Exercise 2. Show that $h(z) = x^3+3xy^2-3x+i(y^3+3x^2y-3y)$ is differentiable on the coordinate axes but is nowhere analytic.

Proof. Let $h(z) = u(x, y) + iv(x, y)$

$$u(x, y) = x^3 + 3xy^2 - 3x \qquad v(x, y) = y^3 + 3x^2y - 3y$$

$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 3 \qquad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = 6xy \qquad \frac{\partial v}{\partial y} = 3y^2 + 3x^2 - 3$$

In order for $h(z)$ to be analytic, there must be a domian in which the Cauchy-Riemann equations hold.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ for all } z$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \implies 6xy = -6xy \implies x = 0 \text{ or } y = 0$$

Therfore $h(z)$ is differentiable on the coordinate axes, however there is no domain contained by the coordinate axes as we can not draw a disc around any point on the coordinate axes which contains only points on the coordinate axes. Thus $h(z)$ is nowhere analytic.

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Exercise 8. Show that if f is analytic in a domain D and either $\Re f(z)$ or $\Im f(z)$ is constant in D, then $f(z)$ must be constant in D.

Proof. WLOG Let $\Re f(z)$ be constant in D. Then for all $z \in D$

$$\Re f^{'}(z) = 0 \implies \frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$$

and since f is analytic on D, by the Cauchy-Riemann equations we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

Therfore

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 \text{ for all } z \in D$$

and $f(z)$ is constant in D.

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Exercise 11. Suppose that $f(z)$ and $\overline{f(z)}$ are analytic in a domain D. Show that $f(z)$ is constant in D.

Proof. Let $f(z) = u(x,y) + iv(x,y)$. Then $\overline{f(z)} = u(x,y) - iv(x,y)$ and $z \in D$. By the Cauchy-Riemann equations we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} & \qquad \frac{\partial u}{\partial x} &= -\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} & \qquad \frac{\partial u}{\partial y} &= \frac{\partial v}{\partial y} \end{aligned}$$

so

$$\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y} \implies \frac{\partial v}{\partial y} = 0 \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x} \implies \frac{\partial v}{\partial x} = 0$$
$$\implies \Im f^{'}(z) = 0$$

Therfore $\Im f(z)$ is constant in D and by the previous exercise $f(z)$ is constant in D. ∎