Exercise 17. Find $\max_{|z|\leq 1} |(z-1)(z+1/2)|$.

By the triangle inequality we get

$$\begin{aligned} |(z-1)(z+1/2)| &= \left|z^2 - z/2 - 1/2\right| \\ &\leq |z^2| + |z/2| + |1/2| \leq 2. \end{aligned}$$

However, because we are taking the modulus of a product of analytic functions on the disc $|z| = 1$, the maximun must occur on the boundary of the disc, so let $z = e^{it}$, $0 \leq t \leq 2\pi$. Then

$$\begin{aligned} \left|z^2 - z/2 - 1/2\right|^2 &= \left(e^{i2t} - \frac{e^{it}}{2} - \frac{1}{2}\right)\left(e^{-i2t} - \frac{e^{-it}}{2} - \frac{1}{2}\right) \\ &= 1 - \frac{e^{it}}{2} - \frac{e^{2it}}{2} - \frac{e^{-it}}{2} + \frac{1}{4} + \frac{e^{it}}{4} - \frac{e^{-2it}}{2} + \frac{e^{-it}}{4} + \frac{1}{4} \\ &= \frac{3}{2} - \frac{e^{it} + e^{-it}}{2} - \frac{e^{2it} + e^{-2it}}{2} + \frac{e^{it} + e^{-it}}{4} \\ &= \frac{3}{2} - \cos(t) - \cos(2t) + \cos(t)/2 \\ &= \frac{3}{2} - \cos(t)/2 - \cos(2t). \end{aligned}$$

We shall use calculus to find the maximun of this function. First we find the critical points.

$$\begin{aligned} \frac{d}{dt}\left(\frac{3}{2} - \cos(t)/2 - \cos(2t)\right) &= \sin(t)/2 + 2\sin(2t) = 0 \\ \sin(t)(1/2 + 4\cos(t)) &= 0 \\ \sin(t) &= 0 \qquad\qquad 4\cos(t) = -1/2 \\ t &= \arcsin(0) \qquad \cos(t) = -1/8 \\ t &= 0, \pi, 2\pi \qquad\quad t = \arccos(-1/8) \end{aligned}$$

To determine if a critical point is a max or min, we can check the sign of the second derivative at that point.

$$\begin{aligned} \frac{d^2}{dt^2}\left(\sin(t)/2 + 2\sin(2t)\right) &= \cos(t)/2 + 4\cos(2t) \\ \cos(\arcsin(0))/2 + 4\cos(2\arcsin(0)) &= 9/2 > 0 \\ \cos(\arccos(-1/8)) + 4\cos(2\arccos(-1/8)) &= -1/8 - 4c < 0 \end{aligned}$$

when $t = \arcsin(0)$, the second derivative is positive so we have a minimum, and when $t = \arccos(-1/8)$ the second derivative is negative so we have a maximun. Thus our max is

$$\sqrt{\frac{3}{2} + \frac{1}{16} - \cos(2\arccos(-1/8))} \approx 1.591$$

and occurs at $z = e^{i\arccos(-1/8)}$ for $0 \leq \arccos(-1/8) \leq 2\pi$ of which there are 2 values:

$$z \approx e^{i1.696},\ e^{i4.587}$$