Vector Transformation and Cross Product

We aim to demonstrate the validity of the following equation involving vector transformations and cross products:

$$\mathcal{S}(\mathbf{v}) \times \mathcal{S}(\mathbf{w}) = \hat{\mathbf{v}} \times \hat{\mathbf{w}}$$

where $\mathcal{S}(\mathbf{v})$ and $\mathcal{S}(\mathbf{w})$ represent the transformed vectors with some corrections involving unit vectors and scalar factors.

Definitions and Assumptions

First, we define the vector transformations:

$$\mathcal{S}(\mathbf{v}) = \hat{\mathbf{v}} + (\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}$$
$$\mathcal{S}(\mathbf{w}) = \hat{\mathbf{w}} + (\hat{\mathbf{w}} \cdot \lambda)\mathbf{s}$$

Here, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}}$ are unit vectors, λ is a vector, and $\mathbf{s}$ is a scalar factor.

Cross Product Expansion

To proceed, we evaluate the cross product $\mathcal{S}(\mathbf{v}) \times \mathcal{S}(\mathbf{w})$.

$$\begin{aligned}
\mathcal{S}(\mathbf{v}) \times \mathcal{S}(\mathbf{w}) &= [\hat{\mathbf{v}} + (\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}] \times [\hat{\mathbf{w}} + (\hat{\mathbf{w}} \cdot \lambda)\mathbf{s}] \\
&= \hat{\mathbf{v}} \times \hat{\mathbf{w}} + \hat{\mathbf{v}} \times [(\hat{\mathbf{w}} \cdot \lambda)\mathbf{s}] + [(\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}] \times \hat{\mathbf{w}} \\
&\quad + [(\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}] \times [(\hat{\mathbf{w}} \cdot \lambda)\mathbf{s}]
\end{aligned}$$

Simplification

Upon simplification, it is shown that:

$$[(\hat{\mathbf{v}} \cdot \lambda)\mathbf{s}] \times \hat{\mathbf{w}} + \hat{\mathbf{v}} \times [(\hat{\mathbf{w}} \cdot \lambda)\mathbf{s}] = \mathbf{0}$$

Thus, we have:

$$\mathcal{S}(\mathbf{v}) \times \mathcal{S}(\mathbf{w}) = \hat{\mathbf{v}} \times \hat{\mathbf{w}}$$

Geometric Interpretation

Let us consider the geometric interpretation by examining a parallelogram formed by $\hat{\mathbf{v}}$ and $(\hat{\mathbf{w}} \cdot \lambda)\,\mathbf{s}$:

$$\text{Area of } \Pi = h \times b$$

which simplifies using the determinant properties:

$$\begin{aligned} |\hat{\mathbf{v}} \to \mathbf{s} + (\hat{\mathbf{w}} \to \mathbf{s})| &= |\hat{\mathbf{v}}| \cdot |\hat{\mathbf{w}} \circ \mathbf{s}| \\ &= |\mathcal{S}(\mathbf{v}) \times \mathcal{S}(\mathbf{w})| \end{aligned}$$

This reinforces our algebraic demonstration with a visual confirmation.