Exercise 5.8.5 (3pt). Prove the following theorem:

Theorem 5.8.12. Brook's Theorem

If G is a graph other than K_n or C_{2n+1}, $\chi \leq \Delta$.

Proof. By corollary 5.8.11 we need to consider only regular graphs. Regular graphs of degree 2 are easy, so we consider only regular graphs of degree at least 3.

Suppose G is not 2 connected, then we have a bridge at some point seperating the graphs. Consider one block seperated by the bridge G_1. If $\Delta(G_1) < \Delta(G)$ then we can color G_1 with $\Delta(G)$ colors. If $\Delta(G_1) = \Delta(G)$, then the node with maximun degree can only be the vertex on the bridge, in which we can color the graph with $\Delta(G)$ colors since the color we color the other side of the bridge with can be repeated. Then, we can color the other block with G_2 with $\Delta(G)$ colors and if we choose different colors on the bridge the whole graph can be colored with $\Delta(G)$ colors.

Suppose G is 2 connected, we want to show that there exists vertices u, v, and w such that u is adjacent to both v and w but v and w are not adjacent, and $G - v - w$ is connected. Let x be a vertex not adjacent to all points. Since $G \neq K_n$, such a point exists.

If $G - x$ is 2 connected, let $v = x$ and w be any point of distance 2 from x. This is possible since any point not adjacent to x must be connected to x by a path of distance n and must be adjacent to a vertex with distance $n - 1$ from x. We can keep walking backwards along this path until we are at a point of distance 2 from x. Then, let a path of length 2 be v, u, w. Since G is 2 connected and v and w are not adjacent to one another, By theorem 5.7.4 there is a path from u to any other vertices not containing v and not containing w. Therefore, $G - v - w$ is connected.

If $G - x$ is not 2 connected, then $G - x$ has a bridge with 2 blocks both connected to x as well. Let $u = x$, and choose w and v adjacent to x in two different blocks of $G - x$. Since w and v are in seperate blocks of $G - x$ and both blocks are still 2 connected, v and w are not adjacent an $G - u - v$ is connected.

Given these vertices, color v and w with color 1, then use the greedy algorithm to color the rest. Since u has maximum degree and we are able to repeat a color, namely color 1 for v and w, we will end up with at most $\Delta(G)$ colors. ■

Note: Lovasz's proof from 1975 used as an outline.

Exercise 5.8.1 (2pt). Suppose G has n vertices and chromatic number k. Prove that G has at least $\binom{k}{2}$ edges.

Proof. Since G has chromatic number k, every color have an edge connecting every other color. This mapping can be represented by K_k which has $\binom{k}{2}$. G must have at least as many edges as K_k otherwise every color wouldn't connect to every other color as there wouldn't be enough edges to connect every color.

■

Exercise 5.8.3 (2pt). Show that $\chi(G-v)$ is either $\chi(G)$ or $\chi(G)-1$

Proof. Let G be a graph with n vertices and each vertex v_i has coloring c_i for where each color is not neccessairly distinct. Suppose we remove v_j either $c_j = c_i$ for some $i \neq j$, in which case $\chi(G-v_j) = \chi(G)$, or $c_j \neq c_i$ for some $i \neq j$, in which case $\chi(G-v_j) = \chi(G)-1$.

■

Exercise 5.8.4 (2pt). Prove theorem 5.8.10 without assuming any particular properties of the order $v_1, \ldots, v_n$.

Theorem 5.8.10.

In any graph G, $\chi \leq \Delta + 1$.

Proof. Consider a graph with vertices $v_1, \ldots, v_n$. To color any vertex v_i, check all neighbors of v_i. Since v_i has at most Δ neighboors and we have up to $\Delta + 1$ colors, we will always be able to find a color for v_i by the pigeonhole principle.

■

Exercise 5.9.1 (2pt). Show that the leading coefficient of $P(G)$ is 1.

Proof. Induction of number of edges n in G. When $n = 0$, $P_G(k) = k^{V(G)}$, which has a leading coefficient of 1. Assume the leading coefficient for a graph with n edges be 1. Consider a graph G with $n+1$ edges. Then

$$P_G(k) = P_{G-e}(k) - P_{G/e}.$$

Since $P_{G-e}(k)$ has n edges, by our hypothesis it has a leading coefficient of 1. Also, since $V(G) = V(G/e) + 1$, P_{G-e} and by theorem 5.9.3 P_G is a polynomial of degree $V(G)$,

$$P_G(k) = k^{V(G)} + \ldots - (k^{V(G)-1} + \ldots)$$

and thus as a leading coefficient of 1.

■

Exercise 5.9.3 (2pt). Show that the constant term of $P_G(k)$ is 0. Show that the coefficient of k is $P_G(k)$ in non-zero if and only if G is connected.

Proof. If we let $k = 0$, we then have 0 colors to color G with, so we should expect $P_G(0) = 0$. This can only happen when the constant coefficient is 0.

Suppose G is not connected. Then there are at least 2 different blocks of G which we can color independantly. In either block the first vertex we color can have k colors, so

$$P_G(k) = k^2 Q(k)$$

for some polynomial $Q(k)$. Therefore k in $P_G(k)$ has a coefficient of 0.

■