Question 1

$$y'' - 2y' - 3y = 3e^{2x}$$

We begin by first solving the homogeneous equation:

$$y'' - 2y' - 3y = 0$$

We can solve this by using the characteristic equation:

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0$$

$$r = 3, -1$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^{3x} + c_2 e^{-x}$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = Ae^{2x}$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 2Ae^{2x}$$

$$y_p'' = 4Ae^{2x}$$

Plugging these into the original equation, we get:

$$(4Ae^{2x}) - 2(2Ae^{2x}) - 3(Ae^{2x}) = 3e^{2x}$$

$$4Ae^{2x} - 4Ae^{2x} - 3Ae^{2x} = 3e^{2x}$$

$$-3Ae^{2x} = 3e^{2x}$$

$$A = -1$$

Therefore, the particular solution is:

$$y_p = -e^{2x}$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = c_1 e^{3x} + c_2 e^{-x} - e^{2x}$$

$$y = e^{3x}(c_1 + c_2 e^{-4x} - e^{-x})$$

This is the general solution to the nonhomogeneous equation.

Question 2

$$y'' + 6y' + 9y = e^{-3x}$$

We begin by first solving the homogeneous equation:

$$y'' + 6y' + 9y = 0$$

We can solve this by using the characteristic equation:

$$r^2 + 6r + 9 = 0$$

$$(r + 3)^2 = 0$$

$$r = -3$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^{-3x} + c_2 x e^{-3x}$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = Ae^{-3x}$$

This guess won't work because e^{-3x} is a solution to the homogeneous equation ($c_1 e^{-3x}$).

Therefore, we must multiply our guess by x:

$$y_p = Axe^{-3x}$$

This guess won't work either because e^{-3x} is also a solution to the homogeneous equation ($c_2 x e^{-3x}$).
Therefore, we must multiply our guess by x^2:

$$y_p = Ax^2 e^{-3x}$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 2Axe^{-3x} - 3Ax^2 e^{-3x}$$

$$y_p'' = 9Ax^2 e^{-3x} - 12Axe^{-3x} + 2Ae^{-3x}$$

Plugging these into the original equation, we get:

$$(9Ax^2 e^{-3x} - 12Axe^{-3x} + 2Ae^{-3x}) + 6(2Axe^{-3x} - 3Ax^2 e^{-3x}) + 9(Ax^2 e^{-3x}) = e^{-3x}$$

$$9Ax^2 e^{-3x} - 12Axe^{-3x} + 2Ae^{-3x} + 12Axe^{-3x} - 18Ax^2 e^{-3x} + 9Ax^2 e^{-3x} = e^{-3x}$$

$$2Ae^{-3x} = e^{-3x}$$

$$A = \frac{1}{2}$$

Therefore, the particular solution is:

$$y_p = \frac{1}{2}x^2e^{-3x}$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = c_1e^{-3x} + c_2xe^{-3x} + \frac{1}{2}x^2e^{-3x}$$

$$y = e^{-3x}(c_1 + c_2x + \frac{1}{2}x^2)$$

This is the general solution to the nonhomogeneous equation.

Question 3

$$2y'' + 3y' + y = x^2 + 3sin(x)$$

We begin by first solving the homogeneous equation:

$$2y'' + 3y' + y = 0$$

We can solve this by using the characteristic equation:

$$2r^2 + 3r + 1 = 0$$

$$(2r + 1)(r + 1) = 0$$

$$r = -\frac{1}{2}, -1$$

Therefore, the homogeneous solution is:

$$y_h = c_1e^{-\frac{1}{2}x} + c_2e^{-x}$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = Ax^2 + Bx + C + Dsin(x) + Ecos(x)$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 2Ax + B + Dcos(x) - Esin(x)$$

$$y_p'' = 2A - Dsin(x) - Ecos(x)$$

Plugging these into the original equation, we get:

$$2(2A - D\sin(x) - E\cos(x)) + 3(2Ax + B + D\cos(x) - E\sin(x)) + Ax^2 + Bx + C + D\sin(x) + E\cos(x) = x^2 + 3\sin(x)$$

$$4A - 2D\sin(x) - 2E\cos(x) + 6Ax + 3B + 3D\cos(x) - 3E\sin(x) + Ax^2 + Bx + C + D\sin(x) + E\cos(x) = x^2 + 3\sin(x)$$

$$Ax^2+(6A+B)x+(-2D-3E+D)\sin x+(-2E+3D+E)\cos x+(4A+3B+C) = x^2+3\sin x$$
$$Ax^2+(6A+B)x+(-D-3E)\sin x+(-E+3D)\cos x+(4A+3B+C) = x^2+3\sin x$$
$$A = 1$$
$$6A + B = 0$$
$$B = -6$$
$$4A + 3B + C = 0$$
$$C = 14$$
$$-3E - D = 3$$
$$-E + 3D = 0$$
$$D = -\frac{3}{10}$$
$$E = -\frac{9}{10}$$

Therefore, the particular solution is:

$$y_p = x^2 - 6x + 14 - \frac{3}{10}sin(x) - \frac{9}{10}cos(x)$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$
$$y = c_1e^{-\frac{1}{2}x} + c_2e^{-x} + x^2 - 6x + 14 - \frac{3}{10}sin(x) - \frac{9}{10}cos(x)$$
$$y = e^{-\frac{1}{2}x}(c_1 + c_2e^{-\frac{1}{2}x}) + x^2 - 6x + 14 - \frac{3}{10}sin(x) - \frac{9}{10}cos(x)$$

This is the general solution to the nonhomogeneous equation.

Question 4

$$y'' - 2y' + 5y = e^{-x}sin(2x)$$

We begin by first solving the homogeneous equation:

$$y'' - 2y' + 5y = 0$$

We can solve this by using the characteristic equation:

$$r^2 - 2r + 5 = 0$$

$$(r-1)^2 + 4 = 0$$

$$r = 1 \pm 2i$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^x cos(2x) + c_2 e^x sin(2x)$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = Ae^{-x}sin(2x) + Be^{-x}cos(2x)$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = -Ae^{-x}sin(2x) + 2Ae^{-x}cos(2x) - Be^{-x}cos(2x) - 2Be^{-x}sin(2x)$$

$$y_p'' = -3Ae^{-x}sin(2x) - 4Ae^{-x}cos(2x) - 3Be^{-x}cos(2x) + 4Be^{-x}sin(2x)$$

Plugging these into the original equation, we get:

$$\begin{aligned}(&-3Ae^{-x}\sin(2x) - 4Ae^{-x}\cos(2x) - 3Be^{-x}\cos(2x) + 4Be^{-x}\sin(2x)) \\ &- 2(-Ae^{-x}\sin(2x) + 2Ae^{-x}\cos(2x) - Be^{-x}\cos(2x) - 2Be^{-x}\sin(2x)) \\ &+ 5(Ae^{-x}\sin(2x) + Be^{-x}\cos(2x)) = e^{-x}\sin(2x)\end{aligned}$$

$$4Ae^{-x}\sin(2x) - 8Ae^{-x}\cos(2x) + 4Be^{-x}\cos(2x) + 8Be^{-x}\sin(2x) = e^{-x}\sin(2x)$$

$$4A + 8B = 1$$

$$-8A + 4B = 0$$

$$A = \frac{1}{20}$$

$$B = \frac{1}{40}$$

Therefore, the particular solution is:

$$y_p = \frac{1}{20}e^{-x}sin(2x) + \frac{1}{40}e^{-x}cos(2x)$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = c_1 e^x cos(2x) + c_2 e^x sin(2x) + \frac{1}{20} e^{-x} sin(2x) + \frac{1}{40} e^{-x} cos(2x)$$

$$y = e^x (c_1 cos(2x) + c_2 sin(2x)) + \frac{1}{20} e^{-x} sin(2x) + \frac{1}{40} e^{-x} cos(2x)$$

This is the general solution to the nonhomogeneous equation.

Question 5

$$y'' + y' - 2y = x^3 + x$$

We begin by first solving the homogeneous equation:

$$y'' + y' - 2y = 0$$

We can solve this by using the characteristic equation:

$$r^2 + r - 2 = 0$$

$$(r + 2)(r - 1) = 0$$

$$r = -2, 1$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^{-2x} + c_2 e^x$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation.

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B$$

Plugging these into the original equation, we get:

$$(6Ax + 2B) + (3Ax^2 + 2Bx + C) - 2(Ax^3 + Bx^2 + Cx + D) = x^3 + x$$

$$6Ax + 2B + 3Ax^2 + 2Bx + C - 2Ax^3 - 2Bx^2 - 2Cx - 2D = x^3 + x$$

$$x^3(3A - 2A) + x^2(2B - 2B - 2C) + x(6A + 2B - 2D) + (2B + C) = x^3 + x$$

$$3A - 2A = 1$$

$$A = 1$$

$$2B - 2B - 2C = 0$$
$$-2C = 0$$
$$C = 0$$
$$6A + 2B - 2D = 0$$
$$2B - 2D = -6$$
$$B - D = -3$$
$$2B + C = 1$$
$$2B = 1$$
$$B = \frac{1}{2}$$
$$D = \frac{7}{2}$$

Therefore, the particular solution is:

$$y_p = x^3 + \frac{1}{2}x^2 + \frac{7}{2}$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$
$$y = c_1 e^{-2x} + c_2 e^{x} + x^3 + \frac{1}{2}x^2 + \frac{7}{2}$$
$$y = e^{-2x}(c_1 + c_2 e^{3x}) + x^3 + \frac{1}{2}x^2 + \frac{7}{2}$$

This is the general solution to the nonhomogeneous equation.

Question 6

$$y'' - y' - 2y = 2\sin(2x); y(0) = 1, y'(0) = -1$$

We begin by first solving the homogeneous equation:

$$y'' - y' - 2y = 0$$

We can solve this by using the characteristic equation:

$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0$$
$$r = 2, -1$$

Therefore, the homogeneous solution is:

$$y_h = c_1 e^{2x} + c_2 e^{-x}$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation. We guess that the particular solution is of the form:

$$y_p = A\sin(2x) + B\cos(2x)$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 2A\cos(2x) - 2B\sin(2x)$$

$$y_p'' = -4A\sin(2x) - 4B\cos(2x)$$

Plugging these into the original equation, we get:

$$(-4A\sin(2x) - 4B\cos(2x)) - (2A\cos(2x) - 2B\sin(2x)) - 2(A\sin(2x) + B\cos(2x)) = 2\sin(2x)$$

$$-4A\sin(2x) - 4B\cos(2x) - 2A\cos(2x) + 2B\sin(2x) - 2A\sin(2x) - 2B\cos(2x) = 2\sin(2x)$$

$$2\sin(2x)(-2A + 2B - 1) + 2\cos(2x)(-2B - 2A - 2) = 2\sin(2x)$$

$$-2A + 2B - 1 = 0$$

$$-2B - 2A - 2 = 0$$

$$A = -\frac{1}{2}$$

$$B = \frac{1}{2}$$

Therefore, the particular solution is:

$$y_p = -\frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x)$$

Now, we can find the general solution by adding the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = c_1e^{2x} + c_2e^{-x} - \frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x)$$

Now, we can find the particular solution by plugging in the initial conditions:

$$1 = c_1 + c_2 - \frac{1}{2}\sin(0) + \frac{1}{2}\cos(0)$$

$$1 = c_1 + c_2 - \frac{1}{2}(0) + \frac{1}{2}(1)$$

$$1 = c_1 + c_2 + \frac{1}{2}$$

$$c_1 + c_2 = \frac{1}{2}$$

$$-1 = 2c_1 - c_2 - \frac{1}{2}\cos(0) - \frac{1}{2}\sin(0)$$

$$-1 = 2c_1 - c_2 - \frac{1}{2}(1) - \frac{1}{2}(0)$$
$$-1 = 2c_1 - c_2 - \frac{1}{2}$$
$$2c_1 - c_2 = -\frac{1}{2}$$
$$c_1 = \frac{1}{4}$$
$$c_2 = \frac{1}{4}$$

Therefore, the particular solution is:

$$y = \frac{1}{4}e^{2x} + \frac{1}{4}e^{-x} - \frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x)$$

This is the solution to the nonhomogeneous equation.

Question 7

$$y'' - 2y' + y = xe^x + 4; y(0) = 1, y'(0) = 1$$

We begin by first solving the homogeneous equation:

$$y'' - 2y' + y = 0$$

We can solve this by using the characteristic equation:

$$r^2 - 2r + 1 = 0$$
$$(r - 1)^2 = 0$$
$$r = 1$$

Therefore, the homogeneous solution is:

$$y_h = c_1e^x + c_2xe^x$$

Now, we can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous equation.
We guess that the particular solution is of the form:

$$y_p = (Ax + B)e^x + C$$

Since ce^x is a solution to the homogeneous equation, we must multiply our guess by x:

$$y_p = (Ax^2 + Bx)e^x + C$$

Since cxe^x is also a solution to the homogeneous equation, we must multiply our guess by x^2 again:

$$y_p = Ax^3e^x + Bx^2e^x + C$$

Now, we can find the first and second derivatives of y_p:

$$y_p' = 3Ax^2e^x + Ax^3e^x + 2Bxe^x + Bx^2e^x$$

$$y_p'' = 6Axe^x + 6Ax^2e^x + Ax^3e^x + 2Be^x + 4Bxe^x + Bx^2e^x$$

Plugging these into the original equation, we get:

$$\begin{aligned}&(6Axe^x + 6Ax^2e^x + Ax^3e^x + 2Be^x + 4Bxe^x + Bx^2e^x)\\ &\quad - 2(3Ax^2e^x + Ax^3e^x + 2Bxe^x + Bx^2e^x) + (Ax^3e^x + Bx^2e^x + C) = xe^x + 4\end{aligned}$$

$$\begin{aligned}&6Axe^x + 6Ax^2e^x + Ax^3e^x + 2Be^x + 4Bxe^x + Bx^2e^x\\ &\quad - 6Ax^2e^x - 2Ax^3e^x - 4Bxe^x - 2Bx^2e^x + Ax^3e^x + Bx^2e^x + C = xe^x + 4\end{aligned}$$

$$\begin{aligned}&x^3e^x(A - 2A + A) + x^2e^x(6A - 2B + B - 2B + B)\\ &\quad + xe^x(6A + 2B - 4B) + e^x(2B + C) = xe^x + 4\end{aligned}$$

$$6A - 2B = 0$$

$$2B + C = 4$$

$$A = \frac{1}{3}$$

$$B = \frac{1}{9}$$

$$C = \frac{26}{9}$$