Signals & Systems

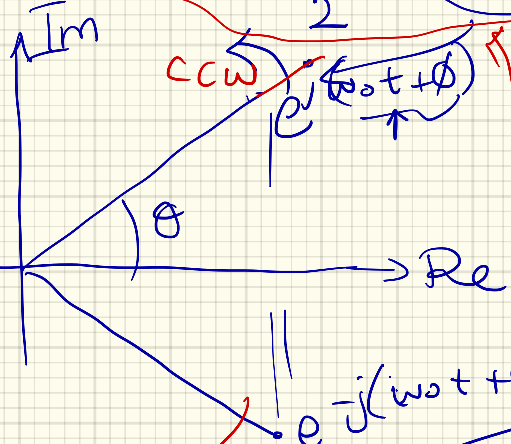
week 2

12.02.2018

A sinusoid signal:

$$\boxed{A\cos(\omega_0 t+\phi) = \frac{A}{2}\left(\underbrace{e^{j(\omega_0 t+\phi)}} + \underbrace{e^{-j(\omega_0 t+\phi)}}\right)}$$

Also write it $A\sin(\omega_0 t+\phi)$ as addition of two complex exponential signals



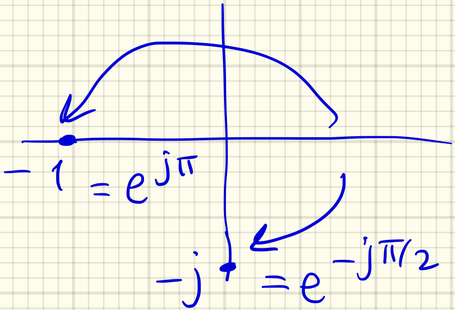
Know this very well:

Euler formula:

$$x(t) = Ae^{j(\omega_0 t+\phi)} = \underbrace{A\cos(\omega_0 t+\phi)} + j\,\underbrace{A\sin(\omega_0 t+\phi)}$$

$$Re\{x(t)\} + j\,Im\{x(t)\}$$

ex: Evaluate $x(t) = \mathrm{Re}\{-3\underbrace{j}\, e^{j\omega t}\}$



$-j = e^{-j\pi/2}$

$= \mathrm{Re}\{3\, \underbrace{e^{-j\pi/2}\, e^{j\omega t}}\}$

$= \mathrm{Re}\{3 e^{j(\omega t - \pi/2)}\}$

$x(t) = 3\cos(\omega t - \frac{\pi}{2})$

exercise: $\underbrace{\cos(a+b)} = \mathrm{Re}\{e^{j(a+b)}\} = \ldots$

Derive

exercise $\sin(2a) = ?$ $\mathrm{Im}\{\underbrace{e^{j2a}}_{e^{ja}\cdot e^{ja}}\} \ldots$

$= \ldots$

Adding Sinusoidal Signals w/ SAME FREQUENCY using Phasors

ex: $x_1(t) = 3\cos\left(2\pi t + \frac{\pi}{4}\right)$

$x_2(t) = 4\cos\left(2\pi t - \frac{\pi}{3}\right)$

same freq.

$x(t) = x_1(t) + x_2(t) = ?$

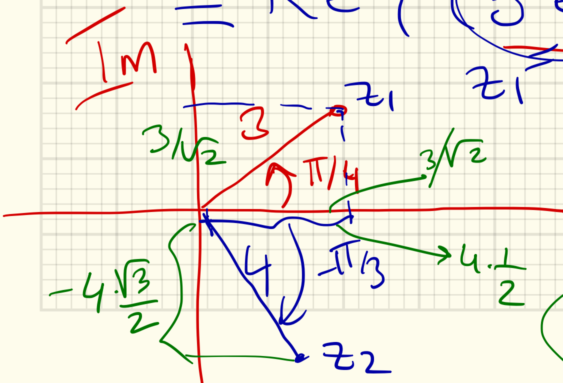
$$x(t) = \underbrace{\boxed{Ae^{j\phi}}}_{\text{phasor: complex amplitude}} e^{j\omega t}$$

Use their phasor representation

$$x(t) = \mathrm{Re}\left\{3e^{j(2\pi t + \pi/4)} + 4e^{j(2\pi t - \pi/3)}\right\}$$

$$= \mathrm{Re}\left\{\left(\underbrace{3e^{j\pi/4}}_{z_1} + \underbrace{4e^{-j\pi/3}}_{z_2}\right)e^{j2\pi t}\right\}$$

Addition of 2 phasors



$$z_1 + z_2 \simeq 4.12 - j1.34 \simeq 4.3e^{-j\theta}$$

$$\theta = \operatorname{atan}\left(\frac{-1.34}{4.12}\right) \simeq -\frac{\pi}{10}$$

$$\Rightarrow x(t) \stackrel{\sim}{=} \text{Re}\{4.3\, e^{-j\frac{\pi}{10}}\, e^{j2\pi t}\}$$

$$x(t) = \underbrace{4.3}_{\text{different magnitude}} \cos\Big(\underbrace{2\pi t}_{\text{same frequency}} - \frac{\pi}{10}\Big) \longrightarrow \text{different phase}$$

<u>In general:</u>

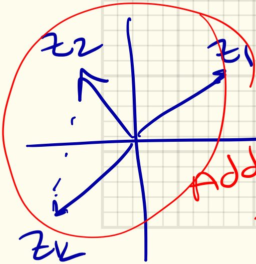
$$\sum_{k=1}^{N} A_k \underbrace{\cos(\omega_0 t + \phi_k)}_{} = A\cos(\omega_0 t + \phi)$$

$$\Big[A_k \cos(k\omega_0 t + \phi_k)\Big] \text{ later}$$

How to find A, ϕ?

$$x(t) = \sum_{k=1}^{N} \text{Re}\{A_k e^{j(\omega_0 t + \phi_k)}\}$$

$$= \text{Re}\Big\{\sum_{k=1}^{N} \underbrace{\{A_k e^{j\phi_k}\}}_{\sum z_k = z} e^{j\omega_0 t}\Big\} \text{ phasor repres. of each sinusoid}$$



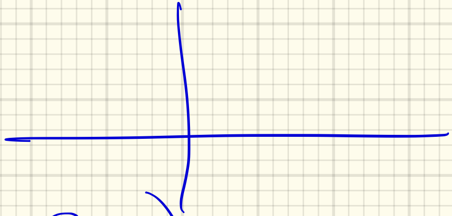
Add them up
find z

$$x(t) = \text{Re}\{A e^{j\phi} e^{j\omega_0 t}\}$$

exercise: $x(t) = 2\cos\left(300\pi t + \frac{3\pi}{4}\right) + 2\sqrt{2}\cos\left(300\pi(t + 0.005)\right)$

Do Phasor addition:

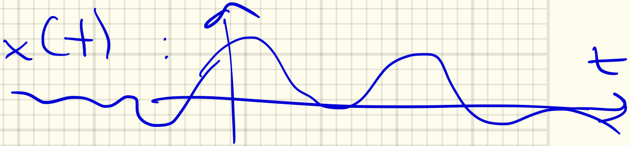
z-plane

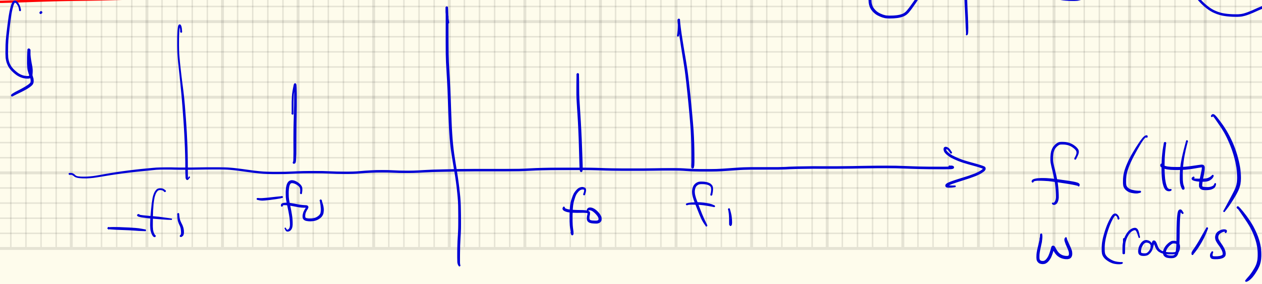


Result → $x(t) = 2\cos\left(300\pi t + 5\pi/4\right)$

CHAPTER 3 SPECTRUM REPRESENTATION

Spectrum: Given a signal $x(t)$:





(Frequency) Spectrum: (Def): A compact representation of "frequency content" of a signal.

added a constant signal

Ex: $x(t) = \underbrace{3\cos\left(4\pi t + \frac{\pi}{3}\right)}_{x_1(t)} + \underbrace{4\cos(6\pi t)}_{x_2(t)} + \underbrace{A}_{x_3(t)}$

$f_3 = 0\ \text{Hz}$

constant signals have ZERO freq. (DC component)

① $x_1(t) = \frac{3}{2} e^{+j(4\pi t + \pi/3)} + \frac{3}{2} e^{-j(4\pi t + \pi/3)}$

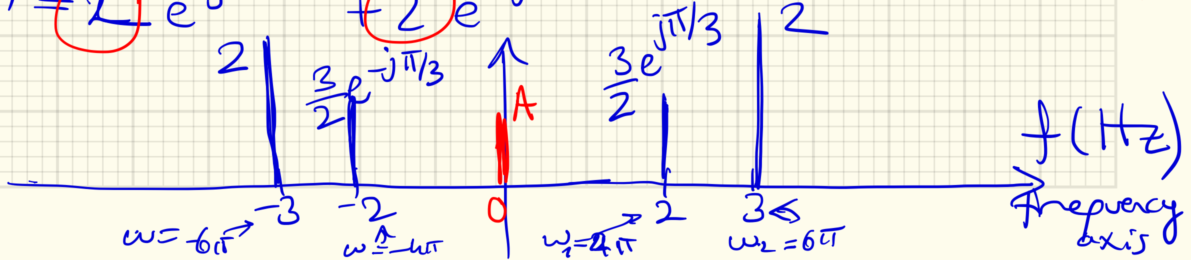
$f_1 = 2\ \text{Hz}$ $\qquad$ $f_{-1} = -2\ \text{Hz}$

$= \frac{3}{2} e^{j\pi/3} e^{j4\pi t} + \frac{3}{2} e^{-j\pi/3} e^{-j4\pi t}$

phasors

② $x_2(t) = 2 e^{j6\pi t} + 2 e^{-j6\pi t}$ $\qquad$ $2e^{j0}$

Plot the spectrum of $x(t)$

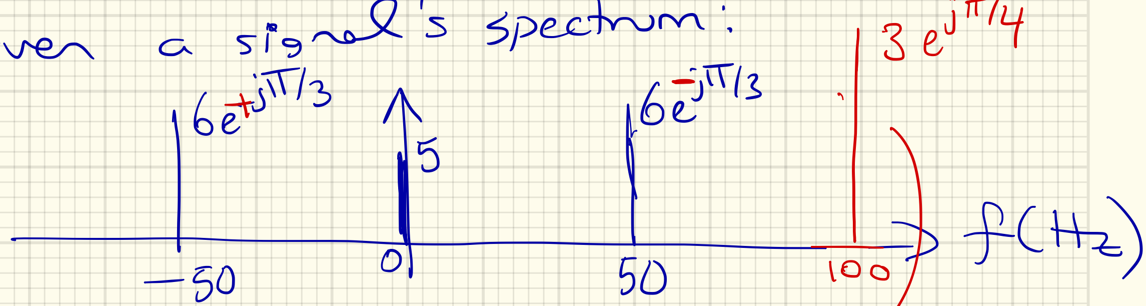


Note: Magnitude of the phasor $(Ae^{j\phi})$ should be positive.

$$-5e^{j\pi/3} = 5e^{j4\pi/3} = 5e^{-j2\pi/3}$$

replace w/ $e^{j\pi}$ or $e^{-j\pi}$

Ex: Given a signal's spectrum:



$$x(t) = ? \quad \underbrace{5 + 12\cos\left(100\pi t - \frac{\pi}{3}\right)}_{\text{real signal.}} \underbrace{+ 3e^{j\pi/4} e^{j2\pi \cdot 100t}}_{\text{complex signal}}$$

$x(t) = 12\sin(100\pi t - \pi/3)$

$$= 12\left(\frac{e^{j(100\pi t - \pi/3)} - e^{-j(100\pi t - \pi/3)}}{2j}\right)$$

exercise.

plot the spectrum of this signal.

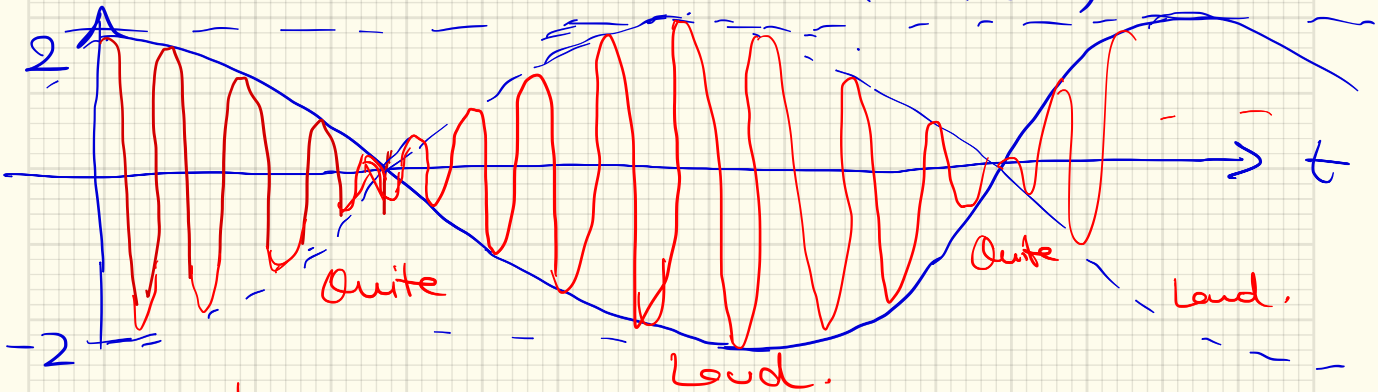
Beat Notes:

ex: $x(t) = \underbrace{2\cos(2\pi\, 20\, t)}\cdot \cos(2\pi\, 200\, t)$

$$x(t) = \cancel{2}\left(\frac{1}{\cancel{2}}e^{j2\pi 20t} + \frac{1}{\cancel{2}}e^{-j2\pi 20t}\right)\cdot\left(\frac{1}{2}e^{j2\pi 200t} + \frac{1}{2}e^{-j2\pi 200t}\right)$$

$$= \underbrace{\frac{1}{2}e^{j2\pi 220t} + \frac{1}{2}e^{-j2\pi 220t}} + \underbrace{\frac{1}{2}e^{j2\pi 180t} + \frac{1}{2}e^{-j2\pi 180t}}$$

$$x(t) = \cos(2\pi\, 220\, t) + \cos(2\pi\, 180\, t)$$



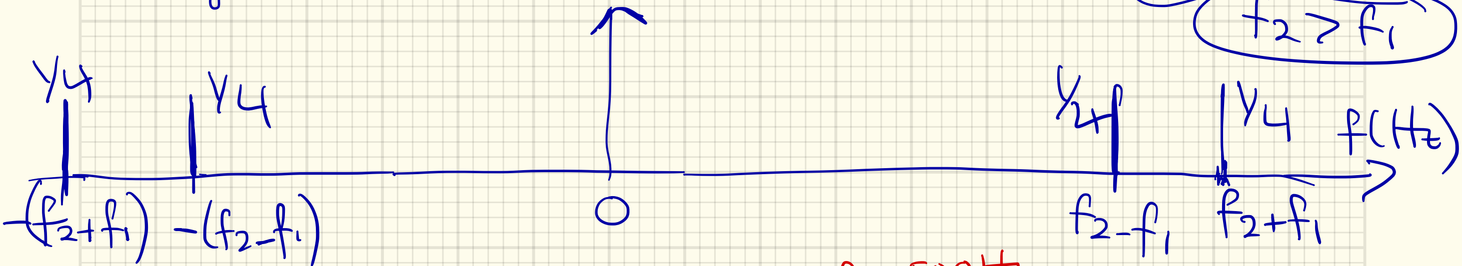
Beat notes

Summary of beat notes: A multiplication of 2 sinusoids or Addition of 2 sinusoids of slightly different frequency.

$$\underbrace{\cos(2\pi f_1 t)\cdot\cos(2\pi f_2 t)}\ = \frac{1}{2}\Big[\cos\big(2\pi(f_1+f_2)t\big) + \cos\big(2\pi(f_1-f_2)t\big)\Big]$$

$f_2 > f_1$

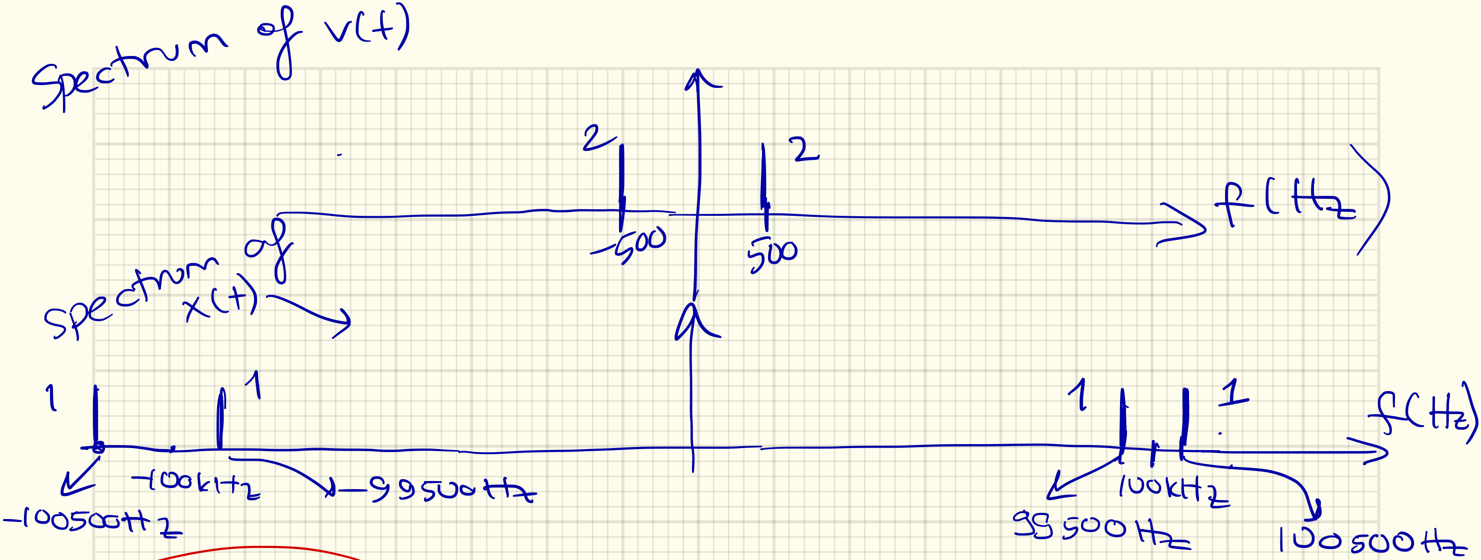
plot the Spectrum of beat note.



ex: $v(t) = 4\cos(1000\pi t)$ ← $f = 500\,Hz$ low freq. signal e.g. audio

$x(t) = v(t)\cos(200000\pi t)$ ← high freq. signal carrier signal

$f_c = 100\,kHz$



exercise: Do the math on x(t) use Euler identities, derive x(t).

Idea: Modulation: We carried frequency content of v(t) (low frequency) to around 100 kHz by modulation.

Concept of Fundamental Frequency:

Periodic signal

$$x(t+T_0) = x(t), \quad \forall t$$

w/ period T_0 (sec)

T_0: smallest period.

$$x(t+kT_0) = x(t), \quad \forall t$$

kT_0 is also a period of the signal

k integer.

Fundamental freq:

$$f_0 \triangleq \frac{1}{T_0} = \frac{1}{\text{sec}}$$

↳ We'll continue next time; Read relevant sections from Chapter 3 of the textbook