Extended Essay: Physics

How does the angular velocity of the rotor affect the stability of massive objects?

Words count: 3987

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1 INTRODUCTION

When I was young, I used to play a space exploration game called KSP. This intensified my curiosity about exploring space. To be able to participate in this game I had to dedicate some time to studying physics. Orbital Mechanics was needed to create precise trajectories.

Interested in aerospace physics, I watched a video of a booster of a Space x attempting to land vertically and failing. After I watched this video, I had a question which is how to increase the stability of the spacecraft. This truly captivated my imagination was Spin Stabilization.

My interest in the stability of spacecraft deepened and I began to study the methods of stabilizing spacecraft, including spin stabilization and three-axis stabilization. that truly captivated my imagination was Spin Stabilization.

1.1 LITERATURE REVIEW AND RESEARCH MODEL

Spin stabilization uses spinning, like a gyroscope, to maintain stability. A gyroscope has a rotor and gimbal, allowing its wheel to move in any direction, preserving momentum for stability. When a gyroscope spins rapidly and encounters an external force, its direction only slightly shifts due to the resistance offered by its angular momentum. This characteristic is known as directional stability. It is the main theme of this extended essay.

The research question of this extended essay is **"HOW DOES THE ANGULAR VELOCITY OF THE ROTOR AFFECT THE STABILITY OF MASSIVE OBJECTS?"**.

While one research paper [12] discusses how gravity affects gyroscope angular momentum, causing precession, another research paper [1] explains using a rotor for boat stabilization. However, the directional stability of gyroscopes remains underexplored. This expanded paper employs a unique research model to further study the directional stability of rotating objects, specifically measuring how angular velocity of rotating objects affects the force required to tilt the axis of rotation.

1.2 BACKGROUND KNOWLEDGE

1.2.1 Stability

In terms of mechanics, stability can be interpreted as the intrinsic quality that an object attempts to return to its original state and maintaining that state when the equilibrium gets disturbed by interference.

Stability can be classified as static stability or dynamic stability. Static stability refers to an object's initial tendency that returns to the original equilibrium.

Dynamic stability is how an object reacts to be stable against a given interference. It can be said that all objects with dynamic stability have positive static stability.

By combining these types of stability, it is possible to comprehensively describe directional stability, which maintains the direction in which an object is moving or rotating.

1.2.2 Stabilization for massive object (spacecraft)

Stability is a principal factor of spacecraft. Spacecraft employ orbit control systems to reach their final orbits, after which posture control systems take over to achieve the desired orientation.

Posture control methods fall into two categories: passive and active. Passive control relies on simple rotation or natural forces like gravity, while active control utilizes powered mechanisms.

Spin stabilization is a technique that uses the gyro effect to continuously rotate the axis of the satellite at a constant speed to maintain its posture.

This method includes both full-body rotation and dual-body rotation, where only one part of the satellite rotates to stabilize posture. An example of spin stabilization is the Pioneer 10 and 11 spacecraft. [13]

1.2.1 Angular momentum and torque

Angular momentum is a vector quantity, and it can be described as the rotational analog of linear momentum Angular momentum is given as following formula:

$$L = I\omega$$

$I \rightarrow rotational\ inertia$

$\omega \rightarrow \text{angular velocity}$

Rotational inertia is defined as following:

> Resistance to changes to its rotation [10]

Angular velocity (ω) can be defined as the rate of change in angular displacement with respect to time.

$$\omega = \frac{\Delta\theta}{\Delta t}$$

$\Delta\theta \rightarrow angular\ displacement$

$\Delta t \rightarrow time\ taken$

An arrow can be used to symbolize angular momentum.

To quote from Khan Academy's article about Torque,

> *'If a hand is curled around the axis of rotation with the fingers pointing in the direction of the force, then the torque vector points in the direction of the thumb.' [7].*

So, the angular momentum arrow for a rotating motor can be visualized as Figure 1.

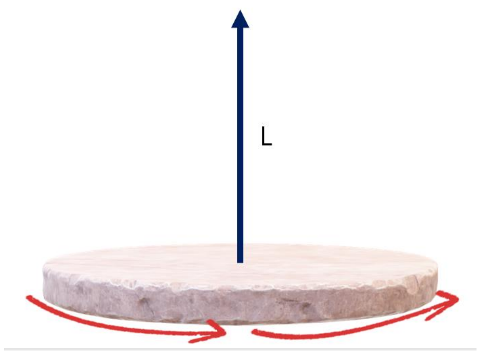

Figure 1: depiction of the angular momentum.

The magnitude angular momentum (L) is represented by the length of the arrow. Torque is expressed as the product of force and perpendicular distance (r) from the origin.

$$\tau = r \times F$$

The torque (τ) also can be represented by an arrow. It is shown in *Figure 2*.

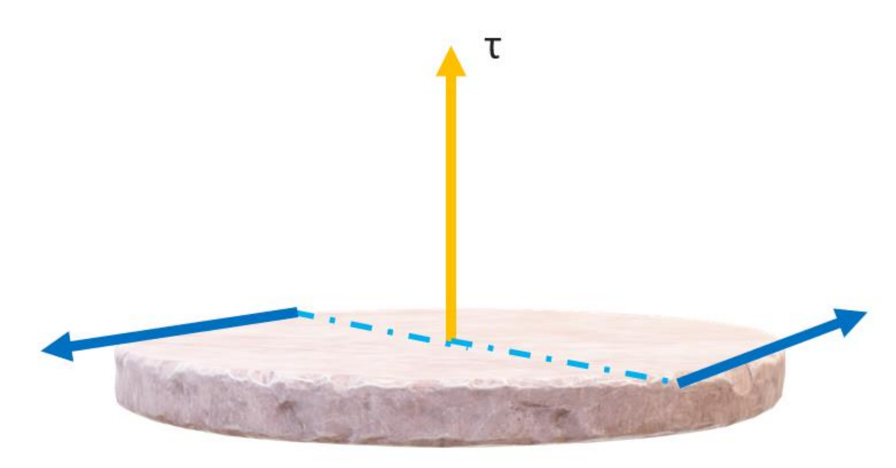

Figure 2: depiction of the torque

Torque also represents the rate of change of angular momentum.

$$\tau = \frac{dL}{dt}$$

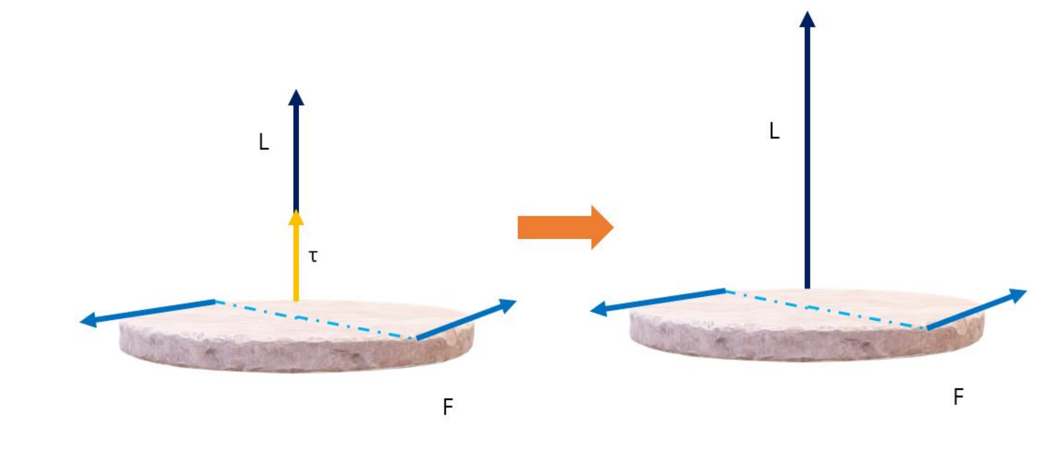

Figure 3: depiction of the relationship between torque and angular momentum.

To quote from "Vector Nature of Rotational Kinematics" which a book dealing with physical concepts and principles related to rotational motion,

> The torque produced is perpendicular to the angular momentum, thus the direction of the angular momentum is changed, but not its magnitude. [11]

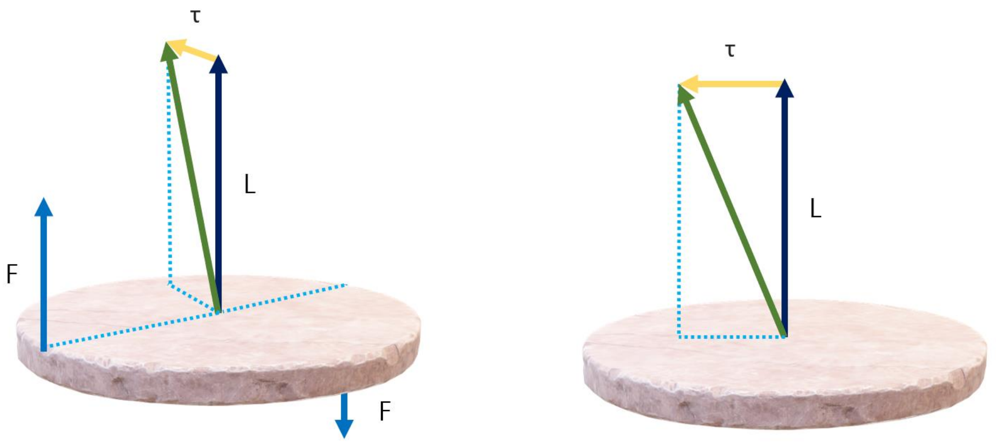

Figure 4: depiction of the changing direction of angular momentum due to perpendicular torque.

1.2.2 Gyroscope directional stability

In the realm of massive objects, the angular momentum vector remains an unwavering constant. This steadfastness is owed to the innate inertia of such objects, akin to their resolute determination to persist in their current momentum state.

Picture a spinning object tenaciously clinging to its angular velocity and direction. Any external force audacious enough to tamper with this trajectory faces formidable opposition. Enter gyro torque, the guardian of the object's angular momentum, steadfastly resisting change and preserving the object's original rotation direction.

This bedrock stability in the object's movement arises from the harmonious interplay of two stalwart principles: the conservation of angular momentum and the unyielding nature of rotational inertia. Throughout the entire rotation, this stability reigns supreme.

1.3 THEORY

Directional stability in spinning objects relies on factors like angular momentum and rotational inertia. To confirm this, we'll investigate the link between rotor angular velocity and the force needed to tilt its axis.

While crafting a full-scale experiment with massive objects proved unfeasible, a scaled-down model was constructed to bridge theory and practice.

Here are the key variables at play:

DV	force required to tilt the axis of rotation of the spinning object. (F)
IV	angular velocity of the rotor (ω)
CV	angular velocity of the rotating axis tilt. Mass, shape, and dimensions of the rotor system.

Figure 5: table of the dependent variable, the independent variable and control variables

2 METHODOLOGY

2.1 EXPERIMENTAL SET UP - ROTOR SYSTEM

Securing an expanse suitable for the creation of an experimental milieu featuring a colossal object presents a formidable challenge. Furthermore, my available human resources fall short of the requisite manpower. Therefore, A 6V electric motor and a hard disk were used for making Miniature version instead of massive experiment to ensure reliability and accessibility.

To set up, a PVC pipe ring covered a quarter of the motor's length, securing it. A hard disk from a computer drive was attached to the motor, and a hook added to the disk's back for axis tilt control with a string.

Holes were drilled at both ends of a steel structure to accommodate screws that face each other. Needles on the screws gripped the PVC tube ring to reduce friction between the motor and the structure. The motor's position was adjusted to align with the center of mass.

A string of specified length was attached to the hook at the back of the motor, and a force meter was hung from it. A support was created to allow vertical movement of the force meter while limiting horizontal motion.

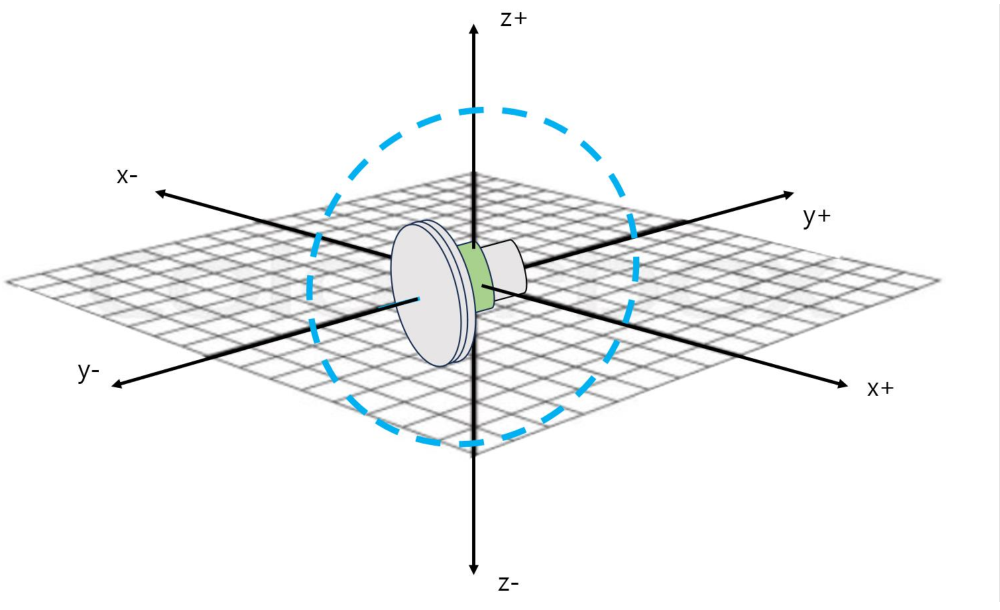

Figure 6: rotor system in 3D plane

In a three-dimensional plane, the rotor system rotates within the xz plane, allowing the rotational axis to tilt solely within the yz plane.

The angular displacement, Δθ, between the initial and final rotational axes is 60 degrees, which converts to 1.04 radians.

With a preset time of 1.5 seconds for this degree of rotation, the angular velocity of the rotating axis tilt(ω_{tilt}) is calculated as following:

$$\omega_{tilt} = \frac{\Delta\theta}{\Delta t} = \frac{1.04}{1.5} = 0.69\, rad\ s^{-1}$$

This motion is evident in Figure 7, where the force meter responds to the external force resulting from the rotation.

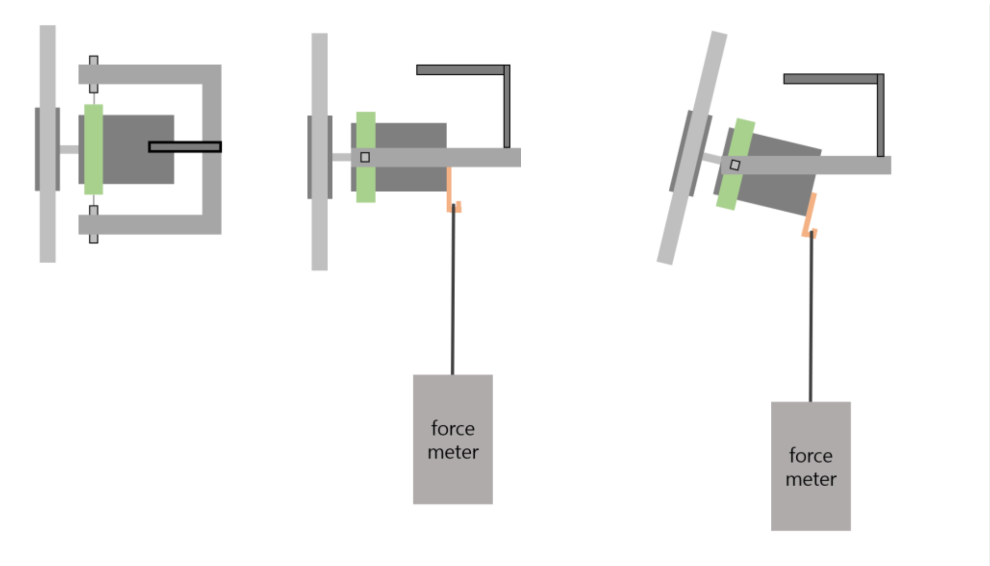

Figure 7: Rotor experiment system diagram

Figure 8 and Figure 9 are the labeled images of the experimental set up.

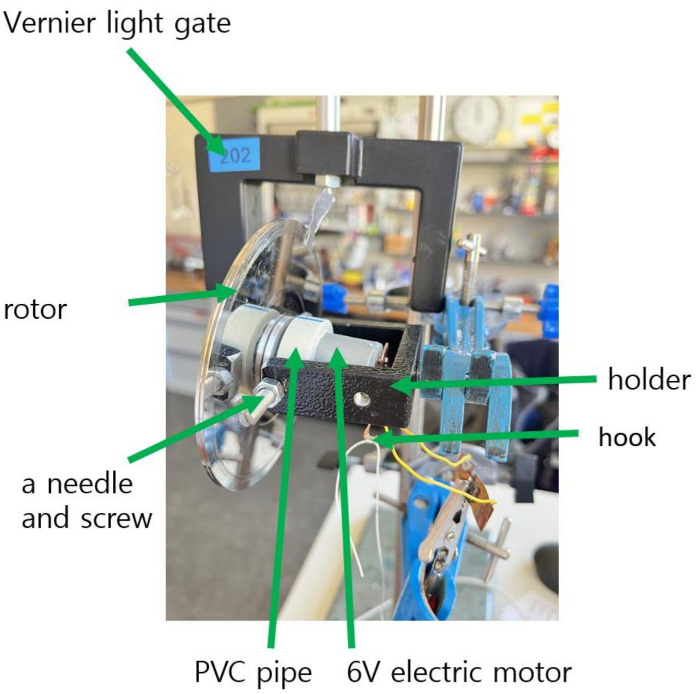

Figure 8: experiment picture 1

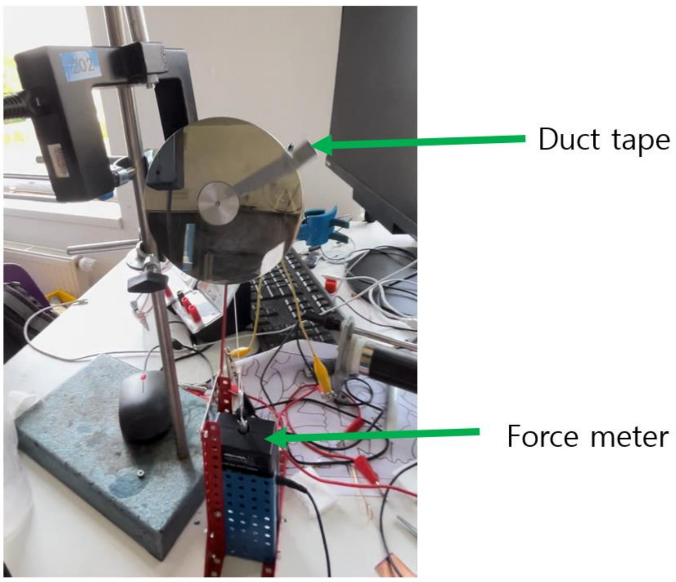

Figure 9: experiment picture 2

2.2 MEASURING ROTATIONAL FREQUENCY

A piece of tape on the hard disk passed through a Vernier light gate for data collection. Logger Pro, with a custom plugin, recorded the motor's rotational frequency as RPM with a 1 RPM reading limit.

2.3 MEASURING FORCE

Force measurement involved a fixed-length string connected to a force meter behind the motor, recording repulsive force during motion. The force meter moved at a constant speed, averaging a 1.5-second duration. Logger Pro was employed for graphing, with a force reading limit of 0.01N.

2.4 ROTATIONAL INERTIA OF THE ROTOR

Rotational inertia is the property of an object that resists changes in rotation while it is rotating. It acts like mass in translational motion.

To calculate the rotational inertia of the entire rotor system, it is necessary to calculate the rotational inertia of each part.

$$I_{total} = \sum I_{parts} \quad [4]$$

Calculation is performed using I_{parts}, as the rotor system comprises multiple components, as shown in Figure 10.

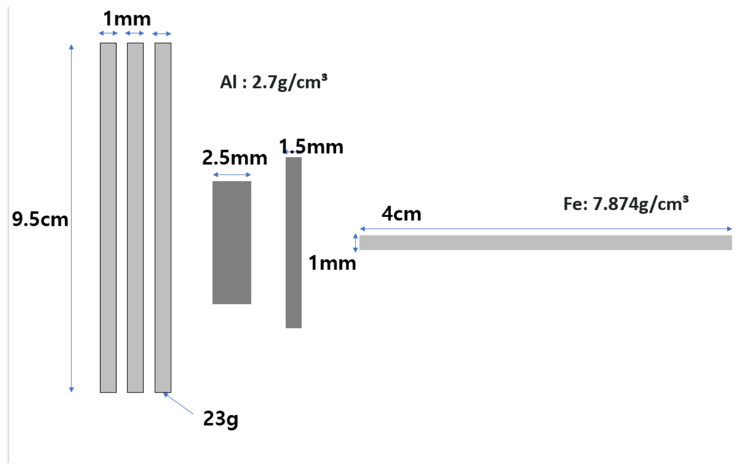

Figure 10: Specifications of the rotor part

2.4.1 Calculation of I_{parts}.

2.4.1.1 Rotational inertia of 3 discs

The rotor system consists of three discs, each weighing 23g. The diameter is shown in figure 11.

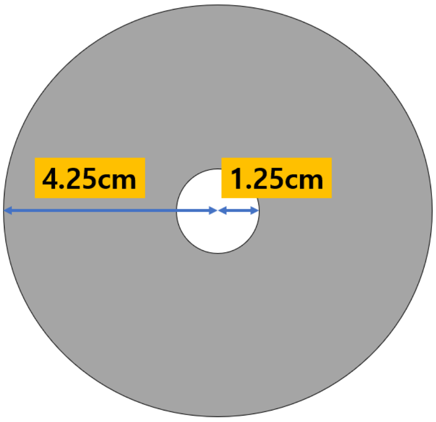

Figure 11: Specifications of disc

The rotational inertia of the hollow cylinder is given by the following formula.

$$I = \frac{1}{2}m({r_o}^2 - {r_i}^2)$$

where r_o represents the outer radius and r_i represents the inner radius.

Firstly, A rotor has three disks, which have the following specifications.

$$r_{o\ disc} = 4.25cm$$
$$r_{i\ disc} = 1.25cm$$
$$m_{disc} = 23g$$

Using the given values, find the rotational inertia of one disc.

$$I_d = \frac{1}{2} \cdot 23 \cdot (4.25^2 - 1.25^2)$$
$$I_d = 189.75\ gcm^2$$

Therefore, I_d of three discs

$$3 \cdot I_d = 3 \cdot 189.75\ =\ 569.25\ gcm^2$$

2.4.1.2 Rotational inertia of aluminum part

Second, this aluminum part is divided into small and large diameter parts. Below displays the exact figures.

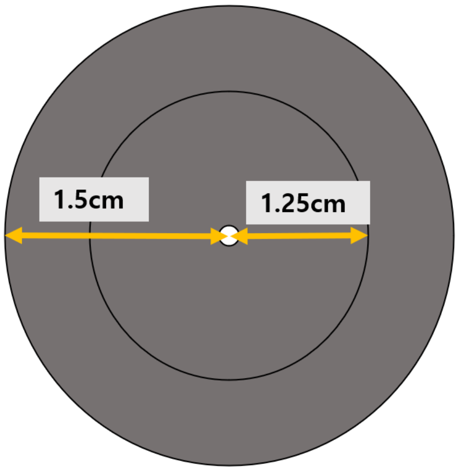

Figure 12: Specifications of the aluminum part

To calculate the rotational inertia of an aluminum part with two different diameters (1 and 2) and a central hole with a radius of 0.5 mm, The volumes of these parts must be determined.

$$V_1 = \pi {r_{o1}}^2 h_1 - \pi {r_{i1}}^2 h_1 = \pi h_1 (r_{o1}^2 - r_{i1}^2)$$
$$h_1 = 1.5mm = 0.15\ cm$$
$$r_{o1} = 1.5cm$$
$$r_{i1} = 0.5mm = 0.05cm$$
$$= \pi \cdot 0.15 \cdot (1.5^2 - 0.05^2)$$
$$= 1.05\ cm^3$$

$$V_2 = \pi {r_{o2}}^2 h_2 - \pi {r_{i2}}^2 h_2 = \pi h_2 (r_{o2}^2 - r_{i2}^2)$$
$$h_2 = 2.5mm = 0.25\ cm$$
$$r_{o2} = 1.25cm$$
$$r_{i2} = 0.5mm = 0.05cm$$
$$= \pi \cdot 0.25 \cdot (1.25^2 - 0.05^2)$$
$$= 1.22cm^3$$

Calculate mass using the given volume and density. Aluminum has a density of 2.7gcm^{-3}.

$$\rho_{Al} = 2.7gcm^{-3}$$
$$m_1 = \rho_{Al} V_1 = 2.7 \cdot 1.05 = 2.85g$$
$$m_2 = \rho_{Al} V_2 = 2.7 \cdot 1.22 = 3.30g$$

Calculates rotational inertia from the calculated mass.

$I_1 = \frac{1}{2} m_1 ({r_{o1}}^2 - {r_{i1}}^2)$ $= \frac{1}{2} \cdot 2.85 \cdot (1.5^2 - 0.05^2)$ $= 3.21 gcm^2$
$I_2 = \frac{1}{2} m_2 ({r_{o2}}^2 - {r_{i2}}^2)$ $= \frac{1}{2} \cdot 3.30 \cdot (1.25^2 - 0.05^2)$ $I_2 = 2.58 gcm^2$

2.4.1.3 Rotational inertia of rod

Lastly, Calculate the rotational inertia of a motor rod. volume and mass must be calculated by using the given volume and density.

$$V_{rod} = \pi r^2 h$$
$$h_{rod} = 4cm$$
$$r_{rod} = 0.5mm = 0.05cm$$
$$V_{rod} = \pi \cdot (0.05)^2 \cdot 4 = 0.031 cm^3$$

Iron has a density of $7.874 gcm^{-3}$.

$$\rho_{Fe} = 7.874 gcm^{-3}$$
$$m_{rod} = \rho_{Fe} V_{rod} = 7.874 \cdot 0.031$$
$$m_{rod} = 0.24 g$$

Calculate rotational inertia from the calculated mass.

$$I_{rod} = \frac{1}{2} m_{rod} r^2$$
$$= \frac{1}{2} \cdot 0.24 \cdot 0.05^2$$
$$I_{rod} = 3.092 \cdot 10^{-4}$$

2.4.1.4 Calculate I_{total}

Then let's find the rotational inertia of the entire rotor system. The total rotational inertia is defined by the following equation

$$I_{total} = I_1 + I_2 + I_3 \cdots = \sum I_{parts}$$

Total rotational inertia is obtained by summing the rotational inertia of all individual parts.

$$I_{total} = I_{3disc} + I_1 + I_2 + I_{rod}$$
$$= 569.25 + 3.2134 + 2.580 + 3.092 \cdot 10^{-4}$$
$$I_{total} = 575\ gcm^2$$

Therefore, the rotational inertia of the entire rotor system is $575\ gcm^2$.

3 DATA COLLECTION

3.1 RAW DATA

The graph shows force samples at 2060 RPM with a 0.3N upward shift. The peak on this graph represents the force required at preset angular velocity, as it takes time for the axis to reach this angular velocity. Measurements should be taken from this peak in Figure 13.

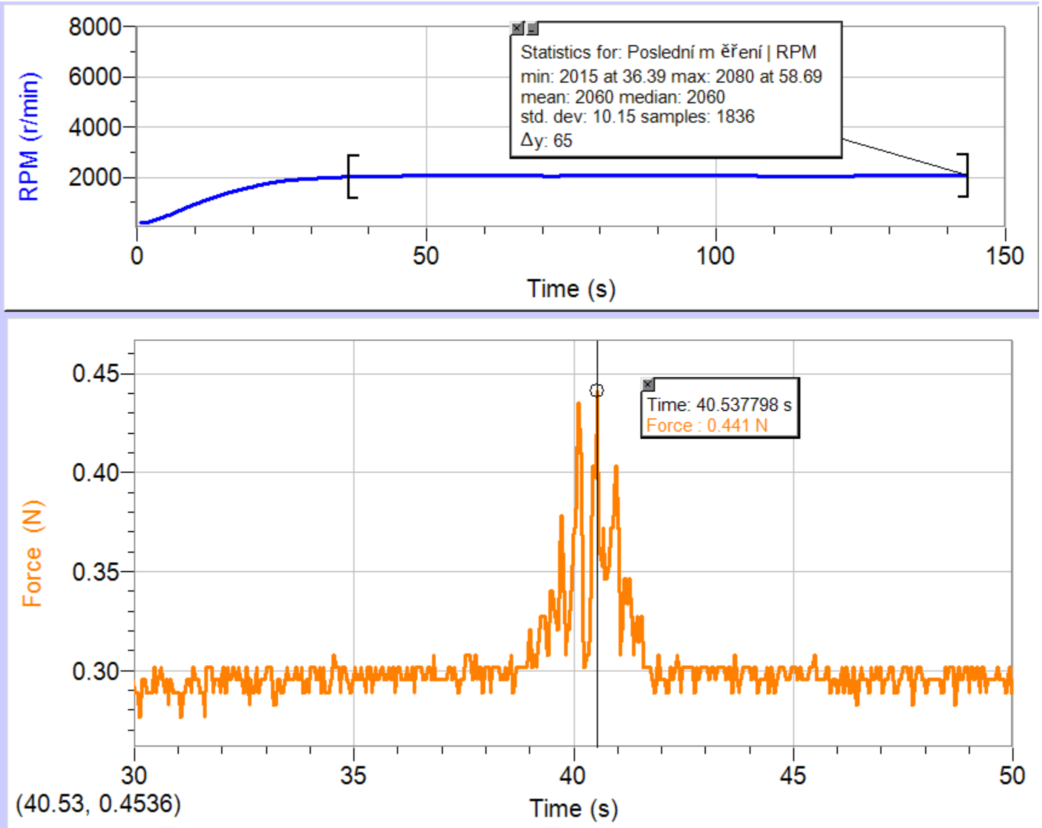

Figure 13: force graph and RPM graph at 2060RPM from logger pro

For precision and reduced uncertainty, the RPM setup was used for five repeated force measurements. RPM and time graphs from Figure 13 were used to derive RPM values and uncertainties. Logger Pro software was employed to calculate the average RPM and its uncertainty.

This data represents the raw measurements of force and rotational frequency (RPM), as described in the 'MEASURING ROTATIONAL FREQUENCY' and 'MEASURING FORCE' chapters above.

RPM $(r \cdot min^{-1})$	ΔRPM $(r \cdot min^{-1})$	$Force\ (N)$				
		Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
2060	65	0.49	0.47	0.43	0.44	0.43
2935	70	0.54	0.54	0.56	0.63	0.58
4012	85	0.71	0.67	0.63	0.64	0.61
4872	119	0.75	0.79	0.77	0.74	0.72
5945	99	0.87	0.84	0.83	0.86	0.89
6599	101	0.96	1.00	0.93	0.98	0.95

Figure 14: raw data table with RPM, RPM uncertainty Force samples

3.2 PROCESSED DATA

3.2.1 Force

The force was measured using a force meter. The average of the five force samples acquired during the experiment at 2060 RPM was calculated.

$Force\ (N)$				
0.49	0.47	0.43	0.44	0.43

$$Average\ Force = \frac{0.49 + 0.47 + 0.43 + 0.44 + 0.43}{5} = 0.45\ (N)$$

The following table shows five force samples and mean force by RPM setting.

RPM $(r \cdot min^{-1})$	$\bar{F}\ (N)$
2060	0.45
2935	0.57
4012	0.65
4872	0.75
5945	0.86
6599	0.96

Figure 15: table with RPM, mean force

3.2.2 Angular velocity

The rotational speed is measured as RPM (Revolutions Per Minute) with the Vernier light gate. RPM can be converted to angular velocity $(rad \cdot s^{-1})$ with the following formula.

$$f_{RPM} = 2060$$

$$\omega = \frac{2 \cdot \pi \cdot f_{RPM}}{60}$$

$$= \frac{2 \cdot \pi \cdot 2060}{60}$$

$$= 68.6\pi$$

$$\approx 215.6\ (rad \cdot s^{-1})$$

The following table shows RPM values and their corresponding angular velocity values in radians per second (3 significant figures).

$RPM(r \cdot min^{-1})$	$\omega\ (rad \cdot s^{-1})$
2060	215.6
2935	307.2
4012	419.9
4872	509.9
5945	622.2
6599	690.7

Figure 16: RPM, angular velocity conversion table

3.3 UNCERTAINTY CALCULATION

In this extended essay, the determination of uncertainty plays an important role in increasing accuracy.

3.3.1 Force

The calculation of uncertainty values is required for five force samples. The table below shows the force measurements and their mean values:

$Force\ (N)$					$\bar{F}\ (N)$
0.49	0.47	0.43	0.44	0.43	0.45

Uncertainty of the force can be obtained by following formula:

$$Uncertainty = \frac{\text{standard deviation of data}}{\sqrt{number\ of\ data\ points}}$$

And the standard deviation of data can be calculated with following formula:

$$s = \sqrt{\frac{\sum(x_i - \bar{x})^2}{N - 1}}$$

$s \rightarrow$ Standard Deviation of data

$x_i \rightarrow$ *Each individual data point*

$\bar{x} \rightarrow$ *the sample mean*

$N \rightarrow$ *total number of data points*

Let's calculate the standard deviation:

$$(x_i - \bar{x})^2$$

$$(0.49 - 0.45)^2 = (0.04)^2 = 0.0016$$
$$(0.47 - 0.45)^2 = (0.02)^2 = 0.0004$$
$$(0.43 - 0.45)^2 = (-0.02)^2 = 0.0004$$
$$(0.44 - 0.45)^2 = (-0.01)^2 = 0.0001$$
$$(0.43 - 0.45)^2 = (-0.02)^2 = 0.0004$$

$$\sum(x_i - \bar{x})^2$$

$$0.0016 + 0.0004 + 0.0004 + 0.0001 + 0.0004$$
$$= 0.0029$$

$$s = \sqrt{\frac{\sum(x_i - \bar{x})^2}{N - 1}}$$
$$= \sqrt{\frac{0.0029}{5 - 1}}$$
$$= 0.02696$$

Now, uncertainty can be calculated as:

$$Uncertainty = \frac{\text{standard deviation of data}}{\sqrt{number\ of\ data}}$$

$$= \frac{0.02696}{\sqrt{5}}$$

$$\approx 0.012$$

The following table summarizes the mean values and uncertainties for the entire mean force:

$\bar{F}$ (N)	ΔF (N)
0.45	0.012
0.57	0.017
0.65	0.017
0.75	0.012
0.86	0.010
0.96	0.012

Figure 17: mean force and uncertainty of force table

3.3.2 Angular velocity

The original RPM has been converted to angular velocity; hence the related uncertainties must be calculated.

$RPM(r \cdot min^{-1})$	$\Delta RPM(r \cdot min^{-1})$	$\omega\ (rad \cdot s^{-1})$	$\Delta\omega\ (rad \cdot s^{-1})$
2060	65	215.6	?

The uncertainty of the converted value can be calculated using the formula:

$$absolute\ uncertainty\ =\ converted\ value\ \times relative\ \text{uncertainty}$$

The fractional uncertainty can be obtained using the following formula:

$$relative\ uncertainty = \frac{absolute\ error}{measured\ value}$$

In this case,

$$\frac{\Delta RPM}{RPM} = \frac{65}{2060}$$

$$\approx 0.0315$$

Now, the converted uncertainty can be calculated as:

$$\Delta\omega = \omega \times \left(\frac{\Delta RPM}{RPM}\right)$$

$$= 215.6 \times 0.0315$$

$$\approx 6.8 \quad (rad \cdot s^{-1})$$

The following table presents the RPM to angular velocity conversion along with uncertainties:

$RPM (r \cdot min^{-1})$	$\Delta RPM (r \cdot min^{-1})$	$\omega\ (rad \cdot s^{-1})$	$\Delta\omega\ (rad \cdot s^{-1})$
2060	65.0	215.6	6.80
2935	70.0	307.2	7.30
4012	85.0	419.9	8.90
4872	119	509.9	12.5
5945	99.0	622.2	10.4
6599	101	690.7	10.6

Figure 18: RPM to angular velocity Conversion Table with uncertainty

3.4 FINAL DATA

The following table shows the calculated angular velocity and force with uncertainty.

$\omega\ (rad \cdot s^{-1})$	$\Delta\omega\ (rad \cdot s^{-1})$	$F\ (N)$	$\Delta F\ (N)$
215.6	6.80	0.45	0.012
307.2	7.30	0.57	0.017
419.9	8.90	0.65	0.017
509.9	12.5	0.75	0.012
622.2	10.4	0.86	0.010
690.7	10.6	0.96	0.012

Figure 19: angular velocity and force table

3.5 GRAPHING AND DATA VISUALIZATION

Related data was collected and visualized using graphing software, especially Logger Pro.

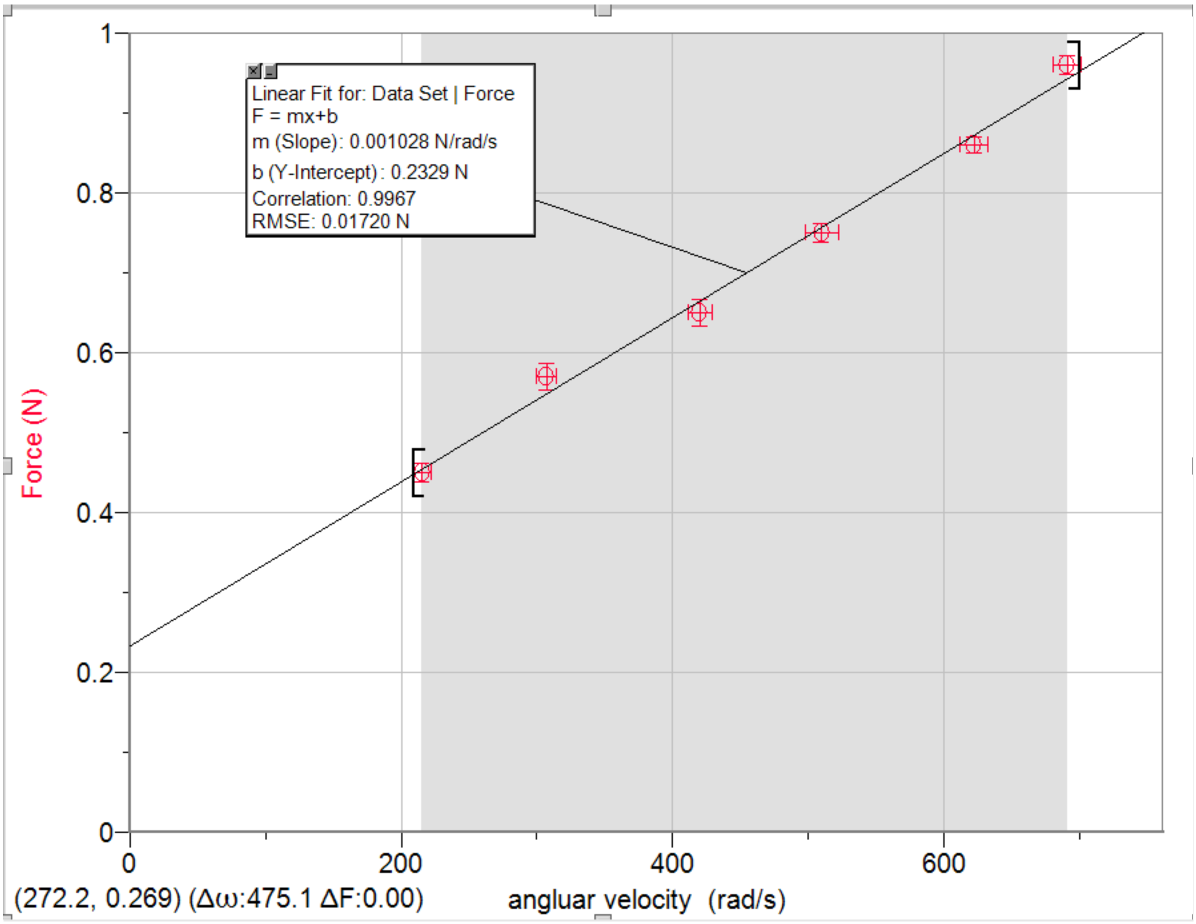

Figure 20: angular velocity and force graph

The graph shows the linear relationship between the force and the angular velocity, and the y-intercept represents the fundamental resistive force, including friction between the needle and the PVC pipe, as approximately 0.23 N. The following equation illustrates the relationship between force (F) and angular velocity in Figure 20.

$$F = 1.028 \cdot 10^{-3}\ \omega + 0.23$$

$$F \propto \omega$$

For a more detailed analysis of this graph, see Section 4.6 1.1 "GRAPH ANALYSIS".

4 ANALYSES

4.1 OPPOSING FORCE

To overcome resistance and move an object, the applied force must surpass the resistance force. Movement begins when these forces balance. The system's resistive force comprises mainly the gyroscopic force, linked to gyroscope stability, and friction between the PVC rotor pipe and the steel needle. The relationship between the force acting when the axis of rotation is tilted, and the angular velocity can be expressed as a linear relationship (refer to Chapter 3.5 "GRAPHING AND DATA VISUALIZATION").

This relationship follows specific principles related to the directional stability of the gyroscope. In terms of directional stability, tilting the rotational axis will generate the opposition force which is the gyroscopic force against force applied. Also, Increasing the angular velocity of the rotor amplifies angular momentum resulting in increased opposite force. The explanation is as follows:

4.2 GYROSCOPIC TORQUE AND ANGULAR MOMENTUM

Gyroscopic torque, often referred to as resistance torque, is best defined as following:

> "A gyroscopic torque will result if the axis of the flywheel is rotated, and it acts perpendicular to the rotor axis." [8]

Gyroscopic torque, the stalwart defender against external forces, heralds the gyroscope's directional stability. Torque represents the dynamic change of angular momentum with respect to time.

$$\tau = \frac{dL}{dt}$$

In conventional scenarios, force application leads to torque generation, subsequently altering angular momentum. However, during the tilting of the rotational axis, it's noteworthy that the angular momentum undergoes initial change. Consequently, torque is induced, giving rise to the sequential generation of forces.

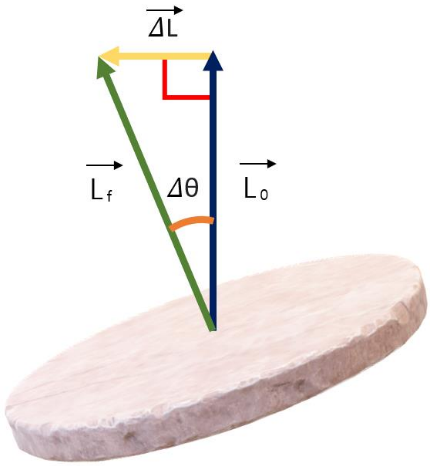

Figure 21: description of changing of direction angular momentum.

To initiate this axis tilt, precision reigns supreme. The external force must mirror the resistive force, ensuring an unswerving, constant-rate axis tilt. In the meticulously orchestrated experiment, a steadfast commitment to minimal acceleration ensured a near-zero net force.

4.2.1 Tiny angular momentum change

In the experiment, the rotational axis tilted perpendicularly, so the angle between changing angular momentum (dL) and initial angular momentum (L_0) is 90 degrees (refer to *Figure 21*. This forms a right-angle triangle which has final angular momentum (L_f) as hypotenuse.

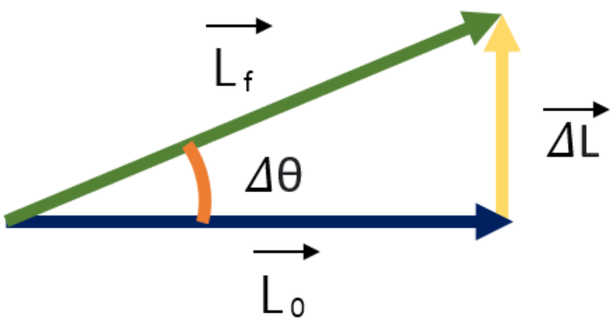

Figure 22: changing angular momentum with momentary small right-angle impulse.

By applying trigonometric functions to this triangle, $\tan(\Delta\theta) = \frac{\Delta L}{L_0}$ can be derived. Therefore, ΔL can be defined as $\Delta L = L_0 \tan(\Delta\theta)$. However, in actual experiment where $\Delta\theta$ is not infinitesimally small, an alternative approach becomes necessary.

4.2.2 Deriving relationships for rotational motion

Given that the above method is not applicable when $\Delta\theta$ is significant, an alternative approach is required to calculate dL, considering the vector nature of angular momentum. The vector Arrow vertices of the angular moment(L) change in a circle path.

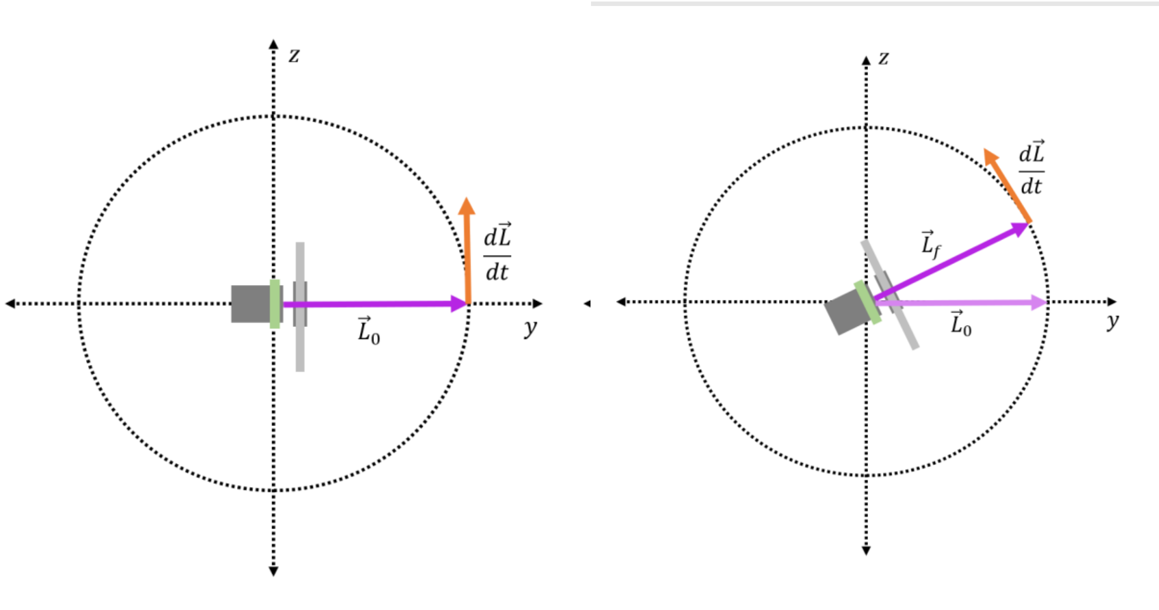

Figure 23: angular momentum varies with respect to time.

Assume vector arrow vertices is a point L on the coordinate plane. The point L rotates constantly by ω $(rad\ s^{-1})$ anticlockwise on a circular path where the center is the origin, and the length of the radius is L_0.

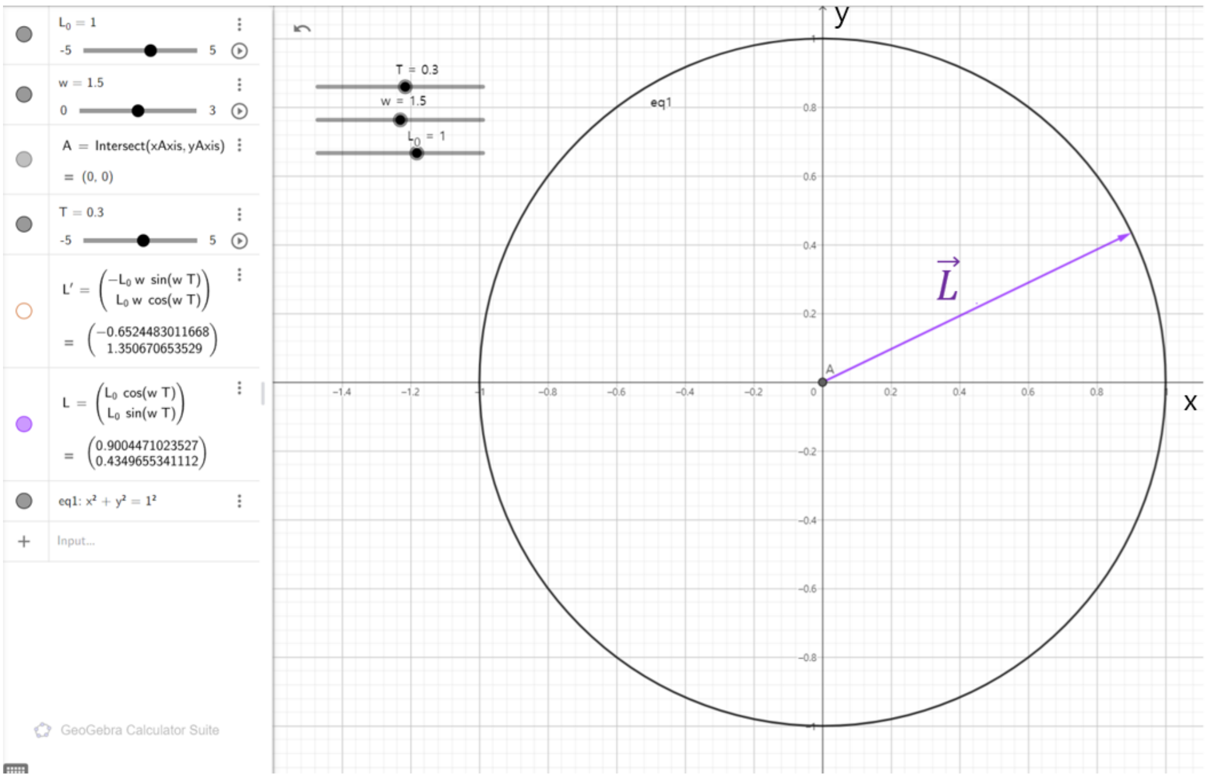

Figure 24: The vector L circular motioning as changing time in the mathematical model

Mark the vector from the origin to point L as $\vec{L}$. The reason for solving these mathematical problems is to find the instantaneous change in vector L over the instantaneous change in time $\left(\frac{d\vec{L}}{dt}\right)$ which is torque.

The vector from the origin to point L can be represented as a combination of x and y vectors.

$$\vec{L} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} L_0 \cos \omega t \\ L_0 \sin \omega t \end{pmatrix}$$

The derivative of the L vector with respect to t is represented as $\frac{d\vec{L}}{dt}$ which can be calculated as follows:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} \frac{d\vec{x}}{dt} \\ \frac{d\vec{y}}{dt} \end{pmatrix} = \begin{pmatrix} -L_0 \omega \sin \omega t \\ L_0 \omega \cos \omega t \end{pmatrix}$$

It's noteworthy that the deduced $\frac{d\vec{L}}{dt}$ vector is consistently perpendicular to the L vector.

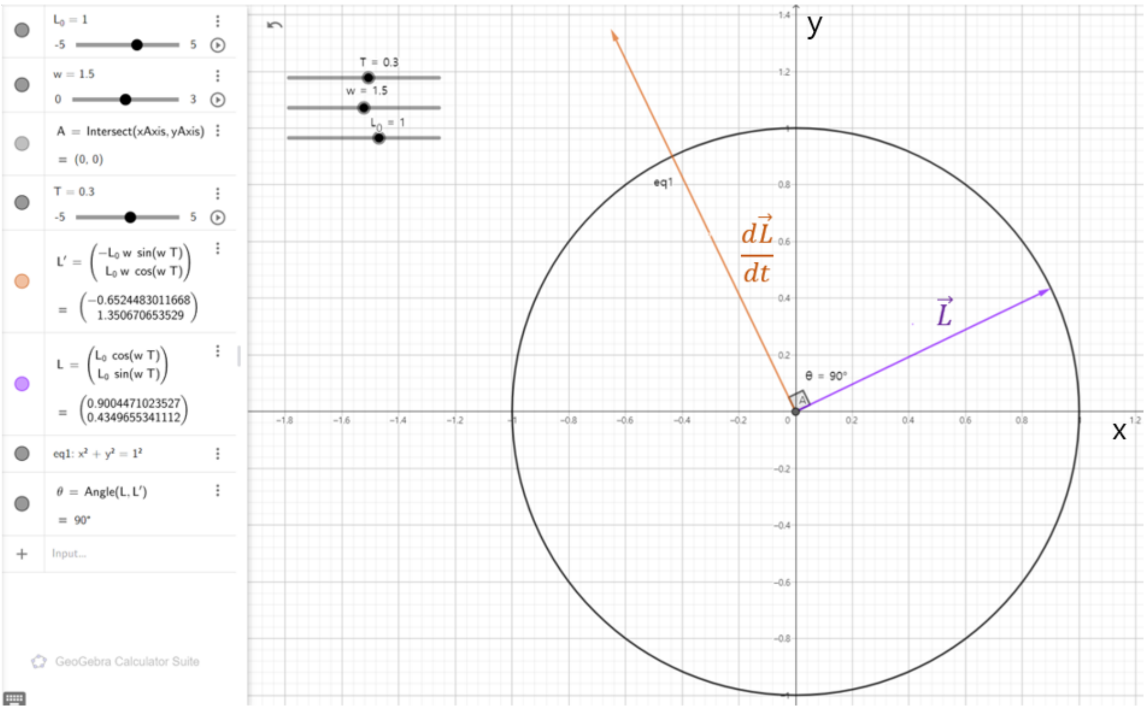

Figure 25: The $\frac{d\vec{L}}{dt}$ *vector in the mathematical model*

For a deeper understanding, please use the following graphical calculator model.

(https://www.geogebra.org/calculator/qxxg7rfv)

To find the absolute value of $\frac{d\vec{L}}{dt}$ The magnitude of vector can be utilized.

$$\frac{dL}{dt} = \left|\frac{d\vec{L}}{dt}\right| = \sqrt{\left(\frac{d\vec{x}}{dt}\right)^2 + \left(\frac{d\vec{y}}{dt}\right)^2}$$

In this context, the absolute value of the differential dL/dt represents the rate of change of angular momentum, which is the torque.

To compute this, the expressions for $\frac{d\vec{x}}{dt}$ and $\frac{d\vec{y}}{dt}$ can be substituted:

$$\frac{dL}{dt} = \left|\frac{d\vec{L}}{dt}\right| = \sqrt{(-L_0\omega\sin\omega t)^2 + (L_0\omega\cos\omega t)^2}$$

Simplifying further:

$$= L_0\omega\sqrt{\sin^2\omega t + \cos^2\omega t}$$

Since $\sin^2\omega t + \cos^2\omega t$ (a trigonometric identity) always equals 1:

$$\frac{dL}{dt} = \left|\frac{d\vec{L}}{dt}\right| = L_0 \cdot \omega$$

The absolute value of the differential is $L_0\omega$, representing the rate of change of angular momentum, which corresponds to torque.

The value of ω above will be expressed as ω_{tilt} This is because ω in the formula means angular velocity that the rotational axis is tilted. The magnitude value of $\frac{d\vec{L}}{dt}$ value shows the rate of change of the angular momentum which is torque.

$$\frac{dL}{dt} = \tau$$

$$\frac{dL}{dt} = L_0 \cdot \omega_{tilt}$$

$$\tau = L_0 \cdot \omega_{tilt}$$

4.3 TORQUE CHASES FORCE

A torque (τ) can be defined as a turning force that tends to cause rotation around an axis. Mathematically, it can be expressed as the product of force (F) and the perpendicular distance (r_{tilt}) between the point of application of force and the pivot point.

$$\tau = F \cdot r_{tilt}$$

The torque introduced in Chapter 4.2.2 'Deriving Relationships for Rotational Motion' can also be related to the expression.

$$\tau = L_0 \cdot \omega_{tilt}$$

$$\tau = F \cdot r_{tilt}$$

$$F \cdot r_{tilt} = L_0 \cdot \omega_{tilt}$$

r_{tilt} indicates the distance between the pivot point of the hook behind the motor and the center point in the experimental environment.

By rearranging the previous equations, The following equation for force (F) is obtained:

$$F = \frac{L_0\omega_{tilt}}{r_{tilt}}$$

Angular momentum can be expressed as the product of rotational inertia and the angular velocity of the rotor.

$$L_0 = I\omega_r$$

ω_r represents the angular velocity of the rotor. This gives the following expression for linear force.

$$F = \frac{I\omega_r\omega_{tilt}}{r_{tilt}}$$

Force (F) in this formula should be classified.

In the experiment, the torque produced by the change of the angular momentum chases a force that causes the rotor to tilt.

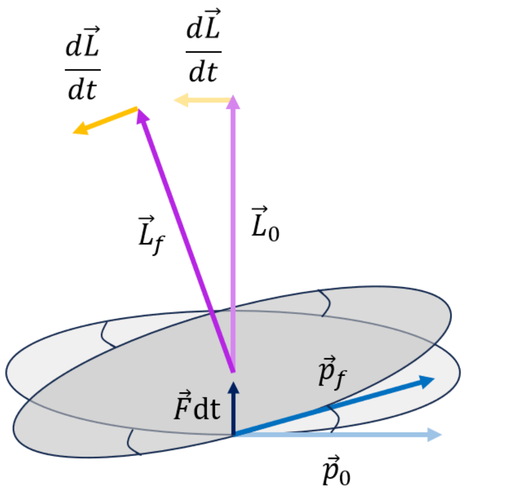

Figure 26: changing angular momentum and linear momentum.

In Figure 26, a linear force alters the direction of a rotating rotor's linear momentum. This force only affects the rotor's spin direction when it's in motion because it applied perpendicularly to linear momentum. Rotating objects show increased directional stability compared to stationary ones due to greater resistance to tilting along the rotational axis.

This can also be interpreted as a rotational axis resists to be tilt. So, this linear force can be defined as a linear resistive force.

It can be expressed in the following way. this linear resistive force can be denoted as F_r.

$$F_r = \frac{I\omega_r\omega_{tilt}}{r_{tilt}}$$

This expression shows a direct relationship between the resistance force due to the linear resistance force (F_r) and the angular velocity of the rotor (ω_r).

$$F_r \propto \omega_r$$

4.4 ROTATIONAL INERTIA

As the actual rotational inertia of the rotor is intricate and challenging to determine, for simplicity, the rotor system is approximated as a single disc. Detailed measurements of angular inertia are presented in chapter 2.2 'ROTATIONAL INERTIA OF THE ROTOR'.

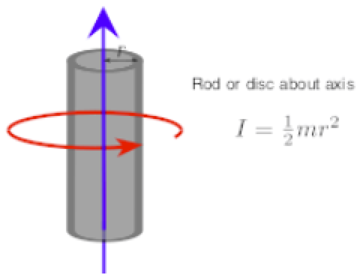

Figure 27: rotational inertia of rod or disc about axis [4]

Rotational inertia is calculated using the following formula:

$$I = \frac{1}{2} m {r_r}^2$$

This formula is incorporated into the expression for resistive force:

$$F_r = \frac{m\omega_{tilt}{r_r}^2}{2r_{tilt}}\omega_r$$

Linear resistive force is related to the mass of the rotor and the square ratio to the radius of the rotor.

4.5 TOTAL FORCE REQUIRED FOR THE ENTIRE SYSTEM

When tilting the axis, both resistive and external forces (F_e) are at play. External forces are caused by factors such as friction between the PVC pipe and the motor stationary needle. The combined resistive force can be defined as follows:

$$F_w = F_r + F_e$$

$$F_w = \frac{I\omega_{tilt}}{r_{tilt}}\omega_r + F_e$$

Within this expression, it is evident that the total resistance force (F_w) of the entire rotor system exhibits a direct proportionality to the angular velocity of the rotor.

$$F_w \propto \omega_r$$

4.6 GRAPH ANALYSIS

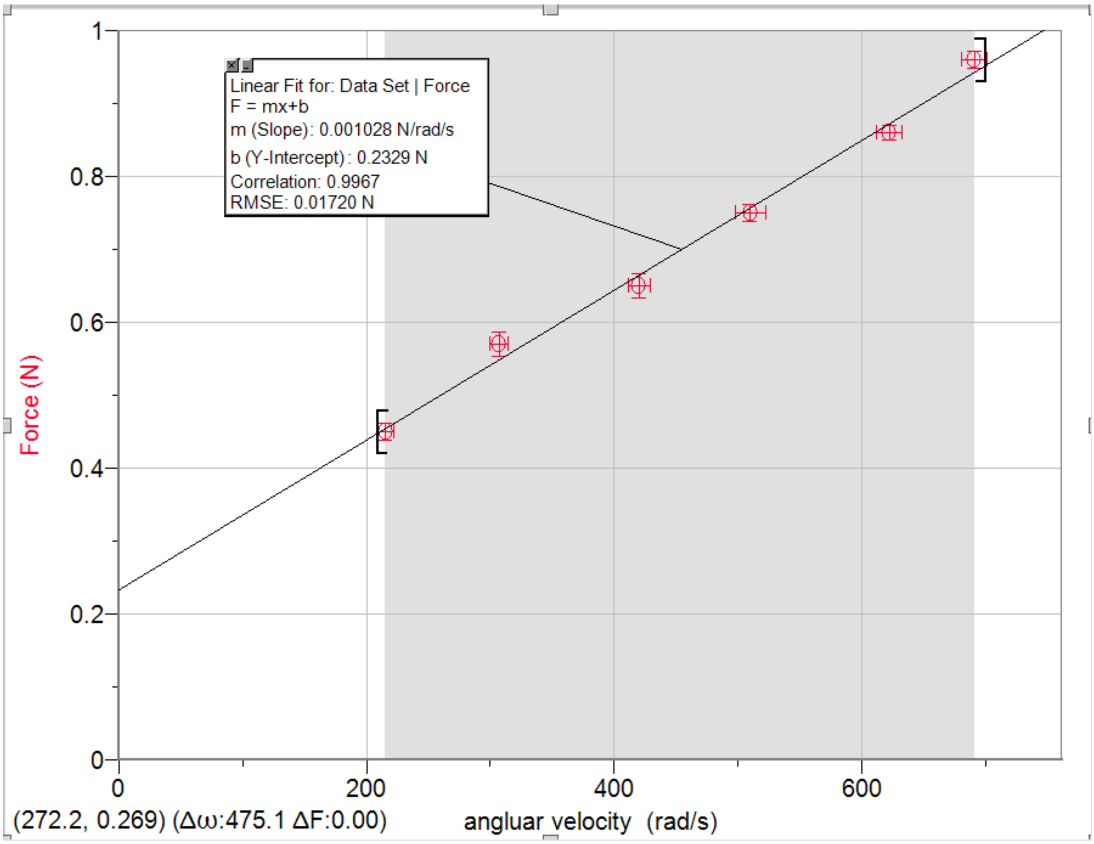

Figure 20: angular velocity and force graph

In Figure 20, which represents the angular velocity and force relationship, A linear graph is described by the equation: ($F = 1.028 \cdot 10^{-3}\omega_r + 0.23$). This equation mirrors the format of the resistive force expression for the entire rotor system ($F_w = \frac{I\omega_{tilt}}{r_{tilt}}\omega_r + F_e$)

In the experimental environment, $\frac{I\omega_{tilt}}{r_{tilt}}$ represents the gradient and F_e, represents y-intercept. Firstly, the value of $\frac{I\omega_{tilt}}{r_{tilt}}$ is $1.028 \cdot 10^{-3}$. Here, 'I' is the rotational inertia of the rotor system, with a value of 575 gcm^2 convertible to $5.75 \cdot 10^{-5}$kgm^2. ω_{tilt} is known as $0.69\, rad\, s^{-1}$ by the discuss in the methodology section, and r_{tilt} is the distance between the point of force application and the pivot point, set at 0.04m.

Calculating $\frac{I\omega_{tilt}}{r_{tilt}}$ as following:

$$\frac{I\omega_{tilt}}{r_{tilt}} = \frac{5.75 \cdot 10^{-5} \cdot 0.69}{0.04} = 9.92 \cdot 10^{-4}\, (kg\, m\, s^{-1})$$

The unit here is $kg\, m\, s^{-1}$, which is the unit of impulse. This indicates Changes in the angular velocity of the rotor result in variations in the resistive force, causing the rotor system to experience impulse.

Experimental conditions indicate that the values in the graph are similar, with errors of approximately 3.5%. Secondly, the y-intercept implies that the external force (F_e) equals $0.23N$.

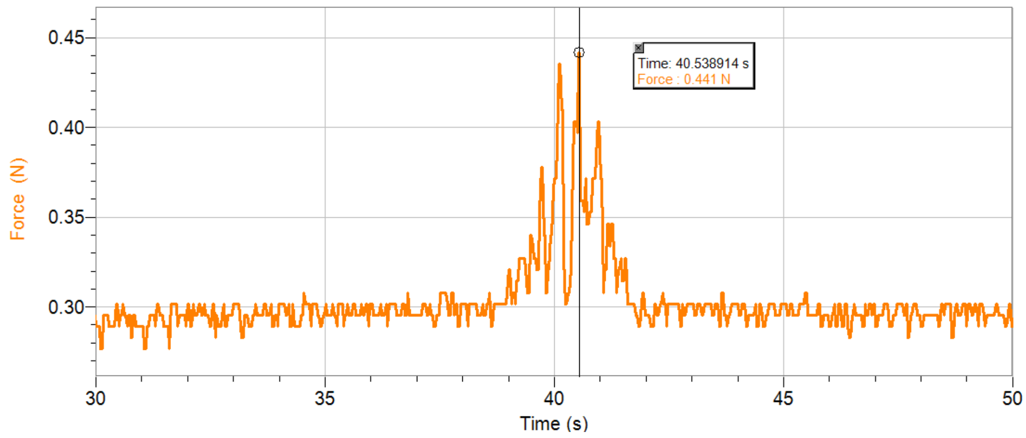

Figure 28: force graph at 2060RPM from logger pro

In this graph, the measurement is taken at approximately 40 seconds. When no measurement is taken, the graph consistently registers 0.3 N, representing an external resistive force. This closely aligns with the y-intercept value of 0.23N. The error of 0.07N between these two values suggests that it can be attributed to experimental variability.

4.7 APPLICATION FOR MASSIVE OBJECT

As mentioned in Chapter 1.2.2 Stabilization for massive object (spacecraft), the stabilization of aerospace objects such as satellites is important. The analysis of this extended essay is the study of the influence of rotor angular velocity on the stability of massive objects for spin stabilization. The rotation stabilization mechanisms of spacecraft, satellites, and other aerospace vehicles can be customized by applying a formula derived from this extended essay.

$$F_w = \frac{I\omega_{tilt}}{r_{tilt}}\omega_r$$

After anticipating external forces that may hinder stability during space missions, it is possible to optimize the angular velocity accordingly to ensure that these massive objects remain stable at various stages of the mission. This contributes to the success and safety of space missions by improving their reliability.

5 CONCLUSION

This extended essay explores the complex domain of gyroscopic stability, highlighting its critical role in spacecraft spin stabilization. In this extended essay, as part of the research on the directional stability of gyroscopes, it is aimed to address the research question 'How does the angular velocity of the rotor impact the stability of massive objects?'

The model under consideration focuses on the gyroscope's rotational axis, which has the unique capability of tilting into the yz plane, distinct from the xy or xz planes, as depicted in Figure 6. The investigation revolves around the collection and analysis of data pairs containing forces and angular velocities within this experimental model. This analysis demonstrates directional stability by establishing a relationship between the changing angular momentum and the angular velocity associated with tilting the axis of rotation into the yz plane.

In the data collection section (chapter 3), A direct proportionality between resistive force and the angular velocity of the rotor is revealed by the expression: F=0.001028ω+0.23. Subsequently, in the analysis section (chapter 4), The relationship between these variables is explored by deriving a differentiating angular momentum vector: $\frac{dL}{dt} = L_0\omega_{tilt}$. This exploration culminates in the expression ($F_w = \frac{I\omega_{tilt}}{r_{tilt}}\omega_r + F_e$), which is validated by comparing the values of $\frac{I\omega_{tilt}}{r_{tilt}}$ and F_e with the specified constants of $1.028 \cdot 10^{-3}$ and 0.23 after inserting the correct parameters into the formula. The derived formula is then shown to be applied to spin stabilization of aerospace vehicles.

Gyroscope theory is highly sophisticated, particularly from a mathematical perspective, given the need to reconcile angular moment vectors and angular velocities in three-dimensional space. Mathematical models accurately describing the directional stability of gyroscopes are often elusive in typical research papers. Therefore, this extended essay presents a unique perspective by presenting a mathematical model that reflects the concept of directional stability.

6 EVALUATION

Throughout this extended essay, several errors were identified concerning the methodology and certain assumptions:

6.1 MINIATURE VERSION OF A MASSIVE OBJECT

Using a small rotor system can lead to differences in behavior due to shrinkage. This is the difference between the structural and material properties of the small rotor system and the large object, resulting in limitation. Therefore, experimental data collected in these experimental environments have uncertainties in describing the properties of massive objects. To overcome this limitation, an experiment made of huge objects must be conducted and show the actual relevance of this extended essay.

6.2 SYSTEMATIC ERROR IN DATA COLLECTION

in the first experiment, a slight unconscious increase in the angular velocity of tilting occurred as the angular velocity of the rotor increased, significantly impacting the experimental results. This is an error caused by poor control of the angular velocity, which tilts the rotational axis, because the force required by the angular velocity or rotor increases. Mathematical analysis later revealed a linear relationship. Subsequent experiments were carefully performed to mitigate these systematic errors while maintaining a constant angular velocity while measuring force samples to obtain accurate data.

6.3 RANDOM ERROR

Various random errors occur in the experimental environment.

6.3.1 Initial vibration

Noise in the Vernier light gate and force meter can cause random fluctuations in the data. And the defects in the apparatus system are also the cause. A flaw was observed in the test apparatus system, where the rotor was constructed by overlapping three hard disks using instant adhesive. Unfortunately, this method led to uneven gaps between the hard disks, resulting in an unbalanced rotor system and initial vibrations.

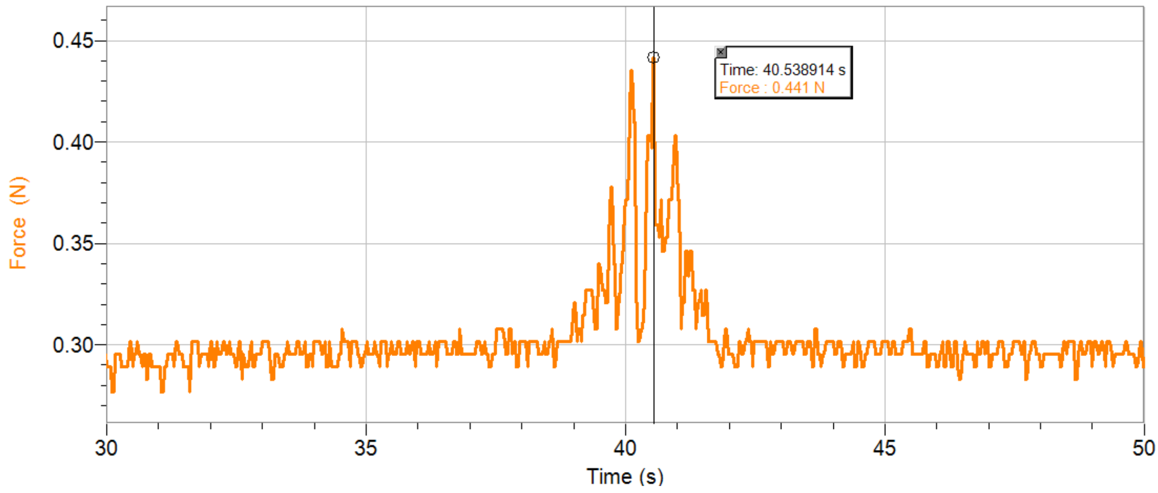

Figure 28: force graph at 2060RPM from logger pro

In Figure 28, the vibration is evident in the force sample at 2060RPM, showcasing the imbalance due to varying gap sizes between the disks. This oscillation may prevent smooth graphs from being obtained, affecting the accuracy and reliability of experimental data.

6.3.2 Change in external resistive force according to wire position change.

A potential source of error stems from the variability in wire positions connecting the motor and power pack. These wires may experience shifts and external forces as the motor turns, potentially influencing the reliability of the experimental setup.

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8 APPENDIX

8.1 DIAGRAM AND TABLE

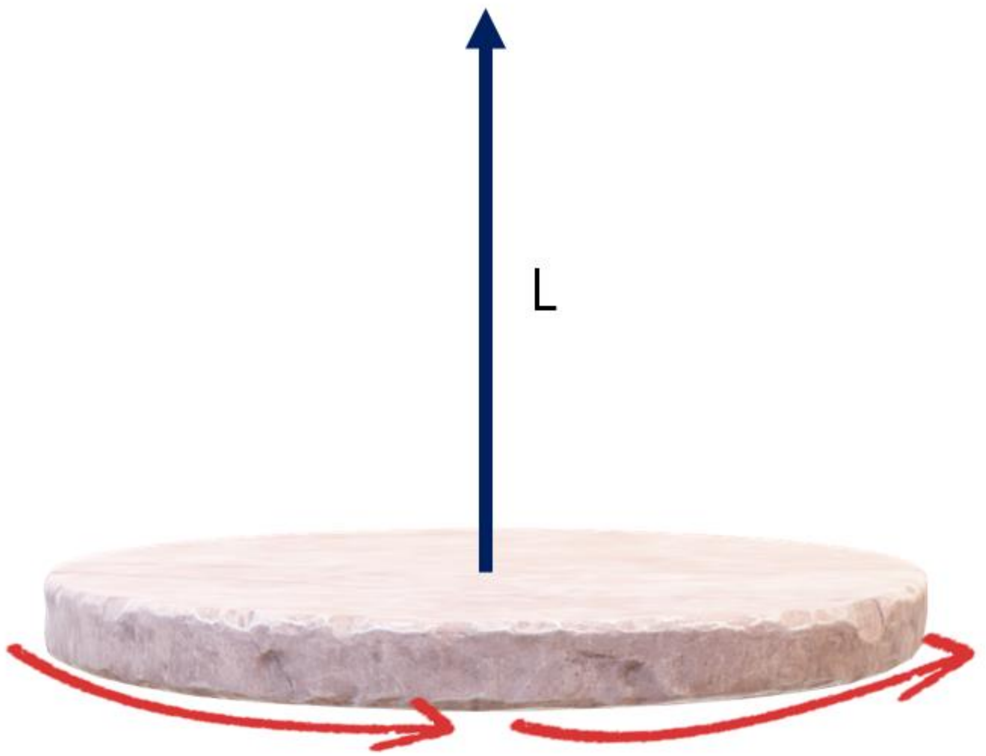

Figure 1: depiction of the angular momentum.

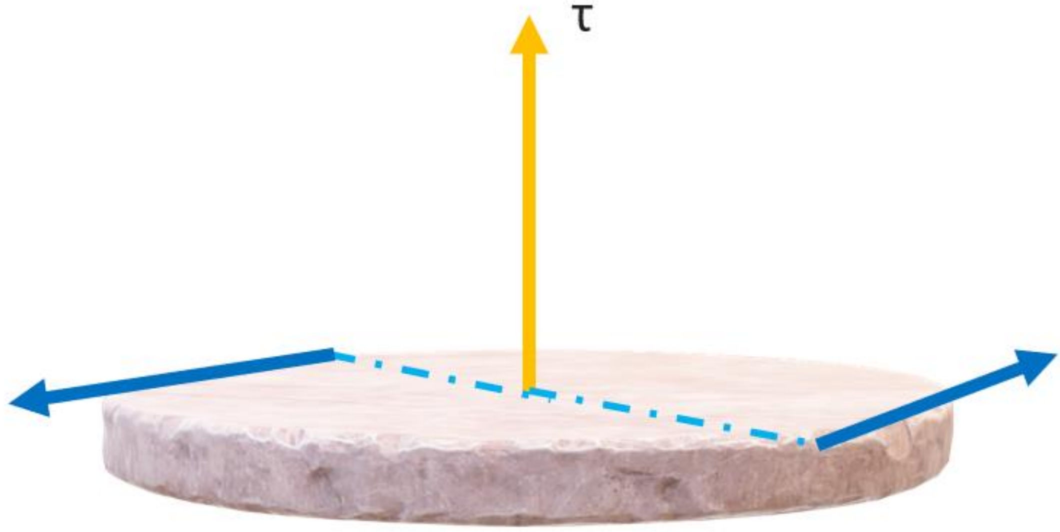

Figure 2: depiction of the torque

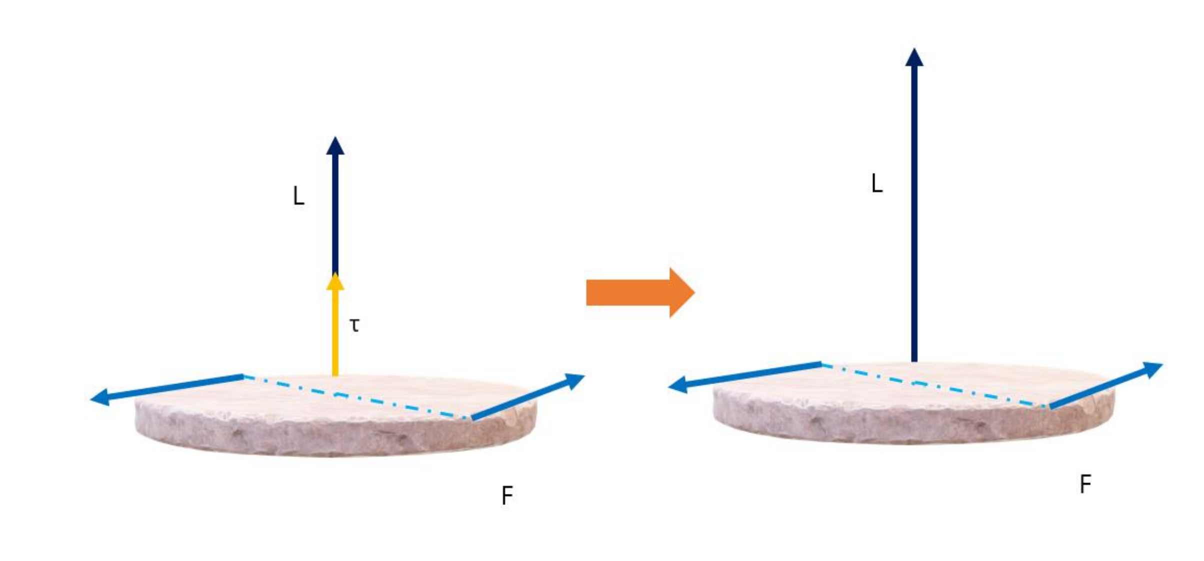

Figure 3: depiction of the relationship between torque and angular momentum.

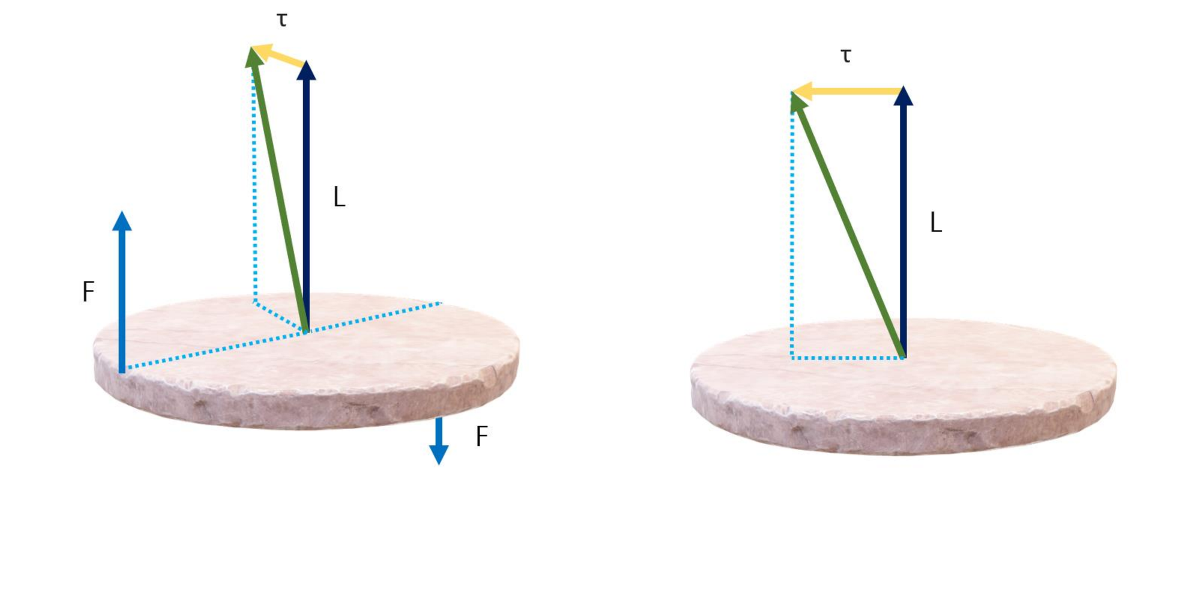

Figure 4: depiction of the changing direction of angular momentum due to perpendicular torque.

DV	force required to tilt the axis of rotation of the spinning object. (F)
IV	angular velocity of the rotor (ω)
CV	angular velocity of the rotating axis tilt. Mass, shape, and dimensions of the rotor system.

Figure 5: table of the dependent variable, the independent variable and control variables

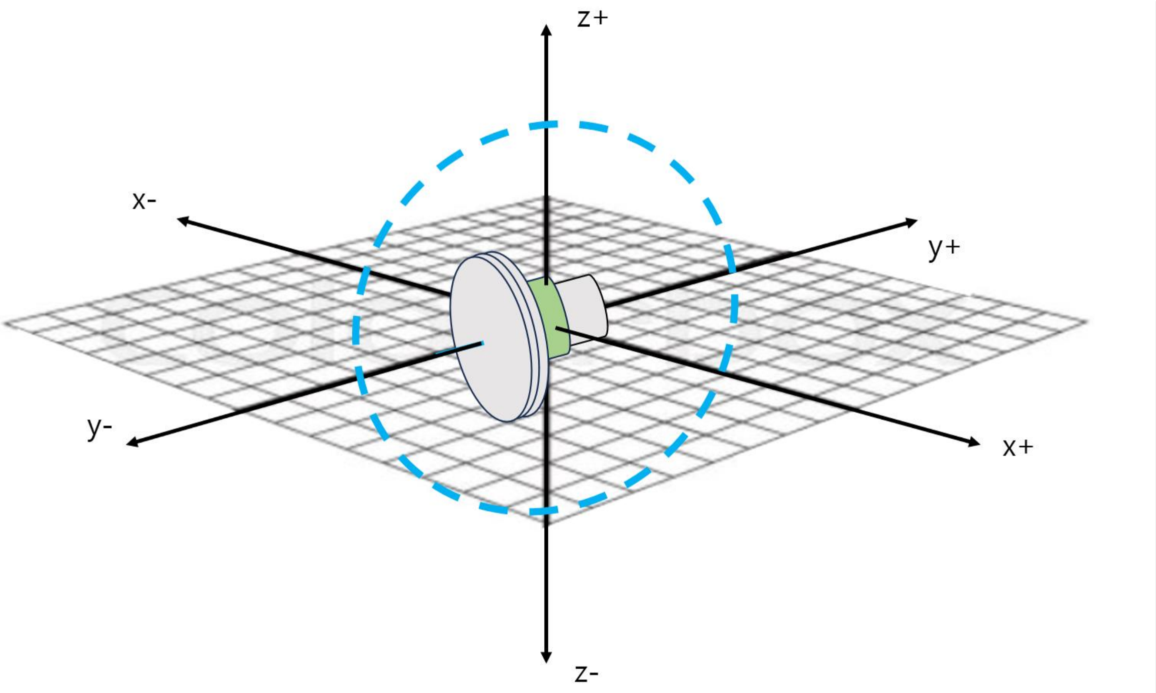

Figure 6: rotor system in 3D plane

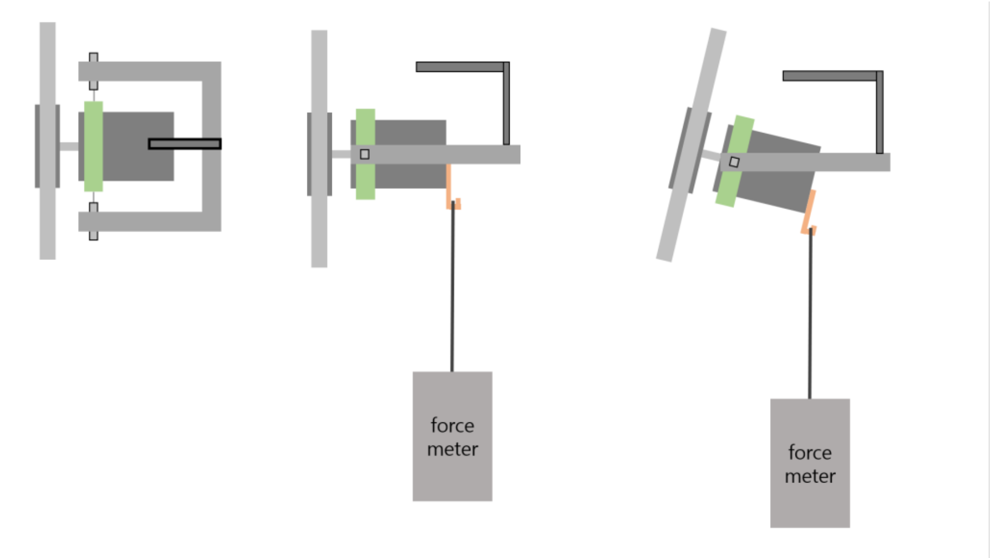

Figure 7: Rotor experiment system diagram

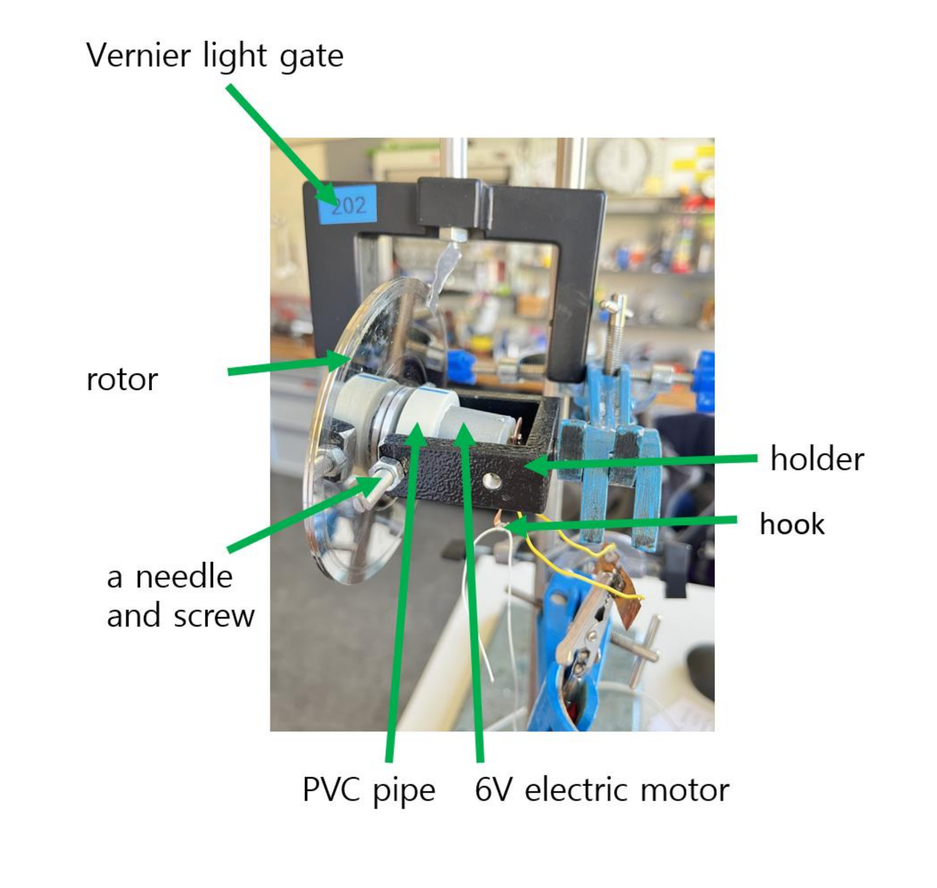

Figure 8: experiment picture 1

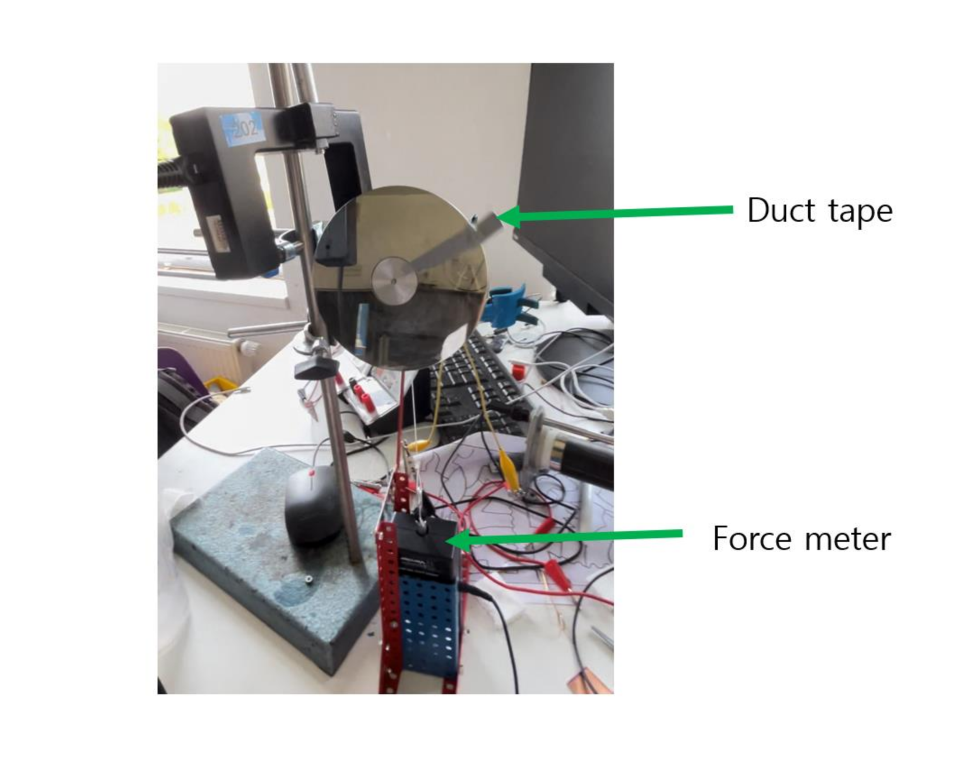

Figure 9: experiment picture 2

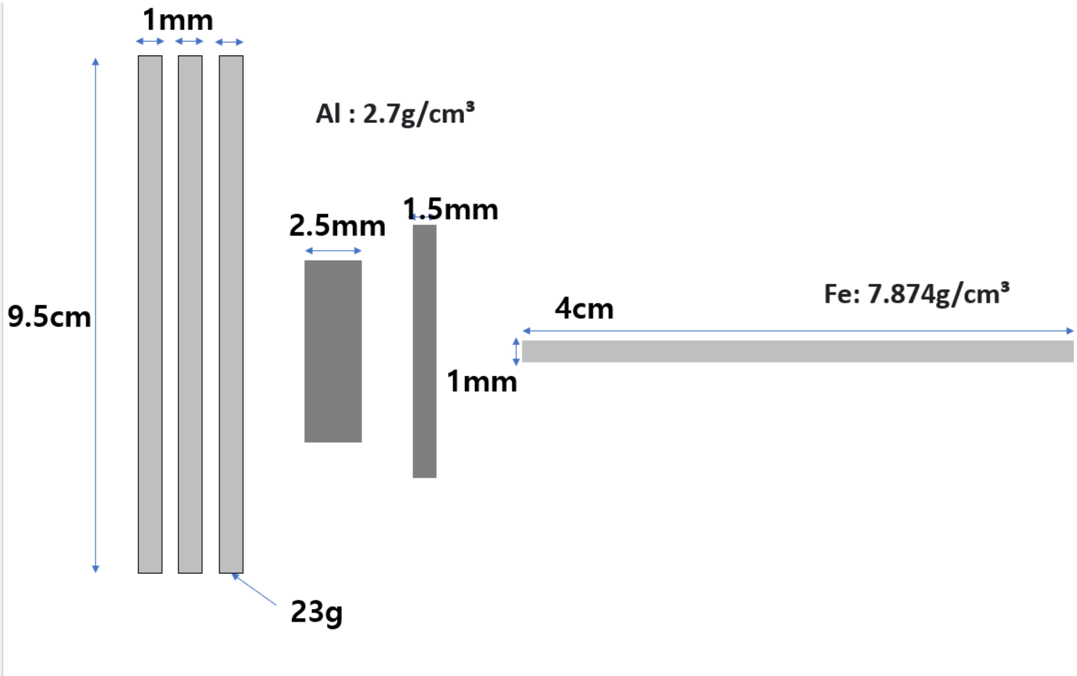

Figure 10: Specifications of the rotor part

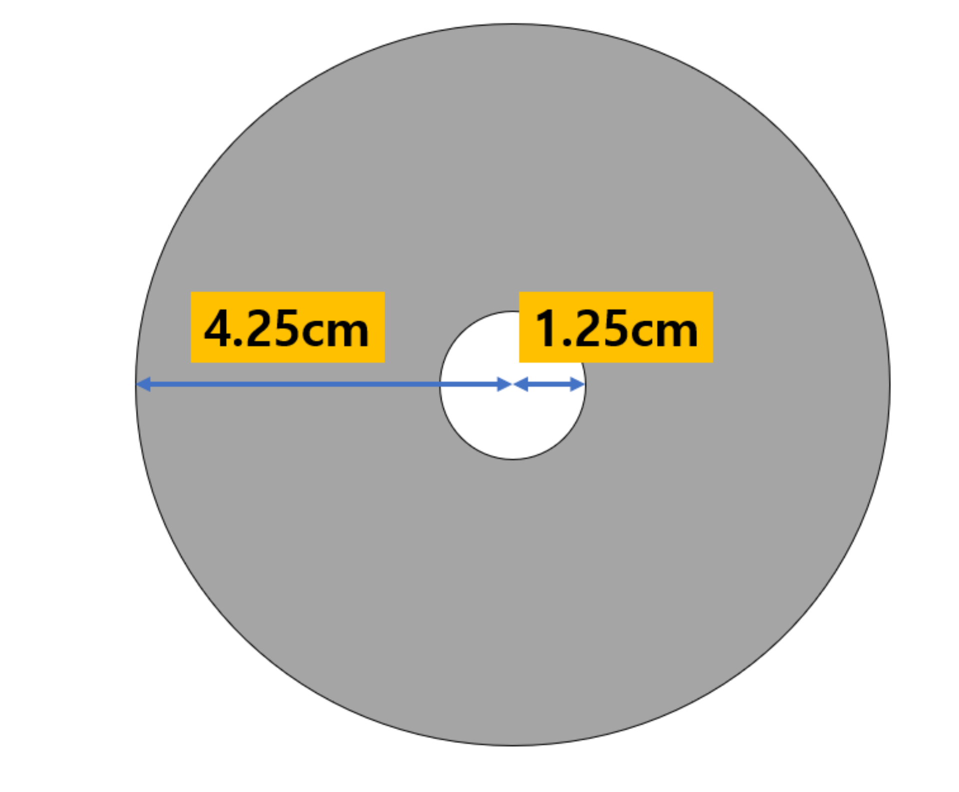

Figure 11: Specifications of disc

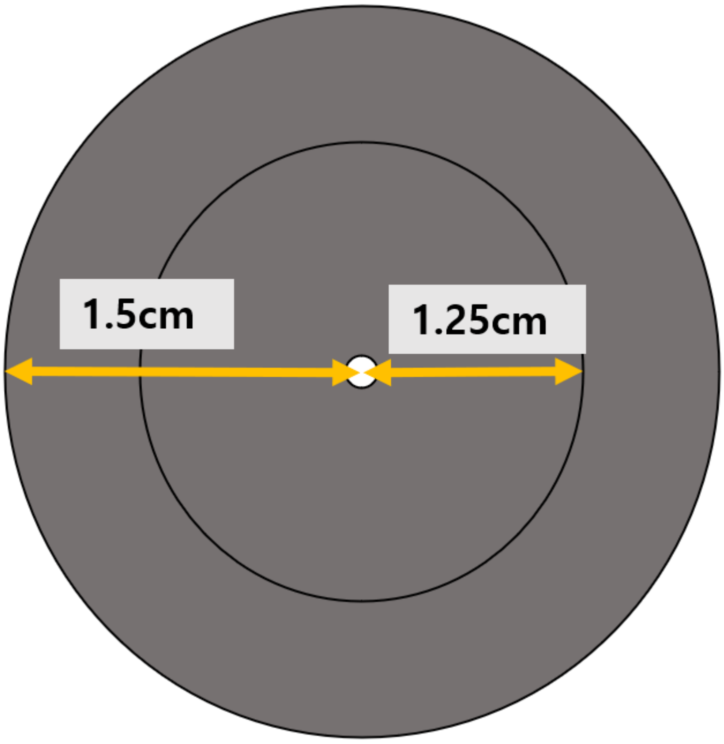

Figure 12: Specifications of the aluminum part

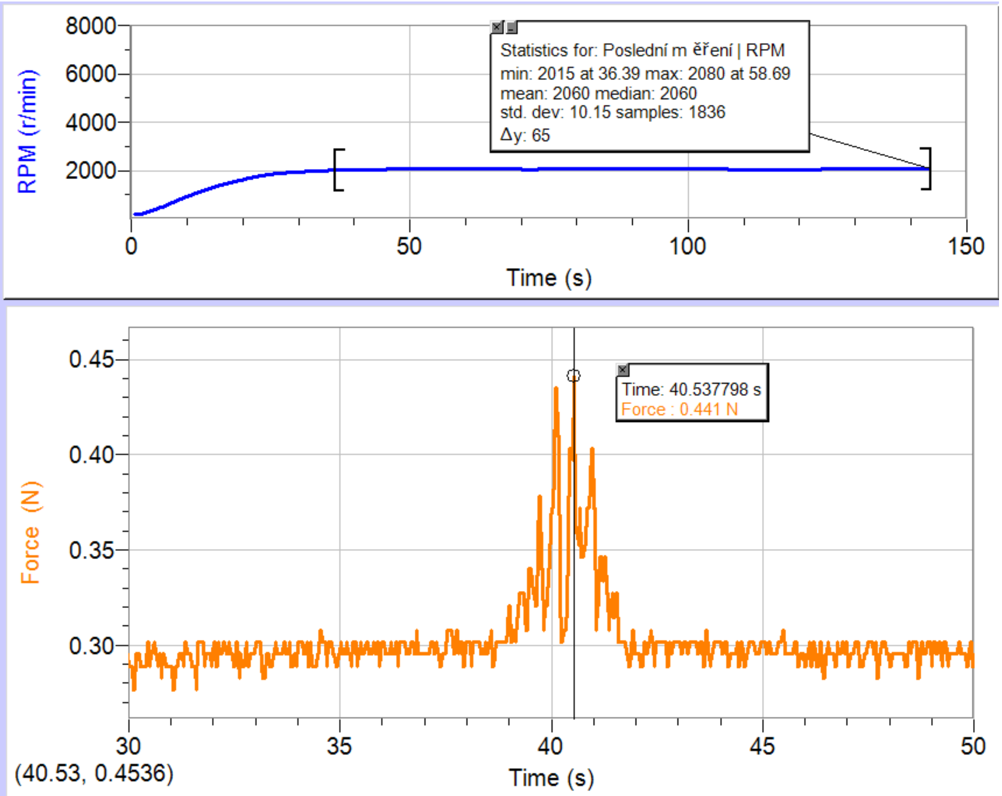

Figure 13: force graph and RPM graph at 2060RPM from logger pro

RPM $(r \cdot min^{-1})$	ΔRPM $(r \cdot min^{-1})$	$Force\ (N)$				
		Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
2060	65	0.49	0.47	0.43	0.44	0.43
2935	70	0.54	0.54	0.56	0.63	0.58
4012	85	0.71	0.67	0.63	0.64	0.61
4872	119	0.75	0.79	0.77	0.74	0.72
5945	99	0.87	0.84	0.83	0.86	0.89
6599	101	0.96	1.00	0.93	0.98	0.95

Figure 14: raw data table with RPM, RPM uncertainty Force samples

RPM $(r \cdot min^{-1})$	$\bar{F}\ (N)$
2060	0.45
2935	0.57
4012	0.65
4872	0.75
5945	0.86
6599	0.96

Figure 15: table with RPM, mean force

$RPM(r \cdot min^{-1})$	$\omega\ (rad \cdot s^{-1})$
2060	215.6
2935	307.2
4012	419.9
4872	509.9
5945	622.2
6599	690.7

Figure 16: RPM, angular velocity conversion table

$\bar{F}\ (N)$	$\Delta F\ (N)$
0.45	0.012
0.57	0.017
0.65	0.017
0.75	0.012
0.86	0.010
0.96	0.012

Figure 17: mean force and uncertainty of force table

$RPM(r \cdot min^{-1})$	$\Delta RPM(r \cdot min^{-1})$	$\omega\ (rad \cdot s^{-1})$	$\Delta\omega\ (rad \cdot s^{-1})$
2060	65.0	215.6	6.80
2935	70.0	307.2	7.30
4012	85.0	419.9	8.90
4872	119	509.9	12.5
5945	99.0	622.2	10.4
6599	101	690.7	10.6

Figure 18: RPM to angular velocity Conversion Table with uncertainty

$\omega\ (rad \cdot s^{-1})$	$\Delta\omega\ (rad \cdot s^{-1})$	$F\ (N)$	$\Delta F\ (N)$
215.6	6.80	0.45	0.012
307.2	7.30	0.57	0.017
419.9	8.90	0.65	0.017
509.9	12.5	0.75	0.012
622.2	10.4	0.86	0.010
690.7	10.6	0.96	0.012

Figure 19: angular velocity and force table

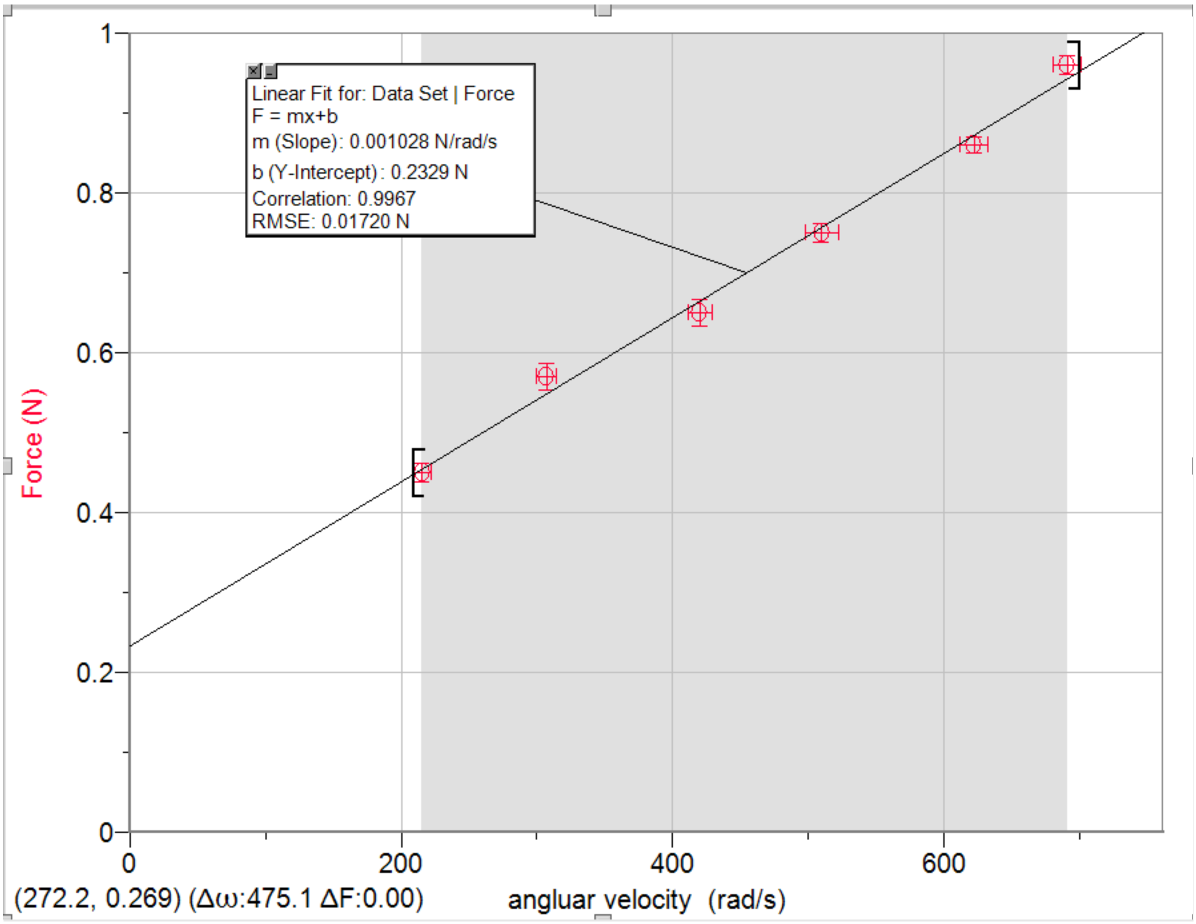

Figure 20: angular velocity and force graph

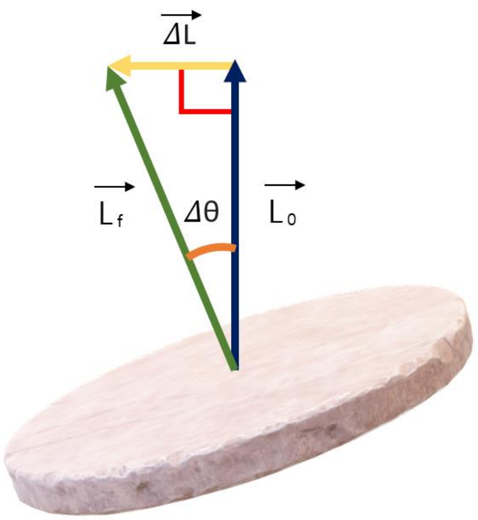

Figure 21: description of changing of direction angular momentum.

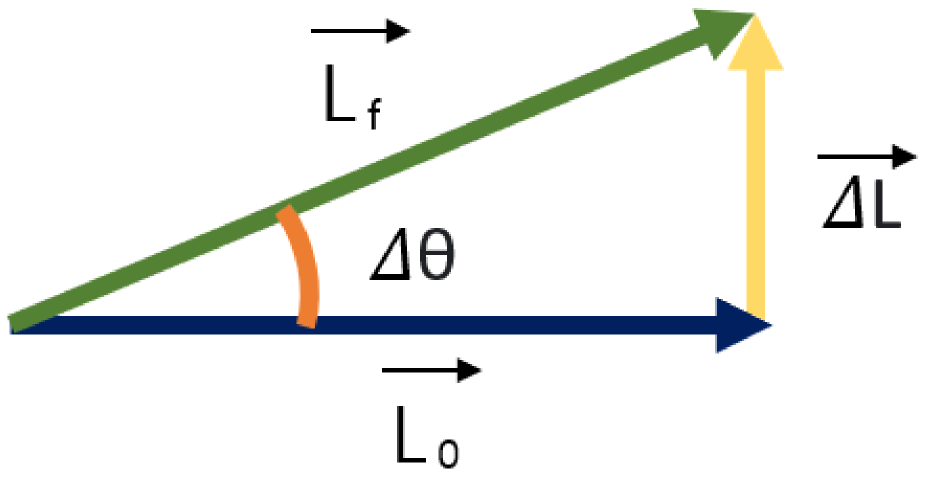

Figure 22: changing angular momentum with momentary small right-angle impulse.

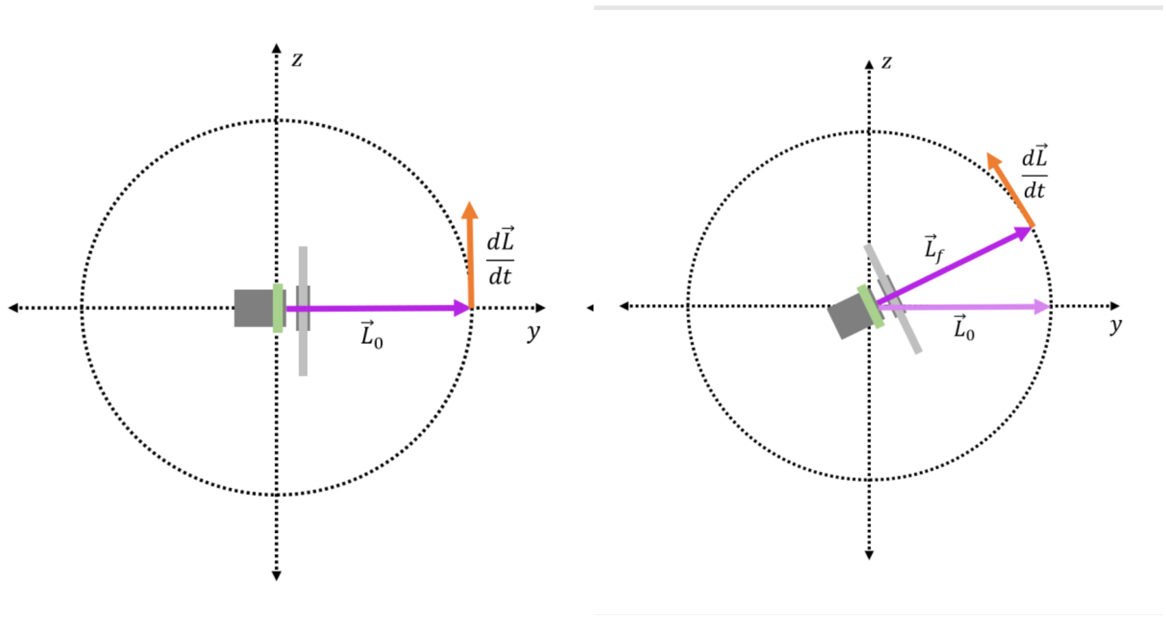

Figure 23: angular momentum varies with respect to time.

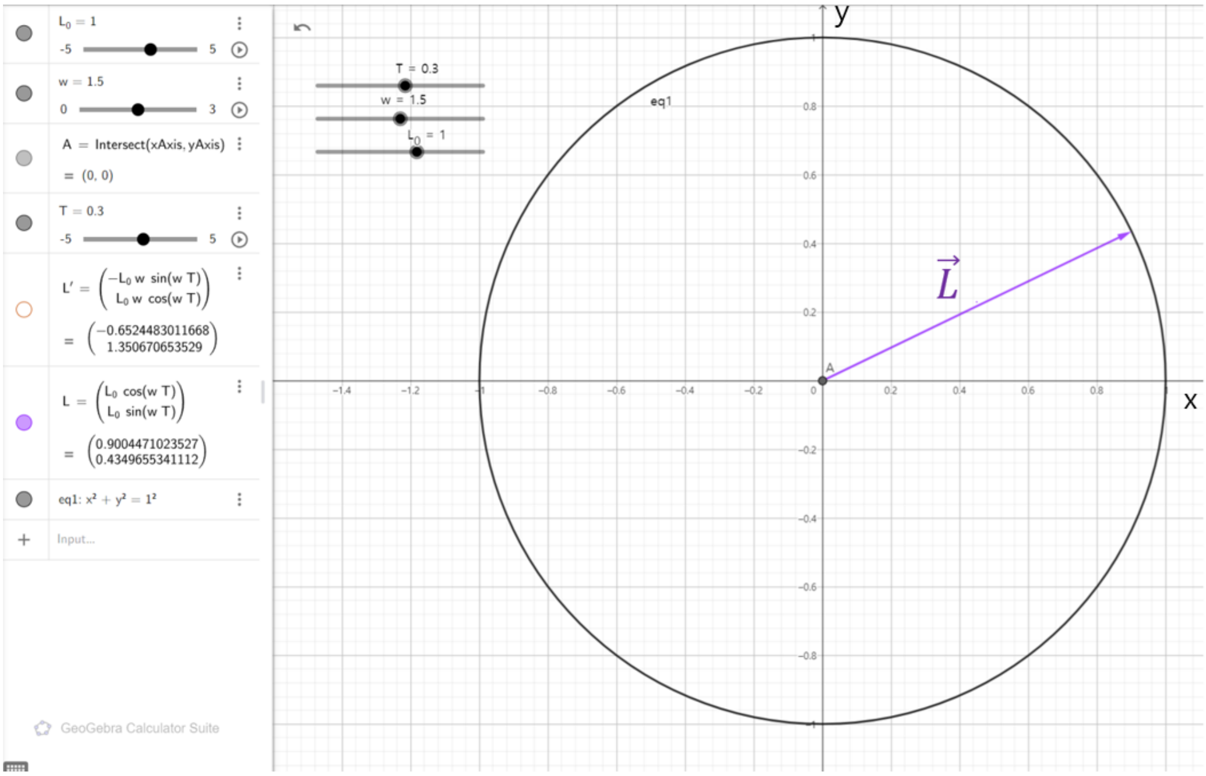

Figure 24: The vector L circular motioning as changing time in the mathematical model

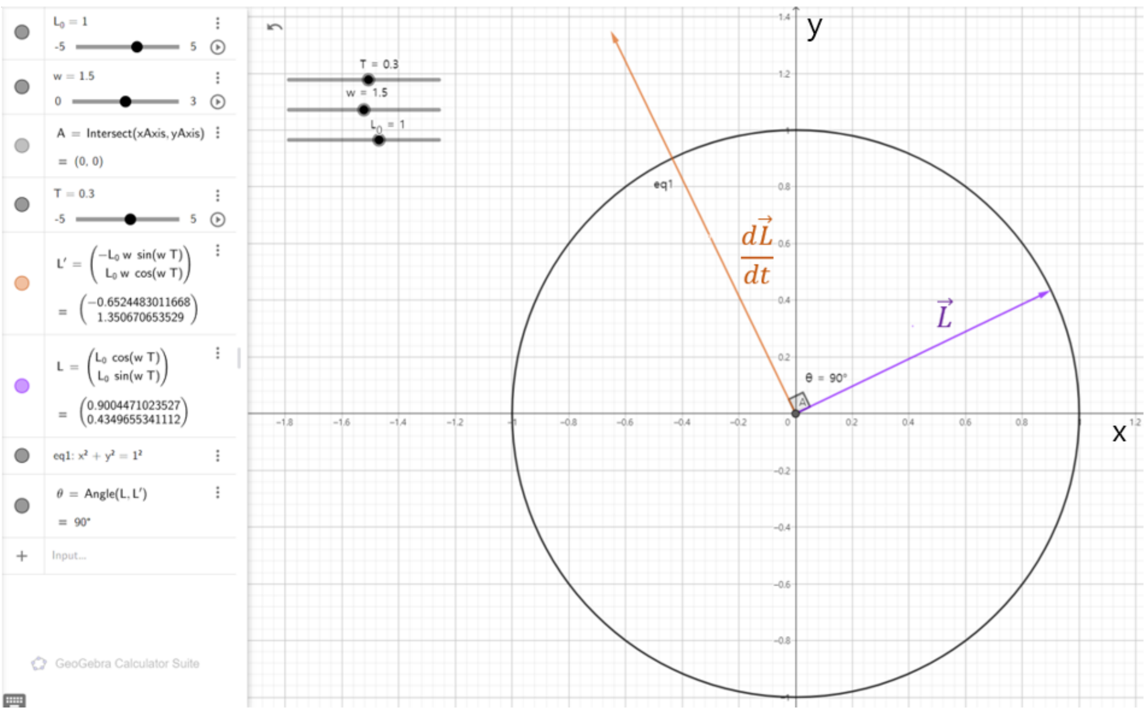

Figure 25: The $\frac{d\vec{L}}{dt}$ vector in the mathematical model

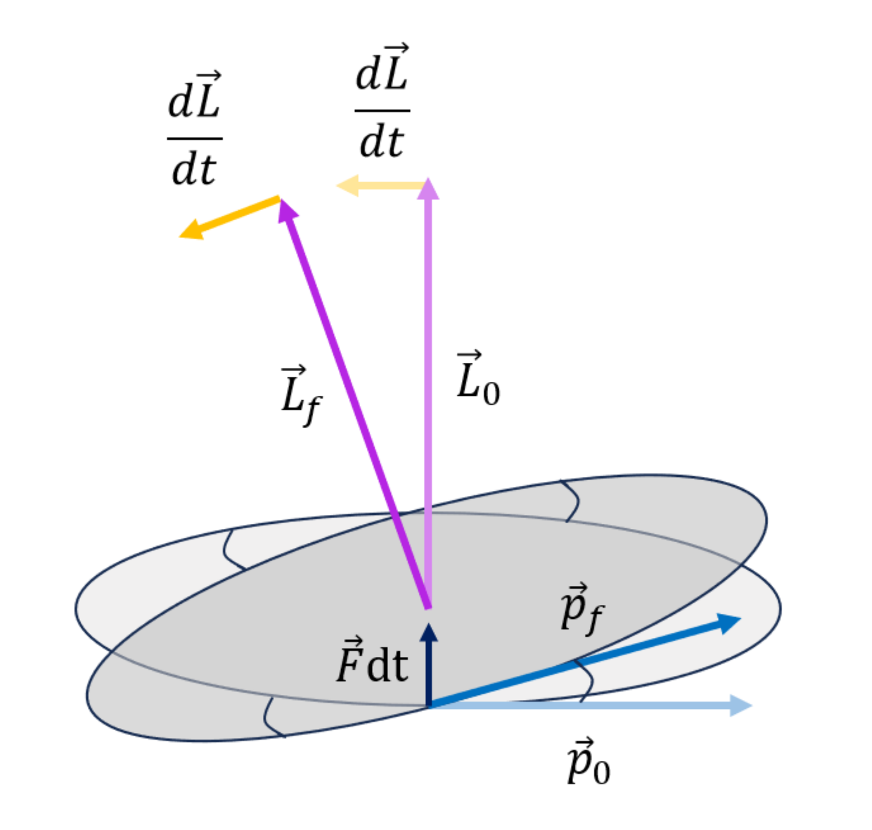

Figure 26: changing angular momentum and linear momentum.

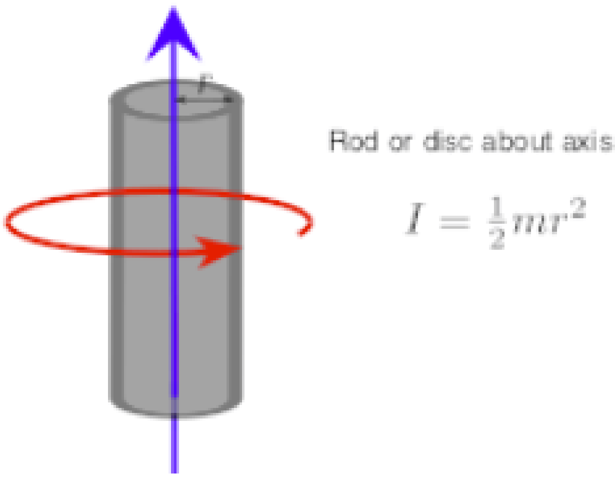

Figure 27: rotational inertia of rod or disc about axis [4]

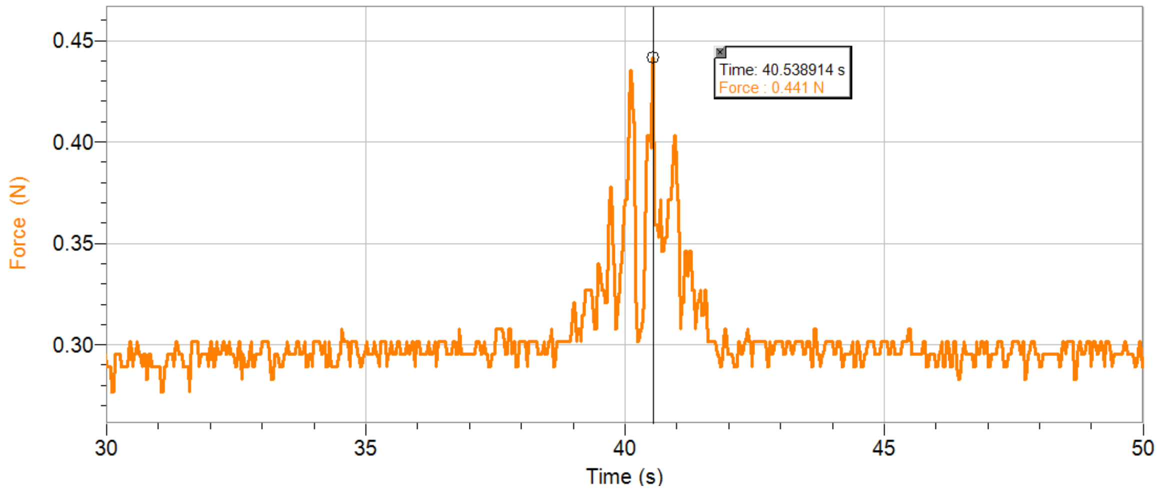

Figure 28: force graph at 2060RPM from logger pro

8.2 VARIABLE NAME AND UNITS

Chapter 1

$L:$ $angular\ momentum\ (kg\ m^2\ s^{-1})$

$I:$ $moment\ of\ inertia\ (kg\ m^2)$

$\omega:$ $angular\ velocity\ (rad\ s^{-1})$

$\Delta\theta:$ $changing\ angular\ displacement\ (rad)$

$dt, \Delta t:$ $changing\ in\ time\ (s)$

$\tau:$ $torque\ (Nm)$

$F:$ $force\ (N)$

$r:$ $perpendicular\ distance\ from\ the\ origin (m)$

$dL, \Delta L:$ $changing\ in\ angular\ momentum\ (kg\ m^2\ s^{-1})$

Chapter 2

ω_{tilt} : $the\ angular\ velocity\ of\ the\ rotating\ axis\ tilt\ (rad\ s^{-1})$

$\Delta\theta$: $changing\ in\ angle\ (rad)$

$dt, \Delta t$: $changing\ in\ time\ (s)$

I_{total} : $total\ moment\ of\ inertia\ of\ whole\ rotor\ system\ (kg\ m^2\ or\ g\ cm^2)$

I_{parts} : $each\ moment\ of\ inertia\ of\ parts\ (kg\ m^2\ or\ gcm^2)$

r_o: $outer\ radius\ (m)$

r_i: $inner\ radius\ (m)$

m : $mass\ (kg\ or\ g)$

Rotational inertia of 3 discs

$r_{o\ disc}$: $outer\ radius\ of\ disc\ (m)$

$r_{i\ disc}$: $inner\ radius\ of\ disc(m)$

m_{disc} : $mass\ of\ disc\ (g)$

I_d : $momentum\ of\ inertia\ of\ disc\ (kg\ m^2)$

Rotational inertia of aluminum part

V_1 : $volume\ of\ large\ part\ of\ aluminum\ part\ (m^3)$

r_{o1} : $outer\ radius\ large\ part\ of\ aluminum\ part\ (m)$

r_{i1} : $inner\ radius\ large\ part\ of\ aluminum\ part\ (m)$

h_1 : $height\ of\ large\ part\ of\ aluminum\ part\ (m)$

V_2 : $volume\ of\ small\ part\ of\ aluminum\ part\ (m^3)$

r_{o2} : $outer\ radius\ small\ part\ of\ aluminum\ part\ (m)$

r_{i2} : $inner\ radius\ small\ part\ of\ aluminum\ part\ (m)$

h_2 : $height\ of\ small\ part\ of\ aluminum\ part\ (m)$

ρ_{Al} : $density\ of\ aluminum\ (gcm^{-3})$

m_1 : $mass\ of\ large\ part\ of\ aluminum\ part\ (g)$

m_2 : $mass\ of\ small\ part\ of\ aluminum\ part\ (g)$

I_1 : $momentum\ of\ inertia\ of\ large\ part\ of\ aluminum\ part\ (kgm2)$

I_2 : $momentum\ of\ inertia\ of\ small\ part\ of\ aluminum\ part\ (kgm2)$

Rotational inertia of rod

V_{rod} : $volume\ of\ rod\ (m^3)$

h_{rod} : $height\ of\ rod\ (m)$

r_{rod} : $radius\ of\ rod\ (m)$

ρ_{Fe} : $density\ of\ Iron\ (gcm^{-3})$

m_{rod} : $mass\ of\ rod\ (g)$

I_{rod} : $momentum\ of\ inertia\ of\ rod\ (kgm^2)$

Chapter 4

τ : $torque\ (Nm)$

$dL, \Delta L$: $changing\ in\ angular\ momentum\ (kgm^2\ s^{-1})$

Δt : $time\ taken\ (s)$

t : $time\ (s)$

$\Delta\theta$: $changing\ angular\ displacement\ (rad)$

ω : $angular\ velocity\ of\ changing\ vector\ \vec{L}$

L_0 : $initial\ angular\ momentum\ (kgm^2 s^{-1})$

L_f : $final\ angular\ momentum\ (kgm^2\ s^{-1})$

$\vec{L}$: $angular\ momentum\ vector$

$d\vec{L}$: $instantaneous\ change\ in\ vector\ \vec{L}$

$\frac{d\vec{L}}{dt}$: instantaneous change in vector $\vec{L}$ over the instantaneous change in time

$\vec{x}$: $x - componentof\ vector\ \vec{L}$

$\vec{y}$: $y - componentof\ vector\ \vec{L}$

$\frac{d\vec{x}}{dt}$: $derivertive\ of\ \vec{x}$

$\frac{d\vec{y}}{dt}$: $derivertive\ of\ \vec{y}$

ω_{tilt} : $angular\ velocity\ that\ the\ rotational\ axis\ is\ tilted\ (rad\ s^{-1})$

F : $linear\ force\ (N)$

r_{tilt}: $perpendicular\ distance\ to\ the\ point\ of\ application\ from\ the\ cneter\ of\ mass\ (m)$

I: $moment\ of\ inertia\ of\ rotor\ (kgm^2)$

ω_r: $angular\ velocity\ of\ rotor\ (rad\ s^{-1})$

F_r : linear resistive force (N)

m: $mass\ of\ rotor\ (kg)$

r_r: $radius\ of\ rotor\ (m)$

F_w: $total\ force\ required\ for\ the\ entire\ system\ (N)$

F_e: $external\ force\ (N)$