Matrix Differential Equation Solution

Consider the matrix function $\Phi(t)$ given by:

$$\Phi = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-t} & 2e^{-5t} \end{bmatrix}$$

The determinant of Φ is computed as:

$$\det(\Phi) = 2e^{-6t}$$

Inverse of the Matrix

The inverse of the matrix Φ, denoted Φ^{-1}, can be calculated using the formula for the inverse of a 2x2 matrix:

$$\Phi^{-1} = \frac{1}{3} \begin{bmatrix} 2e^{2t} & -e^{-5t} \\ -e^{-t} & e^{2t} \end{bmatrix}$$

Matrix U and Particular Solution

With Φ^{-1}, we compute the matrix U as the integral of Φ^{-1}:

$$U = \int \left[\tfrac{1}{3} e^{2t} \right] = \begin{bmatrix} \frac{100}{6} e^{2t} \\ \frac{100}{15} e^{50t} \end{bmatrix}$$

The particular solution χ_p is then given by:

$$\chi_p = \Phi \cdot U = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-t} & 2e^{-5t} \end{bmatrix} \begin{bmatrix} \frac{50}{3} e^{\frac{5}{2}t} \\ 2e^{6t} \end{bmatrix}$$

Evaluating Matrix Products

Calculations lead to:

$$\begin{bmatrix} \frac{50}{3} = \frac{20}{3} \\ 50 \end{bmatrix} + \left[\tfrac{40}{3} \right] = \begin{bmatrix} \frac{30}{3} \\ 90 \end{bmatrix}$$

Consequently, the particular solution is:

$$\chi_p = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$$

This concludes the calculations for solving the given differential equation using matrix methods.