Group Theory: Normal Subgroups and Quotient Groups

1 Introduction

In this note, we explore fundamental concepts in group theory, particularly focusing on normal subgroups and quotient groups. The key idea is to understand when a subgroup K of a group G is normal, denoted $K \triangleleft G$, and how this relates to the structure of the quotient group G/K.

2 Preliminaries

Let G be a group and K a subgroup of G. Recall that K is normal in G if and only if for every $g \in G$ and $k \in K$, we have $gkg^{-1} \in K$. Equivalently, K is normal if $gK = Kg$ for all $g \in G$, meaning every left coset is also a right coset.

3 Characterization of Normal Subgroups

Theorem 1 *Let G be a group and $K \leq G$. Then the following are equivalent:*

1. *$K \triangleleft G$*
2. *For all $a \in G$, $aKa^{-1} \subseteq K$*
3. *For all $a \in G$, $aKa^{-1} = K$*

We will prove the equivalence of these statements.

First, assume $K \triangleleft G$. Then for any $a \in G$ and $k \in K$, we have $aka^{-1} \in K$ by definition of normality. This shows $aKa^{-1} \subseteq K$.

Now assume $aKa^{-1} \subseteq K$ for all $a \in G$. To show equality, we also need $K \subseteq aKa^{-1}$. But this follows by applying the assumption to a^{-1}: $a^{-1}K(a^{-1})^{-1} = a^{-1}Ka \subseteq K$, which implies $K \subseteq aKa^{-1}$. Therefore $aKa^{-1} = K$.

Finally, if $aKa^{-1} = K$ for all $a \in G$, then clearly $K \triangleleft G$ by definition.

4 Quotient Groups and the Natural Projection

Let G be a group and $K \triangleleft G$. The quotient group G/K consists of the cosets of K in G with the group operation defined by $(aK)(bK) = (ab)K$.

There is a natural projection homomorphism $\pi : G \to G/K$ defined by $\pi(g) = gK$. This homomorphism is surjective and has kernel $\ker(\pi) = K$.

5 Key Observations

Let's examine some important properties:

If $K \triangleleft G$, then for any $a \in G$ and $k \in K$, we have $ak = ka'$ for some $a' \in K$. This follows from the fact that $aK = Ka$.

Also, if $k \in K$ and $a \in G$, then $aka^{-1} \in K$ by normality. This implies that $\pi(aka^{-1}) = \pi(k) = K$, the identity element in G/K.

6 Conclusion

The condition $K \triangleleft G$ is fundamental in group theory as it ensures that the quotient G/K has a well-defined group structure. The equivalence between various characterizations of normality provides multiple perspectives for understanding and verifying this important property.