1 Quotients in Vector Spaces

Let $U \subseteq V$ be a subspace.

1.1 Equivalence Relation

We define an equivalence relation $\sim$ on V by declaring $v \sim v'$ if $v - v' \in U$.

1.2 The Quotient Space V/U

The set of equivalence classes is denoted by

$$V/U = \{[v] \mid v \in V\}$$

This creates a new vector space, the quotient space, where each element is a coset $[v]$.

1.3 Basis and Dimension

Suppose $\{u_i\}$ is a basis of U, and $\{u_1, \ldots, u_k, v_1, \ldots, v_\ell\}$ is a basis of V.

Then, the set

$$\{[v_i] \mid 1 \leq i \leq \ell\}$$

forms a basis of the quotient space V/U.

Therefore, the dimension of the quotient space is given by

$$\dim V/U = \ell = \dim V - \dim U$$

This result shows that the dimension of the quotient space is equal to the dimension of the original space minus the dimension of the subspace.