Exercise 4. Using techniques of residues, verify the following integral formula.

$$\int_0^\infty \frac{\sin(2x)}{x(x^2+1)^2}dx = \pi\left(\frac{1}{2}-\frac{1}{e^2}\right)$$

Since $\sin(x)$ is even, we have

$$\int_0^\infty \frac{\sin(2x)}{x(x^2+1)^2}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin(2x)}{x(x^2+1)^2}dx$$

which is the imaginary part of

$$\int_{-\infty}^\infty \frac{e^{2ix}}{x(x^2+1)^2}dx.$$

Let $f(z) = \dfrac{e^{2iz}}{z(z^2+1)^2}$. Since we have a singularity at $z=0$, we have

$$\left(\int_{-\rho}^{r} + \int_{S_r} + \int_r^\rho + \int_{C_p^+}\right) f(z)dz = 2\pi i \text{Res}(f;i)$$

By Jordan's lemma we have

$$\lim_{\rho\to\infty}\int_{C_p^+} f(z)dz = 0$$

and

$$\lim_{r\to 0^+}\int_{S_r} f(z)dz = -i\pi\text{Res}(f;0)$$

so

$$\int_{-\infty}^\infty f(x)dx = 2\pi i\text{Res}(f;i) + \pi i\text{Res}(f;0) - 0.$$

Now we can find the residues. $\text{Res}(f;0) = \lim_{z\to 0} f(z) = 1$, and

$$\begin{aligned}\text{Res}(f;i) &= \lim_{z\to i}\frac{d}{dz}\left(\frac{e^{2iz}}{z(z+i)^2}\right)\\ &= \frac{-4i(2ie^{-2}) - e^{-2}(-8)}{16}\\ &= \frac{-1}{e^2}.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^\infty \frac{\sin(2x)}{x(x^2+1)^2}dx &= \frac{1}{2}\text{Im}\left(\frac{-2\pi i}{e^2} + \pi i\right)\\ &= \pi\left(\frac{1}{2}-\frac{1}{e^2}\right)\end{aligned}$$

Exercise 5. Using techniques of residues, verify the following integral formula.

$$\int_0^\infty \frac{\cos(x)-1}{x^2}dx = \frac{\pi}{2}$$

since $\cos(x) - 1$ is and even function we have

$$\int_0^\infty \frac{\cos(x) - 1}{x^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos(x) - 1}{x^2} dx$$

which is the real part of

$$\int_{-\infty}^\infty \frac{e^{ix} - 1}{x^2} dx.$$

Let $f(z) = \dfrac{e^{iz} - 1}{z^2}$. Since we have a singularity at $z = 0$ we have

$$\left(\int_{-\rho}^r + \int_{S_r} + \int_r^\rho + \int_{C_p^+} \right) f(z) dz = 0$$

By Jordan's lemma we have

$$\lim_{\rho \to \infty} \int_{C_p^+} f(z) dz = 0$$

and

$$\lim_{r \to 0^+} \int_{S_r} f(z) dz = -i\pi \mathrm{Res}(f; 0)$$

so

$$\int_{-\infty}^\infty f(x) dx = \pi i \mathrm{Res}(f; 0) - 0.$$

Now we can find the residues.

$$\mathrm{Res}(f; 0) = \lim_{z \to 0} \frac{d}{dz} (e^{iz} - 1) = i$$

Thus

$$\int_0^\infty \frac{\cos(x) - 1}{x^2} = \frac{1}{2} \mathrm{Re} \left(-\pi i(i) \right) = \frac{\pi}{2}$$