An Alternative Formulation for Transportation Problem

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1 Notations

We are going to need the following notations:

Feas($\cdot$): Feasible region (set of feasible solutions) for the given problem.
Opt($\cdot$): Set of optimal solutions for the given problem.

It is obvious that for any optimization problem P, $\text{Opt}(P) \subseteq \text{Feas}(P)$.

2 Transportation Problem

Recall that we had 3 factories whose electricity should be supplied by 2 power plants for the Transportation Problem we discussed in the class. We formulated this problem as follows:

$$
\begin{aligned}
(H_1) \quad \min \quad & f(x) := \; 50x_{11} + 100x_{12} + 60x_{13} + 30x_{21} + 20x_{22} + 35x_{23} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} \leq 1000 && (1) \\
& x_{21} + x_{22} + x_{23} \leq 2200 && (2) \\
& x_{11} + x_{21} \geq 1500 && (3) \\
& x_{12} + x_{22} \geq 750 && (4) \\
& x_{13} + x_{23} \geq 750 && (5) \\
& x_{ij} \geq 0 \quad i = 1, 2; \; j = 1, 2, 3. && (6)
\end{aligned}
$$

Recall that (1) and (2) are capacity constraints and (3), (4) and (5) are demand constraints. We ask the following question.

Question 1. *Can we rewrite demand constraints (3), (4) and (5) as equalities (=) but still have the same optimal value?*

The answer to Question 1 is "Yes", but how can we show it mathematically? If we rewrite our new problem, it will be

$$
\begin{array}{lll}
(H_2) & \min & 50x_{11} + 100x_{12} + 60x_{13} + 30x_{21} + 20x_{22} + 35x_{23} \\
& \text{s.t.} & (1), (2) \text{ and } (3) \\
& & x_{11} + x_{21} = 1500 \qquad (7) \\
& & x_{12} + x_{22} = 750 \qquad (8) \\
& & x_{13} + x_{23} = 750 \qquad (9) \\
& & x_{ij} \geq 0 \quad i = 1, 2; \; j = 1, 2, 3.
\end{array}
$$

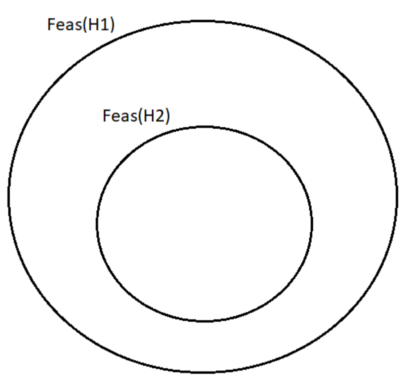
Now, we are going to need the following propositions.

Proposition 1. $\text{Feas}(H_2) \subseteq \text{Feas}(H_1)$, *i.e., a feasible solution of* H_2 *will always be a feasible solution to* H_1*.*

Proof. This can easily be proven. Let us pick an arbitrary $\hat{x}$ such that $\hat{x} \in \text{Feas}(H_2)$. Observe that $\hat{x}$ already satisfies all constraints of (H_1). Therefore $\hat{x} \in \text{Feas}(H_1)$. Hence, we showed that $\text{Feas}(H_2) \subseteq \text{Feas}(H_1)$. □

Question 2. *What is the implication of* Proposition 1*?*

It implies that feasible region of H_2 is included in the feasible region of H_1. We can easily illustrate this relationship by Venn diagrams as follows:



Proposition 2. $\text{Opt}(H_1) \subseteq \text{Feas}(H_2)$, *i.e., an optimal solution of* H_1 *will always be a feasible solution to* H_2*.*

Proof. Let x^* be an optimal solution of (H_1), i.e., $x^* \in \text{Opt}(H_1)$, such that

$$x^* = (x^*_{11},\ x^*_{21},\ x^*_{12},\ x^*_{22},\ x^*_{13},\ x^*_{23}).$$

We want to show that $x^* \in \text{Feas}(H_2)$, i.e., x^* is a feasible solution to H_2. Observe that x^* already satisfies all constraints in (H_1). We need to show x^*

satisfies (7), (8) and (9) as well. We are going to use proof by contradiction. For a contradiction, suppose x^* does not satisfy at least one of the constraints (7), (8) or (9). Then we will have the following possible cases:

(i) only (7) is not satisfed.

(ii) only (8) is not satisfied.

(iii) only (9) is not satisfied.

(iv) only (7) and (8) are not satisfied.

(v) only (7) and (9) are not satisied.

(vi) only (8) and (9) are not satisfied.

(vii) all (7), (8) and (9) are not satisfied.

However, since (7), (8) and (9) have completely different variables, by proving one case, the same argumentation can be extended to all other cases. Therefore, without loss of generality assume case (i) holds, i.e., only (7) is not satisfied. Then, since x^* satisfies (3), we can write

$$x_{11}^* + x_{21}^* = 1500 + \varepsilon,$$

where $\varepsilon > 0$. Now observe that we can always pick $\alpha \geq 0$, $\beta \geq 0$, $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ in such a way that

$$\begin{aligned} x_{11}^* &= \alpha + \varepsilon_1 \\ x_{21}^* &= \beta + \varepsilon_2, \end{aligned}$$

where $\alpha + \beta = 1500$ and $\varepsilon_1 + \varepsilon_2 = \varepsilon$. Now, let us define another solution as follows:

$$\hat{x} := (\alpha,\ \beta,\ x_{12}^*,\ x_{22}^*,\ x_{13}^*,\ x_{23}^*).$$

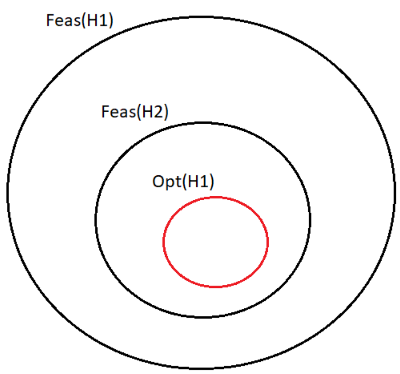
We will show that $\hat{x} \in \mathrm{Feas}(H_1)$. Observe that $\hat{x}$ satisfies capacity constraints (1) and (2) and demand constraints (4) and (5). It also satisfies (3) since $\alpha+\beta = 1500$. Since all entries of $\hat{x}$ are nonnegative, it also satisfies nonnegativity constraint (6). Therefore, we showed that $\hat{x} \in \mathrm{Feas}(H_1)$. Now, if we compare the objective values of $\hat{x}$ and x^*

$$f(\hat{x}) = f(x^*) - \underbrace{50\varepsilon_1}_{>0} - \underbrace{100\varepsilon_2}_{>0} \implies f(\hat{x}) < f(x^*).$$

This is not possible since, in a minimization problem, a feasible solution cannot give less objective function value than an optimal solution. Therefore, this contradicts with the fact that x^* is an optimal solution for (H_1). The similar argumentation can be done for all other cases. Therefore, by contradiction, we showed that any optimal solution H_1 will be a feasible solution to H_2. $\square$

Question 3. *What is the implication of* Proposition 2*?*

After Proposition 2, we will have the following relationship:



The following corollary will be a natural result of Proposition 1 and 2, but its proof is left as an exercise.

Corollary 1. $\mathrm{Opt}(H_1) = \mathrm{Opt}(H_2)$.

Proof. Prove it as an exercise. $\square$

Question 4. *After proving the above propositions and corollary, which problem would you prefer to solve,* (H_1) *or* (H_2)*?*

To answer Question 4 easily, think of the following scenario:

- You are given the task to search for the highest mountain in the Turkey.
- One of your trustworthy friends gave you the following tip: "Highest mountain in Turkey is in city Ağrı".

Do you still search for the highest mountain in the whole region of Turkey, or do you search for it only in the city Ağrı?