Homework Set 1.2

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January 21, 2024

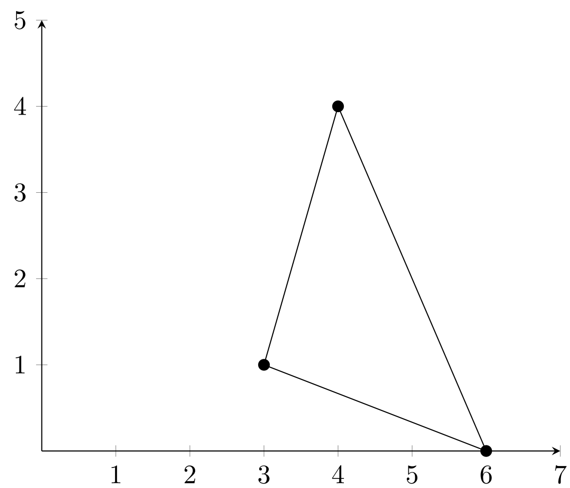
Exercise 6

Show that the points $3+i$, 6, and $4+4i$ are the vertices of a right triangle.

Proof. If the edges of the triangle formed by these vertices satisfy

$$A^2 + B^2 = C^2$$

Then they form a right a right triangle.



Let edge A be the distance from $3+i$ to $4+4i$, B be the distance from $3+i$ to 6, and C be the distance from $4+4i$ to 6. Then

$$\begin{aligned}
A &= \sqrt{(3-4)^2 + (1-4)^2} = \sqrt{10} \\
B &= \sqrt{(3-6)^2 + (1-0)^2} = \sqrt{10} \\
C &= \sqrt{(4-6)^2 + (4-0)^2} = \sqrt{20}
\end{aligned}$$

and

$$\sqrt{10}^2 + \sqrt{10}^2 = \sqrt{20}^2$$

$\square$

Exercise 16

Prove that if $|z| = 1 (z \neq 1)$, then $\text{Re}[1/(1-z)] = \frac{1}{2}$

Proof. Recall that $\text{Re}[z] = (z + \overline{z})/2$ and $z\overline{z} = |z|^2$

$$\begin{aligned}
\text{Re}[1/(1-z)] = \frac{1}{2}\left(\frac{1}{1-z} + \overline{\left(\frac{1}{1-z}\right)}\right) &= \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1-\overline{z}}\right) \\
&= \frac{1}{2}\left(\frac{1-\overline{z}+1-z}{1-\overline{z}-z+z\overline{z}}\right) \\
&= \frac{1}{2}\left(\frac{1-\overline{z}+1-z}{1-\overline{z}-z+|z|^2}\right) \\
&= \frac{1}{2}\left(\frac{1-\overline{z}-z+1}{1-\overline{z}-z+1}\right) \\
&= \frac{1}{2}
\end{aligned}$$

$\square$

Exercise 17

Let $a_1, a_2, \ldots, a_n$ be real constants. Show that if z_0 is a root of the polynomial equation $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n = 0$, then so is $\overline{z_0}$.

Proof. Let $f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n$. Since $a_1, a_2, \ldots, a_n \in \mathbb{R}$, $a_i = \overline{a_i}$. Then

$$
\begin{aligned}
f(\overline{z_0}) &= a_0 \overline{z_0}^n + a_1 \overline{z_0}^{n-1} + \ldots + a_n \\
&= (\overline{a_0})(\overline{z_0})^n + (\overline{a_1})(\overline{z_0})^{n-1} + \ldots + \overline{a_n} \\
&= \overline{a_0 z_0^n} + \overline{a_1 z_0}^{n-1} + \ldots + \overline{a_n} \\
&= \overline{a_0 z_0^n + a_1 z_0^{n-1} + \ldots + a_n} \\
&= \overline{0} \\
&= 0
\end{aligned}
$$

$\square$