Exercise 2. What is the order of the pole of

$$f(z) = \frac{1}{(2\cos z - 2 + z^2)^2}$$

at $z = 0$?

Consider

$$\frac{1}{f(z)} = \left(2\cos z - 2 + z^2\right)^2.$$

Since

$$\cos(z) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \dots,$$

we have

$$\begin{aligned}\frac{1}{f(z)} &= \left(2\left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \dots\right) - 2 + z^2\right)^2 \\ &= \left(\frac{1}{12}z^4 - \frac{2}{6!}z^6 + \dots\right)^2 \\ &= z^8\left(\frac{1}{12^2} - \frac{2^2}{(6!)^2}z^4 + \dots\right).\end{aligned}$$

Let $g(z) = \dfrac{1}{1/12^2 - 2^2/(6!)^2 + \dots}$, which as analytic around $z = 0$. Then

$$f(z) = \frac{g(z)}{z^8}$$

and since $g(0) \neq 0$, by Lemma 7 $z = 0$ is a pole of order 8.

Exercise 12. Prove that if $f(z)$ has a pole of order m at z_0, then $g(z) := f'(z)/f(z)$ has a simple pole at z_0. What is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion for $g(z)$?

Proof. Becuase $f(z)$ has a pole of order m at z_0, by Lemma 7 we can write

$$f(z) = \frac{h(z)}{(z - z_0)^m}$$

where $h(z)$ is analytic on the punctured neighborhood of z_0 and $h(z_0) \neq 0$. Then

$$\begin{aligned}g(z) = \frac{f'(z)}{f(z)} &= \frac{(z - z_0)^m h'(z) - m(z - z_0)^{m-1}h(z)}{(z - z_0)^{2m}} \cdot \frac{(z - z_0)^m}{h(z)} \\ &= \frac{(z - z_0)^{m-1}((z - z_0)h'(z) - mh(z))}{h(z)(z - z_0)^m} \\ &= \frac{(z - z_0)h'(z) - mh(z)}{h(z)(z - z_0)}.\end{aligned}$$

Let

$$l(z) = \frac{(z - z_0)h'(z) - mh(z)}{h(z)}$$

so

$$g(z) = \frac{l(z)}{z - z_0}.$$

Clearly $l(z)$ is analytic around z_0, and $l(z_0) = -m \neq 0$. Therefore z_0 is a simple pole of $g(z)$ and the coefficient of $(z - z_0)^{-1}$ in the laurent expansion for $g(z)$ is $-m$.

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