1 Group Actions and Fixed Fields

Let us consider a Galois extension K/F with Galois group $G = \mathrm{Gal}(K/F)$. In this section, we explore the correspondence between subgroups of G and intermediate fields of the extension K/F.

1.1 The Fixed Field Correspondence

Given a subgroup $H \leq G$, we define the fixed field of H as:

$$K^H = \{x \in K \mid \sigma(x) = x \text{ for all } \sigma \in H\}.$$

This is indeed a subfield of K containing F. Conversely, for an intermediate field $F \subseteq L \subseteq K$, we define the Galois group of K over L as:

$$\mathrm{Gal}(K/L) = \{\sigma \in G \mid \sigma(x) = x \text{ for all } x \in L\}.$$

This is a subgroup of G.

1.2 Key Properties of the Correspondence

We now prove an important property relating these constructions.

Lemma 1 *Let $H \leq G$ and $L = K^H$. Then* $\mathrm{Gal}(K/L) = H$.

By definition, $H \leq \mathrm{Gal}(K/L)$ since every element of H fixes L. To show equality, suppose $\sigma \in \mathrm{Gal}(K/L)$. Then σ fixes L, but since $L = K^H$, this means $\sigma \in H$. Thus $\mathrm{Gal}(K/L) \subseteq H$, and we conclude $\mathrm{Gal}(K/L) = H$.

1.3 Action on Cosets and Intermediate Fields

Now consider an element $a \in K$ and its orbit under the action of G. Let $H = \mathrm{Gal}(K/F(a))$, the subgroup fixing $F(a)$. Then the orbit of a under G is in bijection with the set of left cosets G/H.

For any $\sigma \in G$, the element $\sigma(a)$ is a conjugate of a over F. The stabilizer of a in G is exactly H, so the orbit-stabilizer theorem gives:

$$|G \cdot a| = [G : H] = [F(a) : F],$$

where the last equality follows from the Galois correspondence.

1.4 Normal Subgroups and Galois Subextensions

An important special case occurs when H is a normal subgroup of G. In this case, the fixed field $L = K^H$ is a Galois extension of F, and we have:

$$\mathrm{Gal}(L/F) \cong G/H.$$

To see this, consider the restriction homomorphism:

$$\varphi : G \to \mathrm{Gal}(L/F), \quad \sigma \mapsto \sigma|_L.$$

The kernel of φ is exactly $H = \mathrm{Gal}(K/L)$, since these are the automorphisms that act trivially on L. By the first isomorphism theorem, we obtain:

$$\mathrm{Gal}(L/F) \cong G/H.$$

1.5 Application to the Current Context

In our specific situation, we have K/F Galois with group G, and we consider an intermediate field L with $H = \mathrm{Gal}(K/L)$. The key observation is that for any $\sigma \in G$, the conjugate field $\sigma(L)$ corresponds to the conjugate subgroup $\sigma H \sigma^{-1}$.

In particular, L/F is Galois if and only if H is normal in G, in which case:

$$\mathrm{Gal}(L/F) \cong G/H.$$

This completes our discussion of the fundamental Galois correspondence between subgroups and intermediate fields.