Notes - January 15th

Math 32A

Given two vectors $\mathbf{v}$ and $\mathbf{w}$, we will learn how to find a third vector $\mathbf{u}$ that is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

Very brief introduction to determinants.

An $n \times n$ matrix is an array consisting of n rows and n columns of numbers.

The determinant of a 2×2 matrix is denoted and calculated as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant of a 3×3 matrix can be computed in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Definition. The cross product of two vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is the vector

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

Expanded, this is the vector $\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\mathbf{i} - (v_1 w_3 - v_3 w_1)\mathbf{j} + (v_1 w_2 - v_2 w_1)\mathbf{k}$.

Geometric properties of the cross product.

Given two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, such that they are not parallel and with an angle θ between them, the cross product $\mathbf{v} \times \mathbf{w}$ is the unique vector with the following three properties:

1. $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$;
2. $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$;
3. $\{\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w}\}$ satisfies the right-hand rule.

Properties of the cross product.

- $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$
- $\mathbf{v} \times \mathbf{v} = \mathbf{0}$
- $\mathbf{v} \times \mathbf{w} = 0$ if and only if $\mathbf{w} = \lambda\mathbf{v}$ for some scalar λ, or $\mathbf{v} = \mathbf{0}$.
- $(\lambda\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda\mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$.
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

Cross product of standard basis vectors.

$$
\begin{aligned}
\mathbf{i} \times \mathbf{j} &= \mathbf{k} & \mathbf{j} \times \mathbf{k} &= \mathbf{i} & \mathbf{k} \times \mathbf{i} &= \mathbf{j} \\
\mathbf{j} \times \mathbf{i} &= -\mathbf{k} & \mathbf{k} \times \mathbf{j} &= -\mathbf{i} & \mathbf{i} \times \mathbf{k} &= -\mathbf{j} \\
\mathbf{i} \times \mathbf{i} &= \mathbf{0} & \mathbf{j} \times \mathbf{j} &= \mathbf{0} & \mathbf{k} \times \mathbf{k} &= \mathbf{0}
\end{aligned}
$$

Exercise. Find a vector perpendicular to both $\mathbf{v} = \langle 1, 2, -1\rangle$ and $\mathbf{w} = \langle 0, 3, 2\rangle$.

Exercise. Let $\mathbf{v} = \langle 0, 0, 6\rangle$ and $\mathbf{w} = \langle 1, 1, 0\rangle$. Calculate $\mathbf{v} \times \mathbf{w}$.

Applications.

Area of a parallelogram. Let $\mathbf{u}$ and $\mathbf{v}$ two vectors nonzero vectors. The area of the parallelogram generated by $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} \times \mathbf{v}\|$. The area of the triangle generated by $\mathbf{u}$ and $\mathbf{v}$ is $\dfrac{\|\mathbf{u} \times \mathbf{v}\|}{2}$.

Volume of a parallelepiped. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ three nonzero vectors in $\mathbb{R}^3$. The volume of the parallelepiped generated by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. It is also the absolute value of the determinant of the matrix with rows $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Exercise. Let $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 1 \rangle$. Find the area of the parallelogram generated by $\mathbf{v}$ and $\mathbf{w}$.

Exercise. Let Let $\mathbf{u} = \langle 1, 4, -2 \rangle$, $\mathbf{v} = \langle 2, 3, 0 \rangle$ and $\mathbf{w} = \langle -1, 1, 0 \rangle$. Find the volume of the parallelepiped defined by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.