函数极限及连续性

Limits and Continuity

DEFINITION

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

DEFINITIONS Let c be a real number on the x-axis.

The function f is **continuous at c** if

$$\lim_{x \to c} f(x) = f(c).$$

The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x \to c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** if

$$\lim_{x \to c^-} f(x) = f(c).$$

函数在一点c处连续的三个条件(缺一不可)

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).

即：

1、 函数在该点处有定义

2、 函数在该点处极限存在

3、 函数在该点处的极限等于函数值

间断点分类

第一类间断点:

$\lim\limits_{x\to x_0^-} f(x)$ 及 $\lim\limits_{x\to x_0^+} f(x)$ 均存在，

若 $\lim\limits_{x\to x_0^-} f(x) = \lim\limits_{x\to x_0^+} f(x)$, 称 x_0 为**可去间断点**

若 $\lim\limits_{x\to x_0^-} f(x) \neq \lim\limits_{x\to x_0^+} f(x)$, 称 x_0 为**跳跃间断点**

第二类间断点:

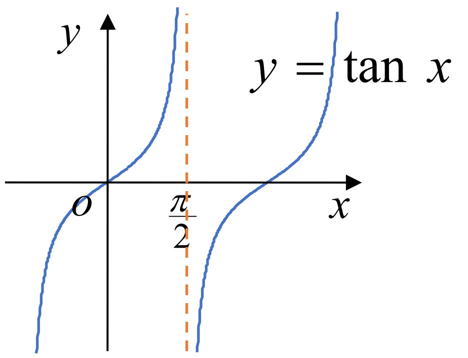
$\lim\limits_{x\to x_0^-} f(x)$ 及 $\lim\limits_{x\to x_0^+} f(x)$ 中至少一个不存在，

若其中有一个为 ∞，称 x_0 为**无穷间断点**

若其中有一个为振荡，称 x_0 为**振荡间断点**

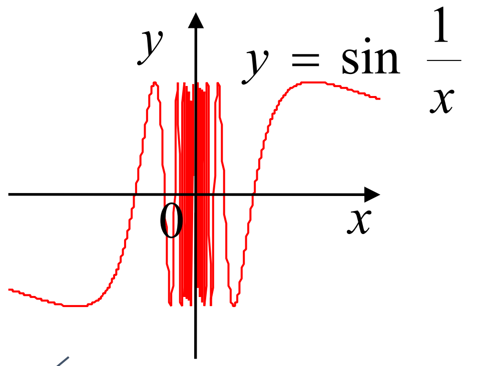
$$(1)\quad y = \tan x$$

$x = \dfrac{\pi}{2}$ 为其无穷间断点 .



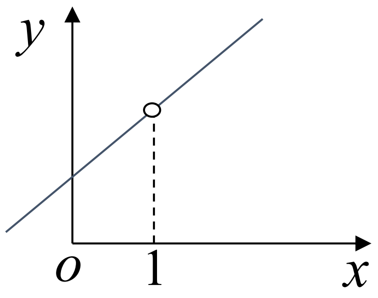
$$(2)\quad y = \sin\frac{1}{x}$$

$x = 0$ 为其振荡间断点 .



$$(3)\quad y = \frac{x^2 - 1}{x - 1}$$

$x = 1$ 为可去间断点 .



例1. 讨论函数 $f(x)=\dfrac{x^2-1}{x^2-3x+2}$ **间断点的类型.**

答案: $\boldsymbol{x=1}$ **是第一类可去间断点 ,**

$\boldsymbol{x=2}$ **是第二类无穷间断点 .**

连续函数的性质

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

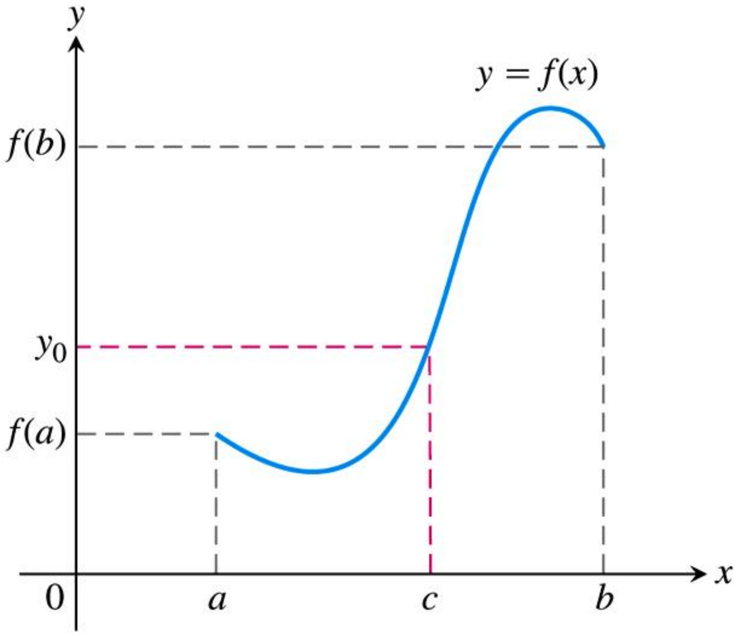
连续函数的极限

THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

连续函数的中值定理

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



根的存在性的判定

例　证明方程$x^3 -15x+1=0$在区间$[-4,4]$上有三个根.

利用连续函数的中值定理可以判断函数的根是否存在：

(1）在$f(x)$函数定义域内有两点a,b(不妨设$a<b$)

满足$f(a)f(b)<0$;

(2)条件(1)说明0包含在$f(x)$值域内，根据连续函数

的中值定理，存在一点c,$(a<c<b), s.t, f(c)=0$.

函数的连续延拓

Continuous Extension to a Point

If $f(c)$ is not defined, but $\lim_{x\to c} f(x) = L$ exists, a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is in the domain of } f \\ L, & \text{if } x = c. \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension of** f

怎样对一个函数进行连续性延拓:

1、求出函数在其无定义的点处的极限;

2、定义一个新的函数，在原来函数无定义的点处定义新函数等于原来函数在该点处的极限，而在其余点处新函数等于原函数。

x趋于无穷时的极限

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write
$$\lim_{x\to\infty} f(x) = L$$
if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x
$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write
$$\lim_{x\to-\infty} f(x) = L$$
if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x
$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

THEOREM 12 All the limit laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

DEFINITIONS

1. We say that ***f*(*x*) approaches infinity as *x* approaches x_0**, and write
$$\lim_{x\to x_0} f(x) = \infty,$$
if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) > B.$$

2. We say that ***f*(*x*) approaches minus infinity as *x* approaches x_0**, and write
$$\lim_{x\to x_0} f(x) = -\infty,$$
if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) < -B.$$

练习 求下面极限

(1) $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{x^2 - a^2}}$　　(2) $\lim\limits_{x \to -\infty} \dfrac{x}{\sqrt{x^2 - a^2}}$

注意求极限是正负无穷的差别

1.水平渐近线

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph the line asymptotically and that the line is an *asymptote*

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

2.垂直渐近线

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

3.斜渐近线

$$f(x) \qquad y = kx + b,$$

$$\lim_{x\to\infty}[f(x)-(kx+b)] = 0,$$

$$\Rightarrow \lim_{x\to\infty}[f(x)-kx] = b$$

$$\lim_{x\to\infty}\left[\frac{f(x)}{x}-k\right] = \lim_{x\to\infty}\frac{1}{x}[f(x)-kx] = 0 \bullet b = 0,$$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x)}{x} = k$$

$$\Rightarrow \lim_{x\to\infty}[f(x)-kx] = b$$

例3 讨论曲线$f(x)=\dfrac{x^3}{x^2+2x-3}$ 的渐近线

导数
Derivatives

导数的定义

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

关于差商极限的几种不同解释

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$

2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$

3. The rate of change of $f(x)$ with respect to x at $x = x_0$

4. The derivative $f'(x_0)$ at a point

1、函数$y = f(x)$的图像在$x = x_0$处的斜率
2、曲线$y = f(x)$在$x = x_0$处切线的斜率
3、函数$y = f(x)$在$x = x_0$处关于x的变化率
4、函数在某一点处导数

导函数的定义

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

导数的一个等价定义

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

导数的两种不同的记号

$$\frac{d}{dx} f(x) \qquad\qquad f'(x)$$

函数在闭区间上可导

A function $y = f(x)$ is **differentiable on a closed interval** $[a, b]$ if it

is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

$$\lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exist at the endpoints

a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

定理　　函数$f(x)$在其定义域一点x_0处可导的充要条件（if and only if）是其在x_0处的左右导数都存在且相等(exist and equal)。

例　设$f(x)=\begin{cases}1-cosx & x\geq 0\\ x & x<0\end{cases}$，讨论$f(x)$在$x=0$处的左右导数.

例（1）设$f(x)=\begin{cases} x\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$,讨论$f(x)$在$x=0$是否可导?

（2）设$f(x)=\begin{cases} x^2\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$,讨论$f(x)$在$x=0$是否可导?

注：左右导数一般用来讨论分段函数在分段点处是否可导

例 设函数$f(x)$ 在区间$[-1,1]$满足$|f(x)|\leq x^2$,证明函数$f(x)$在$x=0$处可导，并求$f'(0)$.

例　设$g(0) = g'(0) = 0$,

$$f(x) = \begin{cases} g(x) sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

求$f'(0)$.

可导必连续

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

注：若$f(x)$在$x = c$处的左导数存在，则$f(x)$在$x = c$处左连续；

若$f(x)$在$x = c$处的右导数存在，则$f(x)$在$x = c$处右连续；

反之，函数$f(x)$在$x = c$处连续并不一定可导。

例 1 设 $f(x)=\begin{cases}\sin x, & x<0\\ ax, & x\geq 0\end{cases}$，问 a 取何时，$f'(x)$ 在 $(-\infty,+\infty)$ 都存在，并求 $f'(x)$.

解: 显然该函数在 $x=0$ 处连续.

$$f'_-(0)=\lim_{x\to 0^-}\frac{\sin x-0}{x-0}=1 \qquad f'_+(0)=\lim_{x\to 0^+}\frac{ax-0}{x-0}=a$$

故 $a=1$ **时，** $f'(0)=1$**，此时** $f'(x)$ **在** $(-\infty,+\infty)$ **都存在，**

$$f'(x)=\begin{cases}\cos x, & x<0\\ 1, & x\geq 0\end{cases}$$

关于导数的定义几个需要注意的问题:

1. $\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}=\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}=\lim\limits_{h\to 0}\dfrac{f(x_0)-f(x_0-h)}{h}$

上述$h\to 0$包含$h\to 0^+$和$h\to 0^-$两种变化过程.

2. $\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0-h)}{2h}$与$\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$并不等价.

即$\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0-h)}{2h}$存在，但$f'(x_0)=\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$未必存在；而若$\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$存在，$\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0-h)}{2h}$一定存在.

例 $f(x)=\begin{cases}x+1, & x\neq 0\\ 2, & x=0\end{cases}$

3. $\lim\limits_{x \to x_0^+} \dfrac{f(x)-f(x_0)}{x-x_0}$与$\lim\limits_{x \to x_0^+} f'(x)$是两个不同的概念：

$\lim\limits_{x \to x_0^+} \dfrac{f(x)-f(x_0)}{x-x_0}$指的是函数$f(x)$在$x_0$处的右导数，而$\lim\limits_{x \to x_0^+} f'(x)$指的是导函数$f'(x)$在$x_0$处的右极限，因此$\lim\limits_{x \to x_0^+} \dfrac{f(x)-f(x_0)}{x-x_0} = \lim\limits_{x \to x_0^+} f'(x)$并不总是成立.

例2、$f(x)=\begin{cases} \dfrac{2}{3}x^3, x \le 1 \\ x^2, x > 1 \end{cases}$在点$x=1$处，有$\lim\limits_{x \to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x \to 1^+} \dfrac{x^2-\frac{2}{3}}{x-1} = +\infty$,

可见$f'_+(1)$不存在.但$\lim\limits_{x \to 1^+} f'(x) = \lim\limits_{x \to 1^+} (x^2)' = 2$.

例3、$f(x)=\begin{cases} x^2 \sin\dfrac{1}{x}, x \ne 0 \\ 0, x = 0 \end{cases}$,

$$\lim_{x \to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x \to 0^+} \frac{x^2 \sin\frac{1}{x} - 0}{x} = \lim_{x \to 0^+} x \sin\frac{1}{x} = 0,$$

即$f'_+(0)$存在，但$\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^+} (x^2 \sin\dfrac{1}{x})' = \lim\limits_{x \to 0^+} (2x \sin\dfrac{1}{x} - \cos\dfrac{1}{x})$不存在.

4.$y=f(x)$在点x_0处的导数与曲线$y=f(x)$在点$(x_0, f(x_0))$处的切线的关系：

(1).$f'(x_0)$存在$\Rightarrow$曲线$y=f(x)$在点$(x_0, f(x_0))$处有斜率为$f'(x_0)$的切线;

(2).$f'(x_0)$不存在，不意味着$y=f(x)$在点$(x_0, f(x_0))$处没有切线

例4、$f(x)=\sqrt[3]{x}$在点$x=0$处不可导，但$f(x)=\sqrt[3]{x}$在点$(0,0)$处有与x轴垂直的切线.

一般地，若$\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}=+\infty$（或-∞）,则$y=f(x)$在点$(x_0, f(x_0))$处具有垂直于$x$轴的切线.

导数极限定理

设函数$f(x)$在点x_0的某邻域$U(x_0)$内连续，在$U^0(x_0)$内可导，且极限$\lim\limits_{x\to x_0} f'(x)$存在，则$f$在点$x_0$可导，且

$$f'(x_0)=\lim_{x\to x_0} f'(x)$$

求导法则

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

基本初等函数的导数 公式

$$(C)' = 0 \qquad (x^{\mu})' = \mu x^{\mu-1}$$

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x \qquad (\csc x)' = -\csc x \cot x$$

$$(a^x)' = a^x \ln a \qquad (e^x)' = e^x$$

$$(\log_a x)' = \frac{1}{x \ln a} \qquad (\ln x)' = \frac{1}{x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2} \qquad (\operatorname{arc}\cot x)' = -\frac{1}{1+x^2}$$

(对数求导法)

例　设$y=\frac{(x+5)^2(x-4)^{\frac{1}{3}}}{(x+2)^5(x+4)^{\frac{1}{2}}}(x>4)$, 求$y'$.

解:　　先对上述函数式两端同时取对数,

$$lny=ln\frac{(x+5)^2(x-4)^{\frac{1}{3}}}{(x+2)^5(x+4)^{\frac{1}{2}}}$$

$$=2\ln(x+5)+\frac{1}{3}\ln(\mathrm{x}-4)-5\ln(\mathrm{x}+2)-\frac{1}{2}\ln(\mathrm{x}+4)$$

再对上式两边分别对x求导，可得

$$\frac{y'}{\mathrm{y}}=\frac{2}{\mathrm{x}+5}+\frac{1}{3(\mathrm{x}-4)}-\frac{5}{\mathrm{x}+2}-\frac{1}{2(\mathrm{x}+4)}$$

整理后，可得

$$y'=\frac{(x+5)^2(x-4)^{\frac{1}{3}}}{(x+2)^5(x+4)^{\frac{1}{2}}}\left(\frac{2}{\mathrm{x}+5}+\frac{1}{3(\mathrm{x}-4)}-\frac{5}{\mathrm{x}+2}-\frac{1}{2(\mathrm{x}+4)}\right)$$

例　设$y=u(x)^{v(x)}$,其中$u(x)>0$, 且$u(x)$和$v(x)$均可导，试求y'.

解:

$$y=u(x)^{v(x)}\left(v'(x)lnu(x)+v(x)\frac{u'(x)}{u(x)}\right)$$

练习　对下面的函数求导.

（1）$y = \ln(lnx)$

（2）$y = \sin\left(\sqrt{1 + x^2}\right)$

（3）$y = \arcsin\left(\frac{1}{x}\right)$

（4）$y = (arctanx^3)^2$

（5）$y = \arcsin(\sin^2 x)$

（6）$y = x^{sinx}$

（7）$y = x^{x^x}$

（8）$y = \frac{1}{\sqrt{a^2 - b^2}}\arcsin\left(\frac{asinx+b}{a+bsinx}\right)$

二阶及高阶导数

Second- and Higher-Order Derivatives

$f'' = (f')'$ the **second derivative** of f

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x)$$

$y''' = dy''/dx = d^3y/dx^3$, is the **third derivative**

the ***n*th derivative** of y with respect to x for any positive integer n.

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

导数的应用

DEFINITION The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

注：当提到变化率，我们指的都是瞬时变化率

速度

DEFINITION **Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

速率

DEFINITION **Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

加速度与加速度变化率

DEFINITIONS **Acceleration** is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

❑ **DEFINITION** If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The

point $x = a$ is the **center** of the approximation.

Differential 微分

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential** $\boldsymbol{dx}$ is an independent variable. The **differential** $\boldsymbol{dy}$ is

$$dy = f'(x)\,dx.$$

注：函数在一点x_0处可微与函数在这一点x_0处可导是等价的。

Estimating with differentials

例 利用微分求近似值

（1） $1.02^{\frac{1}{3}}$

(2) $\sqrt{26}$