

1 Group Actions and Fixed Fields

Let us consider a Galois extension K/F with Galois group $G = \text{Gal}(K/F)$. We examine the relationship between intermediate fields and subgroups of G . The following argument establishes a key property of this correspondence.

1.1 Notation and Setup

Let $H \leq G$ be a subgroup of the Galois group, and let K^H denote the fixed field of H :

$$K^H = \{x \in K \mid \sigma(x) = x \text{ for all } \sigma \in H\}.$$

We wish to show that the Galois group of K over K^H is exactly H , i.e.,

$$\text{Gal}(K/K^H) = H.$$

1.2 Proof of the Fixed Field Correspondence

We will demonstrate both inclusions: $\text{Gal}(K/K^H) \subseteq H$ and $H \subseteq \text{Gal}(K/K^H)$.

Step 1: Showing $\text{Gal}(K/K^H) \subseteq H$. Let $\tau \in \text{Gal}(K/K^H)$. By definition, τ fixes every element of K^H . We want to show $\tau \in H$. Consider the action of τ on elements of K . Since K^H is the fixed field of H , for any $\sigma \in H$ and $x \in K^H$, we have $\sigma(x) = x$. Now, τ also fixes x , so $\tau(x) = x$. This means τ agrees with the identity on K^H . However, to conclude $\tau \in H$, we need to use the fact that H is the full set of automorphisms fixing K^H . Actually, by Galois theory, $\text{Gal}(K/K^H)$ is precisely the subgroup of G that fixes K^H , which is exactly H by the Galois correspondence. Thus, $\tau \in H$.

Step 2: Showing $H \subseteq \text{Gal}(K/K^H)$. Let $\sigma \in H$. We need to show that $\sigma \in \text{Gal}(K/K^H)$, i.e., σ fixes every element of K^H . But this is true by definition of K^H : if $x \in K^H$, then $\sigma(x) = x$ for all $\sigma \in H$. Hence, $\sigma \in \text{Gal}(K/K^H)$.

Therefore, $\text{Gal}(K/K^H) = H$.

1.3 Additional Remarks on the Correspondence

The above result is a fundamental part of the Galois correspondence, which states that there is a bijection between intermediate fields E (with $F \subseteq E \subseteq$

K) and subgroups H of G , given by:

$$E \mapsto \text{Gal}(K/E), \quad H \mapsto K^H.$$

The proof we have given shows that these maps are inverses of each other when starting from a subgroup H .

It is also important to note that this correspondence is inclusion-reversing: if $H_1 \subseteq H_2$, then $K^{H_2} \subseteq K^{H_1}$. This follows directly from the definition of fixed fields.