# Determinants

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# DETERMINANTS

## Definition (Permutation)

Let $S = \{1, 2, ..., n\}$ be a set of integers, arranged in ascending order. A rearrangement $j_1 j_2 ... j_n$ of the elements of $S$ is called a **permutation** of $S$. A permutation $j_1 j_2 ... j_n$ is said to have an **inversion** if a larger integer, $j_r$, precedes a smaller one, $j_s$.

The set of all permutations of $S =: S_n$, and the number permutations of $S_n = n!$. If the total number of inversions is even(odd), the permutation is called even(odd) permutation.

$S_1 = \{\mathbf{1}\}$

$$S_2 = \left\{ \underbrace{\mathbf{12}}_{no\ inv}, \underbrace{21}_{1\text{-}inv} \right\}$$

$$S_3 = \left\{ \underbrace{\mathbf{123}}_{no\ inv}, \underbrace{132}_{1\text{-}inv}, \underbrace{213}_{1\text{-}inv}, \underbrace{\mathbf{231}}_{2\text{-}inv}, \underbrace{\mathbf{312}}_{2\text{-}inv}, \underbrace{321}_{3\text{-}inv} \right\}$$

**Definition (Determinant)**

Let $A=[a_{ij}]$ be an $n\times n$ matrix. The determinant function is defined by

$$\det(A)=\sum_{j_1j_2\ldots j_n\in S_n}(\pm)\,a_{1j_1}a_{2j_2}\ldots a_{nj_n}.$$

(If the permutation $j_1j_2\ldots j_n$ is even, then the sign is taken $+$, otherwise $-$)

**Examples:**

**1.** $A=[a_{11}]\Rightarrow|A|=a_{11}.$

**2.** $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\Rightarrow$

$$|A|=\sum_{j_1j_2\in S_2=\{12,21\}}(\pm)\,a_{\mathbf{1}j_1}a_{\mathbf{2}j_2}=a_{\mathbf{1}1}a_{\mathbf{2}2}-a_{\mathbf{1}2}a_{\mathbf{2}1}.$$

**3.**

$$
\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow |A| \\
&= \sum_{j_1 j_2 j_3 \in S_3 = \{123,231,312,132,213,321\}} (\pm)\, a_{\mathbf{1}j_1} a_{\mathbf{2}j_2} a_{\mathbf{3}j_3} \\
&= a_{\mathbf{1}1} a_{\mathbf{2}2} a_{\mathbf{3}3} + a_{\mathbf{1}2} a_{\mathbf{2}3} a_{\mathbf{3}1} + a_{\mathbf{1}3} a_{\mathbf{2}1} a_{\mathbf{3}2} \\
&\quad - a_{\mathbf{1}1} a_{\mathbf{2}3} a_{\mathbf{3}2} - a_{\mathbf{1}2} a_{\mathbf{2}1} a_{\mathbf{3}3} - a_{\mathbf{1}3} a_{\mathbf{2}2} a_{\mathbf{3}1}.
\end{aligned}
$$

# Properties of determinants

**Theorem:** Let $A$ be an $n \times n$ matrix.

1. $\det\left(A_{r_i \leftrightarrow r_j}\right) = -\det(A)$
2. $\det\left(A_{kr_i \to r_i}\right) = k\det(A)$
3. $\det\left(A_{kr_i + r_j \to r_j}\right) = \det(A)$
4. $\det(A) = \det\left(A^T\right)$
5. If two rows(columns) of $A$ are equal$\Rightarrow$ $\det(A) = 0$
6. If $A$ consists a zero row(column)$\Rightarrow$ $\det(A) = 0$
7. If A is upper(lower) triangular matrix$\Rightarrow$ $\det(A) = a_{11}a_{22}...a_{nn}$
8. $\det(I_n) = 1$
9. $\det(AB) = \det(A)\det(B)$
10. $\det(kA) = k^n \det(A)\,, k \in \mathbb{R}.$

**Remark:** By using (1),(2),(3), we transform a matrix A to the triangular form, then we computed the determinant by using (7). This method is called as computation of determinant via reduction to triangular form.

# Properties of determinants

### Theorem

*If $A$ is an $n \times n$ nonsingular matrix* $\Leftrightarrow \det(A) \neq 0$.

### Corollary

1. $Ax = b$ *has a unique solution* $\Leftrightarrow \det(A) \neq 0$.
2. $Ax = 0$ *has a nontrivial solution* $\Leftrightarrow \det(A) = 0$.
3. *If $A$ is $n \times n$ nonsingular matrix* $\Leftrightarrow \det\left(A^{-1}\right) = \frac{1}{\det(A)}$.
   $\left(\det(I_n) = \det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right) = 1\right.$
   $\left.\Rightarrow \det\left(A^{-1}\right) = \frac{1}{\det(A)}\right)$.

# Properties of determinants

**Examples:**

**1.** $\begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 4.2.3 = 24$

**2.** $\begin{vmatrix} 4 & 1 & 3 \\ 2 & 3 & 0 \\ 1 & 3 & 2 \end{vmatrix} \overset{r_1 \leftrightarrow r_3}{=} - \begin{vmatrix} 1 & 3 & 2 \\ 2 & 3 & 0 \\ 4 & 1 & 3 \end{vmatrix} \underset{-4r_1+r_3 \to r_3}{\overset{-2r_1+r_2 \to r_2}{=}} - \begin{vmatrix} 1 & 3 & 2 \\ 0 & -3 & -4 \\ 0 & -11 & -5 \end{vmatrix}$

$\overset{-\frac{1}{3}r_2 \to r_2}{=} (-1)(-3) \begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{4}{3} \\ 0 & -11 & -5 \end{vmatrix} \overset{11r_2+r_3 \to r_3}{=} 3 \begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & \frac{29}{3} \end{vmatrix} = 29.$

# Properties of determinants

**3.** If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 4 \Rightarrow$ compute $\begin{vmatrix} a_1 & a_2 & 4a_3 - 2a_2 \\ b_1 & b_2 & 4b_3 - 2b_2 \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & 2c_3 - c_2 \end{vmatrix}$.

$$\begin{vmatrix} a_1 & a_2 & 4a_3 - 2a_2 \\ b_1 & b_2 & 4b_3 - 2b_2 \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & 2c_3 - c_2 \end{vmatrix} \overset{2r_3 \to r_3}{=} \frac{1}{2} \begin{vmatrix} a_1 & a_2 & 4a_3 - 2a_2 \\ b_1 & b_2 & 4b_3 - 2b_2 \\ c_1 & c_2 & 4c_3 - 2c_2 \end{vmatrix}$$

$$\overset{2c_2 + c_3 \to c_3}{=} \frac{1}{2} \begin{vmatrix} a_1 & a_2 & 4a_3 \\ b_1 & b_2 & 4b_3 \\ c_1 & c_2 & 4c_3 \end{vmatrix} \overset{\frac{1}{4}c_3 \to c_3}{=} (4)\left(\frac{1}{2}\right) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 4.\frac{1}{2}.4 = 8.$$

**Remark:** We can use elementary row and column operations simultaneously.

**4.** If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3 \Rightarrow$ compute

$$\begin{vmatrix} a_1 + 2b_1 - 3c_1 & a_2 + 2b_2 - 3c_2 & a_3 + 2b_3 - 3c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} .(\mathrm{Hw})$$

**5.** $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$ .(Hw) Here $a, b, c$ are distinct, nonzero numbers. This determinant is called the Vandermonde determinant.

# Remark

For $n \times n$ matrix $A$, the followings are equivalent:

1. $A$ is nonsingular, that is, $A^{-1}$ exists.
2. $A$ is row equivalent to $I_n$.
3. The linear system $Ax = b$ has a unique solution.
4. The homogenous linear system $Ax = 0$ has only zero (trivial) solution.
5. $A$ is a product of elementary matrices.
6. $\det(A) \neq 0$.

Now we show an another method to evaluate the determinant of an $n \times n$ matrix $A$.

**Definition (Minor-Cofactor)**

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **minor** of $a_{ij}$ is defined as $\det(M_{ij})$, where $M_{ij}$ is $(n-1) \times (n-1)$ submatrix of $A$ obtained by deleting the $i$-th row and $j$-th column of $A$. The **cofactor** of $a_{ij}$ is defined as $A_{ij} = (-1)^{i+j} \det(M_{ij})$.

**Example:** $A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \Rightarrow \det(M_{12}) = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6,$

$A_{12} = (-1)^{1+2} \det(M_{12}) = 6.$

# Cofactor Expansion

**Theorem**

*Let* $A = [a_{ij}]$ *be an* $n \times n$ *matrix. Then*

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} a_{ij} A_{ij} \ \textit{(expansion of } \det(A) \textit{ along the } i\textit{-th row)} \\ \det(A) &= \sum_{i=1}^{n} a_{ij} A_{ij} \ \textit{(expansion of } \det(A) \textit{ along the } j\textit{-th column)} \end{aligned}$$

**Remark:** It is useful to expand along the row (column) which has more zero.

**Example:** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$. If we expand $\det(A)$ along the $1st$ row, we get

$$\begin{aligned}
\det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\
&= 1(-1)^{1+1}\begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} \\
&\quad +3(-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\
&= -21.
\end{aligned}$$

**Example:**

$$
\begin{aligned}
A &= \begin{vmatrix} 1 & 1 & 3 & 4 \\ 5 & 3 & 6 & -2 \\ 2 & 0 & 0 & -2 \\ 2 & 0 & -2 & 1 \end{vmatrix} \overset{c_1+c_4\to c_4}{=} \begin{vmatrix} 1 & 1 & 3 & 5 \\ 5 & 3 & 6 & 3 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -2 & 3 \end{vmatrix} \\
&= (-1)^{3+1}(2)\begin{vmatrix} 1 & 3 & 5 \\ 3 & 6 & 3 \\ 0 & -2 & 3 \end{vmatrix} \overset{\frac{1}{3}r_2\to r_2}{=} (2)(3)\begin{vmatrix} 1 & 3 & 5 \\ 1 & 2 & 1 \\ 0 & -2 & 3 \end{vmatrix} \\
&= {}^{(-r_1+r_2\to r_2)} 6 \begin{vmatrix} 1 & 3 & 5 \\ 0 & -1 & -4 \\ 0 & -2 & 3 \end{vmatrix} = (-1)^{1+1}\, 6 \begin{vmatrix} -1 & -4 \\ -2 & 3 \end{vmatrix} = -66.
\end{aligned}
$$

# Applications of determinants: Finding inverse of a matrix

## Definition (Adjoint)

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **adjoint** of A is defined as

$$adj\,(A) = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

where $A_{ji}$ is the cofactor of $a_{ji}$.

**Theorem:** Let $A = [a_{ij}]_{n\times n}$. If $\det(A) \neq 0$, then $A^{-1} = \frac{1}{\det(A)} adj\,(A)$.

**Proof:** Since $A adj\,(A) = \det(A)\, I_n$, we have

$\frac{1}{\det(A)}(A adj\,(A)) = \frac{1}{\det(A)}(\det(A)\, I_n) \Rightarrow A\left(\frac{1}{\det(A)} adj\,(A)\right) = I_n \Rightarrow$
$A^{-1} = \frac{1}{\det(A)} adj\,(A)$.

**Remark:** If $A$ is nonsingular, then $(adj\,(A))^{-1} = \frac{1}{\det(A)} A$.

# Applications of determinants: Finding inverse of a matrix

**Example**

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}$. Find $A^{-1}$, if it exists.

**Solution:** Since $\det(A) = -21 \neq 0$, the matrix $A$ has an inverse.

If we evaluate the matrix $adj(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, we get

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} = -2, \quad A_{21} = -5, \quad A_{31} = -9$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5, \qquad A_{22} - 2, \qquad A_{32} = 9$$

$$A_{12} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3, \qquad A_{23} = 3, \qquad A_{33} = -3.$$

## Applications of determinants: Finding inverse of a matrix

$$
\begin{aligned}
A^{-1} &= \frac{1}{\det(A)} adj(A) \\
&= \frac{-1}{21} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} \frac{2}{21} & \frac{5}{21} & \frac{9}{21} \\ \frac{5}{21} & \frac{2}{21} & -\frac{9}{21} \\ \frac{3}{21} & -\frac{3}{21} & \frac{3}{21} \end{bmatrix}.
\end{aligned}
$$

**Homework:** Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Find $A^{-1}$(if it exists), by using $adj(A)$.

# Applications of determinants: Cramer's rule

**Theorem**

*Consider the linear system of $n$ equations in $n$ unknowns,*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n. \end{aligned}$$

*Let $A$ be the coefficient matrix of given linear system. If* $\det(A) \neq 0$, *then the linear system has a unique solution as*

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)},$$

*where $A_i$ is the matrix obtained from $A$ by replacing $i$-th column of $A$ by $b$.*

# Applications of determinants: Cramer's rule

**Example:** Consider the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 1 \\ x_1 - x_2 + x_3 &= 3. \end{aligned}$$

Since $\det(A) = -21 \neq 0$, the linear system has a unique solution, and the solution can be obtained by using Cramer's rule as:

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 5 & 2 & 3 \\ 1 & 1 & -3 \\ 3 & -1 & 1 \end{vmatrix}}{-21} = 2,$$

$$
\begin{aligned}
x_2 &= \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 5 & 3 \\ 2 & 1 & -3 \\ 1 & 3 & 1 \end{vmatrix}}{-21} = 0, \\
x_3 &= \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix}}{-21} = 1.
\end{aligned}
$$

**Remark:** The Cramer's rule can be used when we have $n$ equations in $n$ unknowns and the coefficient matrix $A$ has an inverse. Otherwise we can use the Gauss-elimination or Gauss-Jordan method. We should also note that the Cramer's rule becomes computationally inefficient when $n$ is large.

# Applications of determinants: Cramer's rule

**Homework:** Solve the following linear system by Cramer's rule, if possible.

$$\begin{aligned}
3x_1 + x_2 - 2x_3 + x_4 &= -2 \\
x_1 + x_2 + x_3 - x_4 &= 2 \\
x_1 - x_2 - x_3 - 5x_4 &= 0 \\
2x_1 + x_2 + x_4 &= 1.
\end{aligned}$$

Answer: $x_1 = 1, x_2 = -1, x_3 = 2, x_4 = 0.$

# Exercises

**1.** For what values of $t$, $\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t & -1 \\ 0 & 0 & t+1 \end{vmatrix} = 0.$

Lets expand the determinant along the 3th row.

$$\begin{aligned} \begin{vmatrix} t-1 & 0 & 1 \\ -2 & t & -1 \\ \mathbf{0} & \mathbf{0} & \mathbf{t+1} \end{vmatrix} &= 0 \Leftrightarrow (t+1)\begin{vmatrix} t-1 & 0 \\ -2 & t \end{vmatrix} = 0 \\ &\Leftrightarrow (t+1)\,t\,(t-1) = 0 \\ &\Leftrightarrow t = 0 \text{ or } t = 1 \text{ or } t = -1. \end{aligned}$$

**2.** For what values of $t$, $\begin{vmatrix} t-1 & -1 & -2 \\ 0 & t & 2 \\ 0 & 0 & t-3 \end{vmatrix} = 0.$

Answer: $t = 0$ or $t = 1$ or $t = 3$.

# Exercises

**3.** Determine values of $t$ for which the homogenous linear system

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 4x_1 - x_2 + 7x_3 &= 0 \\ \mathbf{t}x_1 + x_2 + x_3 &= 0 \end{aligned}$$

has infinitely many solutions (nontrivial solutions).
Recall that $det\,(A) \neq 0 \Leftrightarrow Ax = 0$ has unique(zero)solution.

$$\begin{aligned} det\,(A) &= \begin{vmatrix} \mathbf{1} & -1 & 3 \\ \mathbf{4} & -1 & 7 \\ \mathbf{t} & 1 & 1 \end{vmatrix} \\ &= 1\,(-1)^{1+1} \begin{vmatrix} -1 & 7 \\ 1 & 1 \end{vmatrix} + 4\,(-1)^{2+1} \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} + t\,(-1)^{3+1} \begin{vmatrix} -1 \\ -1 \end{vmatrix} \\ &= 8 - 4t \end{aligned}$$

Thus for $t = 2$, the system has infinitely many solutions.

**4.** Determine values of $t$ for which the linear system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 - \mathbf{t}x_3 &= 0 \\ x_1 - x_2 - 4x_3 &= -1 \end{aligned}$$

has a unique solution.

Answer: For $t \neq -14$, the system has a unique solution.

# Remark

For $n \times n$ matrix $A$, the followings are equivalent:

1. $A$ is nonsingular, that is, $A^{-1}$ exists.
2. $A$ is row equivalent to $I_n$.
3. The linear system $Ax = b$ has a unique solution.
4. The homogenous linear system $Ax = 0$ has only zero (trivial) solution.
5. $A$ is a product of elementary matrices.
6. $\det(A) \neq 0$.