

1 Group Actions and Fixed Fields

Let G be a finite group acting on a field K , and let F be the fixed field of this action. That is,

$$F = \{a \in K \mid g(a) = a \text{ for all } g \in G\}.$$

We are interested in understanding the relationship between subgroups of G and intermediate fields between F and K .

1.1 The Galois Correspondence

Given a subgroup $H \leq G$, we define the fixed field of H as:

$$K^H = \{a \in K \mid h(a) = a \text{ for all } h \in H\}.$$

Conversely, given an intermediate field $F \subseteq L \subseteq K$, we define the Galois group of L over F as:

$$\text{Gal}(K/L) = \{g \in G \mid g(a) = a \text{ for all } a \in L\}.$$

The following theorem establishes a fundamental correspondence:

Theorem 1 *Let K/F be a Galois extension with Galois group G . Then there is a bijection between subgroups $H \leq G$ and intermediate fields $F \subseteq L \subseteq K$, given by:*

$$H \mapsto K^H, \quad L \mapsto \text{Gal}(K/L).$$

Moreover, this correspondence is inclusion-reversing.

1.2 Proof of the Correspondence

We now prove one direction of this correspondence. Suppose $H \leq G$ is a subgroup. We want to show that the fixed field K^H is an intermediate field and that the Galois group of K^H is exactly H .

Let $a \in K^H$, so $h(a) = a$ for all $h \in H$. For any $g \in G$, consider the element $g(a)$. We want to show that $g(a) \in K^H$, i.e., $h(g(a)) = g(a)$ for all $h \in H$.

Since H is a subgroup, for each $h \in H$, there exists $h' \in H$ such that $hg = gh'$. Then:

$$h(g(a)) = (hg)(a) = (gh')(a) = g(h'(a)) = g(a),$$

where the last equality follows because $h'(a) = a$ (since $a \in K^H$). This shows that $g(a) \in K^H$, so K^H is invariant under the action of G .

Now, let $L = K^H$. We claim that $\text{Gal}(K/L) = H$. By definition, $H \leq \text{Gal}(K/L)$ because every element of H fixes L . To show the reverse inclusion, suppose $g \in \text{Gal}(K/L)$. Then g fixes every element of L , so in particular g fixes $a \in K^H$. But this means $g(a) = a$ for all $a \in K^H$, which implies $g \in H$. Therefore, $\text{Gal}(K/L) = H$.

This completes the proof that the map $H \mapsto K^H$ is injective and that the Galois group of the fixed field is exactly the subgroup we started with.

1.3 Remarks on the Correspondence

The above argument shows that for any subgroup $H \leq G$, we have:

$$\text{Gal}(K/K^H) = H.$$

This is a key step in establishing the Galois correspondence. The reverse direction (showing that for any intermediate field L , we have $K^{\text{Gal}(K/L)} = L$) requires additional arguments, typically using the primitive element theorem or the linear independence of characters.

The inclusion-reversing property is also evident: if $H_1 \leq H_2 \leq G$, then $K^{H_2} \subseteq K^{H_1}$. Similarly, if $F \subseteq L_1 \subseteq L_2 \subseteq K$, then $\text{Gal}(K/L_2) \leq \text{Gal}(K/L_1)$.

This correspondence is fundamental in Galois theory and allows us to translate problems about field extensions into problems about group theory, which are often easier to solve.