Bootstrapping Default Probabilities from CDS spreads

This article provides a methodology to derive default curves from CDS spreads (quotes) using python.

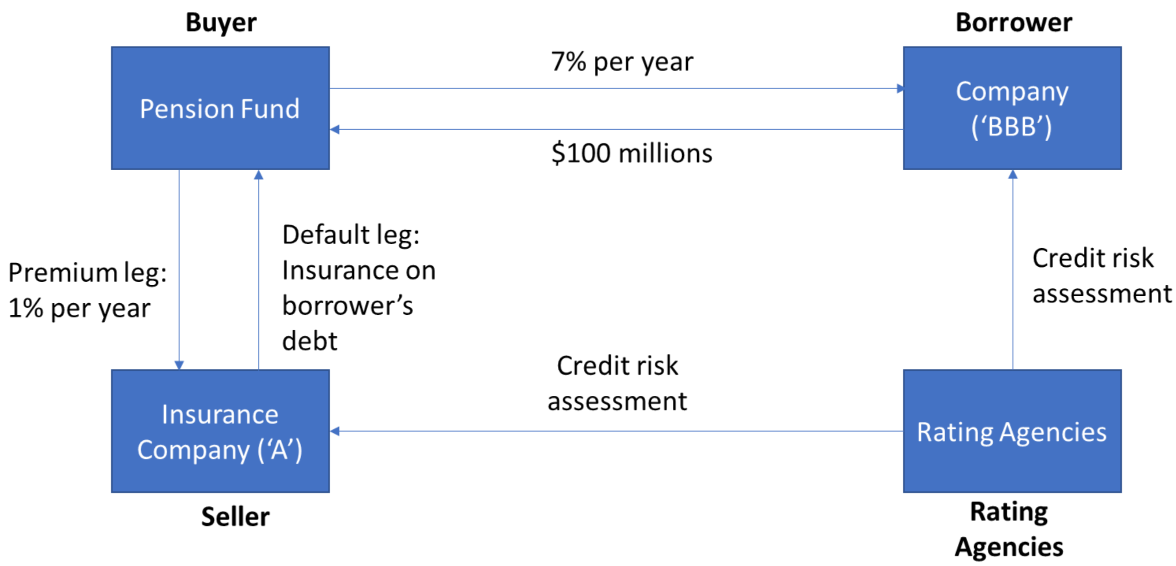
Definition

A Credit Default Swap (CDS) is a credit derivative product that insures the buyer against the default risk of an asset in exchange of the payment of premiums.

There are four parties involved in the transaction: the buyer, the seller, the borrower and the rating agencies.

For example, let's assume a company (the borrower) issues debts rated 'BBB' by rating agencies.

Now, let's consider a pension fund (the buyer) that can invest in bonds issued by companies with a minimum rating of 'A' as per its investment policy. This pension fund cannot invest in bonds issued by the borrower because the credit risk of this company is too high. However, this investment becomes if the pension fund buys a CDS to a financial institution rated at least 'A' to hedge the default risk.



Essentially, a CDS transfers the credit risk from the borrower to the CDS seller. If there is a credit event (default on payments, credit downgrade or spread widening) the insurance company will have to pay the amount that is not recovered to the buyer.

They can also be used for speculation through naked CDS, where the buyer does not hold the underlying debt.

Mathematical Model

The default structure can be modelled using a Poisson process. A Poisson process is often used to model rare and countable events such as defaults.

It gives the probability of default during within a time interval conditional on surviving until the initial time point.

The survival probability is given by the following formula:

$$P(t,T) = e^{-\lambda(T-t)}$$

Where λ is a constant representing intensity or the hazard rate.

In order to fit the term structure of CDS spreads, the model should incorporate a time dependent intensity rate function $\lambda(t)$.

Thus, the survival probability of default can be written:

$$P(t,T) = e^{-\int_t^T \lambda(s)ds}$$

CDS Pricing

In the standard CDS pricing model the fair spread quotes are determined by calculating the expected present value of the cash flows from the premium leg and from the default leg.

Premium leg

The expected present value of the payment made from the buyer to the seller is calculated using this formula:

$$PL_N = S_N \sum_{n=1}^{N} D(0,T_n)P(T_n)\Delta t_n$$

Where:

- $D(0,T_n)$ is the discount factor
- $P(T_n)$ is the survival probability up to time T_n
- $\Delta t_n = T_n - T_{n-1}$

The discount factors are a function of forward rates fr_n:

$$D(0,T_n) = e^{-\sum_{k=1}^{n} fr_k \Delta \mathrm{t}}$$

The probability of survival's formula for a given time and a piecewise hazard rate is:

$$P(T_n) = e^{-\sum_{k=1}^{n} \lambda_k \Delta \mathrm{t}}$$

Assuming that the borrower does not default straight away we have $P(T_0) = 1$

Default leg

The default leg's present value of expected cash flow is given by:

$$DL_N = (1-R)\sum_{n=1}^{N} D(0,T_n)(P(T_{n-1}) - P(T_n))$$

$P(T_{n-1}) - P(T_n)$ gives the probability of defaulting at T_n conditional to surviving until T_{n-1}.

R is the recovery rate.

<u>Fair spread</u>

The fair spread S_N is such that the expected present value of payments of the premium leg and the default leg are equal (i.e. $PL_N = DL_N$).

$$S_N = \frac{(1-R)\sum_{n=1}^{N} D(0,T_n)(P(T_{n-1}) - P(T_n))}{\sum_{n=1}^{N} D(0,T_n)\, P(T_n)\Delta t_n}$$

Default Curve Calculation

Let's assume that we know CDS spreads for increasing maturities are $S_1, S_2 \dots, S_N$. We can then derive the associated survival probabilities $P(T_1), P(T_2), \dots, P(T_n)$

For the first two spreads are calculated using the following formulas:

$$P(T_1) = \frac{1-R}{1-R+\Delta t_1 S_1}$$

Let's put $L = 1 - R$

$$P(T_2) = \frac{D(0,T_1)[L - (L + \Delta t_1 S_2)P(T_1)]}{D(0,T_2)(L + \Delta t_2 S_2)} + \frac{P(T_1)L}{L + \Delta t_2 S_2}$$

Then, for $N > 2$

$$P(T_N) = \frac{\sum_{n=1}^{N-1} D(0,T_n)[LP(T_{n-1}) - (L + \Delta \mathrm{t}_n S_N)P(T_N)]}{D(0,T_N)(L + \Delta t_n S_N)} + \frac{P(T_{N-1})L}{L + \Delta t_N S_N}$$

Implementation

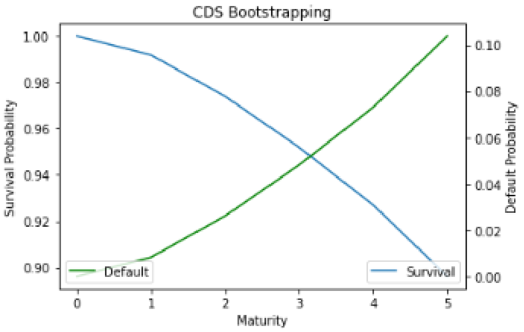
```
# inputs
maturities = range(6)
discount_factors = [0, 0.97, 0.94, 0.92, 0.89, 0.86]
spreads = [0, 50, 79, 98, 112.5, 129]
recovery = 0.40
```

```
# Function that computes the survival probabilities
def Survival_Probabilities(spreads, discount_factors, maturities, recovery):

    # Put all the input into a dataframe
    df = pd.DataFrame({'Maturity': maturities, 'Df':discount_factors, 'Spread':spreads})

    # convert spreads into basis points
    df['Spread'] = df['Spread']/10000

    # create a time step column
    df['Δt'] = df['Maturity'].diff().fillna(0)

    # L = 1-R
    L = 1-recovery

    # number of periods
    N_Period = df.shape[0]

    # create a survival probability column
    df['SP'] = [0]*N_Period

    # first 2 periods
    # SP: Survival Probability
    df['SP'].iloc[0] = 1
    df['SP'].iloc[1] = L/(L+df['Δt'].iloc[1]*df['Spread'].iloc[1])

    # Survival probabilities for N >=2
    for N in range(2,N_Period):

        numerator = 0

        for n in range(1,N):

            numerator = numerator + df['Df'].iloc[n]*(L*df['SP'].iloc[n-1]-\
                                        (L+df['Δt'].iloc[n]*df['Spread'].iloc[N])* \
                                        df['SP'].iloc[n])

        denominator = df['Df'].iloc[N]*(L+df['Δt'].iloc[N]*df['SP'].iloc[N])

        df['SP'].iloc[N] = numerator/denominator + (L*df['SP'].iloc[N-1])/ \
                           (L+df['Δt'].iloc[N]*df['Spread'].iloc[N])

    # Calculate the probability of default
    df['PD'] = 1 - df['SP']

    df.rename(columns = {'PD':'Default', 'SP':'Survival'}, inplace = True)

    return df
```

```
# call the user defined function
SP = Survival_Probabilities(spreads, discount_factors, maturities, recovery)
SP
```

	Maturity	Df	Spread	Δt	Survival	Default
0	0	0.00	0.00000	0.0	1.000000	0.000000
1	1	0.97	0.00500	1.0	0.991736	0.008264
2	2	0.94	0.00790	1.0	0.973901	0.026099
3	3	0.92	0.00980	1.0	0.951854	0.048146
4	4	0.89	0.01125	1.0	0.926967	0.073033
5	5	0.86	0.01290	1.0	0.896158	0.103842

```python
# Plot Survival probability and Default probability
ax = SP.plot(x = 'Maturity', y = 'Survival', legend = True)
ax2 = ax.twinx()
SP.plot(x="Maturity", y="Default", ax=ax2, legend=True, color="g")
ax.legend(loc='lower right')
ax2.legend(loc='lower left')
ax.set_ylabel('Survival Probability')
ax2.set_ylabel('Default Probability')
plt.title('CDS Bootstrapping')
plt.show()
```

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ax2.legend(loc='lower left')
ax.set_ylabel('CDS Spread (bps)')
ax2.set_ylabel('Default Probability')
plt.title('Spread vs Default Probability')
plt.show()
```