Homework Set 1.4

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Exercise 21

Prove that if n is a positive integer, then

$$\left|\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right| \leq n \quad (\theta \neq 0, \pm 2\pi, \pm 4\pi, \ldots)$$

Proof. If $z = e^{i\theta}$ ($|z| = 1$), then

$$\begin{aligned}
\left|\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right| &= \left|\frac{z^{n/2} - z^{-n/2}}{z^{1/2} - z^{-1/2}}\right| \\
&= \left|\frac{z^{n/2} - z^{-n/2}}{z^{1/2} - z^{-1/2}}\right| \cdot \left|\frac{z^{n/2}}{z^{1/2}}\right| \\
&= \left|\frac{z^n - 1}{z - 1}\right| \\
&= |1 + z + z^2 + \ldots + z^{n-1}| \\
&\leq 1 + |z| + |z^2| + \ldots + |z^{n-1}| \\
&= n
\end{aligned}$$

□

Exercise 4b

Write in the form of $re^{i\theta}$

$$\frac{2+2i}{-\sqrt{3}+i}$$

$$2+2i = 2\sqrt{2}\text{cis}(\pi/4) \qquad\qquad -\sqrt{3}+i = 2\text{cis}(5\pi/6)$$

$$\begin{aligned}\frac{2+2i}{-\sqrt{3}+i} &= \frac{2\sqrt{2}\text{cis}(\pi/4)}{2\text{cis}(5\pi/6)} \\ &= \sqrt{2}\text{cis}(-7\pi/12) \\ &= \sqrt{2}e^{-i7\pi/12}\end{aligned}$$