1 Group Homomorphisms and Quotients

In this discussion, we explore the relationship between subgroup homomorphisms and quotient groups.

1.1 Statement and Setup

Let $\pi : G \to G/H$ be the canonical projection from a group G to the quotient group G/H. We examine the condition for a subgroup $K \leq G$ such that $\pi(K) = K/H$. We assume $K/H \trianglelefteq G/H$, that is, K/H is a normal subgroup of G/H.

1.2 Implication: $(\Rightarrow)$

Assume $\pi(K) = K/H$. We need to show $K \trianglelefteq G$.

$$\pi(g)\pi(K)\pi(g^{-1}) = \pi(K)$$

This implies:

$$gKg^{-1} = K$$

Hence, for $k' \in K$, there exists $h \in H$ such that:

$$gkg^{-1} = k'h \quad \text{for some} \quad k' \in K,\, h \in H$$

Since $H \leq \ker \pi$, we have:

$$h \in \ker \pi \quad \Rightarrow \quad H \leq K$$

Thus, $K \trianglelefteq G$.

1.3 Converse: $(\Leftarrow)$

Assume $K \trianglelefteq G$, i.e., $gKg^{-1} = K$ for all $g \in G$.

$$\implies \pi(g)\pi(K)\pi(g^{-1}) = \pi(gKg^{-1}) = \pi(K)$$

This implies:

$$\pi(K) \leq \pi(\ker \pi) \quad \Rightarrow \quad K \leq \ker \pi$$

Thus, $\pi(K) = K/H \implies K/H \trianglelefteq G/H$.

2 Conclusion

We have shown that $K \trianglelefteq G$ if and only if $\pi(K) = K/H \trianglelefteq G/H$. This demonstrates a fundamental correspondence between subgroups of a group and its quotient.