Quotient Group and Normal Subgroup

The problem involves proving a statement about group homomorphisms and normal subgroups. Let $\pi : G \to G/H$ be the natural projection where H is a normal subgroup of G. We want to examine the implications of $K/H \trianglelefteq G/H$ and how it affects the subgroup K of G.

Forward Implication ($\Rightarrow$)

Assume $\pi(K) \trianglelefteq G/H$. Let us demonstrate that this implies $K \trianglelefteq G$.

Given:

$$K/H \trianglelefteq G/H$$

It follows that for any coset $gH \in G/H$, with $\pi(g) \in G/H$, we have:

$$\pi(g)\pi(K)\pi(g^{-1}) = \pi(K')$$

Explanation: Since π is a homomorphism:

$$\pi(gkg^{-1}) = \pi(k')$$

This implies:

$$gkg^{-1} \in K$$

Thus, $GKH \subseteq K$, and so $K \trianglelefteq G$.

Reverse Implication ($\Leftarrow$)

Now consider the reverse, assuming $K \trianglelefteq G$, we need to show that $K/H \trianglelefteq G/H$.

Assume $K \trianglelefteq G$, therefore:

$$gKg^{-1} = K \quad \text{for all} \quad g \in G$$

This implies:

$$\pi(g)\pi(K)\pi(g^{-1}) = \pi(K)$$

Explanation: Since K is normal in G, any conjugate gkg^{-1} remains in K. This property transfers through the homomorphism π implying:

$$K/H \trianglelefteq G/H$$

Thus, proving $\pi(K) = K/H \trianglelefteq G/H$.