

**Exercise 10.** Show that if  $p_n(z)$  has degree  $n$ , then for all  $z$  with  $|z|$  sufficiently large, there are positive constants  $c_1$  and  $c_2$  such that  $c_1|z|^n < |p_n(z)| < c_2|z|^n$ .

*Proof.* Let  $p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ . Then

$$p_n(z) = z^n \left( a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n} \right).$$

Since

$$\lim_{z \rightarrow \infty} \left| \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n} \right| = 0,$$

there exists  $\rho > 1$  such that

$$\left| \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n} \right| < \frac{|a_n|}{2}$$

whenever  $|z| \geq \rho$ . Then

$$\begin{aligned} |p_n(z)| &= |z^n| \left| a_n + \left( \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n} \right) \right| \\ &\geq |z^n| \left( |a_n| - \left| \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n} \right| \right) \\ &> |z|^n \cdot \frac{|a_n|}{2} \end{aligned}$$

and

$$\begin{aligned} |p_n(z)| &= |z^n| \left| a_n + \left( \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n} \right) \right| \\ &\leq |z^n| \left( |a_n| + \left| \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n} \right| \right) \\ &< |z|^n \cdot \frac{3|a_n|}{2} \end{aligned}$$

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**Exercise 16.** Show that if  $R_{m,n}(z)$  is a rational function numerator degree  $m$  and denominator degree  $n$ , then for all  $|z|$  sufficiently large, there are positive constants  $c_1$  and  $c_2$  such then  $c_1|z|^{m-n} < |R_{m,n}(z)| < c_2|z|^{m-n}$ .

*Proof.* Let  $R_{m,n} = \frac{p_m(z)}{p_n(z)}$ . By the results from exercise 10, we have

$$c_1|z|^m < |p_m(z)| < c_2|z|^m \quad \text{and} \quad c_3|z|^n < |p_n(z)| < c_4|z|^n$$

for any  $|z| \geq \rho_1 > 1$  and any  $|z| \geq \rho_2 > 1$  respectively where  $c_i$  is positive. Then, for any  $|z| \geq \max(\rho_1, \rho_2)$ , we can divide the two inequalities which yields the desired result.

$$\begin{aligned} \frac{c_1|z|^m}{c_3|z|^n} &< \frac{|p_m(z)|}{|p_n(z)|} < \frac{c_2|z|^m}{c_4|z|^n} \\ \frac{c_1}{c_3}|z|^{m-n} &< |R_{m,n}(z)| < \frac{c_2}{c_4}|z|^{m-n} \end{aligned}$$

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**Problem 1.** Find the partial fraction decomposition for  $R(z) = \frac{z^2 + 2iz + 3 + i}{z^4 + iz^3}$ .

$$R(z) = \frac{z^2 + 2iz + 3 + i}{z^3(z + i)} = \frac{A_0^1}{z^3} + \frac{A_1^1}{z^2} + \frac{A_2^1}{z} + \frac{A_0^2}{z + i}$$

We can use equation (21) which is as follows.

$$A_s^{(j)} = \lim_{z \rightarrow \zeta_j} \frac{1}{s!} \frac{d^s}{dz^s} [(z - \zeta_j)^{d_j} R_{m,n}(z)].$$

$$A_0^1 = \lim_{z \rightarrow 0} z^3 R(z) = \lim_{z \rightarrow 0} \frac{z^2 + 2iz + 3 + i}{z + i} = \frac{3 + i}{i} = 1 - 3i$$

$$\begin{aligned} A_1^1 &= \lim_{z \rightarrow 0} \frac{d}{dz} (z^3 R(z)) = \lim_{z \rightarrow 0} \frac{d}{dz} \left( \frac{z^2 + 2iz + 3 + i}{z + i} \right) = \lim_{z \rightarrow 0} \frac{(z + i)(2z + 2i) - (z^2 + 2iz + 3 + i)}{(z + i)^2} \\ &= \frac{i(2i) - (3 + i)}{i^2} = 5 + i \end{aligned}$$

$$\begin{aligned} A_2^1 &= \lim_{z \rightarrow 0} \frac{1}{2} \frac{d^2}{dz^2} (z^3 R(z)) = \lim_{z \rightarrow 0} \frac{1}{2} \frac{d}{dz} \left( \frac{z^2 + 2iz - 5 - i}{(z + i)^2} \right) \\ &= \lim_{z \rightarrow 0} \frac{(z + i)^2(2z + 2i) - (z + 2iz - 5 - i)(2z + 2i)}{2(z + i)^4} = \frac{i^2(2i) - (-5 - i)(2i)}{2i(i^3)} = \frac{4 + i}{-i} = -1 + 4i \end{aligned}$$

$$A_0^2 = \lim_{z \rightarrow -i} (z + i) R(z) = \lim_{z \rightarrow -i} \frac{z^2 + 2iz + 3 + i}{z^3} = \frac{4 + i}{i} = 1 - 4i$$

Which gives us our partial fraction decomposition

$$R(z) = \frac{1 - 3i}{z^3} + \frac{5 + i}{z^2} + \frac{-1 + 4i}{z} + \frac{1 - 4i}{z + i}$$