# Notes - January 24th

## Math 32A

**Limit of a vector-valued function.** We say that the limit of a vector-valued function $\mathbf{r}(t)$, as $t$ goes to $t_0$, is the vector $\mathbf{u}$ if

$$\lim_{t \to t_0} \|\mathbf{r}(t) - \mathbf{u}\| = 0.$$

In this case, we write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{u}.$$

To compute the limit of a vector-valued function, it is enough to compute the limit of each of its components. If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then

A vector-valued function $\mathbf{r}(t)$ is said to be continuous at $t_0$ if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

Because we can compute limits using the components of the vector, then the function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is continuous if and only if $x(t)$, $y(t)$ and $z(t)$ are continuous.

**Derivative of a vector-valued function.** The derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \frac{d}{dt}\,\mathbf{r}(t) = \lim_{h\to 0}\,\frac{1}{h}\Big(\mathbf{r}(t+h) - \mathbf{r}(t)\Big)$$

We say that $\mathbf{r}$ is differentiable at $t$ if the limit exists, and $\mathbf{r}$ is differentiable if it is differentiable at all $t$ in its domain. As before, if $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then

We define higher order derivatives as expected:

$$\mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t), \quad \mathbf{r}'''(t) = \frac{d}{dt}\mathbf{r}''(t), \quad \text{and so on...}$$

**Differentiation rules.** Assume that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two differentiable vector-valued function, and $f(t)$ a differentiable scalar-valued function.

- **Sum rule**: $\dfrac{d}{dt}\big(\mathbf{r}_1(t)+\mathbf{r}_2(t)\big)=\mathbf{r}_1'(t)+\mathbf{r}_2'(t)$
- **Constant scalar multiplication rule**: $\dfrac{d}{dt}\big(c\mathbf{r}_1(t)\big)=c\mathbf{r}_1'(t)$
- **Scalar product rule**:
- **Chain rule**:
- **Dot product rule**: $\dfrac{d}{dt}\big(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\big)=\big(\mathbf{r}_1'(t)\cdot\mathbf{r}_2(t)\big)+\big(\mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)\big)$
- **Cross product rule**: $\dfrac{d}{dt}\big(\mathbf{r}_1(t)\times\mathbf{r}_2(t)\big)=\big(\mathbf{r}_1'(t)\times\mathbf{r}_2(t)\big)+\big(\mathbf{r}_1(t)\times\mathbf{r}_2'(t)\big)$

**Exercise 1.** Let $\mathbf{r}(t)=\langle\cos t,\sin t,t\rangle$. Show $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}''(t)$.

The vector $\mathbf{r}'(t_0)$ corresponds to the tangent vector to the path traced by $\mathbf{r}(t)$ at $t_0$. We also refer to $\mathbf{r}'(t_0)$ as the velocity vector at $\mathbf{r}(t_0)$.

If $\mathbf{r}'(t_0)$ exists and it is nonzero, the equation of the tangent line to the path at $\mathbf{r}(t_0)$ is

$$\mathbf{l}(s) = \mathbf{r}(t_0) + s\mathbf{r}'(t_0).$$

**Exercise 2.** Consider the function $\mathbf{r}(t) = \langle t^2, t^3 \rangle$. Find the values of $t$ where you cannot define the tangent line.

**Exercise 3.** Suppose that $\mathbf{r}(t)$ is nonzero and assume that $\mathbf{r}(t)$ has constant length. Show that $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for any $t$. Use this fact to solve again Exercise 1.

**Integral of vector-valued functions.** The integral of a vector-valued function can be defined componentwise:

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

We can define the indefinite integral of $\mathbf{r}(t)$ in the same way. From this we get that, if $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are differentiable, with $\mathbf{r}_1'(t) = \mathbf{r}_2'(t)$, then

$$\mathbf{r}_1(t) = \mathbf{r}_2(t) + \mathbf{c},$$

with $\mathbf{c}$ a constant vector.

**Fundamental theorem of calculus for vector-valued functions.**

1. Assume that $\mathbf{r}(t)$ is continuous on the interval $[a, b]$, and $\mathbf{R}(t)$ is a function such that $\mathbf{R}'(t) = \mathbf{r}$ on $[a, b]$, then

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(b) - \mathbf{R}(a)$$

2. Assume that $\mathbf{r}'(t)$ is continuous on an open interval $I$, and let $a$ be in $I$. Then

$$\frac{d}{dt}\int_a^t \mathbf{r}(s)\,ds = \mathbf{r}(t)$$

**Exercise.** A particle starts out at the point $(1, 1, 1)$ at time $t = 0$ and travels through the space with velocity vector $\mathbf{r}'(t) = \langle 1 - 4\sin 2t, 6t^2, 3\rangle$. Determine the vector-valued function $\mathbf{r}(t)$ corresponding to the position of the particle.