**Problem 1.** Show that all orbits of $F(z) = \lambda z$ are cycles if $\lambda = e^{2\pi i \frac{p}{q}}$, where $p$, $q$ are integers.

*Proof.* Let $z_0 = re^{i\theta}$, then

$$\begin{aligned} z_1 &= re^{i\theta + 2\pi i \frac{p}{q}} \\ z_2 &= re^{i\theta + 2\pi i \frac{2p}{q}} \\ &\vdots \\ z_n &= re^{i\theta + 2\pi i \frac{np}{q}} \end{aligned}$$

At some point, $n = q$ and we have

$$z_q = re^{i\theta + 2\pi i p} = re^{i\theta} = z_0$$

since $p$ and $q$ are integers. Thus, all orbits of $F(z)$ are cycles. ∎

**Problem 2.** Using Octave or Matlab, reproduce the figure at the top of page 139. For specificity, use $c = -1 - 2i$.

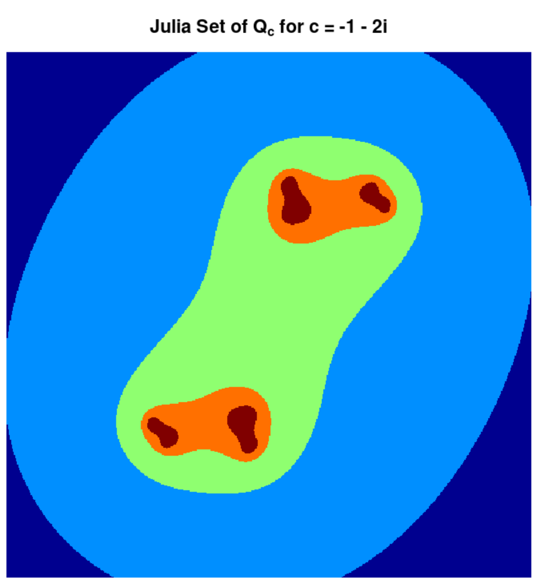


Figure 1: Filled Julia set for $c = -1 - 2i$

**Problem 3a.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Which of the following prime periods is possible without the other three and why?

$$72, \; 114, \; 168, \; 172$$

First we can do a prime decomposition of the four numbers.

$$72 = 2^3 \cdot 3^2 \qquad 114 = 2 \cdot 3 \cdot 19 \qquad 168 = 2 \cdot 3 \cdot 7 \qquad 172 = 2^2 \cdot 43$$

Now we can apply Sharkovsky's Theorem which is as follows:

**Sharkovsky's Theorem.** Suppose $F : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose $F$ has a periodic point of prime period $n$ and that $n$ preceeds $k$ in the Sharkovsky ordering. Then $F$ also has a periodic point of prime period $k$.

and the Sharkovsky ordering is

$$\begin{array}{cccc} 3 & 5 & 7 & \dots \\ 2 \cdot 3 & 2 \cdot 5 & 2 \cdot 7 & \dots \\ 2^2 \cdot 3 & 2^2 \cdot 5 & 2^2 \cdot 7 & \dots \\ & \vdots & & \\ \dots & 2^n & \dots & 2^0 \end{array}$$

The prime which appears last in the Sharkovsky ordering doesn;t imply the rest, so 72 is the possilbe prime period which is possilbe without the rest.

**Problem 3b.** Show that if $F(x_1) < a$ and $F(x_2) > a$ then Period 4 implies Period 2.

*Proof.* First we have that $F(x_1) = x_2$. Let $I_0 = [x_3, x_4]$ and $I_1 = [x_2, x_3]$. Then we have that $F(I_0) \supset I_1$ and $F(I_1) \supset I_0 \cup I_1$ since. From here he have the same situation with Period 3 points, so we have a Period 2 cycle

■

**Problem 4a.** Compute the Schwarzian derivative of a polynomial $P$ if $P'(x) = -2(x+1)(x-2)(x-3)$, by computing the derivative of $\ln |P'(x)|$.

First we have

$$\ln(P'(x)) = \ln(-2(x+1)(x-2)(x-3)) = \ln(-2) + \ln(x+1) + \ln(x-2) + \ln(x-3)$$

We can take the derivative using the chain rule to get

$$\frac{P''(x)}{P'(x)} = \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x-3}$$

Differentiating again we find

$$\frac{P'''(x)P'(x) - (P''(x))^2}{(P'(x))^2} = \frac{P'''(x)}{P'(x)} - \left(\frac{P''(x)}{P'(x)}\right)^2$$
$$= -\frac{1}{(x+1)^2} - \frac{1}{(x-2)^2} - \frac{1}{(x-3)^2}$$

Now we can compute the Schwarzian.

$$SP(x) = \frac{P'''(x)}{P'(x)} - \left(\frac{P''(x)}{P'(x)}\right)^2 - \frac{1}{2}\left(\frac{P''(x)}{P'(x)}\right)$$
$$= -\frac{1}{(x+1)^2} - \frac{1}{(x-2)^2} - \frac{1}{(x-3)^2} - \frac{1}{2}\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x-3}\right)^2$$

**Problem 4b.** In a short essay, summerize the importance of the Schwarzian derivative in our construction of the Mandlebrot set.

The Schwarzian derivative has many principles that make the can simplify the analysis of dynamic systems. One important principle is the Schwarzian min-max Principle: if $SF < 0$, then $F'$ cannot have a positive local minimum or a negative local maximum, so if a function has a negative Schwarzian derivative then it can only have a certain shape. Another property that follows from this the following: If $SF < 0$ and $x_0$ is an attracting periodic point of $F$, then either the imediate basin of attraction of $x_0$ extends to $+\infty$ or $-\infty$, or else there is a critical point of $F$ whose orbit is attracted to $x_0$.

This property can be used to analyze the local dynamics around critical points for any given $c$ of the Mandlebrot set. We can classify critical points as either stable or neutral with this theorem which can help determine regions belonging to the Mandlebrot set without much computation.

**Problem 5.** Let $F(x) = x/(x-1)$. Find the fixed points for Newton's iteration function and determine if they are attracting or repelling. Does your conlusion match what you expect?

First we find the derivative of $F$.

$$F'(x) = \frac{(x-1) - x}{(x-1)^2} = -\frac{1}{(x-1)^2}$$

Then we compute Newton's iteration function for $F$.

$$N(x) = x - \frac{F(x)}{F'(x)} = x - \frac{x/(x-1)}{-1/(x-1)^2} = x + x(x-1)$$

Now we can find the fixed points.

$$x = x + x(x-1) \implies x(x-1) = 0$$

so $x = 0$ and $x = 1$ are fixed points for $N(x)$. To determine if these points are attracting or repelling, we can find the derivative of $N(x)$.

$$N'(x) = 1 + 2x - 1 = 2x$$

$|N'(0)| = 0$ so we have an attracting point at $x = 0$, and $|N'(1)| = 2$ so we have a repelling point at $x = 1$. This makes sense becuase at $F(0) = 0$, so we'd want that to be an attracting fixed point, and at $x = 1$ $F$ goes to infinity, so we'd want that to be a repelling fixed point.

**Problem 6.** Consider $Q_c(z) = z^2 + i$.

(a) Show that the orbit of 0 is eventaully periodic.

(b) Show that the periodic orbit you found is either attracting or repelling

(c) Use the random iteration algorithm to produce a sketch of the Julia set. Should it be connected or disconnected?

(a) When $z = 0$, the orbit of $Q_i$ is as follows.

$$0 \to i \to -1 + i \to -2i + i = -i \to -1 + i$$

As we can see the orbit of 0 is eventaully periodic to the Period 2 cycle of $z_0 = -1 + i$ or $z_0 = -i$.

(b) Since we know the period 2 cycle we want, we can find $|(Q_c^2)'(-i)|$ as follows.

$$\begin{aligned} |(Q_c^2)'(-i)| &= |Q_c'(-i) \cdot Q_c'(-1 + i)| \\ &= |2(-i) \cdot 2(-1 + i)| \\ &= |4(i + 1)| \\ &> 1 \end{aligned}$$

so this cycle is repelling.

(c) Here is the Julia set of $Q_i$. Since we had an eventaully repelling fixed point at $z = 0$ which is a critical point, it makes sense that the Julia set would be disconnected, or a dusting of points

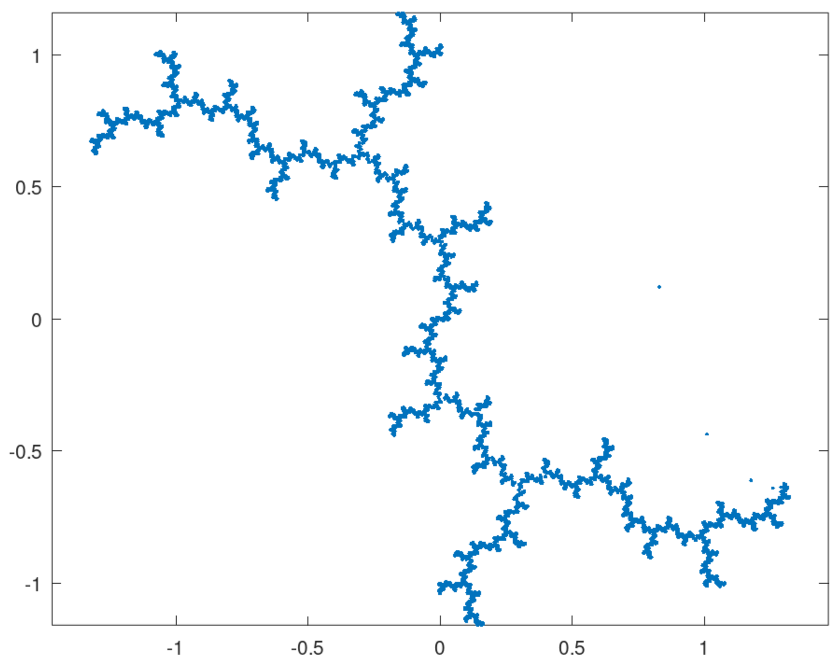


Figure 2: Julia set of $c = i$ using random iteration