Problem 1. Suppose that $|\alpha| < 1$ and $\left|\frac{z-\alpha}{1-\overline{\alpha}z}\right| \leq 1$ where $z,\ \alpha \in \mathbb{C}$. Prove that $|z| \leq 1$.

Proof.

$$\left|\frac{z-\alpha}{1-\overline{\alpha}z}\right| \leq 1 \implies |z-\alpha| \leq |1-\overline{\alpha}z|$$

Recall that $|z| - |w| \leq |z-w|$ and $|z-w| \leq |z| + |w|$ (1.3.15) by the triangle inequality.

$$\begin{aligned}
|z| - |\alpha| &\leq |z-\alpha| \\
&\leq |1-\overline{\alpha}z| \\
&\leq 1 + |\overline{\alpha}z| \\
&\leq 1 + |\alpha||z| \\
|z| - |\alpha||z| &\leq 1 + |\alpha| \\
|z|(1-|\alpha|) &\leq 1 + |\alpha| \\
|z||1-\alpha| &\leq 1 + |\alpha| \\
|z|(1+|\alpha|) &\leq 1 + |\alpha| \\
|z| &\leq 1
\end{aligned}$$

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Problem 2. Let P be a polynomial defined by $P(x) = z^2 + iz + 3 - 2i$. Find $P(w)$, where $w = 10i/(2+i)$. Write the answer in $a + bi$ form and include intermediate steps.

$$w = \frac{10i}{2+i} = \frac{10i(2-i)}{5} = 2i(2-i) = 2+4i$$

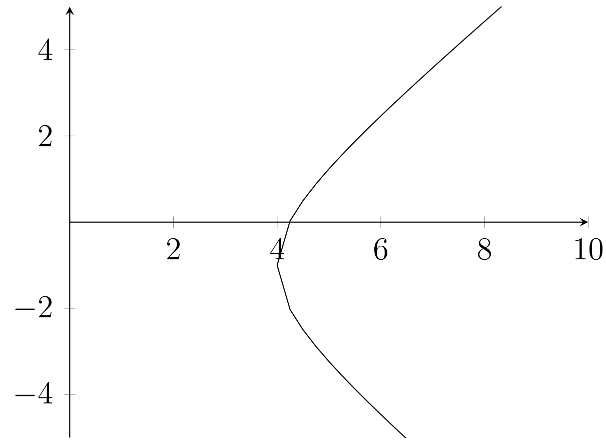
$$\begin{aligned}
P(w) &= (2+4i)^2 + i(2+4i) + 3 - 2i \\
&= (-12+16i) + (-4+2i) + 3 - 2i \\
&= -13 + 16i
\end{aligned}$$

Problem 3. Describe carefully and fully the set of all complex numbers that satisfy the equation $|z-i| = 2|z-1|$.

Let $z = x + iy$.

$$\begin{aligned}
|x+iy-i| &= 2|x+iy-1| \\
x^2 + (y-1)^2 &= 2(x-1)^2 + 2y^2 \\
x^2 - (2x^2 - 4x + 2) &= 2y^2 - (y^2 - 2y + 1) \\
-x^2 + 4x - 2 &= y^2 + 2y - 1 \\
x(4-x) &= y^2 + 2y + 1 \\
&= (y+1)^2 \\
y &= \pm\sqrt{x(4-x)} - 1
\end{aligned}$$

All complex numbers which lie on this parabola satisfy the equation $|z-i| = 2|z-1|$.



Problem 4. Prove the following identity.

$$\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}, \text{ where } 0 < \theta < 2\pi$$

Proof. Let $z = e^{i\theta}$.

$$\begin{aligned} 1 + z + z^2 + \ldots + z^n &= \frac{z^{n+1}-1}{z-1} \\ &= \frac{z^{(n+1)/2}\left(z^{(n+1)/2} - \overline{z}^{(n+1)/2}\right)}{z^{1/2}\left(z^{1/2} - \overline{z}^{1/2}\right)} \\ &= \frac{z^{(n+1)/2}}{z^{1/2}} \cdot \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \end{aligned}$$

$$\begin{aligned} \sin\theta + \sin 2\theta + \ldots + \sin n\theta &= \frac{z-\overline{z}}{2i} + \frac{z^2-\overline{z}^2}{2i} + \ldots + \frac{z^n - \overline{z}^n}{2i} \\ &= \frac{z + z^2 + \ldots + z^n - (\overline{z} + \overline{z}^2 + \ldots + \overline{z}^n)}{2i} \\ &= \left(\frac{z^{n+1}-1}{z-1} - 1 - \frac{\overline{z}^{n+1}-1}{\overline{z}-1} + 1\right)/2i \\ &= \left(\frac{z^{(n+1)/2}}{z^{1/2}} \cdot \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} - \frac{\overline{z}^{(n+1)/2}}{\overline{z}^{1/2}} \cdot \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)}\right)/2i \\ &= \left(\frac{\sin((n+1)\theta/2)}{\sin(\theta/2)}\right)\left(\frac{z^{(n+1)/2}}{z^{1/2}} - \frac{\overline{z}^{(n+1)/2}}{\overline{z}^{1/2}}\right)/2i \\ &= \left(\frac{\sin((n+1)\theta/2)}{\sin(\theta/2)}\right)\left(\frac{z^{-1/2}z^{(n+1)/2} - z^{1/2}z^{-(n+1)/2}}{z^{1/2}z^{-1/2}}\right)/2i \\ &= \left(\frac{\sin((n+1)\theta/2)}{\sin(\theta/2)}\right)\left(\frac{z^{n/2} - z^{-n/2}}{2i}\right) \\ &= \frac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)} \end{aligned}$$

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