

# 1 Group Actions and Fixed Fields

Let  $G$  be a finite group acting on a field  $K$ , and let  $F$  be the fixed field of this action. That is,

$$F = \{a \in K \mid g(a) = a \text{ for all } g \in G\}.$$

We are interested in understanding the relationship between subgroups of  $G$  and intermediate fields between  $F$  and  $K$ .

## 1.1 The Galois Correspondence

Given a subgroup  $H \leq G$ , we define the fixed field of  $H$  as:

$$K^H = \{a \in K \mid h(a) = a \text{ for all } h \in H\}.$$

Conversely, given an intermediate field  $F \subseteq L \subseteq K$ , we define the Galois group of  $L$  over  $F$  as:

$$\text{Gal}(K/L) = \{g \in G \mid g(a) = a \text{ for all } a \in L\}.$$

The following theorem establishes a fundamental correspondence:

**Theorem 1** *Let  $K/F$  be a Galois extension with Galois group  $G$ . Then there is a bijection between subgroups  $H \leq G$  and intermediate fields  $F \subseteq L \subseteq K$ , given by:*

$$H \mapsto K^H, \quad L \mapsto \text{Gal}(K/L).$$

*Moreover, this correspondence is inclusion-reversing.*

## 1.2 Proof of the Correspondence

We now prove one direction of this correspondence. Suppose  $H \leq G$  is a subgroup. We want to show that the fixed field  $K^H$  is an intermediate field and that the Galois group of  $K^H$  is exactly  $H$ .

Let  $a \in K^H$ , so  $h(a) = a$  for all  $h \in H$ . For any  $g \in G$ , consider the element  $g(a)$ . We want to show that  $g(a) \in K^H$ , i.e.,  $h(g(a)) = g(a)$  for all  $h \in H$ .

Since  $H$  is a subgroup, for each  $h \in H$ , there exists  $h' \in H$  such that  $hg = gh'$ . Then:

$$h(g(a)) = (hg)(a) = (gh')(a) = g(h'(a)) = g(a),$$

where the last equality follows because  $h'(a) = a$  (since  $a \in K^H$ ). This shows that  $g(a) \in K^H$ , so  $K^H$  is invariant under the action of  $G$ .

Now, let  $L = K^H$ . We claim that  $\text{Gal}(K/L) = H$ . By definition,  $H \leq \text{Gal}(K/L)$  because every element of  $H$  fixes  $L$ . To show the reverse inclusion, suppose  $g \in \text{Gal}(K/L)$ . Then  $g$  fixes every element of  $L$ , so in particular  $g$  fixes  $a \in K^H$ . But this means  $g(a) = a$  for all  $a \in K^H$ , which implies  $g \in H$ . Therefore,  $\text{Gal}(K/L) = H$ .

This completes the proof that the map  $H \mapsto K^H$  is injective and that the Galois group of the fixed field is exactly the subgroup we started with.

### 1.3 Remarks on the Correspondence

The above argument shows that for any subgroup  $H \leq G$ , we have:

$$\text{Gal}(K/K^H) = H.$$

This is a key step in establishing the Galois correspondence. The reverse direction (showing that for any intermediate field  $L$ , we have  $K^{\text{Gal}(K/L)} = L$ ) requires additional arguments, typically using the primitive element theorem or the linear independence of characters.

The inclusion-reversing property is also evident: if  $H_1 \leq H_2 \leq G$ , then  $K^{H_2} \subseteq K^{H_1}$ . Similarly, if  $F \subseteq L_1 \subseteq L_2 \subseteq K$ , then  $\text{Gal}(K/L_2) \leq \text{Gal}(K/L_1)$ .

This correspondence is fundamental in Galois theory and allows us to translate problems about field extensions into problems about group theory, which are often easier to solve.