Exercise 11e. Discuss the analyticicity of each of the following functions

(e) $x^2 + y^2 + y - 2 + ix$. Let $z = x + iy$

$$\begin{aligned}
x^2 + y^2 + y - 2 + ix &= \frac{(z+\overline{z})^2}{4} + \frac{(z-\overline{z})^2}{-4} + \frac{z-\overline{z}}{2i} - 2 + i\left(\frac{z+\overline{z}}{2}\right) \\
&= \frac{(z+\overline{z})^2 - (z-\overline{z})^2 - 2i(z-\overline{z}) - 8 + 2i(z+\overline{z})}{4} \\
&= \frac{4z\overline{z} - 2i(z - \overline{z} - (z+\overline{z})) - 8}{4} \\
&= z\overline{z} + i\overline{z} - 2 \\
&= \overline{z}\,(z+i) - 2
\end{aligned}$$

Since we are multiplying by $\overline{z}$, this function is not analytical.

(f) $\left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right)$. Let $z = x + iy$

$$\begin{aligned}
\left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right) &= \left(\frac{z+\overline{z}}{2} + \frac{z+\overline{z}}{2z\overline{z}}\right) + i\left(\frac{z-\overline{z}}{2i} - \frac{z-\overline{z}}{2iz\overline{z}}\right) \\
&= \frac{(z\overline{z})(z+\overline{z}) + z + \overline{z}}{2z\overline{z}} + \frac{(z\overline{z})(z-\overline{z}) - z + \overline{z}}{2z\overline{z}} \\
&= \frac{(z\overline{z})(2z) + 2\overline{z}}{2z\overline{z}} \\
&= z + \frac{1}{z}
\end{aligned}$$

This function is analytical, since we can write it in terms of z alone.

Theorem Quotient Rule.

$$\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)f^{'}(z) - f(z)g^{'}(z)}{g(z)^2}$$

For differentiable functions $f(z)$ and $g(z)$ where $g(z) \neq 0$

Proof.

$$\begin{aligned}
\lim_{z\to z_0} \frac{(f(z)/g(z)) - (f(z_0)/g(z_0))}{z - z_0} &= \lim_{z\to z_0} \frac{f(z)g(z_0) - f(z_0)g(z)}{(z - z_0)} \cdot \lim_{z\to z_0} \frac{1}{g(z)g(z_0)} \\
&= \lim_{z\to z_0} \frac{f(z)g(z_0) - f(z)g(z) - f(z_0)g(z) + f(z)g(z)}{z - z_0} \cdot \frac{1}{g(z)^2} \\
&= \lim_{z\to z_0} \frac{f(z)(g(z_0) - g(z)) - g(z)(f(z_0) - f(z))}{g(z)^2(z - z_0)} \\
&= \lim_{z\to z_0} \frac{-f(z)(g(z) - g(z_0)) + g(z)(f(z) - f(z_0))}{g(z)^2(z - z_0)} \\
&= \lim_{z\to z_0} \frac{g(z)(f(z) - f(z_0)) - f(z)(g(z) - g(z_0))}{g(z)^2(z - z_0)} \\
&= \frac{g(z)f^{'}(z) - f(z)g^{'}(z)}{g(z)^2}
\end{aligned}$$

Note: we can evalute $\lim_{z\to z_0} \frac{1}{g(z)g(z_0)} = \frac{1}{g(z)^2}$ since we know $g(z)$ is continuous.

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