Exercise 3.1.8 (3pt). Use generating functions to show that every positive integer can be written in exactly one way as a sum of distinct powers of 2.

Proof. First we can write any number as the following sum

$$x_1 + x_2 + x_3 + \ldots = n$$

where x_i is a distinct power of 2. A generating function for the above equation is as follows,

$$(1+x)(1+x^2)(1+x^{2^2})\cdots(1+x^{2^k})\cdots$$

since we can either have a certain power of 2 in our sum or not. Consider

$$\begin{aligned}
(1-x)(1+x)(1+x^2)(1+x^{2^2})\cdots &= (1-x^2)(1+x^2)(1+x^{2^2})\cdots \\
&= (1-x^{2^2})(1+x^{2^2})(1+x^{2^3})\cdots \\
&\vdots \\
&= (1-x^{2^m})(1+x^{2^m})(x+2^{m+1})\cdots \\
&\vdots \\
&= 1
\end{aligned}$$

This converges to 1 since every coefficent for $x^{2^{m-1}}$ becomes zero and we are left with x^0, in which case the coefficent is 1. We can then choose m to be as large as we want so all coefficents of above x^0 go to zero. Then by dividing by $x-1$ we get

$$(1+x)(1+x^2)(1+x^{2^2})\cdots = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

so there is exavtly one way to write a sum of distinct powers of 2.

■

Exercise 3.4.6 (3pt). Find the generating function for the solutions to $h_n = 9h_{n-1} - 26h_{n-2} + 24h_{n-3}, h_0 = 0, h_1 = 1, h_2 = -1$ and use it to find a formula for h_n.

Let

$$A(x) = h_0 + h_1x + h_2x^2 + \ldots.$$

Then

$$\begin{aligned}
A(x) &= h_0 + h_1x + h_2x^2 + (9h_2 - 26h_1 + 24h_0)x^3 + (9h_3 - 26h_2 + 24h_1)x^4 + \ldots \\
&= h_0 + h_1x + h_2x^2 + (9h_2x^3 + 9h_3x^4 + \ldots) - (26h_1x^3 + 26h_2x^4 + \ldots) + (24h_0x^3 + 24h_1x^4 + \ldots) \\
&= h_0 + h_1x + h_2x^2 + 9x(A(x) - h_0 - h_1x) - 26x^2(A(x) - h_0) + 24x^3A(x) \\
&= A(x)(24x^3 - 26x^2 + 9x) + h_0(26x^2 - 9x + 1) - h_1(9x^2 - x) + h_2x^2
\end{aligned}$$

$$
\begin{aligned}
A(x)(-24x^3 + 26x^2 - 9x + 1) &= x - 10x^2 \\
A(x) &= \frac{10x^2 - x}{24x^3 - 26x^2 + 9x - 1}
\end{aligned}
$$

Using a computer to find the zeros of the demoninator,

$$
\begin{aligned}
A(x) &= \frac{10x^2 - x}{(4x-1)(3x-1)(2x-1)} \\
&= \frac{A}{4x-1} + \frac{B}{3x-1} + \frac{C}{2x-1}
\end{aligned}
$$

Using the cover up method, we get

$$
A = \frac{10(1/4)^2 - (1/4)}{(3(1/4)-1)(2(1/4)-1)} = 3 \qquad B = \frac{10(1/3)^2 - (1/3)}{(4(1/3)-1)(2(1/3)-1)} = -7 \qquad C = \frac{10(1/2)^2 - (1/2)}{(4(1/3)-1)(3(1/2)-1)} = 4
$$

so

$$
\begin{aligned}
A(x) &= \frac{3}{4x-1} - \frac{7}{3x-1} + \frac{4}{2x-1} = -\frac{3}{1-4x} + \frac{7}{1-3x} - \frac{4}{1-2x} \\
&= -3(1 + 4x + (4x)^2 + \ldots) + 7(1 + 3x + (3x)^2 + \ldots) - 4(1 + 2x + (2x)^2 + \ldots)
\end{aligned}
$$

Using this, we can determine

$$
h_n = -3 \cdot 4^n + 7 \cdot 3^n - 4 \cdot 2^n
$$

Exercise 3.4.7 (2pt). Find the generating function for the solution to $h_n = 3h_{n-1} + 4h_{n-2}$, $h_0 = 0$, $h_1 = 1$, and use it to find a formula for h_n.

let

$$
A(x) = h_0 + h_1 x + h_2 x^2 + \ldots
$$

Then

$$
\begin{aligned}
A(x) &= h_0 + h_1 x + (3h_1 + 4h_0)x^2 + (3h_2 + 4h_1)x^3 + \ldots \\
&= h_0 + h_1 x + (3h_1 x^2 + 3h_2 x^3 + \ldots) + (4h_0 x^2 + 4h_1 x^3 + \ldots) \\
&= h_0 + h_1 x + 3x(A(x) - h_0) + 4x^2 A(x) \\
&= A(x)(4x^2 + 3x) + h_0(-3x + 1) + h_1 x \\
A(x)(-4x^2 - 3x + 1) &= x \\
A(x) &= \frac{x}{-4x^2 - 3x + 1} = \frac{x}{(-4x+1)(x+1)} = \frac{A}{1-4x} + \frac{B}{x+1}
\end{aligned}
$$

Using the cover up method, we get

$$
A = \frac{(1/4)}{1 + (1/4)} = \frac{1}{5} \qquad B = \frac{(-1)}{-4(-1) + 1} = \frac{-1}{5}
$$

so

$$\begin{aligned} A(x) &= \frac{1}{5(-4x+1)} - \frac{1}{5(x+1)} = \frac{1}{5(1-4x)} - \frac{1}{5(1-(-x))} \\ &= \frac{1}{5}(1+4x+(4x)^2+\dots) - \frac{1}{5}(1-x+x^2+(-x)^3+\dots) \end{aligned}$$

Using this we can determine

$$h_n = \frac{4^n - (-1)^n}{5}$$

Exercise 3.4.5 (2pt). Find the generating function for the solution to $h_{n-1} + h_{n-2}$, $h_0 = 1$, $h_1 = 3$ and use it to find a formula for h_n.

Let

$$A(x) = h_0 + h_1 x + h_2 x^2 + \dots$$

Then

$$\begin{aligned} A(x) &= h_0 + h_1 x + (h_1 + h_0)x^2 + (h_2 + h_1)x^3 + \dots \\ &= h_0 + h_1 x + (h_1 x^2 + h_2 x^3 + \dots) + (h_0 x^2 + h_1 x^3 + \dots) \\ &= h_0 + h_1 x + x(A(x) - h_0) + x^2 A(x) \\ &= A(x)(x^2 + x) + h_0(1-x) + h_1 x \\ A(x)(-x^2 - x + 1) &= 1 + 2x \\ A(x) &= \frac{1+2x}{-x^2-x+1} = \frac{1+2x}{\left(-x + \dfrac{-1-\sqrt{5}}{2}\right)\left(x - \dfrac{-1+\sqrt{5}}{2}\right)} \end{aligned}$$

Let $a = \dfrac{-1-\sqrt{5}}{2}$ and $b = \dfrac{-1+\sqrt{5}}{2}$ Then

$$A(x) = \frac{1+2x}{(-x+a)(x-b)} = \frac{A}{a-x} + \frac{B}{x-b}$$

Using the cover up method we get

$$A = \frac{1+2a}{a-b} \qquad\qquad B = \frac{1+2b}{a-b}$$

so

$$A(x) = \frac{A}{a} \cdot \frac{1}{1-x/a} + \frac{B}{b} \cdot \frac{1}{1-x/b}$$

Using this we can determine

$$h_n = \frac{A}{a}\left(\frac{1}{a}\right)^n + \frac{B}{b}\left(\frac{1}{b}\right)^n$$

Exercise 3.4.3 (2pt). Find the generating function for the solutions to $h_n = 2h_{n-1} + 3^n$, $h_0 = 0$

Let

$$A(x) = h_0 + h_1x + h_2x^2 + \dots$$

Then

$$\begin{aligned}
A(x) &= h_0 + (2h_0 + 3^n)x + (2h_1 + 3^n)x^2 + \dots \\
&= h_0 + (2h_0x + 2h_1x^2 + \dots) + 3^n(x + x^2 + \dots) \\
&= h_0 + 2xA(x) + \frac{3^n x}{1-x} \\
A(x)(1-2x) &= h_0 + \frac{3^n x}{1-x} \\
A(x) &= \frac{h_0}{1-2x} + \frac{3^n x}{(1-2x)(1-x)} \\
&= \frac{3^n x}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}
\end{aligned}$$

Using the cover up method

$$A(x) = \frac{(1/2)3^n}{1-(1/2)} = 3^n \qquad\qquad B(x) = \frac{3^n}{1-2} = -3^n$$

so

$$A(x) = \frac{3^n}{1-2x} - \frac{3^n}{1-x}$$

Using this, we get

$$h_n = 3^n 2^n - 3^n = 6^n - 3^n$$

Exercise 3.4.8 (2pt). Find a recursion for the number of ways to place flags on an n foot pole, where we have red flags that are 2 feet high, blue flags that are 1 foot high, and yellow flags that are 1 foot high; the heights of the flags must add up to n. Solve the recursion.

If we have a $1ft$ pole, then there are 2 ways to do this, so $h_1 = 2$. For a $2ft$ Pole, there are 5 ways to do this so $h_2 = 5$. From there we can either add a blue or yellow flag to a pole of lenght $n-1$ or we can add a red flag from a pole of lenght $n-2$, so

$$h_n = 2h_{n-1} + h_{n-2}$$

Let

$$A(x) = h_1 + h_2x + h_3x^2 + \dots$$

Then

$$\begin{aligned}
A(x) &= h_1 + h_2x + (2h_2 + h_1)x^2 + (2h_3 + h_2)x^3 + \dots \\
&= h_1 + h_2x + (2h_2x^2 + 2h_3x^2 + \dots) + (h_1x^2 + h_2x^3 + \dots) \\
&= h_1 + h_2x + 2x(A(x) - h_1) + x^2A(x) \\
&= A(x)(x^2 + 2x)h_1(-2x + 1) + h_2x \\
A(x)(-x^2 - 2x + 1) &= -4x + 2 + 5x \\
A(x) &= \frac{x+2}{(-x^2 - 2x + 1)} = \frac{-x-2}{x^2 + 2x - 1} = \frac{-x-2}{(x + 1 + \sqrt{2})(x + 1 - \sqrt{2})}
\end{aligned}$$

Let $a = -1 - \sqrt{2}$ and $b = -1 + \sqrt{2}$. Then

$$A(x) = \frac{-x-2}{(x-a)(x-b)} = \frac{x+2}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{a-x}$$

Using the cover up method

$$A = \frac{a+2}{b-a} \qquad B = \frac{b+2}{a-b}$$

so

$$A(x) = \frac{A}{a} \cdot \frac{1}{1-x/a} + \frac{B}{b} \cdot \frac{1}{1-x/b}$$

Using this we can determine

$$h_n = \frac{A}{a}\left(\frac{1}{a}\right)^n + \frac{B}{b}\left(\frac{1}{b}\right)^n$$

Exercise 3.4.2 (1pt). Find the generating function for the solutions to $h_n = 3h_{n-1} + 4h_{n-2}$, $h_0 = h_1 = 1$. and use it to find a formula for h_n.

Let

$$A(x) = h_0 + h_1 x + h_2 x^2 + \ldots$$

Then

$$\begin{aligned}
A(x) &= h_0 + h_1 x + (3h_1 + 4h_0)x^2 + (3h_2 + 4h_1)x^3 + \ldots \\
&= h_0 + h_1 x + (3h_1 x^2 + 3h_2 x^3 + \ldots) + (4h_0 x^2 + 4h_1 x^3 + \ldots) \\
&= h_0 + h_1 x + 3x(A(x) - h_0) + 4x^2 A(x) \\
&= A(x)(4x^2 + 3x)h_0(-3x+1) + h_1 x \\
A(x)(-4x^2 - 3x + 1) &= -3x + 1 + x \\
A(x) &= \frac{-2x+1}{-4x^2 - 3x + 1} = \frac{2x-1}{(4x-1)(x+1)} = \frac{A}{4x-1} + \frac{B}{x+1}
\end{aligned}$$

Using the cover up method

$$A = \frac{2(1/4)-1}{(1/4)+1} = \frac{-2}{5} \qquad B = \frac{-2-1}{-4-1} = \frac{3}{5}$$

so

$$\begin{aligned}
A(x) &= \frac{3}{5(x+1)} - \frac{2}{5(4x-1)} = \frac{3}{5(1-(-x))} + \frac{2}{5(1-4x)} \\
&= \frac{3}{5}(1 - x + x^2 + (-x)^3 + \ldots) + \frac{2}{5}(1 + 4x + (4x)^2 + \ldots)
\end{aligned}$$

Using this

$$h_n = \frac{3}{5}(-1)^n + \frac{2}{5}4^n$$