# Derivatives

By ZJ

# Differential 微分

**DEFINITION** Let $y = f(x)$ be a differentiable function. The **differential** $\boldsymbol{dx}$ is an independent variable. The **differential** $\boldsymbol{dy}$ is

$$dy = f'(x)\,dx.$$

注：函数在一点$x_0$处可微与函数在这一点$x_0$处可导是等价的。

# Estimating with differentials

例 利用微分求近似值

（1） $1.02^{\frac{1}{3}}$

(2) $\sqrt{26}$

练习 求下面函数的微分

(1)$y = x + 2x^2 - \frac{1}{3}x^3 + x^4$

(2)$y = xlnx - x$

(3)$y = x^2 cosx$

(4)$y = \frac{x}{1-x^2}$

(5)$y = e^{ax} sinbx$

(6)$y = \arcsin(\sqrt{1 - x^2})$

**Change in $y = f(x)$ near $x = a$**

If $y = f(x)$ is differentiable at $x = a$ and $x$ changes from $a$ to $a + \Delta x$, the change $\Delta y$ in $f$ is given by

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

# 最大值与最小值

**DEFINITIONS** Let $f$ be a function with domain $D$. Then $f$ has an **absolute maximum** value on $D$ at a point $c$ if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on $D$ at $c$ if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

极值点

Maximum and minimum values are called **extreme values** of the function $f$. Absolute maxima or minima are also referred to as **global** maxima or minima.

全局极值点(最大值点或最小值点)

# 最大、最小值定理

**THEOREM 1—The Extreme Value Theorem** If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains both an absolute maximum value $M$ and an absolute minimum value $m$ in $[a, b]$. That is, there are numbers $x_1$ and $x_2$ in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other $x$ in $[a, b]$. 闭区间上连续函数在该区间上能够取得最大值和最小值

# 局部极值

**DEFINITIONS** A function $f$ has a **local maximum** value at a point $c$ within its domain $D$ if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing $c$.

A function $f$ has a **local minimum** value at a point $c$ within its domain $D$ if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing $c$.

Local extrema are also called **relative extrema**.

相对极值点

# 费马定理

**THEOREM 2—The First Derivative Theorem for Local Extreme Values** If $f$ has a local maximum or minimum value at an interior point $c$ of its domain, and if $f'$ is defined at $c$, then

$$f'(c) = 0.$$

# 临界点

**DEFINITION** An interior point of the domain of a function $f$ where $f'$ is zero or undefined is a **critical point** of $f$.

A function may have a critical point at $x = c$ without having a local extreme value there.

### How to Find the Absolute Extrema of a Continuous Function $f$ on a Finite Closed Interval

1. Evaluate $f$ at all critical points and endpoints.
2. Take the largest and smallest of these values.

# Monotonic Functions 函数单调性的判定

**COROLLARY 3** Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. 充分条件

If $f'(x) > 0$ at each point $x \in (a, b)$, then $f$ is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then $f$ is decreasing on $[a, b]$.

### First Derivative Test for Local Extrema

Suppose that $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$ except possibly at $c$ itself. Moving across $c$ from left to right,

1. if $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$;
2. if $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$;
3. if $f'$ does not change sign at $c$ (that is, $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local extremum at $c$.

**定理** 1 (**极值第一判别法**)

设函数 $f(x)$ 在 $x_0$ 的某邻域内连续, **且在空心邻域内有导数**, 当$x$由小到大通过 $x_0$ 时,

(1) $f'(x)$ **“左正右负”** ,则 $f(x)$ 在 $x_0$ 取极大值.

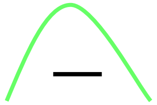
(2) $f'(x)$ **“左负右正”** ,则 $f(x)$ 在 $x_0$ 取极小值;

注：若在$x_0$的这两个区间里$f'(x)$符号不变，则$x_0$不是极值点，例如$f(x)=x^3$和$f(x)=x^{\frac{1}{3}}$.

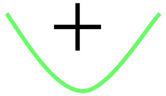
**定理**2 (**极值第二判别法**)　设函数 $f(x)$ 在点 $x_0$ 处具有

$$f'(x_0)=0,\ f''(x_0)\neq 0$$

(1) 若 $f''(x_0)<0$, 则 $f(x)$ 在点 $x_0$ 取极大值；




(2) 若 $f''(x_0)>0$, 则 $f(x)$ 在点 $x_0$ 取极小值 .

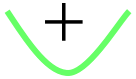



注：　本定理只用来判定一阶导数为零的点是否为极值点

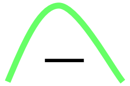
**定理3** (判别法的推广) 若函数 $f(x)$ 在 $x_0$ 点有直到 $n$ 阶导数，且 $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$,

则: 1) 当 $n$ 为偶数时, $x_0$ 为极值点，且

$f^{(n)}(x_0) > 0$ 时, $x_0$ 是极小点；




$f^{(n)}(x_0) < 0$ 时, $x_0$ 是极大点 .




2) 当 $n$ 为奇数时, $x_0$ 不是极值点 .

1.求下面函数的极值点

(1)$f(x) = 2x^3 - x^4$;

(2)$f(x) = \frac{2x}{x^2+1}$;

(3)$f(x) = \frac{(lnx)^2}{x}$;

(4)$f(x) = arctanx - \frac{1}{2}\ln(x^2 + 1)$.

2.求下面函数在给定区间上的最大最小值.

(1)$y = x^5 - 5x^4 + 5x^3 + 1, [-1,2]$;

(2)$y = 2tanx - tan^2 x, \left[0, \frac{\pi}{2}\right)$;

(3)$y = \sqrt{x}lnx, (0, +\infty)$

3.求下面函数的极值.

(1)$f(x) = |x(x^2 - 1)|$;

(2)$f(x) = \frac{x\left(x^2+1\right)}{x^4-x^2+1}$;

(3)$f(x) = (x-1)^2(x+1)^3$.

# 微分中值定理

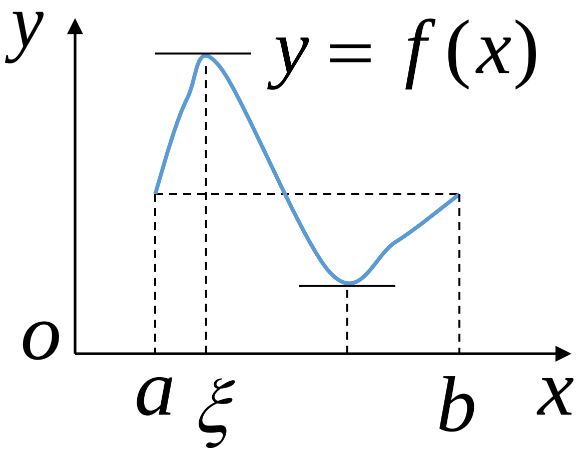
**THEOREM 3—Rolle's Theorem** Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior $(a, b)$. If $f(a) = f(b)$, then there is at least one number $c$ in $(a, b)$ at which $f'(c) = 0$.

**THEOREM 4—The Mean Value Theorem** Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

**COROLLARY 1** If $f'(x) = 0$ at each point $x$ of an open interval $(a, b)$, then $f(x) = C$ for all $x \in (a, b)$, where $C$ is a constant.

# 罗尔（Rolle）定理




$y = f(x)$ 满足:

(1) 在区间 $[a\,,b]$ 上连续

(2) 在区间 $(a\,,b)$ 内可导

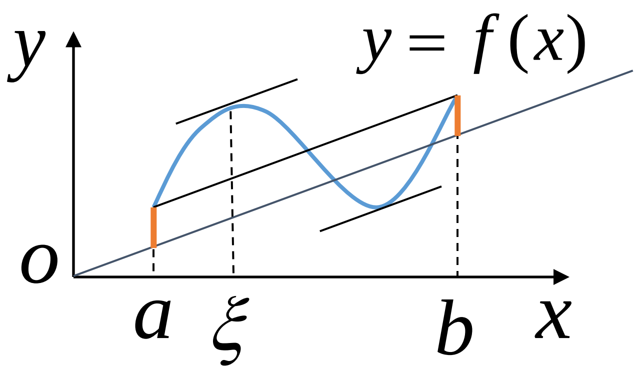
(3) $f(a) = f(b)$

$\Longrightarrow$ 在 $(a\,,b)$ 内至少存在一点 $\xi$，使 $f'(\xi) = 0.$

以下设所提到的导数存在，则

- 如果$f'(x)$没有零点，则$f(x)$至多有(　　)个零点。
- 如果$f'(x)$至多有1个零点，则$f(x)$至多有(　　)个零点。
- $\cdots$
- 如果$f'(x)$至多有$k$个零点，则$f(x)$至多有(　　)个零点。

# 拉格朗日中值定理




$y=f(x)$ 满足:

(1) 在区间 $[a,b]$ 上连续

(2) 在区间 $(a,b)$ 内可导

$\Longrightarrow$ 至少存在一点 $\xi\in(a,b)$, 使 $f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$.

**证:** 问题转化为证 $f'(\xi)-\dfrac{f(b)-f(a)}{b-a}=0$

作辅助函数 $\phi(x)=f(x)-\dfrac{f(b)-f(a)}{b-a}x$

**或作辅助函数** $F(x)=f(x)-f(a)-\dfrac{f(b)-f(a)}{b-a}(x-a)$

**例1.** 设 $f(x) \in C[0,\pi]$, 且在 $(0,\pi)$内可导, 证明至少存

$\xi \in (0,\pi)$, 使 $f'(\xi) = -f(\xi)\cot\xi$.

**提示:** 由结论可知, 只需证

$$f'(\xi)\sin\xi + f(\xi)\cos\xi = 0$$

即 $$[\, f(x)\sin x\,]'\Big|_{x=\xi} = 0$$

设 $$F(x) = f(x)\sin x$$

验证 $F(x)$ 在 $[0,\pi]$ 上满足罗尔定理条件.

**例2.** 证明不等式 $\dfrac{x}{1+x} < \ln(1+x) < x \quad (x > 0).$

**证:** 设 $f(t) = \ln(1+t)$，则$f(t)$在$[0,x]$上满足拉格朗日

因此应有 $f(x) - f(0) = f'(\xi)(x-0), \quad 0 < \xi < x$

$$\ln(1+x) = \frac{x}{1+\xi}, 0 < \xi < x$$

$$\frac{x}{1+x} < \frac{x}{1+\xi} < x$$

$$\frac{x}{1+x} < \ln(1+x) < x \qquad (x > 0)$$

# Concavity and Curve Sketching

**DEFINITION** The graph of a differentiable function $y = f(x)$ is

**(a)** **concave up** on an open interval $I$ if $f'$ is increasing on $I$;

**(b)** **concave down** on an open interval $I$ if $f'$ is decreasing on $I$.

**The Second Derivative Test for Concavity**

Let $y = f(x)$ be twice-differentiable on an interval $I$.

1. If $f'' > 0$ on $I$, the graph of $f$ over $I$ is concave up.
2. If $f'' < 0$ on $I$, the graph of $f$ over $I$ is concave down.

# 拐点

**DEFINITION** A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

连续曲线上有切线的凹凸分界点称为拐点

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

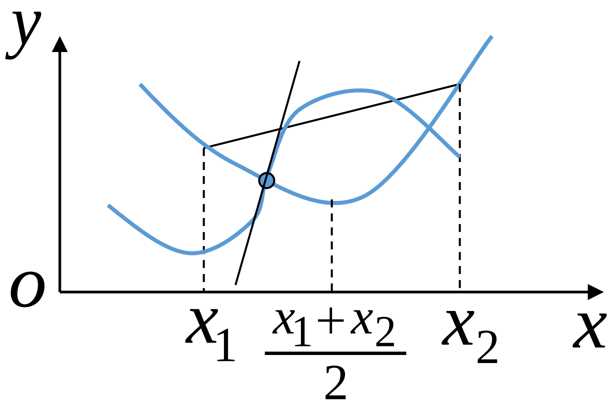
拐点是函数二阶导数为零或二阶导数不存在的点.

# 曲线的凹凸与拐点

**定义**. 设函数 $f(x)$ 在区间 I 上连续，$\forall x_1, x_2 \in I$,

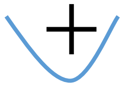
(1) 若恒有 $f(\frac{x_1+x_2}{2}) < \frac{f(x_1)+f(x_2)}{2}$, 则称 $f(x)$ 的图形是上**凹(凸)**的;

(2) 若恒有 $f(\frac{x_1+x_2}{2}) > \frac{f(x_1)+f(x_2)}{2}$, 则称 $f(x)$ 的图形是**下凹**的.



**定理**2.(凹凸判定法) 设函数 $f(x)$在区间$I$ 上有二阶导数

(1) 在 $I$ 内$f''(x)>0$, 则 $f(x)$在 $I$ 内图形是上凹的；




(2) 在 $I$ 内 $f''(x)<0$,则 $f(x)$在 $I$ 内图形是下凹的 .




**说明:**

1) 若在某点二阶导数为 0， 在其两侧二阶导数不变号, 则曲线的凹凸性不变 .

2)根据拐点的定义及上述定理, 可得拐点的判别法如下:

若曲线 $y=f(x)$ 在点$x_0$ 连续且有切线,
但 $f''(x)$ 在 $x_0$ 两侧**异号**, 则点$(x_0, f(x_0))$ 是曲线
$y=f(x)$的一个**拐点**. **以及二阶导数不存在的点.**

# 内容小结

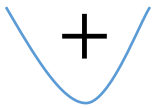
1. 可导函数单调性判别

$f'(x)>0,\ x\in I \Longrightarrow f(x)$ 在 $I$ 上单调递增

$f'(x)<0,\ x\in I \Longrightarrow f(x)$ 在 $I$ 上单调递减

2.曲线凹凸与拐点的判别

$f''(x)>0, x\in I \Longrightarrow$ 曲线 $y=f(x)$ 在 $I$ 上向上凹的




$f''(x)<0, x\in I \Longrightarrow$ 曲线 $y=f(x)$ 在 $I$ 上向下凹的




拐点 — 连续曲线上有切线的凹凸分界点

1.若曲线 $y=ax^3+bx^2+cx$ 在其拐点 $(1,1)$ 处有水平的切线，求系数 $a,b,c$.

2. 若实数 $a_n,a_{n-1},\Lambda,a_0$ 满足 $\frac{a_n}{n+1}+\frac{a_{n-1}}{n}+\Lambda+a_0=0$，证明方程 $a_n x^n+a_{n-1}x^{n-1}+\Lambda\ a_1x+a_0=0$ 在 $(0,1)$ 内必有实根.

3.证明：当 $0<x<\frac{\pi}{2}$ 时 $\sin x+\tan x>2x$.

4.确定方程 $x^2=x\sin x+\cos x$ 有几个实根？

5.判断下列命题的正确性。

（1）若 $f'(c)>0$，则 $f(x)$ 在 $c$ 的某个邻域内递增.

（2）若 $f'(c)>0$，则 $f(x)$ 在 $c$ 的某个右邻域内满足 $f(x)>f(c)$.

（3）若 $f(x)$ 在 $(a,b)$ 内递增且可导，则 $f'(x)>0, x\in(a,b)$.

（4）若 $f'(x_0)$ 存在，且 $\lim\limits_{x\to x_0} f''(x)(x-x_0)=1$，则 $(x_0,f(x_0))$ 是曲线 $y=f(x)$ 的拐点.

## Procedure for Graphing $y = f(x)$

1. Identify the domain of $f$ and any symmetries the curve may have.
2. Find the derivatives $y'$ and $y''$.
3. Find the critical points of $f$, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

**例3** 作函数 $f(x)=\dfrac{(x+1)^3}{(x-1)^2}$ 的图形.

**解** $D: x\neq 1,$

$$f'(x)=\frac{(x+1)^2(x-5)}{(x-1)^3},\qquad f''(x)=\frac{24(x+1)}{(x-1)^4}.$$

令 $f'(x)=0,$ 得 $x=-1,5,$

令 $f''(x)=0,$ 得 $x=-1.$

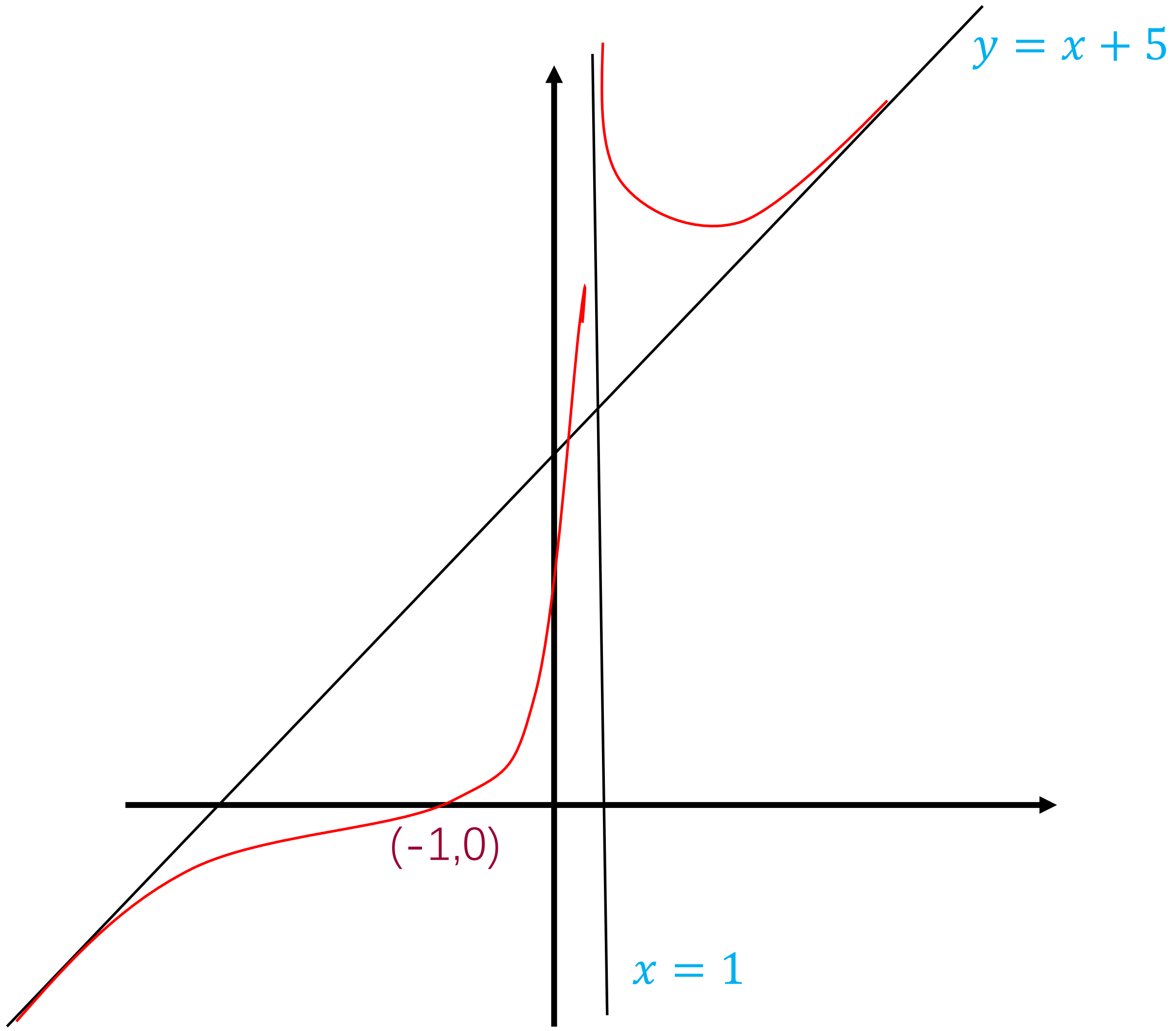
列表　$f'(x)=\dfrac{(x+1)^2(x-5)}{(x-1)^3}$，　$f''(x)=\dfrac{24(x+1)}{(x-1)^4}$.

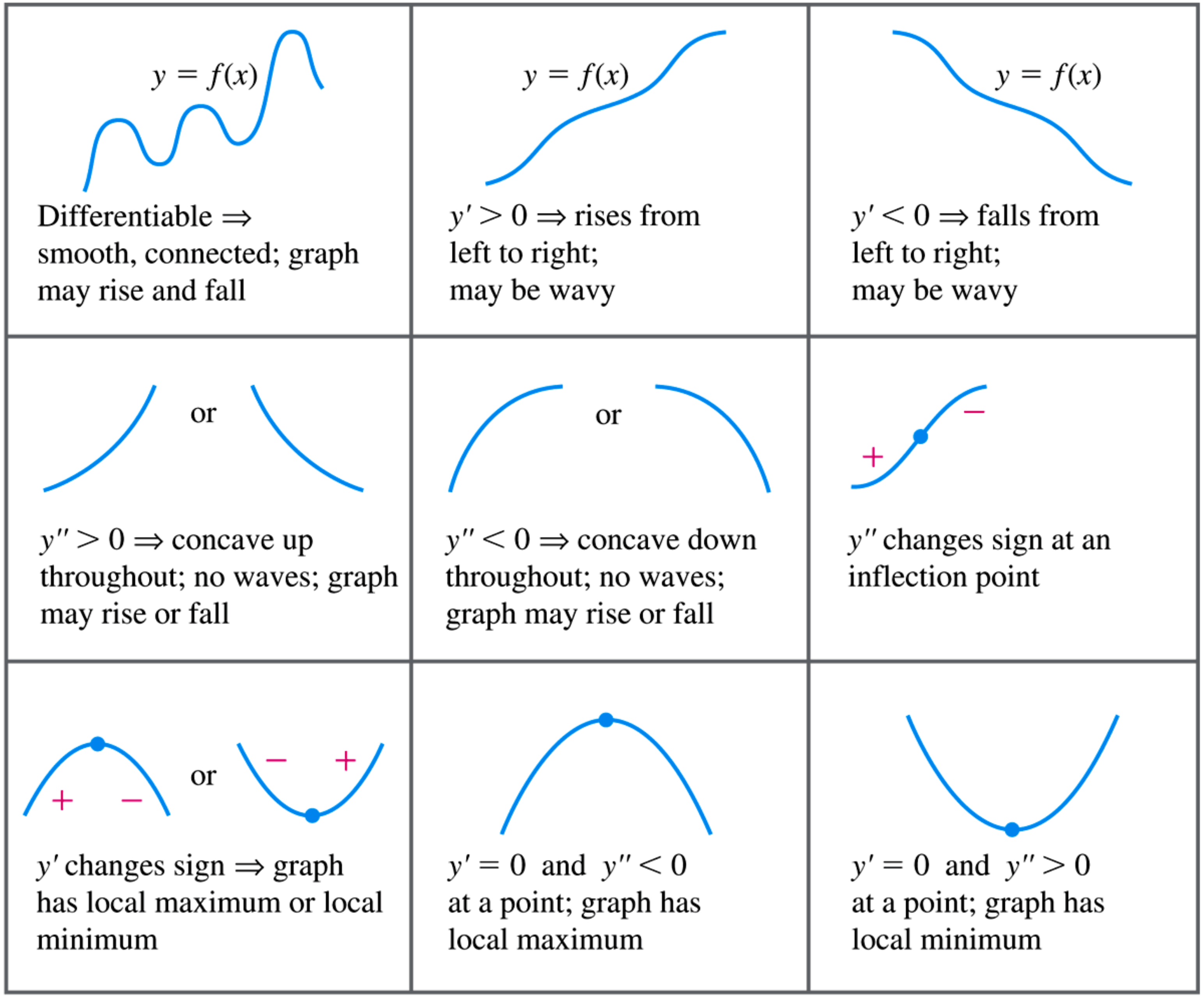
| $x$ | $(-\infty,-1)$ | $-1$ | $(-1,1)$ | $1$ | $(1,5)$ | $5$ | $(5,+\infty)$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $+$ | | $+$ | | $-$ | | $+$ |
| $f''(x)$ | $-$ | | $+$ | | $+$ | | $+$ |
| $f(x)$ | | 拐点 $(-1,0)$ | | | | 极值 $\dfrac{27}{2}$ | |

$\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}\dfrac{(x+1)^3}{(x-1)^2}=+\infty$　　得垂直渐近线 $x=1$;

$a=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\infty}\dfrac{(x+1^3)}{x(x-1)^2}=1$,

$\lim\limits_{x\to\infty}(f(x)-ax)=\lim\limits_{x\to\infty}(\dfrac{(x+1)^3}{(x-1)^2}-x)=5$,　　得斜渐近线 $y=x+5$.


$y = x + 5$
(-1,0)
$x = 1$


$y = f(x)$
Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall
$y = f(x)$
$y' > 0 \Rightarrow$ rises from left to right; may be wavy
$y = f(x)$
$y' < 0 \Rightarrow$ falls from left to right; may be wavy
or
$y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall
or
$y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall
$+$
$-$
$y''$ changes sign at an inflection point
$+$
$-$
or
$-$
$+$
$y'$ changes sign $\Rightarrow$ graph has local maximum or local minimum
$y' = 0$ and $y'' < 0$ at a point; graph has local maximum
$y' = 0$ and $y'' > 0$ at a point; graph has local minimum

# 求解最优化问题

## Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.