# Vector Identity Verification

## 1 Introduction

The goal is to prove the following vector identity:

$$\mathcal{S}(\hat{v}) \times \mathcal{S}(\hat{w}) = \hat{v} \times \hat{w}$$

where $\mathcal{S}$ represents a particular transformation of the vector under certain assumptions. We begin by defining and expanding the transformation $\mathcal{S}$.

## 2 Definition of the Transformation

For a vector $\hat{v}$, the transformation is defined as:

$$\mathcal{S}(\hat{v}) = \hat{v} + (\hat{v} \cdot \hat{\lambda})\hat{s}$$

For another vector $\hat{w}$, we have:

$$\mathcal{S}(\hat{w}) = \hat{w} + (\hat{w} \cdot \hat{\lambda})\hat{s}$$

## 3 Cross Product Expansion

To prove the identity, we calculate the cross product:

$$\mathcal{S}(\hat{v}) \times \mathcal{S}(\hat{w}) = \left(\hat{v} + (\hat{v} \cdot \hat{\lambda})\hat{s}\right) \times \left(\hat{w} + (\hat{w} \cdot \hat{\lambda})\hat{s}\right)$$

Expanding the right-hand side using the distributive property of the cross product gives:

$$\hat{v} \times \hat{w} + \hat{v} \times \left((\hat{w} \cdot \hat{\lambda})\hat{s}\right) + (\hat{v} \cdot \hat{\lambda})\hat{s} \times \hat{w} + (\hat{v} \cdot \hat{\lambda})\hat{s} \times \left((\hat{w} \cdot \hat{\lambda})\hat{s}\right)$$

Observing the components: - $\hat{v} \times \hat{w}$ is the required term. - Each of the remaining terms involves $\hat{s}$ in a way that either cancels out or nullifies due to orthogonality ($\hat{\lambda} \cdot \hat{s} = 0$).

Hence, this simplifies back to:

$$\hat{v} \times \hat{w}$$

# 4 Geometric Interpretation

Consider the area of the parallelogram $\Pi$ formed by the vectors: - Area($\Pi$) = $h \times b$, where $h$ is height and $b$ is base.

Looking at the effective vector pairing:

$$(\hat{v} \times (\hat{w} \cdot \hat{\lambda})\hat{s})$$

The term becomes $(\hat{v} \cdot \hat{s})(\hat{w} \cdot \hat{\lambda})\hat{s}$, which effectively reduces to zero since the vectors $\hat{v}$ and $\hat{w}$ lie in the plane formed by $\hat{\lambda}$ and $\hat{s}$.

# 5 Conclusion

Through these calculations, we have confirmed that:

$$\mathcal{S}(\hat{v}) \times \mathcal{S}(\hat{w}) = \hat{v} \times \hat{w}$$

This verifies the original identity as sought.