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Introduction to Data Assimilation, Subgrid-Scale Parameterization and Predictability

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Evaluation

- https://uhh.de/evasys
- Losung: PS86R

Outline

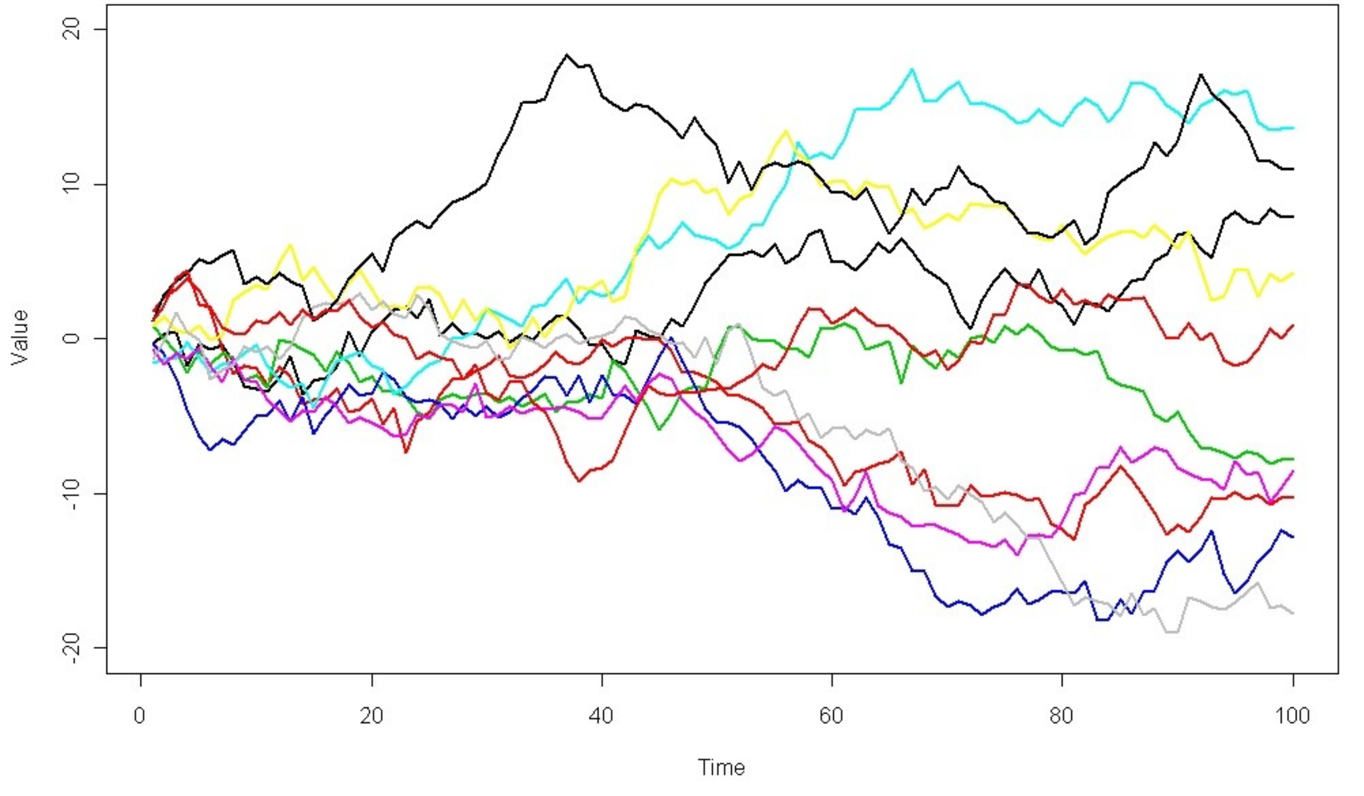
- Stochastic Differential Equations (SDE)
- Integration of SDEs

Random Walk

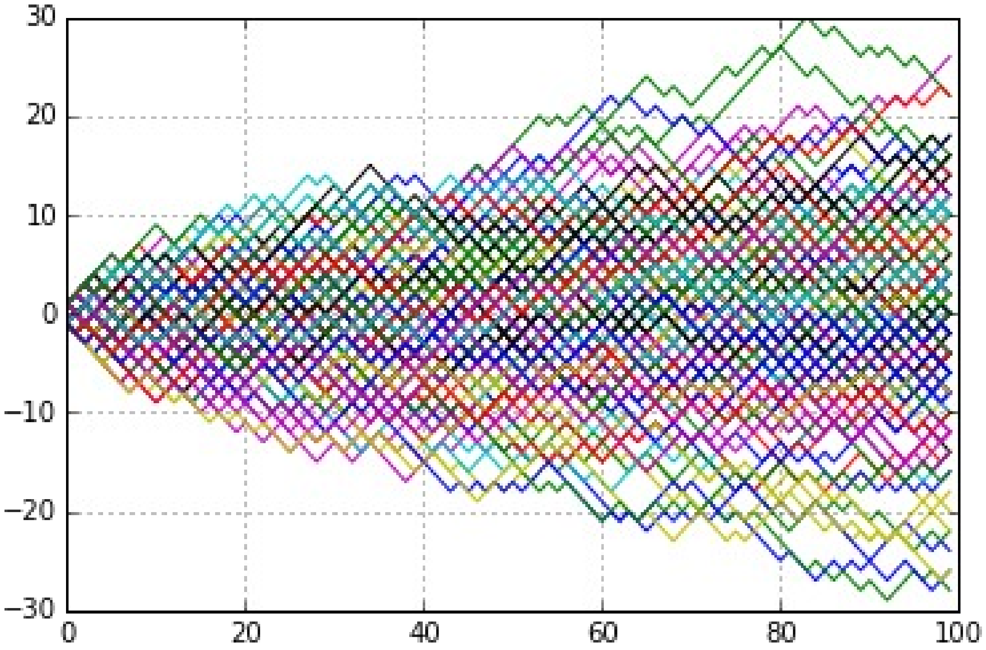
Random Walk:

x(t+1)=x(t)+ξ(t)

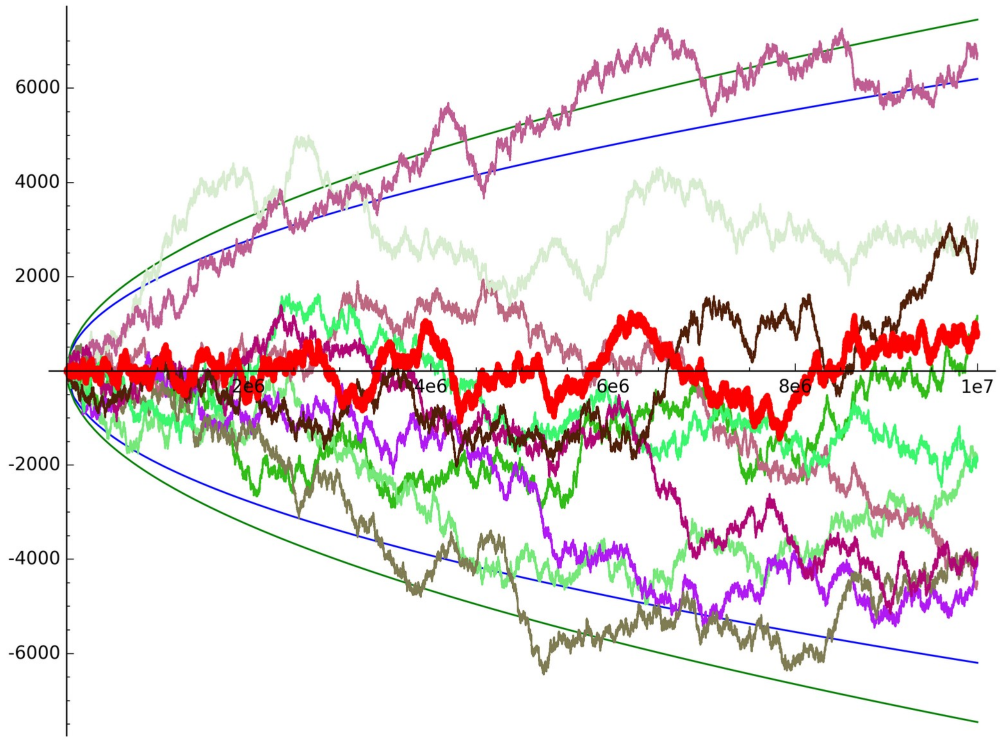
$$s(t)=\sum_{i=1}^{N}\xi_i$$



Random Walk



30
20
10
0
−10
−20
−30
0
20
40
60
80
100



6000
4000
2000
-2000
-4000
-6000
2e6
4e6
6e6
8e6
1e7

Random Walk

How is the variance of a Random Walk growing?

Random Walk

How is the variance of a Random Walk growing?

Random walk: $s(t)=\sum_{i=1}^{n}\xi_i$

Sum of Gaussian variables: s = X + Y

$$N\left(\mu_x+\mu_y\,,\,\sigma_x{}^2+\sigma_y{}^2\right)$$

We have $\mu_x = \mu_y = 0$ and $\sigma_x = \sigma_y = \sigma$

Thus $N\left(0\,,2\,\sigma^2\right)$

Random Walk

How is the variance of a Random Walk growing?

By induction for n steps: $N(0, n\sigma^2)$

Variance after n steps: $\sqrt{E({s_n}^2)} = \sigma\sqrt{n}$

Stochastic Differential Equation

$$dx = a(x,t)\,dt + b(x,t)\,dW_t$$

W_t: Wiener Process

1)$W_0 = 0$

2)W_t is almost surely everywhere continuous

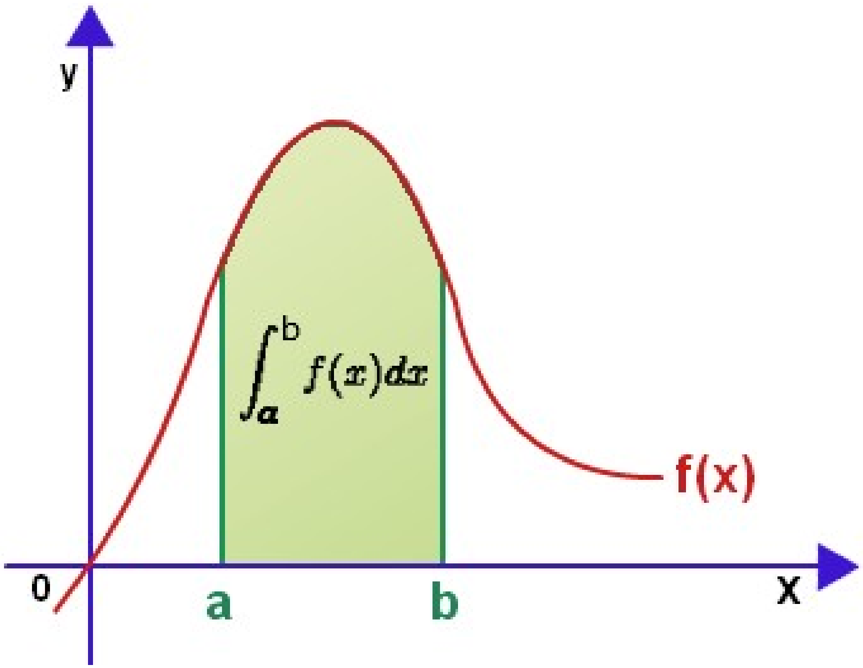
3)W_t has independent increments:

$W_t - W_s \sim N(0, t-s)$ for $0 \leq s < t$

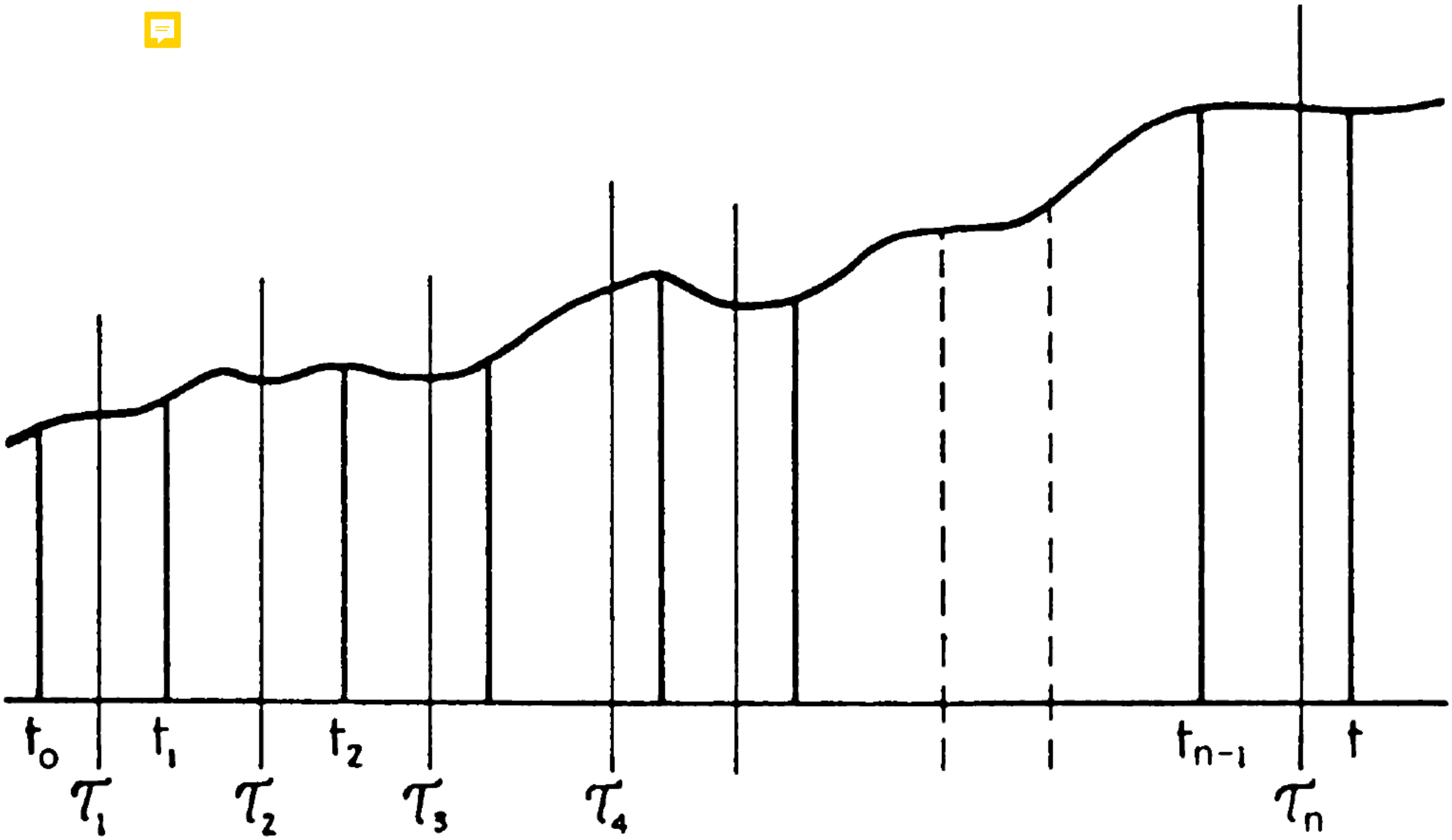
$dW_t = W(t+dt) - W(t) = \xi(t)\,(dt)^{1/2}$

$\langle \xi(t)\xi(t') \rangle = \delta(t-t')$

Integrals



Integrals



Integrals

Riemann Integral

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i)$$

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

Integrals

Riemann Sum

$$S=\sum_{i=1}^{n} f\left(x_i^r\right)\left(x_i-x_{i-1}\right), x_i \leqslant x_i^r \leqslant x_i$$

Left Riemann Sum: $x_i^r = x_{i-1}$

Right Riemann Sum: $x_i^r = x_i$

Middle Riemann Sum: $x_i^r = 0.5*(x_i+x_{i-1})$

Trapezoidal Sum: Average of Left and Right Riemann sums

Integrals

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Trapezoidal Sum: Average of Left and Right Riemann sums

For $\Delta x=x_i-x_{i-1} \rightarrow 0$ they all converge to the same value

Integrals

Riemann-Stieltjes Integral

$$\int_a^b f(x)\,dg(x) = \sum_{i=0}^{n-1} f(c_i)\left(g(x_{i+1}) - g(x_i)\right)$$

where c_i is in the i-th subinterval $[x_i, x_{i+1}]$.

$a = x_0 < x_1 < x_2 < \ldots < x_n = b$

Stochastic Integrals

We define $\int b(t)\,dW(t')$ as a kind of Riemann-Stieltjes integral.

Define intermediate points $t_{i-1} < \tau_i < t_i$

$$S_n = \sum_{i=1}^{n} b(\tau_i)\left(W(t_i) - W(t_{i-1})\right)$$

S_n depends on the choice of intermediate point τ_i!

Let's define: $\tau_i = \alpha t_i + (1-\alpha) t_{i-1}$ $(0<\alpha<1)$

Ito Stochastic Integral

$\alpha=0;\ \tau_i=t_{i-1}$

$$\int_{t_0}^{t} b(t')\,dW'_t = ms-\lim_{n\to\infty}\sum_{i=1}^{n} b(t_{t-1})\left(W_{t_i}-W_{t_{i-1}}\right)$$

ms-lim: mean square limit

Stratonovich Stochastic Integral

α=0.5; τ_i=0.5*(t_{i-1} + t_i)

$$\int_{t_0}^{t} b(t')\,dW'_t = ms-\lim_{n\to\infty}\sum_{i=1}^{n} b\left(\frac{t_{i-1}+t_i}{2}\right)\left(W_{t_i}-W_{t_{i-1}}\right)$$

b(t) and dW(t) are no longer independent

Stochastic Integrals

1) Ito $\int_{t_0}^{t} W(t')dW(t')$

$$S_n = \sum_{i=1}^{n} W_{i-1}(W_i - W_{i-1}) \equiv \sum_{i=1}^{n} W_{i-1}\Delta W_i$$

$$= \tfrac{1}{2}\sum_{i=1}^{n} [(W_{i-1} + \Delta W_i)^2 - (W_{i-1})^2 - (\Delta W_i)^2]$$

$$= \tfrac{1}{2}[W(t)^2 - W(t_0)^2] - \tfrac{1}{2}\sum_{i=1}^{n} (\Delta W_i)^2.$$

Stochastic Integrals

2) Stratonovich

$$S\int_{t_0}^{t} W(t')\,dW(t')$$

$$= \underset{n\to\infty}{\text{ms-lim}} \sum_{i=1}^{n} \frac{W(t_i) + W(t_{i-1})}{2}[W(t_i) - W(t_{i-1})]$$

$$= \tfrac{1}{2}[W(t)^2 - W(t_0)^2]\,.$$

Stochastic Integral

Physical interpretation:

Ito:

- independent noise increments e.g. in finance
- Ito lemma (new rules of calculus)
- Martingaile property $\mathbf{E}(X_{n+1} \mid X_1, \ldots, X_n) = X_n.$

Stratonovich:

- dependent noise increments → 'red noise' e.g. in physical systems like climate
- Normal rules of calculus apply

Stochastic Integral

Relationship between Ito and Stratonovich calculus:

$$\int_{t_0}^{t} \beta(x,t') \circ dW = \int_{t_0}^{t} \beta(x,t')\, dW + \frac{1}{2}\int_{t_0}^{t} b(x,t')\,\partial_x \beta(x,t')\, dt'$$

Denotes Stratonovich calculus

Stochastic Integral

Relationship between Ito and Stratonovich calculus:

Ito SDE: $dx = a\,dt + b\,dW$

$$\Longleftrightarrow \quad dx = \left(a - \frac{1}{2} b\,\partial_x b\right) dt + b \circ dW$$

Stratonovich SDE: $dx = \alpha\,dt + \beta \circ dW$

$$\Longleftrightarrow \quad dx = \left(\alpha + \frac{1}{2} \beta\,\partial_x \beta\right) dt + \beta\,dW$$

Stochastic Integral

Relationship between Ito and Stratonovich calculus:

Ito SDE: $dx = a\,dt + b\,dW$

$$\Longleftrightarrow \quad dx = \left(a - \frac{1}{2} b\,\partial_x b\right) dt + b \circ dW$$

Noise induced drift

Stratonovich SDE: $dx = \alpha\,dt + \beta \circ dW$

$$\Longleftrightarrow \quad dx = \left(\alpha + \frac{1}{2} \beta\,\partial_x \beta\right) dt + \beta\,dW$$

Stochastic Time Discrete Approximation

Ito SDE: $dx = a(x)dt + b(x)dW$

Euler Approximation:

$$x_{n+1} = x_n + a(x_n)\Delta t + b(x_n)(W_{n+1} - W_n)$$

Exercise

$dx = ax(t)dt + bx(t)\,dW$

Discretization: $x_{n+1} = x_n + ax_n\Delta t + bx_n\,\Delta W_n$

ΔW_n is $N(0,\Delta t) = N(0,1)*\mathrm{sqrt}(\Delta t)$

Analytical solution: $x(t)=x(0)\exp((a-0.5b^2)t+bW(t))$

Generate solutions on time interval [0,1] with time steps $\Delta t=2^{-2}, 2^{-4}, 2^{-6}$
Use $x(0)=1.0$, $a=1.5$, $b=1.0$
Use same noise sequence for analytical and numerical solutions!

Exercise

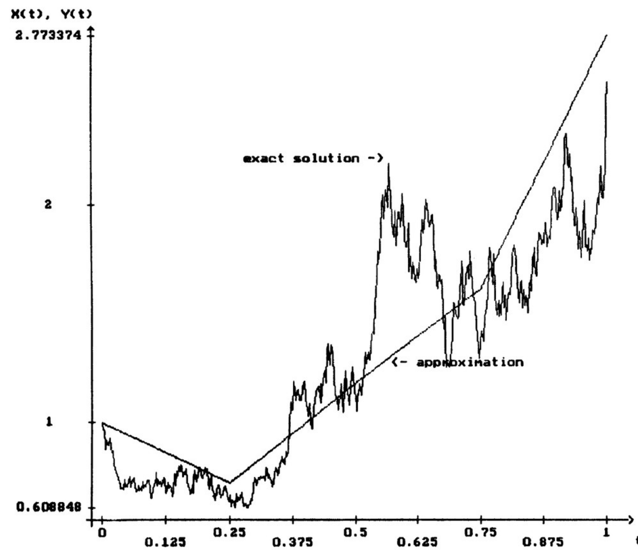

Figure 9.2.1 Euler approximation and exact solution from PC-Exercise 9.2.1.

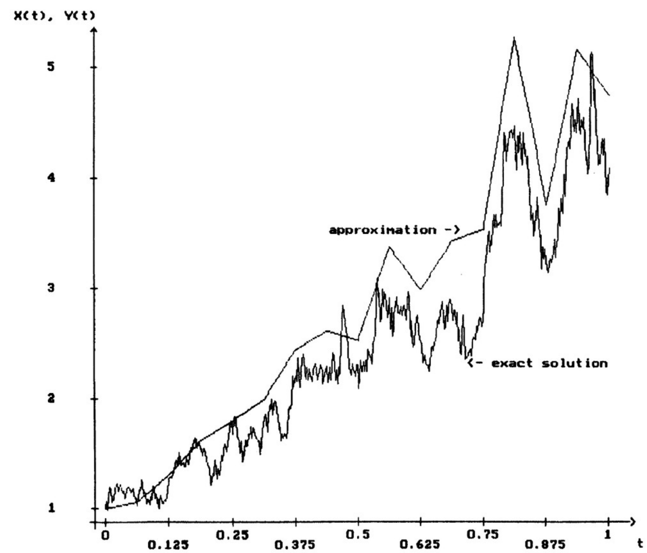

Figure 9.2.2 The Euler approximation for the smaller step size $\Delta = 2^{-4}$.