Exercise 8. Prove that all the roots of the equation $z^6 - 5z^2 + 10 = 0$ lie in the annulus $1 < |z| < 2$.

Proof. Let C_1 be the circle $|z| = 1$ and C_2 the circle $|z| = 2$. Consider $f(z) = 10$ which has no zeros inside and on C_1, and $h(z) = z^6 - 5z^2$. Then on C_1

$$|h(z)| = |z^6 - 5z^2| \leq |z|^6 + 5|z|^2 = 1 + 5 = 6$$

which is striclty less than $|f(z)| = 10$. Therefore $z^6 - 5z^2 + 10$ has no roots inside and on C_1. Now consider $f(z) = z^6$ which clearly has 6 zeros striclty inside C_2, and $h(z) = -5z^2 + 10$. Then on C_2

$$|h(z)| = | - 5z^2 + 10| \leq 5|z|^2 + 10 = 5 \cdot 4 + 10 = 30$$

which is striclty less than $|f(z)| = |z|^6 = 2^6 = 64$. Therefore $z^6 - 5z^2 + 10$ has six roots striclty inside C_2 thus has all six roots in the region $1 < |z| < 2$.

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