State Equations for OCP Parameterized by Arc Length

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1 Problem Definition

The goal is to find the derivative for each state in an Optimal Control Problem (OCP). The states are the position (x, y, z), the inclination I, the azimuth A, and the time t. The full state vector is $\mathbf{Z}(s) = (x(s), y(s), z(s), I(s), A(s), t(s))^T$.

The key idea is to re-parameterize the system from being a function of time, t, to a function of its arc length, s.

2 State Derivations

2.1 Parameterization and the Position States (x, y, z)

The derivation begins with the time derivative of the position vector $\mathbf{X} = (x, y, z)^T$:

$$\dot{\mathbf{X}} = \frac{d\mathbf{X}}{dt} = \mathrm{Rop}(t)\begin{pmatrix}\cos(A)\sin(I)\\ \sin(A)\sin(I)\\ \cos(I)\end{pmatrix}$$

To switch from the independent variable t to s, we use the chain rule: $\frac{d\mathbf{X}}{ds} = \frac{d\mathbf{X}}{dt} \cdot \frac{dt}{ds}$.

If we now takes the inverse ROP as the control $\mathrm{u}_3(s), one gets, because of$

By making this specific choice for our control:

$$u_3(s) = \frac{dt}{ds}(s) := \frac{1}{\mathrm{Rop}(t(s))}$$

the chain rule simplifies dramatically. The 'Rop' terms cancel out:

$$\begin{aligned}\frac{d\mathbf{X}}{ds}(s) &= \left(\mathrm{Rop}(t(s))\begin{pmatrix}\cos(A)\sin(I)\\ \sin(A)\sin(I)\\ \cos(I)\end{pmatrix}\right)\cdot\left(\frac{1}{\mathrm{Rop}(t(s))}\right)\\ &= \begin{pmatrix}\cos(A(s))\sin(I(s))\\ \sin(A(s))\sin(I(s))\\ \cos(I(s))\end{pmatrix}\end{aligned}$$

The consequence of this choice for $u_3(s)$ is that the resulting vector derivative has a magnitude of exactly 1:

$$\left\| \frac{d\mathbf{X}}{ds}(s) \right\| = \sqrt{(\cos^2 A \sin^2 I) + (\sin^2 A \sin^2 I) + (\cos^2 I)} = 1$$

This is the mathematical definition of a curve parameterized by its arc length.

2.2 Angular and Time States

The remaining state derivatives are now straightforwardly defined by the control variables:

$$\begin{aligned} \frac{dI}{ds}(s) &= u_1(s) \\ \frac{dA}{ds}(s) &= u_2(s) \\ \frac{dt}{ds}(s) &= u_3(s) \end{aligned}$$

3 Summary of State Equations

The complete set of differential equations with respect to the arc length s is:

$$\begin{aligned} \frac{dx}{ds} &= \cos(A(s))\sin(I(s)) \\ \frac{dy}{ds} &= \sin(A(s))\sin(I(s)) \\ \frac{dz}{ds} &= \cos(I(s)) \\[1em] \frac{dI}{ds} &= u_1(s) \\ \frac{dA}{ds} &= u_2(s) \\[1em] \frac{dt}{ds} &= u_3(s) \quad \left(\text{where } u_3(s) = \frac{1}{\mathrm{Rop}(t(s))}\right) \end{aligned}$$