Theorem: Random search needs with probability $1 - e^{-\Omega(n)}$ at least $2^{n/2}$ evaluations to find the optimal search point in OneMax

1 Problem Setup and Definition

We use bit strings of length n and so our search space is $\{0,1\}^n$. The objective function is $f(x) = \sum_{i=1}^{n} x_i$, which counts the number of ones. The unique global optimum is 1^n with $f(1^n) = n$.

Our random search algorithm repeatedly samples strings uniformly at random from $\{0,1\}^n$ until we either find the optimum or have exceeded the limit of T evaluations. We stop immediately if we find the best solution.

2 Lemmas

There are 2^n possible strings, but only one of them is optimal.

For each draw, the probability that the sample equals the optimum is 2^{-n}.

Lemma 1. *Under uniform random sampling,* $\mathbb{P}[\textit{sample} = 1^n] = 2^{-n}$.

After T attempts, the event "at least one success" is the union $\bigcup_{t=1}^{T}\{X_t = 1^n\}$. By the union bound from the lectures,

$$\mathbb{P}(\text{at least one success}) \;\leq\; \sum_{t=1}^{T} \mathbb{P}(X_t = 1^n) \;=\; T \cdot 2^{-n}.$$

Lemma 2 (Union Bound)**.** *For events* $E_1, \ldots, E_T$, $\mathbb{P}\Big(\bigcup_{t=1}^{T} E_t\Big) \leq \sum_{t=1}^{T} \mathbb{P}(E_t)$.

3 Results

3.1 Single Run

Proof. Set $T = 2^{n/2}$. By the union bound,

$$\mathbb{P}(\text{success within } 2^{n/2} \text{ evaluations}) \;\leq\; 2^{n/2} \cdot 2^{-n} \;=\; 2^{-n/2}.$$

Therefore,

$$\mathbb{P}(\text{need more than } 2^{n/2} \text{ evaluations}) \;\geq\; 1 - 2^{-n/2}.$$

Since $2^{-n/2} = e^{-(\ln 2)\, n/2}$ decreases exponentially in n, we obtain the stated bound $1 - e^{-\Omega(n)}$. □

The bound also shows that after examining $2^{n/2}$ uniformly random strings, we have covered only a fraction of the search space equal to $(2^{n/2})/2^n = 2^{-n/2}$, so the expected number of hits of the optimum equals $2^{-n/2}$, which vanishes exponentially as n grows.

3.2 Multiple Runs

Running the algorithm multiple times does not change the picture. With R independent runs, each allotted $2^{n/2}$ evaluations, the probability that any run succeeds is at most $R \cdot 2^{-n/2}$ by another application of the union bound.

4 Conclusion

Within $2^{n/2}$ evaluations, the success probability of uniform random search is at most $2^{-n/2}$; equivalently, after $2^{n/2}$ evaluations, only a $2^{-n/2}$ fraction of the space $\{0,1\}^n$ has been inspected. Hence the failure probability is at least $1 - 2^{-n/2} = 1 - e^{-\Omega(n)}$. Therefore, with overwhelmingly high probability as n grows, uniform random search requires more than $2^{n/2}$ evaluations to reach the unique optimum 1^n of OneMax.