Linear Transformations and Matrices

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Definition (Coordinate)

Let $S = \{v_1, v_2, ..., v_n\}$ be an **ordered basis** for the n-dimensional vector space $(V, \oplus, \odot)$. Then every vector v in V can be uniquely expressed in the form

$$v = c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_n \odot v_n$$

where $c_1, c_2, ..., c_n$ are scalars. The **coordinate vector of** v **with respect to the ordered basis** S is defined by

$$[v]_S := \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

The entries of $[v]_S$ are called the **coordinates** of v with respect to the basis S. Note that there is a one-to-one correspondence between v and $[v]_S$.

Example: $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for the vector space $\mathbb{R}^2$.

Every vector $v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ can be written uniquely as

$$\begin{bmatrix} x \\ y \end{bmatrix} = (2y - x) \odot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \oplus (x - y) \odot \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Thus the coordinates of v with respect to the basis S is

$$[v]_S := \begin{bmatrix} 2y - x \\ x - y \end{bmatrix}.$$

If we change the order of the basis, then the coordinates of v change as $\begin{bmatrix} x - y \\ 2y - x \end{bmatrix}$.

Definition (Transition Matrix)

Let $S = \{v_1, v_2, ..., v_n\}$ and $T = \{w_1, w_2, ..., w_n\}$ be an ordered basis for the n-dimensional vector space $(V, \oplus, \odot)$. The **transition matrix** from the basis T to S is defined by

$$P_{S \leftarrow T} = \left[[w_1]_S \, [w_2]_S \ldots [w_n]_S \right]_{n \times n}$$

and the coordinate vector of v wrt S can be written as

$$[v]_S = P_{S \leftarrow T} \, [v]_T \, .$$

Note that the transition matrix is nonsingular matrix and we have

$$P^{-1}_{S \leftarrow T} = P_{T \leftarrow S}.$$

Example

Consider the ordered basis for $\mathbb{R}^3$

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, w_3 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

Find the transition matrix from the basis T to S.

$$w_1 = a_1 \odot v_1 \oplus a_2 \odot v_2 \oplus a_3 \odot v_3 \Rightarrow [w_1]_S = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$w_2 = b_1 \odot v_1 \oplus b_2 \odot v_2 \oplus b_3 \odot v_3 \Rightarrow [w_2]_S = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$w_3 = c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus c_3 \odot v_3 \Rightarrow [w_3]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

$$P_{S \leftarrow T} = [[w_1]_S \, [w_2]_S \, [w_3]_S] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}.$$

Example: Consider the ordered basis for $\mathbb{R}^3$

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$T = \{w_1, w_2, w_3\}.$$

If the transition matrix from the basis T to S is

$$P_{S \leftarrow T} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix},$$

then find the basis T.

Since $P_{S \leftarrow T} = [[w_1]_S \, [w_2]_S \, [w_3]_S] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$,

$$\begin{aligned} w_1 &= a_1 \odot v_1 \oplus a_2 \odot v_2 \oplus a_3 \odot v_3 \Rightarrow [w_1]_S = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\ \Rightarrow \quad & w_1 = 1 \odot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \oplus 2 \odot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \oplus -1 \odot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \end{aligned}$$

Similarly $w_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $w_3 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$.

Linear Transformations

Definition (Linear Transformation)

Let $(V, \oplus, \odot)$ and $(W, \boxplus, \boxdot)$ be real vector spaces. $L : V \longrightarrow W$ is called a **linear transformation** if the following conditions holds:

$$(i)\ L(u \oplus v) = L(u) \boxplus L(v) \text{ for all } u, v \in V$$

$$(ii)\ L(c \odot u) = c \boxdot L(u) \text{ for all } u \in V \text{ and all } c \in \mathbb{R}.$$

Definition

A linear transformation $L : V \to W$ is called **one-to-one** if $L(v_1) = L(v_2)$ implies that $v_1 = v_2$ for $v_1, v_2 \in V$.
A linear transformation $L : V \to W$ is called **onto** if for each $w \in W, \exists v \in V$ such that $L(v) = w$.

Definition

Let $(V, \oplus, \odot)$ and $(W, \boxplus, \boxdot)$ be real vector spaces. L is called an **isomorphism** if $L : V \to W$ is a linear transformation that is one-to-one and onto. In this case the vector spaces V and W are called isomorphic and denoted by $V \cong W$.

Linear Transformations

Definition

Let $L : V \rightarrow W$ be a linear transformation.

- The kernel of L is defined by

$$KerL = \{v \in V \mid L(v) = 0_W\}.$$

- The range of L is defined by

$$RangeL = L(V) = \{w \in W \mid \exists v \in V; L(v) = w\}.$$

Theorem

Let $L : V \rightarrow W$ be a linear transformation. Then we have the following results:

1. $L(0_V) = 0_W$
2. $KerL < V$
3. *L is one-to-one* $\Leftrightarrow$ $KerL = \{0_V\}$
4. $RangeL < W$
5. *L is onto* $\Leftrightarrow$ $L(V) = W$.

Theorem (Rank-Nullity Theorem)

Let $L : V \rightarrow W$ be a linear transformation with $\dim V = n$, then

$$\dim V = \underbrace{\dim KerL}_{\text{"}\textbf{nullity}\text{"} \ of \ L} + \underbrace{\dim RangeL}_{\text{"}\textbf{rank}\text{"} \ of \ L}.$$

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$, $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + x_2 - 3x_3 \end{bmatrix}$ be a linear transformation. Find the rank of L.

Solution:

$$\begin{aligned} KerL &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = 0_W \right\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} 3x_3 \\ -3x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\}. \end{aligned}$$

Thus $\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for $KerL$ and $\dim KerL = 1$. Since

$$\dim V = \dim KerL + \dim RangeL,$$

then we have

$$3 = 1 + rankL.$$

Therefore $rankL = 2$.

Theorem

1. *Let V be an n-dimensional vector space. Then* $V \cong \mathbb{R}^n$.
2. *Let V and W be finite dimensional vector spaces.*
   $V \cong W \Leftrightarrow \dim V = \dim W$.

Example: $L : M_{22} \to \mathbb{R}^4$, $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is an isomorphism. So

$$M_{22} \cong \mathbb{R}^4.$$

Definition (Matrix representation of a linear transformation)

Let $L : V \to W$ be a linear transformation and consider the ordered basis $S = \{v_1, v_2, ..., v_n\}$ and $T = \{w_1, w_2, ..., w_m\}$ for the vector spaces V and W, respectively. The matrix representation of the linear transformation L with respect to the basis S and T is defined by

$$A = [[L(v_1)]_T \, [L(v_2)]_T \ldots [L(v_n)]_T]_{m \times n} .$$

Also for $v \in V$, we have

$$[L(v)]_T = A \, [v]_S .$$

Linear Transformations and Matrices

Theorem

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and consider the standard basis $\{e_1, e_2, ..., e_n\}$ for $\mathbb{R}^n$. Let $A = [L(e_1)\, L(e_2) \ldots L(e_n)]_{m \times n}$. The matrix A is the only matrix satisfiying the property;

$$L(x) = Ax, \text{ for } x \in \mathbb{R}^n.$$

It is called **the standard matrix representation of the linear transformation** *L.*

Remark: Linear transformation $L \leftrightarrow A$

- If A is $m \times n$ matrix, then there is a corresponding linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $L(x) = Ax$, for $x \in \mathbb{R}^n$.
- If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there is a corresponding $m \times n$ matrix A which is defined by $A = [L(e_1)\, L(e_2) \ldots L(e_n)]_{m \times n}$.

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$, $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + x_2 - 3x_3 \end{bmatrix}$ be a linear transformation and consider the standard basis

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

for the vector spaces $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.
Find the matrix representation of the linear transformation L with respect to the basis S and T.

Solution:

$$\begin{aligned}
L(v_1) &= a_1 \odot w_1 \oplus a_2 \odot w_2 \Rightarrow [L(v_1)]_T = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
L(v_2) &= b_1 \odot w_1 \oplus b_2 \odot w_2 \Rightarrow [L(v_2)]_T = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
L(v_3) &= c_1 \odot w_1 \oplus c_2 \odot w_2 \Rightarrow [L(v_3)]_T = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}
\end{aligned}$$

$$A = \left[[L(v_1)]_T \; [L(v_2)]_T \; [L(v_3)]_T \right]_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix}.$$

Example

Find the linear transformation which corresponds to the matrix
$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix}_{2\times 3}$.

Solution: The corresponding linear transformation is defined by
$L : \mathbb{R}^3 \to \mathbb{R}^2$,

$$\begin{aligned} L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) &= A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + x_2 - 3x_3 \end{bmatrix}. \end{aligned}$$

Linear Transformations and Matrices

- To find the rank of the linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, it is enough to check the rank of the matrix A. The rank of an $m \times n$ matrix A is the number of nonzero rows in the reduced row echelon form of the matrix A.

$$\begin{aligned} \dim \mathbb{R}^n &= \dim KerL + \dim L\left(\mathbb{R}^n\right) \\ n &= nullity\ \ A + rank\ \ A \end{aligned}$$

Linear Transformations and Matrices

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$$\begin{aligned} \dim \mathbb{R}^n &= \dim KerL + \dim L(\mathbb{R}^n) \\ n &= nullity\ A + rank\ A \end{aligned}$$

- If A is an $n \times n$ matrix, then we have

$$rank\ A = n \Leftrightarrow nullity\ A = 0 \Leftrightarrow \det A \neq 0 \Leftrightarrow A^{-1} \text{ exists.}$$

Example

Find the rank and the nullity of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix}$.

Solution: If we transform the matrix A to the reduced row echelon form, we have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} rankA &= \text{number of nonzero rows of the matrix A in the reduced ref.} \\ &= 2 \end{aligned}$$

$nullityA = n - rankA = 3 - 2 = 1.$

Linear Transformations and Matrices

Theorem

Let $L : V \to W$ be a linear transformation and consider the ordered basis $S = \{v_1, v_2, ..., v_n\}$ and $S^{'} = \{v_1', v_2', ..., v_n'\}$ for the vector space V, and $T = \{w_1, w_2, ..., w_m\}$ and $T' = \{w_1', w_2', ..., w_m'\}$ for the vector space W. Let the transition matrix from basis S' to S be P, and the transition matrix from basis T' to T be Q. If A is the matrix representation for the linear transformation L with respect to the basis S and T, then $Q^{-1}AP$ is the matrix representation for the linear transformation L with respect to the basis S' and T'.

Definition

Let A and B are $n \times n$ matrices, if there exist nonsingular matrix P such that $B = P^{-1}AP$, then it is called B is **similar** to A.

Theorem

If A and B are similar $n \times n$ matrices, then $rankA = rankB$.

Remark

For $n \times n$ matrix A, the followings are equivalent:

1. A is nonsingular, that is, A^{-1} exists.
2. A is row equivalent to I_n.
3. The linear system $Ax = b$ has a unique solution.
4. The homogenous linear system $Ax = 0$ has only zero (trivial) solution.
5. A is a product of elementary matrices.
6. $\det(A) \neq 0$.
7. The rank of A is n.
8. The nullity of A is zero.(Then the corresponding linear transformation is $1-1$)
9. The columns of A form a linearly independent set of vectors in $\mathbb{R}^n$.

Exercises

1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Then $L : \mathbb{R}^3 \to \mathbb{R}^2, L(x) = Ax$ is a linear transformation.
Since

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix},$$

The columns of A are not linearly independent. So L is not $1-1$.

2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $L : \mathbb{R}^2 \to \mathbb{R}^3, L(x) = Ax$ is a linear transformation.
Since

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

The columns of A are linearly independent. So L is $1-1$.

Exercises

3. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $L : \mathbb{R}^3 \to \mathbb{R}^3, L(x) = Ax$ is a linear transformation.

Since A is in reduced row echelon form, it can easily be seen that The columns of A are linearly dependent. So L is not $1-1$.