Problem 1. This problem refers to Example 4 in Section 4.2. Explain why

$$\int_\Gamma \bar{z}^2 dz = \int_\Gamma (z^2 - 4xyi)dz = -4i\int_\Gamma xydz$$

and evaluate the last integral using the parametrization given by the author.

Let $z = x + iy$

$$\begin{aligned}
\int_\Gamma \bar{z}^2 dz &= \int_\Gamma (x - iy)^2 dz \\
&= \int_\Gamma (x^2 - y^2 - 2xyi)dz \\
&= \int_\Gamma [(x+iy)^2 - 4xyi]dz \\
&= \int_\Gamma (z^2 - 4xyi)dz \\
&= \int_\Gamma z^2 dz - 4i\int_\Gamma xydz
\end{aligned}$$

Since z^2 is continous on $\mathbb{C}$, has an antiderivative throughout $\mathbb{C}$, and Γ is a closed loop, by corollary 2 of Theorem 6 we know the our first integral is 0, leaving us with the desired result.

$$= -4i\int_\Gamma xydz$$

The given parametrization for the 3 line segments of Γ are

$$\begin{aligned}
\gamma_1 : &\quad z_1(t) = x_1(t) + iy_1(t) = t && (0 \le t \le 2), \\
\gamma_2 : &\quad z_2(t) = x_2(t) + iy_2(t) = 2 + ti && (0 \le t \le 2), \\
\gamma_3 : &\quad z_3(t) = x_3(t) + iy_3(t) = -t(1+i) && (-2 \le t \le 0),
\end{aligned}$$

so we have

$$\begin{aligned}
-4i\int_\Gamma xydz &= -4i\left(\int_{\gamma_1} xydz + \int_{\gamma_2} xydz + \int_{\gamma_3} xydz\right) \\
&= -4i\left(\int_0^2 x_1(t)y_1(t)z_1'(t)dt + \int_0^2 x_2(t)y_2(t)z_2'(t)dt + \int_{-2}^0 x_3(t)y_3(t)z_3'(t)dt\right) \\
&= -4i\left(0 + \int_0^2 2tidt + \int_{-2}^0 (-t)^2(-1-i)dt\right) \\
&= -4i\left(2i\int_0^2 tdt - (1+i)\int_{-2}^0 t^2 dt\right) \\
&= -4i\left(2i\left[\frac{t^2}{2}\right]_0^2 - (1+i)\left[\frac{t^3}{3}\right]_{-2}^0\right) \\
&= -4i\left(4i - (1+i)\cdot\frac{8}{3}\right) \\
&= 16 + \frac{32i}{3} - \frac{32}{3} \\
&= \frac{16}{3} + \frac{32i}{3}
\end{aligned}$$

Problem 2. Using the result from Exercise 6 in Section 4.3, we know that $\int_C \frac{1}{z-1} dz = 2\pi i$, where C is the circle $|z| = 2$ traversed once in the positive direction. By considering real and imaginary parts, use this result to evaluate

$$\int_0^{2\pi} \frac{2 - \cos t}{5 - 4\cos t} dt$$

Let $z = 2e^{it}$ for $0 \leq t \leq 2\pi$, then $dz = 2ie^{it} dt$ and

$$\begin{aligned}
\int_C \frac{1}{z-1} dz &= \int_0^{2\pi} \frac{2ie^{it}}{2e^{it} - 1} dt \\
&= \int_0^{2\pi} \frac{2ie^{it}}{2e^{it} - 1} \cdot \frac{2e^{-it} - 1}{2e^{-it} - 1} dt \\
&= \int_0^{2\pi} \frac{4i - 2ie^{it}}{5 - 2e^{it} - 2e^{-it}} dt \\
&= \int_0^{2\pi} \frac{4i - 2i(\cos(t) + i\sin(t))}{5 - 4\cos(t)} \\
&= \int_0^{2\pi} \frac{2\sin(t)}{5 - 4\cos t} dt + 2i \int_0^{2\pi} \frac{2 - \cos t}{5 - 4\cos t} dt = 2\pi i \\
\implies \int_0^{2\pi} \frac{2 - \cos t}{5 - 4\cos t} dt = \pi
\end{aligned}$$

Problem 3. Use partial fractions and the result of Exercise 4.3.6 to evaluate

$$\int_C \frac{3z + 1}{(z-1)(z^2+1)} dz$$

where C is the circle $|z - i| = \sqrt{3}$ traversed once in the positive direction.

First we find the partial fractions of our function.

$$\frac{3z+1}{(z-1)(z+i)(z-i)} = \frac{A}{z-1} + \frac{B}{z+i} + \frac{C}{z-i}$$

$$\begin{aligned}
A &= \frac{3(1) + 1}{1^2 + 1} = \frac{4}{2} = 2 \\
B &= \frac{3(-i) + 1}{(-i-1)(-i-i)} = \frac{1 - 3i}{-2 + 2i} \cdot \frac{-2 - 2i}{-2 - 2i} = \frac{-8 + 4i}{8} = \frac{-2 + i}{2} \\
C &= \frac{3(i) + 1}{(i-1)(i+i)} = \frac{1 + 3i}{-2 - 2i} \cdot \frac{-2 + 2i}{-2 + 2i} = \frac{-8 - 4i}{8} = \frac{-2 - i}{2}
\end{aligned}$$

Since $|1 - i| = \sqrt{2} < \sqrt{3}$ and $|i - i| = 0 < \sqrt{3}$, both $z = 1$ and $z = i$ are in C, however $|-i-i| = 2 > \sqrt{3}$, so $z = -i$ is not in C. Therefore we only integrate around the points $z = 1$ and $z = i$. Let C_0 be a small circle around $z = 1$ and C_1 be the circle around $z = i$. Now we can evaluate our integral using our partial fraction decomposition and integrating around our small circles in the positive direction (which Theorem 13 allows us to do).

$$\begin{aligned}
\int_C \frac{3z+1}{(z-1)(z^2+1)}dz &= \int_C \frac{3z+1}{(z-1)(z+i)(z-i)}dz \\
&= \int_{C_0} \frac{2}{z-1}dz + \int_C \frac{-2+i}{2(z+i)}dz + \int_{C_1} \frac{-2-i}{2(z-i)}dz \\
&= 4\pi i + 0 + \pi i(-2-i) \\
&= \pi + 2\pi i \\
&= \pi(1+2i)
\end{aligned}$$