Real Vector Spaces

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Abstract

- Real Vector Spaces and Subspaces
- Spanning Set and Linear Independency
- Basis and Dimension

REAL VECTOR SPACES

- A vector in the plane is a 2×1 matrix (2-vector)

$$x = \begin{bmatrix} x \\ y \end{bmatrix}; x, y \in \mathbb{R}.$$

- A vector in the space is a 3×1 matrix (3-vector)

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; x, y, z \in \mathbb{R}.$$

- We also represent a vector in the plane as a directed line segment $\nearrow$ for physical applications. In $\mathbb{R}^2$,

$$\begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow (x, y)$$

- In algebraically, all these representations behave in a same manner.

Definition: A **real vector space** is a set of V of elements on which have two operations $\oplus$ and $\odot$ satisfy the following properties:

(i) $\oplus : \underset{(u,v) \longrightarrow u \oplus v}{V \times V \longrightarrow V}$ $\quad u \oplus v \in V$, for all $u, v \in V$

(1) (Commutative) $u \oplus v = v \oplus u$, for all $u, v \in V$
(2) (Associative) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$, for all $u, v, w \in V$
(3) (Identity) For any $u \in V, \exists \mathbf{0} \in V; u \oplus \mathbf{0} = \mathbf{0} \oplus u = u$
(4) (Inverse) For each $u \in V, \exists -u \in V; u \oplus -u = -u \oplus u = \mathbf{0}$

(ii) $\odot : \underset{(c,u) \longrightarrow c \odot u}{\mathbb{R} \times V \longrightarrow V}$ $\quad c \odot u \in V$,for all $u \in V$ and all $c \in \mathbb{R}$

(5) $c \odot (u \oplus v) = c \odot u \oplus c \odot v$, for all $u, v \in V$ and all $c \in \mathbb{R}$
(6) $(c + d) \odot u = c \odot u \oplus d \odot u$, for all $u \in V$ and all $c, d \in \mathbb{R}$
(7) $c \odot (d \odot u) = (c.d) \odot u$, for all $u \in V$ and all $c, d \in \mathbb{R}$
(8) $1 \odot u = u$, for all $u \in V$ and $1 \in \mathbb{R}$

- We denote $(V, \oplus, \odot)$ is a **real vector space**.
- The elements of $(V, \oplus, \odot)$ are called as **vectors**.(**0** is zero vector, $-u$ is negative of u**)**
- The elements of $\mathbb{R}$ are called as **scalars.**
- The operations $\oplus$ and $\odot$ are called as **vector addition** and **scalar multiplication**, respectively.

Examples:

1. $(\mathbb{R}, +, .)$ is a real vector space.
2. $(\mathbb{R}^+, +, .)$ is not a real vector space, because the identity element of the operation $+$ doesn't exist.
3. Let $\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1} \mid a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}$. Then $(\mathbb{R}^n, \oplus, \odot)$ is a real vector space.
4. The set of $m \times n$ matrices with matrix addition and scalar multiplication $(M_{mn}, \oplus, \odot)$ is a vector space.
5. The set of polynomials with degree $\leq n$

$$P_n = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in \mathbb{R} \right\}$$

is a vector space with the usual polynomial addition and scalar multiplication.

5. The set $V=\left\{A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in M_{22} \mid Tr(A)=a+d=0\right\}$ with usual matrix addition and scalar multiplication is a vector space.

(i) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix}\in V$. Then $a+d=0$ and $e+h=0$.

Since $a+d+e+h=0$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\oplus\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\in V$

(1) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\oplus\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}=$
$\begin{bmatrix} e+a & f+b \\ g+c & h+d \end{bmatrix}=\begin{bmatrix} e & f \\ g & h \end{bmatrix}\oplus\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(2) Associative (verify)

(3) Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\oplus\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{0}:=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\in V$ is the zero element.

(4) Since $-(a+d)=0$, $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}\in V$ is the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(ii) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ and $k \in \mathbb{R}$. Then $k(a+d) = 0$. Thus

$$k \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \in V$$

(5) $k \odot \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) = k \odot \left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \right) =$
$\begin{bmatrix} k(a+e) & k(b+f) \\ k(c+g) & k(d+h) \end{bmatrix} = \begin{bmatrix} ka+ke & kb+kf \\ kc+kg & kd+kh \end{bmatrix} =$
$\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \oplus \begin{bmatrix} ke & kf \\ kg & kh \end{bmatrix} = k \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus k \odot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

(6) $(k+t) \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (k+t)a & (k+t)b \\ (k+t)c & (k+t)d \end{bmatrix} =$
$\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \oplus \begin{bmatrix} ta & tb \\ tc & td \end{bmatrix} = k \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus t \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$(7)\ k \odot \left(t \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = k \odot \begin{bmatrix} ta & tb \\ tc & td \end{bmatrix} = \begin{bmatrix} k\,(ta) & k\,(tb) \\ k\,(tc) & k\,(td) \end{bmatrix} =$$
$$\begin{bmatrix} (kt)\,a & (kt)\,b \\ (kt)\,c & (kt)\,d \end{bmatrix} = kt \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$(8)\ 1 \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

6. $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$ with operations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \boxplus \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} := \begin{bmatrix} x + x' \\ y + y' \\ z + z' \end{bmatrix}$$

$$k \boxdot \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} := \begin{bmatrix} x' \\ 1 \\ z' \end{bmatrix}$$

is not a vector space. Since from (5),

$$k \boxdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \boxplus \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \right) = \begin{bmatrix} x + x' \\ 1 \\ z + z' \end{bmatrix}$$

$$\neq k \boxdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \boxplus k \boxdot \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x + x' \\ 2 \\ z + z' \end{bmatrix}.$$

7. $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$ with operations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \oplus \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} := \begin{bmatrix} x' \\ y + y' \\ z + z' \end{bmatrix}$$

$$k \odot \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} := \begin{bmatrix} kx' \\ ky' \\ kz' \end{bmatrix}$$

is not a vector space. Properties (1), (3), (4), (6) fail to hold.

Theorem

The inverse of a vector is unique.

Theorem

Let $(V, \oplus, \odot)$ *be a real vector space. For* $u \in V$ *and* $c \in \mathbb{R}$,
(i) $0 \odot u = \mathbf{0}$
(ii) $c \odot \mathbf{0} = \mathbf{0}$
(iii) $c \odot u = \mathbf{0} \Rightarrow c = 0$ *or* $u = \mathbf{0}$
(iv) $(-1) \odot u = -u.$

Definition (Subspace)

Let $(V, \oplus, \odot)$ be a real vector space and $\emptyset \neq W \subseteq V$. If W is a real vector space with the operations in V, then W is called a **subspace** of V $(W \leq V)$.

To verify that a subset W of a vector sace V is a subspace, it is enough to check the following conditions.

Theorem

Let $(V, \oplus, \odot)$ be a real vector space and $\emptyset \neq W \subseteq V$. Then

$$W \leq V \iff \begin{array}{ll} (i) & u \oplus v \in W \text{ for all } u, v \in W \\ (ii) & c \odot u \in W \text{ for all } u \in W \text{ and all } c \in \mathbb{R}. \end{array}$$

Examples:

1. Let V be a vector space and W be the subset consisting of the zero vector $\{\mathbf{0}\}$.
Since $\mathbf{0} \oplus \mathbf{0} = \mathbf{0}$, $c \odot \mathbf{0} = \mathbf{0}$, W is a subspace of V.
$\{\mathbf{0}\}$ is called the **zero subspace.**
Every vector space has at least two subspace, itself and zero subspace.

2. Let $V = M_{22}$ and $W = \{A \in M_{22} \mid \det(A) = 0\}$.
For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$,
$A \oplus B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin W$, since $det(A \oplus B) = 1 \neq 0$.
Thus $W \nleq V$.

Subspaces

3. $V = \mathbb{R}^3$ is a real vector space with the standart operations $\oplus$ and $\odot$.

- $W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; x + z = 0 \right\} \leq V.$

- $W_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; x + z = 7 \right\} \nleq V.$

- $W_3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; x + y = 0, z > 0 \right\} \nleq V$, since

$$(-1) \odot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \notin W_3.$$

SPANNING SET and LINEAR INDEPENDENCY

A simple way to construct a subspace in a vector space $(V, \oplus, \odot)$ is as follows:

- Let v_1 and v_2 be fixed vectors in V.
- Let $W = \{a_1 \odot v_1 \oplus a_2 \odot v_2 \mid a_1, a_2 \in \mathbb{R}\}$.
- For $w_1 = a_1 \odot v_1 \oplus a_2 \odot v_2$, $w_2 = b_1 \odot v_1 \oplus b_2 \odot v_2 \in W$,

$$w_1 \oplus w_2 \in W \text{ and } c \odot w_1 \in W.$$

Thus $W \leq V$.

It can be performed more than 2 vectors. To do this, now we give the definiton of "**linear combination**".
To describe a vector space, "*linear combination*" plays an important role.

Linear Combination

Definition (Linear Combination)

Let $v_1, v_2, ..., v_k$ be vectors in $(V, \oplus, \odot)$ a vector space. A vector $v \in V$ is called a **linear combination** of $v_1, v_2, ..., v_k$ if

$$v = c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_k \odot v_k,$$

for the scalars $c_1, c_2, ..., c_k$.

Example: Let $V = \mathbb{R}^3$. The vector $v = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ is a linear combination of $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, since

$$v = 1 \odot v_1 \oplus 2 \odot v_2 \oplus (-1) \odot v_3.$$

Spanning Set

Definition (Spannig Set)

Let $S = \{v_1, v_2, ..., v_k\}$ be a set of vectors in a vector space V. Then the set of all vectors in V that are linear combinations of the vectors in S is called the **span of S**, denoted by *Span* S; that is,

$$Span\ S := \{c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_k \odot v_k \mid c_1, c_2, ..., c_k \in \mathbb{R}\}.$$

If every vector in V is a linear combination of the vectors in S, the set S is said to **span** V, or V is **spanned by** S; that is, $Span\ S = V$.

Theorem

Let $S = \{v_1, v_2, ..., v_k\}$ be a set of vectors in a vector space V. Then

$$Span\ S \leq V.$$

Spanning Set

Examples:

1. Let $V = \mathbb{R}^2$ and $S = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, then

$$\begin{aligned} Span\ S &= \{a \odot v_1 \oplus b \odot v_2 \mid a, b \in \mathbb{R}\} \\ &= \left\{ a \odot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \oplus b \odot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} a \\ a+b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}. \end{aligned}$$

2. Let $V = M_{23}$ and

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

$$\begin{aligned}
Span\ S &= \{a \odot v_1 \oplus b \odot v_2 \oplus c \odot v_3 \oplus d \odot v_4 \mid a, b \in \mathbb{R}\} \\
&= \left\{ a \odot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus b \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right. \\
&\quad \left. \oplus c \odot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus d \odot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}.
\end{aligned}$$

Spanning Set

To determine whether a vector v of V is in $Span\ S$, we investigate the consistency of the corresponding linear system.

- If the corresponding system is consistent, then $v \in Span\ S$. That is, if there exist scalars $c_1, c_2, \ldots, c_k$ such that $v = c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \cdots \oplus c_k \odot v_k$, then $v \in Span\ S$.
- If the corresponding system is inconsistent, then $v \notin Span\ S$. That is, if there does not exist scalars $c_1, c_2, \ldots, c_k$ such that $v = c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \cdots \oplus c_k \odot v_k$, then $v \notin Span\ S$.

Spanning Set

Example

Let $V = \mathbb{R}^3$, consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Determine whether $Span\{v_1, v_2, v_3\} = \mathbb{R}^3$.

Solution: Let $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$. If we find the scalars c_1, c_2, c_3 such that $v = c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus c_3 \odot v_3$, then $Span\{v_1, v_2, v_3\} = \mathbb{R}^3$.

Spanning Set

The corresponding linear system

$$\begin{aligned} c_1 + c_2 + c_3 &= x \\ 2c_1 + c_3 &= y \\ c_1 + 2c_2 &= z \end{aligned}$$

is consistent. Since

$$\left[\begin{array}{ccccc} 1 & 1 & 1 & : & x \\ 2 & 0 & 1 & : & y \\ 1 & 2 & 0 & : & z \end{array}\right] \approx \cdots \approx \left[\begin{array}{ccccc} 1 & 0 & 0 & : & \frac{-2x+2y+z}{3} \\ 0 & 1 & 0 & : & \frac{x-y+z}{3} \\ 0 & 0 & 1 & : & \frac{4x-y-2z}{3} \end{array}\right],$$

$$\begin{aligned} c_1 &= \frac{-2x+2y+z}{3} \in \mathbb{R} \\ c_2 &= \frac{x-y+z}{3} \in \mathbb{R} \\ c_3 &= \frac{4x-y-2z}{3} \in \mathbb{R}. \end{aligned}$$

Thus, $Span\{v_1, v_2, v_3\} = \mathbb{R}^3$.

Spanning Set

Consider the set $\left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$, then $Span\{v_1, v_2\} \neq \mathbb{R}^3$.

Since

$$\begin{bmatrix} 1 & 1 & : & x \\ 2 & 0 & : & y \\ 1 & 2 & : & z \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & : & x \\ 0 & 1 & : & \frac{2x-y}{2} \\ 0 & 0 & : & \frac{4x+y+2z}{2} \end{bmatrix}.$$

If $\frac{4x+y+2z}{2} \neq 0$, the system is inconsistent.

Spanning Set

Remark: A vector space may have many spanning sets and these spanning sets need not have the same number of vectors.
Let $S = \{v_1, v_2, ..., v_j, ..., v_k\}$ and $Span S = V$. If v_j is a linear combination of the preceding vectors in S, then the set $S - \{v_j\}$ also spans V.

Example: Let $V = \mathbb{R}^3$ and $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$.

- If $S_1 = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, then $Span \{v_1, v_2, v_3\} = \mathbb{R}^3$.
- If $S_2 = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, then $Span \{v_1, v_2, v_3, v_4\} = \mathbb{R}^3$.

Spanning Set

- If $S_3 = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$, then $Span\{v_1, v_2, v_3, v_4\} = \mathbb{R}^3$.
  Since the corresponding linear system is consistent:

$$\begin{bmatrix} 1 & 1 & 1 & 2 & : & x \\ 2 & 0 & 1 & 1 & : & y \\ 1 & 2 & 0 & 2 & : & z \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 0 & 0 & : & \frac{-2x+2y+z}{3} \\ 0 & 1 & 0 & 1 & : & \frac{x-y+z}{3} \\ 0 & 0 & 1 & 1 & : & \frac{4x-y-2z}{3} \end{bmatrix},$$

$$\begin{aligned} c_1 &= \frac{-2x+2y+z}{3} - t \in \mathbb{R}, \; c_2 = \frac{x-y+z}{3} - t \in \mathbb{R}, \\ c_3 &= \frac{4x-y-2z}{3} - t \in \mathbb{R}, \; c_4 = t \in \mathbb{R}. \end{aligned}$$

Our goal is to find the minimum number of vectors for a spanning set.

Exercises

1. Let $u = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$ and $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ \mathbf{t} \end{bmatrix} \in \mathbb{R}^3$. Find the value of $\mathbf{t}$ which makes $u \in Span\{v_1, v_2, v_3\}$.

Solution: If $\exists\ c_1, c_2, c_3 \in \mathbb{R}$ such that $u = c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus c_3 \odot v_3$, then $u \in Span\{v_1, v_2, v_3\}$.

$$\begin{aligned} c_1 + 3c_2 + c_3 &= 1 \\ 4c_2 + c_3 &= 6 \\ 2c_2 + \mathbf{t}c_3 &= 8 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & 1 & : & 1 \\ 0 & 4 & 1 & : & 6 \\ 0 & 2 & t & : & 8 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 3 & 1 & : & 1 \\ 0 & 1 & 1/4 & : & 3/2 \\ 0 & 0 & (2t-1)/4 & : & 5/2 \end{bmatrix}.$$

If $\frac{2t-1}{4} \neq 0$, then the system is consistent. Thus $t \in \mathbb{R} - \left\{\frac{1}{2}\right\}$.

2. Let $u = \begin{bmatrix} 1 & 0 \\ 2 & t \end{bmatrix}$ and

$$v_1 = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, v_3 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

be vectors in M_{22}. Find the value of $\mathbf{t}$ which makes $u \in Span\{v_1, v_2, v_3\}$.
Answer: $t = \frac{17}{2}$.

3. Find a vector in $\mathbb{R}^3$ which can not be spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \; and \; v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Hint: If we could not find any scalars $c_1, c_2 \in \mathbb{R}$ such that $u = c_1 \odot v_1 \oplus c_2 \odot v_2$, then $u \notin Span\{v_1, v_2\}$. Corresponding linear system will be inconsistent.

Linear dependency & linear independency

Definition

The vectors $v_1, v_2, ..., v_k$ in a vector space $(V, \oplus, \odot)$ are said to be **linearly dependent** if there exist scalars $c_1, c_2, ..., c_k$, not all zero, such that

$$c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_k \odot v_k = \mathbf{0}.$$

Otherwise, the vectors $v_1, v_2, ..., v_k$ are called **linearly independent,** that is, the vectors $v_1, v_2, ..., v_k$ are called linearly independent, if $c_1 = c_2 = \cdots = c_k = 0$ such that

$$c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_k \odot v_k = \mathbf{0}.$$

If $S = \{v_1, v_2, ..., v_k\}$, then we say that the set S is linearly dependent (independent).

Linear dependency & linear independency

To determine whether a set of vectors is linearly independent (dependent), we investigate the trivial (nontrivial) solution of the corresponding homogenous linear system.

- If the system has only trivial (zero) solution, then the vectors are linearly independent.
- If the system has a nontrivial solution, then the vectors are linearly dependent.

Example

Determine whether the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent.

Solution: Let

$$c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus c_3 \odot v_3 = \mathbf{0}.$$

Then the corresponding homogenous linear system is

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 + c_3 &= 0 \\ c_1 + 2c_2 &= 0 \end{aligned}$$

is consistent.

Since

$$\left[\begin{array}{ccccc} 1 & 1 & 1 & : & 0 \\ 2 & 0 & 1 & : & 0 \\ 1 & 2 & 0 & : & 0 \end{array}\right] \approx \cdots \approx \left[\begin{array}{ccccc} 1 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{array}\right],$$

we get $c_1 = c_2 = c_3 = 0$ which indicates the linear system has only zero solution. Thus the vectors are linearly independent.

Theorem

Let $S = \{v_1, v_2, ..., v_n\}$ be a set of vectors in a vector space $\mathbb{R}^n$. Let A be the matrix whose columns are the elements of S. Then

$$S \text{ is linearly independent} \Leftrightarrow \det(A) \neq 0.$$

Example: From the former example, since

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 3 \neq 0$$

which indicates the vectors are linearly independent.

Linear dependency & linear independency

Remarks:

1. Let S_1 and S_2 be finite subset of a vector space V and $S_1 \subset S_2$. Then

$$(i)\ \ S_1 \text{ is linearly dependent} \Rightarrow S_2 \text{ is linearly dependent}$$

$$(ii)\ \ S_2 \text{ is linearly independent} \Rightarrow S_1 \text{ is linearly independent.}$$

2. If $S = \{\mathbf{0}\}$, then S is linearly dependent. (Exp: $5 \odot \mathbf{0} = \mathbf{0}$)
3. If $S = \{v_1, v_2, ..., \mathbf{0}, ..., v_k\}$, then S is linearly dependent.
4. If $S = \{\mathbf{v}\}$ with $\mathbf{0} \neq v$, then S is linearly independent.
5. If $S = \{v_1, v_2, ..., v_k\}$ is linearly independent, then the vectors $v_1, v_2, ..., v_k$ must be distinct and nonzero. Also neither vector is a multiple of the other.

Linear dependency & linear independency

Example: In $\mathbb{R}^3$,

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is linearly dependent. Since

$$1 \odot v_1 \oplus 1 \odot v_2 \oplus 0 \odot v_3 \oplus (-1) \odot v_4 = \mathbf{0}.$$

Also observe that $v_4 = v_1 \oplus v_2$.

Linear dependency & linear independency

Example: In $\mathbb{R}^2$,

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

is linearly independent. Since unique nonzero vector is always linearly independent.

$$S_2 = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent. Since $c_1 \odot v_1 \oplus c_2 \odot v_2 = \mathbf{0} \Rightarrow c_1 = c_2 = 0$.

$$S_3 = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent. Since $v_3 = 2 \odot v_1 \oplus 1 \odot v_2$.

Linear dependency & linear independency

Remark: Lets construct the set of linear independent vectors in a given vector space V.

- $S_1 = \{v_1\}$ where $v_1 \neq \mathbf{0}$
- $S_2 = \{v_1, v_2\}$ where $v_2 \notin SpanS_1$
- $S_3 = \{v_1, v_2, v_3\}$ where $v_3 \notin SpanS_2$

  $\vdots$
- $S_n = \{v_1, v_2, \ldots, v_n\}$ where $v_n \notin SpanS_{n-1}$ and $SpanS_n = V$.

Our goal is to find the maximum number of linearly independent vectors.

Exercises

1. For what values of **t**, the vectors $v_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 8 \\ -6 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ -7 \\ \mathbf{t} \end{bmatrix} \in \mathbb{R}^3$ are linearly dependent?

2. Are the vectors

$$v_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \in M_{22}$$

linearly independent?

3. Are the vectors

$$v_1 = x^3 + x + 2, v_2 = 2x^2 + x, v_3 = 3x^2 + 2x + 2 \in P_2$$

linearly independent?

BASIS and DIMENSION

Motivation: We want to describe every vector in V. To do this, we consider the set of vectors $S = \{v_1, v_2, ..., v_k\}$ of V.

$$SpanS = \{c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_k \odot v_k \mid c_1, c_2, ..., c_k \in \mathbb{R}\}$$

If it spans (generates) all the vectors in V, then $SpanS = V$
A spanning set can have many different spanning sets and these sets need not have the same number of vectors.

- Our goal is to find a more **efficient** spanning set, that is, we are looking for **the minimum number of vectors** for a spanning set that generate the whole vector space.
- On the other hand, we are looking for **the maximum number of linearly independent vectors** in V.

Definition (Basis and Dimension)

Let $v_1, v_2, ..., v_k$ be vectors in a vector space $(V, \oplus, \odot)$. The vectors $v_1, v_2, ..., v_k$ are said to form a **basis** for V if

(i) $Span\{v_1, v_2, ..., v_k\} = V$

(ii) $\{v_1, v_2, ..., v_k\}$ is linearly independent.

The number of vectors in a basis for the vector space V is called as a **dimension** of V $(\dim V)$. The dimension of the zero vector space $\{0\}$ is defined as zero.

A vector space can have many different basis but the dimension of the vector space is always the same.

Examples:

1. $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis (**standard basis**) for the vector space $\mathbb{R}^2$ and $\dim \mathbb{R}^2 = 2$.

$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for the vector space $\mathbb{R}^2$.

2. $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is the standard basis for the vector space $\mathbb{R}^3$ and $\dim \mathbb{R}^3 = 3$.

Generally, the standard basis for the vector space $\mathbb{R}^n$ is defined by $B = \{e_1, e_2, ..., e_n\}$, where e_j is an $n \times 1$ matrix whose j-th row is 1 and 0 elsewhere. Also $\dim \mathbb{R}^n = n$.

3.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is the standard basis for M_{22} and $\dim M_{22} = 4$.

In general,

$$\dim M_{mn} = m \times n.$$

4. $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + d = 0 \right\} < M_{22}$.

$B = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ is a basis for W.

(i) Since any vector $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ in W can be written as

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = b \odot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus c \odot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus a \odot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

then $SpanB = W$.

(ii)

$$c_1 \odot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus c_2 \odot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus c_3 \odot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_3 & c_1 \\ c_2 & -c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow B \text{ is linearly ind.}$$

Thus B is a basis for W and dim $W = 3$.

Theorem

If $B = \{v_1, v_2, ..., v_n\}$ is a basis for a vector space V, then every vector in V can be written uniquely as a linear combination of the vectors in B.

Example: $B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for the vector space $\mathbb{R}^2$.

Every vector $v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ can be written uniquely as

$$\begin{bmatrix} x \\ y \end{bmatrix} = (2y - x) \odot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \oplus (x - y) \odot \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Basis and Dimension

Theorem

Let $V = \mathbb{R}^m$, $S = \{v_1, v_2, ..., v_n\}$, $(n \geq m)$ be a set of nonzero vectors in V and Span$S = W$. Then some subset of S is a basis for W. The procedure for finding this basis is in the following:

1. *Form equation* $c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus \ldots \oplus c_n \odot v_n = \mathbf{0}$
2. *Construct the augmented matrix associated with the corresponding homogenous linear system and transform it to the reduced row echelon form.*
3. *The vectors corresponding to the columns containing the* **leading 1**'*s form a basis for* W.

Basis and Dimension

Example: $V = \mathbb{R}^3$,

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

It is easy to show that $SpanS = V$. Lets find a subset of S that is a basis for $\mathbb{R}^3$.

$$c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus c_3 \odot v_3 \oplus c_4 \odot v_4 \oplus c_5 \odot v_5 = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & -1 & 5 \\ 2 & 1 & -3 & 7 & -2 \\ 1 & -1 & 1 & 1 & 0 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} \mathbf{1} & 0 & 0 & 2 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 1 \\ 0 & 0 & \mathbf{1} & -1 & 1 \end{bmatrix}.$$

Then, the **leading 1**'s appears in columns 1,2,3, so $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

$$v_4 = 2 \odot v_1 \oplus -1 \odot v_3, \;\; v_5 = v_2 \oplus v_3.$$

Basis and Dimension

Let $S = \{v_1, v_2, ..., v_\mathbf{n}\}$ and $W = \{w_1, w_2, ..., w_\mathbf{r}\}$ be sets of vectors in V.

Theorem

1. *If S is a basis and W is linearly independent $\Rightarrow r \leq n$.*
2. *If S is a basis and $SpanW = V \Rightarrow r \geq n$.*
3. *Let $\dim V = n$. If S is linearly independent $\Rightarrow S$ is a basis for V.*
4. *Let $\dim V = n$. If $SpanS = V \Rightarrow S$ is a basis for V.*

Example: $V = \mathbb{R}^3$,

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

$SpanS = V$. Since $\dim \mathbb{R}^3 = 3$, S can not be a basis for $\mathbb{R}^3$.

Exercises

1. Find a basis for the subspace $U = \left\{ \begin{bmatrix} 0 \\ 2a \\ a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ of $\mathbb{R}^4$.

$$U = \left\{ a \odot \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}}_{v_1} \oplus b \odot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_2} \mid a, b \in \mathbb{R} \right\}$$

which implies $Span\{v_1, v_2\} = U$. Also these vectors are linearly independent.

Thus $\{v_1, v_2\}$ is a basis for U and $\dim U = 2$.

2. Find a basis for the subspace $U = \left\{ \begin{bmatrix} a+c \\ a-b \\ b+c \\ b-a \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ of $\mathbb{R}^4$.

$$U = \left\{ a \odot \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{v_1} \oplus b \odot \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}}_{v_2} \oplus c \odot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_3} \mid a, b, c \in \mathbb{R} \right\}$$

then $Span\left\{v_1, v_2, v_3\right\} = U$.

Exercises

But $\{v_1, v_2, v_3\}$ is linearly dependent, since

$$c_1 \odot v_1 \oplus c_2 \odot v_2 \oplus c_3 \odot v_3 = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} \mathbf{1} & 0 & 1 \\ 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If we remove the linearly dependent vector

$$v_3 = v_1 \oplus v_2$$

from our set, then $\{v_1, v_2\}$ is a basis for U and $\dim U = 2$.