Exercise 5.1.13 (3pt). Prove a sequence $d_1 \geq d_2 \geq \ldots \geq d_n$ is graphical if $\sum_{i=1}^{n} d_i$ is even and for all $k \in \{1, 2, \ldots, n\}$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k).$$

Proof. Induction on $s = \sum_{i=1}^{n} d_i$. This is easy to see when $s = 2$, so suppose $s > 2$. WLOG, let $d_n > 0$. Let t be the least integer such that $d_t > d_{t+1}$, or $t = n-1$ if there is no such integer. Let $d'_t = d_t - 1$, $d'_n = d_n - 1$, and $d'_i = d_i$ for all other i. Note that $d'_1 \geq d'_2 \geq \ldots \geq d'_n$. We want to show that $\{d'_i\}$ satisfies the condition of the theorem that for all $k \in \{0, \ldots, n\}$

$$\sum_{i=1}^{k} d'_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d'_i, k)$$

of which there are five cases:

1. $k \geq t$. Note that $\min(a, b) - 1 \leq \min(a-1, b)$. When $k < n$.

$$\begin{aligned}
\sum_{i=1}^{k} d'_i &= \sum_{i=1}^{k} d_i - 1 \\
&\leq k(k+1) + \sum_{i=k+1}^{n} \min(d_i, k) - 1 \\
&= k(k+1) + \sum_{i=k+1}^{n-1} \min(d'_i, k) + \min(d_n, k) - 1 \\
&\leq k(k+1) + \sum_{i=k+1}^{n-1} \min(d'_i, k) + \min(d_n - 1, k) \\
&= k(k+1) + \sum_{i=k}^{n} \min(d'_i, k)
\end{aligned}$$

2. $k < t, d_k < k$. Note that $d_k \leq k-1$

$$\sum_{i=1}^{k} d'_i = kd_k \leq k(k-1) \leq k(k-1) + \sum_{i=k+1}^{n} \min(d'_i, k)$$

3. $k < t, d_k = k$. Consider $d_{k+2} + d_{k+3} + \ldots + d_n$. If $k < n-2$, then we have

$$\ldots + d_{n-1} + d_n \geq 2$$

since $d_{n-2} \geq d_n \geq 1$. If $k \geq n-2$, then $t = n-1$, and our minimum case for our sequence ($d_i = d_k = k = n-2$ for $1 \leq i < n$) is

$$\{d_i\} = (n-2), (n-2), \ldots, d_n.$$

But then, $s = (n-1)(n-2) + d_n$ must be even, so d_n must be even and therefor $d_n \geq 2$. Also, since $k < t$ we have $d_i = d_k = k$ for $1 \leq i \leq k+1$, so

$$\begin{aligned}
\sum_{i=1}^{k} d'_i = kd'_k = k^2 - k + k = k^2 - k + d_{k+1} \\
&\leq k(k-1) + d_{k+1} + d_{k+2} + \dots d_n - 2 \\
&\leq k(k-1) + \sum_{i=k+1, i \neq t}^{n-1} \min(d_i, k) + d_t - 1 + d_n - 1 \\
&= k(k-1) + \sum_{i=k+1, i \neq t}^{n-1} \min(d'_i, k) + d'_t + d'_n \\
&\leq k(k-1) + \sum_{i=k+1}^{n} \min(d'_i, k)
\end{aligned}$$

4. $k < t, d_n > k$. Then $\min(d_i, k) = \min(d_i - 1, k) = k$ for all $1 \leq i \leq n$, so

$$\begin{aligned}
\sum_{i=1}^{n} d'_i = \sum_{i=1}^{n} d_i \\
&\leq k(k+1) + \sum_{j=k+1}^{n} \min(d_i, k) \\
&\leq k(k+1) + \sum_{j=k+1}^{n} \min(d'_i, k)
\end{aligned}$$

5. $k < t, d_k > k, d_n \leq k$

$$\dots$$

So $\{d'_i\}$ satisfies the if part of the theorem. By induction, we can assume $\{d'_i\}$ is graphical. Let G be the graph formed by $\{d'_i\}$ with vertices $v_1, v_2, \dots, v_n$. If there is no edge between v_t and v_n, then we can add this edge to G and this appended graph has degree sequence $\{d_i\}$. Otherwise, we are still 2 edges short, so there exists some v_i where v_t and v_m have no edge. Also, since $d_i \geq d_n$, there is some v_j where v_i and v_j share an edge, and v_j and v_n don't. We can remove the edges from v_t to v_n and v_i to v_j, and add the edges v_i to v_t and v_j to v_n to get another graph G' formed by $\{d'_i\}$ without edges from v_t to v_n. G' is still of degree sequence $\{d'_i\}$, so adding the edge from v'_t to v'_n gives us a graph with degree sequence $\{d_i\}$

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Note: I ended up using the proof by S.A. Choudum as an outline which was referenced in the textbook. Since it was referenced I figured this was okay, but I still tried to state each step as I understood it and fill in some gaps the Choudum found trivial. There are some things about the proof I still have questions on, mainly on the 5th case but for the most part sifting through this elegant proof was very interesting.

Exercise 5.1.2 (2pt). Prove that if $\sum_{i=1}^{n} d_i$ is even, there is a graph with degree sequence $d_1, d_2, \ldots, d_n$.

Proof. Induction on n. When $n = 1$, d_1 must be even, so we can form a graph with one vertex and with $\deg(d_1)/2$ loops. Let $\sum_{i=1}^{n} d_i$ be even, and G be a graph with degree sequence $\{d_i\}$. Now let $\sum_{i=1}^{n+1} d_i$ be even.

Suppose d_{n+1} is even, then $\sum_{i=1}^{n} d_i$ must be even and by our inductive hypothesis form a graph G. Let G' be a new graph. Since d_{n+1} is even, we can add vertex to G' with $\deg(d_{n+1})/2$ loops, and then add the graph all vertices and edges from G to form a graph with the sequence $d_1, d_2, \ldots, d_{n+1}$.

Now suppose d_{n+1} is odd, then $\sum_{i=1}^{n} d_i$ must be odd. From the sequence $d_1, d_2, \ldots, d_n$, choose some d_i for $1 \leq i \leq n$ and let $d_i' = d_i - 1$. Then the sequence $d_1, d_2, \ldots, d_i', \ldots, d_n$ is even and forms a graph G. Let G' be a new graph. add a vertex with $\deg(d_{n+1} - 1)/2$ loops add all vertices and edges from G to G'. Finally add one edge from the vertex with degree d_i' to d_{n+1} to form a graph of sequence $d_1, d_2, \ldots, d_{n+1}$.

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Exercise 5.1.3 (2pt). Suppose $d_1 \geq d_2 \geq \ldots \geq d_n$ and $\sum_{i=1}^{n} d_i$ is even. Prove that there is a multi-graph (no loops) with degree sequence $d_1, d_2, \ldots, d_n$ if and only if $d_1 \leq \sum_{i=2}^{n} d_i$.

Proof. Induction on $s = \sum_{i=1}^{n} d_i$. When $s = 0$ this is trivial, when $s = 2$ we have $d_1 = 1 = d_k$ for some $2 \leq k \leq n$ and $d_i = 0$ for $i \neq k$ and $2 \leq i \leq n$ (n vertices with one edge between two points). Now let $d_1 \geq d_2 \geq \ldots \geq d_n$, $\sum_{i=1}^{n} d_i = s$ and the degree sequence $d_1, d_2, \ldots, d_n$ form a loop-less graph G. Now let $d_1' \geq d_2' \geq \ldots \geq d_n'$ and $\sum_{i=1}^{n} d_i' = s + 2$.

Suppose $d_1' = \sum_{i=2}^{n}$. Then we can draw a graph with vertices $v_1, v_2, \ldots, v_n$. From v_1 to v_2, draw d_2' edges, from v_1 to v_3, draw d_3' edges, and etc. This gives us a loop-less graph with the desired degree sequence.

Suppose $d_1' < \sum_{i=2}^{n}$. Because d_1' is the greatest value in the sequence, there must be two non-zero d_j' and d_k' such that $1 \leq j < k \leq n$. Then, $\sum_{i=1, i \neq i,j}^{n} d_i' + d_k - 1 + d_j - 1 = s$, so then there is a loop-less graph G with degree sequence $d_1', d_2', \ldots, d_j' - 1, \ldots, d_k' - 1, \ldots, d_n'$. Let G' be a new graph with the same vertices and edges as G, but with a extra edge between the vertices with degree $d_j' - 1$ and degree $d_k' - 1$. This gives us a loop-less graph with the desired sequence.

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Exercise 5.2.3 (2pt). Prove that if vertices v and w are joined by a walk they are joined by a path.

Proof. Construction. Let $W = \{v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}\}$ form a walk. We can give an algorithm with a finite amount of steps that given W will return a path P. Since a walk is also a graph, and v_1 is connected to v_{k+1} by definition, we can always take the shortest path in our graph given by the walk from v_1 to v_k. Attached to this assignment is a python script which given a walk/graph, it generates all possible walks of a certain length and finds a path (using depth first search). Here is the demo graph I included in the script

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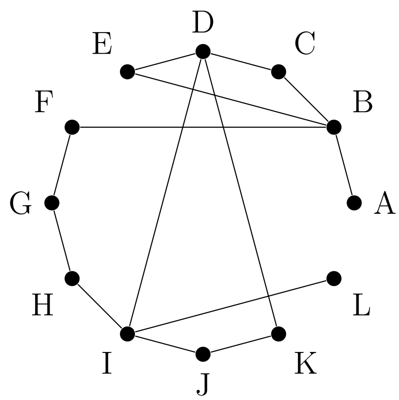
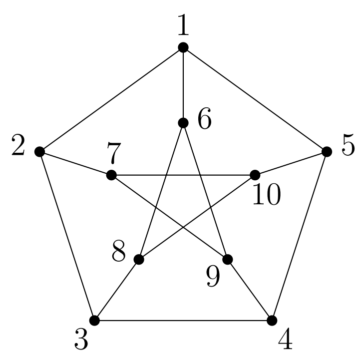


Exercise 5.3.3 (2pt). The graph shown below is the Peterson graph. Does it have a Hamilton cycle? Does it have a Hamilton path?



The Peterson graph has 10 vertices and 15 edges, and has no cycles less than 5. A Hamiltonian cycle needs at least as many edges as vertices, which we have, but then we have an extra 5 left over. With only 5 more edges, we can only construct cycles less than 5, so that Peterson graph has no Hamiltonian cycles. However, there is a Hamilton path, namely

$$(1) \to (2) \to (3) \to (8) \to (6) \to (9) \to (4) \to (5) \to (10) \to (7)$$

Exercise 5.1.4 (1pt). Prove that 0,1,2,3,4 is not graphical.

Proof. If there are 5 vertices and one must have degree 4, then no vertex can have degree 0 as the vertex of degree 4 would be connected to the rest of the vertices.

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