1 Group Actions and Fixed Fields

Let us consider a Galois extension K/F with Galois group $G = \mathrm{Gal}(K/F)$. In this section, we explore the correspondence between subgroups of G and intermediate fields of the extension K/F.

1.1 The Fixed Field Correspondence

Given a subgroup $H \leq G$, we define the fixed field of H as:

$$K^H = \{x \in K \mid \sigma(x) = x \text{ for all } \sigma \in H\}.$$

This is a subfield of K containing F. Conversely, for an intermediate field $F \subseteq L \subseteq K$, we define the Galois group of K over L as:

$$\mathrm{Gal}(K/L) = \{\sigma \in G \mid \sigma(x) = x \text{ for all } x \in L\}.$$

This is a subgroup of G. The fundamental theorem of Galois theory establishes a bijection between subgroups of G and intermediate fields of K/F.

1.2 An Important Lemma on Subgroups and Fixed Fields

We now prove a key lemma relating the fixed fields of conjugate subgroups.

Lemma 1 *Let K/F be a Galois extension with Galois group G, and let $H \leq G$. For any $\sigma \in G$, we have:*

$$K^{\sigma H \sigma^{-1}} = \sigma(K^H).$$

We will show both inclusions.

Proof of $K^{\sigma H \sigma^{-1}} \subseteq \sigma(K^H)$: Let $x \in K^{\sigma H \sigma^{-1}}$. Then for all $\tau \in H$, we have:

$$(\sigma \tau \sigma^{-1})(x) = x.$$

Applying σ^{-1} to both sides yields:

$$\tau(\sigma^{-1}(x)) = \sigma^{-1}(x).$$

This shows that $\sigma^{-1}(x) \in K^H$, since it is fixed by every $\tau \in H$. Therefore, $x = \sigma(\sigma^{-1}(x)) \in \sigma(K^H)$.

Proof of $\sigma(K^H) \subseteq K^{\sigma H \sigma^{-1}}$: Let $y \in K^H$, and consider $x = \sigma(y)$. For any $\tau \in H$, we compute:

$$(\sigma\tau\sigma^{-1})(x) = (\sigma\tau\sigma^{-1})(\sigma(y)) = \sigma(\tau(y)) = \sigma(y) = x,$$

since $\tau(y) = y$ as $y \in K^H$. Hence, $x \in K^{\sigma H \sigma^{-1}}$, proving the inclusion.

Therefore, $K^{\sigma H \sigma^{-1}} = \sigma(K^H)$, as required.

1.3 Consequences for Normal Subgroups

An important special case occurs when H is a normal subgroup of G.

Theorem 1 *If $H \trianglelefteq G$, then the fixed field K^H is a Galois extension of F, and $\mathrm{Gal}(K^H/F) \cong G/H$.*

Since $H \trianglelefteq G$, we have $\sigma H \sigma^{-1} = H$ for all $\sigma \in G$. By the previous lemma, this implies:

$$\sigma(K^H) = K^H \quad \text{for all } \sigma \in G.$$

This means that K^H is invariant under the action of G, which implies that K^H/F is a Galois extension. The restriction map:

$$\sigma \mapsto \sigma|_{K^H}$$

defines a surjective homomorphism from G to $\mathrm{Gal}(K^H/F)$ with kernel H, so by the first isomorphism theorem:

$$\mathrm{Gal}(K^H/F) \cong G/H.$$

This completes our discussion of the fundamental correspondence in Galois theory between subgroups and intermediate fields, with special attention to the behavior under conjugation and normality.