J Sci Comput (2009) 41: 483–509
DOI 10.1007/s10915-009-9311-3

Semi-discrete Entropy Satisfying Approximate Riemann Solvers. The Case of the Suliciu Relaxation Approximation

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Received: 20 May 2008 / Revised: 7 April 2009 / Accepted: 19 June 2009 / Published online: 14 July 2009

Abstract In this work we establish conditions for an approximate simple Riemann solver to satisfy a semi-discrete entropy inequality. The semi-discrete approach is less restrictive than the fully-discrete case and allows to grant some other good properties for numerical schemes. First, conditions are established in an abstract framework for simple Riemann solvers to satisfy a semi-discrete entropy inequality and then the results are applied, as a particular case, to the Suliciu system. This will lead in particular to the definition of schemes for the isentropic gas dynamics and the full gas dynamics system that are stable and preserve the stationary shocks.

Keywords Entropy satisfying schemes · Riemann solvers · Conservation laws · Suliciu solver

1 Introduction

The aim of this paper is to establish conditions for a simple Riemann solver to satisfy a semi-discrete entropy inequality. Classically, a discrete entropy inequality allows to fix a criteria in order to define weak solutions and to analyze the stability of a numerical scheme for a conservative system.

Discrete-entropy inequalities have largely been studied for different conservative systems, see for example [4, 8]. Here we use a semi-discrete approach, where the time variable t is kept continuous and only the space variable x is discretized. This allows to have some

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stability properties for the scheme, formally when $\Delta t \to 0$, and gives a less restrictive approach. This leaves more choices in order to ask other properties like the conservation of stationary shocks by the scheme.

Some schemes have already been presented for the full gas dynamics system that preserve the stationary shocks in [5, 12]. Here we aim entropy satisfying schemes that also preserve stationary shocks.

In Sect. 2 the basic definitions and results are given in a theoretical framework. We study then the Suliciu relaxation system in Sect. 3 as a practical application to the results given by the previous section. The Suliciu relaxation system is attached to the resolution of the isentropic gas dynamics but can also be adapted to handle full gas dynamics, which will be shown in Sect. 4. Finally, in Sect. 5 we show some numerical results for the scheme described in Sect. 3 and we compare to the results given by the fully discrete approach.

2 Entropy Satisfying Simple Riemann Solvers

We introduce here the basic definitions and results we are going to use in next sections. We shall only give the details of those results that are new and we refer the reader to [4] for further details.

We focus here on the study of one-dimensional conservative systems. We consider a one-dimensional system of conservation laws

$$\partial_t U + \partial_x (F(U)) = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{2.1}$$

where $U(t,x)$ is a vector with p components and F is a function that takes values in $\mathbb{R}^p$. We assume that the system is hyperbolic, that is, $A(U) = F'(U)$ has only real eigenvalues and a full set of eigenvectors.

As usual, we define a first-order finite volume conservative scheme for (2.1):

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{\Delta x_i}(F_{i+1/2} - F_{i-1/2}) = 0, \tag{2.2}$$

where $F_{i+1/2} = F(U_i^n, U_{i+1}^n)$.

It is a known fact that system (2.1) can have many weak solutions and entropies allow to fix a criteria in order to select a unique solution to the system.

Definition 2.1 An entropy for the system of conservation laws (2.1) is a convex function $\eta(U)$ with real values such that there exists another real valued function $G(U)$, called the entropy flux, satisfying

$$G'(U) = \eta'(U) F'(U). \tag{2.3}$$

Definition 2.2 A weak solution $U(t,x)$ of (2.1) is said to be entropy satisfying for the pair entropy-entropy flux (η, G) if

$$\partial_t(\eta(U)) + \partial_x(G(U)) \le 0. \tag{2.4}$$

2.1 Entropy Satisfying Numerical Schemes

Defining a numerical scheme that satisfies (2.4) in *some way* is an interesting but a difficult task. Indeed, one may observe shocks in numerical solutions that are not entropy satisfying. At the same time a companion notion of stability for the numerical scheme is deduced from the existence of an entropy.

In order to define numerical schemes that satisfy the entropy inequality, we introduce the following definitions.

Definition 2.3 We say that the scheme (2.2) satisfies a discrete entropy inequality associated to the convex entropy η for (2.1), if there exists a numerical entropy flux function $G(U_l, U_r)$ which is consistent with the exact entropy flux (in the sense that $G(U,U) = G(U)$), such that, under some Courant-Friedrichs-Lewy (CFL) condition, the discrete values computed by (2.2) automatically satisfy

$$\eta(U_i^{n+1}) - \eta(U_i^n) + \frac{\Delta t}{\Delta x_i}(G_{i+1/2} - G_{i-1/2}) \leq 0, \tag{2.5}$$

with

$$G_{i+1/2} = G(U_i^n, U_{i+1}^n). \tag{2.6}$$

Lemma 2.4 *If a numerical scheme satisfies a discrete entropy inequality* (*Definition* 2.3), *then the following inequalities hold*,

$$G(U_r) + \eta'(U_r)(F(U_l, U_r) - F(U_r)) \leq G(U_l, U_r), \tag{2.7}$$

$$G(U_l, U_r) \leq G(U_l) + \eta'(U_l)(F(U_l, U_r) - F(U_l)). \tag{2.8}$$

Definition 2.5 We say that the scheme (2.2) satisfies a semi-discrete entropy inequality associated to the convex entropy η for (2.1), if there exists a numerical entropy flux function $G(U_l, U_r)$ which is consistent with the exact entropy flux (in the sense that $G(U,U) = G(U)$), such that

$$G(U_r) + \eta'(U_r)(F(U_l, U_r) - F(U_r)) \leq G(U_l, U_r), \tag{2.9}$$

$$G(U_l, U_r) \leq G(U_l) + \eta'(U_l)(F(U_l, U_r) - F(U_l)). \tag{2.10}$$

Remark 2.1 Definition 2.5 is related to the *natural* definition of an entropy inequality for a semi-discrete scheme where the time variable t is kept continuous and only the space variable x is discretized. We refer to [4] for further details.

Remark 2.2 Definition 2.3 seems the natural way to adapt (2.4) for numerical schemes and one would like to define numerical schemes that satisfies this property. Nevertheless, this is not an easy task and it can be too restrictive. Sometimes it is interesting to satisfy just the weaker conditions (2.9) and (2.10) which will allow us to ask the scheme to verify some other *good properties* as we will see in Sect. 3.

2.2 Riemann Solvers

We focus here in numerical schemes based on the notion of Riemann solvers and in particular we shall consider simple Riemann solvers.

Definition 2.6 An approximate Riemann solver for (2.1) is a vector function $R(x/t, U_l, U_r)$ that is an approximation of the solution to the Riemann problem with initial data U_l, U_r, in the sense that it must satisfy the consistency relation

$$R(x/t, U, U) = U, \tag{2.11}$$

and the conservative identity

$$F_l(U_l, U_r) = F_r(U_l, U_r), \tag{2.12}$$

where

$$F_l(U_l, U_r) = F(U_l) - \int_{-\infty}^{0} (R(v, U_l, U_r) - U_l)dv, \tag{2.13}$$

$$F_r(U_l, U_r) = F(U_r) + \int_{0}^{\infty} (R(v, U_l, U_r) - U_r)dv. \tag{2.14}$$

Given an approximate Riemann solver R, we can define a conservative numerical scheme with the numerical flux $F = F_l = F_r$.

Definition 2.7 An approximate Riemann solver for (2.1) is said to preserve the entropy shocks of the system if whenever U_l, U_r can be connected by an entropy shock, then $R(x/t, U_l, U_r)$ is the exact solution to the Riemann problem with initial data U_l, U_r.

Lemma 2.8 *Let R be an approximate Riemann solver and define the left and right discrete entropy fluxes by*

$$G_l(U_l, U_r) = G(U_l) - \int_{-\infty}^{0} (\eta(R(v, U_l, U_r)) - \eta(U_l))dv, \tag{2.15}$$

$$G_r(U_l, U_r) = G(U_r) + \int_{0}^{\infty} (\eta(R(v, U_l, U_r)) - \eta(U_r))dv. \tag{2.16}$$

Suppose that

$$G_r(U_l, U_r) - G_l(U_l, U_r) \le 0. \tag{2.17}$$

Then, under a half CFL condition $|\sigma_l(U_i, U_{i+1})|\Delta t \le \Delta x_i/2$, $\sigma_r(U_{i-1}, U_i)\Delta t \le \Delta x_i/2$, the scheme associated to the approximate Riemann solver R satisfies (2.5) *for any function G such that $G_r(U_l, U_r) \le G(U_l, U_r) \le G_l(U_l, U_r)$.*

Lemma 2.9 *Let R be an approximate Riemann solver. Define the semi-discrete left and right entropy fluxes by*

$$G_l^s(U_l, U_r) = G(U_l) - \eta'(U_l) \int_{-\infty}^{0} (R(v, U_l, U_r) - U_l)dv, \tag{2.18}$$

$$G_r^s(U_l, U_r) = G(U_r) + \eta'(U_r) \int_{0}^{\infty} (R(v, U_l, U_r) - U_r)dv, \tag{2.19}$$

which are linearizations of (2.15)–(2.16).

The scheme associated to R is semi-discrete entropy satisfying with respect to the convex entropy η if, and only if,

$$G_r^s(U_l, U_r) - G_l^s(U_l, U_r) \le 0. \tag{2.20}$$

Proof Suppose that (2.20) holds. Then, writing (2.13)–(2.14) into (2.9)–(2.10) we recover Definition 2.5 for any numerical entropy flux function $G(U_l, U_r)$ such that

$$G_r^s(U_l, U_r) \le G(U_l, U_r) \le G_l^s(U_l, U_r). \tag{2.21}$$

Conversely, if the scheme associated to R is semi-discrete entropy satisfying, from (2.9)–(2.10) and using (2.13)–(2.14) we recover (2.20). □

Definition 2.10 We call simple solver any approximate Riemann solver consisting of a set of finitely many simple discontinuities. This means that there exists a finite number $m \ge 1$ of speeds

$$\sigma_0 = -\infty < \sigma_1 < \cdots < \sigma_m < \sigma_{m+1} = +\infty, \tag{2.22}$$

and intermediate states

$$U_0 = U_l, U_1, \dots, U_{m-1}, U_m = U_r \tag{2.23}$$

such that

$$R(x/t, U_l, U_r) = U_i \quad \text{if } \sigma_i < x/t < \sigma_{i+1}. \tag{2.24}$$

Remark 2.3 Given a simple solver defined by (2.22)–(2.24), define $k \in \{0, \dots, m\}$ such that $\sigma_k \le 0 < \sigma_{k+1}$. Then inequality (2.20) reads

$$\begin{aligned} G(U_r) - G(U_l) \le -\eta'(U_r)\left(\sigma_{k+1}(U_k - U_r) + \sum_{i=k+1}^{m-1}(\sigma_{i+1} - \sigma_i)(U_i - U_r)\right) \\ - \eta'(U_l)\left(-\sigma_k(U_k - U_l) + \sum_{i=1}^{k-1}(\sigma_{i+1} - \sigma_i)(U_i - U_l)\right). \end{aligned} \tag{2.25}$$

Proposition 2.11 *Consider a simple solver R with speeds (2.22) and intermediate states (2.23). Let $k \in \{0, \dots, m\}$ such that $\sigma_k \le 0 < \sigma_{k+1}$. A sufficient condition for (2.20) to hold is that there exist quantities E_i, $i = 0, \dots, m$ satisfying*

$$G(U_r) - G(U_l) = \sum_{i=0}^{m-1} \sigma_{i+1}(E_{i+1} - E_i), \tag{2.26}$$

$$E_m = \eta(U_r), \qquad E_0 = \eta(U_l), \tag{2.27}$$

$$\begin{aligned} &E_i \ge \eta'(U_l)(U_i - U_l) + \eta(U_l), \quad \text{for } i = 1, \dots, k-1, k, \\ &E_i \ge \eta'(U_r)(U_i - U_r) + \eta(U_r), \quad \text{for } i = k, k+1, \dots, m-1. \end{aligned} \tag{2.28}$$

Proof We rewrite

$$G(U_r) - G(U_l) = \sigma_m E_m - \sigma_1 E_0 - \sum_{i=1}^{m-1} (\sigma_{i+1} - \sigma_i) E_i. \tag{2.29}$$

Now, using the inequalities (2.28) we get

$$\begin{aligned} G(U_r) - G(U_l) \le & - \sum_{i=1}^{k-1} (\sigma_{i+1} - \sigma_i) \eta'(U_l)(U_i - U_l) \\ & - \sum_{i=k+1}^{m-1} (\sigma_{i+1} - \sigma_i) \eta'(U_r)(U_i - U_r) \\ & + \sigma_k \eta'(U_l)(U_k - U_l) - \sigma_{k+1} \eta'(U_r)(U_k - U_r) \\ & + \left(\sigma_k - \sum_{i=1}^{k-1} (\sigma_{i+1} - \sigma_i) \right) \eta(U_l) \\ & + \left(-\sigma_{k+1} - \sum_{i=k+1}^{m-1} (\sigma_{i+1} - \sigma_i) \right) \eta(U_r) + \sigma_m E_m - \sigma_1 E_0 \\ = & -\eta'(U_r) \left(\sigma_{k+1}(U_k - U_r) + \sum_{i=k+1}^{m-1} (\sigma_{i+1} - \sigma_i)(U_i - U_r) \right) \\ & - \eta'(U_l) \left(-\sigma_k (U_k - U_l) + \sum_{i=1}^{k-1} (\sigma_{i+1} - \sigma_i)(U_i - U_l) \right). \end{aligned} \tag{2.30}$$

Thus, (2.25) holds and the result follows. □

Remark 2.4 There is an analogous result for fully discrete entropy inequalities. If we replace (2.28) by

$$E_i \ge \eta(U_i), \quad \text{for } i = 1, \dots, m-1, \tag{2.31}$$

then (2.17) holds.

η being convex, this condition is stronger than (2.28) as expected.

3 Modified Suliciu Relaxation System

In this section we focus in the isentropic gas dynamics system

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) = 0, \end{cases} \tag{3.1}$$

where $\rho \ge 0$ is the density of the gas, u is the speed, and $p(\rho)$ is the pressure. We shall assume $p(\rho)$ a strictly increasing function of ρ. We denote $U = (\rho, \rho u)^t$ and we define

$$\eta(U) = \rho u^2/2 + \rho e(\rho), \qquad G(U) = (\rho u^2/2 + \rho e(\rho) + p(\rho)) u, \tag{3.2}$$

where $e'(\rho) = \frac{p(\rho)}{\rho^2}$. (G, η) is a pair entropy flux-entropy for system (3.1).

Using the so-called *Suliciu relaxation system*, which is introduced later, one may define a simple Riemann solver for (3.1) which is discrete entropy satisfying and preserves the positivity of the density ρ. This Suliciu system is described in [2–4, 6, 10, 11] and we refer to [4] for a detailed study. Nevertheless a discrete entropy satisfying condition can be too restrictive when we seek for some other properties. For example, one would like the Riemann solver to solve exactly the shocks of the original system. The aim is to modify the simple Riemann based on Suliciu system so that it satisfies just a semi-discrete entropy inequality, preserves positivity of ρ, and solves exactly the shocks of the system.

Consider a Riemann problem for (3.1) with initial data

$$(\rho,\rho u)(t=0,x)=\begin{cases}(\rho_l,\rho_l u_l), & \text{if } x<0,\\ (\rho_r,\rho_r u_r), & \text{if } x>0.\end{cases} \tag{3.3}$$

We introduce the Suliciu system

$$\begin{cases}\partial_t\rho+\partial_x(\rho u)=0,\\ \partial_t(\rho u)+\partial_x(\rho u^2+\pi)=0,\\ \partial_t(\rho\pi/c^2)+\partial_x(\rho\pi u/c^2)+\partial_x u=0,\\ \partial_t(\rho c)+\partial_x(\rho c u)=0,\end{cases} \tag{3.4}$$

and we consider the Riemann problem with initial data

$$(\rho,\rho u,\rho\pi/c^2,\rho c)=\begin{cases}(\rho_l\rho_l u_l,\rho_l\pi_l/c_l^2,\rho_l c_l), & \text{if } x<0,\\ (\rho_r,\rho_r u_r,\rho_r\pi_r/c_r^2,\rho_r c_r), & \text{if } x>0,\end{cases} \tag{3.5}$$

where

$$\pi_l=p(\rho_l),\qquad \pi_r=p(\rho_r), \tag{3.6}$$

and $c_\alpha\equiv c_\alpha(U_\alpha)\,\alpha=l,r$ are positive and will be defined later on.

The solution of (3.1)–(3.3) can be in general a tedious task. This is not the case of the solution of (3.4)–(3.5), which can be easily calculated as all the eigenvalues of the system are linearly degenerated. We shall denote $f=(\rho,\rho u,\rho\pi/c^2,\rho c)$ and define $\mathcal{R}(x/t,f_l,f_r)$ the exact solution to the Riemann problem for (3.4)–(3.5). This solution consists in three wave speeds

$$\sigma_1=u_l-c_l/\rho_l,\qquad \sigma_2=u_l^*=u_r^*,\qquad \sigma_3=u_r+c_r/\rho_r, \tag{3.7}$$

with two intermediate states that we shall index l^* and r^*. The states are obtained from the relations

$$\begin{aligned}c_l^*&=c_l, & c_r^*&=c_r,\\ u_l^*&=u_r^*, & \pi_l^*&=\pi_r^*,\\ (\pi+cu)_l^*&=(\pi+cu)_l, & (\pi-cu)_r^*&=(\pi-cu)_r,\\ (1/\rho+\pi/c^2)_l^*&=(1/\rho+\pi/c^2)_l, & (1/\rho+\pi/c^2)_r^*&=(1/\rho+\pi/c^2)_r.\end{aligned} \tag{3.8}$$

The solution is easily found to be

$$\begin{aligned}&u_l^*=u_r^*=\frac{c_l u_l+c_r u_r+\pi_l-\pi_r}{c_l+c_r}, && \pi_l^*=\pi_r^*=\frac{c_r\pi_l+c_l\pi_r-c_l c_r(u_r-u_l)}{c_l+c_r},\\ &\frac{1}{\rho_l^*}=\frac{1}{\rho_l}+\frac{c_r(u_r-u_l)+\pi_l-\pi_r}{c_l(c_l+c_r)}, && \frac{1}{\rho_r^*}=\frac{1}{\rho_r}+\frac{c_l(u_r-u_l)+\pi_r-\pi_l}{c_r(c_l+c_r)}.\end{aligned} \tag{3.9}$$

We define the approximate simple Riemann solver for the system (3.1)

$$R_{c_l,c_r}(v, U_l, U_r) = L\mathcal{R}(v, M(U_l, c_l), M(U_r, c_r)), \tag{3.10}$$

where $L : \mathbb{R}^4 \to \mathbb{R}^2$ is the canonical projection and

$$M(U, c) = (\rho, \rho u, \rho p(\rho)/c^2, \rho c). \tag{3.11}$$

The *good properties* of this solver depend on the definition of c_l, c_r which are assumed to be positive and depend on U_l, U_r.

Some conditions have already been established in [4] for c_l and c_r in order to define a discrete entropy satisfying scheme. We intend to ask weaker conditions so that R_{c_l,c_r} satisfies a semi-discrete entropy inequality and it preserves the shocks of system (3.1).

We can consider a new variable e and add the equation

$$\partial_t(\rho u^2/2 + \rho e) + \partial_x((\rho u^2/2 + \rho e + \pi)u) = 0 \tag{3.12}$$

to system (3.4). The solution of the Riemann problem for this new system is of the same type as the previous one consisting in three wave speeds with two intermediate states, determined by the relations (3.7)–(3.8) together with

$$(e - \pi/2c^2)_l^* = (e - \pi/2c^2)_l, \quad (e - \pi/2c^2)_r^* = (e - \pi/2c^2)_r. \tag{3.13}$$

We shall note by index 0, 1, 2, 3 the states l, l^*, r^*, r respectively.

We notice then, according to this new equation, that

$$\sigma_i(E_i - E_{i-1}) = \xi_i - \xi_{i-1}, \quad \text{for } i = 1, 2, 3, \tag{3.14}$$

where $E = \rho u^2/2 + \rho e$ and $\xi = (\rho u^2/2 + \rho e + \pi)u$.

Recall that here e is an independent variable.

Thus, from (3.2), we get

$$G(U_r) - G(U_l) = \xi_3 - \xi_0 = \sum_{i=0}^{2} \sigma_{i+1}(E_{i+1} - E_i), \tag{3.15}$$

and we may apply Proposition 2.11 for this decomposition. Thus, (2.28) is a sufficient condition for the scheme to verify a semi-discrete entropy inequality for the pair entropy-entropy flux (η, G).

Remark 3.1

- In [4] it is shown that under the hypothesis $E_l^* \geq \eta(U_l^*)$, $E_r^* \geq \eta(U_r^*)$, the approximate Riemann solver R_{c_l,c_r} satisfies an entropy inequality by interface. As η is a convex function, this hypothesis implies (2.28) as expected.
- We will replace conditions (2.28) by

$$E_i \geq \eta(U) + \eta'(U)(U_i - U), \tag{3.16}$$

where U stands for either U_l or U_r, and U_i stands for either U_l^* or U_r^*. While this condition is stronger than (2.28), it will not impose too many limitations and simplifies further calculations.

- As $\eta(U) = \rho u^2/2 + \rho e(\rho)$, one gets

$$\begin{aligned}
&\eta(U) - \eta(U_i) + \eta'(U)(U - U_i) \\
&\quad = \rho u^2/2 + \rho e(\rho) - \rho_i u_i^2/2 - \rho_i e(\rho_i) \\
&\qquad + (-u^2/2 + \Big(e + p/\rho\Big)(\rho))(\rho_i - \rho) + u(\rho_i u_i - \rho u) \\
&\quad = \rho_i \left(-\frac{(u_i - u)^2}{2} + e(\rho) - e(\rho_i) + p(\rho)\left(\frac{1}{\rho} - \frac{1}{\rho_i}\right)\right). \qquad (3.17)
\end{aligned}$$

Thus, if we suppose $\rho_i \geq 0$, condition (3.16) is equivalent to

$$e(\rho) - e_i + p(\rho)\left(\frac{1}{\rho} - \frac{1}{\rho_i}\right) - \frac{(u_i - u)^2}{2} \leq 0, \qquad (3.18)$$

where (ρ, u) stands for either (ρ_l, u_l), (ρ_r, u_r) and (ρ_i, u_i, e_i) for either (ρ_l^*, u_l^*, e_l^*), (ρ_r^*, u_r^*, e_r^*).

Theorem 3.1 *Let*

$$c_l^2 = \frac{(p_r - p_l)^2}{(p_r - p_l)(\frac{1}{\rho_l} - \frac{1}{\rho_r}) - [-(2(e_l - e_r) + (p_r + p_l)(\frac{1}{\rho_l} - \frac{1}{\rho_r}))]_+}, \qquad (3.19)$$

$$c_r^2 = \frac{(p_r - p_l)^2}{(p_r - p_l)(\frac{1}{\rho_l} - \frac{1}{\rho_r}) - [2(e_l - e_r) + (p_r + p_l)(\frac{1}{\rho_l} - \frac{1}{\rho_r})]_+}. \qquad (3.20)$$

Then, for

$$\widetilde{c}_l = \max\big(c_l, \rho_l(u_l - u_r)_+, \sqrt{\rho_l(p_r - p_l)_+}\big), \qquad (3.21)$$

$$\widetilde{c}_r = \max\big(c_r, \rho_r(u_l - u_r)_+, \sqrt{\rho_r(p_l - p_r)_+}\big), \qquad (3.22)$$

the simple solver $R_{\widetilde{c}_l, \widetilde{c}_r}$ described by (3.10) *with wave speeds* (3.7) *and intermediate states* (3.9), *is an approximate Riemann solver for the isentropic gas dynamics system* (3.1) *and has the properties*:

(i) *it preserves the non negativity of ρ*,
(ii) *it satisfies a semi-discrete entropy inequality*,
(iii) *it preserves the entropy shocks for the isentropic gas dynamics system.*

Remark 3.2 We recall that two different states U_l, U_r can be connected by a shock for system (3.1) if, and only if,

$$(u_r - u_l)^2 = (p_r - p_l)\left(\frac{1}{\rho_l} - \frac{1}{\rho_r}\right). \qquad (3.23)$$

An entropy 1-shock for (3.1) satisfies

$$u_l \geq u_r, \qquad \rho_r \geq \rho_l. \qquad (3.24)$$

An entropy 2-shock for (3.1) satisfies

$$u_l \geq u_r, \qquad \rho_r \leq \rho_l. \tag{3.25}$$

We shall give some lemmas in order to prove this theorem.

Lemma 3.2 *Under the assumptions*

$$\frac{(p_r - p_l)^2}{2(e_l - e_r + p_r(\frac{1}{\rho_l} - \frac{1}{\rho_r}))} \leq c_l^2, \quad \frac{(p_r - p_l)^2}{2(e_r - e_l + p_l(\frac{1}{\rho_r} - \frac{1}{\rho_l}))} \leq c_r^2, \tag{3.26}$$

condition (3.18) *is satisfied.*

Remark 3.3 Using Taylor's theorem, for any $\rho, \rho_0 \geq 0$, there exists $\widetilde{\rho}$ between ρ, ρ_0 such that

$$e(\rho) = e(\rho_0) - p(\rho_0)\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) + \frac{1}{2}\frac{d^2 e}{d(\frac{1}{\rho})^2}(\widetilde{\rho})\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)^2. \tag{3.27}$$

Assume p strictly increasing function of ρ. As $\frac{d^2 e}{d(1/\rho)^2} = \rho^2 \frac{dp}{d\rho}$, we get

$$e_l - e_r + p_r\left(\frac{1}{\rho_l} - \frac{1}{\rho_r}\right) \geq 0, \quad e_r - e_l + p_l\left(\frac{1}{\rho_r} - \frac{1}{\rho_l}\right) \geq 0. \tag{3.28}$$

Proof In the case $(U, U_i) = (U_l, U_l^*)$ or $(U, U_i) = (U_r, U_r^*)$, from (3.9)–(3.13) it follows that

$$\begin{aligned}
&e(\rho) - e_i + p(\rho)\left(\frac{1}{\rho} - \frac{1}{\rho_i}\right) - \frac{(u - u_i)^2}{2} \\
&\quad = \frac{p(\rho)^2}{2c^2} - \frac{\pi_i^2}{2c^2} + p(\rho)\left(\frac{\pi_i^2}{c^2} - \frac{p(\rho)}{c^2}\right) - \frac{1}{2c^2}(p(\rho) - \pi_i)^2 \\
&\quad = -\frac{1}{c^2}(p(\rho) - \pi_i)^2 \leq 0
\end{aligned} \tag{3.29}$$

and (3.18) is automatically satisfied.

Consider now the case $(U, U_i) = (U_r, U_l^*)$ (the case $(U, U_i) = (U_l, U_r^*)$ being analogous).

Using again the identities (3.9)–(3.13) we get

$$\begin{aligned}
&e(\rho_r) - e_l^* + p(\rho_r)\left(\frac{1}{\rho_r} - \frac{1}{\rho_l^*}\right) - \frac{(u_l^* - u_r)^2}{2} \\
&\quad = e_r - e_l + \frac{p_l^2 - (\pi_l^*)^2}{2c_l^2} + p_r\left(\frac{1}{\rho_r} - \frac{1}{\rho_l} + \frac{p_r - p_l - c_r(u_r - u_l)}{c_l(c_l + c_r)}\right) \\
&\qquad - \frac{c_r^2}{2}\left(\frac{p_r - p_l + c_l(u_r - u_l)}{c_r(c_l + c_r)}\right)^2 \\
&\quad = e_r - e_l + p_r\left(\frac{1}{\rho_r} - \frac{1}{\rho_l}\right) + (p_r - p_l)\frac{p_r - p_l - c_r(u_r - u_l)}{c_l(c_l + c_r)}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\frac{p_r-p_l-c_r(u_r-u_l)}{c_l+c_r}\right)^2-\frac{1}{2}\left(\frac{p_r-p_l+c_l(u_r-u_l)}{c_l+c_r}\right)^2\\
&=e_r-e_l+p_r\left(\frac{1}{\rho_r}-\frac{1}{\rho_l}\right)\\
&\quad+\frac{c_l^2+c_r^2}{(c_l+c_r)^2}\left(-\frac{(u_r-u_l)^2}{2}+\frac{c_r(p_r-p_l)^2}{c_l(c_l^2+c_r^2)}-\frac{(p_r-p_l)(u_r-u_l)}{c_l}\right)\\
&=e_r-e_l+p_r\left(\frac{1}{\rho_r}-\frac{1}{\rho_l}\right)-\frac{c_l^2+c_r^2}{2(c_l+c_r)^2}\left(u_r-u_l+\frac{p_r-p_l}{c_l}\right)^2\\
&\quad+\frac{1}{2c_l^2}(p_r-p_l)^2\\
&\le e_r-e_l+p_r\left(\frac{1}{\rho_r}-\frac{1}{\rho_l}\right)+\frac{1}{2c_l^2}(p_r-p_l)^2. \qquad (3.30)
\end{aligned}
$$

Now, from Remark 3.3 and assuming

$$\frac{(p_r-p_l)^2}{2(e_l-e_r+p_r(\frac{1}{\rho_l}-\frac{1}{\rho_r}))}\le c_l^2, \qquad (3.31)$$

we get

$$e(\rho_r)-e_l^*+p(\rho_r)\left(\frac{1}{\rho_r}-\frac{1}{\rho_l^*}\right)-\frac{(u_l^*-u_r)^2}{2}\le 0. \qquad (3.32)$$

□

Lemma 3.3 *Consider two different states $(\rho_l,\rho_l u_l)$ and $(\rho_r,\rho_r u_r)$ such that they can be connected by an entropy 1-shock (resp. 2-shock) for system (3.1) with shock speed σ. Then the following conditions are equivalent.*

(i) *The Suliciu relaxation system preserves the 1-shock (resp. preserves the 2-shock),*
(ii) $c_l=\frac{p_r-p_l}{u_l-u_r}$ (*resp.* $c_r=\frac{p_l-p_r}{u_l-u_r}$).

Proof First, we remark that the approximate Riemann solver (3.10) preserves a shock if and only if one of the following cases is satisfied:

(A) $U_l^*=U_r^*=U_l$ and $\sigma=u_r+\frac{c_r}{\rho_r}$,
(B) $U_l^*=U_r^*=U_r$ and $\sigma=u_l-\frac{c_l}{\rho_l}$,
(C) $U_l^*=U_l$, $U_r^*=U_r$ and $\sigma=u_l^*=u_r^*$.

From (3.9), we see that the third case is not possible. Indeed, this would imply $c_r(u_r-u_l)=p_r-p_l=-c_l(u_r-u_l)$. The constants c_l, c_r being positive, this would mean $u_l=u_r$ and thus $U_l=U_r$.

Now, from Remark 3.2 we get

$$
\begin{aligned}
u_l-\frac{c_l}{\rho_l}-\sigma&=u_l-\frac{c_l}{\rho_l}-\frac{\rho_r u_r-\rho_l u_l}{\rho_r-\rho_l}=\frac{\rho_r(u_l-u_r)}{\rho_r-\rho_l}-\frac{c_l}{\rho_l}\\
&=\frac{1}{\rho_l}\left(\frac{p_r-p_l}{u_l-u_r}-c_l\right), \qquad (3.33)
\end{aligned}
$$

and

$$u_r + \frac{c_r}{\rho_r} - \sigma = u_r + \frac{c_r}{\rho_r} - \frac{\rho_r u_r - \rho_l u_l}{\rho_r - \rho_l} = \frac{\rho_l(u_l - u_r)}{\rho_r - \rho_l} + \frac{c_r}{\rho_r}$$
$$= \frac{1}{\rho_r}\left(\frac{p_r - p_l}{u_l - u_r} + c_r\right). \tag{3.34}$$

c_l and c_r being positive, we get from Remark 3.2 that case (A) corresponds to 1-shocks and case (B) corresponds to 2-shocks.

It is now clear that (i) implies (ii).

Conversely, suppose that $c_l = (p_r - p_l)/(u_l - u_r)$ (the case $c_r = (p_l - p_r)/(u_l - u_r)$ being analogous). From Remark 3.2,

$$c_l = \frac{p_r - p_l}{u_l - u_r} = \frac{u_l - u_r}{\frac{1}{\rho_l} - \frac{1}{\rho_r}}. \tag{3.35}$$

Now, using this last identity in (3.9), it is easy to check that

$$u_l^* = u_r^* = u_r, \qquad \pi_l^* = \pi_r^* = p_r, \tag{3.36}$$

$$\rho_l^* = \rho_r, \qquad \rho_r^* = \rho_r, \tag{3.37}$$

and the result follows. □

Lemma 3.4 *Suppose ρ_l, $\rho_r > 0$ and p a strictly increasing convex function of ρ. Consider the definitions given by* (3.19)–(3.20).

(i) *Assume $\rho_l \neq \rho_r$. Then c_l, c_r are well defined in the sense that they are non negative and the denominator is non null.*
(ii) $\lim_{\rho_l,\rho_r\to\rho} c_l = \lim_{\rho_l,\rho_r\to\rho} c_r = \rho\sqrt{p'(\rho)}$,
(iii) *If u_l, u_r are such that the states $(\rho_l, \rho_l u_l)$, $(\rho_r, \rho_r u_r)$ can be connected by an entropy 1-shock for system* (3.1), *then $c_l = \frac{p_r - p_l}{u_l - u_r}$.*
(iv) *If u_l, u_r are such that the states $(\rho_l, \rho_l u_l)$, $(\rho_r, \rho_r u_r)$ can be connected by an entropy 2-shock for system* (3.1), *then $c_r = \frac{p_l - p_r}{u_l - u_r}$.*
(v) *c_l, c_r satisfy inequality* (3.26).

Proof We shall only give the proofs for c_l, the case for c_r being similar.

In order to prove (i), consider the two possible cases:

(A) $2(e_l - e_r) + (p_r + p_l)(\frac{1}{\rho_l} - \frac{1}{\rho_r}) \geq 0$.
In this case the denominator in (3.19) reduces to $(p_r - p_l)(\frac{1}{\rho_l} - \frac{1}{\rho_r})$. p being an strictly increasing function of ρ it is now clear that this last expression is positive
(B) $2(e_l - e_r) + (p_r + p_l)(\frac{1}{\rho_l} - \frac{1}{\rho_r}) < 0$.
The denominator coincides with $2(e_l - e_r + p_r(\frac{1}{\rho_l} - \frac{1}{\rho_r}))$, and from Remark 3.3 it follows that it is non-negative and non-null as long as $\rho_l \neq \rho_r$.

This proves (i).

Now, consider the function

$$f(\rho) = 2(e(\rho) - e(\rho_r)) + (p(\rho) + p(\rho_r))\left(\frac{1}{\rho} - \frac{1}{\rho_r}\right), \tag{3.38}$$

for any $\rho \in (0, \infty)$.

Using Taylor's theorem we get

$$f(\rho) = \frac{1}{2}\frac{d^2 p}{d(1/\rho)^2}(\widetilde{\rho})\left(\frac{1}{\widetilde{\rho}} - \frac{1}{\rho_r}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_r}\right)^2, \tag{3.39}$$

for some $\widetilde{\rho}$ between ρ and ρ_r.

Thus

$$2(e_l - e_r) + (p_r + p_l)\left(\frac{1}{\rho_l} - \frac{1}{\rho_r}\right) = \frac{1}{2}\frac{d^2 p}{d(1/\rho)^2}(\widetilde{\rho})\left(\frac{1}{\widetilde{\rho}} - \frac{1}{\rho_r}\right)\left(\frac{1}{\rho_l} - \frac{1}{\rho_r}\right)^2, \tag{3.40}$$

for some $\widetilde{\rho}$ between ρ_l and ρ_r.

Thus,

$$c_l^2 = \frac{(\frac{dp}{d(1/\rho)}(\frac{1}{\rho_r} - \frac{1}{\rho_l}))^2 + o(\frac{1}{\rho_l} - \frac{1}{\rho_r})^2}{-\frac{dp}{d(1/\rho)}(\frac{1}{\rho_r} - \frac{1}{\rho_l})^2 + o(\frac{1}{\rho_l} - \frac{1}{\rho_r})^2}, \tag{3.41}$$

and (ii) follows.

Moreover, as the pressure p is a convex function of ρ, $2(e_l - e_r) + (p_r + p_l)(\frac{1}{\rho_l} - \frac{1}{\rho_r})$ has the same sign as $\rho_r - \rho_l$.

Thus, from Remark 3.2, (iii) follows.

(v) is trivial from the definitions of c_l, c_r. □

Proof of Theorem 3.1 First we remark that $\rho_l^* \geq 0$ as long as

$$\frac{\widetilde{c}_l}{\rho_l} \geq \frac{\widetilde{c}_r}{\widetilde{c}_l + \widetilde{c}_r}(u_l - u_r) + \frac{p_r - p_l}{\widetilde{c}_l + \widetilde{c}_r}, \tag{3.42}$$

which is equivalent to

$$\widetilde{c}_r\left(\frac{\widetilde{c}_l}{\rho_l} - (u_l - u_r)\right) \geq p_r - p_l - \frac{\widetilde{c}_l^2}{\rho_l}. \tag{3.43}$$

Thus, the condition $\widetilde{c}_l \geq \max(\rho_l(u_l - u_r)_+, \sqrt{\rho_l(p_r - p_l)_+})$ implies the non negativity of ρ_l^*. The analogous can be said for ρ_r^* and this proves (i).

From Lemma 3.4 we get that $\widetilde{c}_l, \widetilde{c}_r$ satisfy the semi-discrete entropy inequality (3.18), and this proves (ii).

Finally, given a 1-shock for system (3.1), we have

$$c_l^2 - \rho_l(p_r - p_l) = \frac{(p_r - p_l)^2}{(u_l - u_r)^2} - \rho_l(p_r - p_l) = \frac{p_r - p_l}{1/\rho_l - 1/\rho_r}\frac{1}{\rho_r} \geq 0, \tag{3.44}$$

and

$$\rho_l(u_l - u_r) = \rho_l\sqrt{(p_r - p_l)\left(\frac{1}{\rho_l} - \frac{1}{\rho_r}\right)} < \sqrt{\rho_l(p_r - p_l)}. \tag{3.45}$$

Thus $\widetilde{c}_l = \frac{p_r - p_l}{u_l - u_r}$ for 1-shocks and it is preserved by the Suliciu system. The same can be done for the 2-shocks and this proves (iii). □

Remark 3.4 While the property of entropy-satisfying has been established in the semi-discrete sense, the non-negativity of ρ has been established in a discrete sense. Thus, the Riemann solver that has been introduced preserves the convex invariant domain $\mathcal{U} = \{\rho \geq 0\}$ in the sense that

$$U_i^n \in \mathcal{U} \quad \text{for all } i \quad \Rightarrow \quad U_i^{n+1} \in \mathcal{U} \quad \text{for all } i. \tag{3.46}$$

4 Suliciu Relaxation Solver for Full Gas Dynamics

It is also possible to generalize the Suliciu solver to the full gas dynamics system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho(u^2/2 + e)) + \partial_x((\rho(u^2/2 + e) + p)u) = 0, \end{cases} \tag{4.1}$$

where $\rho(t,x) \geq 0$ is the density, $u(t,x)$ is the velocity, $e(t,x) \geq 0$ is the internal energy, and $p = p(\rho, e)$ is the pressure law.

Thermodynamic considerations lead to assume that

$$de + pd(1/\rho) = T ds, \tag{4.2}$$

for some temperature $T(\rho, e) > 0$, and specific entropy $s(\rho, e)$.

We can define a family of entropies

$$\mathcal{H} = \rho\phi(s), \tag{4.3}$$

with entropy fluxes

$$\mathcal{G} = \rho\phi u, \tag{4.4}$$

where ϕ is an arbitrary function such that $\mathcal{H}$ is convex with respect to the conservative variables $(\rho, \rho u, \rho(u^2/2 + e))$.

Remark 4.1 A necessary condition for $\mathcal{H}$ in (4.3) to be convex with respect to $(\rho, \rho u, \rho(u^2/2 + e))$ is that $\phi' \leq 0$. Conversely, if—s is a convex function of $(1/\rho, e)$ and if $\phi' \leq 0$ and $\phi'' \geq 0$, then $\mathcal{H}$ is convex.

The full gas dynamics system and the definition of an entropy satisfying scheme can be handled via a general idea introduced in [6] adapted here to the semi-discrete case.

4.1 Reduction to an Almost Isentropic System

The idea consists in reversing the role of energy conservation and entropy inequality. We shall consider the system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho s) + \partial_x(\rho s u) = 0, \end{cases} \tag{4.5}$$

with the entropy inequality

$$\partial_t(\rho(u^2/2 + e)) + \partial_x((\rho(u^2/2 + e) + p)u) \leq 0. \tag{4.6}$$

The convexity of this entropy with respect to $(\rho, \rho u, \rho s)$ is ensured by the physically relevant condition

$$-s \text{ is a convex function of } (1/\rho, e). \tag{4.7}$$

Now suppose that we have a conservative numerical scheme for solving (4.5),

$$\begin{cases} \rho_i^{n+1} - \rho_i + \dfrac{\Delta t}{\Delta x}(F_{i+1/2}^{\rho} - F_{i-1/2}^{\rho}) = 0, \\ \rho_i^{n+1} u_i^{n+1} - \rho_i u_i + \dfrac{\Delta t}{\Delta x}(F_{i+1/2}^{\rho u} - F_{i-1/2}^{\rho u}) = 0, \\ \rho_i^{n+1} s_i^{n+1} - \rho_i s_i + \dfrac{\Delta t}{\Delta x}(F_{i+1/2}^{\rho s} - F_{i-1/2}^{\rho s}) = 0, \end{cases} \tag{4.8}$$

satisfying a semi-discrete entropy inequality for

$$\eta(U, \rho s) = \rho u^2/2 + \rho e(\rho, s), \qquad G(U, \rho s) = (\rho u^2/2 + \rho e(\rho, s) + p(\rho, s))u, \tag{4.9}$$

which means

$$\begin{aligned} &G(U_r, \rho_r s_r) - G(U_l, \rho_l s_l) \\ &\quad + \nabla_{(U,\rho s)}\eta(U_r, \rho_r s_r)(F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l, U_r, \rho_r s_r) - F^{(\rho,\rho u,\rho s)}(U_r, \rho_r s_r)) \\ &\quad - \nabla_{(U,\rho s)}\eta(U_l, \rho_l s_l)(F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l, U_r, \rho_r s_r) - F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l)) \le 0. \end{aligned} \tag{4.10}$$

We assume moreover that the scheme satisfies for any ϕ convex the inequalities

$$\begin{aligned} &\mathcal{G}(U_r, \rho_r s_r) - \mathcal{G}(U_l, \rho_l s_l) \\ &\quad + \nabla_{(U,\rho s)}\mathcal{H}(U_r, \rho_r s_r)(F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l, U_r, \rho_r s_r) - F^{(\rho,\rho u,\rho s)}(U_r, \rho_r s_r)) \\ &\quad - \nabla_{(U,\rho s)}\mathcal{H}(U_l, \rho_l s_l)(F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l, U_r, \rho_r s_r) - F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l)) \le 0. \end{aligned} \tag{4.11}$$

Define now the scheme for the gas dynamics system (4.1)

$$\begin{cases} \rho_i^{n+1} - \rho_i + \dfrac{\Delta t}{\Delta x}(F_{i+1/2}^{\rho} - F_{i-1/2}^{\rho}) = 0, \\ \rho_i^{n+1} u_i^{n+1} - \rho_i u_i + \dfrac{\Delta t}{\Delta x}(F_{i+1/2}^{\rho u} - F_{i-1/2}^{\rho u}) = 0, \\ E_i^{n+1} - E_i + \dfrac{\Delta t}{\Delta x}(F_{i+1/2}^{E} - F_{i-1/2}^{E}) = 0, \end{cases} \tag{4.12}$$

where $E = \rho \frac{u^2}{2} + \rho e$, and F^E is any flux verifying

$$\begin{aligned} &G(U_r, \rho_r s_r) \\ &\qquad + \nabla_{(U,\rho s)}\eta(U_r, \rho_r s_r)(F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l, U_r, \rho_r s_r) - F^{(\rho,\rho u,\rho s)}(U_r, \rho_r s_r)) \\ &\quad \le F^E(U_l, \rho_l s_l, U_r, \rho_r, s_r) \\ &\quad \le G(U_l, \rho_l s_l) \\ &\qquad + \nabla_{(U,\rho s)}\eta(U_l, \rho_l s_l)(F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l, U_r, \rho_r s_r) - F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l)). \end{aligned} \tag{4.13}$$

Of course, we take initially $s_i = s(\rho_i, e_i)$.

Proposition 4.1 *The scheme* (4.12) *verifies a semi-discrete entropy inequality for the entropy pair*

$$\mathcal{H}(U,E)=\rho\phi(s(\rho,e)),\quad \mathcal{G}(U,E)=\rho\phi(s(\rho,e))u, \tag{4.14}$$

i.e.,

$$\begin{aligned}&\mathcal{G}(U_r,E_r)-\mathcal{G}(U_l,E_l)\\&\quad+\nabla_{(U,E)}\mathcal{H}(U_r,E)(F^{(\rho,\rho u,E)}(U_l,E_l,U_r,E_r)-F^{(\rho,\rho u,E)}(U_r,E_r))\\&\quad-\nabla_{(U,E)}\mathcal{H}(U_l,E_l)(F^{(\rho,\rho u,E)}(U_l,\rho_l s_l,U_r,E_r)-F^{(\rho,\rho u,E)}(U_l,E_l))\le 0.\end{aligned} \tag{4.15}$$

Proof Let $\mathcal{S}(U,E)=\rho s(\rho,e)$.

Using the relations

$$\begin{aligned}&\mathcal{H}(U,E)=\mathcal{H}(U,\rho s(\rho,e)),\quad \mathcal{G}(U,E)=\mathcal{G}(U,\rho s(\rho,e)),\\&\nabla_{(U,E)}\mathcal{H}(U,E)=(\nabla_U\mathcal{H}(U,\rho s(\rho e)),0)\\&\qquad\qquad+\frac{\partial\mathcal{H}}{\partial(\rho s)}(U,\rho s(\rho,e))\nabla_{(U,E)}\mathcal{S}(U,E),\\&\nabla_{(U,E)}\mathcal{S}(U,\eta(U,\rho s))=-\left(\frac{\partial\eta}{\partial(\rho s)}\right)^{-1}\cdot(\nabla_U\eta(U,\rho s),-1),\end{aligned} \tag{4.16}$$

we decompose the left hand side of (4.15) in three sum terms (I), (II), (III),

$$\begin{aligned}(I)&=\mathcal{G}(U_r,\rho_r s_r)-\mathcal{G}(U_l,\rho_l s_l)\\&\quad+\nabla_{(U,\rho s)}\mathcal{H}(U_r,\rho_r s_r)\big(F^{(\rho,\rho u,\rho s)}(U_l,\rho_l s_l,U_r,\rho_r s_r)-F^{(\rho,\rho u,\rho s)}(U_r,\rho_r s_r)\big)\\&\quad-\nabla_{(U,\rho s)}\mathcal{H}(U_l,\rho_l s_l)\big(F^{(\rho,\rho u,\rho s)}(U_l,\rho_l s_l,U_r,\rho_r s_r)\\&\quad-F^{(\rho,\rho u,\rho s)}(U_l,\rho_l s_l)\big),\end{aligned} \tag{4.17}$$

$$\begin{aligned}(II)&=\frac{\partial\mathcal{H}}{\partial(\rho s)}(U_r,\rho_r s_r)\nabla_{(U,E)}\mathcal{S}(U_r,E_r)\big(F^{(\rho,\rho u,E)}(U_l,\rho_l s_l,U_r,\rho_r s_r)-F^{(\rho,\rho u,E)}(U_r,\rho_r s_r)\big)\\&\quad-\frac{\partial\mathcal{H}}{\partial(\rho s)}(U_r,\rho_r s_r)(F^{\rho s}(U_l,\rho_l s_l,U_r,\rho_r s_r)-F^{\rho s}(U_r,\rho_r s_r)),\end{aligned} \tag{4.18}$$

$$\begin{aligned}(III)&=-\frac{\partial\mathcal{H}}{\partial(\rho s)}(U_l,\rho_l s_l)\nabla_{(U,E)}\mathcal{S}(U_l,E_l)\big(F^{(\rho,\rho u,E)}(U_l,\rho_l s_l,U_r,\rho_r s_r)-F^{(\rho,\rho u,E)}(U_l,\rho_l s_l)\big)\\&\quad+\frac{\partial\mathcal{H}}{\partial(\rho s)}(U_l,\rho_l s_l)(F^{\rho s}(U_l,\rho_l s_l,U_r,\rho_r s_r)-F^{\rho s}(U_l,\rho_r s_l)).\end{aligned} \tag{4.19}$$

We show that each of them is non-positive.

From (4.11), it follows that $(I) \le 0$. We shall prove $(II) \le 0$, the inequality $(III) \le 0$ being similar.

Using (4.13) and (4.16) one gets

$$\begin{aligned}
(II) &= -\phi'(s_r)\frac{\partial s}{\partial e}(\rho_r, e_r)(\nabla_U \eta(U_r, \rho_r s_r), -1)\big(F^{(\rho,\rho u,E)}(U_l, \rho_l s_l, U_r, \rho_r s_r) \\
&\quad - F^{(\rho,\rho u,E)}(U_r, \rho_r s_r)\big) - \phi'(s_r)(F^{\rho s}(U_l, \rho_l s_l, U_r, \rho_r s_r) - F^{\rho s}(U_r, \rho_r s_r)) \\
&= -\phi'(s_r)\frac{\partial s}{\partial e}(\rho_r, e_r)\big(-F^E(U_l, \rho_l s_l, U_r, \rho_r s_r) + F^E(U_r, \rho_r s_r) \\
&\quad + \nabla_{(U,\rho s)}\eta(U_r, \rho_r s_r)\big(F^{(\rho,\rho u,\rho s)}(U_l, \rho_l s_l, U_r, \rho_r s_r) \\
&\quad - F^{(\rho,\rho u,\rho s)}(U_r, \rho_r s_r)\big)\big) \le 0.
\end{aligned} \tag{4.20}$$

And this completes the proof. □

4.2 Resolution of an Extended Suliciu Relaxation System

In order to solve system (4.5) we propose an extension of system (3.4), (3.12). We add the transport of specific entropy,

$$\begin{cases}
\partial_t \rho + \partial_x(\rho u) = 0, \\
\partial_t(\rho u) + \partial_x(\rho u^2 + \pi) = 0, \\
\partial_t(\rho u^2/2 + \rho e) + \partial_x((\rho u^2/2 + \rho e + \pi)u) = 0, \\
\partial_t(\rho \pi/c^2) + \partial_x(\rho \pi u/c^2) + \partial_x u = 0, \\
\partial_t(\rho c) + \partial_x(\rho c u) = 0, \\
\partial_t(\rho s) + \partial_x(\rho s u) = 0.
\end{cases} \tag{4.21}$$

One has to take care that in (4.21), ρ, e, s, π are understood as independent variables. The solution to the Riemann problem for (4.21) is obvious since s is not coupled, and it has the same structure as before: (3.9), (3.13), to which we prescribe $s_l^* = s_l, s_r^* = s_r$. Only the initialization is modified, now

$$\pi_l = p_l = p(\rho_l, s_l), \quad \pi_r = p_r = p(\rho_r, s_r). \tag{4.22}$$

We shall define as before the simple Riemann solver $R_{c_l,c_r}(v, U_l, \rho_l s_l, U_r, \rho_r s_r)$ as the canonical projection on the variables $(U, \rho s)$ of the exact solution of the corresponding Riemann problem for (4.21) with (4.22) and c_l, c_r to be defined later on.

Using the same approach as in the isentropic case, we consider the decomposition

$$G(U_r, \rho_r s_r) - G(U_l, \rho_l s_l) = \xi_3 - \xi_0 = \sum_{i=1}^{3} \sigma_i(E_i - E_{i-1}), \tag{4.23}$$

with $E = \rho u^2/2 + \rho e$ and $\xi = (\rho u^2/2 + \rho e + \pi)u$. Recall that here e is an independent variable. Then, from Proposition 2.11, a sufficient condition for (4.10) to hold is

$$E_i \ge \nabla_{(U,\rho s)}\eta(U, \rho s)\begin{pmatrix} U_i - U \\ \rho_i s_i - \rho s \end{pmatrix} + \eta(U, \rho s), \tag{4.24}$$

where $(U,\rho s)$ stands for either $(U_l,\rho_l s_l)$ or $(U_r,\rho_r s_r)$, and the i-indexed quantities stand for either $(U_l^*,\rho_l^* s_l)$ or $(U_r^*,\rho_r^* s_r)$. Using the expression of η and assuming $\rho_l^*,\rho_r^*\geq 0$, condition (4.24) can be replaced by

$$e(\rho,s)-e_i+p(\rho,s)\left(\frac{1}{\rho}-\frac{1}{\rho_i}\right)-\frac{(u_i-u)^2}{2}+\frac{\partial e}{\partial s}(\rho,s)(s_i-s)\leq 0, \tag{4.25}$$

with

$$\begin{aligned}&(\rho,u,s)\in\{(\rho_l,u_l,s_l),(\rho_r,u_r,s_r)\},\\ &(\rho_i,u_i,e_i,s_i)\in\{(\rho_l^*,u_l^*,e_l^*,s_l^*),(\rho_r^*,u_r^*,e_r^*,s_r^*)\}.\end{aligned} \tag{4.26}$$

We shall note $\tau=1/\rho$ and we rewrite relation (4.2) under the form

$$de=-p\,d\tau+T\,ds, \tag{4.27}$$

being e the internal energy, p the pressure and T the temperature.

Definition 4.2 Define $\overline{\frac{\partial p}{\partial\tau}}|_r$ by the relation

$$\frac{p_r-p_l}{\tau_l-\tau_r}=-\overline{\frac{\partial p}{\partial\tau}}\bigg|_r-\frac{p(\tau_r,s_l)-p(\tau_r,s_r)}{e(\tau_r,s_l)-e(\tau_r,s_r)}\cdot\frac{e_l-e_r-\frac{p_l+p_r}{2}(\tau_r-\tau_l)}{\tau_l-\tau_r}. \tag{4.28}$$

We may also define $\overline{\frac{\partial p}{\partial\tau}}|_l$ by exchanging the labels l,r in the previous definition.

Remark 4.2

$$\lim_{s_r,s_l\to s}\frac{p(\tau_r,s_l)-p(\tau_r,s_r)}{e(\tau_r,s_l)-e(\tau_r,s_r)}=\frac{\partial p}{\partial s}\left(\frac{\partial e}{\partial s}\right)^{-1}(\tau_r,s)=\frac{\partial p}{\partial e}(\tau_r,e(\tau_r,s)). \tag{4.29}$$

Thus,

$$p_l-p_r-\frac{p(\tau_r,s_l)-p(\tau_r,s_r)}{e(\tau_r,s_l)-e(\tau_r,s_r)}(e_l-e_r)=(\tau_l-\tau_r)\varphi,\quad \text{with }\varphi\text{ a regular function}, \tag{4.30}$$

and $\overline{\frac{\partial p}{\partial\tau}}|_r$ is well defined.

Theorem 4.3 *Let*

$$c_l^2=\max\left(-\overline{\frac{\partial p}{\partial\tau}}\bigg|_l,\quad\frac{(p_r-p_l)^2}{2(e_l-e_r+p_r(\frac{1}{\rho_l}-\frac{1}{\rho_r})+\frac{\partial e}{\partial s}(\rho_r,s_r)(s_r-s_l))}\right), \tag{4.31}$$

$$c_r^2=\max\left(-\overline{\frac{\partial p}{\partial\tau}}\bigg|_r,\quad\frac{(p_r-p_l)^2}{2(e_r-e_l+p_l(\frac{1}{\rho_r}-\frac{1}{\rho_l})+\frac{\partial e}{\partial s}(\rho_l,s_l)(s_l-s_r))}\right). \tag{4.32}$$

Then, for

$$\begin{aligned}\widetilde{c}_l=\max\Bigg(&c_l,\rho_l(u_l-u_r)_+,\sqrt{\rho_l(p_r-p_l)_+},\\ &\frac{(u_r-u_l)_++\sqrt{((u_r-u_l)_+)^2+4\frac{e_l}{p_l}(p_l-p_r)_+}}{2e_l/p_l}\Bigg),\end{aligned} \tag{4.33}$$

$$\widetilde{c}_r = \max\Bigg(c_r, \rho_r(u_l - u_r)_+, \sqrt{\rho_r(p_l - p_r)_+},$$

$$\frac{(u_r - u_l)_+ + \sqrt{((u_r - u_l)_+)^2 + 4\frac{e_r}{p_r}(p_r - p_l)_+}}{2e_r/p_r}\Bigg), \tag{4.34}$$

the simple solver $R_{\widetilde{c}_l,\widetilde{c}_r}$ *defined by the wave speeds* (3.7) *and intermediate states* (3.9), (3.13), *with* (4.22) *is an approximate Riemann solver for the full gas dynamics system* (4.1) *and satisfies the following properties*:

(i) *it preserves the non negativity of* ρ,
(ii) *it preserves the non negativity of* e,
(iii) *it satisfies a semi-discrete entropy inequality*,
(iv) *it preserves the entropy* 1 *and* 3 *shocks for the full gas dynamics system* (4.1).

Remark 4.3 Let (ρ_l, u_l, e_l), (ρ_r, u_r, e_r), such that $(u_r - u_l)(1/\rho_r - 1/\rho_l) \neq 0$. We recall that these two states can be connected by a shock for the full gas dynamics system (4.1) if, and only if, the following two relations hold,

$$(u_r - u_l)^2 = (p_r - p_l)\left(\frac{1}{\rho_l} - \frac{1}{\rho_r}\right), \tag{4.35}$$

$$e_r - e_l = \frac{p_r + p_l}{2}\left(\frac{1}{\rho_l} - \frac{1}{\rho_r}\right). \tag{4.36}$$

The entropy 1-shocks for (4.1) satisfy

$$u_l \geq u_r, \quad \rho_r \geq \rho_l, \quad p_r \geq p_l, \quad s_r \geq s_l, \quad e_r \geq e_l. \tag{4.37}$$

The entropy 3-shocks for (4.1) satisfy

$$u_l \geq u_r, \quad \rho_r \leq \rho_l, \quad p_r \leq p_l, \quad s_r \leq s_l, \quad e_r \leq e_l. \tag{4.38}$$

To prove this theorem, we shall prove some lemmas.

Lemma 4.4 *Under the assumptions*

$$\begin{aligned} c_l^2 &\geq \frac{(p_r - p_l)^2}{2(e_l - e_r + p_r(\frac{1}{\rho_l} - \frac{1}{\rho_r}) + \frac{\partial e}{\partial s}(\rho_r, s_r)(s_r - s_l))}, \\ c_r^2 &\geq \frac{(p_r - p_l)^2}{2(e_r - e_l + p_l(\frac{1}{\rho_r} - \frac{1}{\rho_l}) + \frac{\partial e}{\partial s}(\rho_l, s_l)(s_l - s_r))}, \end{aligned} \tag{4.39}$$

condition (4.25) *is satisfied.*

Proof In the cases $(U, \rho s, U_i, \rho_i s_i) = (U_l, \rho_l s_l, U_l^*, \rho_l^* s_l^*)$ and $(U, \rho s, U_i, \rho_i s_i) = (U_r, \rho_r s_r, U_r^*, \rho_r^* s_r^*)$, we have $s - s_i = 0$ and following analogous calculations as in the isentropic case, one gets

$$e(\rho, s) - e_i + p(\rho, s)\left(\frac{1}{\rho} - \frac{1}{\rho_i}\right) - \frac{(u_i - u)^2}{2} + \frac{\partial e}{\partial s}(s_i - s)$$
$$= -\frac{1}{c^2}(p(\rho, s) - \pi_i)^2 \leq 0. \tag{4.40}$$

We show now the case $(U, \rho s, U_i, \rho_i s_i) = (U_r, \rho_r s_r, U_l^*, \rho_l^* s_l^*)$ the other case being similar. Some simple calculations show

$$\begin{aligned} e(\rho_r, s_r) - e_l^* + p(\rho_r, s_r)\left(\frac{1}{\rho_r} - \frac{1}{\rho_l^*}\right) - \frac{(u_l^* - u_r)^2}{2} + \frac{\partial e}{\partial s}(s_l^* - s_r) \\ = e_r - e_l + p_r\left(\frac{1}{\rho_r} - \frac{1}{\rho_l}\right) + \frac{1}{2c_l^2}(p_r - p_l)^2 \\ + \frac{\partial e}{\partial s}(\rho_r, s_r)(s_l - s_r) - \frac{c_l^2 + c_r^2}{2(c_l + c_r)^2}\left(u_r - u_l + \frac{p_r - p_l}{c_l}\right)^2. \end{aligned} \tag{4.41}$$

Using Taylor's theorem,

$$\begin{aligned} e(\tau_l, s_l) = e(\tau_r, s_r) + (-p(\tau_r, s_r), T(\tau_r, s_r))\begin{pmatrix} \tau_l - \tau_r \\ s_l - s_r \end{pmatrix} \\ + D^2_{\tau,s} e(\tau^*, s^*)\begin{pmatrix} \tau_l - \tau_r \\ s_l - s_r \end{pmatrix}^2, \end{aligned} \tag{4.42}$$

for some (τ^*, s^*) in the segment defined by $(\tau_l, s_l), (\tau_r, s_r)$.

As $-s(\tau, e)$ is a convex function, then $e(\tau, s)$ is convex and we get

$$e_r - e_l + p_r\left(\frac{1}{\rho_r} - \frac{1}{\rho_l}\right) + \frac{\partial e}{\partial s}(\rho_r, s_r)(s_l - s_r) \le 0, \tag{4.43}$$

and the result follows. □

Lemma 4.5 *Let $(\rho_l, \rho_l u_l, E_l), (\rho_r, \rho_r u_r, E_r)$ two different states such that they can be connected by an entropy 1-shock (resp. 2-shock) for the full gas dynamics system (4.1) with shock speed σ. Then the following conditions are equivalent.*

(i) *The Suliciu relaxation system preserves the 1-shock (resp. preserves the 3-shock),*
(ii) $c_l = \frac{p_r - p_l}{u_l - u_r}$ *(resp. $c_r = \frac{p_l - p_r}{u_l - u_r}$).*

Proof The proof can be stated in the same way as in the isentropic case. □

Lemma 4.6 *Let c_l, c_r be defined by (4.31)–(4.34). Then the following properties hold:*

(i) $\lim_{\substack{\rho_l, \rho_r \to \rho \\ s_r, s_l \to s}} c_l = \lim_{\substack{\rho_l, \rho_r \to \rho \\ s_r, s_l \to s}} c_r = \sqrt{-\frac{\partial p}{\partial(\frac{1}{\rho})}(\rho, s)}$,
(ii) *If $(\rho_l, u_l, e_l), (\rho_r, u_r, e_r)$ can be connected by an entropy 1-shock for (4.1), then* $c_l = \frac{p_r - p_l}{u_l - u_r}$,
(iii) *If $(\rho_l, u_l, e_l), (\rho_r, u_r, e_r)$ can be connected by an entropy 3-shock for (4.1), then* $c_r = \frac{p_l - p_r}{u_l - u_r}$,
(iv) c_l, c_r *satisfy the inequalities (4.39).*

Proof

$$-\overline{\frac{\partial p}{\partial \tau}}\bigg|_l = -\left(\varphi - \frac{p_l + p_r}{2} \cdot \frac{p(\rho_r, s_r) - p(\rho_r, s_l)}{e(\rho_r, s_r) - e(\rho_r, s_l)}\right), \tag{4.44}$$

where $\varphi = \frac{\partial p}{\partial(\frac{1}{\rho})}(\rho_r, s_l) + p(\rho_r, s_l)\frac{p(\rho_r,s_r)-p(\rho_r,s_l)}{e(\rho_r,s_r)-e(\rho_r,s_l)} + O(\rho_r - \rho_l)$.

Thus, we get the limit $-\overline{\frac{\partial p}{\partial \tau}}|_l \to -\frac{\partial p}{\partial \tau}$ for $\rho_r - \rho_l \to 0$ and $s_r - s_l \to 0$.

The convexity of $e(\tau, s)$ implies

$$\left(\nabla p(\tau, s)\begin{pmatrix} x \\ y \end{pmatrix}\right)^2 \le -\frac{\partial p}{\partial \tau}(\tau, s) \cdot D^2 e(\tau, s)\begin{pmatrix} x \\ y \end{pmatrix}^2, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{4.45}$$

Thus, there exist (τ_1, s_1), (τ_2, s_2) in the segment defined by (τ_l, s_l), (τ_r, s_r) such that,

$$\begin{aligned} 0 &\le \frac{(p_r - p_l)^2}{2(e_l - e_r + p_r(\frac{1}{\rho_l} - \frac{1}{\rho_r}) + \frac{\partial e}{\partial s}(\rho_r, s_r)(s_r - s_l))} \\ &= \frac{(\nabla p(\tau_1, s_1)(\begin{matrix} \tau_r - \tau_l \\ s_r - s_l \end{matrix}))^2}{D^2 e(\tau_2, s_2)(\begin{matrix} \tau_r - \tau_l \\ s_r - s_l \end{matrix})^2} \\ &\le -\frac{\partial p}{\partial \tau}(\rho_1, s_1)\frac{D^2 e(\tau_1, s_1)(\begin{matrix} \tau_r - \tau_l \\ s_r - s_l \end{matrix})^2}{D^2 e(\tau_2, s_2)(\begin{matrix} \tau_r - \tau_l \\ s_r - s_l \end{matrix})^2} \to -\frac{\partial p}{\partial \tau}, \end{aligned} \tag{4.46}$$

and (i) follows.

From Lemma 4.5 and from Remark 4.3, (ii) and (iii) are easily verified.

Finally, (iv) is trivial. □

Proof of Theorem 4.3 As in the isentropic case, the condition $\widetilde{c}_l \ge \max(\rho_l(u_l - u_r)_+, \sqrt{\rho_l(p_r - p_l)})$ implies the non negativity of ρ_l^*. The analogous can be said for ρ_r^*.

We shall show $e_l^* \ge 0$ ($e_r^* \ge 0$ is proved in the same way). This is equivalent to

$$\begin{aligned} 2\widetilde{c}_l^2 e_l \ge &-\left(\frac{\widetilde{c}_l}{\widetilde{c}_l + \widetilde{c}_r}\right)^2 (p_r - p_l - \widetilde{c}_r(u_r - u_l))^2 \\ &- 2p_l \frac{\widetilde{c}_l}{\widetilde{c}_l + \widetilde{c}_r}(p_r - p_l - \widetilde{c}_r(u_r - u_l). \end{aligned} \tag{4.47}$$

Thus, a sufficient condition for $e_l^* \ge 0$ to hold is

$$2\widetilde{c}_l^2 e_l \ge -2p_l \frac{\widetilde{c}_l}{\widetilde{c}_l + \widetilde{c}_r}(p_r - p_l - \widetilde{c}_r(u_r - u_l)), \tag{4.48}$$

and, as we have

$$-p_l \frac{1}{\widetilde{c}_l + \widetilde{c}_r}(p_r - p_l - \widetilde{c}_r(u_r - u_l)) \le p_l\left(\frac{(p_l - p_r)_+}{\widetilde{c}_l} + (u_r - u_l)_+\right), \tag{4.49}$$

a sufficient condition for (4.48) to hold is

$$\widetilde{c}_l e_l \ge p_l\left(\frac{(p_l - p_r)_+}{\widetilde{c}_l} + (u_r - u_l)_+\right) \tag{4.50}$$

which is satisfied if we take

$$\widetilde{c}_l \geq \frac{(u_r - u_l)_+ + \sqrt{((u_r - u_l)_+)^2 + 4\frac{e_l}{p_l}(p_l - p_r)_+}}{2\frac{e_l}{p_l}}, \tag{4.51}$$

and this proves (ii).
(iii) is trivial from Lemma 4.4.

Finally, for an entropy 1-shock for the system (4.1), we have $p_r \geq p_l$, $u_l \geq u_r$, and for an entropy 3-shock we have $p_l \geq p_r$, $u_l \geq u_r$.

Thus, (iv) can be established in the same way as in the isentropic case. □

Remark 4.4 As in the isentropic case, the non-negativity of the variables ρ, e has been established in a discrete sense. (See Remark 3.4.)

5 Numerical Tests

We show here some tests for the scheme described in Sect. 3. For all tests we take $p(\rho) = g\rho^2/2$ with $g = 9.81$. We use 100 point in the specified domain.

First, we consider the following Riemann problem in the interval $[-0.5, 0.5]$ with initial data

$$\rho^0(x) = \begin{cases} 0.5, & \text{if } x < 0, \\ 1, & \text{if } x > 0, \end{cases} \qquad u^0(x) = \begin{cases} 2\sqrt{g\frac{3}{2}}, & \text{if } x < 0, \\ \sqrt{g\frac{3}{2}}, & \text{if } x > 0. \end{cases} \tag{5.1}$$

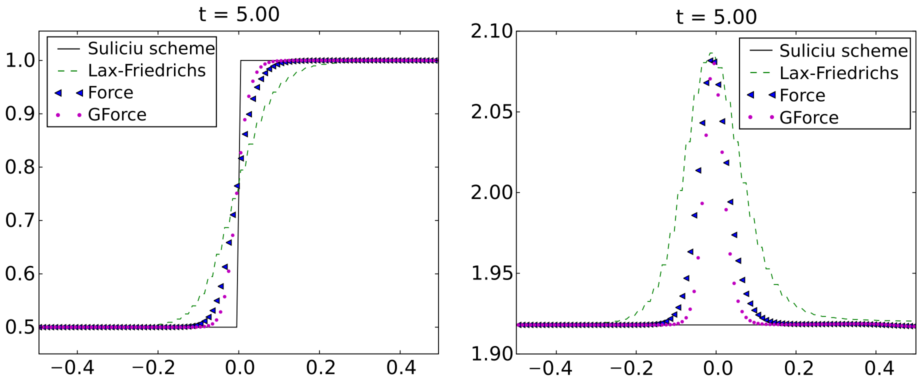
The left and right states can be connected by a stationary entropy shock. We know that the modified Suliciu scheme described in Sect. 3 preserves such shocks as it is shown in Fig. 1 and Fig. 2, while it is not the case for other schemes shown here. We remark that Roe scheme would preserve also such shocks but its computational cost is more expensive than that of Suliciu scheme. Moreover, Roe scheme does not satisfy in general an entropy inequality; as a consequence, an entropy-fix technique has to be added to the numerical scheme in order to capture the entropy solution in the presence of smooth transitions while it is not the case for the Suliciu scheme. Figure 1 compares the scheme defined in Sect. 3 with classical Lax-Friedrichs scheme and Force and Gforce schemes introduced in [13], while Fig. 2 compares the Suliciu scheme to Lax-Friedrichs combined with a third order WENO reconstruction or the hyperbolic reconstruction introduced in [9] as well as the GForce scheme combined with an hyperbolic reconstruction.

Let us consider now the a Riemann problem in the interval [0, 1] with initial data

$$\rho^0(x) = \begin{cases} 1, & \text{if } x < 0.5, \\ 2, & \text{if } x > 0.5, \end{cases} \qquad u^0(x) = \begin{cases} 3.2125, & \text{if } x < 0.5, \\ 0.5, & \text{if } x > 0.5. \end{cases} \tag{5.2}$$

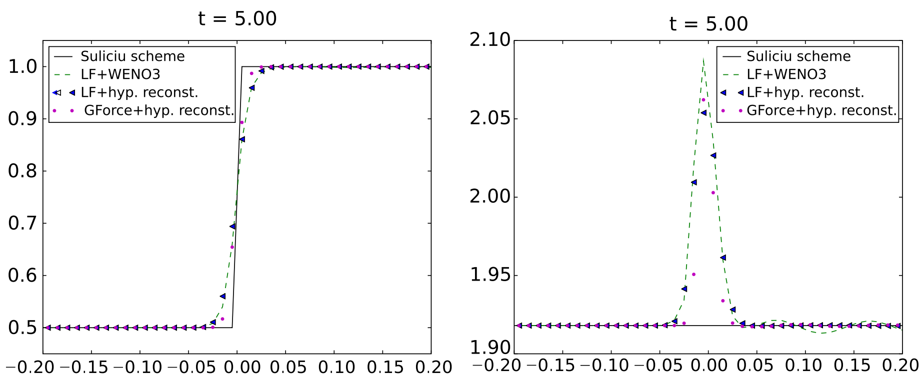
The exact solution to this Riemann consists of a 1-shock of speed $\sigma = -2.2125$.

Numerical results are shown in Fig. 3, Fig. 4 and Fig. 5. Figure 3 compares the modified Suliciu scheme introduced in Sect. 3 with the original discrete entropy satisfying Suliciu scheme. Figure 4 and Fig. 5 compare the scheme with other numerical fluxes. As we can see, the semi-discrete approach give less diffusive shock profiles than the fully discrete approach and the accuracy of the scheme in shocks is comparable to that of high order reconstruction schemes.



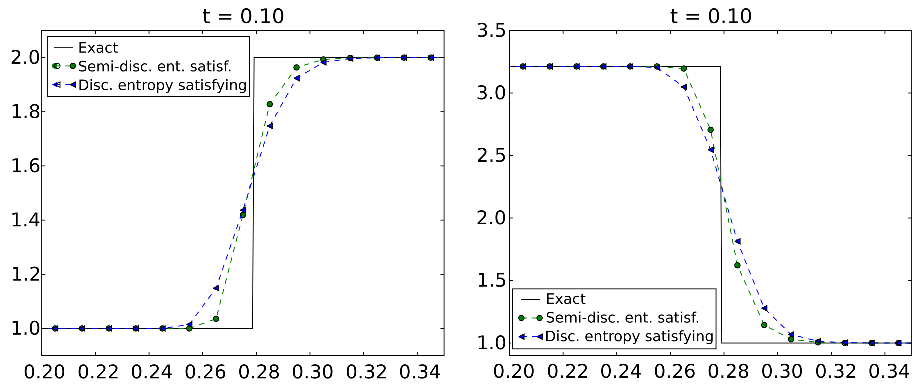
(a) Density (b) Discharge

Fig. 1 Numerical simulations corresponding to (5.1). Comparison with Lax-Friedrichs, Force and GForce schemes



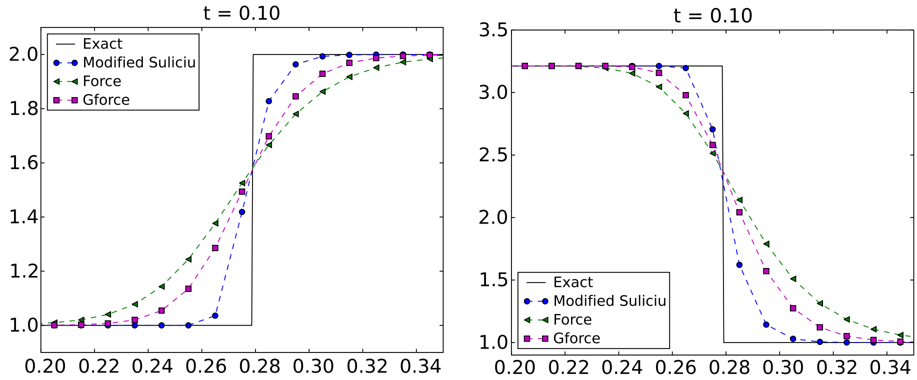
(a) Density (b) Discharge

Fig. 2 Numerical simulations corresponding to (5.1). Comparison with High Order schemes



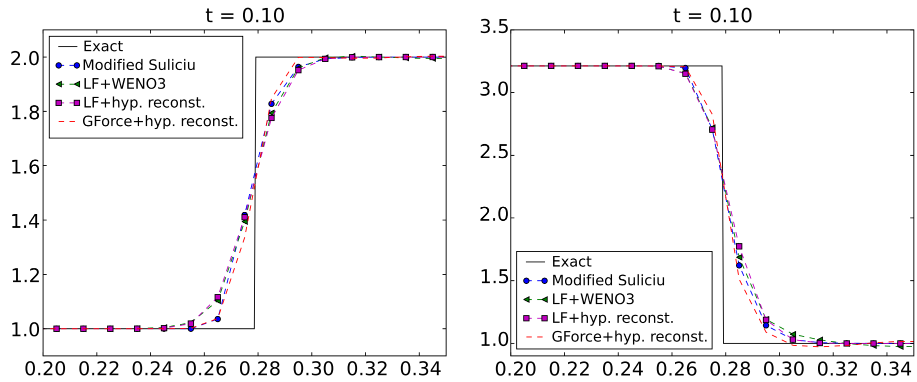
(a) Density (b) Discharge

Fig. 3 Numerical simulations corresponding to (5.2). Comparison of semi-discrete entropy satisfying Suliciu scheme with the fully discrete entropy satisfying Suliciu scheme



(a) Density (b) Discharge

Fig. 4 Numerical simulations corresponding to (5.2). Comparison with Lax-Friedrichs, Force and GForce schemes



(a) Density (b) Discharge

Fig. 5 Numerical simulations corresponding to (5.2). Comparison with High Order schemes

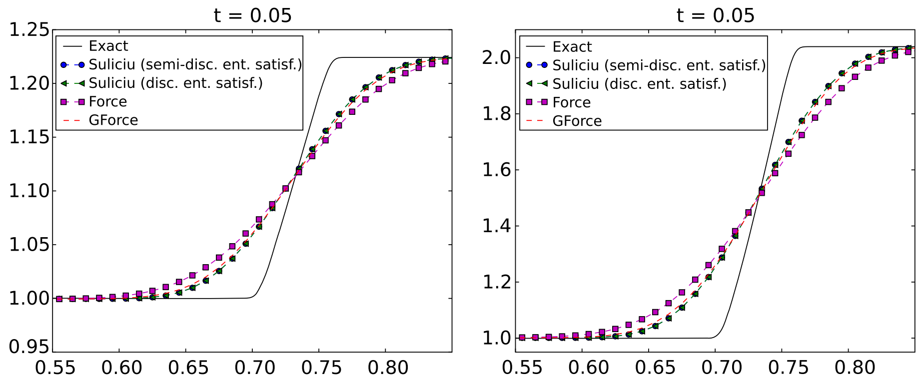
Now, we solve the Riemann problem with initial data

$$\rho^0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 1.2242 & \text{if } x > 0.5, \end{cases} \qquad u^0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 1.666 & \text{if } x > 0.5. \end{cases} \tag{5.3}$$

The solution consist of a simple rarefaction wave. As we can see in Fig. 6, no major differences is shown in this case for the two Suliciu schemes as the improvement of the modified Suliciu scheme is related to shock regions. Figure 7 compares the scheme with different high order reconstruction techniques. As expected, a high order scheme will give better results in smooth regions.

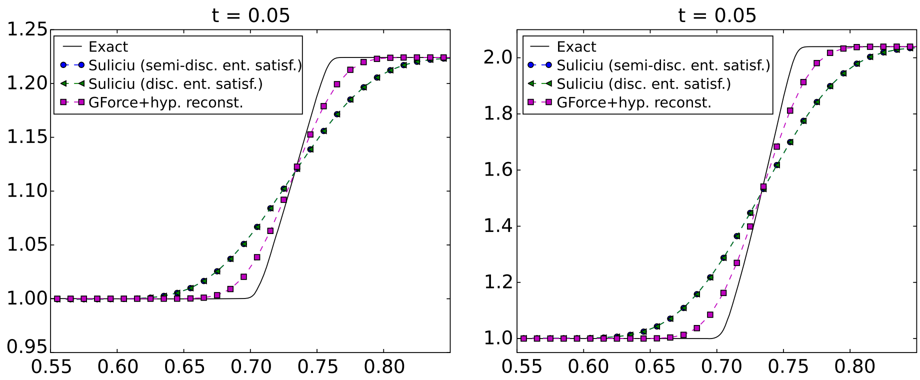
Finally we would like to show an example to enlighten that the modified Suliciu scheme can be useful also in the context of Saint Venant equations with arbitrary topography

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t (hu) + \partial_x (hu^2 + \frac{g}{2}h^2) = -gh\partial_x z, \end{cases} \tag{5.4}$$



(a) Density (b) Discharge

Fig. 6 Nume rical simulations corresponding to (5.3). Comparison with original Suliciu scheme, Force and GForce schemes



(a) Density (b) Discharge

Fig. 7 Numerical simulations corresponding to (5.3). Comparison with High Order schemes

where $h(t,x) \geq 0$ is the water height, $u(t,x)$ is the velocity and $z(x)$ is a known function which represents the topography. When $z(x)$ is constant, the system reduces to a particular case of (3.1) and the scheme introduced in Sect. 3 can be applied. In the more general case where $z(x)$ is a given function, one may apply the hydrostatic reconstruction described in [1] to treat the source term. This hydrostatic reconstruction technique consist roughly in solving a Riemann problem by applying a given numerical flux for the homogeneous problem (with $\partial_x z = 0$) to some reconstructed states related to the original left and right data. By applying this technique, one obtains a numerical flux that is semi-discrete entropy satisfying given that the homogeneous flux is semi-discrete entropy satisfying. Thus, a good choice for this homogenous flux would be the modified Suliciu scheme presented previously.

We show a test taken from [7]. It is designed to assess the long time behavior and convergence to steady state. The space domain is $[0, 25]$, and we introduce a topography given by

$$z(x) = \begin{cases} 0.2 - 0.05(x-10)^2 & \text{if } 8 < x < 12, \\ 0 & \text{otherwise,} \end{cases} \tag{5.5}$$

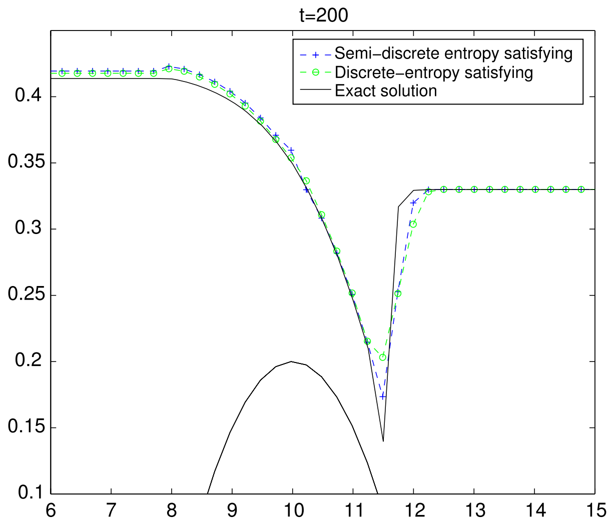


Fig. 8 Density profile for (5.5)–(5.6)

with initial data

$$\rho^0 = 0.33, \qquad u^0 = 0.18/0.33. \tag{5.6}$$

The result is shown in Fig. 8. We remark again that the semi-discrete approach is less diffusive in the shock region.

6 Conclusion

The semi-discrete approach for an approximate Riemann problem lets to define stable schemes while it is less restrictive and allows to ask other properties like the exact resolution of stationary shocks.

As a particular case, we have applied this technique to the Suliciu relaxation system and we have observed that not only the stationary shocks are preserved, but we obtain a slight amelioration for shocks. At the present stage, the schemes shown in Sects. 3 and 4 do not treat vacuum correctly in the sense that if one of the two densities ρ_l or ρ_r tends to 0 while the other remains finite, the propagation speed c/ρ will tend to infinity. Thus, the CFL condition would give a zero time-step for the scheme. Further work is being done by the authors in order to adapt the scheme to treat vacuum correctly.

Acknowledgements This research has been partially supported by the Spanish Government Research projects MTM2006-08075 and P06-RNM-01594.

References

1. Audusse, E., Bouchut, F., Bristeau, M.-O., Klein, R., Perthame, B.: A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows. SIAM J. Sci. Comput. **25**(6), 2050–2065 (2004) (electronic)
2. Baudin, M., Berthon, C., Coquel, F., Masson, R., Tran, Q.H.: A relaxation method for two-phase flow models with hydrodynamic closure law. Numer. Math. **99**(3), 411–440 (2005)

3. Bouchut, F.: Entropy satisfying flux vector splittings and kinetic BGK models. Numer. Math. **94**(4), 623–672 (2003)
4. Bouchut, F.: Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources. In: Frontiers in Mathematics. Birkhäuser, Basel (2004)
5. Colella, P., Glaz, H.M.: Efficient solution algorithms for the Riemann problem for real gases. J. Comput. Phys. **59**(2), 264–289 (1985)
6. Coquel, F., Godlewski, E., Perthame, B., In, A., Rascle, P.: Some new Godunov and relaxation methods for two-phase flow problems. In: Godunov Methods, Oxford, 1999, pp. 179–188. Kluwer/Plenum, New York (2001)
7. Gallouët, T., Hérard, J.-M., Seguin, N.: Some approximate Godunov schemes to compute shallow-water equations with topography. Comput. & Fluids **32**(4), 479–513 (2003)
8. LeFloch, P.G.: Hyperbolic systems of conservation laws. In: Lectures in Mathematics ETH Zürich. Birkhäuser, Basel (2002). The theory of classical and nonclassical shock waves
9. Marquina, A.: Local piecewise hyperbolic reconstruction of numerical fluxes for nonlinear scalar conservation laws. SIAM J. Sci. Comput. **15**(4), 892–915 (1994)
10. Suliciu, I.: On modelling phase transitions by means of rate-type constitutive equations. Shock wave structure. Int. J. Eng. Sci. **28**(8), 829–841 (1990)
11. Suliciu, I.: Some stability-instability problems in phase transitions modelled by piecewise linear elastic or viscoelastic constitutive equations. Int. J. Eng. Sci. **30**(4), 483–494 (1992)
12. Toro, E.F.: A practical introduction. In: Riemann Solvers and Numerical Methods for Fluid Dynamics, 2nd edn. Springer, Berlin (1999)
13. Toro, E., Titarev, V.A.: MUSTA fluxes for systems of conservation laws. J. Comput. Phys. **216**(2), 403–429 (2006)