Problem 1. Verify that

$$-\frac{1}{2}\ln(\tan(x/2)) = \sum_{n=0}^{\infty}\frac{\cos((2n+1)x)}{2n+1}.$$

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\cos((2n+1)x)}{2n+1} &= \text{Re}\left(\sum_{n=0}^{\infty}\frac{(e^{ix})^{2n+1}}{2n+1}\right) = \text{Re}\left(\frac{1}{2}\ln\left[\frac{1+e^{ix}}{1-e^{ix}}\right]\right)\\
&= -\frac{1}{2}\text{Re}\left(\ln\left[\frac{1-e^{ix}}{1+e^{ix}}\right]\right) = -\frac{1}{2}\text{Re}\left(\ln\left[\frac{e^{ix/2}(e^{-ix/2}-e^{ix/2})}{e^{ix/2}(e^{-ix/2}+e^{ix/2})}\right]\right)\\
&= -\frac{1}{2}\text{Re}\left(\ln\left[\frac{-(e^{ix/2}-e^{-ix/2})}{e^{ix/2}+e^{-ix/2}}\right]\right) = -\frac{1}{2}\text{Re}\left(\ln\left[\frac{e^{ix/2}-e^{-ix/2}}{i^2(e^{ix/2}+e^{-ix/2})}\right]\right)\\
&= -\frac{1}{2}\text{Re}\left(\ln(\tan(x/2)) - \ln(i)\right) = -\frac{1}{2}\ln(\tan(x/2))
\end{aligned}$$

for all $0 < x < \pi$. Note that we have found a fourier series for $-\frac{1}{2}\ln(\tan(x/2))$ without any integration, just by stepping into the complex plane and then stepping back out.

Exercise 3.1.18. Show that if

$$R(z) = \frac{d_1}{z-z_1} + \frac{d_2}{z-z_2} + \ldots + \frac{d_r}{z-z_r}$$

where each d_i is real and positive and each z_i lies in the upper half-plane Im $z > 0$, then $R(z)$ has no zeros in the lower half-plane Im $z < 0$.

Proof. Let $R(z_0) = 0$ where $z_0 \neq z_i$. Then $\overline{R(z_0)} = 0$ and

$$\frac{d_1(z_0-z_1)}{|z_0-z_1|^2} + \frac{d_2(z_0-z_2)}{|z_0+z_2|^2} + \ldots + \frac{d_r(z_0-z_r)}{|z_0-z_r|^2} = 0.$$

Let

$$c_i = \frac{d_i}{|z_0-z_i|^2}.$$

Since $z_0 \neq z_i$, $|z_0-z_i|^2 > 0$. Also, since $d_i > 0$, $c_i > 0$. Then

$$\begin{aligned}
c_1(z_0-z_1) + c_2(z_0-z_2) + \ldots + c_r(z_0-z_r) &= 0\\
(c_1+c_2+\ldots+c_r)z_0 &= c_1z_1 + c_2z_2 + \ldots + c_rz_r\\
z_0 &= \frac{c_1z_1+c_2z_2+\ldots+c_rz_r}{c_1+c_2+\ldots+c_r}
\end{aligned}$$

$$\text{Im}(z_0) = \text{Im}\left(\frac{c_1z_1+c_2z_2+\ldots+c_rz_r}{c_1+c_2+\ldots+c_r}\right) = \frac{c_1\text{Im }z_1 + c_2\text{Im }z_2 + \ldots + c_r\text{Im }z_r}{c_1+c_2+\ldots+c_r}$$

Since Im $z_i > 0$ and $c_i > 0$, Im $z_0 > 0$.

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Exercise 3.2.22. Prove that for any m distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ ($\lambda_i \neq \lambda_j$ for $i \neq j$), the functions $e^{\lambda_1 z}, e^{\lambda_2 z}, \ldots, e^{\lambda_m z}$ are linearly independant on $\mathbb{C}$.

Proof. Base case, $m = 1$. Clearly then

$$ce^{\lambda z} = 0 \implies c = 0$$

since $e^{\lambda z}$ is never 0. Let $c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \ldots + c_m e^{\lambda_m z} = 0$ for all z only if $c_1 = c_2 = \ldots = c_m = 0$. Now let

$$f(z) = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \ldots + c_m e^{\lambda_m z} + c_{m+1} e^{\lambda_{m+1} z} = 0$$

for all z. Then

$$c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \ldots + c_m e^{\lambda_m z} = -c_{m+1} e^{\lambda_{m+1} z}.$$

When $z = 0$, then

$$c_1 + c_2 + \ldots + c_m = -c_{m+1}$$

so

$$\begin{aligned}
f(z) &= c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \ldots + c_m e \lambda_m z - (c_1 + c_2 + \ldots + c_m) e^{\lambda_{m+1} z} = 0 \\
f'(z) &= c_1 \lambda_1 e^{\lambda_1 z} + c_2 \lambda_2 e^{\lambda_2 z} + \ldots c_m \lambda_m e^{\lambda_m z} - \lambda_{m+1}(c_1 + c_2 + \ldots c_m) e^{\lambda_{m+1} z} \\
&= c_1(\lambda_1 e^{\lambda_1 z} - \lambda_{m+1} e^{\lambda_{m+1} z}) + c_2(\lambda_2 e^{\lambda_2 z} - \lambda_{m+1} e^{\lambda_{m+1} z}) + \ldots + c_m(\lambda_m e^{\lambda_m z} - \lambda_{m+1} e^{\lambda_{m+1} z}) \\
&= c_1 \left(\lambda_1 - \lambda_{m+1} e^{(\lambda_{m+1} - \lambda_1) z}\right) e^{\lambda_1 z} + \ldots + c_m \left(\lambda_m - \lambda_{m+1} e^{(\lambda_{m+1} - \lambda_m) z}\right) e^{\lambda_m z} = 0.
\end{aligned}$$

By our induction hypothesis, we have

$$c_i \left(\lambda_i - \lambda_{m+1} e^{(\lambda_{m+1} - \lambda_i) z}\right) = 0$$

for all z and $1 \leq i \leq m$. When $z = 0$, then

$$c_i(\lambda_i - \lambda_{m+1}) = 0$$

and since $\lambda_i \neq \lambda_j$, $c_i = 0$ for $1 \leq i \leq m$. Thus, by our previous equality, $c_{m+1} = 0$.

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