函数极限及连续性

Limits and Continuity

Average and Instantaneous Speed

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Limit 极限

If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L."

极限运算法则

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

THEOREM 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

函数极限的定义

DEFINITION Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon .$$

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.

2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

One-Sided Limits单侧极限

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided**.

If the approach is from the right, the limit is a**right-hand limit**. From the left, it is a **left-hand limit**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit

单侧极限的定义

DEFINITIONS We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x \to c^+} f(x) = L \qquad \text{(see Figure 2.28)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c < x < c + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at c**, and write

$$\lim_{x \to c^-} f(x) = L \qquad \text{(see Figure 2.29)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c - \delta < x < c \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

重要极限

THEOREM 7—Limit of the Ratio sin θ/θ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

求极限的方法

(1)函数的连续性
(2)极限的四则运算法则
(3)重要极限
(4)左右极限相等，极限存在(主要用来考察分段函数的极限)
(5)夹逼准则 (The Sandwich Theorem)

利用重要极限来求函数的极限

1. $\lim\limits_{x \to 0} x \cot x$

2. $\lim\limits_{x \to 0} \frac{1-\cos 2x}{x \sin x}$

3. $\lim\limits_{x \to 0} \frac{\sqrt{1-\cos x^2}}{1-\cos x}$

4. $\lim\limits_{x \to a} \frac{\sin^2 x-\sin^2 a}{x-a}$

左右单侧极限

Example 1讨论下列函数在$x \to 0$时的极限或左右极限：

$$(1)\ f(x) = \frac{|x|}{x}$$

$$(2)\ f(x) = [x]$$

$$(3)\ f(x) = \begin{cases} 2x & x > 0 \\ 0 & x = 0 \\ 1 + x^2 & x < 0 \end{cases}$$

Example 2讨论$\lim\limits_{x \to 0} f(x)$的存在性

$$(4)\ f(x) = \begin{cases} x \sin \frac{1}{x} & x > 0 \\ 1 & x < 0 \end{cases}$$

$$(5)\ f(x) = x \sin \frac{1}{x} + \cos \frac{1}{x}$$

注：分段函数分点处的极限，要分别求左极限和右极限.

小结

单侧极限可用来证明函数在某一点处极限**不存在**:

(1)证明左极限与右极限**至少**有一个不存在

(2)或证明左极限和右极限均存在,但**不相等**

函数极限的性质

1.(函数极限的唯一性) 如果 $\lim_{x\to c} f(x) = L$，那么这个极限唯一.

2.(函数极限的局部有界性) 如果 $\lim_{x\to c} f(x) = L$，那么存在常数 $M > 0$ 和 $\delta > 0$，使得当 $0 < |x - c| < \delta$ 时，有 $|f(x)| \le M$.

3.(函数极限的保号性)如果 $\lim_{x\to c} f(x) = L > 0(< 0)$,那么存在 $\delta > 0$, 使得当 $0 < |x - c| < \delta$ 时，有 $f(x) > 0(< 0)$.

1.计算极限 $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ $\qquad \lim\limits_{x\to 1}\dfrac{x^4-1}{x^3-1}$

2.证明 若 $\lim\limits_{x\to 0} g(x)=0$,则 $\lim\limits_{x\to 0} g(x)\sin\dfrac{1}{x}=0$

3.找反例说明命题

"若 $\lim\limits_{x\to c}|f(x)|=|l|$,则 $\lim\limits_{x\to c} f(x)=l$。"

是错误的。

4.找反例说明命题

"若 $\lim\limits_{x\to c}[f(x)+g(x)]$存在,则 $\lim\limits_{x\to c} f(x)$和

$\lim\limits_{x\to c} g(x)$存在"是错误的。

1.判断下列命题

(1)若 $\lim\limits_{x\to c} f(x)=l$,则存在某$\delta>0$,使得$f(x)$在$c$的邻域

$(c-\delta,c)\cup(c,c+\delta)$内有界.

(2)若 $\lim\limits_{x\to c} f(x)=1$,则存在某$\delta>0$,使得在$c$的邻域

$(c-\delta,c)\cup(c,c+\delta)$内$f(x)<\dfrac{3}{2}$.

(3)若$f(x)>0$,且 $\lim\limits_{x\to c} f(x)=l$,则$l>0$.

2.计算极限 $\lim\limits_{x\to 0}\dfrac{\tan 2x}{3x}$ $\quad\lim\limits_{x\to 0}[\dfrac{1}{x}]\sin x$

3.若 $\lim\limits_{x\to 0^+} f(x)=l$, $\lim\limits_{x\to 0^-} f(x)=m$,问下列极限

存在吗？若存在则求出来。

$\lim\limits_{x\to 0} f(-x)$ $\quad\lim\limits_{x\to 0+} f(x^2-x)$

$\lim\limits_{x\to 0^-}(2f(-x)+f(x^2))$