Notes - January 6th

Math 32A

Today we will work in $\mathbb{R}^2 := \{(x, y) \mid x, y \in \mathbb{R}\}$.
A two-dimensional vector $\mathbf{v}$ is determined by two points in the plane: the initial point P and the terminal point Q. We denote it as

$$\mathbf{v} = \overrightarrow{PQ}.$$

One special case is when the vector initial point is the origin O. Every vector $\mathbf{v} = \overrightarrow{PQ}$ is equivalent to a unique vector $\mathbf{v_0} = \overrightarrow{OR}$.

Definition. Let $P = (a_1, b_1)$ and $Q = (a_2, b_2)$ be two points in $\mathbb{R}^2$. The components of $\mathbf{v} = \overrightarrow{PQ}$ are the quantities

$$a = a_2 - a_1 \qquad b = b_2 - b_1.$$

If the components of a vector $\mathbf{v}$ are a and b, we write $\mathbf{v} = \langle a, b\rangle$. Two vectors with the same components are called *equivalent.*

The *lenght* of a vector $\mathbf{v} = \langle a, b\rangle$ is $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$.

The *zero vector* is the vector $\mathbf{0} = \langle 0, 0\rangle$.

Vector algebra.

Given two vectors $\mathbf{v}$ and $\mathbf{w}$, can we find a vector $\mathbf{v}+\mathbf{w}$?

Similarly, we can find $\mathbf{v}-\mathbf{w}$.

We can perform what is called *scalar multiplication*. Given a vector $\mathbf{v}$ and a real number λ, we can define the vector $\lambda\mathbf{v}$ as the vector whose length is $|\lambda|\|v\|$, points in the same direction of $\mathbf{v}$ if $\lambda > 0$, or it points in the opposite direction if $\lambda < 0$. In this case we call λ a *scalar*.

It is easier to compute these using vector components. Let $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{w} = \langle c, d \rangle$.

- $\mathbf{v} + \mathbf{w} = \langle a + c, b + d \rangle$
- $\mathbf{v} - \mathbf{w} = \langle a - c, b - d \rangle$
- $\lambda \mathbf{v} = \langle \lambda a, \lambda b \rangle$

Basic properties. For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars λ,

- $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- $\lambda(\mathbf{v} + \mathbf{w}) = \lambda \mathbf{v} + \lambda \mathbf{w}$

Exercise. Let $\mathbf{v} = \langle 2, 3 \rangle$ and $\mathbf{w} = \langle -1, 4 \rangle$. Calculate $2\mathbf{v} + \mathbf{w}$.

Definition. A *linear combination* of vectors $\mathbf{v}$ and $\mathbf{w}$ is a vector $r\mathbf{v} + s\mathbf{w}$, where r, s are scalars.

Exercise. Express the vector $\mathbf{u} = \langle 7, 2 \rangle$ as a linear combination of the vectors $\mathbf{v} = \langle -1, 2 \rangle$ and $\mathbf{w} = \langle 3, 2 \rangle$.

Definition. A *unit vector* is a vector of length 1. If a unit vector $\mathbf{e}$ is based at the origin, then we can write $\mathbf{e} = \langle \cos\theta, \sin\theta \rangle$, where θ is the angle between $\mathbf{e}$ and the x-axis.

Given a vector $\mathbf{v}$, we can find a unit vector $\mathbf{e}_\mathbf{v}$ pointing in the same direction of $\mathbf{v}$.

$$\mathbf{e}_\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

Any vector $\mathbf{v}$ can be written in terms of its length and direction, as

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{e}_\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle.$$

Exercise. Write the vector $\mathbf{v} = \langle -3, 3 \rangle$ as $\|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle$.

We identify two special unit vectors: $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. Then any vector $\langle a, b \rangle$ can be written as $a\mathbf{i} + b\mathbf{j}$.

Triangle inequality. For any two vectors $\mathbf{v}$ and $\mathbf{w}$ we have that

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$