Exercise 3e. Evaluate the following integral using the Cauchy residue theorem.

$$\oint_{|z|=1} \frac{1}{z^2 \sin z} dz.$$

We have singularities at $z = 0, \pm\pi, \pm 2\pi, \ldots$ but since we are integrating around $|z| = 1$, we only care about the singularity at $z = 0$. Using the Cauchy residue theorem we

$$\oint_{|z|=1} \frac{1}{z^2 \sin z} dz = 2\pi i \mathrm{Res}(f; 0)$$

where $f(z) = \dfrac{1}{z^2 \sin z}$. Using the Maclaurin series we have

$$f(z) = \frac{1}{z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)} = \frac{1}{z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right)}.$$

At $z = 0$ we have a pole of order 3, so

$$\begin{aligned}
\mathrm{Res}(f; 0) &= \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz}(z^3 f(z)) \\
&= \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2} \left(\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots} \right) \\
&= \lim_{z\to 0} \frac{1}{2} \frac{d}{dz} \left(\frac{\frac{z}{3} - \frac{z^3}{30} + \ldots}{\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \ldots\right)^2} \right) \\
&= \frac{1}{2} \cdot \frac{1 \cdot \frac{1}{3} - 0 \cdot 0}{1^4} \\
&= \frac{1}{6}
\end{aligned}$$

and then

$$\oint_{|z|=1} \frac{1}{z^2 \sin z} dz = \frac{\pi i}{3}.$$

Exercise 6. Suppose that f is analytic and has a zero of order m at the point z_0. Show that the function $g(z) = f'(z)/f(z)$ has a simple pole at z_0 with $\mathrm{Res}(g; z_0) = m$

Proof. Since f is analytic and has a zero of order m at z_0, we can write f as

$$f(z) = (z - z_0)^m h(z)$$

where $h(z)$ is analytic and $h(z_0) \neq 0$. Then

$$\begin{aligned} g(z) = \frac{f'(z)}{f(z)} &= \frac{m(z-z_0)^{m-1}h(z) + (z-z_0)^m h'(z)}{(z-z_0)^m h(z)} \\ &= \frac{mh(z) + (z-z_0)h'(z)}{(z-z_0)h(z)} \\ &= \frac{m}{z-z_0} + \frac{h'(z)}{h(z)} \end{aligned}$$

and since $h(z)$ is analytic and $h(z_0) \neq 0$, $g(z)$ has a simple pole at $z = z_0$ and all negative coefficients for $h'(z)/h(z)$ in it's Laurent expansion are 0. Therefore,

$$\text{Res}(g; z_0) = m$$

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