

# Discrete Mathematics

## Relations and Functions

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## Topics

### Relations

Introduction  
Relation Properties  
Equivalence Relations

### Functions

Introduction  
Pigeonhole Principle  
Recursion

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## Relation

### Definition

**relation:**  $\alpha \subseteq A \times B \times C \times \cdots \times N$

- ▶ **tuple:** an element of a relation

- ▶  $\alpha \subseteq A \times B$ : *binary relation*
- ▶  $a\alpha b$  is the same as  $(a, b) \in \alpha$

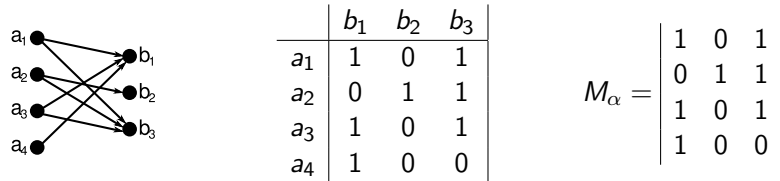
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## Relation Example

### Example

$A = \{a_1, a_2, a_3, a_4\}, B = \{b_1, b_2, b_3\}$

$\alpha = \{(a_1, b_1), (a_1, b_3), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3), (a_4, b_1)\}$



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## Relation Composition

### Definition

**relation composition:**

let  $\alpha \subseteq A \times B, \beta \subseteq B \times C$

$\alpha\beta = \{(a, c) \mid a \in A, c \in C, \exists b \in B [a\alpha b \wedge b\beta c]\}$

### Example



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## Relation Composition

- ▶  $M_{\alpha\beta} = M_\alpha \times M_\beta$
- ▶ using logical operations:  
 $1 : T \quad 0 : F \quad \cdot : \wedge \quad + : \vee$

### Example

$$M_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad M_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad M_{\alpha\beta} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

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## Relation Composition Associativity

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

$$\begin{aligned} & (a, d) \in (\alpha\beta)\gamma \\ \Leftrightarrow & \exists c [(a, c) \in \alpha\beta \wedge (c, d) \in \gamma] \\ \Leftrightarrow & \exists c [\exists b [(a, b) \in \alpha \wedge (b, c) \in \beta] \wedge (c, d) \in \gamma] \\ \Leftrightarrow & \exists b [(a, b) \in \alpha \wedge \exists c [(b, c) \in \beta \wedge (c, d) \in \gamma]] \\ \Leftrightarrow & \exists b [(a, b) \in \alpha \wedge (b, d) \in \beta\gamma] \\ \Leftrightarrow & (a, d) \in \alpha(\beta\gamma) \end{aligned}$$

□

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## Relation Composition Theorems

- ▶ let  $\alpha, \delta \subseteq A \times B$ , and  
let  $\beta, \gamma \subseteq B \times C$
- ▶  $\alpha(\beta \cup \gamma) = \alpha\beta \cup \alpha\gamma$
- ▶  $\alpha(\beta \cap \gamma) \subseteq \alpha\beta \cap \alpha\gamma$
- ▶  $(\alpha \cup \delta)\beta = \alpha\beta \cup \delta\beta$
- ▶  $(\alpha \cap \delta)\beta \subseteq \alpha\beta \cap \delta\beta$
- ▶  $(\alpha \subseteq \delta \wedge \beta \subseteq \gamma) \Rightarrow \alpha\beta \subseteq \delta\gamma$

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## Relation Composition Theorems

$$\alpha(\beta \cup \gamma) = \alpha\beta \cup \alpha\gamma.$$

$$\begin{aligned} & (a, c) \in \alpha(\beta \cup \gamma) \\ \Leftrightarrow & \exists b [(a, b) \in \alpha \wedge (b, c) \in (\beta \cup \gamma)] \\ \Leftrightarrow & \exists b [(a, b) \in \alpha \wedge ((b, c) \in \beta \vee (b, c) \in \gamma)] \\ \Leftrightarrow & \exists b [((a, b) \in \alpha \wedge (b, c) \in \beta) \\ & \vee ((a, b) \in \alpha \wedge (b, c) \in \gamma)] \\ \Leftrightarrow & (a, c) \in \alpha\beta \vee (a, c) \in \alpha\gamma \\ \Leftrightarrow & (a, c) \in \alpha\beta \cup \alpha\gamma \end{aligned}$$

□

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## Converse Relation

### Definition

$$\alpha^{-1} = \{(b, a) \mid (a, b) \in \alpha\}$$

$$\text{▶ } M_{\alpha^{-1}} = M_{\alpha}^T$$

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## Converse Relation Theorems

- ▶  $(\alpha^{-1})^{-1} = \alpha$
- ▶  $(\alpha \cup \beta)^{-1} = \alpha^{-1} \cup \beta^{-1}$
- ▶  $(\alpha \cap \beta)^{-1} = \alpha^{-1} \cap \beta^{-1}$
- ▶  $\overline{\alpha}^{-1} = \overline{\alpha^{-1}}$
- ▶  $(\alpha - \beta)^{-1} = \alpha^{-1} - \beta^{-1}$
- ▶  $\alpha \subset \beta \Rightarrow \alpha^{-1} \subset \beta^{-1}$

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## Converse Relation Theorems

$$\overline{\alpha}^{-1} = \overline{\alpha^{-1}}.$$

$$\begin{aligned} & (b, a) \in \overline{\alpha}^{-1} \\ \Leftrightarrow & (a, b) \in \overline{\alpha} \\ \Leftrightarrow & (a, b) \notin \alpha \\ \Leftrightarrow & (b, a) \notin \alpha^{-1} \\ \Leftrightarrow & (b, a) \in \overline{\alpha^{-1}} \end{aligned}$$

□

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## Converse Relation Theorems

$$(\alpha \cap \beta)^{-1} = \alpha^{-1} \cap \beta^{-1}.$$

$$\begin{aligned} & (b, a) \in (\alpha \cap \beta)^{-1} \\ \Leftrightarrow & (a, b) \in (\alpha \cap \beta) \\ \Leftrightarrow & (a, b) \in \alpha \wedge (a, b) \in \beta \\ \Leftrightarrow & (b, a) \in \alpha^{-1} \wedge (b, a) \in \beta^{-1} \\ \Leftrightarrow & (b, a) \in \alpha^{-1} \cap \beta^{-1} \end{aligned}$$

□

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## Converse Relation Theorems

$$(\alpha - \beta)^{-1} = \alpha^{-1} - \beta^{-1}.$$

$$\begin{aligned} (\alpha - \beta)^{-1} &= (\alpha \cap \overline{\beta})^{-1} \\ &= \alpha^{-1} \cap \overline{\beta}^{-1} \\ &= \alpha^{-1} \cap \overline{\beta^{-1}} \\ &= \alpha^{-1} - \beta^{-1} \end{aligned}$$

□

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## Relation Composition Converse

### Theorem

$$(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$$

### Proof.

$$\begin{aligned} & (c, a) \in (\alpha\beta)^{-1} \\ \Leftrightarrow & (a, c) \in \alpha\beta \\ \Leftrightarrow & \exists b [(a, b) \in \alpha \wedge (b, c) \in \beta] \\ \Leftrightarrow & \exists b [(b, a) \in \alpha^{-1} \wedge (c, b) \in \beta^{-1}] \\ \Leftrightarrow & (c, a) \in \beta^{-1}\alpha^{-1} \end{aligned}$$

□

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## Relation Properties

- ▶ let  $\alpha \subseteq A \times A$
- ▶  $\alpha^n$ :  $\alpha\alpha \cdots \alpha$
- ▶ **identity relation**:  $E = \{(a, a) \mid a \in A\}$

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## Reflexivity

reflexive

$$\alpha \subseteq A \times A$$

$$\forall a [a\alpha a]$$

- ▶  $E \subseteq \alpha$
- ▶ nonreflexive:  
 $\exists a [\neg(a\alpha a)]$
- ▶ irreflexive:  
 $\forall a [\neg(a\alpha a)]$

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## Reflexivity Examples

Example

$$\mathcal{R}_1 \subseteq \{1, 2\} \times \{1, 2\}$$

$$\mathcal{R}_1 = \{(1, 1), (1, 2), (2, 2)\}$$

- ▶  $\mathcal{R}_1$  is reflexive

Example

$$\mathcal{R}_2 \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$$

$$\mathcal{R}_2 = \{(1, 1), (1, 2), (2, 2)\}$$

- ▶  $\mathcal{R}_2$  is nonreflexive

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## Reflexivity Examples

Example

$$\mathcal{R} \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$$

$$\mathcal{R} = \{(1, 2), (2, 1), (2, 3)\}$$

- ▶  $\mathcal{R}$  is irreflexive

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## Reflexivity Examples

### Example

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$\mathcal{R} = \{(a, b) \mid ab \geq 0\}$$

- $\mathcal{R}$  is reflexive

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## Symmetry

### symmetric

$$\alpha \subseteq A \times A$$

$$\forall a, b [(a = b) \vee (a\alpha b \wedge b\alpha a) \vee (\neg(a\alpha b) \wedge \neg(b\alpha a))]$$

$$\forall a, b [(a = b) \vee (a\alpha b \leftrightarrow b\alpha a)]$$

- $\alpha^{-1} = \alpha$
- asymmetric:  
 $\exists a, b [(a \neq b) \wedge ((a\alpha b \wedge \neg(b\alpha a)) \vee (\neg(a\alpha b) \wedge b\alpha a))]$
- antisymmetric:  
 $\forall a, b [(a = b) \vee (a\alpha b \rightarrow \neg(b\alpha a))]$   
 $\Leftrightarrow \forall a, b [(a = b) \vee \neg(a\alpha b) \vee \neg(b\alpha a)]$   
 $\Leftrightarrow \forall a, b [\neg(a\alpha b \wedge b\alpha a) \vee (a = b)]$   
 $\Leftrightarrow \forall a, b [(a\alpha b \wedge b\alpha a) \rightarrow (a = b)]$

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## Symmetry Examples

### Example

$$\mathcal{R} \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$$

$$\mathcal{R} = \{(1, 2), (2, 1), (2, 3)\}$$

- $\mathcal{R}$  is asymmetric

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## Symmetry Examples

### Example

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$\mathcal{R} = \{(a, b) \mid ab \geq 0\}$$

- $\mathcal{R}$  is symmetric

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## Symmetry Examples

### Example

$$\mathcal{R} \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$$

$$\mathcal{R} = \{(1, 1), (2, 2)\}$$

- $\mathcal{R}$  is symmetric and antisymmetric

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## Transitivity

### transitive

$$\alpha \subseteq A \times A$$

$$\forall a, b, c [(a\alpha b \wedge b\alpha c) \rightarrow (a\alpha c)]$$

- $\alpha^2 \subseteq \alpha$
- nontransitive:  
 $\exists a, b, c [(a\alpha b \wedge b\alpha c) \wedge \neg(a\alpha c)]$
- antitransitive:  
 $\forall a, b, c [(a\alpha b \wedge b\alpha c) \rightarrow \neg(a\alpha c)]$

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## Transitivity Examples

### Example

$$\mathcal{R} \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$$

$$\mathcal{R} = \{(1, 2), (2, 1), (2, 3)\}$$

- $\mathcal{R}$  is antitransitive

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## Transitivity Examples

### Example

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$\mathcal{R} = \{(a, b) \mid ab \geq 0\}$$

- $\mathcal{R}$  is nontransitive

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## Converse Relation Properties

### Theorem

*The reflexivity, symmetry and transitivity properties are preserved in the converse relation.*

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## Closures

► reflexive closure:

$$r_\alpha = \alpha \cup E$$

► symmetric closure:

$$s_\alpha = \alpha \cup \alpha^{-1}$$

► transitive closure:

$$t_\alpha = \bigcup_{i=1,2,3,\dots} \alpha^i = \alpha \cup \alpha^2 \cup \alpha^3 \cup \dots$$

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## Special Relations

### predecessor - successor

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$\mathcal{R} = \{(a, b) \mid a - b = 1\}$$

- irreflexive
- antisymmetric
- antitransitive

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## Special Relations

### adjacency

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$\mathcal{R} = \{(a, b) \mid |a - b| = 1\}$$

- irreflexive
- symmetric
- antitransitive

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## Special Relations

### strict order

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$\mathcal{R} = \{(a, b) \mid a < b\}$$

- ▶ irreflexive
- ▶ antisymmetric
- ▶ transitive

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## Special Relations

### partial order

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$\mathcal{R} = \{(a, b) \mid a \leq b\}$$

- ▶ reflexive
- ▶ antisymmetric
- ▶ transitive

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## Special Relations

### preorder

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$\mathcal{R} = \{(a, b) \mid |a| \leq |b|\}$$

- ▶ reflexive
- ▶ asymmetric
- ▶ transitive

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## Special Relations

### limited difference

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}, m \in \mathbb{Z}^+$$

$$\mathcal{R} = \{(a, b) \mid |a - b| \leq m\}$$

- ▶ reflexive
- ▶ symmetric
- ▶ nontransitive

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## Special Relations

### comparability

$$\mathcal{R} \subseteq \mathbb{U} \times \mathbb{U}$$

$$\mathcal{R} = \{(a, b) \mid (a \subseteq b) \vee (b \subseteq a)\}$$

- ▶ reflexive
- ▶ symmetric
- ▶ nontransitive

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## Special Relations

### sibling

- ▶ irreflexive
- ▶ symmetric
- ▶ transitive
- can a relation be symmetric, transitive and irreflexive?

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## Compatibility Relations

### Definition

compatibility relation:  $\gamma$

- ▶ reflexive
- ▶ symmetric
- ▶ when drawing, lines instead of arrows
- ▶ matrix representation as a triangle matrix
- ▶  $\alpha\alpha^{-1}$  is a compatibility relation

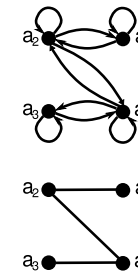
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## Compatibility Relation Example

### Example

$$A = \{a_1, a_2, a_3, a_4\}$$

$$\mathcal{R} = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_1, a_2), (a_2, a_1), (a_2, a_4), (a_4, a_2), (a_3, a_4), (a_4, a_3)\}$$



$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & & & \\ 0 & 0 & & \\ 0 & 1 & 1 & \end{vmatrix}$$

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## Compatibility Relation Example

### Example ( $\alpha\alpha^{-1}$ )

$P$ : persons,  $L$ : languages

$P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$

$L = \{l_1, l_2, l_3, l_4, l_5\}$

$\alpha \subseteq P \times L$

$$M_\alpha = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix}$$

$$M_{\alpha^{-1}} = \begin{vmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}$$

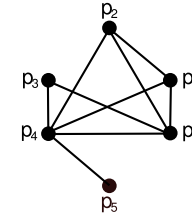
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## Compatibility Relation Example

### Example ( $\alpha\alpha^{-1}$ )

$\alpha\alpha^{-1} \subseteq P \times P$

$$M_{\alpha\alpha^{-1}} = \begin{vmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{vmatrix}$$



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## Compatibility Block

### Definition

**compatibility block:**  $C \subseteq A$

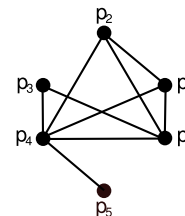
$\forall a, b [a \in C \wedge b \in C \rightarrow a\gamma b]$

- ▶ **maximal compatibility block:**  
not a subset of another compatibility block
- ▶ an element can be a member of more than one MCB
- ▶ **complete cover:**  $C_\gamma$   
set of all MCBs

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## Compatibility Block Example

### Example ( $\alpha\alpha^{-1}$ )



- ▶  $C_1 = \{p_4, p_6\}$
- ▶  $C_2 = \{p_2, p_4, p_6\}$
- ▶  $C_3 = \{p_1, p_2, p_4, p_6\}$  (MCB)

$$C_\gamma = \{ \{p_1, p_2, p_4, p_6\}, \\ \{p_3, p_4, p_6\}, \\ \{p_4, p_5\} \}$$

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## Equivalence Relations

### Definition

**equivalence relation:**  $\epsilon$

- ▶ reflexive
- ▶ symmetric
- ▶ transitive
- ▶ **equivalence classes (partitions)**
- ▶ every element is a member of exactly one equivalence class
- ▶ complete cover:  $C_\epsilon$

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## Equivalence Relation Example

### Example

$$\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$\mathcal{R} = \{(a, b) \mid \exists m \in \mathbb{Z} [a - b = 5m]\}$$

- ▶  $\mathcal{R}$  partitions  $\mathbb{Z}$  into 5 equivalence classes

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## References

### Required Reading: Grimaldi

- ▶ Chapter 5: Relations and Functions
  - ▶ 5.1. **Cartesian Products and Relations**
- ▶ Chapter 7: Relations: The Second Time Around
  - ▶ 7.1. **Relations Revisited: Properties of Relations**
  - ▶ 7.4. **Equivalence Relations and Partitions**

### Supplementary Reading: O'Donnell, Hall, Page

- ▶ Chapter 10: Relations

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## Functions

### Definition

**function:**  $f : X \rightarrow Y$

$$\forall x \in X \forall y_1, y_2 \in Y [(x, y_1), (x, y_2) \in f \rightarrow y_1 = y_2]$$

- ▶  $X$ : **domain**,  $Y$ : **codomain** (or *range*)
- ▶  $y = f(x)$  is the same as  $(x, y) \in f$
- ▶  $y$  is the *image* of  $x$  under  $f$
- ▶ let  $f : X \rightarrow Y$  and  $X_1 \subseteq X$   
*subset image:*  $f(X_1) = \{f(x) \mid x \in X_1\}$

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## Subset Image Examples

### Example

$$f : \mathbb{R} \rightarrow \mathbb{R}$$
$$f(x) = x^2$$

$$f(\mathbb{Z}) = \{0, 1, 4, 9, 16, \dots\}$$

$$f(\{-2, 1\}) = \{1, 4\}$$

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## Function Properties

### Definition

$f : X \rightarrow Y$  is **one-to-one** (or **injective**):

$$\forall x_1, x_2 \in X [f(x_1) = f(x_2) \rightarrow x_1 = x_2]$$

### Definition

$f : X \rightarrow Y$  is **onto** (or **surjective**):

$$\forall y \in Y \exists x \in X [f(x) = y]$$

$$\blacktriangleright f(X) = Y$$

### Definition

$f : X \rightarrow Y$  is **bijective**:

$f$  is one-to-one and onto

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## One-to-one Function Examples

### Example

$$f : \mathbb{R} \rightarrow \mathbb{R}$$
$$f(x) = 3x + 7$$

$$\begin{aligned} f(x_1) &= f(x_2) \\ \Rightarrow 3x_1 + 7 &= 3x_2 + 7 \\ \Rightarrow 3x_1 &= 3x_2 \\ \Rightarrow x_1 &= x_2 \end{aligned}$$

### Counterexample

$$g : \mathbb{Z} \rightarrow \mathbb{Z}$$
$$g(x) = x^4 - x$$

$$\begin{aligned} g(0) &= 0^4 - 0 = 0 \\ g(1) &= 1^4 - 1 = 0 \end{aligned}$$

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## Onto Function Examples

### Example

$$f : \mathbb{R} \rightarrow \mathbb{R}$$
$$f(x) = x^3$$

### Counterexample

$$f : \mathbb{Z} \rightarrow \mathbb{Z}$$
$$f(x) = 3x + 1$$

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## Function Composition

### Definition

let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$

$$g \circ f : X \rightarrow Z$$

$$(g \circ f)(x) = g(f(x))$$

- ▶ function composition is not commutative
- ▶ function composition is associative:  
 $f \circ (g \circ h) = (f \circ g) \circ h$

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## Function Composition Examples

### Example (commutativity)

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = x^2$$

$$g : \mathbb{R} \rightarrow \mathbb{R}$$

$$g(x) = x + 5$$

$$g \circ f : \mathbb{R} \rightarrow \mathbb{R}$$

$$(g \circ f)(x) = x^2 + 5$$

$$f \circ g : \mathbb{R} \rightarrow \mathbb{R}$$

$$(f \circ g)(x) = (x + 5)^2$$

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## Composite Function Theorems

### Theorem

let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$

$f$  is one-to-one  $\wedge$   $g$  is one-to-one  $\Rightarrow g \circ f$  is one-to-one

Proof.

$$\begin{aligned} (g \circ f)(a_1) &= (g \circ f)(a_2) \\ \Rightarrow g(f(a_1)) &= g(f(a_2)) \\ \Rightarrow f(a_1) &= f(a_2) \\ \Rightarrow a_1 &= a_2 \end{aligned}$$

□

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## Composite Function Theorems

### Theorem

let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$

$f$  is onto  $\wedge$   $g$  is onto  $\Rightarrow g \circ f$  is onto

Proof.

$$\forall z \in Z \exists y \in Y g(y) = z$$

$$\forall y \in Y \exists x \in X f(x) = y$$

$$\Rightarrow \forall z \in Z \exists x \in X g(f(x)) = z$$

□

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## Identity Function

### Definition

**identity function:**  $1_X$

$$1_X : X \rightarrow X$$

$$1_X(x) = x$$

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## Inverse Function

### Definition

$f : X \rightarrow Y$  is **invertible**:

$$\exists f^{-1} : Y \rightarrow X [f^{-1} \circ f = 1_X \wedge f \circ f^{-1} = 1_Y]$$

►  $f^{-1}$ : **inverse** of function  $f$

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## Inverse Function Examples

### Example

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = 2x + 5$$

$$f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$$

$$f^{-1}(x) = \frac{x-5}{2}$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(2x + 5) = \frac{(2x+5)-5}{2} = \frac{2x}{2} = x$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{x-5}{2}\right) = 2\frac{x-5}{2} + 5 = (x-5) + 5 = x$$

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## Inverse Function

### Theorem

*If a function is invertible, its inverse is unique.*

### Proof.

let  $f : X \rightarrow Y$

let  $g, h : Y \rightarrow X$  such that:

$$g \circ f = 1_X \wedge f \circ g = 1_Y$$

$$h \circ f = 1_X \wedge f \circ h = 1_Y$$

$$h = h \circ 1_Y = h \circ (f \circ g) = (h \circ f) \circ g = 1_X \circ g = g$$

□

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## Invertible Function

### Theorem

*A function is invertible if and only if it is one-to-one and onto.*

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## Invertible Function

If invertible then one-to-one.

$$f : A \rightarrow B$$

$$\begin{aligned} f(a_1) &= f(a_2) \\ \Rightarrow f^{-1}(f(a_1)) &= f^{-1}(f(a_2)) \\ \Rightarrow (f^{-1} \circ f)(a_1) &= (f^{-1} \circ f)(a_2) \\ \Rightarrow 1_A(a_1) &= 1_A(a_2) \\ \Rightarrow a_1 &= a_2 \end{aligned}$$

If invertible then onto.

$$f : A \rightarrow B$$

$$\begin{aligned} b \\ &= 1_B(b) \\ &= (f \circ f^{-1})(b) \\ &= f(f^{-1}(b)) \end{aligned}$$

□

□

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## Invertible Function

If bijective then invertible.

$$f : A \rightarrow B$$

- ▶  $f$  is onto  $\Rightarrow \forall b \in B \exists a \in A f(a) = b$
- ▶ let  $g : B \rightarrow A$  be defined by  $a = g(b)$
- ▶ is it possible that  $g(b) = a_1 \neq a_2 = g(b)$  ?
- ▶ this would mean:  $f(a_1) = b = f(a_2)$
- ▶ but  $f$  is one-to-one

□

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## Pigeonhole Principle

### Definition

**pigeonhole principle** (Dirichlet drawers):

If  $m$  pigeons go into  $n$  holes and  $m > n$ , then at least one hole contains more than one pigeon.

- ▶ let  $f : X \rightarrow Y$   
 $|X| > |Y| \Rightarrow f$  is not one-to-one
- ▶  $\exists x_1, x_2 \in X [x_1 \neq x_2 \wedge f(x_1) = f(x_2)]$

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## Pigeonhole Principle Examples

### Example

- ▶ Among 367 people, at least two have the same birthday.
- ▶ In an exam where the grades are integers between 0 and 100, how many students have to take the exam to make sure that at least two students will have the same grade?

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## Generalized Pigeonhole Principle

### Definition

**generalized pigeonhole principle:**

If  $m$  objects are distributed to  $n$  drawers, then at least one of the drawers contains  $\lceil m/n \rceil$  objects.

### Example

Among 100 people, at least 9 ( $\lceil 100/12 \rceil$ ) were born in the same month.

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## Pigeonhole Principle Example

### Theorem

*In any subset of cardinality 6 of the set  $S = \{1, 2, 3, \dots, 9\}$  there are two elements which total 10.*

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## Pigeonhole Principle Example

### Theorem

*Let  $S$  be a set of positive integers smaller than or equal to 14, with cardinality 6. The sums of the elements in all nonempty subsets of  $S$  cannot be all different.*

### Proof Trial

$A \subseteq S$

$s_A$  : sum of the elements of  $A$

- ▶ holes:  
 $1 \leq s_A \leq 9 + \dots + 14 = 69$
- ▶ pigeons:  $2^6 - 1 = 63$

### Proof.

look at the subsets for which  $|A| \leq 5$

- ▶ holes:  
 $1 \leq s_A \leq 10 + \dots + 14 = 60$
- ▶ pigeons:  $2^6 - 2 = 62$

□

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## Pigeonhole Principle Example

### Theorem

Among any 101 elements chosen from the set  $S = \{1, 2, 3, \dots, 200\}$  there are at least two numbers such that one divides the other.

### Proof Method

- ▶ we first show that  $\forall n \exists! p [n = 2^r p \wedge r \in \mathbb{N} \wedge \exists t \in \mathbb{Z} [p = 2t + 1]]$
- ▶ then, by using this theorem we prove the main theorem

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## Pigeonhole Principle Example

### Theorem

$\forall n \exists! p [n = 2^r p \wedge r \in \mathbb{N} \wedge \exists t \in \mathbb{Z} [p = 2t + 1]]$

### Proof of existence.

$n = 1$ :  $r = 0, p = 1$

$n \leq k$ : assume  $n = 2^r p$

$n = k + 1$ :

$n = 2$ :  $r = 1, p = 1$

$n$  prime ( $n > 2$ ):  $r = 0, p = n$

$\neg(n \text{ prime})$ :  $n = n_1 n_2$

$n = 2^{r_1} p_1 \cdot 2^{r_2} p_2$

$n = 2^{r_1+r_2} \cdot p_1 p_2$

### Proof of uniqueness.

if not unique:

$$\begin{aligned} n &= 2^{r_1} p_1 = 2^{r_2} p_2 \\ \Rightarrow 2^{r_1-r_2} p_1 &= p_2 \\ \Rightarrow 2 \mid p_2 \end{aligned}$$

□

□

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## Pigeonhole Principle Example

### Theorem

Among any 101 elements chosen from the set  $S = \{1, 2, 3, \dots, 200\}$  there are at least two numbers such that one divides the other.

### Proof.

- ▶  $T = \{t \mid t \in S, \exists i \in \mathbb{Z} [t = 2i + 1]\}, |T| = 100$
- ▶ let  $f : S \rightarrow T$  and  $r \in \mathbb{N}$   
 $s = 2^r t \rightarrow f(s) = t$ 
  - ▶ if 101 elements are chosen from  $S$ , at least two of them will have the same image in  $T$ :  $f(s_1) = f(s_2) \Rightarrow 2^{m_1} t = 2^{m_2} t$

$$\frac{s_1}{s_2} = \frac{2^{m_1} t}{2^{m_2} t} = 2^{m_1-m_2}$$

□

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## Recursive Functions

### Definition

**recursive function**: a function defined in terms of itself

$f(n) = h(f(m))$

- ▶ *inductively defined function*: a recursive function where the size is reduced at every step

$$f(n) = \begin{cases} k & \text{if } n = 0 \\ h(f(n-1)) & \text{if } n > 0 \end{cases}$$

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## Recursion Examples

### Example

$$f_{91}(n) = \begin{cases} n - 10 & \text{if } n > 100 \\ f_{91}(f_{91}(n + 11)) & \text{if } n \leq 100 \end{cases}$$

### Example (factorial)

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n - 1) & \text{if } n > 0 \end{cases}$$

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## Euclid Algorithm

### Example (greatest common divisor)

$$\gcd(a, b) = \begin{cases} b & \text{if } b \mid a \\ \gcd(b, a \bmod b) & \text{if } b \nmid a \end{cases}$$

$$\begin{aligned} \gcd(333, 84) &= \gcd(84, 333 \bmod 84) \\ &= \gcd(84, 81) \\ &= \gcd(81, 84 \bmod 81) \\ &= \gcd(81, 3) \\ &= 3 \end{aligned}$$

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## Fibonacci Sequence

### Fibonacci sequence

$$F_n = \text{fib}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ \text{fib}(n - 2) + \text{fib}(n - 1) & \text{if } n > 2 \end{cases}$$

$F_1$	$F_2$	$F_3$	$F_4$	$F_5$	$F_6$	$F_7$	$F_8$	...
1	1	2	3	5	8	13	21	...

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## Fibonacci Sequence

### Theorem

$$\sum_{i=1}^n F_i^2 = F_n \cdot F_{n+1}$$

### Proof.

$$\begin{aligned} n = 2 : \quad \sum_{i=1}^2 F_i^2 &= F_1^2 + F_2^2 = 1 + 1 = 1 \cdot 2 = F_2 \cdot F_3 \\ n = k : \quad \sum_{i=1}^k F_i^2 &= F_k \cdot F_{k+1} \\ n = k + 1 : \quad \sum_{i=1}^{k+1} F_i^2 &= \sum_{i=1}^k F_i^2 + F_{k+1}^2 \\ &= F_k \cdot F_{k+1} + F_{k+1}^2 \\ &= F_{k+1} \cdot (F_k + F_{k+1}) \\ &= F_{k+1} \cdot F_{k+2} \end{aligned}$$

□

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## Ackermann Function

### Ackermann function

$$ack(x, y) = \begin{cases} y + 1 & \text{if } x = 0 \\ ack(x - 1, 1) & \text{if } y = 0 \\ ack(x - 1, ack(x, y - 1)) & \text{if } x > 0 \wedge y > 0 \end{cases}$$

## References

### Required Reading: Grimaldi

- ▶ Chapter 5: Relations and Functions
  - ▶ 5.2. Functions: Plain and One-to-One
  - ▶ 5.3. Onto Functions: Stirling Numbers of the Second Kind
  - ▶ 5.5. The Pigeonhole Principle
  - ▶ 5.6. Function Composition and Inverse Functions

### Supplementary Reading: O'Donnell, Hall, Page

- ▶ Chapter 11: Functions