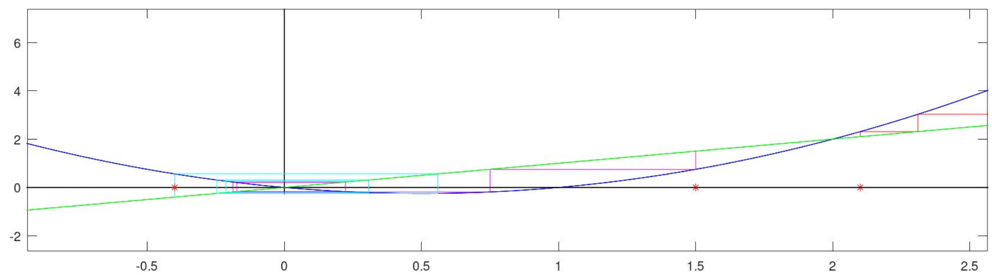
Chapter 4

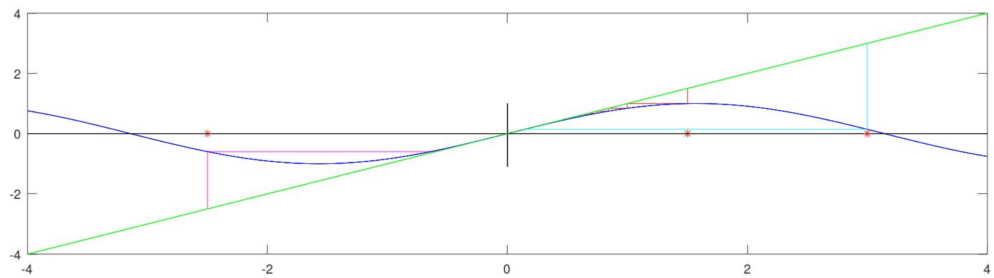
Exercise 1. Use graphical analysis to describe the fate of all orbits for each of the following functions. Use different colors for orbits that behave differently

f. $F(x) = x^2 - x$ (fixed points at $x_0 = 0$ and $x_0 = 2$)



If $x_0 < -1$, $F(x)$ goes to infinity. If $x_0 = -1$, $F(x)$ is eventually fixed. If $-1 < x_0 < 0$, $F(x)$ converges to 0. If $0 < x < 2$, then $F(x)$ converges to 0, and if $x > 2$, $F(x)$ goes to infinity.

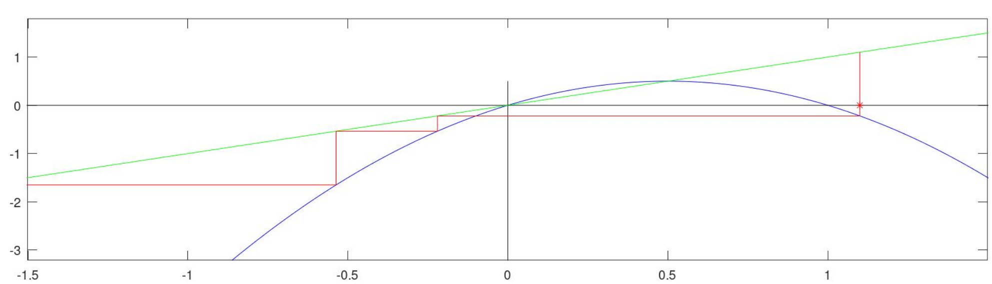
g. $F(x) = \sin x$ (fixed point at $x_0 = 0$)



If $x_0 < 0$, then $F(x)$ converges to 0, and if $x_0 > 0$, $F(x)$ also converges to 0.

Exercise 2. Use graphical analysis to find $\{x_0 \mid F^n(x_0) \to \infty\}$ for each of the following functions

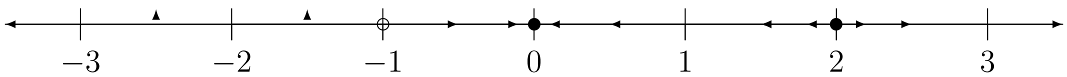
a. $F(x) = 2x(1 - x)$



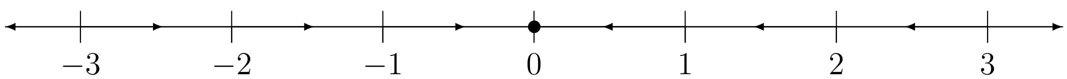
When $x_0 > 1$ or $x_0 > 1$, then $F^n(x_0) \to -\infty$. There is no x_0 s.t. $F^n(x_0) \to +\infty$.

Exercise 3. Sketch the phase portraits for each of the functions in exercise 1.

f. $F(x) = x^2 - x$



g. $F(x) = \sin x$



Chapter 5

Experiment: Rates of Convergence

For each function listed below, find the fixed poit p, $|F^{'}(p)|$, whether or not p is attracting or neutral, The number of iterations necessary for the orbit of 0.2 to "reach" p, (within 10^{-5})

a. $F(x) = x^2 + 0.25$
$p = 1/2$, $|F^{'}(p)| = 1$, p is neutral, 99987 iterations.

b. $F(x) = x^2$
$p = 0$, $|F^{'}(p)| = 0$, p is attracting, 3 iterations.

c. $F(x) = x^2 - 0.24$
$p = -1/5$, $|F^{'}(p)| = 0.4$, p is attracting, 1 iteration.

d. $F(x) = x^2 - 0.75$
$p = -1/2$, $|F^{'}(p)| = 1$, p is neutral, $>$ 500000 iterations.

When $|F^{'}(p)| = 1$, there is a really slow rate of convergence, and when $|F^{'}(p)|$ is close to 0, there is a much faster convergence. (c) had the fastest convergence, even though it didn't have the smallest $|F^{'}(p)|$. (d) had the slowest convergence by far, and p was a neutral point in that case.

Exercise 1. For each of the following functions, find all the fixed points and classify them as attracting, repelling, or neutral.

a. $F(x) = x^2 - x/2$

$$\begin{aligned} x^2 - x/2 &= x \\ x^2 - 3x/2 &= 0 \\ x(x - 3/2) &= 0 \end{aligned}$$

Fixed points occur at $x = 0$ and $x = 3/2$.

$$\begin{aligned} |F^{'}(0)| &= |2(0) - 1/2| \\ &= |-1/2| \\ &= 1/2 \end{aligned} \qquad \begin{aligned} |F^{'}(3/2)| &= |2(3/2) - 1/2| \\ &= |3 - 1/2| \\ &= 5/2 \end{aligned}$$

The fixed point $x = 0$ is attracting and the fixed point $x = 5/2$ is repelling.

c. $F(x) = 3x(1 - x)$

$$\begin{aligned} 3x - 3x^2 &= x \\ -3x^2 + 2x &= 0 \\ x(-3x + 2) &= 0 \end{aligned}$$

Fixed points occur at $x = 0$ and $x = 2/3$.

$$\begin{aligned} |F^{'}(0)| &= |3 - 6(0)| \\ &= 3 \end{aligned} \qquad \begin{aligned} |F^{'}(2/3)| &= |3 - 6(2/3)| \\ &= |3 - 4| \\ &= 1 \end{aligned}$$

The fixed point $x = 0$ is repelling and the fixed point $x = 2/3$ is neutral.

Exercise 2. For each of the following functions, zero lies on a periodic orbit. Classify this orbit as attracting, repelling, or neutral.

d. $F(x) = |x - 2| - 1$

$$\{0, 1, 0, 1, \dots\}$$

0 lies on period 2 orbit.

$$\begin{aligned} |(F^2)^{'}(0)| &= |F^{'}(1)| \cdot |F^{'}(0)| \\ &= 1 \end{aligned}$$

This orbit is neutral.

Exercise 3. Suppose x_0 lies on a cycle of prime period n for the doubling function D. Evaluate $(D^n)^{'}(x_0)$. Is this cycle attracting or repelling?

Let $\{x_0, x_1, \dots, x_{n-1}\}$ be the cycle containing x_0.

$$\begin{aligned} (D^n)^{'}(x_0) &= D^{'}(x_{n-1}) \cdot D^{'}(x_{n-2}) \cdot \dots \cdot D^{'}(x_0) \\ &= 2^n \end{aligned}$$

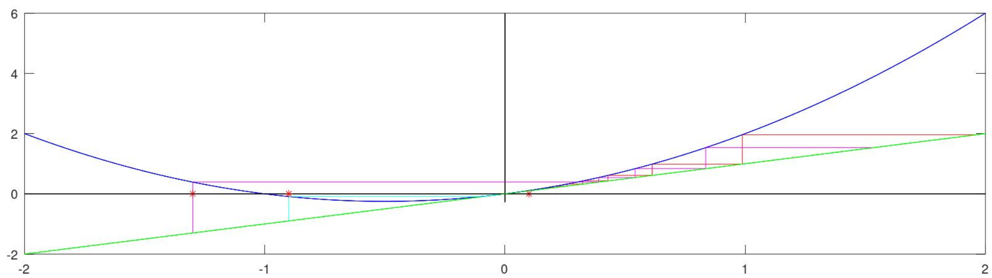
This cycle is repelling.

Exercise 4. Each of the following functions has a neutral fixed point. Find this fixed point and, using graphical analysis with an accurate graph, determine if it is weakly attracting, weakly repelling, or neither.

a. $F(x) = x + x^2$

$$\begin{aligned} x + x^2 &= x \\ x^2 &= 0 \\ x &= 0 \end{aligned}$$

$x = 0$ is our neutral fixed point.



$x_0 = 0$ is weakly attracting on the interval $-1 < x < 0$, and weakly repelling everywher else besides the fixed point $x = 0$ and $x = -1$ which is eventually fixed.