Constrained Low-Rank Quaternion Matrix Completion Fusing Learnable Transforms and Plug-and-Play Denoising Prior for Color Image Inpainting (Supplementary Material)

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Appendix A. The proof of Theorem 5

Theorem 5: Let $\phi = \{\dot{\mathbf{Z}}, \dot{\mathbf{X}}, \dot{\mathbf{D}}_1, \dot{\mathbf{D}}_2, \dot{\mathbf{M}}, \dot{\mathbf{\Lambda}}_1, \dot{\mathbf{\Lambda}}_2\}$ and $\{\phi^t\}_{t=1}^{\infty}$ denote the sequence generated by the CLQNMF algorithm. Assume that the dual sequences $\{\dot{\mathbf{\Lambda}}_1^t\}$ and $\{\dot{\mathbf{\Lambda}}_2^t\}$ are bounded, then the primal variable sequences $\{\dot{\mathbf{Z}}^t\}$, $\{\dot{\mathbf{X}}^t\}$, and $\{\dot{\mathbf{M}}^t\}$ are Cauchy sequences.

Proof. We first prove that the augment Lagrangian function is bounded, it derives:

$$
\begin{aligned}
&\mathcal{L}_{\beta^{t+1}}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1}) \\
&= \frac{1}{2}\|\mathcal{P}_\Omega(\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{Y}})\|_F^2 + \lambda(\|\dot{\mathbf{Z}}^{t+1}\|_* - \alpha\|\dot{\mathbf{Z}}^{t+1}\|_F) + \frac{\beta^{t+1}}{2}\|\dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}\|_F^2 + \sum_{i=1}^{2}\Psi(\dot{\mathbf{D}}_i^{t+1}) \\
&+ \Re\left(\langle \dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1}\rangle\right) + \frac{\beta^{t+1}}{2}\|\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}\|_F^2 + \Re\left(\langle \dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1}\rangle\right) \\
&= \mathcal{L}_{\beta^t}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t}, \dot{\mathbf{\Lambda}}_2^{t}) \\
&+ \Re\left(\langle \dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1} - \dot{\mathbf{\Lambda}}_1^{t}\rangle\right) + \frac{\beta^{t+1} - \beta^t}{2}\|\dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}\|_F^2 \\
&+ \Re\left(\langle \dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^{t}\rangle\right) + \frac{\beta^{t+1} - \beta^t}{2}\|\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}\|_F^2 \\
&= \mathcal{L}_{\beta^t}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t}, \dot{\mathbf{\Lambda}}_2^{t}) + \frac{\beta^{t+1} + \beta^t}{2(\beta^t)^2}(\|\dot{\mathbf{\Lambda}}_1^{t+1} - \dot{\mathbf{\Lambda}}_1^{t}\|_F^2 + \|\dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^{t}\|_F^2).
\end{aligned}
$$

Note that $\{\dot{\mathbf{\Lambda}}_1^t\}$ and $\{\dot{\mathbf{\Lambda}}_2^t\}$ are bounded, it follows that $\{\dot{\mathbf{\Lambda}}_1^{t+1} - \dot{\mathbf{\Lambda}}_1^t\}$ and $\{\dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^t\}$ are bounded. Assume N_1 and N_2 are their respective upper bounds. Then, we have $\|\dot{\mathbf{\Lambda}}_1^{t+1} - \dot{\mathbf{\Lambda}}_1^t\|_F \leq N_1$ and $\|\dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^t\|_F \leq N_2$. Moreover, the following inequality holds:

$$
\mathcal{L}_{\beta^{t+1}}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1}) \leq \mathcal{L}_{\beta^t}(\dot{\mathbf{X}}^{t}, \dot{\mathbf{Z}}^{t}, \dot{\mathbf{M}}^{t}, \dot{\mathbf{D}}_1^{t}, \dot{\mathbf{D}}_2^{t}, \dot{\mathbf{\Lambda}}_1^{t}, \dot{\mathbf{\Lambda}}_2^{t}).
$$

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Since the optimal solutions $\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}$ have been obtained for their respective subproblems, it follows that:

$$
\begin{aligned}
&\mathcal{L}_{\beta^{t+1}}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1}) \\
&\leq \mathcal{L}_{\beta^{t}}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t}, \dot{\mathbf{\Lambda}}_2^{t}) + \frac{\beta^{t+1}+\beta^t}{2(\beta^t)^2}\left(N_1^2+N_2^2\right) \\
&\leq \mathcal{L}_{\beta^{0}}(\dot{\mathbf{X}}^{1}, \dot{\mathbf{Z}}^{1}, \dot{\mathbf{M}}^{1}, \dot{\mathbf{D}}_1^{1}, \dot{\mathbf{D}}_2^{1}, \dot{\mathbf{\Lambda}}_1^{0}, \dot{\mathbf{\Lambda}}_2^{0}) + \left(N_1^2+N_2^2\right)\sum_{t=0}^{\infty}\frac{\beta^{t+1}+\beta^t}{2(\beta^t)^2} \\
&\leq \mathcal{L}_{\beta^{0}}(\dot{\mathbf{X}}^{1}, \dot{\mathbf{Z}}^{1}, \dot{\mathbf{M}}^{1}, \dot{\mathbf{D}}_1^{1}, \dot{\mathbf{D}}_2^{1}, \dot{\mathbf{\Lambda}}_1^{0}, \dot{\mathbf{\Lambda}}_2^{0}) + \left(N_1^2+N_2^2\right)\sum_{t=0}^{\infty}\frac{1+\rho}{2\beta^0\rho^t} \\
&\leq \mathcal{L}_{\beta^{0}}(\dot{\mathbf{X}}^{1}, \dot{\mathbf{Z}}^{1}, \dot{\mathbf{M}}^{1}, \dot{\mathbf{D}}_1^{1}, \dot{\mathbf{D}}_2^{1}, \dot{\mathbf{\Lambda}}_1^{0}, \dot{\mathbf{\Lambda}}_2^{0}) + \frac{N_1^2+N_2^2}{\beta^0}\sum_{t=0}^{\infty}\frac{1}{\rho^{t-1}} < +\infty.
\end{aligned}
$$

Hence, $\mathcal{L}$ is bounded. Next, we prove that the sequences $\{\dot{\mathbf{X}}^{t+1}\}$, $\{\dot{\mathbf{Z}}^{t+1}\}$, $\{\dot{\mathbf{D}}_1^{t+1}\}$, and $\{\dot{\mathbf{D}}_2^{t+1}\}$ are bounded.

$$
\begin{aligned}
&\frac{1}{2}\|\mathcal{P}_\Omega(\dot{\mathbf{X}}^{t+1}-\dot{\mathbf{Y}})\|_F^2 + \lambda(\|\dot{\mathbf{Z}}^{t+1}\|_* - \alpha\|\dot{\mathbf{Z}}^{t+1}\|_F) + \sum_{i=1}^{2}\Psi(\dot{\mathbf{D}}_i^{t+1}) \\
&= \mathcal{L}_{\beta^{t}}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t}, \dot{\mathbf{\Lambda}}_2^{t}) - \Re\left(\langle \dot{\mathbf{\Lambda}}_1^t, \dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}\rangle\right) \\
&- \frac{\beta^t}{2}\|(\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1} - \dot{\mathbf{X}}^{t+1}\|_F^2 - \Re\left(\langle \dot{\mathbf{\Lambda}}_2^t,\ \dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}\rangle\right) - \frac{\beta^t}{2}\|\ \dot{\mathbf{M}}^{t+1} - \dot{\mathbf{X}}^{t+1}\|_F^2 \\
&= \mathcal{L}_{\beta^{t}}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t}, \dot{\mathbf{\Lambda}}_2^{t}) + \frac{1}{2\beta^t}(\|\dot{\mathbf{\Lambda}}_1^t\|_F^2 - \|\dot{\mathbf{\Lambda}}_1^{t+1}\|_F^2 + \|\dot{\mathbf{\Lambda}}_2^t\|_F^2 - \|\dot{\mathbf{\Lambda}}_2^{t+1}\|_F^2).
\end{aligned}
$$

Therefore, it obtains $\{\dot{\mathbf{X}}^{t+1}\}$, $\{\dot{\mathbf{Z}}^{t+1}\}$, $\{\dot{\mathbf{D}}_1^{t+1}\}$, and $\{\dot{\mathbf{D}}_2^{t+1}\}$ are bounded.

$$
\begin{aligned}
&\|\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{X}}^t\|_F \\
&= \|\mathcal{P}_{\Omega^c}\left(\frac{\dot{\mathbf{T}}_1^t+\dot{\mathbf{T}}_2^t}{2}\right) + \mathcal{P}_\Omega\left(\frac{\beta^t(\dot{\mathbf{T}}_1^t+\dot{\mathbf{T}}_2^t)+\dot{\mathbf{Y}}}{1+2\beta^t}\right) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1}}{2\beta^{t-1}} - \frac{(\dot{\mathbf{D}}_1^t)^H\dot{\mathbf{Z}}^t\dot{\mathbf{D}}_2^t}{2} - \frac{\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t-1}}{2\beta^{t-1}} - \frac{\dot{\mathbf{M}}^t}{2}\|_F \\
&= \|\mathcal{P}_{\Omega^c}\left(\frac{\dot{\mathbf{T}}_1^t+\dot{\mathbf{T}}_2^t}{2}\right) + \mathcal{P}_\Omega\left(\frac{\beta^t(\dot{\mathbf{T}}_1^t+\dot{\mathbf{T}}_2^t)+\dot{\mathbf{Y}}}{1+2\beta^t}\right) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1}}{2\beta^{t-1}} - \frac{\dot{\mathbf{T}}_1^t + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t}}{2} - \frac{\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t-1}}{2\beta^{t-1}} - \frac{\dot{\mathbf{T}}_2^t + \frac{\dot{\mathbf{\Lambda}}_2^t}{\beta^t}}{2}\|_F \\
&= \|\mathcal{P}_{\Omega^c}\left(\frac{\dot{\mathbf{T}}_1^t+\dot{\mathbf{T}}_2^t-\dot{\mathbf{T}}_1^t-\dot{\mathbf{T}}_2^t}{2}\right) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1}}{2\beta^{t-1}} - \frac{\dot{\mathbf{\Lambda}}_1^t}{2\beta^t} - \frac{\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t-1}}{2\beta^{t-1}} - \frac{\dot{\mathbf{\Lambda}}_2^t}{2\beta^t} \\
&+ \mathcal{P}_\Omega\left(\frac{2\beta^t(\dot{\mathbf{T}}_1^t+\dot{\mathbf{T}}_2^t)+2\dot{\mathbf{Y}}-(\dot{\mathbf{T}}_1^t+\dot{\mathbf{T}}_2^t)-2\beta^t(\dot{\mathbf{T}}_1^t+\dot{\mathbf{T}}_2^t)}{2(1+2\beta^t)}\right)\|_F \\
&= \|\mathcal{P}_\Omega\left(\frac{2\dot{\mathbf{Y}}-\dot{\mathbf{T}}_1^t-\dot{\mathbf{T}}_2^t}{2(1+2\beta^t)}\right) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1} - \dot{\mathbf{\Lambda}}_2^t + \dot{\mathbf{\Lambda}}_2^{t-1}}{2\beta^{t-1}} - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_2^t}{2\beta^t}\|_F \\
&= \frac{1}{\beta^t}\|\frac{\beta^t}{2(1+2\beta^t)}\mathcal{P}_\Omega(2\dot{\mathbf{Y}}-\dot{\mathbf{T}}_1^t-\dot{\mathbf{T}}_2^t) - \frac{\rho}{2}(\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1} - \dot{\mathbf{\Lambda}}_2^t + \dot{\mathbf{\Lambda}}_2^{t-1}) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_2^t}{2}\|_F.
\end{aligned}
$$

Define $\mathbf{R}_1^t = \frac{\beta^t}{2(1+2\beta^t)}\mathcal{P}_\Omega(2\dot{\mathbf{Y}}-\dot{\mathbf{T}}_1^t-\dot{\mathbf{T}}_2^t) - \frac{\rho}{2}(\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1} - \dot{\mathbf{\Lambda}}_2^t + \dot{\mathbf{\Lambda}}_2^{t-1}) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_2^t}{2}$. It is evident that $\mathbf{R}_1^t$ is bounded, which implies

$||\mathbf{R}_1^t|| \leq N_3$. Consequently, we have

$$
\begin{aligned}
&\|\dot{\mathbf{X}}^{t+l+1} - \dot{\mathbf{X}}^t\|_F \\
&\leq \|\dot{\mathbf{X}}^{t+l+1} - \dot{\mathbf{X}}^{t+l}\|_F + \cdots + \|\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{X}}^t\|_F \\
&= \frac{\|\mathbf{R}_1^{t+l}\|_F}{\beta^{t+l}} + \cdots + \frac{\|\mathbf{R}_1^t\|_F}{\beta^t} \leq N_3 \left(\frac{1}{\beta^{t+l}} + \cdots + \frac{1}{\beta^t}\right) \\
&= \frac{N_3}{\beta^t}\left(1 + \frac{1}{\rho} + \cdots + \frac{1}{\rho^l}\right) \leq \frac{N_3 \rho}{\beta^t(\rho - 1)} \to 0 \ (t \to \infty).
\end{aligned}
$$

Thus, $\{\dot{\mathbf{X}}^t\}$ constitutes a Cauchy sequence.
In the following, we prove that the sequence $\{\dot{\mathbf{Z}}^t\}$ is also a Cauchy sequence.

$$
\begin{aligned}
&\|\dot{\mathbf{Z}}^{t+1} - \dot{\mathbf{Z}}^t\|_F^2 \\
&= \|\dot{\mathbf{U}}^{t+1}\mathbf{\Delta}^{t+1}\dot{\mathbf{V}}^{t+1} - \dot{\mathbf{Z}}^t\|_F^2 \\
&= \|\dot{\mathbf{U}}^{t+1}\mathbf{\Delta}^{t+1}\dot{\mathbf{V}}^{t+1} - \dot{\mathbf{D}}_1^{t+1}(\dot{\mathbf{X}}^{t+1} + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t})\dot{\mathbf{D}}_2^{t+1} + \dot{\mathbf{D}}_1^{t+1}(\dot{\mathbf{X}}^{t+1} + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t})\dot{\mathbf{D}}_2^{t+1} - \dot{\mathbf{Z}}^t\|_F^2 \\
&= \|\dot{\mathbf{U}}^{t+1}\mathbf{\Delta}^{t+1}\dot{\mathbf{V}}^{t+1} - \dot{\mathbf{U}}^{t+1}\mathbf{\Sigma}^{t+1}\dot{\mathbf{V}}^{t+1} + \dot{\mathbf{D}}_1^{t+1}(\dot{\mathbf{X}}^{t+1} + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta})\dot{\mathbf{D}}_2^{t+1} - \dot{\mathbf{Z}}^t\|_F^2 \\
&\leq \|\mathbf{\Sigma}^{t+1} - \mathbf{\Delta}^{t+1}\|_F^2 + \|\dot{\mathbf{D}}_1^{t+1}(\dot{\mathbf{X}}^{t+1} + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t})\dot{\mathbf{D}}_2^{t+1} - \dot{\mathbf{Z}}^t\|_F^2 \\
&= \sum_{i=1}^{r}(\delta_i^{t+1} - \sigma_i^{t+1})^2 + \|\dot{\mathbf{D}}_1^{t+1}(\dot{\mathbf{X}}^{t+1} + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t})\dot{\mathbf{D}}_2^{t+1} - \dot{\mathbf{Z}}^t\|_F^2 \\
&\leq \frac{\lambda^2 r}{(\beta^t)^2} + \|\dot{\mathbf{X}}^{t+1} + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t} - \dot{\mathbf{D}}_1^{t+1}\dot{\mathbf{Z}}^t\dot{\mathbf{D}}_2^{t+1}\|_F^2 \\
&= \frac{\lambda^2 r}{(\beta^t)^2} + \|\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{X}}^t + \dot{\mathbf{X}}^t - \dot{\mathbf{D}}_1^{t+1}\dot{\mathbf{Z}}^t\dot{\mathbf{D}}_2^{t+1} + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t}\|_F^2 \\
&= \frac{\lambda^2 r}{(\beta^t)^2} + \|(\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{X}}^t) + (\dot{\mathbf{X}}^t - \dot{\mathbf{D}}_1^{t+1}\dot{\mathbf{Z}}^t\dot{\mathbf{D}}_2^{t+1}) + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t}\|_F^2 \\
&= \frac{\lambda^2 r}{(\beta^t)^2} + \|\frac{\mathbf{R}_1^t}{\beta^t} + \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1}}{\beta^{t-1}} + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t}\|_F^2 \\
&= \frac{1}{(\beta^t)^2}(\lambda^2 r + \|\mathbf{R}_1^t + \rho(\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1}) + \dot{\mathbf{\Lambda}}_1^t\|_F^2) = \frac{\lambda^2 r + \mathbf{R}_2^t}{(\beta^t)^2},
\end{aligned}
$$

where $\mathbf{R}_2^t = \mathbf{R}_1^t + \rho(\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1}) + \dot{\mathbf{\Lambda}}_1^t$, and $\mathbf{R}_2^t$ is bounded. Let N_4 denote its upper bound, i.e. $\|\mathbf{R}_2^t\|_F \leq N_4$. It follows that $\|\dot{\mathbf{Z}}^{t+1} - \dot{\mathbf{Z}}^t\|_F \leq \frac{\sqrt{\lambda^2 r + ||\mathbf{R}_2^t||_F}}{\beta^t} \leq \frac{\sqrt{\lambda^2 r + N_4}}{\beta^t} = \frac{N_5}{\beta^t}$, where $\sqrt{\lambda^2 r + N_4} = N_5$. The second inequality holds because the condition: If $\delta_i^{t+1} \geq \frac{\lambda}{\beta^t}$, then $\sigma_i^{t+1} = \frac{||z||_2 + \frac{\alpha\lambda}{\beta^t}}{||z||_2} z$, where $z = \max\{\delta_i^{t+1} - \frac{\lambda}{\beta^t},\ 0\}$, which implies that $\delta_i^{t+1} - \sigma_i^{t+1} \leq \frac{\lambda}{\beta^t}$; if $\delta_i^{t+1} < \frac{\lambda}{\beta^t}$, then $\delta_i^{t+1} - \sigma_i^{t+1} = \delta_i^{t+1} < \frac{\lambda}{\beta^t}$. Therefore, the sequence $\{\dot{\mathbf{\Lambda}}_1^t\}$ is bounded. Furthermore, it obtains

$$
\begin{aligned}
&\|\dot{\mathbf{Z}}^{t+l+1} - \dot{\mathbf{Z}}^t\|_F \\
&\leq \|\dot{\mathbf{Z}}^{t+l+1} - \dot{\mathbf{Z}}^{t+l}\|_F + \cdots + \|\dot{\mathbf{Z}}^{t+1} - \dot{\mathbf{Z}}^t\|_F \\
&\leq \frac{N_5}{\beta^{t+l}} + \cdots + \frac{N_5}{\beta^t} \leq N_5 \left(\frac{1}{\beta^{t+l}} + \cdots + \frac{1}{\beta^t}\right) \\
&= \frac{N_5}{\beta^t}\left(1 + \frac{1}{\rho} + \cdots + \frac{1}{\rho^l}\right) \leq \frac{N_5 \rho}{\beta^t(\rho - 1)} \to 0 \ (t \to \infty).
\end{aligned}
$$

Therefore, $\{\dot{\mathbf{Z}}^t\}$ is a Cauchy sequence.

Finally, we prove that the sequence $\{\dot{\mathbf{M}}^t\}$ is also a Cauchy sequence.

$$
\begin{aligned}
&\|\dot{\mathbf{M}}^{t+1} - \dot{\mathbf{M}}^t\|_F \\
&= \|\dot{\mathbf{M}}^{t+1} - \dot{\mathbf{X}}^{t+1} + \dot{\mathbf{X}}^{t+1} - \dot{\mathbf{X}}^t + \dot{\mathbf{X}}^t - \dot{\mathbf{M}}^t\|_F \\
&= \|\frac{\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t+1}}{\beta^t} + \frac{\mathbf{R}_1^t}{\beta^t} + \frac{\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t-1}}{\beta^{t-1}}\|_F \\
&= \frac{1}{\beta^t}\|2\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^{t-1} + \mathbf{R}_1^t\|_F.
\end{aligned}
$$

Let $\mathbf{R}_3^t = 2\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^{t-1} + \mathbf{R}_1^t$. Clearly, $\mathbf{R}_3^t$ is bounded. Assuming that $\|\mathbf{R}_3^t\| \leq N_6$, we obtain:

$$
\begin{aligned}
&\|\dot{\mathbf{M}}^{t+l+1} - \dot{\mathbf{M}}^t\|_F \\
&\leq \|\dot{\mathbf{M}}^{t+l+1} - \dot{\mathbf{M}}^{t+l}\|_F + \cdots + \|\dot{\mathbf{M}}^{t+1} - \dot{\mathbf{M}}^t\|_F \\
&\leq \frac{N_6}{\beta^{t+l}} + \cdots + \frac{N_6}{\beta^t} \leq N_6\left(\frac{1}{\beta^{t+l}} + \cdots + \frac{1}{\beta^t}\right) \\
&= \frac{N_6}{\beta^t}\left(1 + \frac{1}{\rho} + \cdots + \frac{1}{\rho^l}\right) \leq \frac{N_6\rho}{\beta^t(\rho-1)} \to 0 \ (t \to \infty).
\end{aligned}
$$

Thus, the sequence $\{\dot{\mathbf{M}}^t\}$ is a Cauchy sequence.

□

Appendix B. The proof of Theorem 6

Assumption 1: A denoiser is said to be bounded with a nonnegative parameter δ if the function $\mathfrak{D}(\mathcal{X}, \delta) : \mathbb{R}^{I_1 \times I_2 \times 3} \mapsto \mathbb{R}^{I_1 \times I_2 \times 3}$ satisfies, for any input $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times 3}$ and δ,

$$
||\mathfrak{D}(\mathcal{X}, \delta) - \mathcal{X}||_F^2 \leq \delta^2 C \tag{B.1}
$$

for some universal constant C independent of I_1, I_2 and δ.

Theorem 6: Let $\phi = \{\dot{\mathbf{Z}}, \dot{\mathbf{X}}, \dot{\mathbf{D}}_1, \dot{\mathbf{D}}_2, \dot{\mathbf{M}}, \mathcal{S}, \dot{\mathbf{\Lambda}}_1, \dot{\mathbf{\Lambda}}_2, \mathcal{T}\}$ and $\{\phi^t\}_{t=1}^{\infty}$ be the sequence generated by CLQNMF-PnP algorithm. If Assumption 1 holds and the dual variable sequences $\{\dot{\mathbf{\Lambda}}_1^t\}$, $\{\dot{\mathbf{\Lambda}}_2^t\}$ and $\{\mathcal{T}^t\}$ are bounded. Then, the primal variable sequences $\{\dot{\mathbf{Z}}^t\}$, $\{\dot{\mathbf{X}}^t\}$, $\{\dot{\mathbf{M}}^t\}$ and $\{\mathcal{S}^t\}$ are Cauchy sequences.

Proof. Similar to the proof of Theorem 5, we have

$$
\begin{aligned}
&\mathcal{L}_{\beta^{t+1}}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \mathcal{S}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1}, \mathcal{T}^{t+1}) \\
&= \frac{1}{2}\|\mathcal{P}_\Omega(\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{Y}})\|_F^2 + \lambda(\|\dot{\mathbf{Z}}^{t+1}\|_* - \alpha\|\dot{\mathbf{Z}}^{t+1}\|_F) + \sum_{i=1}^{2}\Psi(\dot{\mathbf{D}}_i^{t+1}) + \lambda_2\Phi_{\mathsf{pnp}}(\mathcal{S}) \\
&+ \frac{\beta^{t+1}}{2}\|\dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}\|_F^2 + \Re\left(\langle\dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1}\rangle\right) \\
&+ \frac{\beta^{t+1}}{2}\|\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}\|_F^2 + \Re\left(\langle\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1}\rangle\right) + \frac{\beta^{t+1}}{2}\|\mathcal{X}^{t+1} - \mathcal{S}^{t+1}\|_F^2 + \langle\mathcal{X}^{t+1} - \mathcal{S}^{t+1}, \mathcal{T}^{t+1}\rangle \\
&= \mathcal{L}_{\beta^t}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \mathcal{S}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^t, \dot{\mathbf{\Lambda}}_2^t, \mathcal{T}^t) \\
&+ \Re\left(\langle\dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1} - \dot{\mathbf{\Lambda}}_1^t\rangle\right) + \frac{\beta^{t+1} - \beta^t}{2}\|\dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}\|_F^2 + \Re\left(\langle\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^t\rangle\right) \\
&+ \frac{\beta^{t+1} - \beta^t}{2}\|\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}\|_F^2 + \langle\mathcal{X}^{t+1} - \mathcal{S}^{t+1}, \mathcal{T}^{t+1} - \mathcal{T}^t\rangle + \frac{\beta^{t+1} - \beta^t}{2}\|\mathcal{S}^{t+1} - \mathcal{X}^{t+1}\|_F^2 \\
&= \mathcal{L}_{\beta^t}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \mathcal{S}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^t, \dot{\mathbf{\Lambda}}_2^t, \mathcal{T}^t) + \frac{\beta^{t+1} + \beta^t}{2(\beta^t)^2}(\|\dot{\mathbf{\Lambda}}_1^{t+1} - \dot{\mathbf{\Lambda}}_1^t\|_F^2 + \|\dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^t\|_F^2 + \|\mathcal{T}^{t+1} - \mathcal{T}^t\|_F^2).
\end{aligned}
$$

Since the sequences $\{\dot{\mathbf{\Lambda}}_1^t\}$, $\{\dot{\mathbf{\Lambda}}_2^t\}$ and $\{\mathcal{T}^t\}$ are assume to be bounded, therefore $\{\dot{\mathbf{\Lambda}}_1^{t+1} - \dot{\mathbf{\Lambda}}_1^t\}$, $\{\dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^t\}$ and $\{\mathcal{T}^{t+1} - \mathcal{T}^t\}$ are also bounded. Let N_1, N_2 and N_7 be the corresponding upper bounds, namely $\|\dot{\mathbf{\Lambda}}_1^{t+1} - \dot{\mathbf{\Lambda}}_1^t\|_F \leq N_1$, $\|\dot{\mathbf{\Lambda}}_2^{t+1} - \dot{\mathbf{\Lambda}}_2^t\|_F \leq N_2$ and $\|\mathcal{T}^{t+1} - \mathcal{T}^t\|_F \leq N_7$. Moreover, the following inequality holds:

$$\mathcal{L}_{\beta^{t+1}}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \mathcal{S}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1}, \mathcal{T}^{t+1}) \leq \mathcal{L}_{\beta^t}(\dot{\mathbf{X}}^t, \dot{\mathbf{Z}}^t, \mathcal{S}^t, \dot{\mathbf{M}}^t, \dot{\mathbf{D}}_1^t, \dot{\mathbf{D}}_2^t, \dot{\mathbf{\Lambda}}_1^t, \dot{\mathbf{\Lambda}}_2^t, \mathcal{T}^t)$$

Since the optimal solutions $\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \dot{\mathbf{M}}^{t+1}$ have been obtained for their respective subproblems, it follows that:

$$\begin{aligned}
&\mathcal{L}_{\beta^{t+1}}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \mathcal{S}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^{t+1}, \dot{\mathbf{\Lambda}}_2^{t+1}, \mathcal{T}^{t+1}) \\
&\leq \mathcal{L}_{\beta^t}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \mathcal{S}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^t, \dot{\mathbf{\Lambda}}_2^t, \mathcal{T}^{t+1}) + \frac{\beta^{t+1} + \beta^t}{2(\beta^t)^2}\left(N_1^2 + N_2^2 + N_7^2\right) \\
&\leq \mathcal{L}_{\beta^0}(\dot{\mathbf{X}}^1, \dot{\mathbf{Z}}^1, \mathcal{S}^1, \dot{\mathbf{M}}^1, \dot{\mathbf{D}}_1^1, \dot{\mathbf{D}}_2^1, \dot{\mathbf{\Lambda}}_1^0, \dot{\mathbf{\Lambda}}_2^0, \mathcal{T}^0) + \left(N_1^2 + N_2^2 + N_7^2\right)\sum_{t=0}^{\infty}\frac{\beta^{t+1} + \beta^t}{2(\beta^t)^2} \\
&\leq \mathcal{L}_{\beta^0}(\dot{\mathbf{X}}^1, \dot{\mathbf{Z}}^1, \mathcal{S}^1, \dot{\mathbf{M}}^1, \dot{\mathbf{D}}_1^1, \dot{\mathbf{D}}_2^1, \dot{\mathbf{\Lambda}}_1^0, \dot{\mathbf{\Lambda}}_2^0, \mathcal{T}^0) + \left(N_1^2 + N_2^2 + N_7^2\right)\sum_{t=0}^{\infty}\frac{1+\rho}{2\beta^0\rho^t} \\
&\leq \mathcal{L}_{\beta^0}(\dot{\mathbf{X}}^1, \dot{\mathbf{Z}}^1, \mathcal{S}^1, \dot{\mathbf{M}}^1, \dot{\mathbf{D}}_1^1, \dot{\mathbf{D}}_2^1, \dot{\mathbf{\Lambda}}_1^0, \dot{\mathbf{\Lambda}}_2^0, \mathcal{T}^0) + \frac{N_1^2 + N_2^2 + N_7^2}{\beta^0}\sum_{t=0}^{\infty}\frac{1}{\rho^{t-1}} < +\infty.
\end{aligned}$$

Hence, $\mathcal{L}$ is bounded. Next we prove that the sequences $\{\dot{\mathbf{X}}^{t+1}\}$, $\{\dot{\mathbf{Z}}^{t+1}\}$, $\{\mathcal{S}^{t+1}\}$, $\{\dot{\mathbf{D}}_1^{t+1}\}$, and $\{\dot{\mathbf{D}}_2^{t+1}\}$ are bounded.

$$\begin{aligned}
&\frac{1}{2}\|\mathcal{P}_\Omega(\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{Y}})\|_F^2 + \lambda_1(\|\dot{\mathbf{Z}}^{t+1}\|_* - \alpha\|\dot{\mathbf{Z}}^{t+1}\|_F) + \sum_{i=1}^{2}\Psi(\dot{\mathbf{D}}_i^{t+1}) + \lambda_2\Phi_{\mathsf{pnp}}(\mathcal{S}^{k+1}) \\
&= \mathcal{L}_{\beta^t}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \mathcal{S}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^t, \dot{\mathbf{\Lambda}}_2^t, \mathcal{T}^t) - \Re\left(\langle\dot{\mathbf{\Lambda}}_1^t, \dot{\mathbf{X}}^{t+1} - (\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1}\rangle\right) - \frac{\beta^t}{2}\|\dot{\mathbf{M}}^{t+1} - \dot{\mathbf{X}}^{t+1}\|_F^2 \\
&- \Re\left(\langle\dot{\mathbf{\Lambda}}_2^t,\ \dot{\mathbf{X}}^{t+1} - \dot{\mathbf{M}}^{t+1}\rangle\right) - \frac{\beta^t}{2}\|(\dot{\mathbf{D}}_1^{t+1})^H\dot{\mathbf{Z}}^{t+1}\dot{\mathbf{D}}_2^{t+1} - \dot{\mathbf{X}}^{t+1}\|_F^2 + \langle\mathcal{T}^t, \mathcal{X}^{t+1} - \mathcal{S}^{t+1}\rangle - \frac{\beta^t}{2}\|\mathcal{S}^{t+1} - \mathcal{X}^{t+1}\| \\
&= \mathcal{L}_{\beta^t}(\dot{\mathbf{X}}^{t+1}, \dot{\mathbf{Z}}^{t+1}, \mathcal{S}^{t+1}, \dot{\mathbf{M}}^{t+1}, \dot{\mathbf{D}}_1^{t+1}, \dot{\mathbf{D}}_2^{t+1}, \dot{\mathbf{\Lambda}}_1^t, \dot{\mathbf{\Lambda}}_2^t, \mathcal{T}^t) + \frac{1}{2\beta^t}(\|\dot{\mathbf{\Lambda}}_1^t\|_F^2 - \|\dot{\mathbf{\Lambda}}_1^{t+1}\|_F^2 + \|\dot{\mathbf{\Lambda}}_2^t\|_F^2 - \|\dot{\mathbf{\Lambda}}_2^{t+1}\|_F^2 + \|\mathcal{T}^t\|_F^2 - \|\mathcal{T}^{t+1}\|_F^2).
\end{aligned}$$

Therefore, $\{\dot{\mathbf{X}}^{t+1}\}$, $\{\dot{\mathbf{Z}}^{t+1}\}$, $\{\mathcal{S}^{t+1}\}$, $\{\dot{\mathbf{D}}_1^{t+1}\}$, and $\{\dot{\mathbf{D}}_2^{t+1}\}$ are bounded.

$$\begin{aligned}
&\|\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{X}}^t\|_F \\
&= \|\mathcal{P}_{\Omega^c}\left(\frac{\dot{\mathbf{T}}_1^t + \dot{\mathbf{T}}_2^t + \dot{\mathbf{T}}_3^t}{3}\right) + \mathcal{P}_\Omega\left(\frac{\beta^t(\dot{\mathbf{T}}_1^t + \dot{\mathbf{T}}_2^t + \dot{\mathbf{T}}_3^t) + \dot{\mathbf{Y}}}{1 + 3\beta^t}\right) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1}}{3\beta^{t-1}} - \frac{(\dot{\mathbf{D}}_1^t)^H\dot{\mathbf{Z}}^t\dot{\mathbf{D}}_2^t}{3} \\
&- \frac{\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t-1}}{3\beta^{t-1}} - \frac{\dot{\mathbf{M}}^t}{3} - \frac{\mathcal{T}^t - \mathcal{T}^{t-1}}{3\beta^{t-1}} - \frac{\mathcal{S}^t}{3}\|_F \\
&= \|\mathcal{P}_{\Omega^c}\left(\frac{\dot{\mathbf{T}}_1^t + \dot{\mathbf{T}}_2^t + \dot{\mathbf{T}}_3^t}{3}\right) + \mathcal{P}_\Omega\left(\frac{\beta^t(\dot{\mathbf{T}}_1^t + \dot{\mathbf{T}}_2^t + \dot{\mathbf{T}}_3^t) + \dot{\mathbf{Y}}}{1 + 3\beta^t}\right) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1}}{3\beta^{t-1}} - \frac{\dot{\mathbf{T}}_1^t + \frac{\dot{\mathbf{\Lambda}}_1^t}{\beta^t}}{3} \\
&- \frac{\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t-1}}{3\beta^{t-1}} - \frac{\dot{\mathbf{T}}_2^t + \frac{\dot{\mathbf{\Lambda}}_2^t}{\beta^t}}{3} - \frac{\mathcal{T}^t - \mathcal{T}^{t-1}}{3\beta^{t-1}} - \frac{\dot{\mathbf{T}}_3^t + \frac{\mathcal{T}^t}{\beta^t}}{3}\|_F \\
&= \|\mathcal{P}_{\Omega^c}\left(\frac{\dot{\mathbf{T}}_1^t + \dot{\mathbf{T}}_2^t + \dot{\mathbf{T}}_3^t - \dot{\mathbf{T}}_1^t - \dot{\mathbf{T}}_2^t - \dot{\mathbf{T}}_3^t}{2}\right) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1}}{3\beta^{t-1}} - \frac{\dot{\mathbf{\Lambda}}_1^t}{3\beta^t} - \frac{\dot{\mathbf{\Lambda}}_2^t - \dot{\mathbf{\Lambda}}_2^{t-1}}{3\beta^{t-1}} - \frac{\dot{\mathbf{\Lambda}}_2^t}{3\beta^t} - \frac{\mathcal{T}^t - \mathcal{T}^{t-1}}{3\beta^{t-1}} - \frac{\mathcal{T}^t}{3\beta^t} \\
&+ \mathcal{P}_\Omega\left(\frac{3\beta^t(\dot{\mathbf{T}}_1^t + \dot{\mathbf{T}}_2^t + \dot{\mathbf{T}}_3^t) + 3\dot{\mathbf{Y}} - (\dot{\mathbf{T}}_1^t + \dot{\mathbf{T}}_2^t + \dot{\mathbf{T}}_3^t) - 3\beta^t(\dot{\mathbf{T}}_1^t + \dot{\mathbf{T}}_2^t + \dot{\mathbf{T}}_3^t)}{3(1 + 3\beta^t)}\right)\|_F \\
&= \|\mathcal{P}_\Omega\left(\frac{3\dot{\mathbf{Y}} - \dot{\mathbf{T}}_1^t - \dot{\mathbf{T}}_2^t - \dot{\mathbf{T}}_3^t}{3(1 + 3\beta^t)}\right) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1} - \dot{\mathbf{\Lambda}}_2^t + \dot{\mathbf{\Lambda}}_2^{t-1} - \mathcal{T}^t + \mathcal{T}^{t-1}}{3\beta^{t-1}} - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_2^t - \mathcal{T}^t}{3\beta^t}\|_F \\
&= \frac{1}{\beta^t}\|\frac{\beta^t}{3(1 + 3\beta^t)}\mathcal{P}_\Omega(3\dot{\mathbf{Y}} - \dot{\mathbf{T}}_1^t - \dot{\mathbf{T}}_2^t - \dot{\mathbf{T}}_3^t) - \frac{\rho}{3}(\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1} - \dot{\mathbf{\Lambda}}_2^t + \dot{\mathbf{\Lambda}}_2^{t-1} - \mathcal{T}^t + \mathcal{T}^{t-1}) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_2^t - \mathcal{T}^t}{3}\|_F.
\end{aligned}$$

Let $\mathbf{R}_4^t = \frac{\beta^t}{3(1+3\beta^t)}\mathcal{P}_\Omega(3\dot{\mathbf{Y}} - \dot{\mathbf{T}}_1^t - \dot{\mathbf{T}}_2^t - \dot{\mathbf{T}}_3^t) - \frac{\rho}{3}(\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_1^{t-1} - \dot{\mathbf{\Lambda}}_2^t + \dot{\mathbf{\Lambda}}_2^{t-1} - \mathcal{T}^t + \mathcal{T}^{t-1}) - \frac{\dot{\mathbf{\Lambda}}_1^t - \dot{\mathbf{\Lambda}}_2^t - \mathcal{T}^t}{3}$, then we have that $\mathbf{R}_4^t$ is bounded. Assuming that $\|\mathbf{R}_4^t\|_F \leq N_8$, then it derives

$$
\begin{aligned}
&\|\dot{\mathbf{X}}^{t+l+1} - \dot{\mathbf{X}}^t\|_F \\
&\leq \|\dot{\mathbf{X}}^{t+l+1} - \dot{\mathbf{X}}^{t+l}\|_F + \cdots + \|\dot{\mathbf{X}}^{t+1} - \dot{\mathbf{X}}^t\|_F \\
&= \frac{\|\mathbf{R}_4^{t+l}\|_F}{\beta^{t+l}} + \cdots + \frac{\|\mathbf{R}_4^t\|_F}{\beta^t} \leq N_8\left(\frac{1}{\beta^{t+l}} + \cdots + \frac{1}{\beta^t}\right) \\
&= \frac{N_8}{\beta^t}\left(1 + \frac{1}{\rho} + \cdots + \frac{1}{\rho^l}\right) \leq \frac{N_8\rho}{\beta^t(\rho-1)} \to 0 \ (t \to \infty).
\end{aligned}
$$

Thus, $\{\dot{\mathbf{X}}^t\}$ is a Cauchy sequence. The proof that $\{\dot{\mathbf{Z}}^t\}$ and $\{\dot{\mathbf{M}}^t\}$ are Cauchy sequences is similar to that in Theorem 5.

Next, we prove that the sequence $\{\mathcal{S}^t\}$ is also a Cauchy sequence.

$$
\begin{aligned}
&\|\mathcal{S}^{t+1} - \mathcal{S}^t\|_F \\
&= \|\mathfrak{D}\left(\Gamma(\dot{\mathbf{X}}^{t+1}) + \frac{\mathcal{T}^t}{\beta^t}, \sqrt{\gamma/\beta^t}\right) - \left(\Gamma(\dot{\mathbf{X}}^{t+1}) + \frac{\mathcal{T}^t}{\beta^t}\right) + \left(\Gamma(\dot{\mathbf{X}}^{t+1}) + \frac{\mathcal{T}^t}{\beta^t}\right) - \left(\Gamma(\dot{\mathbf{X}}^t) + \frac{\mathcal{T}^{t-1}}{\beta^{t-1}}\right) + \left(\Gamma(\dot{\mathbf{X}}^t) + \frac{\mathcal{T}^{t-1}}{\beta^{t-1}}\right) - \mathcal{S}^t\|_F \\
&\leq \|\mathfrak{D}\left(\Gamma(\dot{\mathbf{X}}^{t+1}) + \frac{\mathcal{T}^t}{\beta^t}, \sqrt{\gamma/\beta^t}\right) - \left(\Gamma(\dot{\mathbf{X}}^{t+1}) + \frac{\mathcal{T}^t}{\beta^t}\right)\|_F + \|\Gamma(\dot{\mathbf{X}}^{t+1}) + \frac{\mathcal{T}^t}{\beta^t} - \Gamma(\dot{\mathbf{X}}^t) - \frac{\mathcal{T}^{t-1}}{\beta^{t-1}}\|_F + \|\left(\Gamma(\dot{\mathbf{X}}^t) + \frac{\mathcal{T}^{t-1}}{\beta^{t-1}}\right) - \mathcal{S}^t\|_F \\
&\leq \sqrt{\frac{\gamma C}{\beta^t}} + \|\Gamma(\dot{\mathbf{X}}^{t+1}) - \Gamma(\dot{\mathbf{X}}^t) + \frac{\mathcal{T}^t}{\beta^t} - \frac{\mathcal{T}^{t-1}}{\beta^{t-1}}\|_F + \|\mathfrak{D}\left(\Gamma(\dot{\mathbf{X}}^t) + \frac{\mathcal{T}^{t-1}}{\beta^{t-1}}, \sqrt{\gamma/\beta^{t-1}}\right) - \left(\Gamma(\dot{\mathbf{X}}^t) + \frac{\mathcal{T}^{t-1}}{\beta^{t-1}}\right)\|_F \\
&\leq \sqrt{\frac{\gamma C}{\beta^t}} + \|\Gamma(\dot{\mathbf{X}}^{t+1}) - \Gamma(\dot{\mathbf{X}}^t)\|_F + \|\frac{\mathcal{T}^t}{\beta^t} - \frac{\mathcal{T}^{t-1}}{\beta^{t-1}}\|_F + \sqrt{\frac{\gamma C}{\beta^{t-1}}} \\
&\leq \sqrt{\frac{\gamma C}{\beta^t}} + \sqrt{\frac{\gamma C}{\beta^{t-1}}} + \frac{\|\mathbf{R}_4^t\|_F}{\beta^t} + \frac{1}{\beta^t}\|\mathcal{T}^t - \rho\mathcal{T}^{t-1}\|_F \\
&= \frac{1}{\sqrt{\beta_t}}\left(\sqrt{\gamma C} + \sqrt{\rho\gamma C} + \frac{\|\mathbf{R}_4^t\|_F}{\sqrt{\beta^t}} + \frac{1}{\sqrt{\beta^t}}\|\mathcal{T}^t - \rho\mathcal{T}^{t-1}\|_F\right)
\end{aligned}
$$

Let $\mathbf{R}_5^t = \sqrt{\gamma C} + \sqrt{\rho\gamma C} + \frac{\|\mathbf{R}_4^t\|_F}{\sqrt{\beta^t}} + \frac{1}{\sqrt{\beta^t}}\|\mathcal{T}^t - \rho\mathcal{T}^{t-1}\|_F$. Obviously, $\mathbf{R}_5^t$ is bounded, and assuming that $\|\mathbf{R}_5^t\|_F \leq N_9$, we have

$$
\begin{aligned}
&\|\mathcal{S}^{t+l+1} - \mathcal{S}^t\|_F \\
&\leq \|\mathcal{S}^{t+l+1} - \mathcal{S}^{t+l}\|_F + \cdots + \|\mathcal{S}^{t+1} - \mathcal{S}^t\|_F \\
&\leq \frac{N_9}{\sqrt{\beta^{t+l}}} + \cdots + \frac{N_9}{\sqrt{\beta^t}} = N_9\left(\frac{1}{\sqrt{\beta^{t+l}}} + \cdots + \frac{1}{\sqrt{\beta^t}}\right) = \frac{N_9}{\sqrt{\beta^t}}\left(1 + \frac{1}{\rho} + \cdots + \frac{1}{\rho^l}\right) \\
&\leq \frac{N_9\rho}{\sqrt{\beta^t}(\rho-1)} \to 0 \ (t \to \infty).
\end{aligned}
$$

Thus, $\{\mathcal{S}^t\}$ is a Cauchy sequence.

$\square$