Exercise 1. Calculate each of the following integrals along the inducated contours. Observe that a standard table of integrals can be used. Explain why.

Because the following functions are continuous in $\mathbb{C}$ have antiderivatives in $\mathbb{C}$, then for any contour in $\mathbb{C}$

$$\int_\Gamma f(z)dz = F(z_T) - F(z_1)$$

where z_1 is the initail point of a contour and z_T is the terminal point of a contour.

(a) $\int_\Gamma (3z^2 - 5z + i)dz$ along the line segment from $z = i$ to $z = 1$.

$$\begin{aligned}\int_\Gamma (3z^2 - 5z + i) &= \left[z^3 - \frac{5}{2}z^2 + iz\right]_i^1 \\ &= \left(1 - \frac{5}{2} + i\right) - \left(-i + \frac{5}{2} - 1\right) \\ &= -3 + 2i\end{aligned}$$

(e) $\int_\Gamma \sin^2 z \cos(z)dz$ along the contour in Fig. 4.24.
Let $u = \sin(z)$, then $du = \cos(z)dz$.

$$\begin{aligned}\int_\Gamma \sin^2(z)\cos(z)dz &= \int_\Gamma u^2 du \\ &= \left[\frac{u^3}{3}\right]_{\sin(\pi)}^{\sin(i)} \\ &= \frac{\sin^3(i)}{3} \\ &= -\frac{i}{3}\sinh^3(1)\end{aligned}$$

Exercise 7. Show that if C is a positevely oriented circle and z_0 lies outside C, then

$$\int_C \frac{dz}{z - z_0} = 0$$

Proof. Let s be the line from z_0 passing through center of C and extending onward. Since z_0 lies outside of C, this line always exists and has a direction. Now let $-s$ be the line from z_0 in the opposite direction of s extending forever. Then $D = \mathbb{C} - (-s)$ forms a domain in which $\dfrac{1}{z - z_0}$ is continuous and has an antiderivative, namely $\log(z - z_0)$ with branch cut $-s$. Also, clearly $C \in D$, so by Theorem 7 the integral of C.

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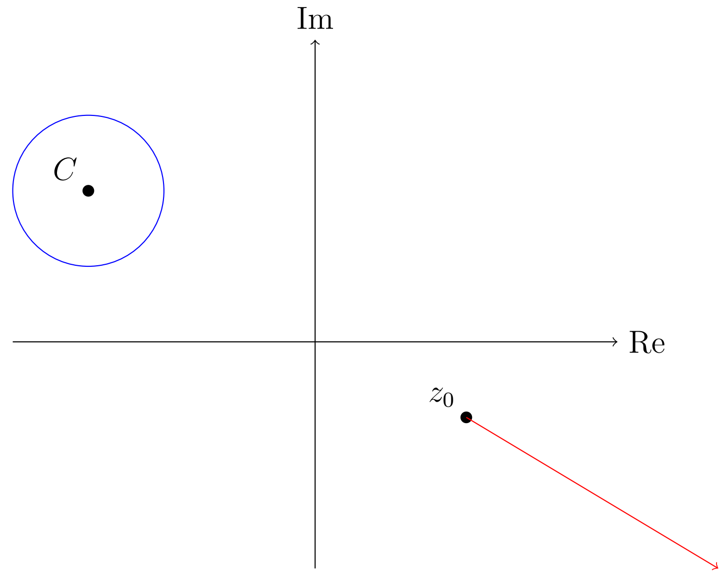

Exercise 12. Let f be an analytic function with a continuous derivative satisfying $|f'(z)| \leq M$ for all z in the disk $D : |z| < 1$. Show that

$$|f(z_2) - f(z_1)| \leq M|z_2 - z_1| \quad (z_1, z_2 \in D).$$

Proof. Let Γ be the sine segment from z_1 to z_2, then

$$\begin{aligned} |f(z_2) - f(z_1)| &= \left| \int_\Gamma f'(z) dz \right| \\ &\leq \int_\Gamma |f'(z)| dz \\ &\leq \int_\Gamma M dz \\ &= M(z_2 - z_1) \\ &\leq M|z_2 - z_1| \end{aligned}$$

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