BST 140.651 Problem Set 1

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Problem 1

$Var(X) = E[X^2] - E[X]^2$

Where $Var(X) = E[(X - \mu)^2]$, so

$E[(X - \mu)^2] = E[X^2] - E[X]^2$

$E[X^2 - 2X\mu + \mu^2] = E[X^2] - E[X]^2$

$E[X^2] + E[-2X\mu] + E[\mu^2] = E[X^2] - E[X]^2$

Since μ is a constant, and constants can be pulled out of the expected value function,

$E[X^2] - 2\mu E[X] + \mu^2 E[1] = E[X^2] - E[X]^2$

Simplifying, and with the knowledge that $E[X] = \mu$:

$-2\mu^2 + \mu^2 = -\mu^2$

From which we find

$-2\mu^2 = -2\mu^2$

Which we know to be true.

Problem 2

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g is a valid density if all values are positive and if the integral across all values is 1.

We know all values are positive because it is the sum of three densities (which themselves must always be positive), each of which is multiplied by a scalar that is also positive.

To show that the integral across all values is equal to 1,

$\int\limits_{-\infty}^{\infty} g(x) = 1$

Since we know

$g(x) = \pi_1 f_1(x) + \pi_2 f_2(x) + \pi_3 f_3(x)$

then

$\int\limits_{-\infty}^{\infty} g(x) = \int\limits_{-\infty}^{\infty} \left(\pi_1 f_1(x) + \pi_2 f_2(x) + \pi_3 f_3(x) \right)$

which is equivalent to

$\int\limits_{-\infty}^{\infty} g(x) = \pi_1 \int\limits_{-\infty}^{\infty} f_1(x) + \pi_2 \int\limits_{-\infty}^{\infty} f_2(x) + \pi_3 \int\limits_{-\infty}^{\infty} f_3(x)$

and since each function f_i is a valid density their integrals must equal 1, giving

$\int\limits_{-\infty}^{\infty} g(x) = \pi_1 + \pi_2 + \pi_3$

Since we are additionally told $\Sigma_{i=1}^3 \pi_i = 1$

$\int\limits_{-\infty}^{\infty} g(x) = 1$ ∎

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We know that mean (or expected value) for a continous distribution is given by

$\int\limits_{-\infty}^{\infty} xg(x)$

Which is

$\int\limits_{-\infty}^{\infty} x\Big(\pi_1 f_1(x) + \pi_2 f_2(x) + \pi_3 f_3(x)\Big)$

which can be expanded to

$\pi_1 \int\limits_{-\infty}^{\infty} xf_1(x) + \pi_2 \int\limits_{-\infty}^{\infty} xf_2(x) + \pi_3 \int\limits_{-\infty}^{\infty} xf_3(x)$

And since we know the definition of the expected value, we can rewrite this as

$E[X] = \pi_1 \mu_1 + \pi_2 \mu_2 + \pi_3 \mu_3$

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We have previously shown that the definition of

$Var(x) = E[X^2] - E[X]^2$

We can represent the variance of g in these terms:

$\int\limits_{-\infty}^{\infty} x^2 g(x) - [\int\limits_{-\infty}^{\infty} xg(x)]^2$

The right term we have previously found - it's simply the square of the expectation value we found in the previous problem. The left terms can be found by first writing g in terms of f

$\int\limits_{-\infty}^{\infty} x^2 g(x) = \int\limits_{-\infty}^{\infty} x^2 \Big(\pi_1 f_1(x) + \pi_2 f_2(x) + \pi_3 f_3(x)\Big)$

or distributed:

$\int\limits_{-\infty}^{\infty} x^2 g(x) = \pi_1 \int\limits_{-\infty}^{\infty} x^2 f_1(x) + \pi_2 \int\limits_{-\infty}^{\infty} x^2 f_2(x) + \pi_3 \int\limits_{-\infty}^{\infty} x^2 f_3(x)$

and then noticing that the variance equation can be written as follows:

$\sigma^2 = E[X^2] + \mu^2$

And solving for $E[X^2]$:

$E[X^2] = \mu^2 + \sigma^2$

which is equivalent to

$\int\limits_{-\infty}^{\infty} x^2 f_i(x) = \mu_i^2 + \sigma_i^2$

Substituting in, we get

$\int\limits_{-\infty}^{\infty} x^2 g(x) = \pi_1(\mu_1^2 + \sigma_1^2) + \pi_2(\mu_2^2 + \sigma_2^2) + \pi_3(\mu_3^2 + \sigma_3^2)$

Combining this left hand term with the right hand term, we get

$Var(x) = \pi_1(\mu_1^2+\sigma_1^2)+\pi_2(\mu_2^2+\sigma_2^2)+\pi_3(\mu_3^2+\sigma_3^2)-(\pi_1\mu_1+\pi_2\mu_2+\pi_3\mu_3)^2$

Problem 3

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$\int\limits_{-\infty}^{\infty} x^2 f_i(x) = \mu_i^2 + \sigma_i^2$