Exercise 4c. Show that the inversion mapping $w = f(z) = 1/z$ maps the circle $|z-1| = 1$ onto the verticle line $x = 1/2$

Proof. Let $z = x + iy$

$$\begin{aligned} 1 = |z-1|^2 &= (z-1)\overline{(z-1)} \\ &= (z-1)(\overline{z}-1) \\ &= z\overline{z} - z - \overline{z} + 1 \\ &= |z|^2 - 2x + 1 \\ \implies |z|^2 &= 2x \end{aligned}$$

$$\begin{aligned} w = \frac{1}{x+iy} &= \frac{x-iy}{x^2+y^2} \\ &= \frac{x}{|z|^2} - \frac{iy}{|z|^2} \\ &= \frac{x}{2x} - \frac{iy}{|z|^2} \\ &= \frac{1}{2} - \frac{iy}{|z|^2} \end{aligned}$$

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Exercise 6c. The *Joukowski mapping* is defined by

$$w = J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Show that J maps the circle $|z| = r$ $(r > 0,\ r \neq 1)$ onto the eclipse

$$\frac{u^2}{\left[\frac{1}{2}\left(r + \frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(r - \frac{1}{r}\right)\right]^2} = 1,$$

where $J(z) = u + iv$.

Proof. Let $z = x + iy$

$$\begin{aligned} J(z) &= \frac{1}{2}\left(x + iy + \frac{1}{x+iy}\right) \\ &= \frac{1}{2}\left(x + iy + \frac{x}{r^2} - \frac{iy}{r^2}\right) \\ &= \frac{x}{2}\left(1 + \frac{1}{r^2}\right) + \frac{iy}{2}\left(1 - \frac{1}{r^2}\right) = u + iv \end{aligned}$$

$$\begin{aligned} u &= \frac{x}{2}\left(1 + \frac{1}{r^2}\right) \\ x &= \frac{u}{\frac{1}{2}\left(1 + \frac{1}{r^2}\right)} \end{aligned}$$

$$\begin{aligned} v &= \frac{y}{2}\left(1 - \frac{1}{r^2}\right) \\ y &= \frac{v}{\frac{1}{2}\left(1 - \frac{1}{r^2}\right)} \end{aligned}$$

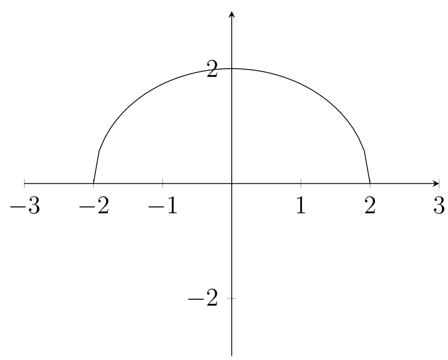
$$|z|^2 = x^2 + y^2 = \frac{u^2}{\left[\frac{1}{2}\left(1 + \frac{1}{r^2}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(1 - \frac{1}{r^2}\right)\right]^2} = r^2$$

$$\frac{u^2}{r^2\left[\frac{1}{2}\left(1 + \frac{1}{r^2}\right)\right]^2} + \frac{v^2}{r^2\left[\frac{1}{2}\left(1 - \frac{1}{r^2}\right)\right]^2} = 1$$

$$\frac{u^2}{\left[\frac{1}{2}\left(r + \frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(r - \frac{1}{r}\right)\right]^2} = 1$$

■

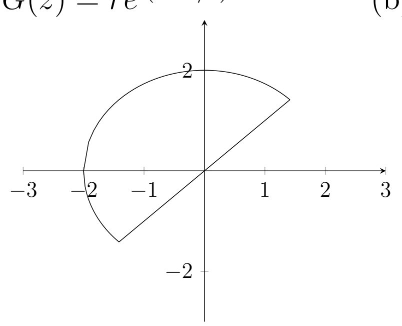
Exercise 8. A function of the form $G(z) = e^{i\phi}z$, where ϕ is a real constant, generates a *rotation mapping*. Sketch the image of the semidisk $|z| \leq 2$, $\Im z \geq 0$, under G when **(a)** $\phi = \pi/4$; **(b)** $\phi = -\pi/4$; **(c)** $\pi = 3\pi/4$.



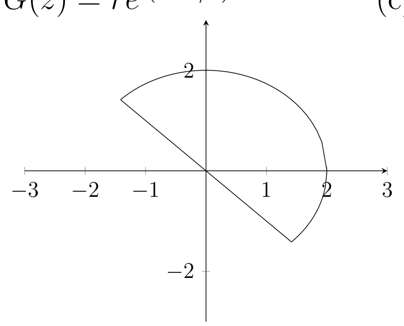
Let $z = re^{i\theta}$, then

$$G(z) = re^{i(\phi+\theta)}$$

(a) $G(z) = re^{i(\theta+\pi/4)}$



(b) $G(z) = re^{i(\theta-\pi/4)}$



(c) $G = re^{i(\theta+3\pi/4)}$