Elementary Matrices

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Matrix representation of a linear system

Consider the linear system of m equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m. \end{aligned}$$

Then this linear system can be written in matrix form as:

$$Ax = b$$

Solving Linear Systems

We can use echelon forms of the matrix to determine the solutions of a linear system. We have two methods for solving a linear sytem:

1. Gauss Elemination Method:

- Transform the augmented matrix $[A : b]$ to the row echelon form $[C : d]$ by using elemantary row operations
- Solve the corresponding linear system $[C : d]$ by using back substitution.

2. Gauss-Jordan Method:

- Transform the augmented matrix $[A : b]$ to the reduced row echelon form $[C : d]$ by using elemantary row operations
- Solve the corresponding linear system $[C : d]$ without back substitution.

If we transform the augmented matrix $[A : b]$ to the row echelon form $[C : d]$, then we have the following cases:

Case 1: If $\operatorname{rank}[C : d] = \operatorname{rank}[C] = r$, then the linear system is consistent.
(i) If $n = r$, then the system has a unique solution.
(ii) If $n > r$, then the system has infinitely many solutions depend on $n - r$ parameters.

Case 2: If $\operatorname{rank}[C : d] \neq \operatorname{rank}\,[C]$, the linear system has no solution (inconsistent).

Finding Inverse of a Matrix

Let A is an $n \times n$ square matrix. To find A^{-1}, if it exists, we transform the augmented matrix $[A : I_n]$ to the reduced row echelon form $[C : D]$.

- If $C = I_n$, then $D = A^{-1}$
- If $C \neq I_n$, then C has a row of zeros. (A is singular)

Definition

An elementary matrix E is an $n \times n$ matrix that can be obtained from the identity matrix I_n by one elementary row operation.

Examples:

1. $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ is an elementary matrix, since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{3r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is not an elementary matrix,

since $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[-r_3 \to r_3]{4r_2 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not an elementary matrix, since we do not have any row operation that gives this matrix.

4. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is not an elementary matrix, since it is not square matrix.

5. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix, since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3r_2 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Properties of Elementary Matrices

- All elementary matrices are invertible.
- The inverse of an elementary matrix is also an elementary matrix.
- Let B be the matrix obtained by using one elementary row operation to matrix A, and E be the corresponding elementary matrix. Then $EA = B$.

Example:

$$\begin{aligned} A \ &:= \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \xrightarrow{-3r_1+r_2 \to r_2} \begin{bmatrix} 1 & 4 \\ 0 & -7 \end{bmatrix} =: B \\ I_2 \ &:= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-3r_1+r_2 \to r_2} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} =: E \end{aligned}$$

Thus $EA = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -7 \end{bmatrix} = B.$

Properties of Elementary Matrices

Theorem

A is invertible $\Leftrightarrow$ *A is a product of elementary matrices.*

Example:

$$A \quad : \quad = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \xrightarrow{-3r_1+r_2 \to r_2} \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & -7 \end{bmatrix}}_{B}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-3r_1+r_2 \to r_2} \underbrace{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}}_{E_1}$$

$$B \quad : \quad = \begin{bmatrix} 1 & 4 \\ 0 & -7 \end{bmatrix} \xrightarrow{-\frac{1}{7}r_2 \to r_2} \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{C}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{7}r_2 \to r_2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix}}_{E_2}$$

$$C \quad : \quad = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \xrightarrow{-4r_2+r_1 \to r_2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-4r_2+r_1 \to r_2} \underbrace{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}}_{E_3}$$

Thus we have $E_1A = B$, $E_2B = C$, $E_3C = I_2$. Hence $E_3E_2E_1A = I_2$.

$$
\begin{aligned}
(E_3 E_2 E_1 A)\, A^{-1} &= I_2 A^{-1} \\
&\Rightarrow E_3 E_2 E_1 = A^{-1} \\
&\Rightarrow (E_3 E_2 E_1)^{-1} = \left(A^{-1}\right)^{-1} \\
&\Rightarrow E_1^{-1} E_2^{-1} E_3^{-1} = A \\
&\Rightarrow \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}.
\end{aligned}
$$

Homework: Express the matrix $A = \begin{bmatrix} -2 & 4 \\ 5 & -14 \end{bmatrix}$ as a product of elementary matrices.

Answer: $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$

Finding Inverse of a Matrix

Remark: For $n \times n$ matrix A, the followings are equivalent:

1. A is nonsingular, that is, A^{-1} exists.
2. A is row equivalent to I_n.
3. The linear system $Ax = b$ has a unique solution.
4. The homogenous linear system $Ax = 0$ has only zero (trivial) solution.
5. A is a product of elementary matrices.