tut3

October 17, 2021

1 Tutorial 3

1.1 Q1. Essential Convolution

1.1.1 (a)

$$u[n] = x[n] * y[n] = [16, 4, -22, 40, -5, -27, 9, -6, -1, 3, -1, 2, 0], \ \ n \in [-8, 4]$$

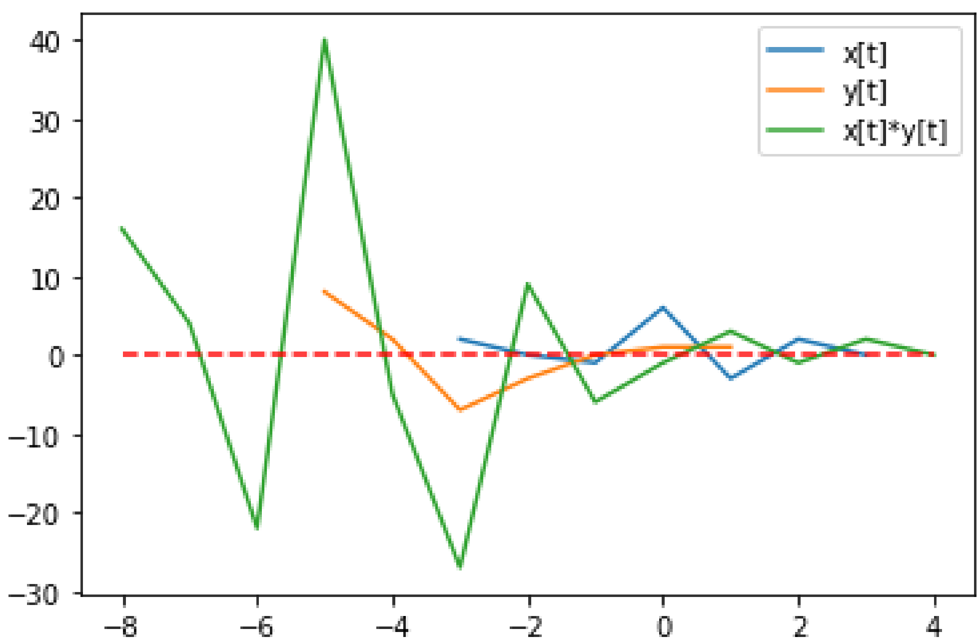
$$v[n] = x[n] * w[n] = [6, 12, -5, 18, 40, -9, 29, 19, 23, 0, 9, 2, 0], \ \ n \in [-5, 7]$$
$$g[n] = w[n] * x[n] = [24, 54, -17, -29, 43, 45, -22, -53, -23, 6, 12, 7, 1], \ \ n \in [-7, 5]$$

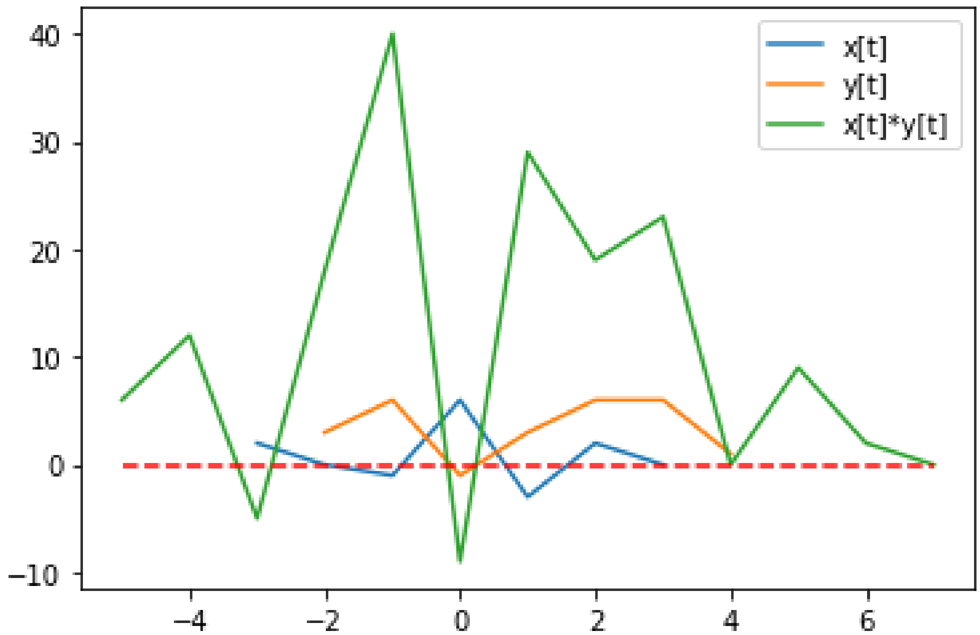
Use Python to verify the result:

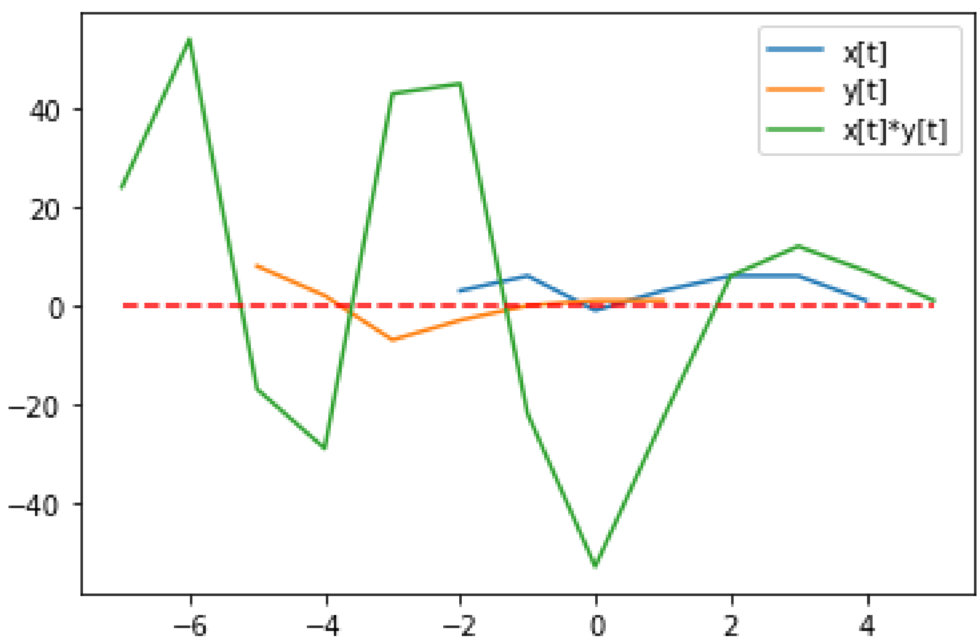
[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
xt=np.array([2,0,-1,6,-3,2,0])
t1=np.array(list(range(0,7)))-3
yt = np.array([8,2,-7,-3,0,1,1])
t2=np.array(list(range(0,7)))-5
zt = np.array([3,6,-1,3,6,6,1])
t3 = np.array(list(range(0,7)))-2

import matplotlib.colors as mcolors
colors = list(mcolors.TABLEAU_COLORS.keys())
ya=np.convolve(xt,yt)
ta=np.array(list(range(len(ya))))+t1[0]+t2[0]
print("u[n]=",ya)
print("t_1 [n]=",ta)
yb=np.convolve(xt,zt)
tb=np.array(list(range(len(yb))))+t1[0]+t3[0]
print("v[n]=",yb)
print("t_2 [n]=",tb)
yc = np.convolve(zt, yt)
tc = np.array(list(range(len(yc)))) + t3[0] + t2[0]
print("g[n]=", yc)
print("t_3 [n]=", tc)
plt.figure(1)
p11,=plt.plot(t1,xt,color=mcolors.TABLEAU_COLORS[colors[0]])
```

```
p21,=plt.plot(t2,yt,color=mcolors.TABLEAU_COLORS[colors[1]])
p31,=plt.plot(ta,ya,color=mcolors.TABLEAU_COLORS[colors[2]])
plt.plot(ta,np.zeros([1,len(ta)])[0],"r--")
plt.legend([p11,p21,p31],["x[t]","y[t]","x[t]*y[t]"])
plt.figure(2)
p12, = plt.plot(t1, xt, color=mcolors.TABLEAU_COLORS[colors[0]])
p22, = plt.plot(t3, zt, color=mcolors.TABLEAU_COLORS[colors[1]])
p32, = plt.plot(tb, yb, color=mcolors.TABLEAU_COLORS[colors[2]])
plt.plot(tb, np.zeros([1, len(tb)])[0], "r--")
plt.legend([p12, p22, p32], ["x[t]", "y[t]", "x[t]*y[t]"])
plt.figure(3)
p13, = plt.plot(t3, zt, color=mcolors.TABLEAU_COLORS[colors[0]])
p23, = plt.plot(t2, yt, color=mcolors.TABLEAU_COLORS[colors[1]])
p33, = plt.plot(tc, yc, color=mcolors.TABLEAU_COLORS[colors[2]])
plt.plot(tc, np.zeros([1, len(tc)])[0], "r--")
plt.legend([p13, p23, p33], ["x[t]", "y[t]", "x[t]*y[t]"])
plt.show()
```

```
u[n]= [ 16   4 -22  40  -5 -27   9  -6  -1   3  -1   2   0]
t_1 [n]= [-8 -7 -6 -5 -4 -3 -2 -1  0  1  2  3  4]
v[n]= [ 6 12 -5 18 40 -9 29 19 23  0  9  2  0]
t_2 [n]= [-5 -4 -3 -2 -1  0  1  2  3  4  5  6  7]
g[n]= [ 24  54 -17 -29  43  45 -22 -53 -23   6  12   7   1]
t_3 [n]= [-7 -6 -5 -4 -3 -2 -1  0  1  2  3  4  5]
```







1.2 Q2.

Use the delay properties of delta functions,

$$x[t] * \delta[t - t_0] = x[t - t_0]$$

to express the $v[n]$ in terms of $y[n]$:

$$\begin{aligned} v[n] &= x_1[n] * x_2[n] * \delta[n - N_1] * \delta[n - N_2] \\ &= x_1[n] * x_2 * \delta[n - N_1 - N_2] \\ &= y[n - N_1 - N_2] \end{aligned}$$

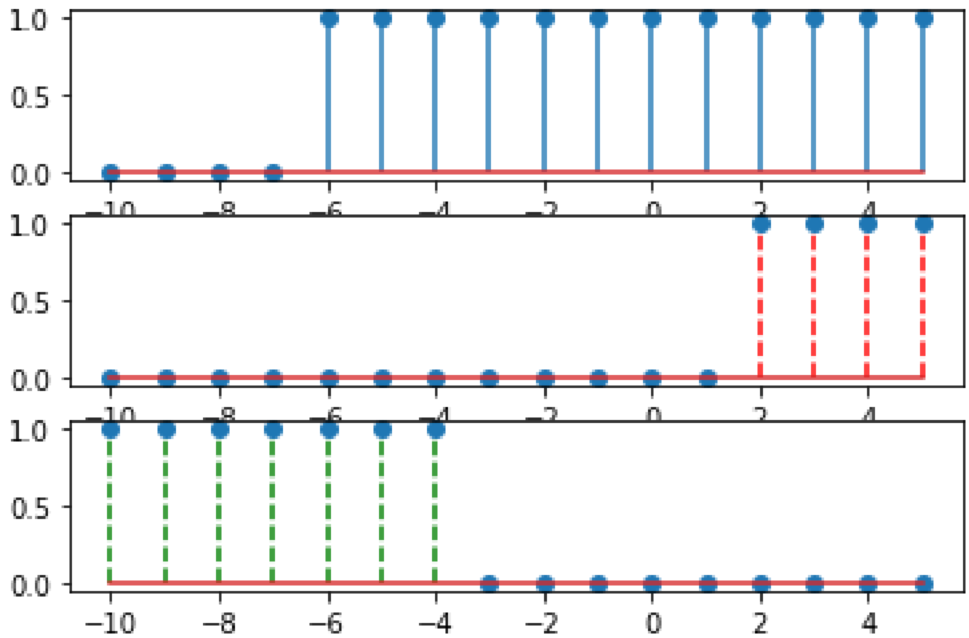
1.3 Q3.

Display the given sequences with example rectangle pulses to get an intuitive view. Take the example parameters below:

$$M > 0, R < M < L \Rightarrow -L = -10, -M = -6, R = -4, K = 2, N = 4$$

Then use Python to plot the sequences:

[ ]:
```
l=10
m=6
r=4
k=2
n=6

t1=np.arange(-1*l,n,step=1)
hn=np.array(t1>=-1*m)*np.array(t1<=n)*1
gn=1*np.array(t1>=k)*np.array(t1<=n)
wn=1*np.array(t1>=-1*l)*np.array(t1<=-1*r)

plt.figure(1)
plt.subplot(311)
plt.stem(t1,hn)
plt.subplot(312)
plt.stem(t1,gn,linefmt='r--')
plt.subplot(313)
plt.stem(t1,wn,linefmt='g--')
plt.show()
```



1.3.1 (a)

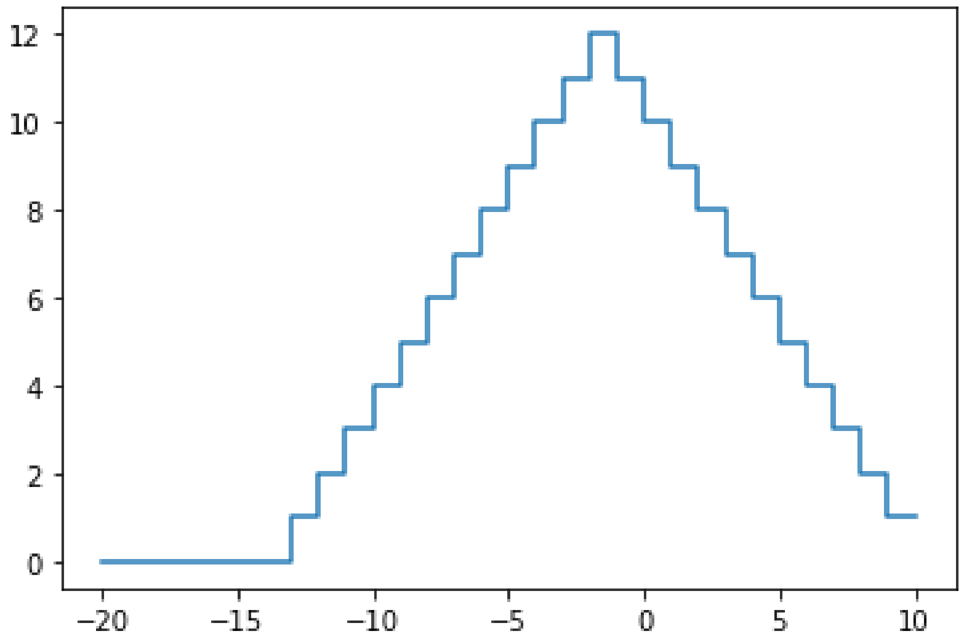
For $y_1[t] = h[t] * h[t]$, the length of convolved sequence should be

`len(numpy.convolve(h,h))=2*len(h)+1`

which equals to $2M + 2N - 1$. And the non-zero region of convolution is $[-2M, 2N]$. The python plot below is evident to the deduction above.

[ ]:
```
y1=np.convolve(hn,hn)
ty1=np.array(list(range(0,len(y1))))+2*t1[0]
print("Length of h[n]=",np.count_nonzero(hn))
print("Length of y1[n]=",np.count_nonzero(y1))

plt.figure(1)
plt.step(ty1,y1)
plt.show()
```

```
Length of h[n]= 12
Length of y1[n]= 23
```



1.3.2 (b)

Similar to (a), the length of $g[n] * g[n]$ is

```
2*len(g[n])+1
```

and equals to $2N - 2K + 1$. The beginning index of the convolution output is

$$I_b = t(g[t]) + t(g[t]) = 2K,$$

and the ending index is

$$I_f = 2N.$$

1.3.3 (c)

Use the conclusion from question (a) and (b), the length and non-zero interval of $y_3[n] = h[n] * g[n]$ is

$$L = len(h[n]) + len(g[n]) + 1 = M + 2N - K + 1,$$

$$I_i = t_i(h[t]) + t_i(g[t]) = -M + K, I_f = t_f(h[t]) + t_f(g[t]) = 2N$$

Therefore

$$T = [-M + K, 2N]$$

1.3.4 (d)

$$L = len(h[t]) + len(w[t]) + 1 = N + M + L - R + 1,$$

$$T = [-L - M, N - R]$$

1.4 Q4.

Evaluate the equation use ZT.

$$\begin{aligned}
\mathcal{Z}(a^t u[t] * u[t]) &= \frac{z/a}{z/a - 1} \cdot \frac{z}{z-1} \\
&= \frac{z^2}{(z-a)(z-1)} \\
&= \frac{a}{a-1}\frac{z}{z-a} + \frac{-1}{a-1}\frac{z}{z-1}
\end{aligned}$$

Therefore

$$\begin{aligned}
a^t u[t] * u[t] &= \mathcal{Z}^{-1}(\frac{a}{a-1}\frac{z}{z-a} + \frac{-1}{a-1}\frac{z}{z-1}) \\
&= \left(\frac{1 - a^{n+1}}{1-a}\right) u[t]
\end{aligned}$$

1.5 Q5.

Validate the conjugate symmetricity by performing the two transforms below:

$$z[t] = x[t] * y[t] \Rightarrow z_1[t] = z^*[-t], D = \frac{z[t]}{z_1[t]}$$

$D = 1$ indicates the $z[t]$ is conjugate symmetric, $D = -1$ indicates the $z[t]$ is conjugate asymmetric.

1.5.1 (a)

For arbitrary function, evaluate its conjugate symmetric and conjugate asymmetric parts:

$$\begin{aligned}
x_{CS}[t] &= \frac{1}{2}(x[t] + x^*[-t]) \\
x_{CA}[t] &= \frac{1}{2}(x[t] - x^*[-t])
\end{aligned}$$

Therefore,

$$\begin{aligned}
z[t] &= \frac{1}{2}(h[t] + h^*[-t]) * \frac{1}{2}(g[t] + g^*[-t]) \\
&= \frac{1}{4}(h[t] * g[t] + h^*[-t] * g^*[-t] + h[t] * g^*[-t] + h^*[-t] * g[t]), \\
z_1[t] &= z^*[-t] \\
&= \frac{1}{4}(h^*[-t] * g^*[-t] + h[t] * g[t] + h^*[-t] * g[t] + h[t] * g^*[-t]) \\
\Rightarrow D &= 1
\end{aligned}$$

The function $h_{CS}[t] * g_{CS[t]}$ is conjugate symmetric.

1.5.2 (b)

Use the identical method to determine the conjugate symmetricity of $h_{CA}[t] * g_{CS}[t]$.

$$D = \frac{z[t]}{z_1[t]} = -1$$

This function is conjugate asymmetric.

1.5.3 (c)

Use the identical method to determine the conjugate symmetricity of $h_{CA}[t] * g_{CA}[t]$.

$$D = \frac{z[t]}{z_1[t]} = 1$$

This function is conjugate symmetric.

1.6 Q6.

1.6.1 (a)

$$x_1[n] = \alpha^n u[n] * \delta[n-1]$$

Given $\alpha < 1$, the convergence of $x_1[n]$ is the same as $\alpha^n u[n]$, which converges to a constant

$$\sum_0^\infty \alpha^n u[n] = \frac{1}{1-\alpha}, \alpha < 1$$

1.6.2 (b)

The converging radius of the expression

$$\sum_0^\infty n\alpha^n x^n, x = 1, \alpha < 1$$

is

$$\lim_{n\to\infty} \left| \frac{n\alpha^n}{(n+1)\alpha^{n+1}} \right| = \frac{1}{\alpha} > 1.$$

Therefore the expression converges absolutely on the interval $|\alpha| < 1$

1.6.3 (c)

$$\sum_{0}^{\infty}\frac{1}{(n+2)(n+3)}<\sum_{0}^{\infty}\frac{1}{n^2},$$

Therefore the expression converges absolutely.

1.7 Q7.

Use the conclusion of Q6.(a), evaluate the convergence range:

$$|\alpha|>1$$

1.8 Q8.

Obtain the impulse response of a LTI system from its step response using differentiation.

$$\begin{aligned}\mathcal{S}(u[t])&=h[t]*u[t],\\ \delta[t]&=u[t]-u[t-1]=u[t]*(\delta[t]-\delta[t-1])\\ \Rightarrow h[t]&=\mathcal{S}(u[t])*(\delta[t]-\delta[t-1])\\ &=s[n]-s[n-1].\end{aligned}$$

Thus the response of arbitary input $x[t]$ is the convolution of input and impulse response:

$$y[t]=x[t]*h[t]=x[t]*(s[t]-s[t-1])$$

1.9 Q9.

Use ZT to obtain the step response.

$$s[t]=u[t]\left(\frac{1-(-a)^{n+1}}{1+a}\right)$$

1.10 Q10.

Use ZT to obtain the system function in Z-domain:

$$\begin{aligned}Y&=H_4(H_1H_2+H_3)(X+H_5Y)\\ (1-H_4(H_1H_2+H_3))Y&=H_4(H_1H_2+H_3)X\\ Y&=\frac{H_4(H_1H_2+H_3)}{1-H_4(H_1H_2+H_3)}X\end{aligned}$$