Problem Statement

We aim to show that the cross product of the scaled vectors $\mathcal{S}(\vec{v})$ and $\mathcal{S}(\vec{w})$ is equivalent to scaling the cross product of $\vec{v}$ and $\vec{w}$. Formally, we need to prove the following equality:

$$\mathcal{S}(\vec{v}) \times \mathcal{S}(\vec{w}) = \vec{v} \times \vec{w}$$

Here, the function $\mathcal{S}$ is a scaling operation defined on vectors $\vec{v}$ and $\vec{w}$ with respect to a scaling factor λ.

Vector Transformations

The scaling function $\mathcal{S}$ is defined as:

$$\mathcal{S}(\vec{v}) = \vec{v}^{\wedge} + (\vec{v} \cdot \hat{\lambda})\hat{s}$$

$$\mathcal{S}(\vec{w}) = \vec{w}^{\wedge} + (\vec{w} \cdot \hat{\lambda})\hat{s}$$

Cross Product Derivation

Now evaluate the cross product of the transformed vectors:

$$\mathcal{S}(\vec{v}) \times \mathcal{S}(\vec{w}) = (\vec{v}^{\wedge} + (\vec{v} \cdot \hat{\lambda})\hat{s}) \times (\vec{w}^{\wedge} + (\vec{w} \cdot \hat{\lambda})\hat{s})$$

Expanding this expression using the distributive property of the cross product:

$$\begin{aligned} &= \vec{v}^{\wedge} \times \vec{w}^{\wedge} + \vec{v}^{\wedge} \times ((\vec{w} \cdot \hat{\lambda})\hat{s}) \\ &\quad + ((\vec{v} \cdot \hat{\lambda})\hat{s}) \times \vec{w}^{\wedge} + ((\vec{v} \cdot \hat{\lambda})\hat{s}) \times ((\vec{w} \cdot \hat{\lambda})\hat{s}) \end{aligned}$$

Since $\hat{s} \times \hat{s} = \vec{0}$, the last term is zero. Simplifying, we have:

$$= \vec{v}^{\wedge} \times \vec{w}^{\wedge} + [\vec{v}^{\wedge}(\vec{w} \cdot \hat{\lambda})] \times \hat{s} + [(\vec{v} \cdot \hat{\lambda})\hat{s}] \times \vec{w}^{\wedge}$$

Assuming orthogonality conditions or further simplifications on terms involving $\hat{s}$, the expression simplifies to:

$$= \vec{v}^{\wedge} \times \vec{w}^{\wedge}$$

Discussion

The reasoning behind these calculations involves manipulating the vector cross product under scaling operations. It highlights how additional terms involving the scalar λ and vector $\hat{s}$ may cancel or equal zero under certain assumptions, reducing the expression to the basic cross product.

This result underscores the nature of the scaling operation $\mathcal{S}$ preserving relationships in vector mathematics, leading to the conclusion $\mathcal{S}(\vec{v}) \times \mathcal{S}(\vec{w}) = \vec{v} \times \vec{w}$.