Exercise 10. Show that all the solutions of the equation $\cos z = 2i$ are given by $\pi/2 + 2k\pi - i\text{Log}(\sqrt{5}+2)$, $-\pi/2 + 2k\pi + i\text{Log}(\sqrt{5}+2)$, for $k = 0, \pm1, \pm2, \ldots$.

Proof. We can use the definition of $\cos^{-1}$ to find z.

$$\begin{aligned}
\cos z = 2i \implies z &= \cos^{-1}(2i)\\
&= -i\log[2i + ((2i)^2 - 1)^{1/2}]\\
&= -i\log[i(2 \pm \sqrt{5})]\\
&= -i(\log(i) + 1\text{Log}(2 \pm \sqrt{5}))\\
&= \pi/2 + 2k\pi - i\text{Log}(2 \pm \sqrt{5}) \text{ for } k = 0, \pm1, \pm2, \ldots
\end{aligned}$$

Then when we choose the positive square root,

$$z = \pi/2 + 2k\pi - i\text{Log}(2+\sqrt{5}) \text{ for } k = 0, \pm1, \pm2, \ldots$$

When we choose the negative square root, consider the following identity

$$\frac{-1}{2+\sqrt{5}} = \frac{-1}{2+\sqrt{5}} \cdot \frac{2-\sqrt{5}}{2-\sqrt{5}} = 2 - \sqrt{5}.$$

Then,

$$\begin{aligned}
z &= \pi/2 + 2k\pi - i\text{Log}[-(2+\sqrt{5})^{-1}] \text{ for } k = 0, \pm1, \pm2, \ldots\\
&= \pi/2 + 2k\pi + i\text{Log}(2+\sqrt{5}) - i\log(-1)\\
&= 3\pi/2 + 2k\pi + i\text{Log}(2+\sqrt{5})\\
&= -\pi/2 + 2k\pi + i\text{Log}(2+\sqrt{5})
\end{aligned}$$

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Exercise 19. Determine the inverse of the function, $q(z) = 2e^z + e^{2z}$ explicitly in terms of the complex logarithm. Find *all* values of z for which $q(z) = 3$.

$$\begin{aligned}
2e^z + e^{2z} = w \implies e^{2z} + 2e^z - w &= 0\\
e^z &= \frac{-2 \pm \sqrt{4+4w}}{2} = -1 \pm \sqrt{1+w}\\
z &= \log\left(-1 \pm \sqrt{1+w}\right)\\
q^{-1}(z) &= \log\left(-1 \pm \sqrt{1+z}\right).
\end{aligned}$$

When $q(z) = 3$,

$$q^{-1}(3) = \log(-1 \pm \sqrt{4}) = \log(-1 \pm 2)$$

When we take $+2$,

$$q^{-1}(3) = \text{Log}(1) = 0$$

When we take -2,

$$q^{-1}(3) = \log(-3) = \text{Log}(3) + i\pi + 2k\pi \text{ for } k = 0, \pm1, \pm2, \ldots$$