Exercise 3d. Let C be the circle $|z| = 2$ traversed once in the positive sense. Compute the following integral.

$$\int_C \frac{5z^2+2z+1}{(z-i)^3}dz$$

Observe that $f(z) = 5z^2+2z+1$ is analytic inside and on C. Therefore from Eq. (14) with $z_0 = i$, which is inside C, and $m = 3$ we have

$$\int_C \frac{5z^2+2z+1}{(z-i)^3}dz = \frac{2\pi i f''(i)}{2!} = 10\pi i$$

Exercise 6. Evaluate

$$\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}dz$$

where Γ is the circle $|z| = 3$ traversed once counterclockwise.

Observe that

$$\left(z^2+1\right)^2 = [(z+i)(z-i)]^2 = (z+i)^2(z-i)^2$$

and

$$\frac{1}{(z+i)^2(z-i)^2} = \frac{A}{z+i} + \frac{B}{(z+i)^2} + \frac{C}{z-i} + \frac{D}{(z-i)^2}.$$

Since the coeffecient for z^3 in the numerator is 0, we know that $A + C = 0$. Also

$$A = \lim_{z\to -i}\frac{d}{dz}\left(\frac{1}{(z-i)^2}\right) = \lim_{z\to -i}\frac{-2(z-i)}{(z-i)^4} = \frac{4i}{(-2i)^4} = \frac{i}{4}$$
$$\implies C = -\frac{i}{4}$$

and to find B and D we can use the cover up method.

$$B = \frac{1}{(-2i)^2} = -\frac{1}{4} \qquad\qquad D = \frac{1}{(2i)^2} = -\frac{1}{4}$$

Then

$$\begin{aligned}\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}dz &= \int_\Gamma e^{iz}\left(\frac{1}{(z+i)^2(z-i)^2}\right)dz \\ &= \int_\Gamma \left(\frac{ie^{iz}}{4(z+i)} - \frac{e^{iz}}{4(z+i)^2} - \frac{ie^{iz}}{4(z-i)} - \frac{e^{iz}}{4(z-i)^2}\right)dz.\end{aligned}$$

Let C_0 be a small circle around $z = -i$ and C_1 be a small circle around $z = i$. Since our function we are integrating is analytic everywhere except for these 2 points and both points are in Γ, we can rewrite our integral as the sum of two integrals around C_0 and C_1 going counterclockwise. We can then use Eq. (14) with the appropriate z_0 and m for each integral

$$\begin{aligned}&= \oint_{C_0}\left(\frac{ie^{iz}}{4(z+i)} - \frac{e^{iz}}{4(z+i)^2}\right)dz - \oint_{C_1}\left(\frac{ie^{iz}}{4(z-i)} + \frac{e^{iz}}{4(z-i)^2}\right)dz \\ &= \frac{1}{4}[2\pi i(ie^{-i^2}) - 2\pi i(ie^{-i^2}) - 2\pi i(ie^{i^2}) - 2\pi i(ie^{i^2})] \\ &= -\pi i^2 e^{i^2} \\ &= \frac{\pi}{e}\end{aligned}$$