Exercise 6.3.2. Verify the following integral formula by comparing it to the improper integral in Example 2.

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2} dx = \frac{\pi}{6}$$

Consider the function

$$g(x) = \frac{x^2}{(x^2+1)^2}.$$

From Example 2 we know p.v.$\int_{\infty}^{\infty} g(x)dx = \pi/2$. Let $x = 3u$, then

$$\begin{aligned}
\frac{x^2}{(x^2+9)^2} &= \frac{9u^2}{(9u^2+1)^2} \\
&= \frac{9u^2}{9(3u^2+1/3)^2} \\
&= \frac{u^2}{(3u^2+1/3)^2} \cdot \frac{(1/3)^2}{(1/3)^2} \\
&= \frac{1}{9} \cdot \frac{u^2}{(u^2+1)} \\
&= \frac{1}{9} g(u)
\end{aligned}$$

and $dx = 3du$. Then we have

$$\begin{aligned}
\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^2} dx &= \frac{1}{3} \text{p.v.} \int_{-\infty}^{\infty} g(u) du \\
&= \frac{1}{3} \cdot \frac{\pi}{2} \\
&= \frac{\pi}{6}
\end{aligned}$$

Exercise 6.4.3. Using the method of residue, verify the integral formula.

$$\text{p.v.} \int_{0}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dx = \frac{\pi}{2e}$$

Since $\cos(x) = \cos(-x)$, we have

$$\text{p.v.} \int_{0}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dx = \text{p.v.} \int_{-\infty}^{0} \frac{\cos(x)}{(x^2+1)^2} dx$$

and

$$\text{p.v.} \int_{0}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dx = \frac{1}{2} \left(\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dx \right)$$

Now we can break up the integral as follows,

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dz = \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{2(x^2+1)^2} dx + \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{-ix}}{2(x^2+1)^2} dx.$$

Let $f_1(z) = \dfrac{e^{iz}}{2(z^2+1)^2}$. Then

$$\int_{\Gamma_\rho} f_1(z)dz = \int_{-\rho}^{\rho} f_1(x)dx + \int_{C_\rho^+} f_1(z)dz.$$

We encounter singularities at $z = \pm i$, and because

$$|f_1(z)| = |f_1(x+iy)| = \frac{|e^{ix} \cdot e^{-y}|}{2|z^2+1|^2} = \frac{e^{-y}}{2|z^2+1|^2}$$

we have in the upper half plane ($y \geq 0$)

$$|f_1(z)| \leq \frac{1}{2|z^2+1|^2}.$$

Thus for any $\rho > 1$, the integral over C_ρ^+ is bounded by

$$\left| \int_{C_\rho^+} f_1(z)dz \right| \leq \frac{\pi\rho}{2(\rho^2-1)^2}$$

which goes to zero as $\rho \to \infty$. Since $+i$ is the only singularity in the upper half plane, we have for $\rho > 1$

$$\int_{-\rho}^{\rho} f_1(x)dz + \int_{-\rho}^{\rho} f_1(z)dz = 2\pi i \text{Res}(f_1; i).$$

Hence on taking the limit as $\rho \to \infty$ we get

$$\text{p.v.} \int_{\infty}^{\infty} \frac{e^{ix}}{2(x^2+1)^2} dx + 0 = 2\pi i \text{Res}(f_1; i).$$

But

$$\begin{aligned}
\text{Res}(f_1; i) &= \lim_{z \to i} \frac{d}{dz} [(z-i)^2 f_1(z)] \\
&= \lim_{z \to i} \frac{d}{dz} \frac{e^{iz}}{2(z+i)^2} \\
&= \frac{-8 \cdot (i/e) - (1/e) \cdot 8i}{64} \\
&= \frac{-i}{4e}
\end{aligned}$$

so

$$\text{p.v.} \int_{\infty}^{\infty} \frac{e^{ix}}{2(x^2+1)^2} dx = \frac{\pi}{2e}$$

Let $f_2(z) = \dfrac{e^{-iz}}{2(z^2+1)^2}$, whose singularities coincide with f_1. Then

$$\left| \int_{C_\rho^-} f_2(z)dz \right| \leq \frac{\pi\rho}{2(\rho^2-1)^2}$$

is also satisfied by the same argument for f_1. However, since we are moving from $\rho \to -\rho$, we are going in the negative direction so

$$\int_{-\rho}^{\rho} f_2(x)dz + \int_{-\rho}^{\rho} f_2(z)dz = -2\pi i \text{Res}(f_2; -i).$$

and as $\rho \to \infty$ we get

$$\begin{aligned}
\text{p.v.} \int_{\infty}^{\infty} \frac{e^{-ix}}{2(x^2+1)^2} dx + 0 &= -2\pi i \text{Res}(f_2; -i). \\
&= -2\pi i \lim_{z \to -i} \frac{d}{dz}[(z+i)^2 f_2(z)] \\
&= -2\pi i \left(\frac{-8 \cdot (-i/e) - (1/e) \cdot (-8i)}{64} \right) \\
&= -2\pi i \left(\frac{i}{4e} \right) \\
&= \frac{\pi}{2e}
\end{aligned}$$

Therefore

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dx = \frac{\pi}{2e} + \frac{\pi}{2e} = \frac{\pi}{e}$$

and

$$\text{p.v.} \int_{0}^{\infty} \frac{\cos(x)}{(x^2+1)^2} dx = \frac{\pi}{2e}$$