Exercise 6c. Compute $\int_\Gamma \bar{z}dz$, where Γ is the circle $|z| = 2$ traversed three times clockwise.

We can paramatize Γ as follows,

$$z(t) = -2\cos(t) + 2i\sin(t) \text{ for } 0 \leq t \leq 6\pi.$$

Then

$$\begin{aligned}
\oint_\Gamma \bar{z}dz &= \int_0^{6\pi} \overline{(-2\cos(t) + i\sin(t))}(2\sin(t) + 2i\cos(t))dt \\
&= \int_0^{6\pi} -2(\cos(t) + i\sin(t))2i(\cos(t) - i\sin(t)) \\
&= -4i\int_0^{6\pi} (\cos^2(t) + \sin^2(t))dt = -4i\int_0^{6\pi} dt = -24i\pi
\end{aligned}$$

Exercise 8. Let C be the perimeter of the square with vertices at the points $z = 0$, $z = 1$, $z = 1 + i$, and $z = i$ traversed once in that order. Show that

$$\oint_C e^z dz = 0$$

Proof. We can split C into 4 smooth curves as follows,

$$\begin{aligned}
c_1 :&\quad z_1(t) = t &\quad 0 \leq t \leq 1 \\
c_2 :&\quad z_2(t) = 1 + it &\quad 0 \leq t \leq 1 \\
c_3 :&\quad z_3(t) = 1 - t + i &\quad 0 \leq t \leq 1 \\
c_4 :&\quad z_4(t) = i(1 - t) &\quad 0 \leq t \leq 1
\end{aligned}$$

Then,

$$\begin{aligned}
\oint_C e^z dz &= \int_{c_1} e^z dz + \int_{c_2} e^z dz + \int_{c_3} e^z dz + \int_{c_4} e^z dz \\
&= \int_0^1 e^t dt + \int_0^1 ie^{1+it}dt + \int_0^1 -e^{1-t+i}dt + \int_0^1 -ie^{i(1-t)}dt \\
&= \left[e^t\right]_0^1 + ie\left[\frac{1}{i}e^{it}\right]_0^1 - e^{1+i}\left[-e^{-t}\right]_0^1 - ie^i\left[-\frac{1}{i}e^{-it}\right]_0^1 \\
&= (e - 1) + ie\left(\frac{e^i - 1}{i}\right) - e^{i+1}\left(\frac{-1}{e} + 1\right) - ie^i\left(\frac{-1}{ie^i} + \frac{1}{i}\right) \\
&= e - 1 + e^{i+1} - e + e^i - e^{i+1} + 1 - e^i \\
&= 0
\end{aligned}$$

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Exercise 10. Compute $\int_C \bar{z}^2 dz$ along the perimeter of the square in Ex. 8.

We can use the paramaterization of C as given in Ex. 8. So,

$$\begin{aligned}
\oint_C \bar{z}^2 dz &= \int_{c_1} \bar{z}^2 dz + \int_{c_2} \bar{z}^2 dz + \int_{c_3} \bar{z}^2 dz + \int_{c_4} \bar{z}^2 dz \\
&= \int_0^1 t^2 dt + \int_0^1 i(1-it)^2 dt + \int_0^1 -(1-t-i)^2 dt + \int_0^1 -i(i(t-1))^2 dt \\
&= \left[\frac{t^3}{3}\right]_0^1 + i\int_0^1 (1-2it-t^2)dt - \int_0^1 (t^2-(2-2i)t-2i)dt - i\int_0^1 -(t^2-2t+1)dt \\
&= \frac{1}{3} + i\left[t - it^2 - \frac{t^3}{3}\right]_0^1 - \left[\frac{t^3}{3} - (1-i)t^2 - 2it\right] + i\left[\frac{t^3}{3} - t^2 + t\right]_0^1 \\
&= \frac{1}{3} + i\left(1 - i - \frac{1}{3}\right) - \left(\frac{1}{3} - (1-i) - 2i\right) + i\left(\frac{1}{3} - 1 + 1\right) \\
&= \frac{1}{3} + i + 1 - \frac{i}{3} - \frac{1}{3} + 1 - i + 2i + \frac{i}{3} \\
&= 2 + 2i
\end{aligned}$$