CAN207 Continuous and Discrete Time Signals and Systems

Lecture 13

Sampling and Reconstruction

Zhao Wang

Zhao.wang@xjtlu.edu.cn

Room EE322

Xi'an Jiaotong-Liverpool University

西交利物浦大學

Content

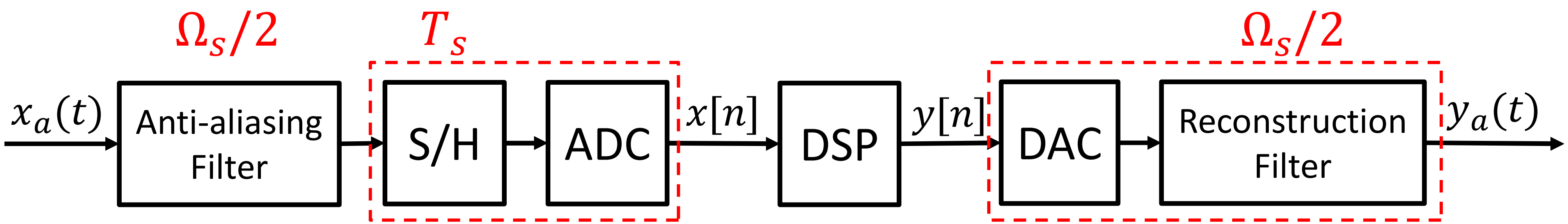
- 1. Sampling
  - 1.0. What and Why?
  - 1.1. Sampling in Time domain (TD)
  - 1.2. Sampling in Frequency domain (FD)
  - 1.3. Nyquist Theorem
- 2. Reconstruction
  - 2.1 Interpolation
  - 2.2 Reconstruction theory
  - 2.3 Reconstruction in FD – filtering
  - 2.4 Reconstruction in TD – interpolation
  - 2.5 Realisation

1.0 Digital Processing of CT Signals

- Most signals in nature are continuous in time

  => Need a way for "digital processing of continuous-time signals" => ***SAMPLING***!

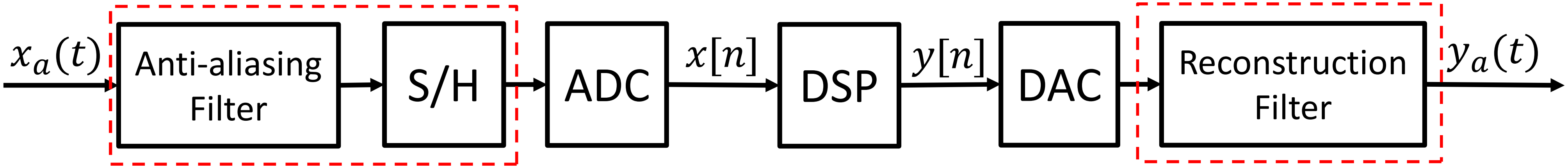
(a) Ideal data flow for the digital processing of continuous-time signals



(b) Practical data flow for the digital processing of continuous-time signals

Xi'an Jiaotong-Liverpool University
西交利物浦大學

1.0 Analog -> Digital ->Analog



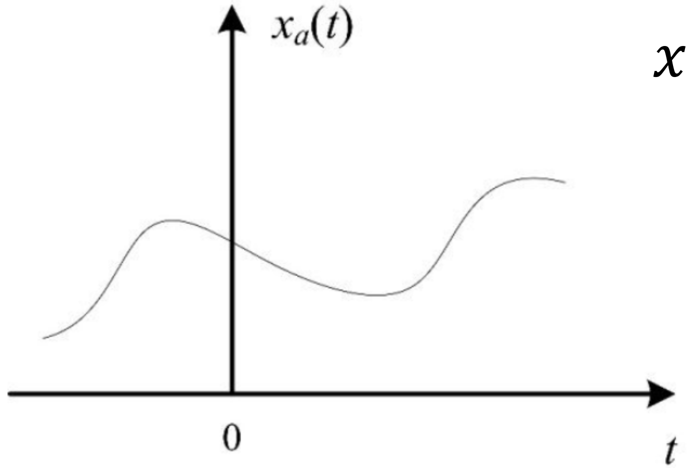
- Conversion of the continuous-time signal into a discrete-time signal
  - Anti-aliasing filter – to prevent potentially detrimental effects of sampling
  - Sample & Hold – discrete in time and keep the sampling values for a while to allow the A/D converter to do its job
  - ***Analog to Digital Converter (A/D) – conversion in amplitude***
- Processing of the discrete-time signal
  - Digital Signal Processing –Filter, digital processor
- Conversion of the processed discrete-time signal back into a cont.-time signal
  - ***Digital to analog converter (D/A) –to obtain the continuous signal***
  - Reconstruction / smoothing filter -smooth out the signal from the D/A

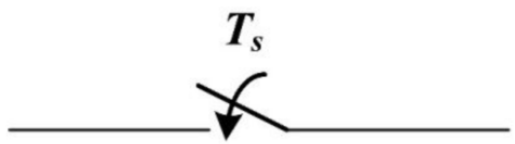
Xi'an Jiaotong-Liverpool University 西交利物浦大學

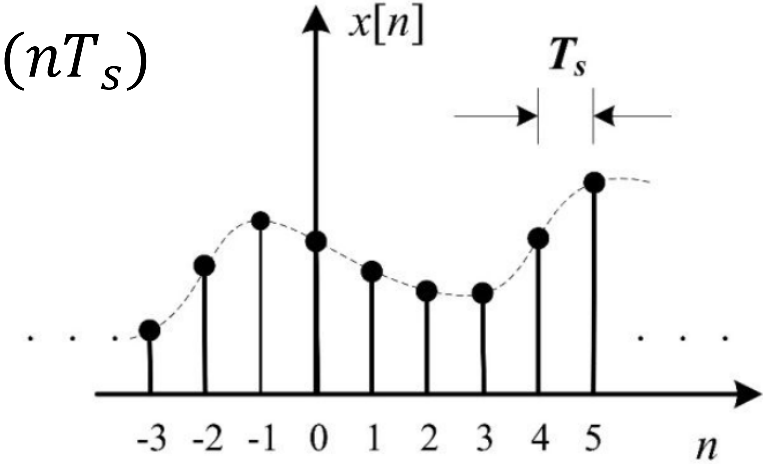
1.1 Sampling in Time domain (TD)

- A discrete-time sequence is developed by uniformly sampling the continuous-time signal $x_a(t)$

$$x[n] = x_a(t)|_{t=nT_s} = x_a(nT_s)$$







- The time variable - time t is related to the discrete time variable n only at discrete-time instants t_n

$$t_n = nT_s = \frac{n}{F_s} = \frac{2\pi n}{\Omega_s} \begin{cases} T_s = 1/F_s \text{ (Sampling period, second, second/sample)} \\ F_s = 1/T_s \text{ (Sampling frequency, Hz, cycles/second)} \\ \Omega_s = 2\pi/T_s \text{ (Sampling angular frequency, radian/second)} \end{cases}$$

1.1 Sampling in Time domain (TD)

- Consider $x_a(t) = A\cos(\Omega_0 t + \phi)$
- Now $x[n] = A\cos(\Omega_0 n T_s + \phi)$

$$= A\cos(\frac{2\pi\Omega_0}{\Omega_s} n + \phi) = A\cos(\omega_0 n + \phi) = x_a(nT)$$

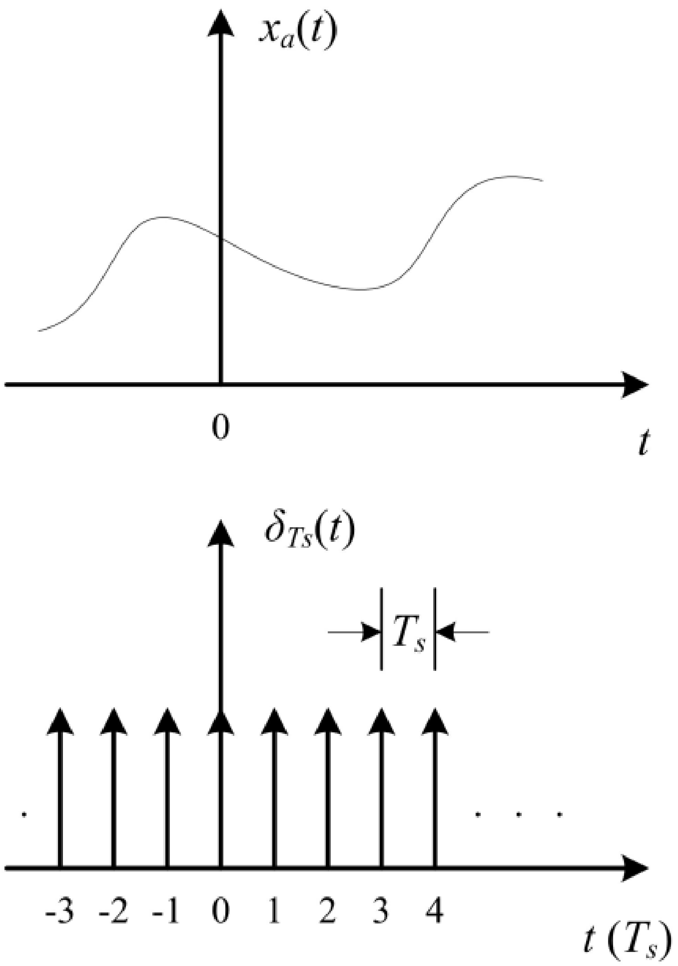
  – Where $\omega_0 = \dfrac{2\pi\Omega_0}{\Omega_s} = \Omega_0 T_s$

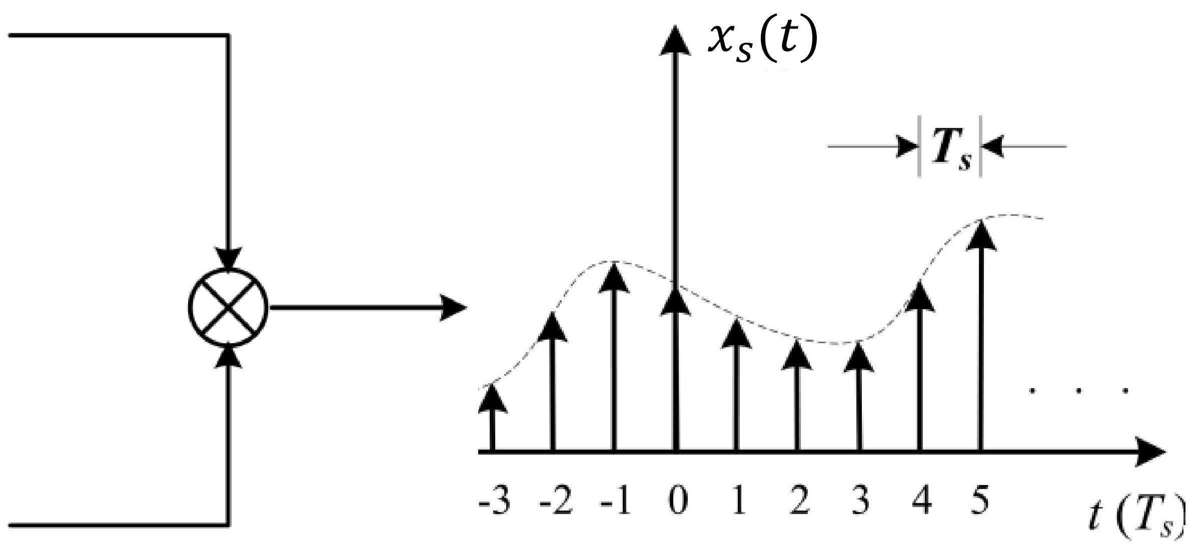
  – ω_0 is the (normalized) digital angular frequency of the signal
    - Unit: radians/sample
  – Ω_0 is the analog angular frequency of signal
    - Unit: radians/second
  – Ω_s is the sampling analog angular frequency
    - Unit: radians/second

1.1 Sampling in Time domain (TD)

- In mathematics, the periodic sampling is modelled as the multiplication of continuous signal and impulse train

$$x_s(t) = x_a(t)\delta_{T_s}(t) = x_a(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s)$$
$$= \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$





Recall Lect. 8 p17

2. $\tilde{x}(t)$ PERIODIC, $x(t)$ REPRESENTS ONE PERIOD

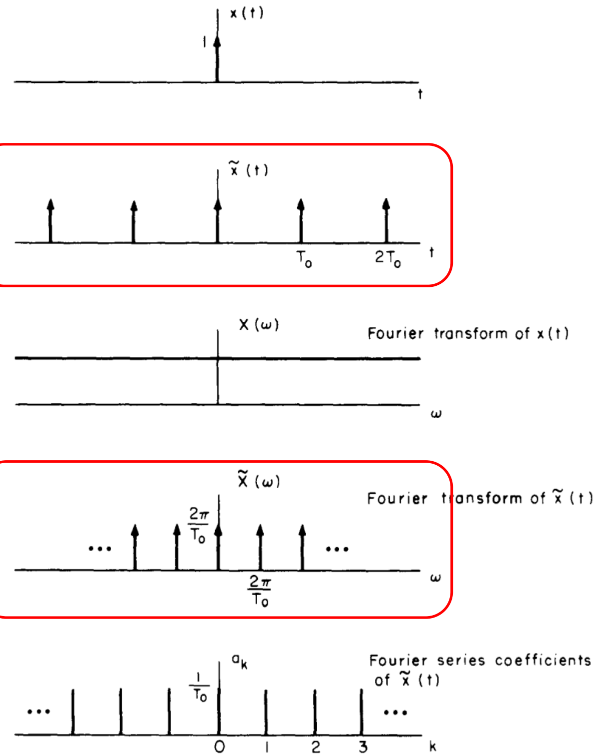
- Fourier series coefficients of $\tilde{x}(t)$

  = $(1/T_o)$ times samples of Fourier <u>transform</u> of $x(t)$

3. $\tilde{x}(t)$ PERIODIC

- Fourier <u>transform</u> of $\tilde{x}(t)$ defined as impulse train:

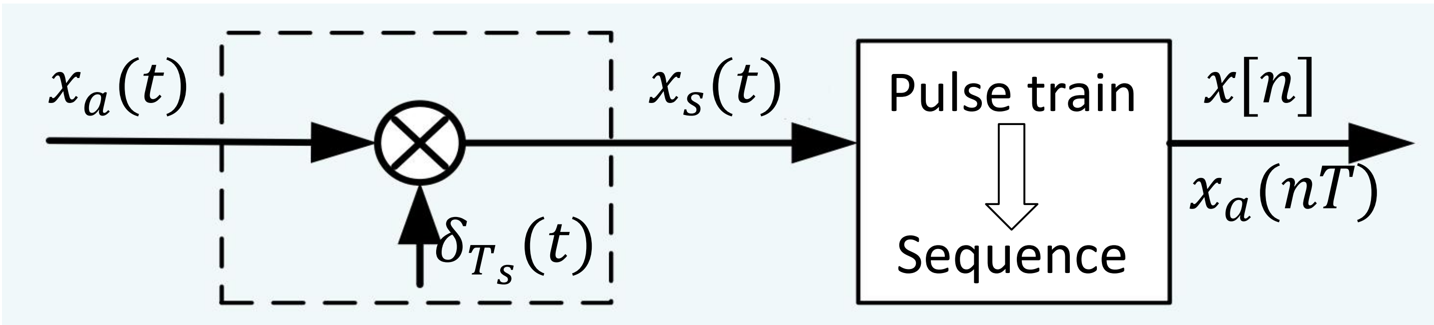
$$\tilde{X}(\omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi a_k \,\delta(\omega - k\omega_o)$$



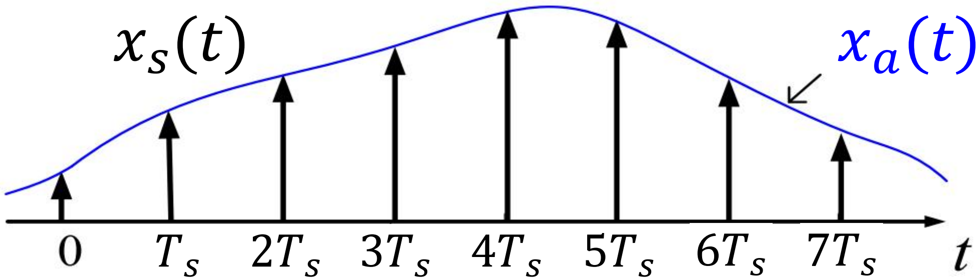
$$\sum_{k=-\infty}^{\infty} \delta(t-kT) \leftrightarrow \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)\,,\ \omega_0 = \frac{2\pi}{T}$$

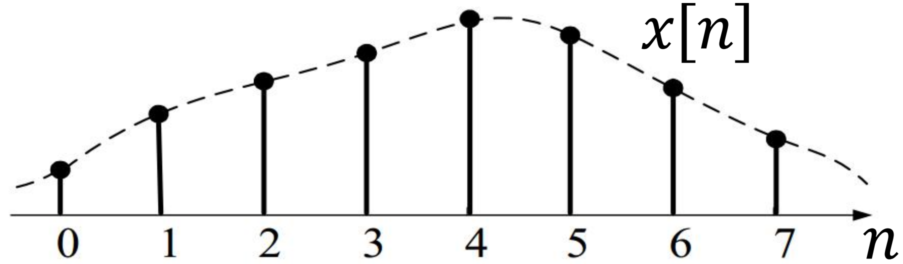
1.1 Sampling in Time domain (TD)

- The system to convert the continuous-time (CT) signal $x_a(t)$ to a discrete-time (DT) signal $x[n]$ is shown:



$$x_s(t) = x_a(t)\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

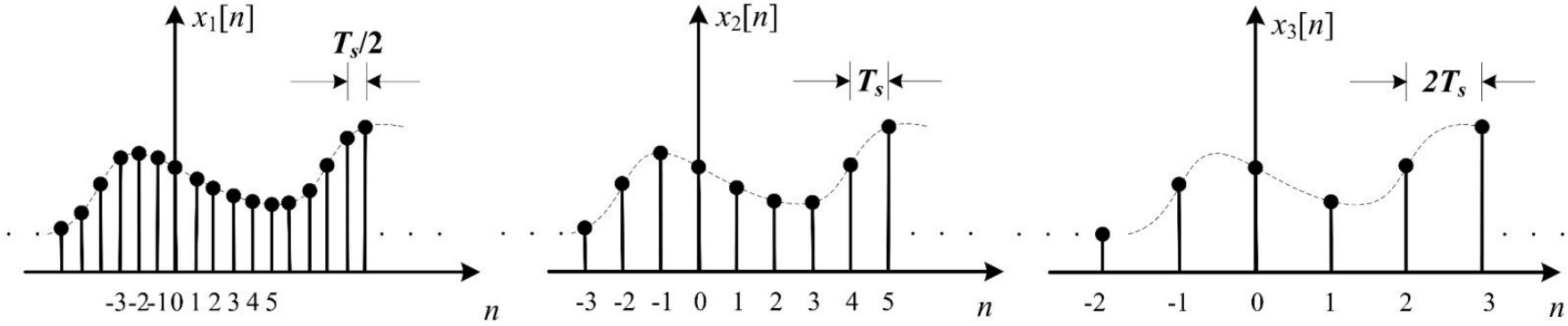




Xi'an Jiaotong-Liverpool University
西交利物浦大學

1.1 How signal changed after sampling

- In time domain, continuous -> discrete



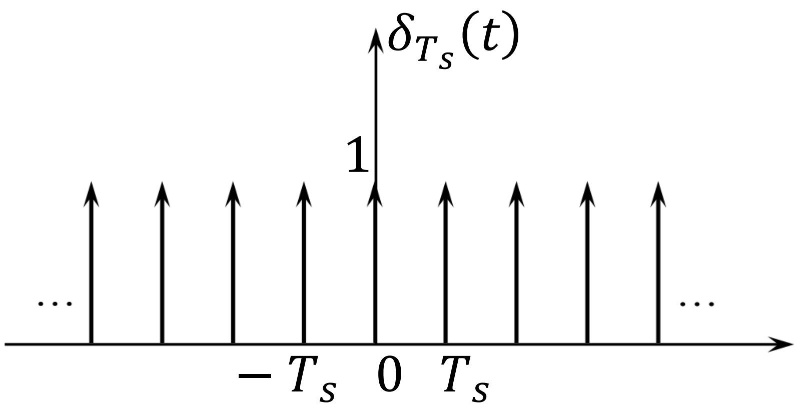
- Different sampling rates, different details
  - More samples = higher sampling rate/frequency = more detail = more information kept = more resource occupation
  - Less samples = lower sampling rate/frequency = less details = more information loss = less resource occupation
- How to choose the sampling period/rate?

1.2 Frequency domain analyses

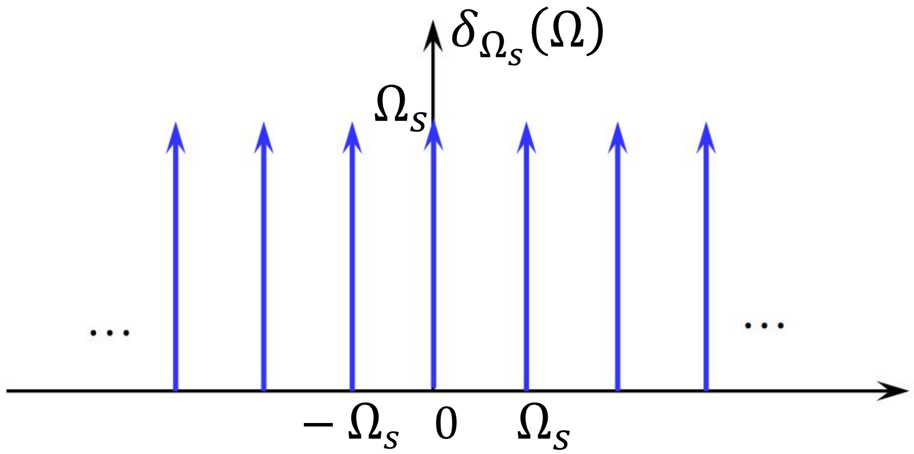
- Review: CTFT of a pulse train $\delta_{T_s}(t)$

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad \delta_{\Omega_s}(\Omega) = \Omega_s \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$\Omega_s = 2\pi f_s = \frac{2\pi}{T_s}$$



TD: Time domain



FD: Frequency domain

1.2 Sampling in Frequency domain (FD)

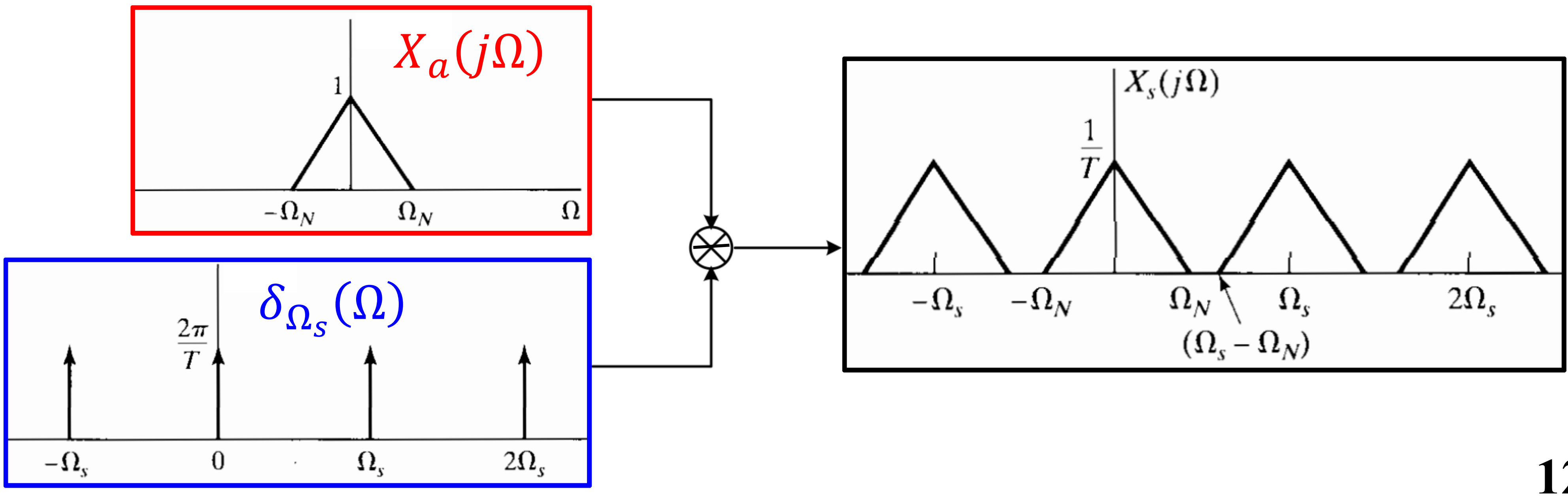
- In TD: multiplication between $x_a(t)$ and $\delta_{T_s}(t)$

$$x_s(t) = x_a(t) \cdot \delta_{T_s}(t)$$

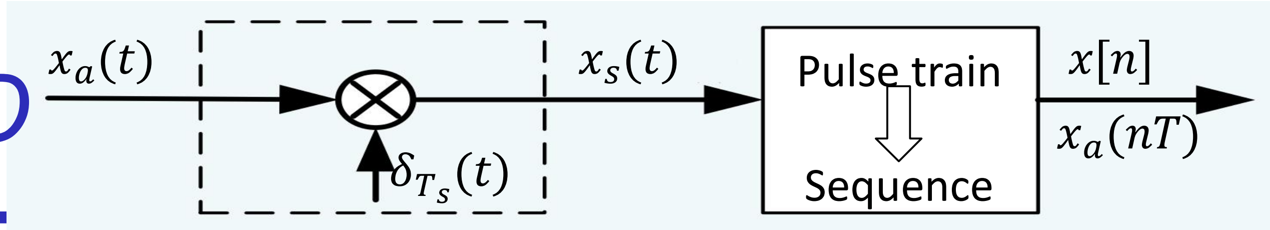
CTFT

- In FD: convolution between $X_a(j\Omega)$ and $\delta_{\Omega_s}(\Omega)$

$$X_s(j\Omega) = \frac{1}{2\pi} X_a(j\Omega) * \Omega_s \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a[j(\Omega - k\Omega_s)]$$



1.2 Sampling in FD



- An alternative expression of $X_s(j\Omega)$ is:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

$$\Big\Updownarrow \text{CTFT}$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_a(nT_s)e^{-j\Omega T_s n}$$

Since: $$x[n] = x_a(nT_s)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X_s(j\Omega) = X\left(e^{j\omega}\right)\Big|_{\omega=\Omega T_s} = X\left(e^{j\Omega T_s}\right)$$

Recall

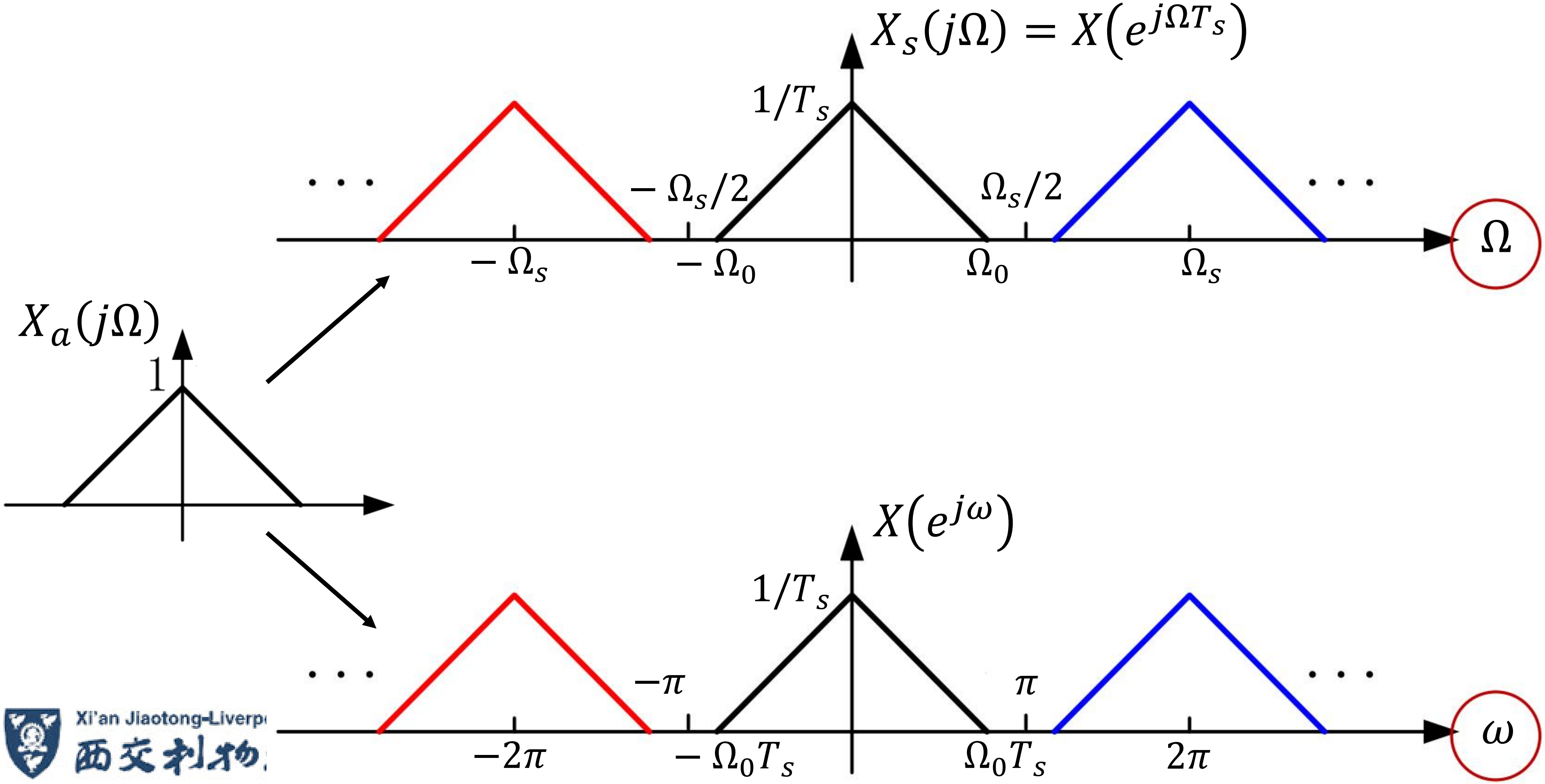
$$X_s(j\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a[j(\Omega - k\Omega_s)]$$

$$X\left(e^{j\Omega T_s}\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a[j(\Omega - k\Omega_s)]$$

$$X\left(e^{j\omega}\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left[j\left(\frac{\omega}{T_s} - \frac{2\pi k}{T_s}\right)\right]$$

1.2 Sampling in Frequency domain (FD)

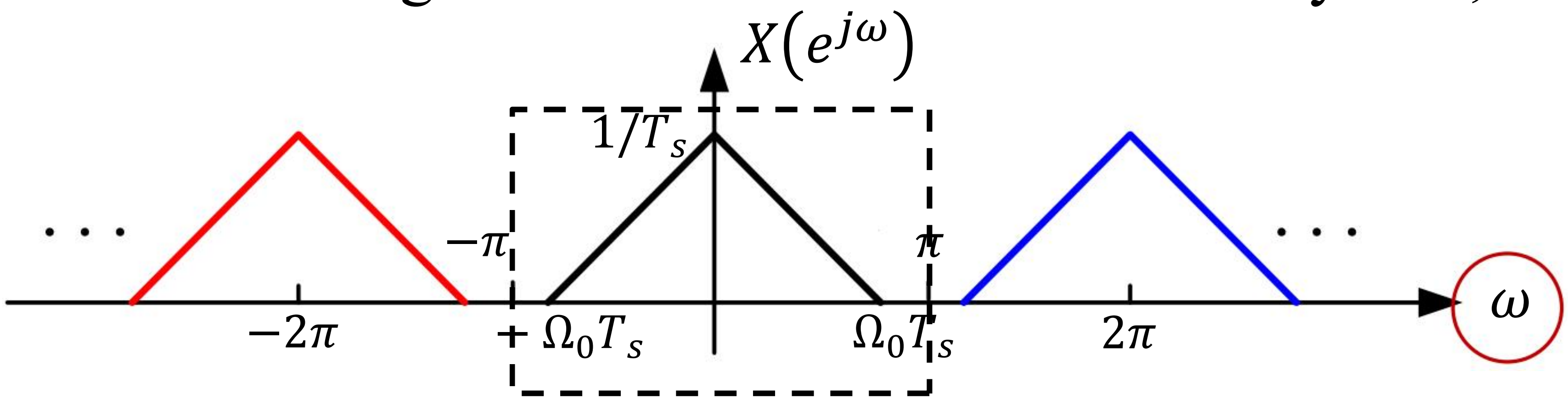
- Discretization in TD => Periodicity in FD



Xi'an Jiaotong-Liverp

1.3 Nyquist-Shannon Theorem

- The spectrum of the sampled signal contains all the information of the original CT signal
  – So the CT signal can be recovered without any loss;



  – But a condition needs to be satisfied:

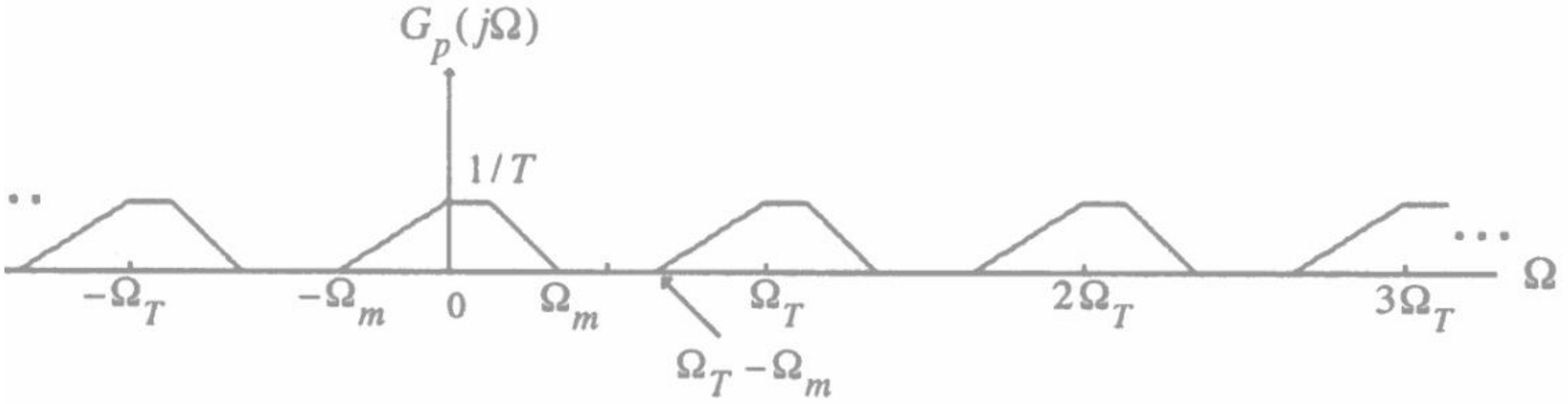
$$\Omega_0 T_s \leq \pi \Longleftrightarrow 2\Omega_0 \leq \Omega_s$$

  – The Nyquist-Shannon theorem!

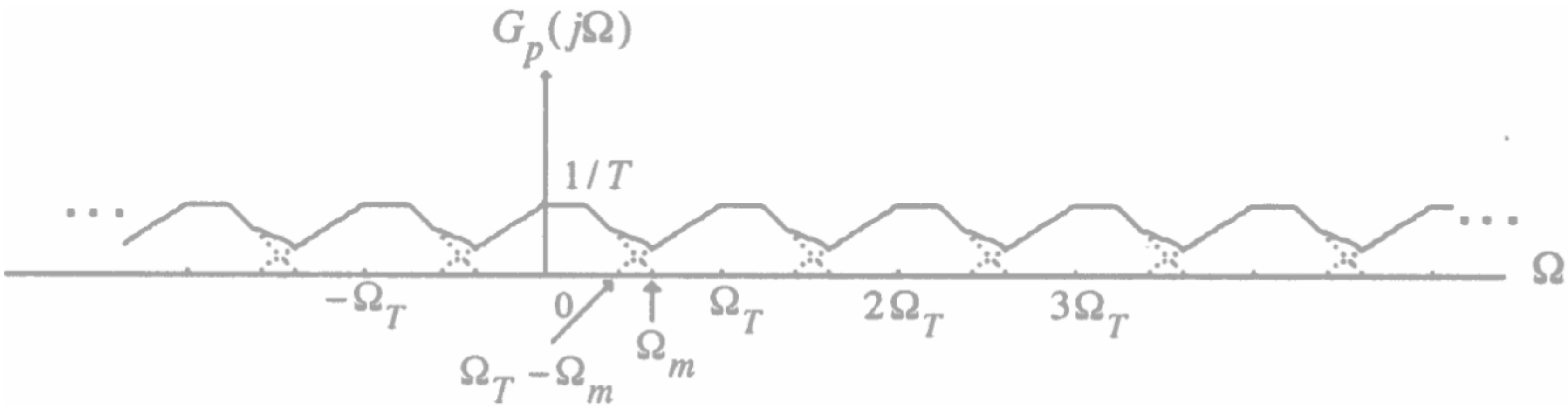
Xi'an Jiaotong-Liverpool University 西交利物浦大学

1.3 Over-sampling and under-sampling

- Over-sampling: $2\Omega_m < \Omega_T$



- Down-sampling: $2\Omega_m > \Omega_T$



Wrap-up of SAMPLING

- What is sampling process?
  - The first step to convert a continuous-time signal to a discrete-time signal;
- In time domain: multiplication the CT signal to a pulse train, then convert the modulated pulse train to sequence;
- In frequency domain: copy and shift (create infinite replica) the spectrum of the CT signal;
  - The CT signal can be recovered from the sampled signal if Nyquist theorem is satisfied, i.e., $2\Omega_0 \leq \Omega_s$
- Nyquist theorem

Example

- A continuous-time signal $x_a(t)$ is the linear combination of the components with 300Hz, 1.2kHz and 3.5kHz frequencies. Sampling $x_a(t)$ with a sampling frequency 2 kHz gets a sequence x[n]. Sending x[n] through an ideal lowpass filter with the cutoff frequency of 900Hz get a continuous-time output $y_a(t)$.
- What are the frequency components in $y_a(t)$?
- What is the critical sampling frequency?

Quiz 1

- The signal $x(t) = \sin(\pi t) + 4\sin(3\pi t)\cos(2\pi t)$, where *t* is in ms, is sampled at a rate of 3 kHz.
- Find the frequency components in $x(t)$.
- Find the Nyquist rate of the signal.

2.1 Interpolation

- What is interpolation?
  - In the mathematical field of numerical analysis, interpolation is a method of ***constructing new data points*** within the range of a discrete set of known data points.
  - In this module, interpolation is a procedure whereby we convert a discrete-time (DT) sequence $x[n]$ to a continuous-time (CT) function $x(t)$.
  - Requirement: for the CT function x(t), its values at multiples of T_s should be equal to the corresponding points of the DT sequence x[n]:

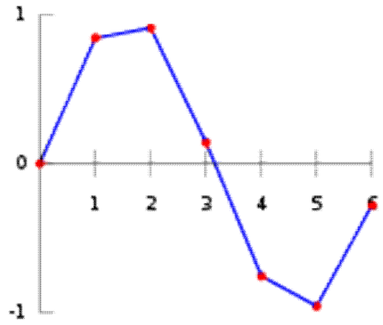
$$x(t)|_{t=nT_s} = x[n]$$

The interpolation problem now reduces to "filling the gap" between these instants.

2.1 Interpolation methods

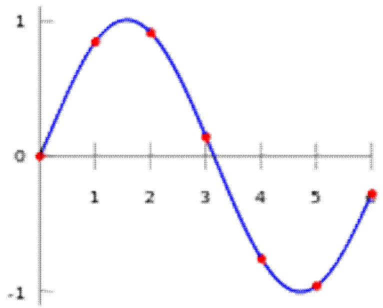
- 1. Zero-order/Local Interpolation

$$I_0(t) = \mathrm{rect}(t)$$

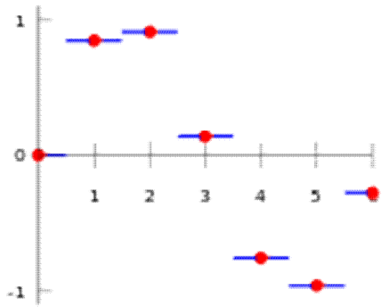


- 2. First-order/Linear Interpolation

$$I_1(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$



- 3. Higher-order/Polynomial Interpolation

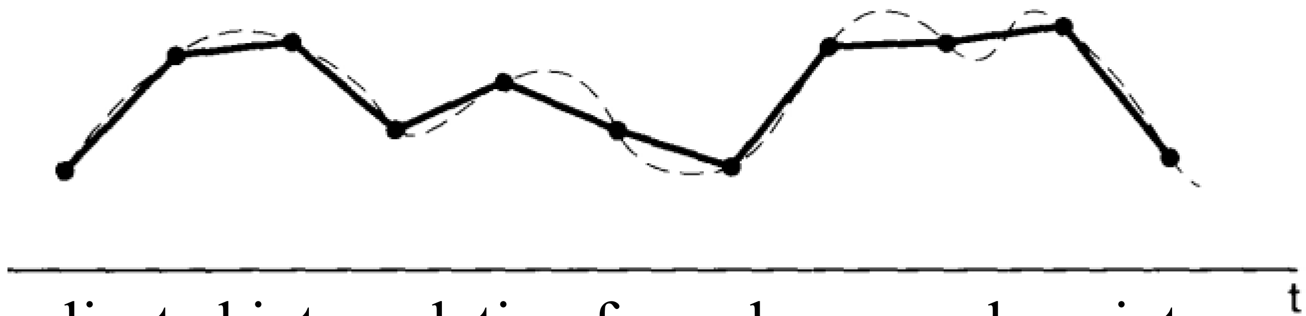


Xi'an Jiaotong-Liverpool University 西交利物浦大学

Which one should we use in signal reconstruction?

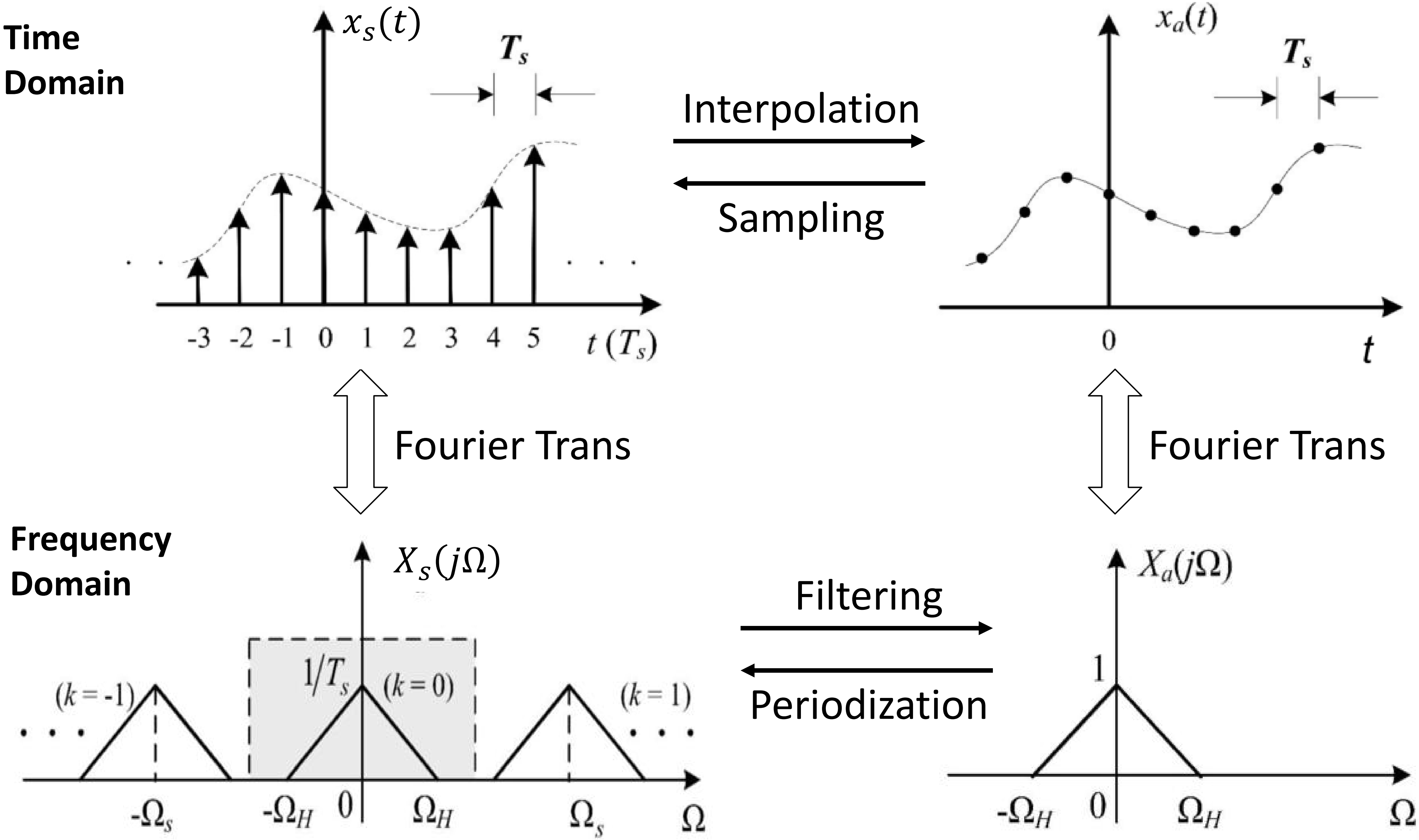
2.1 Reconstruction with interpolation

- Interpolation, the fitting of a continuous signal to a set of sample values, is a commonly used procedure for reconstructing a function, either approximately or exactly, from samples.
  - One simple interpolation procedure is the zero-order hold.
  - Another useful form of interpolation is *linear interpolation,* whereby adjacent sample points are connected by a straight line as shown below.



  - In more complicated interpolation formulas, sample points may be connected by higher order polynomials or other mathematical functions, like sinc().

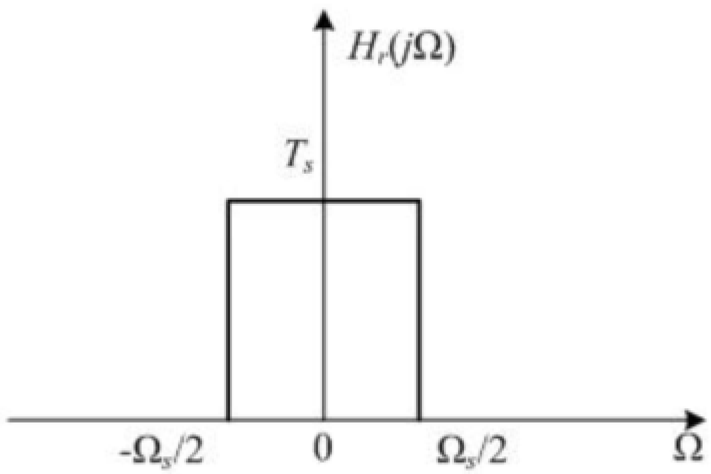
2.2 Reconstruction Theory



2.3 Reconstruction filter in Frequency domain

- Reconstruction or smoothing filter is used to eliminate all the replicas of the spectrum outside the baseband

- Ideal lowpass filter
  - Frequency domain $$H_r(j\Omega) = \begin{cases} T_s, & |\Omega| < \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

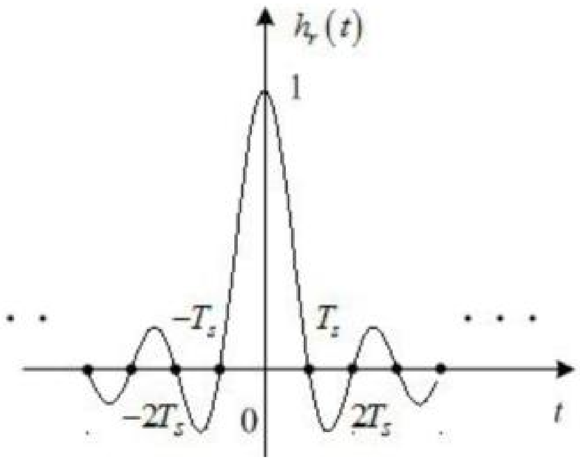


2.4 Reconstruction filter in time domain

- This lowpass filter in time domain is a "sinc" function:
  - Time domain

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{T_s}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}d\Omega$$

$$= \frac{\sin(\Omega_s t/2)}{\Omega_s t/2} = \frac{\sin(\pi t/T_s)}{\pi t/T_s} = \mathrm{sinc}(\frac{t}{T_s})$$



  - Multiply with $H_r(j\Omega)$ (in FD) is equivalent to convolve with $h_r(t)$ (in TD), the recovered signal $x_r(t) = x_s(t) * h_r(t)$

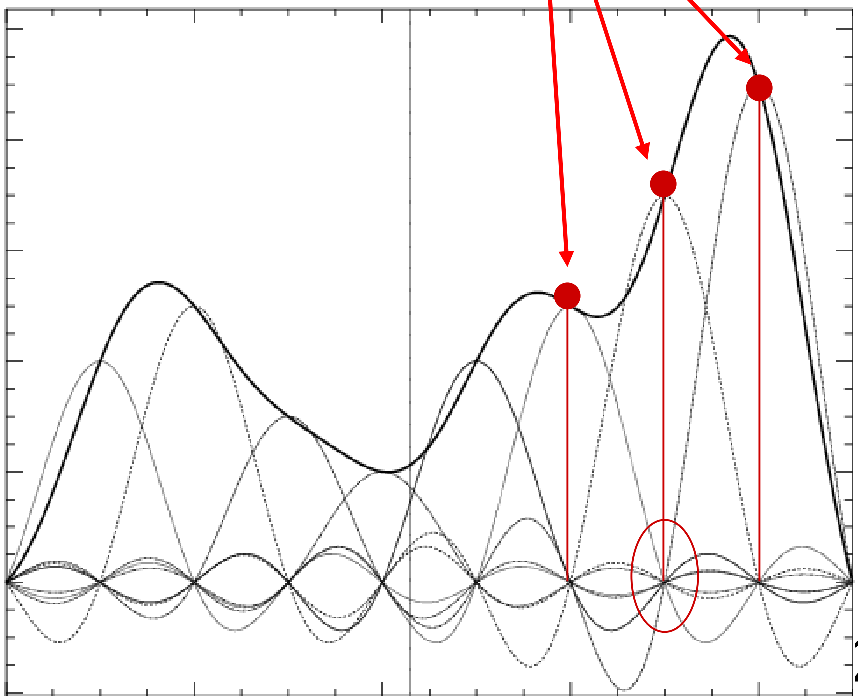
  - Impulse train $x_s(t)$: $x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$

2.4 Reconstruction filter in time domain

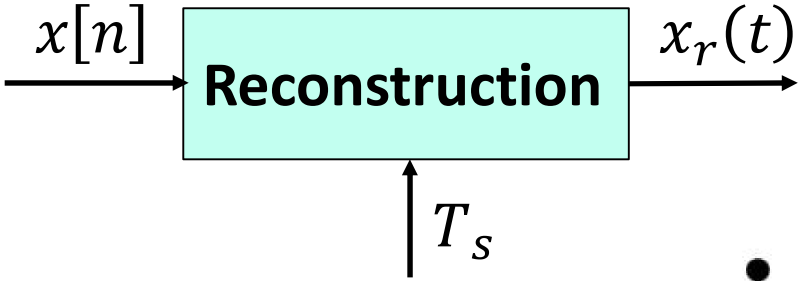
- Convolution between the discretized signal $x_s(t)$ and the reconstruction lowpass filter $h_r(t)$:

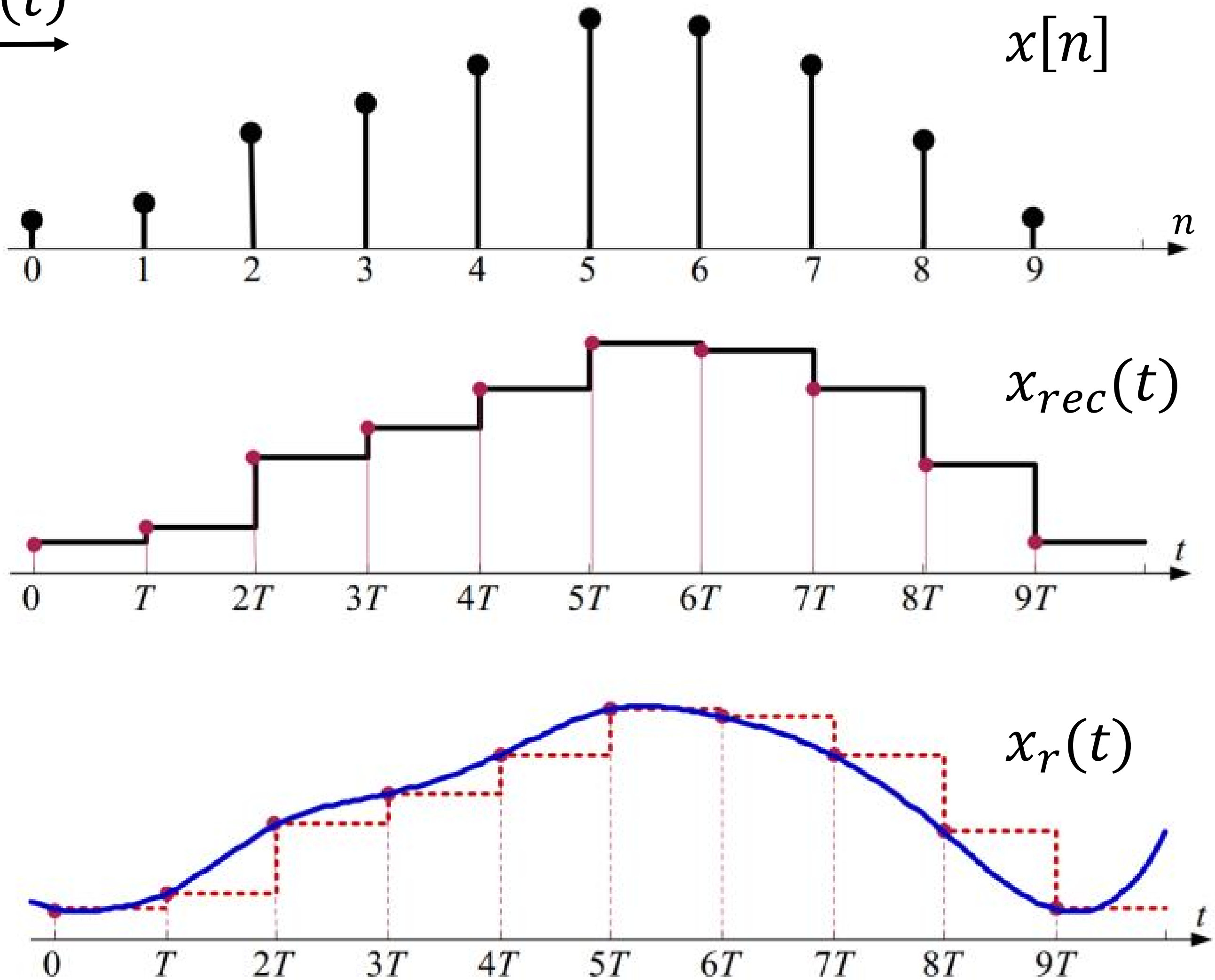
$$x_r(t) = x_s(t) * h_r(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)h_r(t - nT_s) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}(t - nT_s)$$

- The values are interpolated as a linear combination of the time-shifted sinc functions
- The amplitudes are scaled according to the sample values at the center locations of the sinc (the interpolation functions)



2.5 Realization





Xi'an Jiaotong-Liverpool University
西交利物浦大學

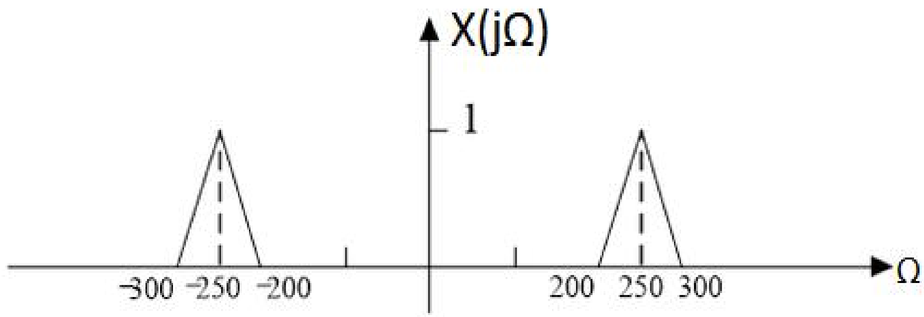
Wrap-up of RECONSTRUCTION

- Continuous-time signal can be reconstructed from the discrete-time sequence;
- Reconstruction can be realized as
  - In time domain: interpolation;
  - In frequency domain: filtering.
- Ideal reconstruction:

Filter with ideal lowpass filter	$\Leftrightarrow$	Interpolate with sinc function
Frequency domain		Time domain

Quiz 2

- The spectrum $X(j\Omega)$ of a continuous-time signal $x(t)$ is shown below. $x(t)$ is sampled with the sampling angular frequency Ωs, and get a discrete-time sequence $x[k]$, whose spectrum is $X(j\omega)$.



  - a) If $\Omega s = 200$ rad/s, sketch $X(j\omega)$ with all the labels for horizontal and vertical axes;
  - b) If $\Omega c = 200$ rad/s, can $x(t)$ be reconstructed from $x[k]$? Explain the reason.

Next ...

- Discrete-time Fourier Series
- Discrete-time Fourier Transform