Notes - January 13th Math 32A

Given two vectors, how can we calculate the angle θ between them? For this, we define the **dot product**.

Definition. The dot product of two vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is the scalar defined as

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Properties.

- $\bullet \ \mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$
- $\bullet \ \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- $(\lambda \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (\lambda \mathbf{w}) = \lambda (\mathbf{v} \cdot \mathbf{w})$
- $\bullet \ \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $\bullet \ (\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$
- $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$

Remark. Notice that the dot product of two vectors is a number, so it is no longer a vector.

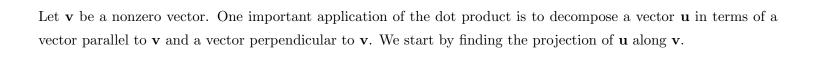
Exercise. Let $\mathbf{v} = \langle 2, 3, -1 \rangle$ and $\mathbf{w} = \langle -5, 1, 2 \rangle$. Calculate $\mathbf{v} \cdot \mathbf{w}$.

Theorem. Let θ be the angle between two nonzero vectors \mathbf{v} and \mathbf{w} . Then

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta \quad \text{or} \quad \cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

- The angle θ between two angles is chosen to satisfy $0 \le \theta \le \pi$.
- Two nonzero vectors \mathbf{v} and \mathbf{w} are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$. We write $\mathbf{v} \perp \mathbf{w}$.
- The angle between two vectors \mathbf{v} and \mathbf{w} is acute if and only if $\mathbf{v} \cdot \mathbf{w} > 0$.
- The angle between two vectors ${\bf v}$ and ${\bf w}$ is acute if and only if ${\bf v}\cdot{\bf w}<0$.

Exercise. Find the angle between the vectors $\mathbf{v} = \langle 1, 1, -2 \rangle$ and $\mathbf{w} = \langle 2, -1, -1 \rangle$.



The projection of \mathbf{u} along \mathbf{v} is the vector

$$\mathbf{u}_{\parallel} = \left(rac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}
ight)\mathbf{v} = \left(rac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}
ight)\mathbf{e}_{\mathbf{v}}$$

Define $\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\parallel}$.

We can think of \mathbf{u}_{\parallel} as the component of \mathbf{u} parallel to \mathbf{v} , and \mathbf{u}_{\perp} as the component of \mathbf{v} perpendicular to \mathbf{v} . The decomposition of \mathbf{u} with respect to \mathbf{v} is

$$\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$$

