## Notes - January 27th Math 32A

Today we will learn how to find the length of a curve traced by a vector-valued function.

Let  $\mathcal{C}$  be a curve parametrized by  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ ,  $a \leq t \leq b$ . We say that a parametrization directly traverses  $\mathcal{C}$  if the path traces  $\mathcal{C}$  from one end to the other without changing direction along the way.

**Definition.** Assume that  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$  directly traverses  $\mathcal{C}$  for  $a \leq t \leq b$ . Assume that  $\mathbf{r}'(t)$  exists and is continuous. Then the arc length s of  $\mathcal{C}$  is equal to

$$s = \int_{a}^{b} \|\mathbf{r}'(t)\| dt = \int_{a}^{b} \sqrt{x'(t)^{2} + y'(t)^{2} + z'(t)^{2}} dt$$

Warning. If the parametrization does not directly traverse the curve, then the above integral measures the distance traveled by a particle whose path is traced by  $\mathbf{r}(t)$ .

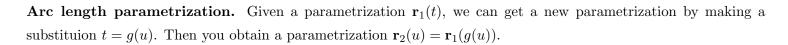
**Exercise.** Find the arc length of the helix given by  $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, \sqrt{5}t \rangle$  for  $0 \le t \le 2\pi$ .

We can define the arc length function

$$s(t) = \int_{a}^{t} \|\mathbf{r}'(u)\| du,$$

so s(t) denotes the distance traveled in the interval [a,t]. Then we can calculate the speed of at time t as

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|.$$



If the t = g(u) increases from a to b as u varies from c to d, then the path  $\mathbf{r}_1(t)$  for  $a \le t \le b$  is also parametrized by  $\mathbf{r}_2(t)$  for  $c \le t \le d$ . Let  $\mathcal{C}$  be the curve parametrized by  $\mathbf{r}(t) = \langle t^2, \sin t, t \rangle$  for  $0 \le t \le d$ . Give a different

parametrization of C using the parameter u, where  $t = g(u) = e^u$ .

A parametrization  $\mathbf{r}(s)$  is called an arc length parametrization if  $\|\mathbf{r}'(s)\| = 1$  for all s. This parametrization satisfies three important properties:

- 1. The parameter s corresponds to the arc length of the curve that is traced from the starting point.
- 2.  $\|\mathbf{r}'(s)\| = 1$ .
- 3. The arc length of the curve traced over any interval [a,b] is equal to b-a, as

$$\int_{a}^{b} \|\mathbf{r}'(s)\| \, ds = \int_{a}^{b} 1 \, ds = b - a$$

How to find an arc length parametrization. Start with  $\mathbf{r}(t)$  such that  $\mathbf{r}'(t) \neq \mathbf{0}$  for all t.

1. Find the arc length function.

$$s = g(t) = \int_a^t \|\mathbf{r}'(u)\| \, du$$

- 2. Determine the inverse  $g^{-1}(s)$  of g(t). This exists because of our assumptions.
- 3. Take the new parametrization  $\mathbf{r}_1(s) = \mathbf{r}(g^{-1}(s))$ .

Exercise. Find an arc length parametrization of the helix that is traced by the parametrization

$$\mathbf{r}(t) = \langle \cos 3t, \sin 3t, 4t \rangle$$

and verify that  $\|\mathbf{r}_1'(s)\| = 1$ .