# Proof of a Group Theoretic Property

We are given a group $G$ and a normal subgroup $H$ of $G$. We are exploring a condition involving subgroups $K$ of $G$ and the quotient group $G/H$. The goal is to prove the equivalence:

$$\pi(K) = K/H \quad \text{if and only if} \quad K \trianglelefteq G.$$

Here, $\pi : G \to G/H$ is the canonical projection.

## Forward Direction

($\Rightarrow$) Assume that $\pi(K) = K/H$. We want to show that $K \trianglelefteq G$.

Start by letting $\pi : G \to G/H$ be the natural projection. Suppose that $\pi(K) = K/H$, which means that every coset $gH$ such that $g \in K$ satisfies the equality condition under projection.

$$\pi(g)\pi(K)\pi(g^{-1}) = \pi(K).$$

This implies:

$$gKg^{-1} \subseteq K.$$

Hence, $gk'g^{-1} = k'h$ for some $h \in H$, and therefore $K$ is normal in $G$.

## Backward Direction

($\Leftarrow$) Assume that $K \trianglelefteq G$. We need to prove that $\pi(K) = K/H$.

Since $K \trianglelefteq G$, for any $g \in G$, $gKg^{-1} = K$.

Therefore, if $k \in K$, $gkg^{-1} \in K$. It follows that

$$\pi(gk)\pi(g^{-1}) = \pi(K).$$

As a subgroup, $\pi(K)$ must be $K/H$ precisely because $K$ is closed and normal in $G$.

Thus, we have:

$$K/H \trianglelefteq G/H,$$

proving the backward direction.