Exercise 10. Show that the sequence of functions $F_n(z) = z^n/(z^n - 3^n)$, $n = 1, 2, \ldots$, converges to zero for $|z| < 3$ and to 1 for $|z| > 3$.

Proof. Consider that

$$\left|\frac{3}{z}\right| = \frac{3}{|z|}.$$

Also

$$F_n(z) = \frac{z^n}{z^n - 3^n} = \frac{z^n}{z^n\left(1 - \dfrac{3^n}{z^n}\right)} = \frac{1}{1 - \left(\dfrac{3}{z}\right)^n}$$

When $|z| < 3$, then $3/|z| = |3/z| > 1$, so

$$\lim_{n\to\infty} \frac{1}{1 - \left(\dfrac{3}{z}\right)^n} = 0$$

since the magnitide of the denominator gets abritrarily large. When $|z| > 3$, then $3/|z| = |3/z| < 1$, so

$$\lim_{n\to\infty} \frac{1}{1 - \left(\dfrac{3}{z}\right)^n} = \frac{1}{1-0} = 1$$

since the magnitide of $\left(\dfrac{3}{z}\right)^n$ goes to zero.

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