Homework Set 1

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Exercise 1.2.2 (1pt)

A poker hand consists of five cards from a standard 52 card deck with four suits and thirteen values in each suit; the order of the cards in a hand is irrelevant. How many hands consist of 2 cards with one value and 3 cards of another value (a full house)? How many consist of 5 cards from the same suit (a flush)

In a deck of cards, there are $\binom{4}{2} = 6$ ways to make a pair and this permutes over the 13 different values. There $\binom{4}{3} = 4$ ways to make 3 of a kind, but since we need a different value then from the pair, this permutes over 12. Thus there are $13 * 6 * 12 * 4$ ways to make a full house.

There are $\binom{13}{5}$ ways to make a flush in a suit, and with 4 different suits this makes $4\binom{13}{5}$ total combinations for a flush.

Exercise 1.2.3 (3pt)

Six men and six women are to be seated around a table, with men and women alternating. The chairs don't matter, only who is next to whom, but right and left are different. How many seating arrangements are possible?

Once we place either a man/woman in a chair, it fixes the 2 genders in an alternating pattern. Since we can rotate the chairs (as the actual chairs don't matter), there is only one way to do this. Say we originally place a man, then there are 5 men left and 6 woman. Now, since direction does matter, we are essentially ordering the 5 men and the 6 woman. There are 5! ways to order the men *and* 6! ways to order the woman, for a total of 5!6! possible combinations.

Exercise 1.2.4 (2pt)

Eight people are to be seated around a table; the chairs don't matter, only who is next to whom, but right and left are different. Two people, X and Y, cannot be seated next to each other. How many seating arrangements are possible?

Once we place X (in which there is one way since chairs don't matter), there are 5 more spots Y can be placed since Y can't be in X's spot, left of X, or right of X. Since the chairs don't matter, we are essentially ordering the other 6 people, in which there are 6! ways. The total number of seating arrangements is $6 * 5!$.

Exercise 1.2.6 (2pt)

Suppose that we want to place 8 non-attacking rooks on a chessboard. In how many ways can we do this if the 16 most 'northwest' squares must be empty? How about if only the 4 most 'northwest' squares must be empty?

In the first file, there is 4 choices for a rook, in the second 3, the third 2, and the fourth file is fixed. For the next 4 files, we then have 4 choices for the 5th rook, 3 for the 6th, 2 for the 7th, and the last rook is also fixed. This gives us $(4!)^2$ total combinations.

Similarly, in the 4 square case, there are 6 ways for the first rook, and 5 for the second, and then 6 for the third and so fourth for a total of $6 * 5 * 6!$ combinations.

Exercise 1.3.4 (1pt)

Use a combinatorial argument to prove that $\binom{k}{2} + \binom{n-k}{2} + k(n-k) = \binom{n}{2}$

Proof. We shall count the left and right side

RHS: choose a subset of 2 elements out of an n-element set

LHS: choose a subset of 2 elements from a k-element subset (A) of an n-element set, or choose a subset of 2 elements from the complement of the k-element subset (A^c), or choose 1 element from A and one element from A^c.

□

Exercise 1.3.8 (3pt)

Verify that $\binom{n+1}{2} + \binom{n}{2} = n^2$.

$$\begin{aligned}
\binom{n+1}{2} + \binom{n}{2} &= \frac{(n+1)!}{2!(n-1)!} + \frac{n!}{2!(n-2)!} \\
&= \frac{(n+1)!}{2(n-1)(n-2)!} + \frac{n!(n-1)}{2(n-1)(n-2)!} \\
&= \frac{(n+1)! + n!(n-1)}{2(n-1)!} \\
&= \frac{(n-1)!(n(n+1) + n(n-1))}{2(n-1)!} \\
&= \frac{n((n+1) + (n-1))}{2} \\
&= \frac{n(2n)}{2} \\
&= n^2
\end{aligned}$$

Find a simple expression for $\sum_{i=1}^{n} i^2$ using $\sum_{k=0}^{n} \binom{k}{i} = \binom{n+1}{i+1}$

$$\begin{aligned}
\sum_{i=0}^{n} i^2 &= \sum_{i=0}^{n} \binom{i+1}{2} + \sum_{i=0}^{n} \binom{i}{2} \\
&= \binom{n+2}{3} + \binom{n+1}{3} \\
&= \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!} \\
&= \frac{(n+2)!}{3!(n-1)!} + \frac{(n-1)(n+1)!}{3!(n-1)!} \\
&= \frac{(n+2)! + (n-1)(n+1)!}{3!(n-1)!} \\
&= \frac{n(n+2)(n+1) + n(n+1)(n-1)}{6} \\
&= \frac{n(n+1)((n+2) + (n-1))}{6} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$