Lecture 3: Solving Linear Systems

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Matrix representation of a linear system

Consider the linear system of m equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m. \end{aligned} \tag{1}$$

Define the matrices;

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}.$$

Then the linear system (1) can be written in matrix form as:

$$Ax = b$$

Matrix representation of a linear system

- The matrix A is called **coefficient matrix** of the linear system (1).
- The matrix $[A : b]$ which is obtained by adjoining column b to A is called **augmented matrix** of the linear system (1).

Example

Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 1 \\ x_1 - x_2 + x_3 &= 3. \end{aligned}$$

The linear system can be written in a matrix form as $Ax = b$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}_{3\times 3}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}, \quad b = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}_{3\times 1}.$$

Echelon form of a matrix

Definition

An $m \times n$ matrix A is said to be in **reduced row echelon form** if it satisfies the followig rules:

1. All zero rows, if there are any, lie at the bottom of the matrix.
2. The first nonzero entry from the left of a non zero row is 1. (This entry is called a leading one of its row)
3. For each nonzero row, the leading one lies to the right and below of any leading ones in preceding row.
4. If a column contains a leading one, then all other entries in that column are zero.

If a matrix satisfies the conditions (1) , (2) and (3) we say that it is in **row echelon form**.

Echelon form of a matrix

Examples

The following matrices are not in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The following matrices are in row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 3 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The following matrices are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Echelon form of a matrix

Every matrix can be transform row (column) echelon form by means of row (column) operations:

Definition

An **elemantary row operations** on a matrix A are

1. Interchange the i-th and j-th rows $(r_i \leftrightarrow r_j)$
2. Multiply a row by a non zero constant $(kr_i \longrightarrow r_i)$
3. Add a multiple of i-th row to the another j-th row $(kr_i + r_j \longrightarrow r_j)$

Echelon form of a matrix

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$.

Echelon form of a matrix

Homework: Find the reduced row echelon form of the following matrices.

1) $$\begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

2) $$\begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 1 & 1 & 3 \end{bmatrix}$$

3) $$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ -2 & 9 & 4 \end{bmatrix}$$

Now we use echelon forms to determine the solutions of a linar system. We have two methods for solving a linear sytem:

1. **Gauss Elemination Method:**
    - Transform the augmented matrix $[A : b]$ to the row echelon form $[C : d]$ by using elemantary row operations
    - Solve the corresponding linear system $[C : d]$ by using back substitution.
2. **Gauss-Jordan Method:**
    - Transform the augmented matrix $[A : b]$ to the reduced row echelon form $[C : d]$ by using elemantary row operations
    - Solve the corresponding linear system $[C : d]$ without back substitution.

Example

Solve the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 1 \\ x_1 - x_2 + x_3 &= 3. \end{aligned}$$

Solution: We can write the above linear system in a matrix form

$$Ax = b \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}.$$

Solving Linear Systems

If we transform the augmented matrix of the linear system to the row echelon form, we get

$$[A:b] \approx \begin{bmatrix} 1 & 2 & 3 & : & 5 \\ 2 & 1 & -3 & : & 1 \\ 1 & -1 & 1 & : & 3 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 2 & 3 & : & 5 \\ 0 & 1 & 3 & : & 3 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}.$$

The corresponding linear system is

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ x_2 + 3x_3 &= 3 \\ x_3 &= 1. \end{aligned}$$

Then by using back substitution, the unique solution of the linear system is $x_1 = 2, x_2 = 0, x_3 = 1$.

To obtain the solution of the linear system by using Gauss-Jordan method, we transform the augmented matrix in a reduced row echlon form:

$$[A:b] \approx \left[\begin{array}{cccc} 1 & 2 & 3 & : & 5 \\ 2 & 1 & -3 & : & 1 \\ 1 & -1 & 1 & : & 3 \end{array}\right] \approx \cdots \approx \left[\begin{array}{ccccc} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 1 \end{array}\right].$$

Then the unique solution of the linear system is $x_1 = 2, x_2 = 0, x_3 = 1$.

Example

Solve the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ x_2 + 2x_3 + 3x_4 &= 1 \\ 2x_2 + 4x_3 + 6x_4 &= 3. \end{aligned}$$

Solution: If we transform the augmented matrix of the linear system to the row echelon form, we get

$$[A : b] \approx \begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 0 & 1 & 2 & 3 & : & 1 \\ 0 & 2 & 4 & 6 & : & 3 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 0 & 1 & 2 & 3 & : & 1 \\ 0 & 0 & 0 & 0 & : & 1 \end{bmatrix}.$$

Since the last equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$ can never be satisfied, the linear system has no solution.(It is inconsistent).

Example

Solve the linear system

$$\begin{aligned} x_1 + 2x_3 + 3x_4 + x_5 &= 5 \\ x_2 - x_3 + 3x_4 + 2x_5 &= 1 \\ x_3 + 3x_4 + 2x_5 &= 2 \\ 2x_4 + 6x_5 &= 2. \end{aligned}$$

Solution: If we transform the augmented matrix of the linear system to the row echelon form, we get

$$[A:b] \approx \begin{bmatrix} 1 & 0 & 2 & 3 & 1 & : & 5 \\ 0 & 1 & -1 & 3 & 2 & : & 1 \\ 0 & 0 & 1 & 3 & 2 & : & 2 \\ 0 & 0 & 0 & 2 & 6 & : & 2 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 2 & 3 & 1 & : & 5 \\ 0 & 1 & -1 & 3 & 2 & : & 1 \\ 0 & 0 & 1 & 3 & 2 & : & 2 \\ 0 & 0 & 0 & 1 & 3 & : & 1 \end{bmatrix}.$$

Solving Linear Systems

The corresponding linear system is

$$\begin{aligned}
x_1 + 2x_3 + 3x_4 + x_5 &= 5 \\
x_2 - x_3 + 3x_4 + 2x_5 &= 1 \\
x_3 + 3x_4 + 2x_5 &= 2 \\
x_4 + 3x_5 &= 1.
\end{aligned}$$

Then the linear system has infinitely many solutions depend on the real parameter r :

$$\begin{aligned}
x_1 &= 4 - 6r \\
x_2 &= -4 - 14r \\
x_3 &= -1 + 7r \\
x_4 &= 1 - 3r \\
x_5 &= r, r \in \mathbb{R}.
\end{aligned}$$

Solving Linear Systems

Remark: Consider the linear system $Ax = b$. When we transform the augmented matrix $[A : b]$ to the row echelon form $[C : d]$;

1. If the number of nonzero rows of $[C : d]$ is equal to the number of nonzero rows of $[C]$, the linear system is consistent.
    - In this case if the number of unknowns (n) is equal to the number of nonzero rows (r), then the system has a unique solution.
    - If the number of unknowns (n) is greater than to the number of nonzero rows (r), then the system has infinitely many solutions depend on $n - r$ parameters.
2. If the number of nonzero rows of $[C : d]$ is not equal to the number of nonzero rows of $[C]$, the linear system has no solution (inconsistent).

Finding Inverse of a Matrix

Let A is an $n \times n$ square matrix. To find A^{-1}, if it exists, we transform the augmented matrix $[A : I_n]$ to the reduced row echelon form $[C : D]$.

- If $C = I_n$, then $D = A^{-1}$
- If $C \neq I_n$, then C has a row of zeros. (A is singular)

Finding Inverse of a Matrix

Example: Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$. Since the reduced row echelon form of the matrix A is I_3, the matrix A is nonsingular.

Finding Inverse of a Matrix

Remark: For $n \times n$ matrix A, the followings are equivalent:

1. A is nonsingular, that is, A^{-1} exists.
2. A is row equivalent to I_n, that is, the reduced row echelon form of A is I_n.
3. The linear system $Ax = b$ has a unique solution for every $n \times 1$ matrix b.
4. The homogenous linear system $Ax = 0$ has only zero (trivial) solution.