Lecture 1: Sytems of Linear Equations

Elif Tan

Ankara University

Preliminaries

- Linear Equations and Matrices
- Solving Linear Systems
- Determinants
- (Real) Vector Spaces
- Linear Transformations
- Eigenvalues and Eigenvectors
- Inner Product Spaces

Textbook:

B. Kolman and D.R. Hill. Elemantery Linear Algebra with Applications. Pearson I.E. (9th Edition)

Systems of Linear Equations

Definition

A linear equation in n unknowns is an equation which has the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b. \tag{1}$$

Here $x_1, x_2, ..., x_n$ are unknowns, $a_1, a_2, ..., a_n$, and b are real or complex constants.

A solution of a linear equation is a sequence of $s_1, s_2, ..., s_n$, such that (1) is satisfied when

$$x_1 = s_1, x_2 = s_2, ..., x_n = s_n.$$

Systems of Linear Equations

Examples

A solution of the linear equation

$$3x_1 + 2x_2 - 5x_3 = -7$$

is $x_1 = 2, x_2 = 1$, and $x_3 = 3$.
The equations

$$2\sqrt{x_1} - 5x_3 = 12$$

and

$$3x_1x_2 + 7x_3 = -1$$

are not linear equations because they are not in the form (1).

Systems of Linear Equations

Definition

A system of linear equations is a set of m linear equations in n unknowns which has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m. \end{aligned} \tag{2}$$

Here a_{ij} are known constants.

A solution of the linear system (2) is a sequence $s_1, s_2, \ldots, s_n$, such that every equation in the system (2) is satisfied when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$.

Systems of Linear Equations

- If the linear system (2) has no solution, then it is called inconsistent.
- If the linear system (2) has one or more solutions, then it is called consistent.
- If $b_1 = b_2 = \cdots = b_n = 0$, then the linear system (2) is called a homogeneous system. Note that a solution to a homogeneous system such that $x_1 = x_2 = \cdots = x_n = 0$ is called a trivial solution, otherwise it is called a nontrivial solution.

Systems of Linear Equations

To find a solution to a linear system, we use the method of elimination which allow us to obtain a new linear system that has exactly the same solutions, but is easy to solve. We need the following 3 manipulations:

1. Interchange the i-th and j-th equations

Systems of Linear Equations

To find a solution to a linear system, we use the method of elimination which allow us to obtain a new linear system that has exactly the same solutions, but is easy to solve. We need the following 3 manipulations:

1. Interchange the i-th and j-th equations
2. Multiply an equation by a non zero constant

Systems of Linear Equations

To find a solution to a linear system, we use the method of elimination which allow us to obtain a new linear system that has exactly the same solutions, but is easy to solve. We need the following 3 manipulations:

1. Interchange the i-th and j-th equations
2. Multiply an equation by a non zero constant
3. Add a multiple of i-th equation to the another j-th equation.

Example

Consider the linear system

$$\begin{aligned} x+2y+3z &= 5 \\ 2x+y-3z &= 1 \\ x-y+z &= 3. \end{aligned}$$

If we add (-2) times the first equation to the second one and (-1) times the first equation to the third one, we get

$$\begin{aligned} -3y-9z &= -9 \\ -3y-2z &= -2. \end{aligned}$$

If we add (-1) times the first equation to the second one, we get $z = 1$.
Then substituting the value of z into the first equation of second linear system, we get $y = 0$.
Finally substituting these values of y and z into the first equation of first linear system, we find that $x = 2$.
So the system has a unique solution $x = 2$, $y = 0$, $z = 1$.

Systems of Linear Equations

Example

$$
\begin{aligned}
x + 2y - 3z &= -4 \\
2x + y - 3z &= 4
\end{aligned}
$$

Systems of Linear Equations

Example

$$\begin{aligned} 2x + y &= 4 \\ 4x + 2y &= 6 \end{aligned}$$

REMARK: These examples show that a linear system may have

- a unique solution

Systems of Linear Equations

REMARK: These examples show that a linear system may have

- a unique solution
- infinitely many solutions

REMARK: These examples show that a linear system may have

- a unique solution
- infinitely many solutions
- no solution

Homework: Solve each given linear systems by the method of elimination.

1)

$$\begin{aligned} 2x - 3y + 4z &= -12 \\ x - 2y + z &= -5 \\ 3x + y + 2z &= 1 \end{aligned}$$

2)

$$\begin{aligned} x + y &= 5 \\ 3x + 3y &= 10 \end{aligned}$$

3)

$$\begin{aligned} x + y - 2z &= 5 \\ 2x + 3y + 4z &= 2 \end{aligned}$$

Systems of Linear Equations

Example

Consider the linear system

$$\begin{aligned} 2x - y &= 5 \\ 4x - 2y &= \mathbf{t}. \end{aligned}$$

Determine the value(s) of **t** so that the system is consistent or inconsistent.

Systems of Linear Equations

Example

Consider the linear system

$$\begin{aligned} x + 2y &= 10 \\ 3x + (6 + \mathbf{t})\, y &= 30. \end{aligned}$$

Determine the value(s) of **t** so that the system has infinitely many solutions.
Determine the value(s) of **t** so that the system has a unique solution.