Exercise 5.7.3 (2pt). Suppose G is simple with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$, and for $k \leq n - d_n - 1$, $d_k \geq k$. Show G is connected

Proof. Suppose G is not connected. Consider a connected subgraph of G containing v_n and all its neighbors, which has $d_n + 1$ vertices. Now Consider another connected subgraph of G, G' not connected to v_n with m vertices, then $m \leq n - (d_n + 1)$ so $d_m \geq m$. Let the largest degree of our subgraph with m vertices be d_i, so $d_i \leq m - 1$. However then

$$m \leq d_m \leq d_i \leq m - 1$$

and we have a contradiction. ■

Exercise 5.7.4 (2pt). Recall that a graph is k-regular if all the vertices have degree k. What is the smallest integer k that makes this true:

If G is simple, has n vertices, $m \geq k$, and G is m-regular, then G is connected.

If n is even, then the smallest k is $n/2$. If n is odd, then the smallest k is $(n-1)/2$. This is because any less and we can split G into 2 graphs of $K_{n/2}$ if n is odd, and $K_{(n+1)/2} + K_{(n-1)/2}$ if n is even, which has is $n/2 - 1$ regular in the even case and $(n-1)/2 - 1$ regular in the odd case.

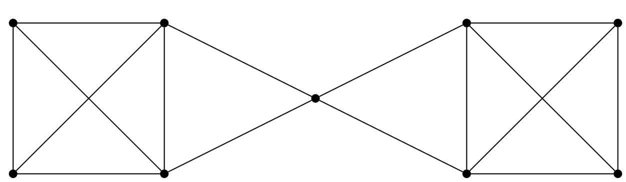
Exercise 5.7.5 (2pt). Suppose G has at least one edge. Show that G is 2-connected if and only if for all vertices v and edges e there is a cycle containing v and e.

Proof. Let G be 2-connected. Suppose there is no cycle containing v and e. Then, v and e must be connected by a path otherwisee containing a bridge, otherwise v and e would not be connected or v and e would form a cycle. However, since the path contains a bridge, then G is not 2 connected and we have a contradiction.

For all vertices v and e in a graph G, let there be a cycle containing v and e. Then in order to disconnect any 2 vertices we would need to remove at least 2 vertices, one for either direction along the cycle, so G is 2-connected. ■

Exercise 5.7.6 (2pt). Find a simple graph with $\kappa(G) < \lambda(G) < \delta(G)$.

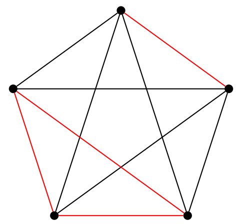
Consider the following graph G:



$$\kappa(G) = 1 < \lambda(G) = 2 < \delta(G) = 3$$

Exercise 5.4.2 (2pt). What is the smallest number of edges that can be removed from K_5 to create a bipartite graph?

We can turn K_5 into $K_{2,3}$ by removing the four red edges:



This is the smallest number of edges we can remove since $K_{2,3}$ has 6 total edges and our only other option $K_{1,4}$ has only 4 edges, so we'd have to remove more edges.

Exercise 5.5.1 (2pt). Suppose that G is a connected graph, and that every spannig tree contains edge e. Show that e is a bridge.

Proof. Suppose e is not a bridge. Then $G - e$ is a connected graph, and it has a spanning tree. However, this spanning tree would also be a spanning tree of G which does not contain e, so we have a contradiction. ■