Question 1

$$y'' - 2y' - 3y = 3e^{2x}$$

We begin by first solving the homogeneous equation:

$$y'' - 2y' - 3y = 0$$

We can solve this by using the characteristic equation:

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$r = 3, -1$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^{3x} + c_2 e^{-x}$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = Ae^{2x}$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 2Ae^{2x}$$

$$y_p'' = 4Ae^{2x}$$

Plugging these into the original equation, we get:

$$(4Ae^{2x}) - 2(2Ae^{2x}) - 3(Ae^{2x}) = 3e^{2x}$$

$$4Ae^{2x} - 4Ae^{2x} - 3Ae^{2x} = 3e^{2x}$$

$$-3Ae^{2x} = 3e^{2x}$$

$$A = -1$$

Therefore, the particular solution is:

$$y_p = -e^{2x}$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = c_1 e^{3x} + c_2 e^{-x} - e^{2x}$$

$$y = e^{3x}(c_1 + c_2 e^{-4x} - e^{-x})$$

This is the general solution to the nonhomogeneous equation.

Question 2

$$y'' + 6y' + 9y = e^{-3x}$$

We begin by first solving the homogeneous equation:

$$y'' + 6y' + 9y = 0$$

We can solve this by using the characteristic equation:

$$r^2 + 6r + 9 = 0$$

$$(r + 3)^2 = 0$$

$$r = -3$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^{-3x} + c_2 x e^{-3x}$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = Ae^{-3x}$$

This guess won't work because e^{-3x} is a solution to the homogeneous equation ($c_1 e^{-3x}$).

Therefore, we must multiply our guess by x:

$$y_p = Axe^{-3x}$$

This guess won't work either because e^{-3x} is also a solution to the homogeneous equation ($c_2 x e^{-3x}$).
Therefore, we must multiply our guess by x^2:

$$y_p = Ax^2 e^{-3x}$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 2Axe^{-3x} - 3Ax^2 e^{-3x}$$

$$y_p'' = 9Ax^2 e^{-3x} - 12Axe^{-3x} + 2Ae^{-3x}$$

Plugging these into the original equation, we get:

$$(9Ax^2 e^{-3x} - 12Axe^{-3x} + 2Ae^{-3x}) + 6(2Axe^{-3x} - 3Ax^2 e^{-3x}) + 9(Ax^2 e^{-3x}) = e^{-3x}$$

$$9Ax^2 e^{-3x} - 12Axe^{-3x} + 2Ae^{-3x} + 12Axe^{-3x} - 18Ax^2 e^{-3x} + 9Ax^2 e^{-3x} = e^{-3x}$$

$$2Ae^{-3x} = e^{-3x}$$

$$A = \frac{1}{2}$$

Therefore, the particular solution is:

$$y_p = \frac{1}{2}x^2 e^{-3x}$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = c_1 e^{-3x} + c_2 x e^{-3x} + \frac{1}{2}x^2 e^{-3x}$$

$$y = e^{-3x}(c_1 + c_2 x + \frac{1}{2}x^2)$$

This is the general solution to the nonhomogeneous equation.

Question 3

$$2y'' + 3y' + y = x^2 + 3sin(x)$$

We begin by first solving the homogeneous equation:

$$2y'' + 3y' + y = 0$$

We can solve this by using the characteristic equation:

$$2r^2 + 3r + 1 = 0$$

$$(2r + 1)(r + 1) = 0$$

$$r = -\frac{1}{2}, -1$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^{-\frac{1}{2}x} + c_2 e^{-x}$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = Ax^2 + Bx + C + Dsin(x) + Ecos(x)$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 2Ax + B + Dcos(x) - Esin(x)$$

$$y_p'' = 2A - Dsin(x) - Ecos(x)$$

Plugging these into the original equation, we get:

$$2(2A - D\sin(x) - E\cos(x)) + 3(2Ax + B + D\cos(x) - E\sin(x))$$
$$+ Ax^2 + Bx + C + D\sin(x) + E\cos(x) = x^2 + 3\sin(x)$$

$$4A - 2D\sin(x) - 2E\cos(x) + 6Ax + 3B + 3D\cos(x) - 3E\sin(x)$$
$$+ Ax^2 + Bx + C + D\sin(x) + E\cos(x) = x^2 + 3\sin(x)$$

$$Ax^2 + (6A+B)x + (-2D-3E+D)\sin x + (-2E+3D+E)\cos x + (4A+3B+C) = x^2 + 3\sin x$$
$$Ax^2 + (6A+B)x + (-D-3E)\sin x + (-E+3D)\cos x + (4A+3B+C) = x^2 + 3\sin x$$
$$A = 1$$
$$6A + B = 0$$
$$B = -6$$
$$4A + 3B + C = 0$$
$$C = 14$$
$$-3E - D = 3$$
$$-E + 3D = 0$$
$$D = -\frac{3}{10}$$
$$E = -\frac{9}{10}$$

Therefore, the particular solution is:

$$y_p = x^2 - 6x + 14 - \frac{3}{10}sin(x) - \frac{9}{10}cos(x)$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$
$$y = c_1e^{-\frac{1}{2}x} + c_2e^{-x} + x^2 - 6x + 14 - \frac{3}{10}sin(x) - \frac{9}{10}cos(x)$$
$$y = e^{-\frac{1}{2}x}(c_1 + c_2e^{-\frac{1}{2}x}) + x^2 - 6x + 14 - \frac{3}{10}sin(x) - \frac{9}{10}cos(x)$$

This is the general solution to the nonhomogeneous equation.

Question 4

$$y'' - 2y' + 5y = e^{-x}sin(2x)$$

We begin by first solving the homogeneous equation:

$$y'' - 2y' + 5y = 0$$

We can solve this by using the characteristic equation:

$$r^2 - 2r + 5 = 0$$

$$(r-1)^2 + 4 = 0$$

$$r = 1 \pm 2i$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^x cos(2x) + c_2 e^x sin(2x)$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = Ae^{-x}sin(2x) + Be^{-x}cos(2x)$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = -Ae^{-x}sin(2x) + 2Ae^{-x}cos(2x) - Be^{-x}cos(2x) - 2Be^{-x}sin(2x)$$

$$y_p'' = -3Ae^{-x}sin(2x) - 4Ae^{-x}cos(2x) - 3Be^{-x}cos(2x) + 4Be^{-x}sin(2x)$$

Plugging these into the original equation, we get:

$$\begin{aligned}&(-3Ae^{-x}\sin(2x) - 4Ae^{-x}\cos(2x) - 3Be^{-x}\cos(2x) + 4Be^{-x}\sin(2x))\\ &\quad - 2(-Ae^{-x}\sin(2x) + 2Ae^{-x}\cos(2x) - Be^{-x}\cos(2x) - 2Be^{-x}\sin(2x))\\ &\quad + 5(Ae^{-x}\sin(2x) + Be^{-x}\cos(2x)) = e^{-x}\sin(2x)\end{aligned}$$

$$4Ae^{-x}\sin(2x) - 8Ae^{-x}\cos(2x) + 4Be^{-x}\cos(2x) + 8Be^{-x}\sin(2x) = e^{-x}\sin(2x)$$

$$4A + 8B = 1$$

$$-8A + 4B = 0$$

$$A = \frac{1}{20}$$

$$B = \frac{1}{40}$$

Therefore, the particular solution is:

$$y_p = \frac{1}{20}e^{-x}sin(2x) + \frac{1}{40}e^{-x}cos(2x)$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = c_1e^x cos(2x) + c_2e^x sin(2x) + \frac{1}{20}e^{-x} sin(2x) + \frac{1}{40}e^{-x} cos(2x)$$

$$y = e^x(c_1cos(2x) + c_2sin(2x)) + \frac{1}{20}e^{-x} sin(2x) + \frac{1}{40}e^{-x} cos(2x)$$

This is the general solution to the nonhomogeneous equation.

Question 5

$$y'' + y' - 2y = x^3 + x$$

We begin by first solving the homogeneous equation:

$$y'' + y' - 2y = 0$$

We can solve this by using the characteristic equation:

$$r^2 + r - 2 = 0$$

$$(r + 2)(r - 1) = 0$$

$$r = -2, 1$$

Therefore, the homogeneous solution is:

$$y_h = c_1e^{-2x} + c_2e^x$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. Since the non homogeneous part is a sum of two terms, we can find the particular solution by finding the particular solution to each term and adding them together.

We guess that the particular solution to the first term is of the form:

$$y_{p1} = Ax^3 + Bx^2 + Cx + D$$

Now, we can find the first and second derivatives of y_{p1}:

$$y'_{p1} = 3Ax^2 + 2Bx + C$$

$$y''_{p1} = 6Ax + 2B$$

Plugging these into the original equation, we get:

$$(6Ax + 2B) + (3Ax^2 + 2Bx + C) - 2(Ax^3 + Bx^2 + Cx + D) = x^3$$

$$6Ax + 2B + 3Ax^2 + 2Bx + C - 2Ax^3 - 2Bx^2 - 2Cx - 2D = x^3$$

$$x^3(-2A) + x^2(3A - 2B) + x(6A + 2B - 2C) + (2B + C - 2D) = x^3$$
$$-2A = 1$$
$$3A - 2B = 0$$
$$6A + 2B - 2C = 0$$
$$2B + C - 2D = 0$$
$$A = -\frac{1}{2}$$
$$B = -\frac{3}{4}$$
$$C = -\frac{3}{4}$$
$$D = -\frac{1}{2}$$

Therefore, the particular solution to the first term is:

$$y_{p1} = -\frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x - \frac{1}{2}$$

We guess that the particular solution to the second term is of the form:

$$y_{p2} = Ex + F$$

Now, we can find the first and second derivatives of y_{p2}:

$$y'_{p2} = E$$
$$y''_{p2} = 0$$

Plugging these into the original equation, we get:

$$0 + E - 2(Ex + F) = x$$
$$-2Ex - 2F = x$$
$$E = -\frac{1}{2}$$
$$F = 0$$

Therefore, the particular solution to the second term is:

$$y_{p2} = -\frac{1}{2}x$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_{p1} + y_{p2}$$
$$y = c_1e^{-2x} + c_2e^x - \frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x - \frac{1}{2} - \frac{1}{2}x$$
$$y = e^{-2x}(c_1 + c_2e^{3x}) - \frac{1}{2}x^3 - \frac{7}{4}x^2 - \frac{5}{4}x - \frac{1}{2}$$

This is the general solution to the nonhomogeneous equation.

Question 6

$$y'' - y' - 2y = 2\sin(2x); y(0) = 1, y'(0) = -1$$

We begin by first solving the homogeneous equation:

$$y'' - y' - 2y = 0$$

We can solve this by using the characteristic equation:

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$r = 2, -1$$

Therefore, the homogeneous solution is:

$$y_h = c_1e^{2x} + c_2e^{-x}$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = A\sin(2x) + B\cos(2x)$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 2A\cos(2x) - 2B\sin(2x)$$

$$y_p'' = -4A\sin(2x) - 4B\cos(2x)$$

Plugging these into the original equation, we get:

$$(-4A\sin(2x) - 4B\cos(2x)) - (2A\cos(2x) - 2B\sin(2x)) - 2(A\sin(2x) + B\cos(2x)) = 2\sin(2x)$$

$$-4A\sin(2x) - 4B\cos(2x) - 2A\cos(2x) + 2B\sin(2x) - 2A\sin(2x) - 2B\cos(2x) = 2\sin(2x)$$

$$2\sin(2x)(-2A + 2B - 1) + 2\cos(2x)(-2B - 2A - 2) = 2\sin(2x)$$

$$-2A + 2B - 1 = 0$$

$$-2B - 2A - 2 = 0$$

$$A = -\frac{1}{2}$$

$$B = \frac{1}{2}$$

Therefore, the particular solution is:

$$y_p = -\frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x)$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = c_1e^{2x} + c_2e^{-x} - \frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x)$$

Now, we can find the particular solution by plugging in the initial conditions:

$$1 = c_1 + c_2 - \frac{1}{2}\sin(0) + \frac{1}{2}\cos(0)$$

$$1 = c_1 + c_2 - \frac{1}{2}(0) + \frac{1}{2}(1)$$

$$1 = c_1 + c_2 + \frac{1}{2}$$

$$c_1 + c_2 = \frac{1}{2}$$

$$-1 = 2c_1 - c_2 - \frac{1}{2}\cos(0) - \frac{1}{2}\sin(0)$$

$$-1 = 2c_1 - c_2 - \frac{1}{2}(1) - \frac{1}{2}(0)$$

$$-1 = 2c_1 - c_2 - \frac{1}{2}$$

$$2c_1 - c_2 = -\frac{1}{2}$$

$$c_1 = \frac{1}{4}$$

$$c_2 = \frac{1}{4}$$

Therefore, the particular solution is:

$$y = \frac{1}{4}e^{2x} + \frac{1}{4}e^{-x} - \frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x)$$

This is the solution to the nonhomogeneous equation.

Question 7

$$y'' - 2y' + y = xe^x + 4; y(0) = 1, y'(0) = 1$$

We begin by first solving the homogeneous equation:

$$y'' - 2y' + y = 0$$

We can solve this by using the characteristic equation:

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^x + c_2 x e^x$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation.
We guess that the particular solution to the first term is of the form:

$$y_{p1} = (Ax + B)e^x$$

Now, we can find the first and second derivatives of y_{p1}:

$$y'_{p1} = Ae^x + (Ax + B)e^x$$

$$y''_{p1} = 2Ae^x + (Ax + B)e^x$$

Plugging these into the original equation, we get:

$$(2Ae^x + (Ax + B)e^x) - 2(Ae^x + (Ax + B)e^x) + (Ax + B)e^x = xe^x$$

$$2Ae^x + Ax^2e^x + Bxe^x - 2Ae^x - 2Ax^2e^x - 2Bxe^x + Ax^2e^x + Bxe^x = xe^x$$

$$2Ax^2e^x - 2Ax^2e^x + Ax^2e^x + Bxe^x - 2Bxe^x = xe^x$$

$$Ax^2e^x - Bxe^x = xe^x$$

$$A = 1$$

$$B = -1$$

Therefore, the particular solution to the first term is:

$$y_{p1} = (x - 1)e^x$$

We guess that the particular solution to the second term is of the form:

$$y_{p2} = C$$

Now, we can find the first and second derivatives of y_{p2}:

$$y'_{p2} = 0$$

$$y''_{p2} = 0$$

Plugging these into the original equation, we get:

$$0 - 2(0) + C = 4$$

$$C = 4$$

Therefore, the particular solution to the second term is:

$$y_{p2} = 4$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_{p1} + y_{p2}$$

$$y = c_1 e^x + c_2 x e^x + (x-1)e^x + 4$$

Now, we can find the particular solution by plugging in the initial conditions: $y(0) = 1$

$$1 = c_1 + c_2(0) + (0-1)(1) + 4$$

$$1 = c_1 + 3$$

$$c_1 = -2$$

$y'(0) = 1$

$$1 = c_1(1) + c_2(1) + c_2(0)(1) + 0$$

$$1 = -2 + c_2$$

$$c_2 = 3$$

Therefore, the particular solution is:

$$y = -2e^x + 3xe^x + (x-1)e^x + 4$$

This is the solution to the nonhomogeneous equation.