Theorem: RLS and (1+1) EA reach a optimal search point in F3 in $O(n \log n)$ fitness evaluations with probability $\Omega(1)$

1 Problem Setup and Definition

We use bit strings of length n where each bit is either a 0 or 1. The function F3 is strictly monotone linear function defined as:

$$f(x) = \sum_{i=1}^{n} w_i x_i \quad \text{with all } w_i > 0$$

Thus the global optimum is 1^n.

Let $d(x)$ denote the number of zeros in x (Hamming distance).

RLS (Randomized Local Search) starts from any x and in each step flips exactly one randomly chosen bit. It accepts the offspring if and only if its fitness is greater or equal (elitist selection).

The (1+1)EA starts from any x and in each iteration flips every bit independently with a probability of $1/n$. This also follows the elitist selection.

2 Lemmas

Lemma 1. *RLS improvement probability: Let, current distance $i = d(x)$*

RLS then improves to distance $i - 1$ with probability $p_i = i/n$ and the expected waiting time for this improvement is $\mathbb{E}[T_i] = n/i$.

Proof. Since all $w_i > 0$, flipping a 0 bit to 1 increases f while flipping a 1 to 0 decreases it.

Since RLS deploys elitism, it improves if and only if it flips one of the i zeros. This happens with a probability i/n.

The mean waiting time is $1/p_i = n/i$. □

Lemma 2. *Coupon-collector sum for RLS: Starting from any $d_0 \leq n$, the expected optimization time of RLS satisfies*

$$\mathbb{E}[T_{\mathrm{RLS}}] \leq \sum_{i=1}^{d_0} \frac{n}{i} \leq n \sum_{i=1}^{n} \frac{1}{i} = nH_n = O(n \log n).$$

Lemma 3. (1+1) *EA success probability from distance i: From distance i, the probability to* decrease *the distance in one step is at least*

$$p_i \;\geq\; \frac{i}{n}\Big(1 - \frac{1}{n}\Big)^{n-1} \;\geq\; \frac{i}{en} \qquad (n \geq 2).$$

Proof. The probability of flipping only one of the i zero bits is $\frac{i}{n}(1-\frac{1}{n})^{n-1}$.

Since $(1-\frac{1}{n})^{n-1} \geq 1/e$ for $n \geq 2$, the bound follows. ☐

Lemma 4. *Multiplicative drift (standard form): Let $(D_t)_{t\geq 0}$ be a non-negative process with $D_t > 0$ implying $\mathbb{E}[D_t - D_{t+1} \mid D_t] \geq \delta D_t$ for some $\delta > 0$.*

Then the hitting time $T = \min\{t : D_t = 0\}$ obeys

$$\mathbb{E}[T] \;\leq\; \frac{1}{\delta}\big(\ln D_0 + 1\big).$$

Lemma 5. *Markov-type constant-probability bound: For any non-negative random variable X, $\mathbb{P}\big[X \leq 2\,\mathbb{E}[X]\big] \geq \frac{1}{2}$.*

3 Results

Proof. **RLS:** By the coupon-collector sum above, $\mathbb{E}[T_{\mathrm{RLS}}] \leq nH_n = O(n \log n)$.

By the constant-probability bound,

$$\mathbb{P}\big[T_{\mathrm{RLS}} \leq 2\,\mathbb{E}[T_{\mathrm{RLS}}]\big] \geq \tfrac{1}{2},$$

so RLS finishes in $O(n \log n)$ evaluations with probability at least $1/2 = \Omega(1)$.

(1+1) EA: Let $D_t = d(X_t)$ be the distance process under elitism. From distance i, Lemma 3 gives $\mathbb{P}(D_{t+1} < D_t \mid D_t = i) \geq i/(en)$; hence

$$\mathbb{E}[D_t - D_{t+1} \mid D_t = i] \;\geq\; \frac{i}{en} \;=\; \frac{1}{en}\,D_t.$$

By multiplicative drift (Lemma 4) with $\delta = \frac{1}{en}$,

$$\mathbb{E}[T_{(1+1)}] \;\leq\; en\big(\ln d_0 + 1\big) \;\leq\; en(\ln n + 1) = O(n \log n).$$

Applying Lemma 5 yields $\mathbb{P}\big[T_{(1+1)} \leq 2\,\mathbb{E}[T_{(1+1)}]\big] \geq \frac{1}{2}$, so the (1+1) EA also finishes in $O(n \log n)$ evaluations with probability $\Omega(1)$. ☐

4 Conclusion

On the function F3, both RLS and the (1+1) EA have optimization time $\Theta(n \log n)$. Therefore, by applying Markov's inequality, each completes in $O(n \log n)$ fitness evaluations with constant probability $\Omega(1)$, uniformly in n. The elitist local methods require $O(n \log n)$ evaluations, scaling almost linearly with n.