Exercise 5.4.3c. Find the radius of convergence of the following power series.

$$\sum_{k=0}^{\infty} \left(2+(-1)^k\right)^k z^k$$

First we can find the sum of the series.

$$\begin{aligned}\sum_{k=0}^{\infty} \left(2+(-1)^k\right)^k z^k &= 1 + z + (3z)^2 + z^3 + (3z)^4 + \dots \\ &= \sum_{k=0}^{\infty}(3z)^{2k} + \sum_{k=0}^{\infty} z^{2k+1} \\ &= \sum_{k=0}^{\infty}(9z^2)^k + z\sum_{k=0}^{\infty}(z^2)^k \\ &= \frac{1}{1-9z^2} + \frac{z}{1-z^2}\end{aligned}$$

for $|9z^2| < 1$ and $|z^2| < 1$. We only care about the one which is more restrictive, so we have

$$9|z|^2 < 1 \implies |z| < \frac{1}{3} \qquad\qquad |z|^2 < 1 \implies |z| < 1$$

which gives us a radius of convergence of 1/3.

Exercise 5.1.21. Let $0 < c < 1$. An increasingly lazy and asymmetric frog leaps one meter (from $z = 0$ to $z = 1$) on his first jump, c meter on his second jump, c^2 meters on his third jump, and so on, each time turning exactly an angle $0 < \alpha < \pi/2$ to the left of his preceding flight path. Show the that frog eventually ends up on a circle that depends on c, but not α.

Proof. Let S be the sum of the jumps of the frog and $z = e^{i\alpha}$. Since $|cz| = |c| < 1$, we have

$$S = 1 + cz + (cz)^2 + \dots = \sum_{k=0}^{\infty}(cz)^k = \frac{1}{1-cz}$$

Consider the following quantity.

$$\begin{aligned}
|(1-c^2)S-1|^2 &= \left(\frac{1-c^2}{1-cz}-1\right)\left(\frac{1-c^2}{1-c/z}-1\right)\\
&= \left(\frac{1-c^2}{1-cz}-1\right)\left(\frac{z-c^2z}{z-c}-1\right)\\
&= \frac{z(1-c^2)(1-c^2)}{(1-cz)(z-c)}-\frac{1-c^2}{1-cz}-\frac{z-c^2z}{z-c}+1\\
&= \frac{z(1-c^2)(1-c^2)}{(1-cz)(z-c)}-\frac{(1-c^2)(z-c)}{(1-cz)(z-c)}-\frac{(z-c^2z)(1-cz)}{(z-c)(1-cz)}+1\\
&= \frac{z-2c^2z+zc^4-z+c+c^2z-c^3-z+cz^2+c^2z-c^3z^2+z-cz^2-c+c^2z}{(z-c)(1-cz)}\\
&= \frac{c^2z+zc^4-c^3-c^3z^2}{(z-c)(1-cz)}\\
&= \frac{c^2(z+zc^2-c-cz^2)}{(z-c)(1-cz)}\\
&= \frac{c^2(z(1-cz)-c(1-cz))}{(z-c)(1-cz)}\\
&= c^2\\
\implies |(1-c^2)S-1| &= c
\end{aligned}$$

Then we have

$$\left|S-\frac{1}{1-c^2}\right|=\frac{c}{1-c^2}$$

which gives us a center at $\frac{1}{1-c^2}$ and a radius of $\frac{c}{1-c^2}$

■

Problem 3. Show that the function f defined by the power series $f(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(k!)^2}z^{2k}$ satisfies the differential equation $zf''(z)+f'(z)+zf(z)=0$.

Proof. Fist consider

$$f'(z)=\sum_{k=0}^{\infty}\frac{2k(-1)^k}{4^k(k!)^2}z^{2k-1}$$
$$f''(z)=\sum_{k=0}^{\infty}\frac{2k(2k-1)(-1)^k}{4^k(k!)^2}z^{2k-2}.$$

so

$$\begin{aligned}
zf''(z) + f'(z) + zf(z) &= z\sum_{k=0}^{\infty} \frac{2k(2k-1)(-1)^k}{4^k(k!)^2} z^{2k-2} + \sum_{k=0}^{\infty} \frac{2k(-1)^k}{4^k(k!)^2} z^{2k-1} + z\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k(k!)^2} z^{2k} \\
&= \sum_{k=0}^{\infty} \left(\frac{2k(2k-1)(-1)^k}{4^k(k!)^2} + \frac{2k(-1)^k}{4^k(k!)^2} \right) z^{2k-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k(k!)^2} z^{2k+1} \\
&= \sum_{k=1}^{\infty} \left(\frac{2k(2k-1)(-1)^k}{4^k(k!)^2} + \frac{2k(-1)^k}{4^k(k!)^2} \right) z^{2k-1} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4^{k-1}((k-1)!)^2} z^{2k-1} \\
&= \sum_{k=1}^{\infty} \left(\frac{2k(2k-1)(-1)^k}{4^k(k!)^2} + \frac{2k(-1)^k}{4^k(k!)^2} + \frac{(-1)^{k-1}}{4^{k-1}((k-1)!)^2} \right) z^{2k-1} \\
&= \sum_{k=1}^{\infty} \left(\frac{2k(2k-1)(-1)^k}{4^k(k!)^2} + \frac{2k(-1)^k}{4^k(k!)^2} - \frac{4k^2(-1)^k}{4^k(k!)^2} \right) z^{2k-1} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k(4k^2 - 2k + 2k - 4k^2)}{4^k(k!)^2} z^{2k-1} \\
&= 0
\end{aligned}$$

■

Problem 4. Consider the function $g(z) = \dfrac{1}{(z+2)(z-3)}$. Find the Laurent series for the function g in each of the following domains:

(a) $0 < |z| < 2$

First note:

$$\frac{1}{(z+2)(z-3)} = \frac{1}{5(z-3)} - \frac{1}{5(z+2)}$$

Then for $|z| < 2$ we have

$$\frac{1}{5(z-3)} = -\frac{1}{15} \cdot \frac{1}{1 - z/3} = -\frac{1}{15} \sum_{k=0}^{\infty} \left(\frac{z}{3}\right)^k = -\sum_{k=0}^{\infty} \frac{z^k}{5 \cdot 3^{k+1}}$$

and

$$\frac{1}{5(z+2)} = \frac{1}{10} \cdot \frac{1}{1 - (-z)/2} = \frac{1}{10} \sum_{k=0}^{\infty} (-1)^k \left(\frac{z}{2}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{5 \cdot 2^{k+1}}$$

so

$$\frac{1}{(z+2)(z-3)} = \sum_{k=0}^{\infty} \left(-\frac{1}{5 \cdot 3^{k+1}} - \frac{(-1)^k}{5 \cdot 2^{k+1}} \right) z^k.$$

(b) $2 < |z| < 3$

For $\dfrac{1}{5(z-3)}$ our equation is still valid, but we have

$$\frac{1}{5(z+2)} = \frac{1}{5z} \cdot \frac{1}{1 - (-2)/z} = \frac{1}{5z} \sum_{k=0}^{\infty} (-1)^k \left(\frac{2}{z}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{5z^{k+1}}$$

so

$$\frac{1}{(z+2)(z-3)} = -\sum_{k=0}^{\infty} \frac{z^k}{5 \cdot 3^{k+1}} - \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{5z^{k+1}}.$$

(c) $|z-3| > 5$

Since we are centering our series at $z = 3$ ...

$$\frac{1}{5(z+2)} = \frac{1}{5} \cdot \frac{1}{(z-3)+5} = \frac{1}{5(z-3)} \cdot \frac{1}{1-(-5)/(z-3)} = \frac{1}{5(z-3)} \sum_{k=0}^{\infty} (-1) \left(\frac{5}{(z-3)} \right)^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k 5^{k-1}}{(z-3)^{k+1}}$$

so

$$\frac{1}{(z+2)(z-3)} = \frac{1}{5(z-3)} - \sum_{k=0}^{\infty} \frac{(-1)^k 5^{k-1}}{(z-3)^{k+1}}$$

(d) $|z| > 3$

$$\frac{1}{5(z-3)} = \frac{1}{5z} \cdot \frac{1}{1-3/z} = \frac{1}{5z} \sum_{k=0}^{\infty} \left(\frac{3}{z} \right)^k = \sum_{k=0}^{\infty} \frac{3^k}{5z^{k+1}}$$