1 Vector Spaces and Quotients

In linear algebra, a subspace U of a vector space V can be used to construct a new vector space called the quotient space, denoted V/U.

1.1 Construction of the Quotient Space

- $U \subseteq V$: Here, U is a subspace of V.

The equivalence relation on V is defined by:

$$v \sim v' \quad \text{if} \quad v - v' \in U$$

where $v, v' \in V$.

The set of equivalence classes under this relation is:

$$V/U = \{[v] \mid v \in V\}$$

This construction forms a new vector space called the *quotient space*.

1.2 Bases and Dimension

- Let $\{u_1, u_2, \ldots, u_k\}$ be a basis of U.
- Extend this basis to a basis of V, denoted $\{u_1, \ldots, u_k, v_1, \ldots, v_\ell\}$.

Then,

$$\{[v_i] \mid 1 \leq i \leq \ell\}$$

is a basis for the quotient space V/U.

The dimension of the quotient space is given by the formula:

$$\dim V/U = \ell = \dim V - \dim U$$

This shows that the dimension of the quotient space V/U is the difference between the dimension of V and the dimension of U.