Notes - January 27th

Math 32A

Today we will learn how to find the length of a curve traced by a vector-valued function.

Let $\mathcal{C}$ be a curve parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. We say that a parametrization directly traverses $\mathcal{C}$ if the path traces $\mathcal{C}$ from one end to the other without changing direction along the way.

Definition. Assume that $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ directly traverses $\mathcal{C}$ for $a \le t \le b$. Assume that $\mathbf{r}'(t)$ exists and is continuous. Then the *arc length* s of $\mathcal{C}$ is equal to

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

Warning. If the parametrization does not directly traverse the curve, then the above integral measures the distance traveled by a particle whose path is traced by $\mathbf{r}(t)$.

Exercise. Find the arc length of the helix given by $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, \sqrt{5}t \rangle$ for $0 \le t \le 2\pi$.

We can define the arc length function

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| \, du,$$

so $s(t)$ denotes the distance traveled in the interval $[a, t]$. Then we can calculate the speed of at time t as

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|.$$

Arc length parametrization. Given a parametrization $\mathbf{r}_1(t)$, we can get a new parametrization by making a substituion $t = g(u)$. Then you obtain a parametrization $\mathbf{r}_2(u) = \mathbf{r}_1(g(u))$.

If the $t = g(u)$ increases from a to b as u varies from c to d, then the path $\mathbf{r}_1(t)$ for $a \le t \le b$ is also parametrized by $\mathbf{r}_2(t)$ for $c \le t \le d$. Let $\mathcal{C}$ be the curve parametrized by $\mathbf{r}(t) = \langle t^2, \sin t, t \rangle$ for $3 \le t \le 9$. Give a different

parametrization of $\mathcal{C}$ using the parameter u, where $t = g(u) = e^u$.

A parametrization $\mathbf{r}(s)$ is called an arc length parametrization if $\|\mathbf{r}'(s)\| = 1$ for all s. This parametrization satisfies three important properties:

1. The parameter s corresponds to the arc length of the curve that is traced from the starting point.

2. $\|\mathbf{r}'(s)\| = 1$.

3. The arc length of the curve traced over any interval $[a, b]$ is equal to $b - a$, as

$$\int_a^b \|\mathbf{r}'(s)\| \, ds = \int_a^b 1 \, ds = b - a$$

How to find an arc length parametrization. Start with $\mathbf{r}(t)$ such that $\mathbf{r}'(t) \neq \mathbf{0}$ for all t.

1. Find the arc length function.
$$s = g(t) = \int_a^t \|\mathbf{r}'(u)\|\, du$$

2. Determine the inverse $g^{-1}(s)$ of $g(t)$. This exists because of our assumptions.

3. Take the new parametrization $\mathbf{r}_1(s) = \mathbf{r}(g^{-1}(s))$.

Exercise. Find an arc length parametrization of the helix that is traced by the parametrization

$$\mathbf{r}(t) = \langle \cos 3t, \sin 3t, 4t \rangle$$

and verify that $\|\mathbf{r}_1'(s)\| = 1$.