Exercise 8. Verify that

$$\frac{d}{dz}\sinh z = \cosh z, \qquad \text{and} \qquad \frac{d}{dz}\cosh z = \sinh z.$$

$$\begin{aligned}
\frac{d}{dz}\sinh z &= \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right) = \frac{e^z - (-e^{-z})}{2} = \cosh z \\
\frac{d}{dz}\cosh z &= \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right) = \frac{e^z - e^{-z}}{2} = \sinh z
\end{aligned}$$

Exercise 11. Explain why the function $\operatorname{Re}\left(\frac{\cos z}{e^z}\right)$ is harmonic in the whole plane.

Since $1/e^z$ and $\cos z$ are entire, or analytic on the whole complex plane, their product must also be entire. Therefore, the real and imaginery parts of $\frac{\cos z}{e^z}$ are harmonic in the whole complex plane. Let $z = x + iy$, then writing $\operatorname{Re}\left(\frac{\cos z}{e^z}\right)$ in $u(x, y)$ yields

$$\begin{aligned}
\operatorname{Re}\left(\frac{\cos z}{e^z}\right) &= \operatorname{Re}\left(\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2e^{x+iy}}\right) \\
&= \operatorname{Re}\left(\frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2e^x e^{iy}} \cdot \frac{e^{-iy}}{e^{-iy}}\right) \\
&= \operatorname{Re}\left(\frac{e^{-y}e^{i(x-y)} + e^{y}e^{-i(x+y)}}{2e^x}\right) \\
&= \frac{e^{-y}\cos(x-y) + e^{y}\cos(x+y)}{2e^x}
\end{aligned}$$

Exercise 12c. Establish the following hyperbolic identity by using the relations (14) and other corresponding trigonometric identities.

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$\begin{aligned}
\cosh(z_1 + z_2) = \cos(iz_1 + iz_2) &= \cos iz_1 \cos iz_2 - \sin iz_1 \sin iz_2 \\
&= \cosh z_1 \cosh z_2 - i^2 \sinh z_1 \sinh z_2 \\
&= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2
\end{aligned}$$

Exercise 14b. Prove that $\tan z$ is periodic with period π.

Proof. It suffices to show that $\tan(z \pm \pi) = \tan z$.

$$\tan(z \pm \pi) = \frac{\sin(z \pm \pi)}{\cos(z \pm \pi)} = \frac{\sin z \cos \pi \pm \sin \pi \cos z}{\cos z \cos \pi \mp \sin z \sin \pi} = \frac{-\sin z}{-\cos z} = \tan z$$

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