Notes - January 13th
Math 32A

Given two vectors, how can we calculate the angle θ between them? For this, we define the **dot product**.

Definition. The dot product of two vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is the scalar defined as

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Properties.

- $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- $(\lambda \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (\lambda \mathbf{w}) = \lambda(\mathbf{v} \cdot \mathbf{w})$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$
- $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

Remark. Notice that the dot product of two vectors is a number, so it is no longer a vector.

Exercise. Let $\mathbf{v} = \langle 2, 3, -1 \rangle$ and $\mathbf{w} = \langle -5, 1, 2 \rangle$. Calculate $\mathbf{v} \cdot \mathbf{w}$.

Theorem. Let θ be the angle between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$. Then

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta \quad \text{or} \quad \cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

- The angle θ between two angles is chosen to satisfy $0 \leq \theta \leq \pi$.
- Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$. We write $\mathbf{v} \perp \mathbf{w}$.
- The angle between two vectors $\mathbf{v}$ and $\mathbf{w}$ is acute if and only if $\mathbf{v} \cdot \mathbf{w} > 0$.
- The angle between two vectors $\mathbf{v}$ and $\mathbf{w}$ is acute if and only if $\mathbf{v} \cdot \mathbf{w} < 0$.

Exercise. Find the angle between the vectors $\mathbf{v} = \langle 1, 1, -2 \rangle$ and $\mathbf{w} = \langle 2, -1, -1 \rangle$.

Let $\mathbf{v}$ be a nonzero vector. One important application of the dot product is to decompose a vector $\mathbf{u}$ in terms of a vector parallel to $\mathbf{v}$ and a vector perpendicular to $\mathbf{v}$. We start by finding the projection of $\mathbf{u}$ along $\mathbf{v}$.

The projection of $\mathbf{u}$ along $\mathbf{v}$ is the vector

$$\mathbf{u}_{\|} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\right)\mathbf{e}_{\mathbf{v}}$$

Define $\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\|}$.
We can think of $\mathbf{u}_{\|}$ as the component of $\mathbf{u}$ parallel to $\mathbf{v}$, and $\mathbf{u}_{\perp}$ as the component of $\mathbf{v}$ perpendicular to $\mathbf{v}$
The decomposition of $\mathbf{u}$ with respect to $\mathbf{v}$ is

$$\mathbf{u} = \mathbf{u}_{\|} + \mathbf{u}_{\perp}$$

Exercise. Let $\mathbf{v} = \langle 1, 2, 1 \rangle$ and $\mathbf{u} = \langle 5, -1, 9 \rangle$. Find the decomposition of $\mathbf{u}$ with respect to $\mathbf{v}$. Verify that $\mathbf{u}_\perp \perp \mathbf{v}$.

Exercise. Suppose a 40 km/h wind $\mathbf{w}$ is blowing from the west toward a bridge that is oriented 60 degrees east of north. How strong is the wind blowing directly toward the bridge?