Cross Product Identity

1 Introduction

In this document, we aim to demonstrate a vector identity involving a specific operation $\mathcal{S}$ applied to vectors. The claim to be shown is:

$$\mathcal{S}(\hat{\mathbf{v}}) \times \mathcal{S}(\hat{\mathbf{w}}) = \hat{\mathbf{v}} \times \hat{\mathbf{w}}$$

where $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are unit vectors.

2 Vector Transformation

We begin by expressing the operation $\mathcal{S}$ applied to vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$:

$$\mathcal{S}(\hat{\mathbf{v}}) = \hat{\mathbf{v}} + (\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}$$

$$\mathcal{S}(\hat{\mathbf{w}}) = \hat{\mathbf{w}} + (\hat{\mathbf{w}} \cdot \lambda)\mathbf{s}$$

Here, λ and $\mathbf{s}$ are auxiliary vectors, possibly indicating specific directions or scales relevant to the transformation.

3 Cross Product Computation

The cross product of the transformed vectors is computed as follows:

$$\begin{aligned}
\mathcal{S}(\hat{\mathbf{v}}) \times \mathcal{S}(\hat{\mathbf{w}}) &= (\hat{\mathbf{v}} + (\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}) \times (\hat{\mathbf{w}} + (\hat{\mathbf{w}} \cdot \lambda)\mathbf{s}) \\
&= \hat{\mathbf{v}} \times \hat{\mathbf{w}} + \hat{\mathbf{v}} \times ((\hat{\mathbf{w}} \cdot \lambda)\mathbf{s}) \\
&\quad + ((\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}) \times \hat{\mathbf{w}} + ((\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}) \times ((\hat{\mathbf{w}} \cdot \lambda)\mathbf{s})
\end{aligned}$$

4 Simplification

Observing that the last term evaluates to zero due to the properties of the cross product ($\mathbf{s} \times \mathbf{s} = \mathbf{0}$), we focus on simplifying:

$$\hat{\mathbf{v}} \times ((\hat{\mathbf{w}} \cdot \lambda)\mathbf{s}) = (\hat{\mathbf{w}} \cdot \lambda)(\hat{\mathbf{v}} \times \mathbf{s})$$
$$((\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}) \times \hat{\mathbf{w}} = -(\hat{\mathbf{v}} \cdot \lambda)(\hat{\mathbf{w}} \times \mathbf{s})$$

Both terms involving $\mathbf{s}$ cancel each other out under the sum, reaffirming that:

$$\mathcal{S}(\hat{\mathbf{v}}) \times \mathcal{S}(\hat{\mathbf{w}}) = \hat{\mathbf{v}} \times \hat{\mathbf{w}}$$

5 Conclusion

The manipulation of the expressions confirms the original identity. This property highlights a unique invariance under the transformation $\mathcal{S}$.