Exercise 1.8.5 (2pt). Show that $x^{\underline{n}} = \prod_{k=0}^{n-1}(x-k) = \sum_{i=0}^{n} s(n,i)x^i$, $n \geq 1$. Find a similar identity for $x^{\overline{n}} = \prod_{k=0}^{n-1}(x+k)$.

Proof. We shall use induction on n. Base case, $n = 1$ and $n = 2$.

$$\prod_{k=0}^{0}(x-k) = x = \sum_{i=0}^{1} s(1,0)x^i = 0 + x$$
$$\prod_{k=0}^{1}(x-k) = x(x-1) = \sum_{i=0}^{2} s(2,0)x^i = 0 + x - x^2$$

Let $x^{\underline{n}} = \sum_{i=0}^{n} s(n,i)x^i$, then

$$\begin{aligned}
x^{\underline{n+1}} = \prod_{k=0}^{n}(x-k) &= \left(\prod_{k=0}^{n-1}(x-k)\right)(x-n) \\
&= \left(\sum_{i=0}^{n} s(n,i)x^i\right)(x-n) \\
&= \sum_{i=0}^{n} s(n,i)x^{i+1} - ns(n,i)x^i \\
&= \sum_{i=1}^{n+1} s(n,i-1)x^i - \sum_{i=0}^{n} ns(n,i)x^i \\
&= \sum_{i=0}^{n+1}(s(n,i-1) - ns(n,i))x^i \\
&= \sum_{i=0}^{n+1} s(n+1,i)x^i
\end{aligned}$$

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For the rising factorial, we just just want the unsigned stirling numbers since we are adding, so

$$x^{\overline{n}} = \prod_{k=0}^{n-1}(x+k) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} x^i$$

Exercise 1.8.6 (2pt). Show that $\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} = x^n$

Proof.

$$\begin{aligned}
\sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}} &= \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \sum_{i=0}^{k} s(k,i)x^i \\
&= \sum_{k=0}^{n} \sum_{i=0}^{n} S(k,n)S(n,i)x^i
\end{aligned}$$

Applying Theorem 1.8.6

$$\begin{aligned}\sum_{k=0}^{n}\sum_{i=0}^{n} S(n,k)S(k,i)x^i &= \sum_{k=0}^{n} \delta_{n,k}x^i \\ &= x^n\end{aligned}$$

■

Exercise 2.2.2 (2pt). Prove D_n is even if and only if n is odd.

Proof. Let $n = 2m$, that is n is even. Then

$$\begin{aligned}D_n &= nD_{n-1} + (-1)^n \\ &= 2mD_{2m-1} + 1\end{aligned}$$

and D_n must be odd. Now let $n = 2m+1$, that is n is odd. Then

$$D_n = (2m+1)D_{2m} - 1$$

and D_{2m} must be odd since $2m$ is even. Since an odd number times an odd number is odd, and an odd number minus 1 must be even, D_n must be even.

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Exercise 2.2.11 (2pt). Suppose n people are seated in $m \geq n$ chairs in a room. At some point there is a break, and everyone leaves the room. When they return, in how many ways can they be seated so that no person occupies the same chair as before the break?

Suppose we pick person at random, call them P_0. There are $m-1$ ways to sit P_0 so they don't sit in their seat. Suppose P_0 picks an legal seat, then call the person originally sitting there P_1. They also have $m-1$ seats to choose from, their original seat is occupied by P_0. Wherever P_1 sits, let P_2 be the person originally occupying P_1, and they will have $m-2$ ways to choose their seats. In general person P_k has $m-k$ ways to choose a seat for $k \geq 1$ for a total of

$$(m-1)(m-1)(m-2)\cdots(m-(n-1)) = \frac{(m-1)(m-1)!}{(m-n)!}$$

possible combinations.

Problem 9 (3pt). In a small town, n married couples attend a town hall meeting. Each of the $2n$ people wants to speak exactly once. In how many ways can we schedule the participants if no married couple can take two consecutive slots?

We are essentially ordering people, in which there are $(2n)!$ total ways. If every couple speaks consecutively then there $2^n n!$ total ways, since there are $n!$ ways to order each couple, and 2 possible ways to arrange each couple. If all but one couple speaks consecutively, there are $2^{n-1}(n+1)!$

ways to do that, as we are ordering $n+1$ objects and we can arrange $n-1$ of them in 2 ways. If all but 2 couples speak back to back, there are $2^{n-2}(n+2)!$. In general, if all but k couples speak, there are $2^{n-k}(n+k)!$ ways to arrange them. In total, there are

$$(2n)! - \sum_{k=0}^{n-1} 2^{n-k}(n+k)!$$

Exercise 2.2.1 (1pt). Prove that $D_n = nD_{n-1} + (-1)^n$ when $n \geq 1$, by induction on n.

Proof. Base case: $n = 1$.

$$D_1 = 1D_0 + 1 = 1 - 1 = 0 = \sum_{k=0}^{1} (-1)^k \frac{1}{k!} = 1 - 1 = 0$$

Let $D_n = nD_{n-1} + (-1)^n$

$$\begin{aligned}
D_{n+1} &= (n+1)! \sum_{k=0}^{n+1} (-1)^k \frac{1}{k!} \\
&= (n+1) \left(n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!} + (-1)^{n+1} \frac{n!}{(n+1)!} \right) \\
&= (n+1)(D_n + (-1)^{n+1} \frac{1}{n+1}) \\
&= (n+1)D_n + (-1)^{n+1}
\end{aligned}$$

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