Exercise 4. Does there exist a power series $\sum_{j=0}^{\infty} a_j z^j$ that converges at $z = 2 + 3i$ and diverges at $z = 3 - i$?

No. From Theorem 10 we know that if a power series converges at a point then it must converge everyhwere inside the circle where the point lies. Since $3 - i$ is inside the circle $|z| \leq |2 + 3i|$, then if a power series converges at $2 + 3i$ it must converge at $3 - i$.

Exercise 13a. The following initial-value problem has a unique solution that is analytic at the origin. Find the power series expansion $\sum_{j=0}^{\infty} a_j z^j$ of the solution by determining a recurrence relation for the coefficients a_j.

$$\begin{cases} \dfrac{d^2 f}{dz^2} - z\dfrac{df}{dz} - f = 0 \\ f(0) = 1, \quad f'(0) = 0 \end{cases}$$

First we have

$$\begin{aligned} f''(0) &= zf'(0) + f(0) = 1 \\ f'''(0) &= zf''(0) + 2f'(0) = 0 \\ f^{(4)}(0) &= zf^{(3)}(0) + 3f^{(2)}(0) = 3 \\ f^{(5)}(0) &= zf^{(4)}(0) + 4f^{(3)}(0) = 0 \\ f^{(6)}(0) &= zf^{(5)}(0) + 5f^{(4)}(0) = 15 \\ &\vdots \end{aligned}$$

and in general

$$f^{(j)}(0) = (j-1)f^{(j-2)}(0) = \begin{cases} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (j-1) & \text{when } j \text{ is even} \\ 0 & \text{when } j \text{ is odd} \end{cases}.$$

Note that:

$$\prod_{k=0}^{j} (2k-1) = \frac{(2j)!}{2^j j!}.$$

Thus, using the Maclaurin series we have

$$f(z) = \sum_{j=0}^{\infty} \frac{(2j)!}{2^j j!} \cdot \frac{z^{2j}}{(2j)!} = \sum_{j=0}^{\infty} \frac{(z^2/2)^j}{j!} = e^{z^2/2}$$