is a tautology. Circle the one that is a tautology.

(*a*) $p \to (q \wedge p)$

(*b*) $p \to \neg(q \wedge p)$

(*c*) $p \to (q \vee p)$

(*d*) $\neg[q \vee (q \to p)]$

4. Let p, q, and r be propositions. Determine if the compound proposition $[(p \oplus q) \wedge r] \to (p \vee q)$ is a tautology, a contradiction, or a contingency. Be sure to justify your answer. [***Instructor's Note:*** Recall that $\oplus$ is the symbol for *exclusive or*.]

5. Suppose $P(x,y)$ is the propositional function "$x + 2y = xy$", where x and y are integers. Determine the truth value of each statement.

(*a*) $P(1,-1)$

(*b*) $P(0,0)$

(*c*) $\exists x\, P(x,2)$

(*d*) $\exists y\, P(3,y)$

(*e*) $\forall x\, \exists y\, P(x,y)$

(*f*) $\exists x\, \forall y\, P(x,y)$

6. Suppose $P(x,y)$ is a propositional function and the universe of discourse for the variables x and y is $\{1,2,3\}$. Suppose $P(1,3)$, $P(2,2)$, $P(2,3)$, $P(3,1)$, $P(3,2)$ are true, and $P(x,y)$ is false otherwise. Determine the truth values of the following statements.

(*a*) $\forall x\, \exists y\, P(x,y)$

(*b*) $\exists x\, \forall y\, P(x,y)$

(*c*) $\exists x\, \exists y\, [P(x,y) \wedge P(y,x)]$

7. Suppose that the domain of discourse for all variables is the set of all animals. Consider the following propositional functions:

$D(x)$: x is a dog $\qquad$ $R(y)$: y is a rabbit $\qquad$ $C(x,y)$: x chases y

Exactly one of the following is an appropriate translation of the statement "Only dogs chase rabbits". Circle the one that is an appropriate translation.

(*a*) $\forall y\, \exists x\, [R(y) \wedge C(x,y) \to D(x)]$

(*b*) $\exists y\, \forall x\, [R(y) \wedge C(x,y) \to D(x)]$

(*c*) $\forall y\, \forall x\, [R(y) \wedge C(x,y) \to D(x)]$

(*d*) $\exists y\, \exists x\, [R(y) \wedge C(x,y) \to D(x)]$

8. Suppose the variable x represents students and the variable y represents courses, and

$A(y)$: y is an advanced course $\qquad$ $F(x)$: x is a freshman $\qquad$ $T(x,y)$: x is taking y

Write the each of the following statements using the predicates and any needed quantifiers.

(*a*) Eric is taking MTH 281.

(*b*) All students are freshman.

(*c*) There is a course that every freshman is taking.

(*d*) No freshman is taking an advanced course.

Discrete Math I
Exam I (1/12/12) – Page 2

Name: ______________________________

Instructions: Provide all steps necessary to solve the problem. Simplify your answer as much as possible. Additionally, clearly indicate the value or expression that is your final answer! Please put your name on both pages. Clearly distinguish answers of 'T' from answers of 'F'.

9. A function f defined on **R** (the set of real numbers) is said to be ***continuous*** at $x = 1$ if it satisfies the following statement:

For each $a > 0$, there exists a $b > 0$ such that, for all $x \in \mathbf{R}$, if $|x - 1| < b$, then $|f(x) - f(1)| < a$.

(*a*) Express this statement using quantifiers.

(*b*) Take your answer for **(*a*)** and negate it. Write this proposition without any negation symbols.

10. Consider the argument given to the right.

- If the argument if valid, provide a valid proof of the result (that is, use the laws of logical equivalences and the rules of inference to demonstrate that the conclusion is valid).
- If the argument is not valid, provide specific truth values of p, q, r, and s in which the premises are true, but the conclusion is false.

$\neg p \to \neg q$
$q \vee r$
$\neg(s \wedge r)$

$\therefore\ p$

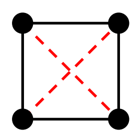
11. Let p and q be propositions. Suppose you wish to prove a theorem of the form "if p then q".

(***a***) If you give a direct proof, what assumption (or assumptions) is (are) made at the beginning of the proof and what needs to be established?

(***b***) If you give a proof of the contrapositive, what assumption (or assumptions) is (are) made at the beginning of the proof and what needs to be established?

(***c***) If you give a proof by contradiction, what assumption (or assumptions) is (are) made at the beginning of the proof and what needs to be established?

Extra Credit: How can four *distinct* points be arranged in the plane so that the six distances between pairs of points take on only two different values? [*Instructor's Note:* One way is given to the right.]



(***a***) Draw all possible arrangements. [WARNING! Ambiguous answers will not be graded. Clearly draw your arrangements and be sure to indicate which sides have equal length.]

(***b***) How many distinctly different arrangements did you draw?

EQUIVALENCES AND IMPLICATION EQUIVALENCES

Double negation law: $\neg(\neg p) \equiv p$

Identity laws: $p \vee \mathbf{F} \equiv p$, $p \wedge \mathbf{T} \equiv p$

Domination laws: $p \vee \mathbf{T} \equiv \mathbf{T}$, $p \wedge \mathbf{F} \equiv \mathbf{F}$

Negation laws: $p \vee \neg p \equiv \mathbf{T}$, $p \wedge \neg p \equiv \mathbf{F}$

Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$

Commutative laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$

Associative laws: $p \vee (q \vee r) \equiv (p \vee q) \vee r$,
$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$,
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption laws: $p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$

DeMorgan's laws: $\neg(p \vee q) \equiv \neg p \wedge \neg q$,
$\neg(p \wedge q) \equiv \neg p \vee \neg q$

1. $p \to q \equiv \neg p \vee q$
2. $p \to q \equiv \neg q \to \neg p$
3. $p \vee q \equiv \neg p \to q$
4. $p \wedge q \equiv \neg(p \to \neg q)$
5. $\neg(p \to q) \equiv p \wedge \neg q$
6. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
7. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
8. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
9. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$
10. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
11. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
12. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
13. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

RULES OF INFERENCE

p

$\therefore\ p \vee q$ (*Addition*)

$p \wedge q$

$\therefore\ p$ (*Simplification*)

p
q

$\therefore\ p \wedge q$ (*Conjunction*)

p
$p \to q$

$\therefore\ q$ (*Modus ponens*)

$\neg q$
$p \to q$

$\therefore\ \neg p$ (*Modus tollens*)

$p \to q$
$q \to r$

$\therefore\ p \to r$ (*Hypothetical syllogism*)

$p \vee q$
$\neg p$

$\therefore\ q$ (*Disjunctive syllogism*)

$p \vee q$
$\neg p \vee r$

$\therefore\ q \vee r$ (*Resolution*)

Discrete Math I – Solutions to Exam 1

1. This statement is false. The statements 1+1=2 and 2+2=4 are true while 1+1=3 and 2+2=3 are false. Therefore, the overall statement is $(T \vee F) \rightarrow (F \wedge T) \equiv T \rightarrow F \equiv F$.

2. "*If I am not rich, then* $0 \neq 1$."

3. The tautology is **(*c*)**.

(*a*) Not a tautology because if $p \equiv T$ and $q \equiv F$, then

$$p \rightarrow (q \wedge p) \equiv T \rightarrow (F \wedge T) \equiv T \rightarrow F \equiv F.$$

(*b*) Not a tautology because if $p \equiv T$ and $q \equiv T$, then

$$p \rightarrow \neg(q \wedge p) \equiv T \rightarrow \neg(T \wedge T) \equiv T \rightarrow \neg T \equiv T \rightarrow F \equiv F.$$

(*d*) Not a tautology because if $p \equiv T$ and $q \equiv T$, then

$$\neg[q \vee (q \rightarrow p)] \equiv \neg[T \vee (T \rightarrow T)] \equiv \neg[T \vee T] \equiv \neg T \equiv F.$$

4. The proposition is a tautology. The only way the proposition can be false is if **(1)** $(p \oplus q) \wedge r \equiv T$ and **(2)** $p \vee q \equiv F$. The only way **(2)** is true is if $p \equiv F$ and $q \equiv F$. However, these truth values imply $p \oplus q \equiv F$ and so **(1)** would be $(p \oplus q) \wedge r \equiv F$, no matter the truth value of r.

5. [***Instructor's Note:*** The problem would have been slightly more difficult by choosing the domain to be the set of reals.]

(*a*) $P(1,-1) \equiv (1 - 2 = -1)$ is TRUE.

(*b*) $P(0,0) \equiv (0 = 0)$ is TRUE.

(*c*) $\exists x\, P(x,2) \equiv \exists x\, (x + 4 = 2x) \equiv \exists x\, (x = 4)$ is TRUE.

(*d*) $\exists y\, P(3,y) \equiv \exists y\, (3 + 2y = 3y) \equiv \exists y\, (3 = y)$ is TRUE.

(*e*) $\forall x\, \exists y\, P(x,y)$ is FALSE when $x = 2$. In this case, $\exists y\, P(2,y) \equiv \exists y\, (2 + 2y = 2y) \equiv \exists y\, (2 = 0)$. This also fails for any $x \neq 3$, $x \neq -1$ because $y \notin \mathbf{Z}$.

(*f*) $\exists x\, \forall y\, P(x,y) \equiv \exists x\, \forall y\, (x+2y = xy) \equiv \exists x\, \forall y\, (y = x / (x - 2))$ is FALSE since y has a unique value (for any $x \neq 2$) so the proposition can't be true for all y.

6. **(*a*)** T; **(*b*)** F; **(*c*)** T

7. The answer is **(*c*)**.

(*a*) Translates to "If there exists an animal that chases every rabbit, then that animal is a dog."

(*b*) Translates to "There exists a rabbit that only dogs chase."

(*c*) Translates to "For every rabbit, if it is being chased, it is being chased by a dog."

(*d*) Translates to "There exists an animal such that if that animal is a rabbit, then there exists a dog that chases it."

[***Instructor's Note:*** Of the 34 students taking the exam, we have the following distribution.

	(*a*)	**(*b*)**	**(*c*)**	**(*d*)**
Number of students that answered	11	7	9	7

Based on this, I would say that problem 7 was, by far, the hardest problem on the exam.]

8.

(*a*) T(Eric, MTH 281)

(*b*) $\forall x\, F(x)$

(*c*) $\exists y\, \forall x\, [F(x) \rightarrow T(x,y)]$

[***Instructor's Note:*** This was the next hardest problem on the exam. Since it was a free response, I wanted to clarify two of the incorrect answers.

- $\forall x\, \exists y\, [F(x) \rightarrow T(x,y)]$ translates as "All freshmen are taking a course (but not necessarily the same course)."

- $\exists y\ \forall x\ [F(x) \wedge T(x,y)]$ translates as "There exists a class such that all students are taking the class and all students are freshmen."

There were lots of other incorrect answers, but these were the ones that occurred most often.]

(*d*) $\forall x\ \forall y\ \neg\ [F(x) \wedge A(y) \wedge T(x,y)]$

[***Instructor's Note:*** There were many answers logically equivalent to the answer provided. They are

$$\begin{aligned} &\forall x\ \forall y\ \neg\ [F(x) \wedge A(y) \wedge T(x,y)] \\ &\quad \equiv \forall x\ \forall y\ \ [(F(x) \wedge A(y)) \rightarrow \neg T(x,y)] \\ &\quad \equiv \forall x\ \forall y\ \ [T(x,y) \rightarrow \neg\ (F(x) \wedge A(y))] \end{aligned}$$

You can produce many more by rearranging any of the propositions $F(x)$, $A(y)$, and $T(x,y)$ in the first line.]

9.

(*a*)

Answer #1: If you give yourself the option to specify the universe of discourse for each variable in the quantifiers, then we get the following proposition.

$$\forall a{\in}\mathbf{R}^+\ \exists b{\in}\mathbf{R}^+\ \forall x{\in}\mathbf{R}\ [|x-1| < b \rightarrow |f(x) - f(1)| < a]$$

Answer #2: Without the presumption of the previous answer, we need to specify that a and b can only be positive by the following.

$$\begin{aligned} &\forall a\ \exists b\ \forall x\ [(a > 0 \wedge b > 0 \wedge |x-1| < b) \rightarrow |f(x) - f(1)| < a] \\ &\quad \equiv \forall a\ \exists b\ \forall x\ [(a > 0 \wedge b > 0) \rightarrow (|x-1| < b) \rightarrow |f(x) - f(1)| < a)] \\ &\quad \equiv \forall a\ \exists b\ [(a > 0 \wedge b > 0) \rightarrow \forall x\ (|x-1| < b) \rightarrow |f(x) - f(1)| < a)] \end{aligned}$$

(*b*) [*Instructor's Note:* For this problem, grading was based not so much as comparing to the correct answer, but from your answer in part **(*a*)**. If you changed the quantifiers and made an attempt at negating the proposition using the laws of logical equivalences and negating the inequalities correctly, then a higher percentage was awarded.]

Answer #1:

$$\begin{aligned} &\neg\ \forall a{\in}\mathbf{R}^+\ \exists b{\in}\mathbf{R}^+\ \forall x{\in}\mathbf{R}\ [|x-1| < b \rightarrow |f(x) - f(1)| < a] \\ &\quad \equiv \exists a{\in}\mathbf{R}^+\ \forall b{\in}\mathbf{R}^+\ \exists x{\in}\mathbf{R}\ \neg[|x-1| < b \rightarrow |f(x) - f(1)| < a] \\ &\quad \equiv \exists a{\in}\mathbf{R}^+\ \forall b{\in}\mathbf{R}^+\ \exists x{\in}\mathbf{R}\ [\ |x-1| < b \wedge \neg(|f(x) - f(1)| < a)\ ] && \text{[Law 5]} \\ &\quad \equiv \exists a{\in}\mathbf{R}^+\ \forall b{\in}\mathbf{R}^+\ \exists x{\in}\mathbf{R}\ [\ |x-1| < b \wedge |f(x) - f(1)| \geq a\ ] \end{aligned}$$

Answer #2: Only the negation of the first proposition in Answer #2 for part **(*a*)** is presented.

$$\begin{aligned} &\neg\forall a\ \exists b\ \forall x\ [(a > 0 \wedge b > 0 \wedge |x-1| < b) \rightarrow |f(x) - f(1)| < a] \\ &\quad \equiv \exists a\ \forall b\ \exists x\ \neg[(a > 0 \wedge b > 0 \wedge |x-1| < b) \rightarrow |f(x) - f(1)| < a] \\ &\quad \equiv \exists a\ \forall b\ \exists x\ [(a > 0 \wedge b > 0 \wedge |x-1| < b) \wedge \neg(|f(x) - f(1)| < a)\ ] && \text{[Law 5]} \\ &\quad \equiv \exists a\ \forall b\ \exists x\ [(a > 0 \wedge b > 0 \wedge |x-1| < b) \wedge |f(x) - f(1)| \geq a\ ] \end{aligned}$$

10. The argument is not valid. The lone counterexample is when $p \equiv q \equiv s \equiv$ F and $r \equiv$ T.

[*Instructor's Note :* This was actually a typo on my part. I had intended this to be a valid proof. The argument should have been as follows:

$\neg p \rightarrow \neg q$
$q \vee r$
$\neg(s \vee r)$

$\therefore\ p$

I messed up one symbol and it changed the proof. Anyway, the solution should have been as follows:

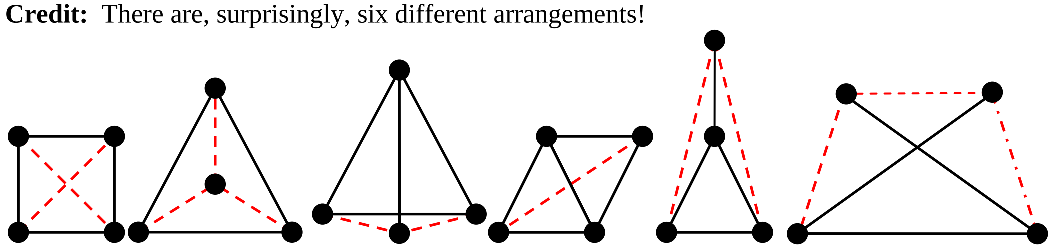
Step	**Reason**
1. $\neg(s \vee r)$	Hypothesis
2. $\neg s \wedge \neg r$	Line 1; DeMorgan's law
3. $\neg r \wedge \neg s$	Line 2; Commutative law
4. $\neg r$	Line 3; Simplification
5. $q \vee r$	Hypothesis
6. $r \vee q$	Line 5; Commutative law
7. q	Lines 4 and 6; Disjunctive Syllogism
8. $\neg p \rightarrow \neg q$	Hypothesis
9. $q \rightarrow p$	Line 8; Implication Equivalence #2 (Contrapositive)
10. p	Lines 5 and 7; Modus Ponens

I'll have the chance to ask something like this again on the final exam.]

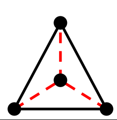
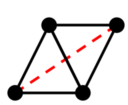
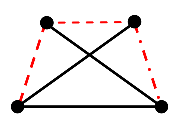
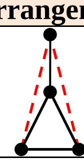
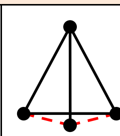
11.

(*a*) Suppose p, establish q.
(*b*) Suppose $\neg q$, establish $\neg p$.
(*c*) Suppose $\neg q$ and p (or simply $\neg q \wedge p$), establish $\neg p$ (or any contradiction).

Extra Credit: There are, surprisingly, six different arrangements!



[*Instructor's Note:* Please note that to receive credit for any drawing, you had to provide an indication of the lengths of the distances (this includes drawing line segments to represent all distances). Any vague answers were disregarded. Even if the intent of the drawing was fairly obvious, you had to follow the directions to receive credit.]

Number of Students that Provided an Allowable Arrangement					
Arrangement					
Number of Students	16	14	3	2	0

Number of Correct Answers Per Student					
Number of Correct Arrangements	1	2	3	4	5
Extra Credit	0.5	1	1.5	2	2.5
Number of Students	10	8	3	0	0

Discrete Math I – Exam #1 (page 1)

P	**Answer/Solution**	*A %*	*M %*	**O**
1	FALSE	*95*	*100*	**2**
2	"*If I am not rich, then* $0 \neq 1$."	*95*	*100*	**4**
3	The tautology is **(*c*)**.	*97*	*100*	**4**
4	The proposition is a tautology.	*90*	*100*	**5**
5a	T	*97*	*100*	**1.5**
5b	T	*100*	*100*	**1.5**
5c	T	*92*	*100*	**2**
5d	T	*97*	*100*	**2**
5e	F	*66*	*100*	**2**
5f	T	*100*	*100*	**2**
6a	T	*92*	*100*	**2**
6b	F	*68*	*100*	**2**
6c	T	*87*	*100*	**2**
7	**(*c*)**	*34*	*10*	**4**
8a	T(Eric, MTH 281)	*78*	*100*	**3.5**
8b	$\forall x\ F(x)$	*97*	*100*	**3.5**
8c	$\exists y\ \forall x\ [F(x) \rightarrow T(x,y)]$	*62*	*73*	**3.5**
8d	$\forall x\ \forall y\ \neg\ [F(x) \wedge A(y) \wedge T(x,y)]$	*61*	*60*	**3.5**
Overall		***82***	***84***	**50**

Discrete Math I – Exam #1 (page 2)

P	Answer/Solution	*A %*	*M %*	O
name		*86*	*100*	**1**
9a	$\forall a{\in}\mathbf{R}^+\ \exists b{\in}\mathbf{R}^+\ \forall x{\in}\mathbf{R}\ [\lvert x-1\rvert < b \rightarrow \lvert f(x)-f(1)\rvert < a]$ or $\forall a\ \exists b\ \forall x\ [(a>0 \wedge b>0 \wedge \lvert x-1\rvert < b) \rightarrow \lvert f(x)-f(1)\rvert < a]$	*86*	*85*	**8**
9b		*62*	*64*	**3**
10	The argument is not valid. The lone counterexample is when $p \equiv q \equiv s \equiv$ F and $r \equiv$ T.	*75*	*71*	**6**
11a	Suppose p, establish q.	*67*	*70*	**4**
11b	Suppose $\neg q$, establish $\neg p$.	*67*	*70*	**3**
11c	Suppose $\neg q$ and p, establish $\neg p$ (or any contradiction).	*55*	*55*	**2**
Overall		***72***	***74***	**27**
ec	{21 students received all or some of the credit available for this problem}			**2.5**