Exercise 10. Show that if $p_n(z)$ has degree n, then for all z with $|z|$ sufficiently large, there are positive constants c_1 and c_2 such that $c_1|z|^n < |p_n(z)| < c_2|z|^n$.

Proof. Let $p_n(z) = a_n z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0$. Then

$$p_n(z) = z^n(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}).$$

Since

$$\lim_{z\to\infty}\left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right| = 0,$$

there exists $\rho > 1$ such that

$$\left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right| < \frac{|a_n|}{2}$$

whenever $|z| \geq \rho$. Then

$$\begin{aligned}
|p_n(z)| &= |z^n|\left|a_n + \left(\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right)\right| \\
&\geq |z^n|\left(|a_n| - \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right|\right) \\
&> |z|^n \cdot \frac{|a_n|}{2}
\end{aligned}$$

and

$$\begin{aligned}
|p_n(z)| &= |z^n|\left|a_n + \left(\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right)\right| \\
&\leq |z^n|\left(|a_n| + \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right|\right) \\
&< |z|^n \cdot \frac{3|a_n|}{2}
\end{aligned}$$

■

Exercise 16. Show that if $R_{m,n}(z)$ is a rational function numerator degree m and denominator degree n, then for all $|z|$ sufficiently large, there are positive constants c_1 and c_2 such then $c_1|z|^{m-n} < |R_{m,n}(z)| < c_2|z|^{m-n}$.

Proof. Let $R_{m,n} = \dfrac{p_m(z)}{p_n(z)}$. By the results from exercise 10, we have

$$c_1|z|^m < |p_m(z)| < c_2|z|^m \qquad \text{and} \qquad c_3|z|^n < |p_n(z)| < c_4|z|^n$$

for any $|z| \geq \rho_1 > 1$ and any $|z| \geq \rho_2 > 1$ respectively where c_i is positive. Then, for any $|z| \geq \max(\rho_1, \rho_2)$, we can divide the two inequalities whcich yields the desired result.

$$\frac{c_1|z|^m}{c_3|z|^n} < \frac{|p_m(z)|}{|p_n(z)|} < \frac{c_2|z|^m}{c_4|z|^n}$$
$$\frac{c_1}{c_3}|z|^{m-n} < |R_{m,n}(z)| < \frac{c_2}{c_4}|z|^{m-n}$$

■

Problem 1. Find the partial fraction decomposition for $R(z) = \dfrac{z^2 + 2iz + 3 + i}{z^4 + iz^3}$.

$$R(z) = \frac{z^2 + 2iz + 3 + i}{z^3(z + i)} = \frac{A_0^1}{z^3} + \frac{A_1^1}{z^2} + \frac{A_2^1}{z} + \frac{A_0^2}{z + i}$$

We can use equation (21) which is as follows

$$A_s^{(j)} = \lim_{z \to \zeta_j} \frac{1}{s!} \frac{d^s}{dz^s} [(z - \zeta_j)^{d_j} R_{m,n}(z)].$$

$$A_0^1 = \lim_{z\to 0} z^3 R(z) = \lim_{z \to 0} \frac{z^2 + 2iz + 3 + i}{z + i} = \frac{3 + i}{i} = 1 - 3i$$
$$A_1^1 = \lim_{z\to 0} \frac{d}{dz}(z^3 R(z)) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^2 + 2iz + 3 + i}{z + i}\right) = \lim_{z \to 0} \frac{(z + i)(2z + 2i) - (z^2 + 2iz + 3 + i)}{(z + i)^2}$$
$$= \frac{i(2i) - (3 + i)}{i^2} = 5 + i$$
$$A_2^1 = \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2}(z^3 R(z)) = \lim_{z\to 0} \frac{1}{2} \frac{d}{dz}\left(\frac{z^2 + 2iz - 5 - i}{(z + i)^2}\right)$$
$$= \lim_{z\to 0} \frac{(z + i)^2(2z + 2i) - (z + 2iz - 5 - i)(2z + 2i)}{2(z + i)^4} = \frac{i^2(2i) - (-5 - i)(2i)}{2i(i^3)} = \frac{4 + i}{-i} = -1 + 4i$$
$$A_0^2 = \lim_{z\to -i} (z + i)R(z) = \lim_{z \to -i} \frac{z^2 + 2iz + 3 + i}{z^3} = \frac{4 + i}{i} = 1 - 4i$$

Which gives us our partial fraction decomposition

$$R(z) = \frac{1 - 3i}{z^3} + \frac{5 + i}{z^2} + \frac{-1 + 4i}{z} + \frac{1 - 4i}{z + i}$$