Introduction to Data Assimilation, Subgrid-Scale Parameterization and Predictability

Christian Franzke

Meteorological Institute

Center for Earth System Research and Sustainability

University of Hamburg

Email: christian.franzke@uni-hamburg.de

Outline for Today

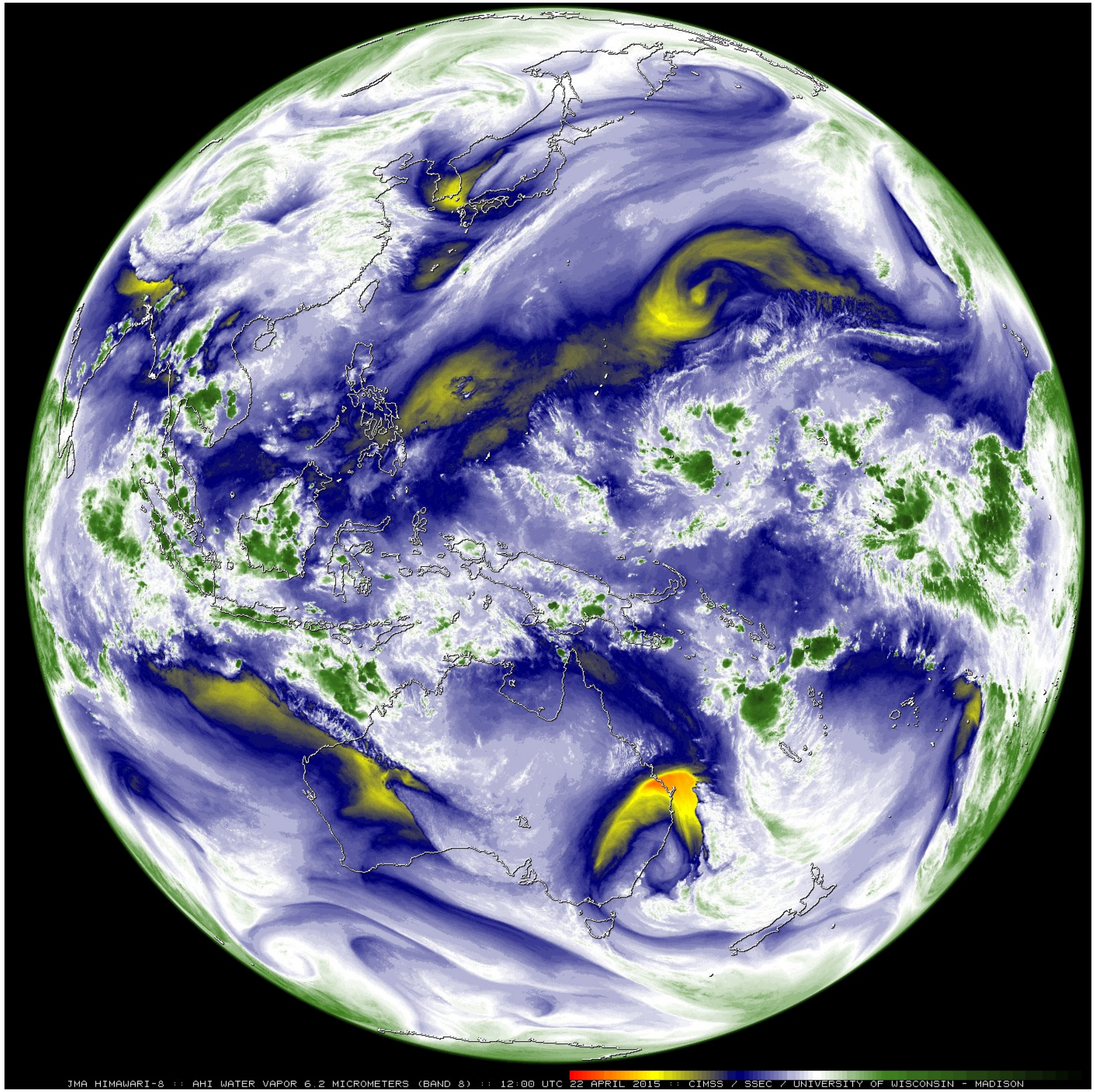
Subgrid-scale Parameterizations

- Motivation
- Reynolds Averaging
- Energy Spectra
- Random Number Generation

Climate Models

Explainity Video

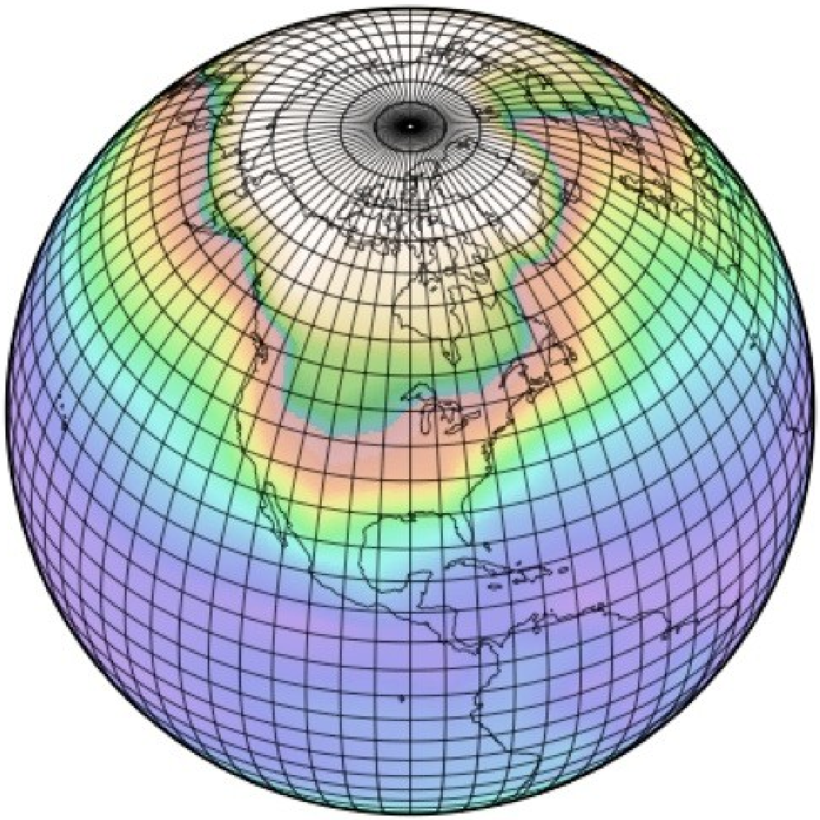
Climate Models

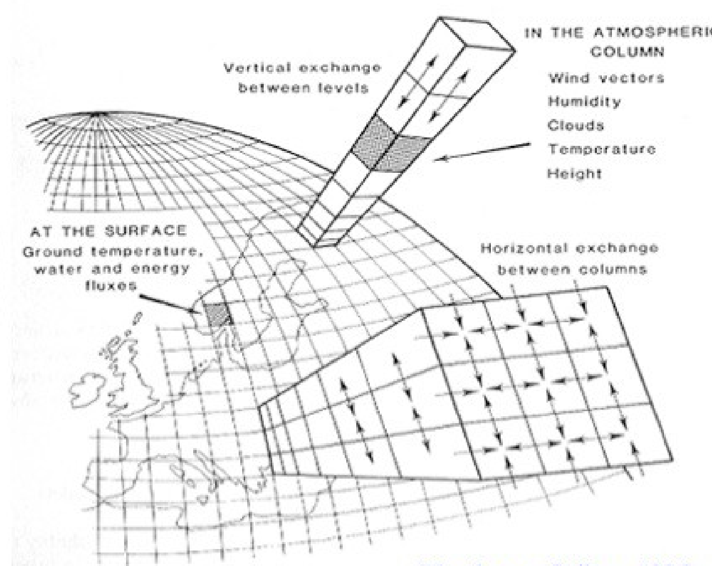


NOAA

Climate Models

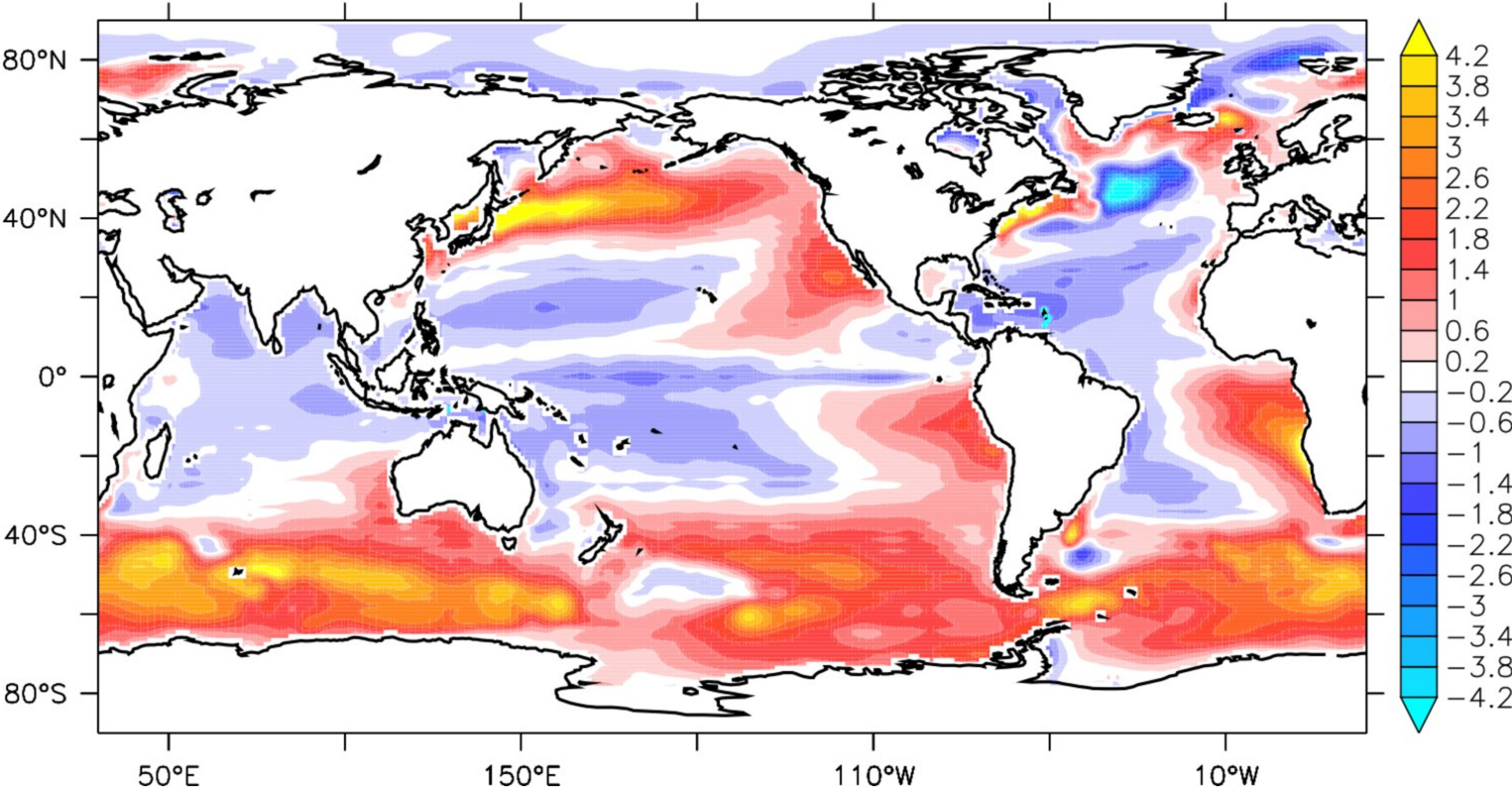
Grid Point Models





(Henderson-Sellers, 1985)

SST Bias



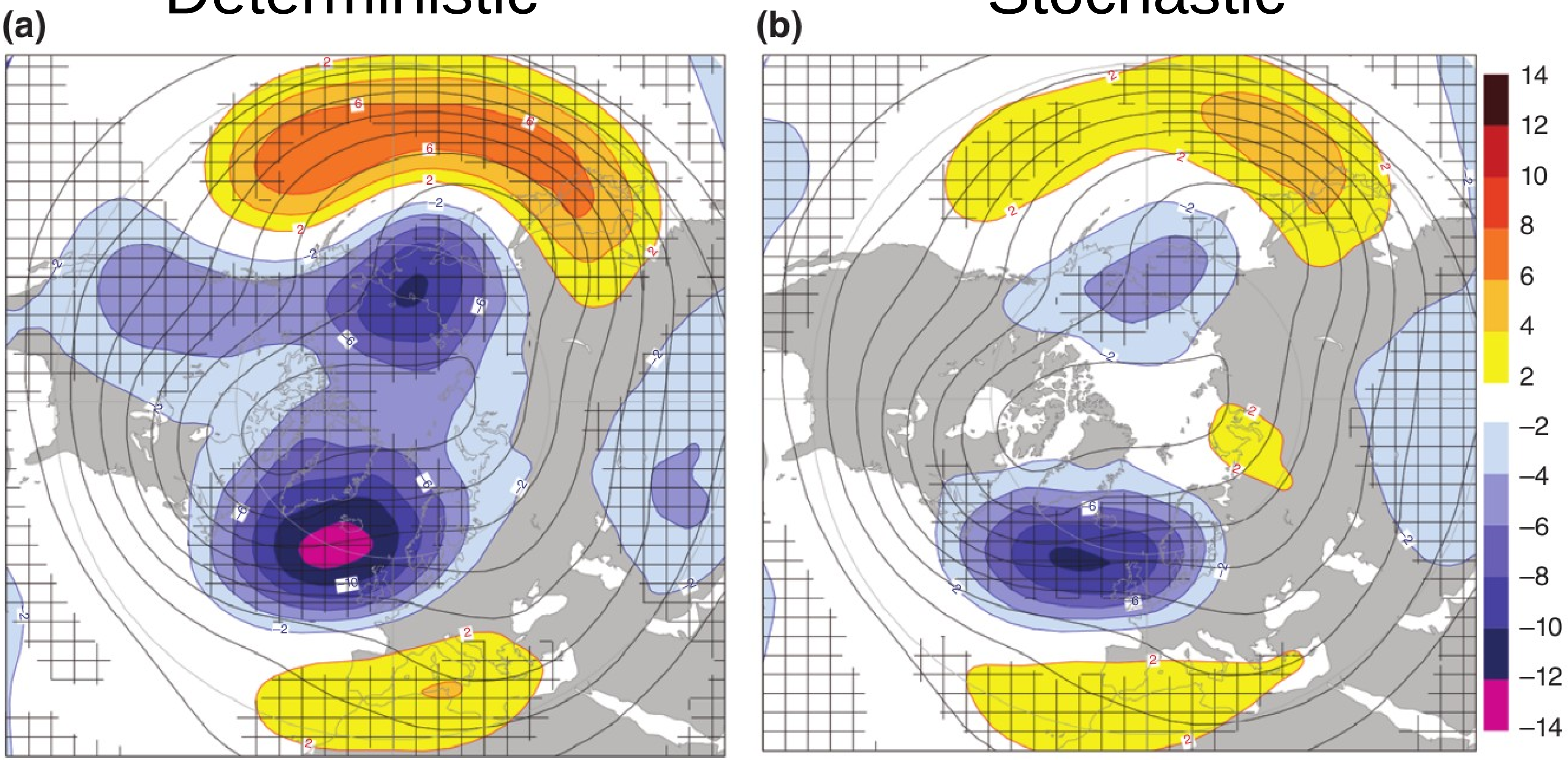
Annual-mean SST difference between a long EC-Earth control run and the WOA09 climatology

Sterl et al. 2011

Atmospheric Bias

Deterministic

Stochastic



Franzke et al. 2015

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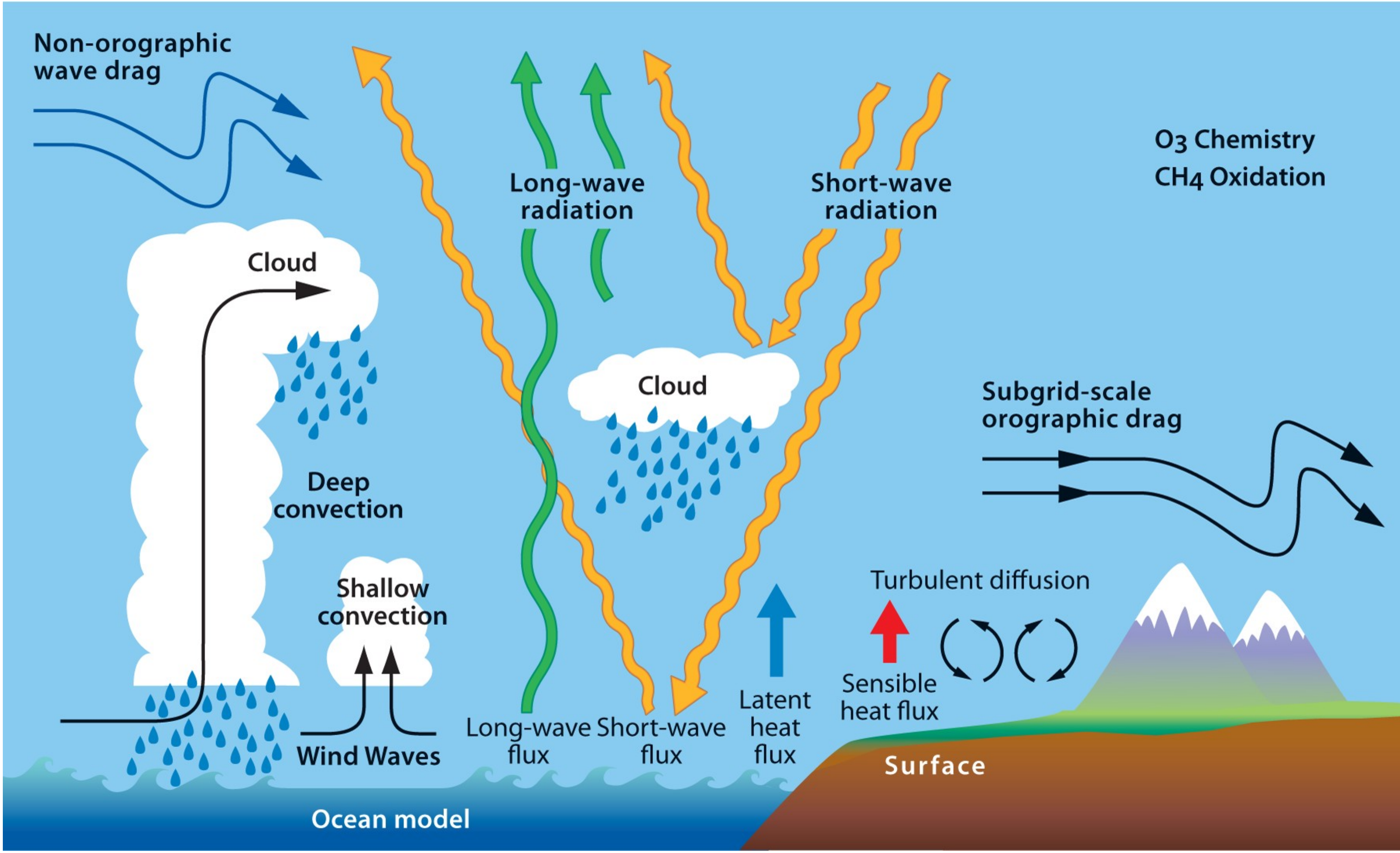
Subgrid-Scale Parameterizations



The closure problem

The closure problem

Decomposing the velocity field into mean and fluctuating components

$$u = \bar{u} + u'$$

$$\overline{u'} = 0$$

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The closure problem

Reynolds Averaging:

Consider
$$\frac{du}{dt}+uu+ru=0$$

Averaging
$$\frac{d\overline{u}}{dt}+\overline{uu}+r\,\overline{u}=0$$

$$\overline{uu}=\overline{u}\,\overline{u}+\overline{u'u'}$$

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The closure problem

Reynolds Averaging:

Consider $$\frac{du}{dt}+uu+ru=0$$

Averaging $$\frac{d\bar{u}}{dt}+\overline{uu}+r\bar{u}=0$$

$$\overline{uu}=\bar{u}\,\bar{u}+\overline{u'u'} \neq 0$$

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The closure problem

Obtain equation for $\overline{uu}$

$$\frac{1}{2}\frac{d\overline{u^2}}{dt}+\overline{uuu}+r\overline{u^2}=0$$

Triple term

Continue ...

$$\frac{d\overline{u^3}}{dt}+\overline{uuuu}+\ldots=0$$

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Deterministic Parameterizations

Zeroth order closure: $\overline{u'u'}=0$

K-Theory: $\overline{u'u'}=K\frac{\partial\bar{u}}{\partial x}$

Second order closure:

$$\overline{u'u'u'}=K'\frac{\partial\overline{u'u'}}{\partial x}$$

Kalnay 2003

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Energy Spectra

Energy Spectra

Triad Interactions

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = F + \nu \nabla^2 \psi, \qquad \zeta = \nabla^2 \psi.$$

Fourier Decomposition

$$\psi(x, y, t) = \sum_{\boldsymbol{k}} \widetilde{\psi}(\boldsymbol{k}, t) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \zeta(x, y, t) = \sum_{\boldsymbol{k}} \widetilde{\zeta}(\boldsymbol{k}, t) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

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Energy Spectra

$$\frac{\partial}{\partial t}\widetilde{\psi}(\boldsymbol{k},t) = \sum_{\boldsymbol{p},\boldsymbol{q}} A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\widetilde{\psi}(\boldsymbol{p},t)\widetilde{\psi}(\boldsymbol{q},t) + \widetilde{F}(\boldsymbol{k}) - \nu k^4\widetilde{\psi}(\mathbf{k},t),$$

Triad Interactions

$$A(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = (q^2/k^2)(p^x q^y - p^y q^x)\delta(\boldsymbol{p}+\boldsymbol{q}-\mathbf{k})$$

Only

$$\boldsymbol{p}+\boldsymbol{q} = \mathbf{k}$$

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Energy Spectra

Triad Interactions

(i) Local interactions, in which $k \sim p \sim q$;

(ii) Nonlocal interactions, in which $k \sim p \gg q$.

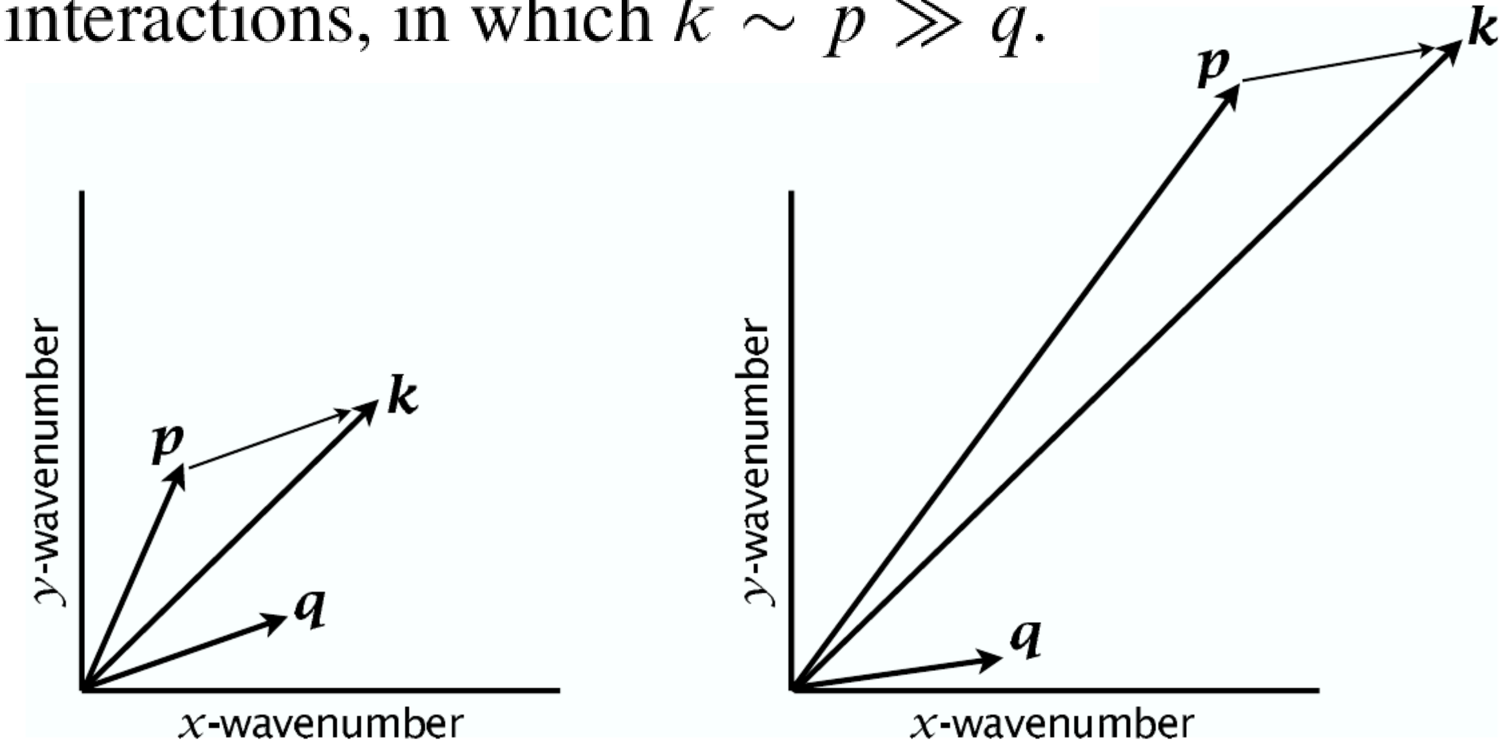

Fig. 8.1 Two interacting triads, each with $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$. On the left, a local triad with $k \sim p \sim q$. On the right, a nonlocal triad with $k \sim p \gg q$.

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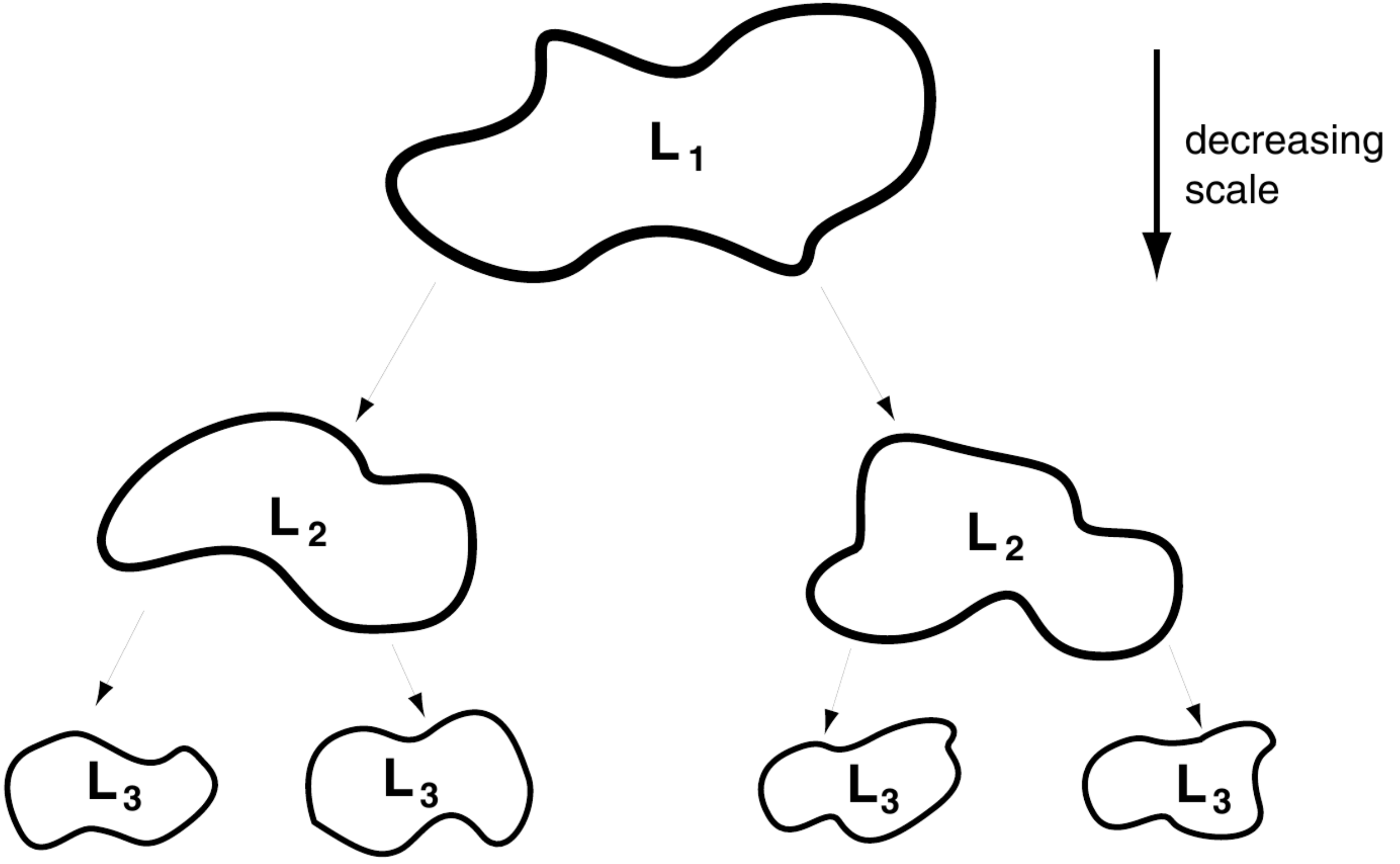
Energy Spectra

Energy cascade



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Energy Spectra

Energy (3d turbulence)

$$\widehat{E} = \int E \, \mathrm{d}V = \frac{1}{2}\int (u^2 + v^2 + w^2)\, dV$$
$$= \frac{1}{2}\sum (|\widetilde{u}|^2 + |\widetilde{v}|^2 + |\widetilde{w}|^2)\, \mathrm{d}\boldsymbol{k}$$

$$\widehat{E} \equiv \int \mathcal{E}(k)\, \mathrm{d}k$$

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Energy Spectra

Dimensions and the Kolmogorov Spectrum

Quantity	Dimension
Wavenumber, k	1/L
Energy per unit mass, E	$U^2 = L^2/T^2$
Energy spectrum, $\mathcal{E}(k)$	$EL = L^3/T^2$
Energy Flux, ε	$E/T = L^2/T^3$

If $\mathcal{E} = f(\varepsilon, k)$ then the only dimensionally consistent relation for the energy spectrum is

$$\mathcal{E} = \mathcal{K}\varepsilon^{2/3}k^{-5/3}$$

where $\mathcal{K}$ is a dimensionless constant.

Buckingham π theorem

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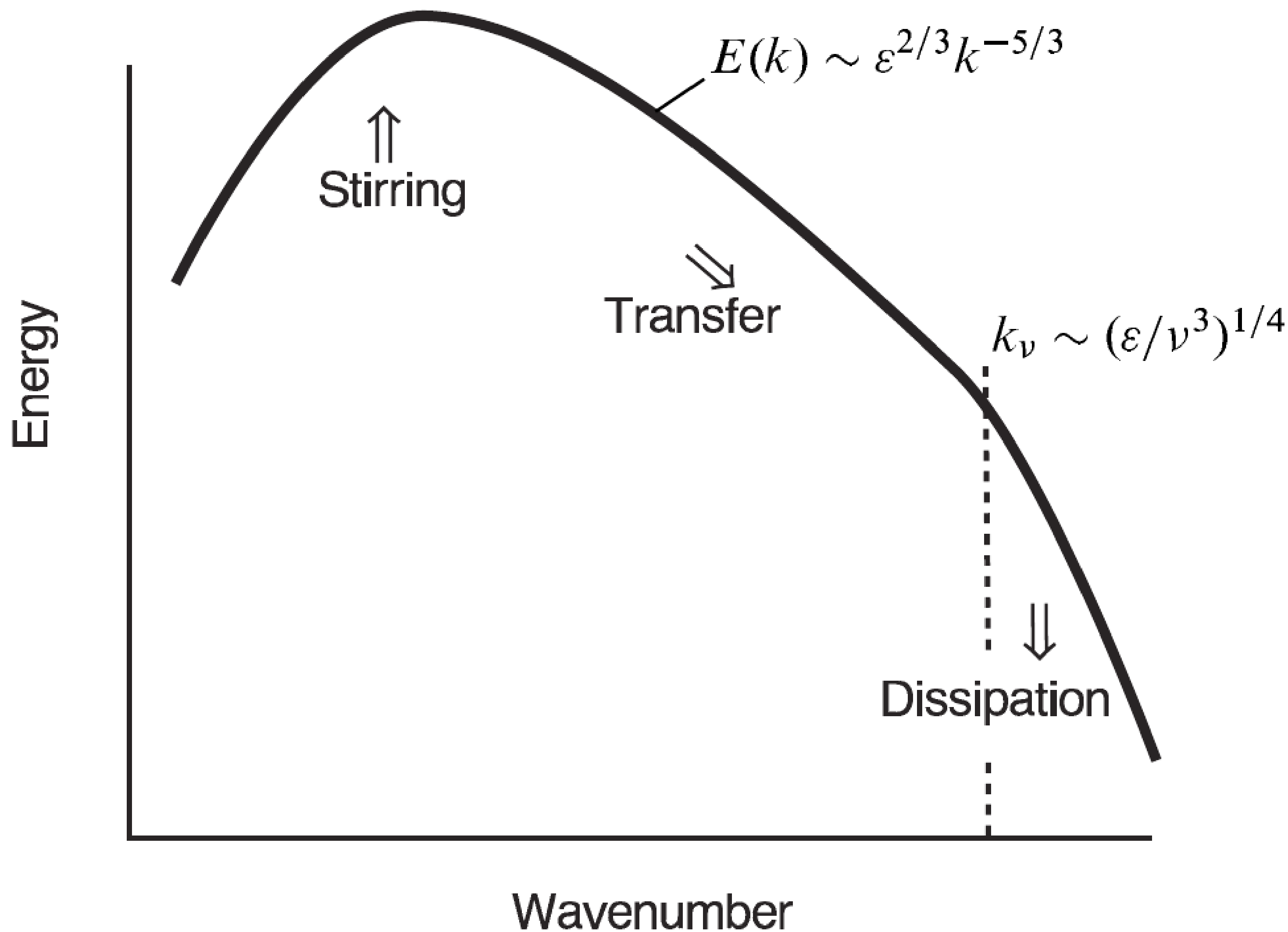
Energy Spectra



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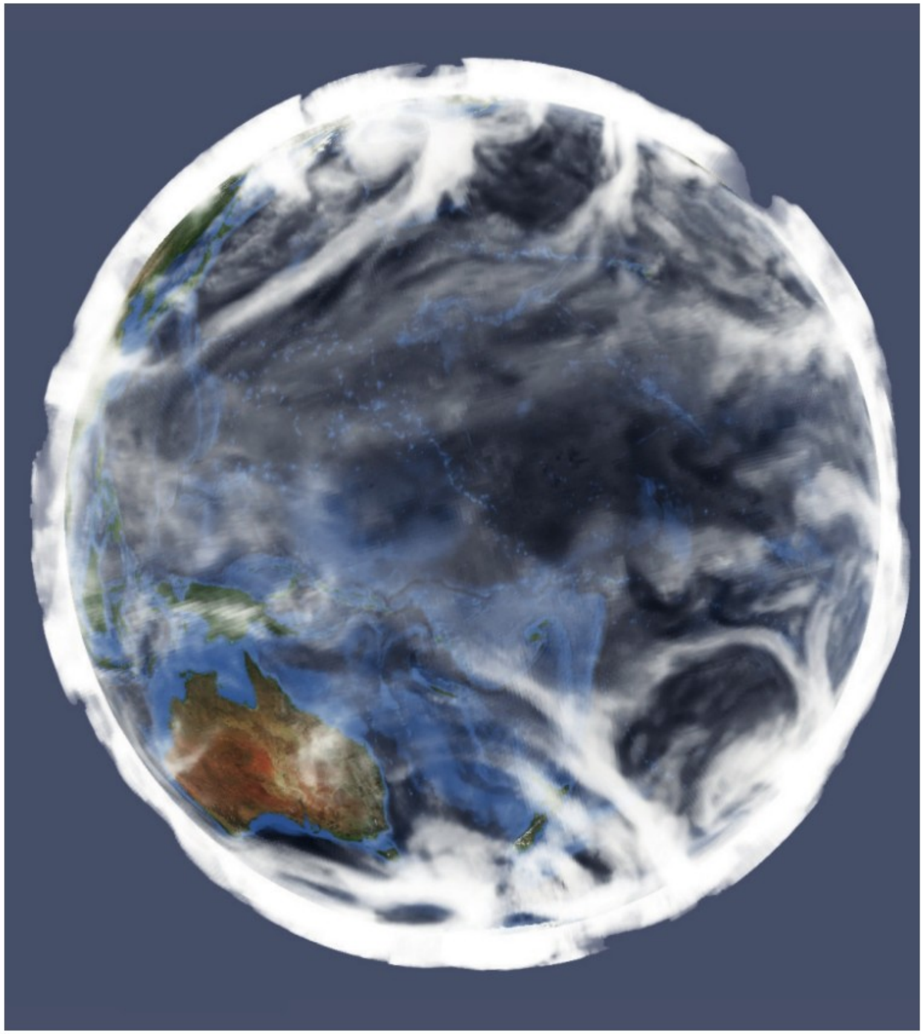
Energy Spectra

Two Dimensional Turbulence



Energy Spectra

Two Dimensional Turbulence

Conservation of Energy and Enstrophy
→ different from 3d turbulence

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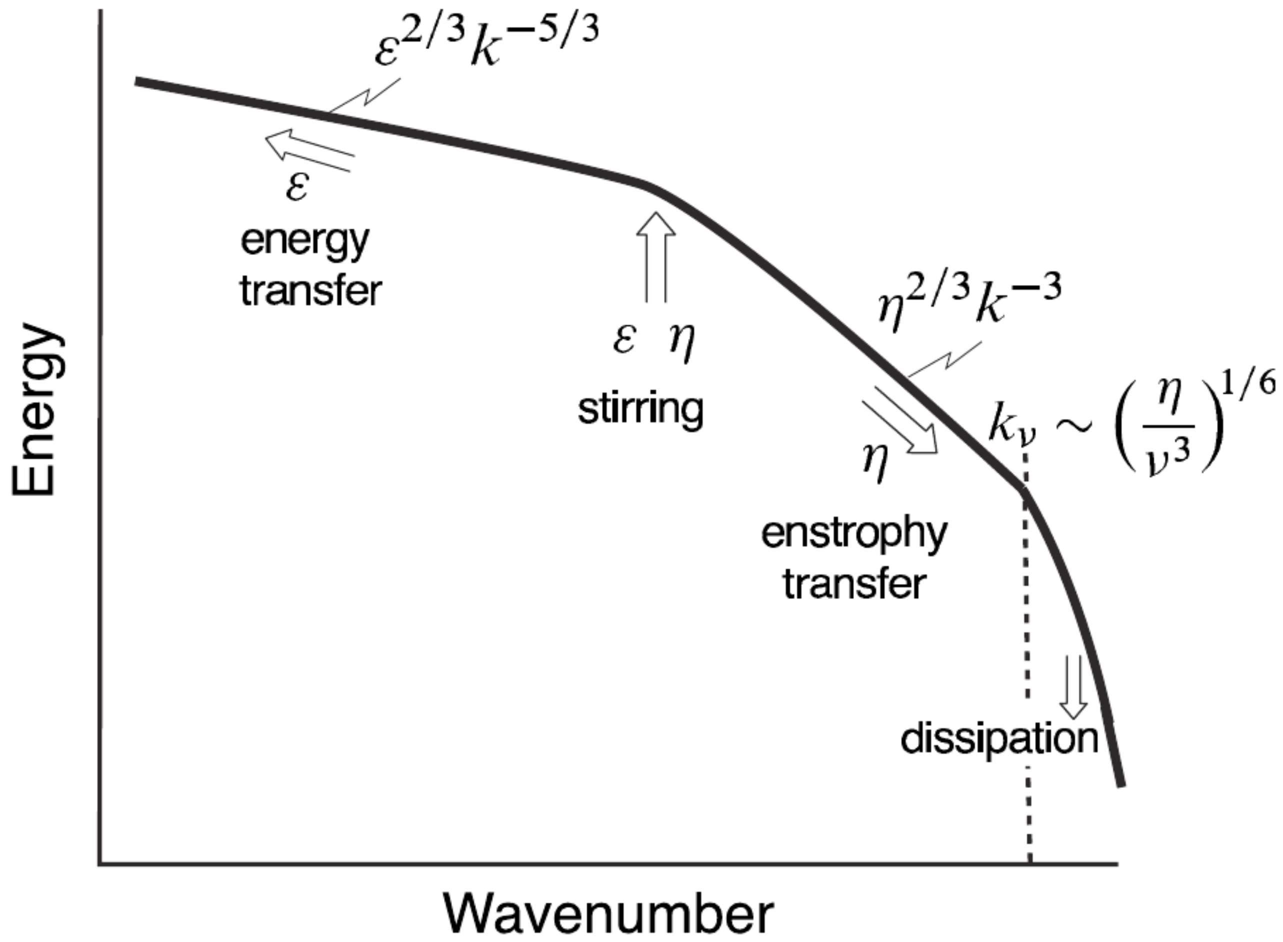
Energy Spectra

Two Dimensional Turbulence



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Energy Spectra

- Most traditional schemes are overly damping
- Neglecting upscale injection of energy
  → Backscatter
- Need for Stochastic Backscatter schemes

Random Number Generator

How do we generate random numbers?

Random Number Generator

Pseudo-Random Numbers:

Linear congruential pseudo random number Generator:
These have the recursive form

$$X_{n+1} = (aX_n + b)(mod\, c)$$

Seed X_0

Uniform distributed: $U_n = \frac{X_n}{c}$

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Random Number Generator

Linear congruential pseudo random number Generator:

$$X_{n+1} = (aX_n + b)(mod\ c)$$

Uniform distributed: $U_n = \frac{X_n}{c}$

c should be chosen as large as possible
Perhaps as power of 2

<u>That is not a recommended RNG!</u>

Random Number Generator

How to generate Gaussian distributed random numbers?

Random Number Generator

Box-Muller Method

$$N_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2)$$

$$N_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2)$$

U1, U2: Uniform random numbers
N1, N2: Gaussian random numbers

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Random Number Generator

How to check the randomness?

Random Number Generator

How to check the randomness?

Check distribution
Check autocorrelation

https://en.wikipedia.org/wiki/Diehard_tests

Random Number Generator

Mersenne Twister

- very long period of $2^{19937} – 1$.
- It passes numerous tests for statistical randomness, including the Diehard tests.

Used by R, Python, Matlab, …

GCC uses different PRNG

Random Number Generator

Exercise:

Write a Random Number Generator

$$X_{n+1} = (aX_n + b) \bmod c$$

$$U_n = \frac{X_n}{c}$$

1) a=11, b=0, c=2^2

2) a=65539, b=0, c=2^{31}