Exercise 4. Show that the range of the function $z(t) = t^3 + it^6$, $-1 \le t \le 1$, is a smooth curve even though the given parametrization is not admissible.

Proof. Let $u = t^3$, then

$$z(u) = u + iu^2, \text{ for } (-1)^3 \le u \le (1)^3$$

Then

$$\frac{dz(u)}{du} = 1 + 2iu \neq 0$$

for all $-1 \le u \le 1$. Therefore the range of $z(u)$ is a smooth curve and thus the range of $z(t)$ is also a smooth curve.

■

Exercise 8. Parametrize the contour Γ indicated in in Fig. 4.14. Also give a parametrization for the opposite contour $-\Gamma$

We have 2 sections to consider, γ_1 and γ_2. γ_1 goes from $(-2, 2i)$ to $(-1, 0)$, which can be describe by

$$z_1(t) = t - 2 + 2i(1-t) \text{ for } 0 \le t \le 1.$$

γ_2 is a semicircle with radius 1 about the origin going clockwise, which can be described by

$$z_2(t) = -\cos(t) + i\sin(t) \text{ for } 0 \le t \le \pi.$$

We can shift the bounds on t for γ_2, and

$$z(t) = \begin{cases} t - 2 + 2i(1-t) & 0 \le t \le 1 \\ -\cos(t-1) + i\sin(t-1) & 1 \le t \le 1+\pi \end{cases}$$

describes Γ. For $-\Gamma$, we reverse the direction, so

$$z(t) = \begin{cases} \cos(t) + i\sin(t) & 0 \le t \le \pi \\ -(t-\pi) - 1 + 2i(t-\pi) & \pi \le t \le \pi+1 \end{cases}$$

describes $-\Gamma$.

Exercise 10. Using an admissible parametrization verify from formula (1) that

(a) the length of the line segment from z_1 to z_2 is $|z_2 - z_1|$;

Proof. Let γ be our contour. We can describe γ as follows:

$$\begin{aligned} z(t) = z_1 + t(z_2 - z_1) &= x_1 + iy_1 + t((x_2 + iy_2) - (x_1 + iy_1)) \\ &= t(x_2 - x_1) + x_1 + i(t(y_2 - y_1) + y_1). \end{aligned}$$

for $0 \leq t \leq t$. Let $s(t)$ be the lenght of γ, then

$$\frac{ds}{dt} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |z_2 - z_1|$$

and

$$s = \int_0^1 |z_2 - z_1| dt = |z_2 - z_1|$$

■

(b) the length of the circle $|z - z_0| = r$ is $2\pi r$

Proof. Let γ be our contour. We can describe γ as follows:

$$z(t) = r\cos(t) + x_0 + i(r\sin(t) + y_0)$$

for $0 \leq t \leq 2\pi$. Let $s(t)$ be the length of γ, then

$$\frac{ds}{dt} = \sqrt{(-r\sin(t))^2 + (r\cos(t))^2} = \sqrt{r^2(\sin^2(t) + \cos^2(t))} = r.$$

so

$$s = \int_0^{2\pi} r dt = 2\pi r$$

■