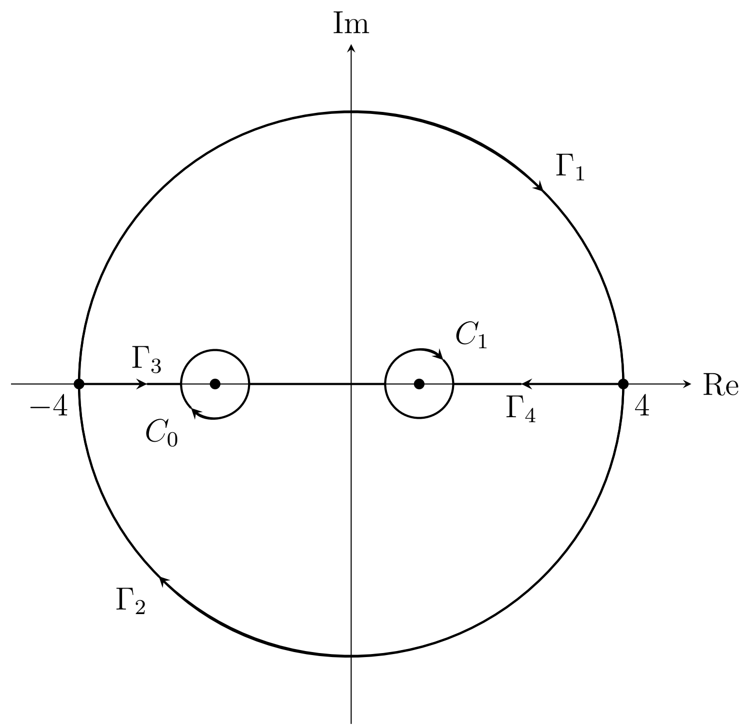
Exercise 15. Evaluate

$$\int_{\Gamma} \frac{z}{(z+2)(z-1)} dz$$

Where Γ is the circle $|z| = 4$ traversed twice in the clockwise direction.

Since $\frac{z}{(z+2)(z-1)}$ is analytic everywhere except the points $z = -2$ and $z = 1$, we can construct a new contour which encloses these points in small circles, C_0 and C_1 respectively and splitting apart Γ into 4 segments as follows



where Γ_1 is the top half of Γ from -4 to 4, Γ_2 is the bottom half of Γ from 4 to -4, Γ_3 is the path from -4 to 4 creating the top half semi-circles around -2 and 1, and Γ_4 is the path from 4 to -4 creating bottom half semi-circles around -2 and 1. Then, $\Gamma = 2(\Gamma_1 + \Gamma_2)$, and integrating along Γ_1 is the same as integrating along Γ_3 and similarly integrating along Γ_2 is the same as integrating along Γ_4. Combining the two segments we and taking into account the cancellations along the real axis we find

$$\begin{aligned}
\int_{\Gamma} \frac{z}{(z+2)(z-1)} dz &= 2\left(\int_{\Gamma_1} + \int_{\Gamma_2}\right) \frac{z}{(z+2)(z-1)} dz \\
&= 2\oint_{C_0} \frac{z}{(z+2)(z-1)} dz + 2\oint_{C_1} \frac{z}{(z+2)(z-1)} \\
&= 2\oint_{C_0} \left(\frac{2}{3}\cdot\frac{1}{z+2} + \frac{1}{3}\cdot\frac{1}{z-1}\right) dz + 2\oint_{C_1} \left(\frac{2}{3}\cdot\frac{1}{z+2} + \frac{1}{3}\cdot\frac{1}{z-1}\right) dz \\
&= \frac{4}{3}(-2\pi i) + \frac{2}{3}(-2\pi i) \\
&= -4\pi i
\end{aligned}$$

Exercise 17. Evaluate
$$\int_\Gamma \frac{2z^2-z+1}{(z-1)^2(z+1)}dz$$
where Γ is the figure-eight contour traversed once as shown in Fig. 4.49.

By the same method from the previous exercise we find that integrating along Γ is the same as integrating along C_0+C_1 where C_0 is a small circle in the counter-clockwise direction around $z=-1$ and C_1 is a small circle in the clockwise direction around $z=1$ for the function $\dfrac{2z^2-z+1}{(z-1)^2(z+1)}$.
Note:
$$\frac{2z^2-z+1}{(z-1)^2(z+1)} = \frac{A}{(z-1)^2}+\frac{B}{z-1}+\frac{C}{z+1}$$
To find A, B, and C we can use our method from section 3.1
$$\begin{aligned}
A &= \lim_{z\to 1}\frac{2z^2-z+1}{z+1} = 1\\
B &= \lim_{z\to 1}\frac{d}{dz}\left(\frac{2z^2-z+1}{z+1}\right) = \lim_{z\to 1}\frac{(z+1)(4z-1)-(2z^2-z+1)}{(z+1)^2} = \frac{4}{4} = 1\\
C &= \lim_{z\to -1}\frac{2z^2-z+1}{(z-1)^2} = \frac{4}{4} = 1
\end{aligned}$$
So
$$\begin{aligned}
\int_\Gamma \frac{2z^2-z+1}{(z-1)^2(z+1)}dz &= \oint_{C_0}\frac{2z^2-z+1}{(z-1)^2(z+1)}dz + \oint_{C_1}\frac{2z^2-z+1}{(z-1)^2(z+1)}dz\\
&= \oint_{C_0}\left(\frac{1}{(z-1)^2}+\frac{1}{z-1}+\frac{1}{z+1}\right)dz + \oint_{C_1}\left(\frac{1}{(z-1)^2}+\frac{1}{z-1}+\frac{1}{z+1}\right)dz\\
&= 2\pi i - 2\pi i\\
&= 0
\end{aligned}$$