Introduction to Data Assimilation, Subgrid-Scale Parameterization and Predictability

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Outline for Today

Linear Stability and Lyapunov exponents

Lyapunov exponents

Linear versus non-linear dynamical systems

Linear System:
Evolution of basic solutions
→ time independent linear combinations
span solution space

Aleksandr Mikhailovich Lyapunov
(1857 – 1918)

Lyapunov exponents

Linear versus non-linear dynamical system

Non-Linear System:
No obvious basic solutions
→ there are no time independent linear combinations that span solution space

Aleksandr Mikhailovich Lyapunov
(1857 – 1918)

Lyapunov exponents

What is the property that makes these systems so different?

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Non-Linear Systems are very often chaotic

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Lyapunov exponents

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Non-Linear Systems are very often chaotic

What is a chaotic dynamical system?

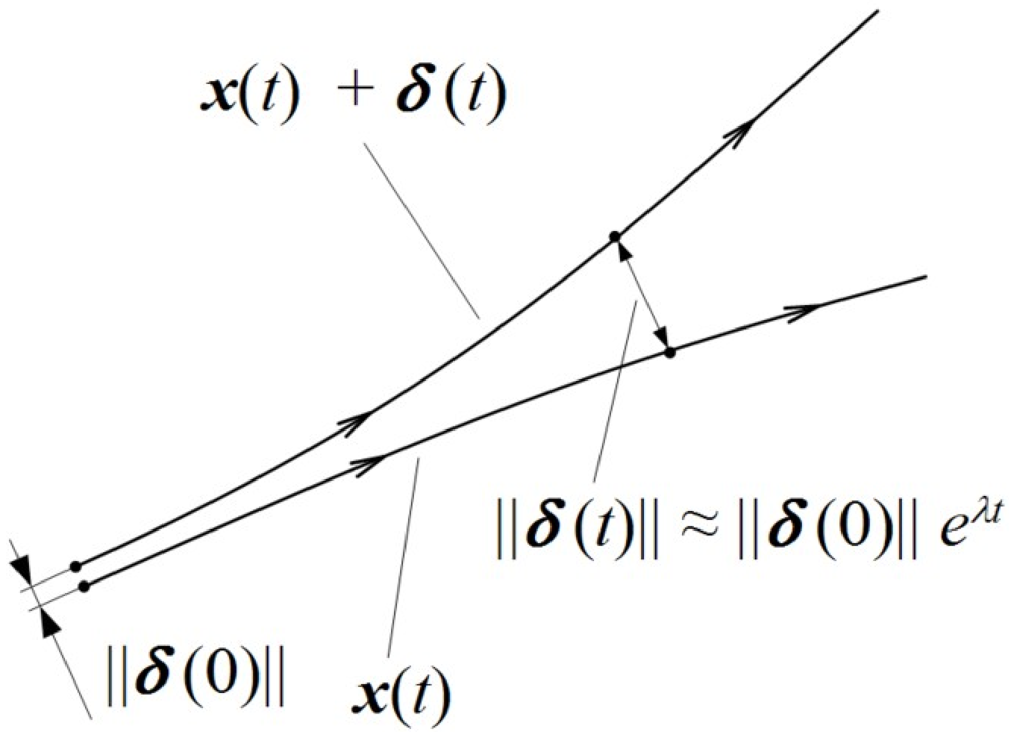
Exponential divergence for small perturbations

Lyapunov exponents

- x(t) evolves freely
- The perturbation grows exponentially
- If the perturbation is rescaled at regular intervals, we get the maximum Lyapunov exponent

- In a high dimensional system there are many directions for growth
- If we repeat this procedure iteratively in orthogonal space we get successively all Lyapunov exponents (Lyapunov spectrum)

Lyapunov exponents



What has this to do with meteorology?

- Classical Lifecycle of Cyclones (Bjerknes & Solberg 1922)
- Small perturbations create variability
- What kind of perturbations are amplifying and which are decaying?

What has this to do with meteorology?

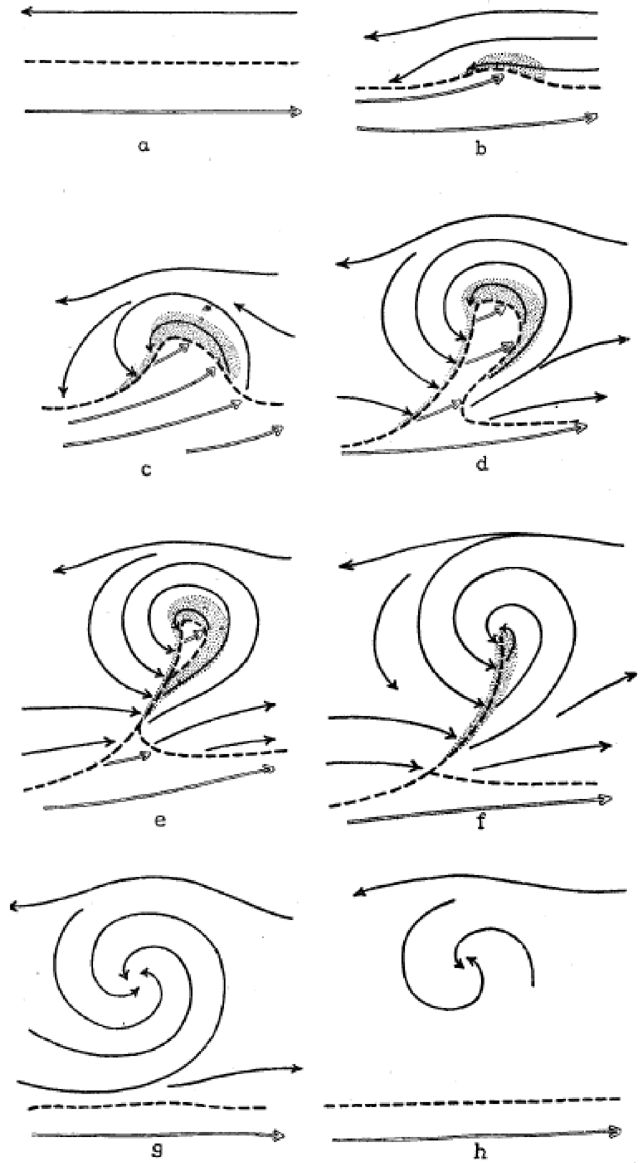


Fig. 2. The »Life cycle« of cyclones.

What has this to do with meteorology?

Formal Decomposition

- Decompose atmospheric field x into eddy field and zonal mean

Classification of solutions

- Stationary balance
- Turbulent balance
- Climate Mean

“Classical” Linear Stability Analysis

Success in classical linear theory with normal modes

(e.g. Charney 1947, Eady 1949, Pedlosky 1964)

Fast growing **linear instabilities** (normal modes) of stationary/mean states qualitatively explain

- **energy conversions** between x^E and $[x]$ (**baroclinic** and **barotropic** conversions)
- **heat** and **momentum transports** of x^E ➔ **Lorenz Energy Cycle**
- most important **spatial scales** and **life times** of mid latitudes cyclones
- Compare **normal modes** vs **empirical orthogonal functions**

Classical Stabiliy analysis vs Lyapunov exponents

Limitations of Classical Approach

- Atmosphere is chaotic and turbulent
  - Cyclones/eddies develop not on stationary state
- Air Parcels are not simply perturbed by themselves (they have collective oscillations)
- Variability is at least initially created by linear instabilities of the turbulent flow

Classical Stabiliy analysis vs Lyapunov exponents

TODAY

- **Lyapunov exponents** (LEs): the average growth rates of linear perturbations

New & Extended Approach

- Covariant Lyapunov Vectors (CLVs): **actual perturbations** which grow with the LEs

Lyapunov exponents – Definiton

Full Solution of dynamical system

$$\frac{\partial}{\partial \mathrm{t}} u_A = f(u_A)$$

Perturbed Solution

$$\frac{\partial}{\partial \mathrm{t}} u_B = f(u_B),\, u_B = u_A + v$$

Lyapunov exponents – Osedelecs Theorem

Essential Content of the Theorem

- n solutions to n dimensional tangent linear equation (n vectors dependent on time)
- Each solution grows exponentially ~ $\exp\left(\int_{t_0}^{t} \lambda(\tau)d\tau\right)$
- The mean of $\lambda(t)$ are the Lyapunov exponents

We for now are only interested in rate of separation, hence the Lyapunov exponents

Lyapunov exponents – Computational Algorithm

Continuous Equation

$$\frac{\partial}{\partial \mathrm{t}}\,\mathbf{v}(t) = \frac{\partial F}{\partial u}\,(u_A)\,\mathbf{v}(t)$$

Discretized Version

$$\mathbf{v}_{n+1} = \mathcal{F}_n v_n = \left(\left(\mathbb{1} + J(x_n)\right) v_n\right)$$

Bennetin et al. 1996 algorithm

$$\mathcal{F}_n \cdot Q_n = V_{n+1} \underbrace{= Q_{n+1} R_{n+1}}_{\text{QR decomposition}}$$

The diagonal of the R matrix contains $\exp(\lambda_n * \Delta t)$

Tangent Linear Model

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$$

$$\mathbf{x}^{n+1} = \mathbf{x}^n + \Delta t \mathbf{F}\left(\frac{\mathbf{x}^n + \mathbf{x}^{n+1}}{2}\right)$$

Nonlinear model that only depends on initial conditions

$$\mathbf{x}(t) = M[\mathbf{x}(t_0)]$$

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Tangent Linear Model

Add small perturbation

$$M[\mathbf{x}(t_0) + \mathbf{y}(t_0)] = M[\mathbf{x}(t_0)] + \frac{\partial M}{\partial \mathbf{x}}\mathbf{y}(t_0) + O[\mathbf{y}(t_0)^2]$$
$$= \mathbf{x}(t) + \mathbf{y}(t) + O[\mathbf{y}(t_0)^2]$$

Evolution model of small perturbation

$$\frac{d\mathbf{y}}{dt} = \mathbf{J}\mathbf{y}$$

where $\mathbf{J} = \partial \mathbf{F}/\partial \mathbf{x}$ is the Jacobian of $\mathbf{F}$.

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Tangent Linear Model

$$\frac{d\mathbf{y}}{dt} = \mathbf{J}\mathbf{y}$$

This system of linear ordinary differential equations is the tangent linear model in differential form. Its solution between t_0 and t can be obtained by integrating (6.3.5) in time using the same time difference scheme used in the nonlinear model (6.3.3):

$$\mathbf{y}(t) = \mathbf{L}(t_0, t)\mathbf{y}(t_0) \tag{6.3.6}$$

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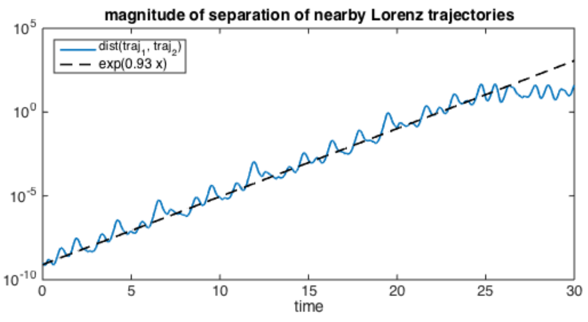
Exercise

Compute largest Lyapunov exponent:

1) Start with initial condition on attractor
2) Select nearby point (separated by d_0 ~10^{-9})
3) Advance both orbits by one time step and calculate new separation d_1.
4) Plot separations over a few time steps
5) Compute linear regression slope
6) Average also over many different initial conditions.



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