Group Theory and Normal Subgroups

In this document, we explore the relationship between groups and their quotient groups with a focus on normal subgroups. Let G be a group and H a normal subgroup of G. The quotient group G/H is defined, and we delve into certain properties and equivalences involving subgroup K.

Equivalence of Subgroup Conditions

We wish to establish the equivalence of the two conditions

$$\pi(K) = K/(K \cap H)$$

and

$$K/H \trianglelefteq G/H.$$

Forward Implication ($\Rightarrow$)

Assume $\pi(K) = K/(K \cap H)$.

- Consider the projection $\pi : G \to G/H$.
- Let us explore the properties of this map π.
- Here, $\pi(g)\pi(k)\pi(g^{-1}) = \pi(gkg^{-1}) = \pi(k')$.

  This implies:

$$gkg^{-1} = k'h, \quad h \in H$$

hence

$$K \trianglelefteq G.$$

  This follows because $gKg^{-1} \subseteq K \Rightarrow gKg^{-1} \in K$. Thus $K \trianglelefteq G$.

Reverse Implication ($\Leftarrow$)

Conversely, assume $K \trianglelefteq G$.

- This means $gKg^{-1} = gKg^{-1}h$ for $h \in H$.
- Thus, $\pi(g)\pi(k)\pi(g^{-1}) = \pi(gkg^{-1}) = \pi(k')$.

We conclude:

$$K \trianglelefteq K \cap \ker \pi$$

and therefore

$$K \subseteq K \cap H \quad \text{so then} \quad \pi(K) = K/H \quad \text{which implies} \quad K/H \trianglelefteq G/H.$$