

Let G be a group. Let HG . Let $K \leq G$ such that $K/H = G/H$. We have:

$$\begin{aligned}|G| &= |H| \cdot |K/H| \\&= |H| \cdot |G/H| \\&= |H| \cdot \frac{|G|}{|H|} \\&= |G|\end{aligned}$$

Now consider:

$$\begin{aligned}N_G(K) &= \{g \in G \mid gKg^{-1} = K\} \\N_H(K) &= \{h \in H \mid hKh^{-1} = K\}\end{aligned}$$

Since $K \leq G$ and HG , we have $K \cap HK$.

Also, since $K/H = G/H$, we have $KH = G$.

Now, if $k \in K$ and $h \in H$, then:

$$khk^{-1} \in H \quad (\text{since } HG)$$

and

$$khk^{-1} \in K \quad (\text{since } K \leq G \text{ and } h \in H \leq G)$$

Therefore $khk^{-1} \in K \cap H$.

So K normalizes H , and H normalizes K .

Thus $N_G(K) \geq H$ and $N_H(K) = H \cap N_G(K)$.

Since $K/H = G/H$, we have $|G/K| = |H/(K \cap H)|$.

Therefore:

$$|N_G(K)| = |H| \cdot |N_G(K)/H|$$