Eigenvalues and Eigenvectors

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Definition (Eigenvalues and Eigenvectors)

Let $L : V \rightarrow V$ be a linear transformation and $dimV = n$. The scalar λ is called an **eigenvalue** of L if $\exists$ $\mathbf{0} \neq v \in V$ such that

$$L(v) = \lambda \odot v,$$

and the vector v is called an **eigenvector** of L associated with the eigenvalue λ.

In $\mathbb{R}^n$, the eigenvalue problem reduces to determine whether $\lambda \odot v$ can be parallel to v.

Eigenvalues and Eigenvectors

The eigenvalue problem for linear transformation can be stated as a matrix representation of this linear transformation.

Definition

A scalar λ is called an **eigenvalue** of the $n \times n$ matrix A if there is a nonzero solution x of

$$Ax = \lambda x.$$

Such an x is called an **eigenvector** of A corresonding to the eigenvalue λ.

The set of all eigenvectors of A coresonding to the eigenvalue λ is called the **eigenspace** of A.

Eigenvalues and Eigenvectors

Let A be $n \times n$ matrix.

$$Ax = \lambda x \Rightarrow \lambda x - Ax = 0$$

$$\Rightarrow (\lambda I_n - A)\, x = 0$$

- The **characteristic polynomial** of A is defined by

$$P_A(\lambda) := \det(\lambda I_n - A).$$

- The equation $P_A(\lambda) = 0$ is called the **characteristic equation** of A. The roots of the characteristic polynomial are **eigenvalues** of A. Nonzero solutions of the homogenous linear system

$$(\lambda I_n - A)\, x = 0$$

  are **eigenvectors** of A associated with the eigenvalue λ.

Example (1)

Find the eigenvalues and corresonding eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I_3 - A) = 0 \\ &\Rightarrow \begin{vmatrix} \lambda - 1 & -4 & 0 \\ 0 & \lambda - 2 & -5 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = 0 \\ &\Rightarrow (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0. \end{aligned}$$

The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = 3$.

Eigenvalues and Eigenvectors

To find the eigenvectors corresonding to the eigenvalues λ, we need to solve the equation $(\lambda I_3 - A)\,x = 0$, *i.e.*

$$\begin{cases} (\lambda - 1)\,x_1 - 4x_2 = 0 \\ (\lambda - 2)\,x_2 - 5x_3 = 0 \\ \qquad\qquad (\lambda - 3)\,x_3 = 0 \end{cases}$$

- For $\lambda_1 = 1$, we obtain

$$x = \left\{ \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \mid r \in \mathbb{R} \right\} = \left\{ r \odot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} \mid r \in \mathbb{R} \right\} = Span\,\{v_1\}\,.$$

That is, the eigenvectors corresonding to the eigenvalue $\lambda_1 = 1$ are precisely the set of scalar multiples of the vector v_1.

Eigenvalues and Eigenvectors

- For $\lambda_1 = 1$, the eigenspace is $\left\{ r \odot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} \mid r \in \mathbb{R} \right\} = Span\{v_1\}$.
- For $\lambda_2 = 2$, the eigenspace is $\left\{ s \odot \underbrace{\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}}_{v_2} \mid s \in \mathbb{R} \right\} = Span\{v_2\}$.
- For $\lambda_3 = 3$, the eigenspace is $\left\{ t \odot \underbrace{\begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}}_{v_3} \mid t \in \mathbb{R} \right\} = Span\{v_3\}$.

Eigenvalues and Eigenvectors

Example (2)

Find the eigenvalues and corresonding eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}.$$

Solution:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I_3 - A) = 0 \\ &\Rightarrow \begin{vmatrix} \lambda - 1 & 1 & 1 \\ 0 & \lambda - 3 & -2 \\ 0 & 1 & \lambda \end{vmatrix} = 0 \\ &\Rightarrow (\lambda - 1)(\lambda(\lambda - 3) + 2) = 0 \\ &\Rightarrow (\lambda - 1)^2 (\lambda - 2) = 0. \end{aligned}$$

The eigenvalues of A are $\lambda_{1,2} = 1$ (the multiplicity is 2) and $\lambda_3 = 2$.

Eigenvalues and Eigenvectors

To find the eigenvectors corresonding to the eigenvalue λ, we need to solve the equation$(\lambda I_3 - A)\, x = 0$.

- For $\lambda_{1,2} = 1$, the eigenspace is

$$x = \left\{ \begin{bmatrix} r \\ s \\ -s \end{bmatrix} \mid r, s \in \mathbb{R} \right\} = \left\{ r \odot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} \oplus s \odot \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{v_2} \mid r, s \in \mathbb{R} \right\}$$

- For $\lambda_3 = 2$, the eigenspace is

$$x = \left\{ t \odot \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{v_3} \mid t \in \mathbb{R} \right\} = Span \left\{ v_3 \right\}.$$

Example (3)

Find the eigenvalues and corresonding eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}.$$

Solution:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I_3 - A) = 0 \\ &\Rightarrow \begin{vmatrix} \lambda-3 & -1 & 1 \\ -2 & \lambda-2 & 1 \\ -2 & -2 & \lambda \end{vmatrix} = 0 \\ &\Rightarrow (\lambda-1)(\lambda-2)^2 = 0. \end{aligned}$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_{2,3} = 2$ (the multiplicity is 2).

To find the eigenvectors corresonding to the eigenvalue λ, we need to solve the equation $(\lambda I_3 - A)\,x = 0$.

- For $\lambda_1 = 1$, the eigenspace is

$$x = \left\{ r \odot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{v_1} \mid r \in \mathbb{R} \right\} = Span\left\{v_1\right\}.$$

- For $\lambda_{2,3} = 2$, the eigenspace is

$$x = \left\{ s \odot \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{v_2} \mid t \in \mathbb{R} \right\} = Span\left\{v_2\right\}.$$

Example (4)

Find the eigenvalues and corresonding eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix}.$$

Solution:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I_3 - A) = 0 \\ &\Rightarrow \begin{vmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda & 0 \\ 1 & -2 & \lambda + 1 \end{vmatrix} = 0 \\ &\Rightarrow \lambda^3 = 0. \end{aligned}$$

The eigenvalues of A are $\lambda_{1,2,3} = 0$ with the multiplicity of 3.

To find the eigenvectors corresonding to the eigenvalue λ, we need to solve the equation $(\lambda I_3 - A)\,x = 0$, *i.e.*

$$\begin{cases} (\lambda - 1)\,x_1 - x_3 = 0 \\ \lambda x_2 = 0 \\ x_1 - 2x_2 + (\lambda + 1)\,x_3 = 0 \end{cases}$$

- For $\lambda_{1,2,3} = 0$, the eigenspace is

$$x = \left\{ r \odot \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{v} \;\middle|\; r \in \mathbb{R} \right\} = Span\,\{v\}\,.$$

Remark:

- It is useful to find a set of linearly independent eigenvectors for a given matrix A.
- If the eigenvectors $v_1, v_2, \ldots, v_k$ correspond to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of A, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Cayley-Hamilton Theorem

Let A be $n \times n$ matrix. If we expand the determinant $P_A(\lambda)$ and collect terms in the same power of λ, we have

$$P_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$

Theorem (Cayley-Hamilton Theorem)

Every square matrix A satisfies its own characteristic equation, i.e.

$$P_A(A) = \mathbf{0}.$$

In the following, we give some applications of the Cayley-Hamilton Theorem.

1. $\det(A) = (-1)^n a_0$.
2. If $a_0 \neq 0$, then A^{-1} exists and $A^{-1} = \frac{-1}{a_0}\left(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n\right)$.
3. $A^n = -\left(a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I_n\right)$.

Cayley-Hamilton Theorem

Example

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ if it exists.

Solution:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I_3 - A) = \begin{vmatrix} \lambda - 1 & -2 & -3 \\ -1 & \lambda - 1 & -2 \\ 0 & -1 & \lambda - 2 \end{vmatrix} \\ &= \lambda^3 - 4\lambda^2 + \lambda + \underbrace{\mathbf{1}}_{a_0 \neq 0} \end{aligned}$$

- Since $a_0 \neq 0$, then A^{-1} exists.
- $\det(A) = (-1)^n a_0 = -1$

Cayley-Hamilton Theorem

- From Cayley-Hamilton Theorem,

$$P_A(A) = \mathbf{0} \Rightarrow A^3 - 4A^2 + A + I_3 = \mathbf{0}$$

$$\begin{aligned}
&\Rightarrow \quad A^3 - 4A^2 + A + I_3 = \mathbf{0} \\
&\Rightarrow \quad A^3 - 4A^2 + A + \mathbf{A}\mathbf{A}^{-1} = \mathbf{0} \\
&\Rightarrow \quad A\left(A^2 - 4A + I_3 + A^{-1}\right) = \mathbf{0} \\
&\Rightarrow \quad \mathbf{A}^{-1} A\left(A^2 - 4A + I_3 + A^{-1}\right) = \mathbf{A}^{-1}\mathbf{0} \\
&\Rightarrow \quad A^2 - 4A + I_3 + A^{-1} = \mathbf{0} \\
&\Rightarrow \quad A^{-1} = -A^2 + 4A - I_3 \\
&\Rightarrow \quad A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}.
\end{aligned}$$

Cayley-Hamilton Theorem

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Find A^5.

From Cayley-Hamilton Theorem, $P_A(A) = \mathbf{0} \Rightarrow A^3 - 4A^2 + A + I_3 = \mathbf{0}$

$$\begin{aligned}
\Rightarrow \quad & A^3 = 4A^2 - A - I_3 \\
\Rightarrow \quad & A^4 = 4A^3 - A^2 - A = 4\left(4A^2 - A - I_3\right) - A^2 - A \\
= \quad & 15A^2 - 5A - 4I_3 \\
\Rightarrow \quad & A^5 = 15A^3 - 5A^2 - 4A \\
= \quad & 15\left(4A^2 - A - I_3\right) - 5A^2 - 4A = 55A^2 - 19A - 15I_3 \\
= \quad & \begin{bmatrix} 131 & 347 & 658 \\ 91 & 241 & 457 \\ 55 & 146 & 277 \end{bmatrix}.
\end{aligned}$$

DIAGONALIZATION

Recall that the $n \times n$ matrices A and B are said to be **similar**, written $A \approx B$, if there exist nonsingular matrix P such that $P^{-1}AP = B$. Similar matrices have the same characteristic polynomial, hence the same eigenvalues.

Definition

The $n \times n$ matrix A is **diagonalizable** if there exits nonsingular matrix P such that $P^{-1}AP = D$, where $D := \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix}$ is diagonal matrix.

That is, the $n \times n$ matrix A is diagonalizable if $A \approx D$.

Remark:

- Let $L : V \to V$ be a linear transformation and $dim V = n$. We say that L is diagonalizable, if its matrix representation A is diagonalizable.
- Let $L : V \to V$ be a linear transformation and $dim V = n$. Then L is diagonalizable $\Leftrightarrow$ V has a basis S which consists of the eigenvectors of L. Moreover, if the matrix representation of L with respect to the basis S is the diagonal matrix D, then the entries on the main diagonal of D are the eigenvalues of L.

Diagonalization

Following theorem gives when an $n \times n$ matrix A can be diagonalized.

Theorem

$A \approx D \Leftrightarrow A$ has n linearly independent eigenvectors.
Moreover, the entries on the main diagonal of D are the eigenvalues of A.

- If the roots of the characteristic polynomial of an $n \times n$ matrix A are distinct, then A is diagonalizable.
- If the roots of the characteristic polynomial of an $n \times n$ matrix A are not all distinct, then A may or may not be diagonalizable.

The procedure for diagonalization

Let A be $n \times n$ matrix.

1. Find the eigenvalues of A. If the eigenvalues of A are all distinct, then A is diagonalizable. If eigenvalues of A are not all distinct, A may or may not be diagonalizable.

The procedure for diagonalization

Let A be $n \times n$ matrix.

1. Find the eigenvalues of A. If the eigenvalues of A are all distinct, then A is diagonalizable. If eigenvalues of A are not all distinct, A may or may not be diagonalizable.
2. Find the eigenvectors associated with the eigenvalues.

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2. Find the eigenvectors associated with the eigenvalues.
3. Compare the size of A and the number of linearly independent eigenvectors. If they are equal, then A is diagonalizable. Otherwise, A is not diagonalizable.

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4. Construct the matrix P whose columns are eigenvectors of A.

The procedure for diagonalization

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1. Find the eigenvalues of A. If the eigenvalues of A are all distinct, then A is diagonalizable. If eigenvalues of A are not all distinct, A may or may not be diagonalizable.
2. Find the eigenvectors associated with the eigenvalues.
3. Compare the size of A and the number of linearly independent eigenvectors. If they are equal, then A is diagonalizable. Otherwise, A is not diagonalizable.
4. Construct the matrix P whose columns are eigenvectors of A.
5. Construct the diagonal matrix D such that $P^{-1}AP = D$.

Diagonalization

Example (1)

Diagonalize the matrix $A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$, if possible.

Solution:

1. The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

2. The eigenvectors associated with the eigenvalues are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}.$$

3. Since the number of linear independent eigenvectors is equal to the dimension of A, A is diagonalizable.

Diagonalization

4. The matrix P consists of the eigenvectors of A, i.e.

$$P = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. The diagonal matrix D is

$$\begin{aligned} P^{-1}AP &= D \\ &= \begin{bmatrix} 1 & -4 & 10 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{2} & 0 \\ 0 & 0 & \mathbf{3} \end{bmatrix}. \end{aligned}$$

Example (2)

Diagonalize the matrix $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}$, if possible.

Solution:

1. The eigenvalues of A are $\lambda_{1,2} = 1$ and $\lambda_3 = 2$.

2. The eigenvectors of A corresponding to $\lambda_{1,2} = 1$ are

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

The eigenvector of A corresponding to $\lambda_3 = 2$ is $v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

3. Since the number of linear independent eigenvectors is equal to the dimension of A, A is diagonalizable.

Diagonalization

4. The matrix P consists of the eigenvectors of A, i.e.

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix}.$$

5. The diagonal matrix D is

$$\begin{aligned} P^{-1}AP &= D \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{2} \end{bmatrix}. \end{aligned}$$

Example (3)

Diagonalize the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$, if possible.

Solution:

1. The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_{2,3} = 2$.

2. The eigenvectors of A corresponding to $\lambda_1 = 1$ are $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

The eigenvector of A corresponding to $\lambda_{2,3} = 2$ is $v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

3. Since the number of linear independent eigenvectors is not equal to the dimension of A, A is not diagonalizable.

Diagonalization

Applications of diagonalization:

1. $A^{-1} = PD^{-1}P^{-1}$, $D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1/d_n \end{bmatrix}$.
2. $A^k = PD^kP^{-1}$, $D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n^k \end{bmatrix}$.

Example

Compute A^5, for the matrix $A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution: Since A is diagonalizable, we have

$$\begin{aligned} A^5 &= PD^5P^{-1} \\ &= \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 3^5 \end{bmatrix} \begin{bmatrix} 1 & -4 & 10 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 124 & 1800 \\ 0 & 32 & 1055 \\ 0 & 0 & 243 \end{bmatrix}. \end{aligned}$$

Jordan Canonical Form

If an $n \times n$ matrix A cannot be diagonalized, then we can often find a matrix J similar to A. The square matrix J is said to be in **Jordan canonical form**, and the square matrix J_i is called a Jordan blok.

$$Q^{-1}AQ = J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{bmatrix}, \text{ where } J_i := \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}.$$