Lecture 2: Matrices

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MATRICES

Definition

An $m \times n$ matrix A is a rectangular array of mn scalars, i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}.$$

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- The scalar a_{ij} which is in the i-th row and j-th column of A is called the (i, j)-th entry of A.
- We write the matrix A as $A = [a_{ij}]_{m \times n}$.
- If all corresponding entries of $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal, then they are called an equal matrix.

Matrix Operations

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- **Transpose:** Let $A = [a_{ij}]_{m \times n}$. Then $A^T = [a_{ji}]_{n \times m}$.

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- **Transpose:** Let $A = [a_{ij}]_{m\times n}$. Then $A^T = [a_{ji}]_{n\times m}$.
- **Matrix multiplication:** Let $A = [a_{ij}]_{m\times p}$ and $B = [b_{ij}]_{p\times n}$. The product of A and B is defined by

$$AB = [c_{ij}]_{m\times n}, \text{ where } c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

Remark: The product of A and B is defined only when the number of columns of A is equal to the number of rows of B.

Properties of Matrix Operations

Example: Let $A = \begin{bmatrix} 1 & \mathbf{x} & 3 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \\ \mathbf{y} \end{bmatrix}$. If $AB = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$,

then find **x** and **y**.

Properties of Matrix Addition

- Let A, B, and C be $m \times n$ matrices.
  1. $A + B = B + A$ (commutative)
  2. $A + (B + C) = (A + B) + C$ (associative)
  3. $A + \mathbf{0} = A$ ($\mathbf{0}$ is $m \times n$ zero matrix)
  4. $A + (-A) = \mathbf{0}$ ($-A$ is the negative of A)

Properties of Scalar Multiplication

- Let A, B are matrices of the appropriate sizes and $r, s \in \mathbb{R}$ (scalars).
  1. $r\left(sA\right) = \left(rs\right)A$
  2. $\left(r+s\right)A = rA + sA$
  3. $r\left(A+B\right) = rA + rB$
  4. $A\left(rB\right) = r\left(AB\right) = \left(rA\right)B$.

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- **Remark 1:** Note that AB need not always equal BA!

  Example: Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \text{ while } BA = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}.$$

Properties of Matrix Multiplication

- **Remark 2:** For $a, b \in \mathbb{R}$, $ab = 0$ can hold only if $a = 0$ or $b = 0$. However, this is not true for matrices.
  Example: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$. Then neither A nor B is the zero matrix, but
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- **Remark 3:** For $a, b, c \in \mathbb{R}$, if $ab = ac$ and $a \neq 0$, then $b = c$. However, the cancellation law does not hold for matrices.
  Example: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$. Then
  $$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}, \text{ but } B \neq C.$$

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- A matrix A is called **symmetric** if $A^T = A$.

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  Example: $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 3 \end{bmatrix}$
- A matrix A is called **skew-symmetric** if $A^T = -A$.

  Example: $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

Non singular Matrix

Definition

Identity matrix: The matrix $I_n = [d_{ij}]_{n\times n}$ is called the $n \times n$ identity matrix whose entries satisfy the following rule:

$$d_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} .$$

Nonsingular matrix: The matrix $A = [a_{ij}]_{n\times n}$is called the nonsingular matrix if there exists a matrix $B = [b_{ij}]_{n\times n}$ such that $AB = BA = I_n$. The matrix B is called an inverse of A.

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$.

Since $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,then $B = A^{-1}$.

Properties of inverse of a matrix

- The inverse of a matrix is unique, if it exists.
- Let A and B be $n \times n$ nonsingular matrices.
  1. $\left(A^{-1}\right)^{-1} = A$
  2. $(AB)^{-1} = B^{-1}A^{-1}$
  3. $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$.

Non singular Matrix

Example: Let $A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$. Find A^{-1}, if it exist.

Non singular Matrix

Homework:

1) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that A is nonsingular if and only if $ad - bc \neq 0$.

2) Find a 2×2 matrix A such that $A^2 = I_2$.

3) Find a 2×2 matrix A such that $A^2 = \mathbf{0}$.

4) Suppose that A is nonsingular. Show that

$$\begin{aligned} AB &= AC \Rightarrow B = C \\ AB &= \mathbf{0} \Rightarrow B = \mathbf{0} \end{aligned}$$

Matrix representation of a linear system

Consider the linear system of m equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m. \end{aligned} \tag{1}$$

Define the matrices;

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}.$$

Then the linear system (1) can be written in matrix form as:

$$Ax = b$$

Matrix representation of a linear system

- The matrix A is called **coefficient matrix** of the linear system (1).
- The matrix $[A : b]$ which is obtained by adjoining column b to A is called **augmented matrix** of the linear system (1).

Example

Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 1 \\ x_1 - x_2 + x_3 &= 3. \end{aligned}$$

The linear system can be written in a matrix form as $Ax = b$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}_{3\times 3}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}, \quad b = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}_{3\times 1}.$$