L33 Stokes' theorem (cont.)

Stokes and path independence.

Definition L33.1 (Simply Connected)**.** A region is simply connected if every closed loop C inside it bounds some surface S inside it.

Example L33.2. The complement of the z axis is not simply connected (shown by considering a loop encircling the z-axis); the complement of the origin is simply connected.

Example L33.3. Topology uses these considerations to classify surfaces in space: e.g., the mathematical proof that a sphere and a torus are "different" surfaces is that the sphere is simply connected, the torus isn't (in fact it has two "independentÓ loops that don't bound).

Recall: if $\vec{F}$ is a gradient field then $\text{curl}(\vec{F}) = 0$.
Conversely, the following theorem:

Theorem L33.4. *If $\nabla \times \vec{F} = 0$ in a simply connected region then $\vec{F}$ is conservative (so $\int_C F \cdot d\vec{r}$ is path-independent and we can find a potential).*

Proof. Assume R simply connected, $\nabla \times \vec{F} = 0$, and consider two curves C_1 and C_2 with same end points. Then $C = C_1 - C_2$ is a closed curve so bounds some S; $\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\ dS = 0.$ □

Idea L33.5 (Orientability)**.** We can apply Stokes' theorem to any surface S bounded by C provided that it has a well-defined normal vector! A counterexample is the M obius strip. It's a one-sided surface, so we can't compute flux through it (no possible consistent choice of orientation of $\hat{n}$). Instead, if we want to apply Stokes to the boundary curve C, we must find a two-sided surface with boundary C.

Stokes and surface independence

In Stokes we can choose any S bounded by C: so if a same C bounds two surfaces S_1 and S_2, then is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{S_1} \text{curl}(\vec{F}) \cdot \hat{n}\ dS = \iint_{S_2} \text{curl}(\vec{F}) \cdot \hat{n}\ dS?$$

Can we prove directly that the two flux integrals are equal? Answer: Change othe rientation of S_2, then $S = S_1 - S_2$ is a closed surface with $\hat{n}$ outwards; so we can apply the divergence theorem: $\iint_S \text{curl}(\vec{F}) \cdot \hat{n}\ dS = \iiint_D \text{div}(\text{curl}(\vec{F}))dV$. But $\text{div}(\text{curl}(\vec{F})) = 0$, always. (Checked by calculating in terms of components of $\vec{F}$; also, symbolically: $\nabla \cdot (\nabla \times \vec{F}) = 0$, much like $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ for vectors in Calculus 2).

Homework

Read pp. Re-read §6.7
Problems: §6.7: 349, 351, 353, 354, 355