Exercise 10. Show that if $p_n(z)$ has degree n, then for all z with $|z|$ sufficiently large, there are positive constants c_1 and c_2 such that $c_1|z|^n < |p_n(z)| < c_2|z|^n$.

Proof. Let $p_n(z) = a_n z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0$. Then

$$p_n(z) = z^n(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}).$$

Since

$$\lim_{z\to\infty}\left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right| = 0,$$

there exists $\rho > 1$ such that

$$\left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right| < \frac{|a_n|}{2}$$

whenever $|z| \geq \rho$. Then

$$\begin{aligned}
|p_n(z)| &= |z^n|\left|a_n + \left(\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right)\right| \\
&\geq |z^n|\left(|a_n| - \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right|\right) \\
&> |z|^n \cdot \frac{|a_n|}{2}
\end{aligned}$$

and

$$\begin{aligned}
|p_n(z)| &= |z^n|\left|a_n + \left(\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right)\right| \\
&\leq |z^n|\left(|a_n| + \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right|\right) \\
&< |z|^n \cdot \frac{3|a_n|}{2}
\end{aligned}$$

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