Introduction to Data Assimilation, Subgrid-Scale Parameterization and Predictability

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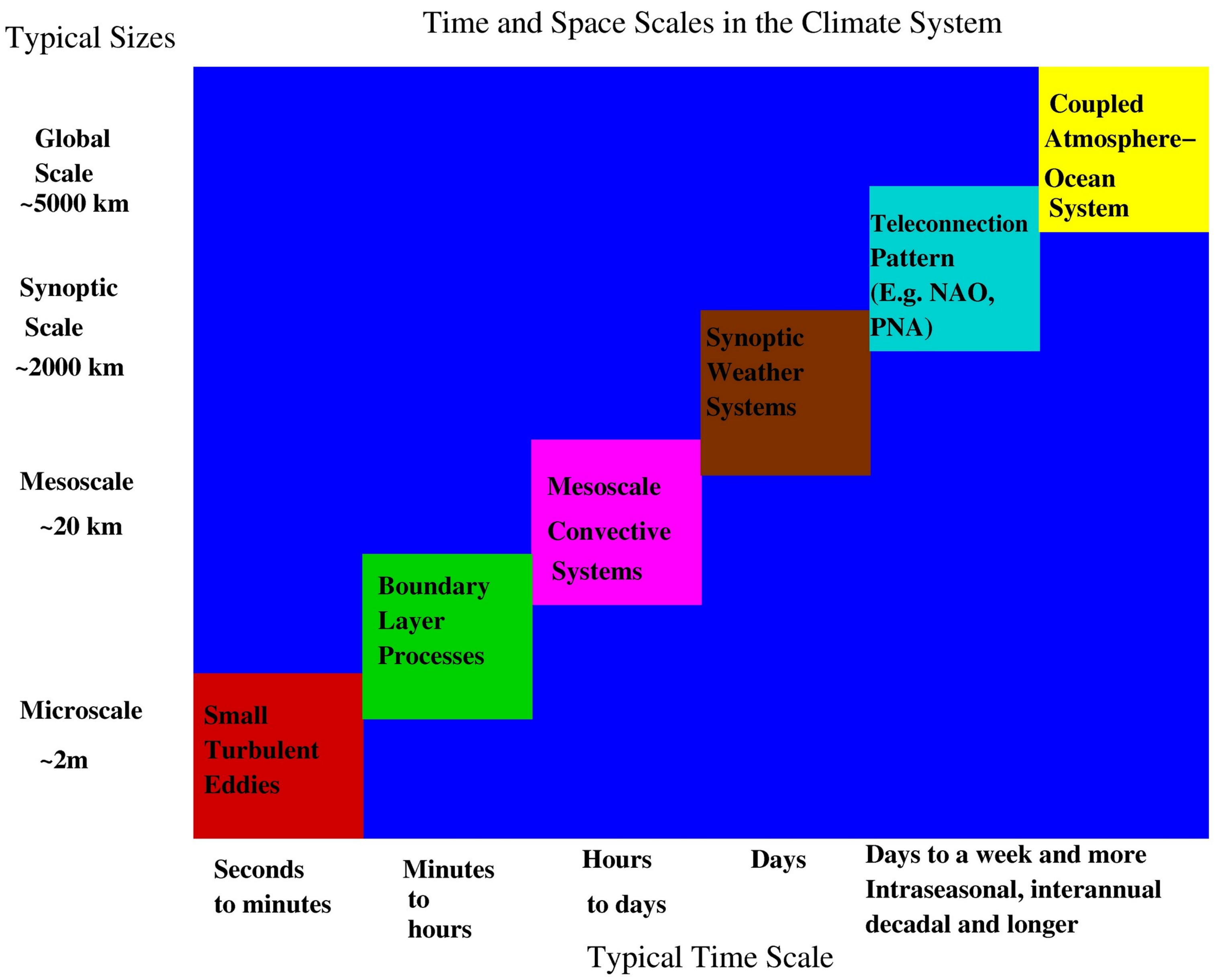
Outline

- Dimension reduction
- Stochastic climate models

Time Scales in the Climate System



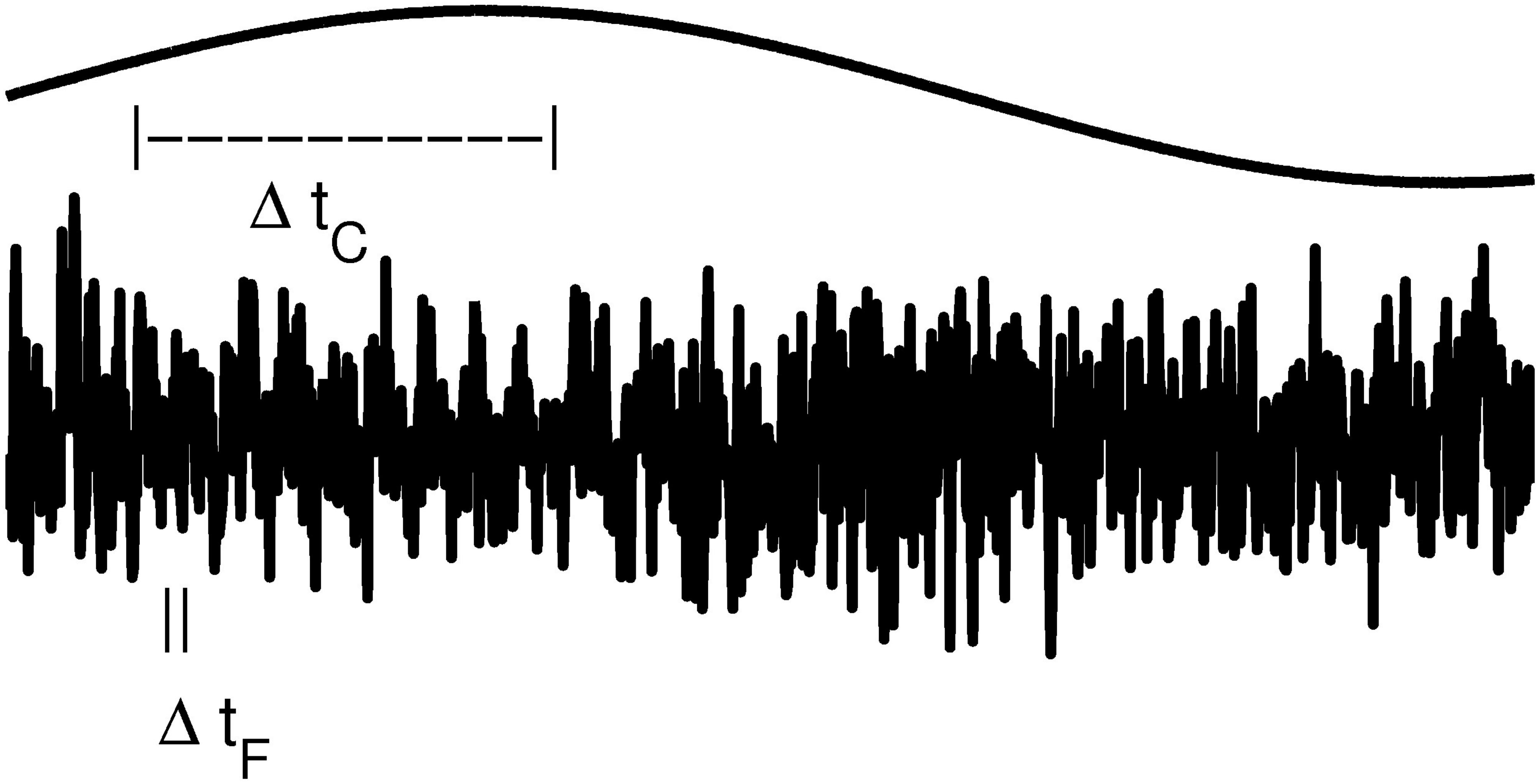
Dimension Reduction

Dimension Reduction

1) Simplification of the equations of motion
via scale analysis:
Primitive Equations
→ Quasi-geostrophic equations
→ Barotropic equations
→ ...

2) Adiabatic elimination of fast modes

Time Scale Separation



Dimension Reduction

$$dx = f(x, y)\,dt$$

$$dy = \frac{1}{\epsilon} g(x, y)\,dt + \frac{1}{\sqrt{\epsilon}} h(x, y)\,dW$$

ε: Measure of time scale separation

Dimension Reduction

$$dx = f(x, y)\,dt$$

$$dy = \frac{1}{\epsilon} g(x, y)\,dt + \frac{1}{\sqrt{\epsilon}} h(x, y)\,dW$$

ε: Measure of time scale separation

$$\Rightarrow dx = \tilde{f}(x)\,dt + \tilde{g}(x)\,dW$$

Example

$$dx = ax\,dt + by\,dt$$

$$dy = -\frac{1}{\epsilon}\,cx\,dt + \frac{1}{\sqrt{\epsilon}}\,k\,dW$$

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$$dx = ax\,dt + by\,dt$$

$$dy = -\frac{1}{\epsilon} cx\,dt + \frac{1}{\sqrt{\epsilon}} k\,dW$$

Solve equation for y(t):

$$y(t) = y(0)\,\mathrm{e}^{\frac{-ct}{\epsilon}} + \frac{1}{\sqrt{\epsilon}} \int_0^t e^{\frac{-c(t-s)}{\epsilon}} k\,dW$$

Example

$$dx = ax\,dt + b\left[\, y(0)\, e^{\frac{-ct}{\epsilon}} + \frac{k}{\sqrt{\epsilon}} \int_0^t e^{\frac{-c(t-s)}{\epsilon}}\, dW \right]$$

Time scale separation: ε-> 0

$y(0)\, e^{\frac{-ct}{\epsilon}} \rightarrow 0$ for $\epsilon \rightarrow 0$

$\frac{k}{\sqrt{\epsilon}} \int_0^t e^{\frac{-c(t-s)}{\epsilon}}\, dW \rightarrow \frac{k}{c} dW$ for $\epsilon \rightarrow 0$

Example

Effective model:

$$dx = ax\,dt + \frac{bk}{c}\,dW$$

Stochastic Climate Modeling

$$d\vec{z} = F\,dt + L\vec{z}\,dt + B(\vec{z},\vec{z})\,dt$$

$$\vec{z} = (\vec{x},\vec{y})$$

Slow modes

Fast modes

Stochastic Climate Modeling

$$d\,\vec{x} = F_1\,dt + L_{11}\,\vec{x}\,dt + L_{12}\,\vec{y}\,dt + B^1_{11}(\vec{x},\vec{x})\,dt + B^1_{12}(\vec{x},\vec{y})\,dt + B^1_{22}(\vec{y},\vec{y})\,dt$$

$$d\,\vec{y} = F_2\,dt + L_{21}\,\vec{x}\,dt + L_{22}\,\vec{y}\,dt + B^2_{11}(\vec{x},\vec{x})\,dt + B^2_{12}(\vec{x},\vec{y})\,dt + B^2_{22}(\vec{y},\vec{y})\,dt$$

Stochastic Modeling Assumption:

$$B^2_{22}(\vec{y},\vec{y}) \approx -\frac{\Gamma}{\epsilon}\,\vec{y} + \frac{\sigma}{\sqrt{\epsilon}}\,d\,\vec{W}$$

Stochastic Process
Ornstein-Uhlenbeck Process

Stochastic Mode Reduction

$$d\,x_1 = \frac{A_1}{\epsilon} x_2\, y\, dt$$

$$d\,x_2 = \frac{A_2}{\epsilon} x_1\, y\, dt$$

$$d\,y = \frac{A_3}{\epsilon} x_1 x_2\, dt - \frac{\gamma}{\epsilon^2} y\, dt + \frac{\sigma}{\epsilon} dW$$

Energy conservation: $A_1 + A_2 + A_3 = 0$

Stochastic Mode Reduction

$$y(t)=y(0)e^{-\frac{\gamma t}{\epsilon^2}}+\frac{A_3}{\epsilon}\int_0^t e^{\frac{-\gamma(t-s)}{\epsilon^2}}x_1 x_2\,ds+g(t)$$

$$g(t)=\frac{\sigma}{\epsilon}\int_0^t e^{\frac{-\gamma(t-s)}{\epsilon^2}}dW$$

$\epsilon \rightarrow 0$

$$y(0)e^{-\frac{\gamma t}{\epsilon^2}}\rightarrow 0$$

$$\frac{A_3}{\epsilon}\int_0^t e^{\frac{-\gamma(t-s)}{\epsilon^2}}x_1 x_2\,ds\rightarrow\frac{A_3}{\gamma}x_1(t)x_2(t)$$

$$\frac{1}{\epsilon}g(t)\rightarrow\frac{\sigma}{\gamma}dW$$

Stochastic Mode Reduction

$$d\,x_1 = \frac{A_1 A_3}{\gamma} x_1 x_2^2\,dt + \frac{\sigma}{\gamma} A_1 x_2 \circ dW$$

Stratonovich

$$d\,x_2 = \frac{A_2 A_3}{\gamma} x_1^2 x_2\,dt + \frac{\sigma}{\gamma} A_2 x_1 \circ dW$$

$$d\,x_1 = \frac{A_1}{\gamma}\left[A_3 x_2^2 + A_2 \frac{\sigma^2}{2\gamma}\right] x_1\,dt + \frac{\sigma}{\gamma} A_1 x_2\,dW$$

Ito

$$d\,x_2 = \frac{A_2}{\gamma}\left[A_3 x_1^2 + A_1 \frac{\sigma^2}{2\gamma}\right] x_2\,dt + \frac{\sigma}{\gamma} A_2 x_1\,dW$$

Reduced Order Models

3 level QG Model (Marshall and Molteni 1993

$$\frac{\partial q_1}{\partial t} = -J(\psi_1, q_1) - D_1(\psi_1, \psi_2) + S_1$$

$$\frac{\partial q_2}{\partial t} = -J(\psi_2, q_2) - D_2(\psi_1, \psi_2, \psi_3) + S_2$$

$$\frac{\partial q_3}{\partial t} = -J(\psi_3, q_3) - D_3(\psi_2, \psi_3) + S_3,$$

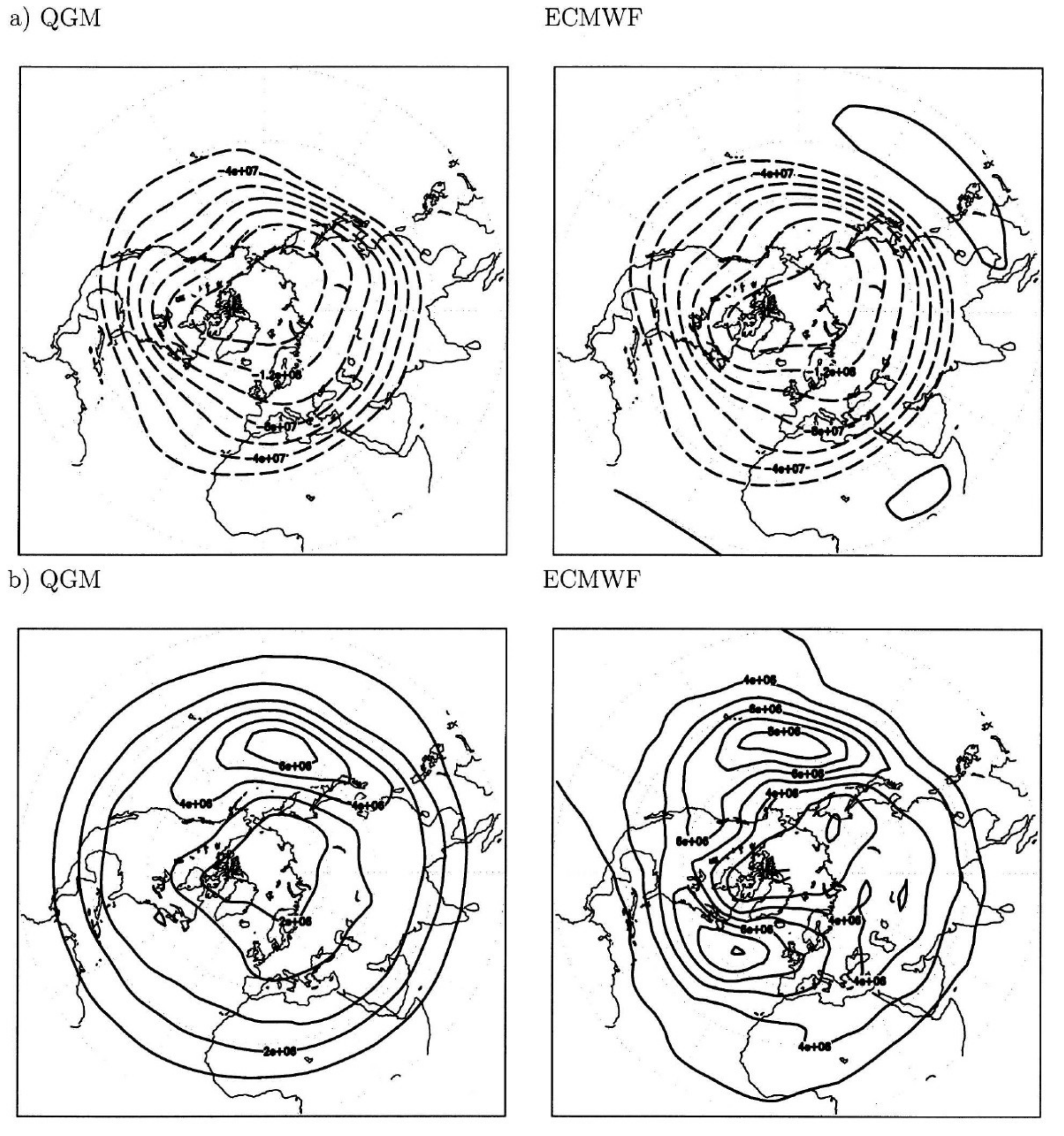
$$q_1 = \nabla^2\psi_1 - R_1^{-2}(\psi_1 - \psi_2) + f$$

$$q_2 = \nabla^2\psi_2 + R_1^{-2}(\psi_1 - \psi_2) - R_2^{-2}(\psi_2 - \psi_3) + f$$

$$q_3 = \nabla^2\psi_3 + R_2^{-2}(\psi_2 - \psi_3) + f\left(1 + \frac{h}{H_0}\right),$$

Franzke and Majda 2006

Reduced Order Models



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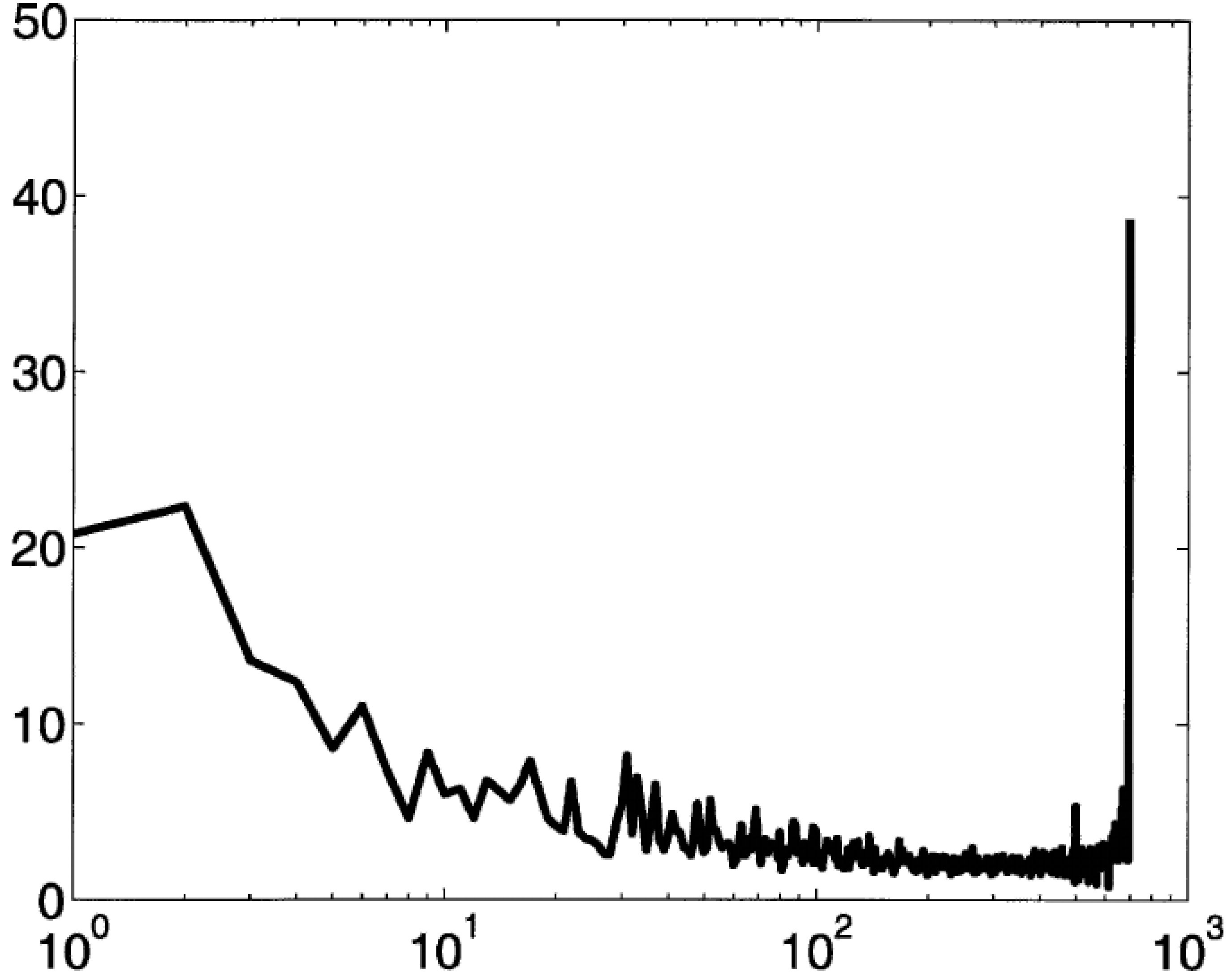
Reduced Order Models



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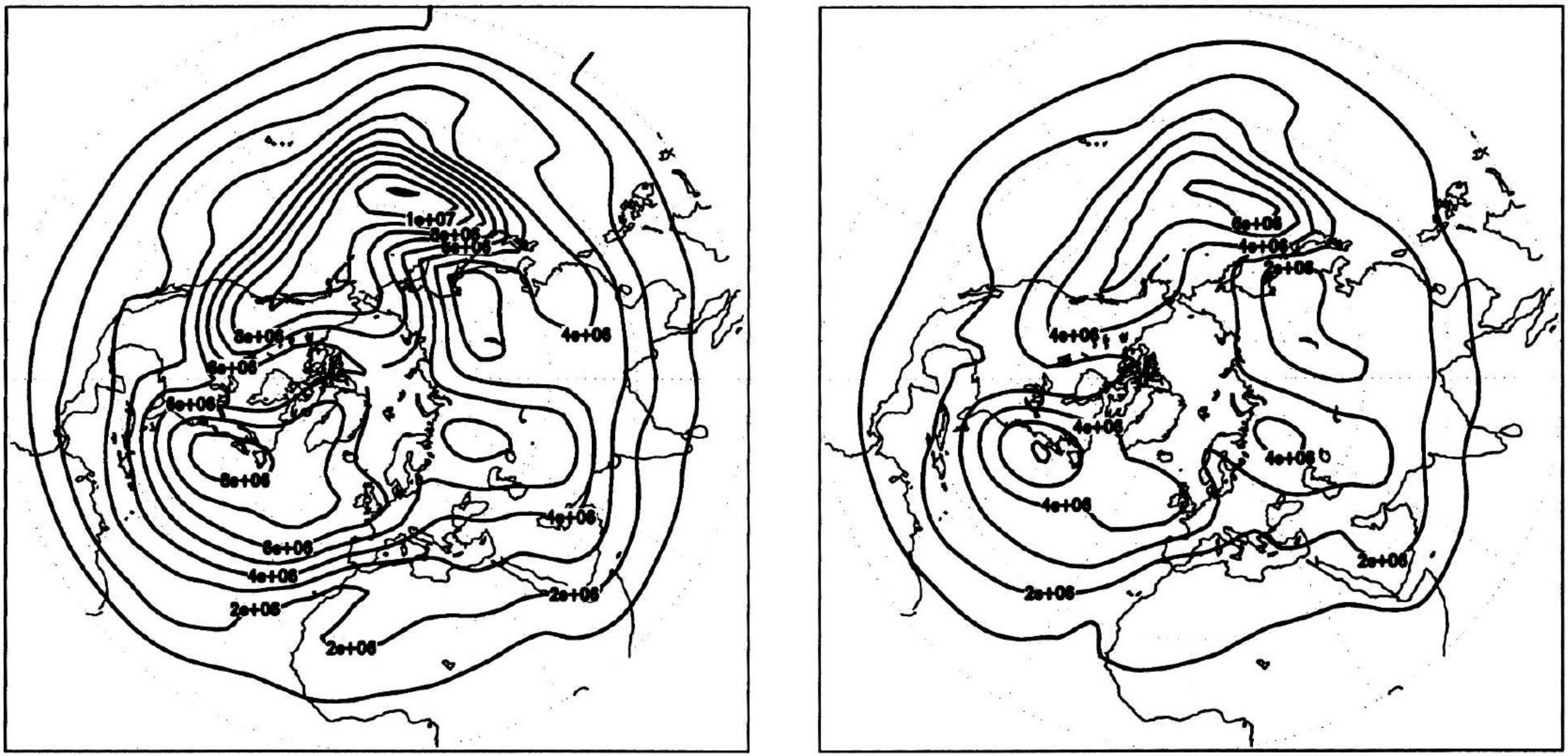
Reduced Order Models

a) QG-Model b) S-MTV



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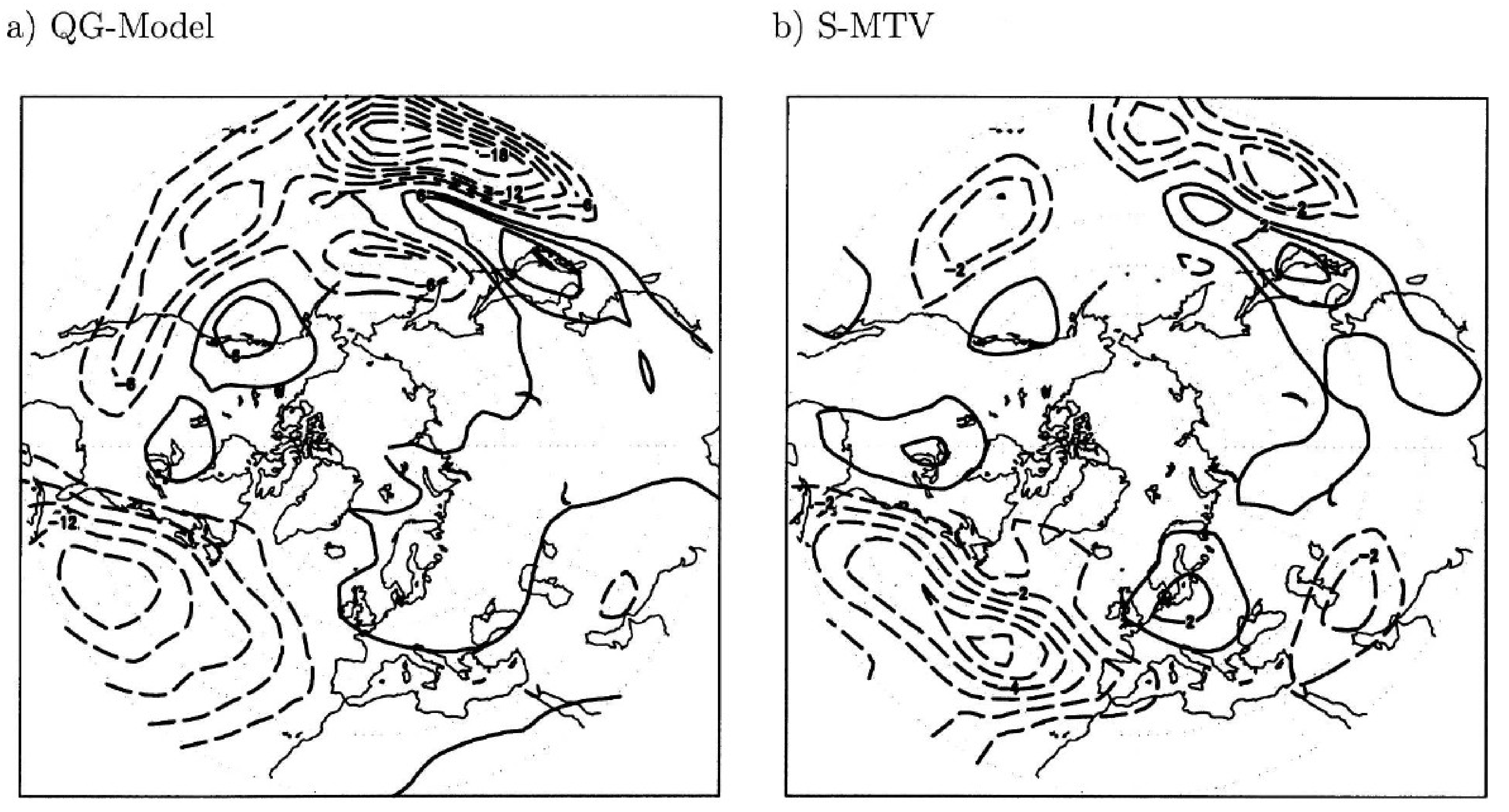
Reduced Order Models



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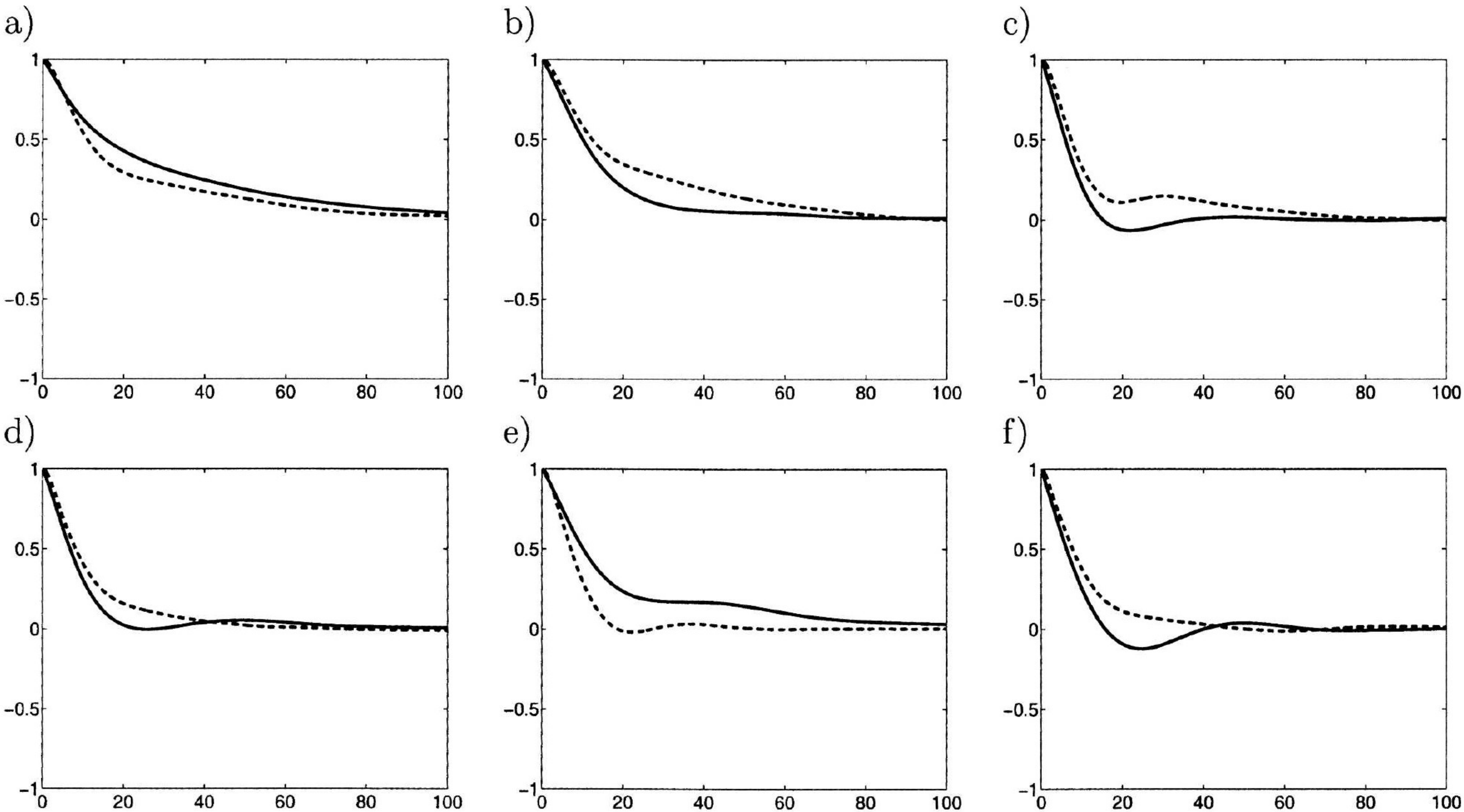
Reduced Order Models



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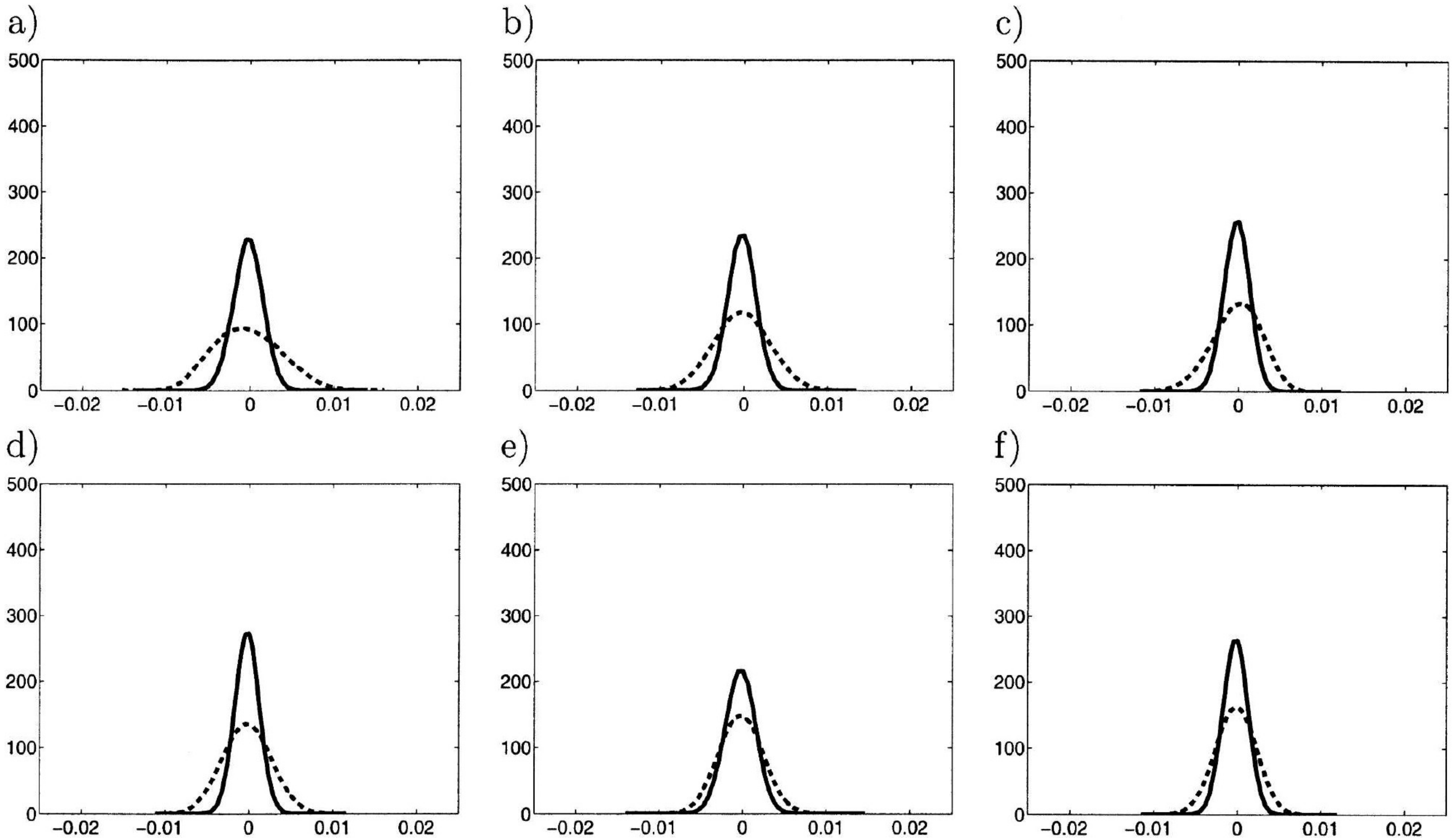
Reduced Order Models



Franzke and Majda 2006

Exercise

$$d\,x_1 = \frac{A_1}{\epsilon} x_2 y\,dt$$

$$d\,x_2 = \frac{A_2}{\epsilon} x_1 y\,dt$$

$$d\,y = \frac{A_3}{\epsilon} x_1 x_2\,dt - \frac{\gamma}{\epsilon^2} y\,dt + \frac{\sigma}{\epsilon} dW$$

$A_1=0.5$ $\gamma=1$

$A_2=0.5$ $\sigma=1$

$A_3=-1$ $\varepsilon=0.1$

$$d\,x_1 = \frac{A_1}{\gamma}\left[A_3 x_2^2 + A_2 \frac{\sigma^2}{2\gamma}\right] x_1\,dt + \frac{\sigma}{\gamma} A_1 x_2\,dW$$

$$d\,x_2 = \frac{A_2}{\gamma}\left[A_3 x_1^2 + A_1 \frac{\sigma^2}{2\gamma}\right] x_2\,dt + \frac{\sigma}{\gamma} A_2 x_1\,dW$$

Use 4-th order Runge-Kutta for deterministic part and Euler forward for stochastic part

Compute Autocorrelation function and Probability Density Function for comparison