Exercise 2.3.12. Let $P_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$. Show by induction on n that

$$\frac{P_n'(z)}{P_n(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \ldots + \frac{1}{z - z_n}$$

[NOTE: $P'(z)/P(z)$ is called the *logarithmic derivation* of $P(z)$.]

Proof. Base case, $n = 1$. $P_1(z) = z - z_1$. Clearly then

$$\frac{P_1'(z)}{P_1(z)} = \frac{1}{z - z_1}.$$

Let $\dfrac{P_k(z)}{P_k(z)} = \dfrac{1}{z - z_1} + \dfrac{1}{z - z_2} + \ldots + \dfrac{z}{z - z_k}$. Then

$$\begin{aligned}
\frac{P_{k+1}'(z)}{P_{k+1}(z)} &= \frac{\frac{d}{dz}(P_k(z)(z - z_{k+1}))}{P_k(z)(z - z_{k+1})} \\
&= \frac{P_k'(z)(z - z_{k+1}) + P_k(z)}{P_k(z)(z - z_{k+1})} \\
&= \frac{P_k'(z)}{P_k(z)} + \frac{1}{z - z_{k+1}} \\
&= \frac{1}{z - z_1} + \frac{1}{z - z_2} + \ldots + \frac{1}{z - z_k} + \frac{1}{z - z_{k+1}}.
\end{aligned}$$

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Problem 2. Verify that if $f(z) = u(r, \theta) + iv(r, \theta)$, then

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r \cdot \frac{\partial v}{\partial r}$$

To show this we shall evaluate the following limit

$$f'(z_0) = \lim_{(r,\theta)\to(r_0,\theta_0)} \frac{(u(r, \theta) + iv(r, \theta)) - (u(r_0, \theta_0) + iv(r_0, \theta_0))}{re^{i\theta} - r_0 e^{i\theta_0}}$$

First we evaluate the limit as $r \to r_0$. Geometrically, this approaches z_0 on the ray given by the angle θ_0.

$$\begin{aligned}
f'(z_0) &= \lim_{(r,\theta_0)\to(r_0,\theta_0)} \frac{(u(r, \theta_0) + iv(r, \theta_0)) - (u(r_0, \theta_0) + iv(r_0, \theta_0))}{re^{i\theta_0} - r_0 e^{i\theta_0}} \\
&= \lim_{(r,\theta_0)\to(r_0,\theta_0)} \left[\left(\frac{u(r, \theta_0) - u(r_0, \theta_0)}{e^{i\theta_0}(r - r_0)} \right) + i \left(\frac{v(r, \theta_0) - v(r_0, \theta_0)}{e^{i\theta}(r - r_0)} \right) \right] \\
&= e^{-i\theta_0} \left[\lim_{(r,\theta_0)\to(r_0,\theta_0)} \left(\frac{u(r, \theta_0) - u(r_0, \theta_0)}{(r - r_0)} \right) + i \lim_{(r,\theta_0)\to(r_0,\theta_0)} \left(\frac{v(r, \theta_0) - v(r_0, \theta_0)}{(r - r_0)} \right) \right] \\
&= e^{-i\theta_0} \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right)
\end{aligned}$$

Now we evaluate the limit as $\theta \to \theta_0$. Geometrically, this approaches z_0 on the circle of radius r centered at the origin.

$$\begin{aligned}
f'(z_0) &= \lim_{(r_0,\theta)\to(r_0,\theta_0)} \frac{(u(r_0,\theta)+iv(r_0,\theta)) - (u(r_0,\theta_0)+iv(r_0,\theta_0))}{r_0e^{i\theta} - r_0e^{i\theta_0}} \\
&= \lim_{(r_0,\theta)\to(r_0,\theta_0)} \frac{(u(r_0,\theta)+iv(r_0,\theta)) - (u(r_0,\theta_0)+iv(r_0,\theta_0))}{r_0(e^{i\theta} - e^{i\theta_0})} \cdot \frac{\theta-\theta_0}{\theta-\theta_0} \\
&= \lim_{(r_0,\theta)\to(r_0,\theta_0)} \left[\left(\frac{u(r_0,\theta)-u(r_0,\theta_0)}{r_0(\theta-\theta_0)} + i\frac{v(r_0,\theta)-v(r_0,\theta_0)}{r_0(\theta-\theta_0)}\right)\left(\frac{\theta-\theta_0}{e^{i\theta}-e^{i\theta_0}}\right)\right] \\
&= \frac{1}{r_0}\left[\lim_{(r_0,\theta)\to(r_0,\theta_0)}\left(\frac{u(r_0,\theta)-u(r_0,\theta_0)}{(\theta-\theta_0)} + i\frac{v(r_0,\theta)-v(r_0,\theta_0)}{(\theta-\theta_0)}\right)\right]\lim_{(r_0,\theta_0)\to(r_0,\theta_0)}\left(\frac{e^{i\theta}-e^{i\theta_0}}{\theta-\theta_0}\right)^{-1} \\
&= \frac{1}{r_0}\left(\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right)\left(\lim_{(r_0,\theta)\to(r_0,\theta_0)}\frac{e^{i\theta}-e^{i\theta_0}}{\theta-\theta_0}\right)^{-1} \\
&= \frac{1}{r_0}\left(\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right)\left(\frac{de^{i\theta}}{d\theta}(\theta_0)\right)^{-1} \\
&= \frac{e^{-i\theta_0}}{ir_0}\left(\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right) \\
&= \frac{e^{-i\theta_0}}{r_0}\left(\frac{\partial v}{\partial\theta} - i\frac{\partial u}{\partial\theta}\right)
\end{aligned}$$

Setting both limits equal to each other then yields the desired results.

$$\begin{aligned}
e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) &= \frac{e^{-i\theta}}{r}\left(\frac{\partial v}{\partial\theta} - i\frac{\partial u}{\partial\theta}\right) \\
\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} &= \frac{1}{r}\left(\frac{\partial v}{\partial\theta} - i\frac{\partial u}{\partial\theta}\right) \\
\implies \frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial\theta} &\text{ and } \frac{\partial u}{\partial\theta} = -r\cdot\frac{\partial v}{\partial r}
\end{aligned}$$

Problem 3. Suppose that f is analytic and nonzero in a domain D. Prove that $\phi(x,y) = \ln(|f(z)|)$ is harmonic in D.

Proof. Let $z = x+iy$ and $f(z) = u(x,y)+iv(x,y)$. Then

$$\begin{aligned}
\phi(x,y) = \ln(|f(z)|) &= \ln((u^2+v^2)^{1/2}) \\
&= \frac{1}{2}\ln(u^2+v^2)
\end{aligned}$$

First we take the first derivative with respect to x.

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1}{2}\ln(u^2+v^2)\right) &= \frac{1}{2}\left(\frac{1}{u^2+v^2}\right)\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right) \\
&= \frac{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}}{u^2+v^2}
\end{aligned}$$

Then we take the second derivative with respect to x.

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x}}{u^2+v^2}\right)$$
$$= \frac{(u^2+v^2)\left[u\dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\partial u}{\partial x}\right)^2 + v\dfrac{\partial^2 v}{\partial x^2} + \left(\dfrac{\partial v}{\partial x}\right)^2\right] - 2\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x}\right)^2}{(u^2+v^2)^2}$$

Now we can take the second derivative with respect to y, which will be the same except our partials will be with respect to y instead of x.

$$\frac{\partial^2\phi}{\partial y^2} = \frac{(u^2+v^2)\left[u\dfrac{\partial^2 u}{\partial y^2} + \left(\dfrac{\partial u}{\partial y}\right)^2 + v\dfrac{\partial^2 v}{\partial y^2} + \left(\dfrac{\partial v}{\partial y}\right)^2\right] - 2\left(u\dfrac{\partial u}{\partial y} + v\dfrac{\partial v}{\partial y}\right)^2}{(u^2+v^2)^2}$$

Finally we add the second partials together and use the Cauchy-Riemann equations as well as the fact that u and v are harmonic to simplify.

$$(u^2+v^2)^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$$
$$= (u^2+v^2)\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)^2\right] -$$
$$2\left[\left(u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y}\right)^2 + \left(u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x}\right)^2\right]$$
$$= 2(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] - 2\left\{u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + v^2\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]\right\}$$
$$= 2(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] - 2(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$
$$= 0$$

Since $f(z)$ is nonzero, so is u^2+v^2

$$\implies \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

and thus $\phi(x,y)$ is harmonic in D.

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