Exercise 4.6.14. Prove the *minimum modulus principle:* Let f be analytic in a bounded domain D and continuous up to and including its boundary. Then *if f is nonzero in D*, the modulus $|f(z)|$ attains its minimum value on the boundary of D. Give an example to show why the italicized condition is essential.

Proof. Consider $g(z) = 1/f(z)$. Because $f(z)$ is analytic, continuous up to and including its boundary, and nonzero on D, then so is $g(z)$. By Theorem 24 $|g(z)|$ must have it's maximum at some point z_0 on the boundary of D. $|g(z)|$ acheives a maximum when $|f(z)|$ is a minimum, so $|f(z_0)|$ is a minimum of $|f(z)|$.

■

Consider the function $f(z) = z$ on $|z| \leq 1$. $|f(z)|$ acheives it's minimum at $z = 0$, which is not on the boundary. However, if we cut out the open circle $|z| < 0.5$ which contains all the zero's of $f(z)$, then $|f(z)|$ contains its minimum on the circle $|z| = 0.5$, which is a boundary of our domain.

Exercise 4.6.16. Show that $\max_{|z|\leq 1} |az^n + b| = |a| + |b|$

Proof. Since $az^n + b$ is entire, we know the maximum must occur on the boundary. Let $z = e^{i\theta}$, $a = re^{i\phi}$ and $b = \rho e^{i\psi}$. Then

$$\begin{aligned}
|az^n + b|^2 &= |re^{i(\phi+n\theta)} + \rho e^{i\psi}|^2 \\
&= (re^{i(\phi+n\theta)} + \rho e^{i\psi})(re^{-i(\phi+n\theta)} + \rho e^{-i\psi}) \\
&= r^2 + r\rho e^{i(\phi-\psi+n\theta)} + r\rho e^{-i(\phi-\psi+n\theta)} + \rho^2 \\
&= r^2 + 2r\rho\cos(\phi - \psi + n\theta) + \rho^2
\end{aligned}$$

Which has a max when $\cos(\phi - \psi + n\theta) = 1$. This happens at $\theta = \dfrac{\psi - \phi + 2\pi k}{n}$ for $k = 0, \pm 1, \pm 2, \ldots$. Therefore

$$\begin{aligned}
\max(r^2 + 2r\rho\cos(\phi - \psi + n\theta) + \rho^2) &= r^2 + 2r\rho + \rho^2 \\
&= (r + \rho)^2 \\
&= (|a| + |b|)^2
\end{aligned}$$

and we have

$$\max_{|z|\leq 1} |az^n + b|^2 = (|a| + |b|)^2 \implies \max_{|z|\leq 1} |az^n + b| = |a| + |b|$$

■

Exercise 5.2.18. Establish each of the following error estimates: for $|z| \leq 1$,

(a) $\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| \leq \frac{1}{(n+1)!} \cdot \left(1 + \frac{1}{n+1}\right)$

Since $e^z - \sum_{k=0}^{n} \frac{z^k}{k!}$ is analytic and continuous on and up to the disk $|z| \leq 1$, the maximum modulus must occor when $|z| = 1$.

$$\begin{aligned}
\left|e^z - \sum_{k=0}^{n} \frac{z^k}{k!}\right| &= \left|\sum_{k=0}^{\infty} \frac{z^k}{k!} - \sum_{k=0}^{n} \frac{z^k}{k!}\right| \\
&= \left|\sum_{k=n+1}^{\infty} \frac{z^k}{k!}\right| \\
&\leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} \\
&\leq \sum_{k=n+1}^{\infty} \frac{1}{k!} \\
&= \frac{1}{(n+1)!}\left(1 + \frac{1}{(n+2)} + \frac{1}{(n+2)(n+3)} + \dots\right) \\
&\leq \frac{1}{(n+1)!}\left(1 + \frac{1}{(n+2)} + \frac{1}{(n+2)^2} + \dots\right) \\
&= \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \left(\frac{1}{n+2}\right)^k \\
&= \frac{1}{(n+1)!}\left(\frac{1}{1 - \frac{1}{n+2}}\right) \\
&= \frac{1}{(n+1)!}\left(\frac{n+2}{n+1}\right) \\
&= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1}\right)
\end{aligned}$$

(b) $\left|\sin z - \sum_{k=0}^{n} \frac{(-1)^k z^{2k+1}}{(2k+1)!}\right| \leq \frac{1}{(2n+3)!}\left(\frac{4n^2+18n+20}{4n^2+18n+19}\right)$

Again, since our function $\sin z - \sum_{k=0}^{n} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$ is analytic and continuous on and up to the disk

$|z| \leq 1$, the maximum modulus must occurs when $|z| = 1$.

$$
\begin{aligned}
\left| \sin z - \sum_{k=0}^{n} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \right| &= \left| \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} - \sum_{k=0}^{n} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \right| \\
&= \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \right| \\
&\leq \sum_{k=n+1}^{\infty} \frac{|z|^{2k+1}}{(2k+1)!} \\
&\leq \sum_{k=n+1}^{\infty} \frac{1}{(2k+1)!} \\
&= \frac{1}{(2n+3)!} \left(1 + \frac{1}{(2n+5)(2n+4)} + \frac{1}{(2n+7)(2n+6)(2n+5)(2n+4)} + \dots \right) \\
&\leq \frac{1}{(2n+3)!} \left(1 + \frac{1}{(2n+5)(2n+4)} + \frac{1}{(2n+5)^2(2n+4)^2} + \dots \right) \\
&= \frac{1}{(2n+3)!} \sum_{k=0}^{\infty} \left(\frac{1}{(2n+5)(2n+4)} \right)^k \\
&= \frac{1}{(2n+3)!} \left(\frac{1}{1 - \frac{1}{(2n+5)(2n+4)}} \right) \\
&= \frac{1}{(2n+3)!} \left(\frac{(2n+5)(2n+5)}{(2n+5)(2n+4) - 1} \right) \\
&= \frac{1}{(2n+3)!} \left(\frac{4n^2 + 18n + 20}{4n^2 + 18n + 19} \right)
\end{aligned}
$$