BST 140.651 Problem Set 3

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Problem 1

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a

The maximum likelihood can be represented generally as

$\mathcal{L}(\theta|x) = \prod_{i=1}^{n} f(x_i, \theta)$

We know that the probability mass function for a Bernoulli random variable x can be written as

$F(X) = \theta^k (1-\theta)^{1-k}$

where k is an indicator value that can take on the value of either 0 or 1, where we define $0 = Tails$ and $1 = Heads$.

Therefore,

$\prod_{i=1}^{n} f(x_i, \theta) = \theta^7 (1-\theta)^3$

The maximum likelihood can be solved for by taking the derivative and setting it to zero, then solving for θ.

$(\theta^7 (1-\theta)^3)' = 7\theta^6 (1-\theta)^3 - 3(1-\theta)^2 = 0$

$\implies \theta = \frac{1}{\frac{3}{7}+1}$ ■

b

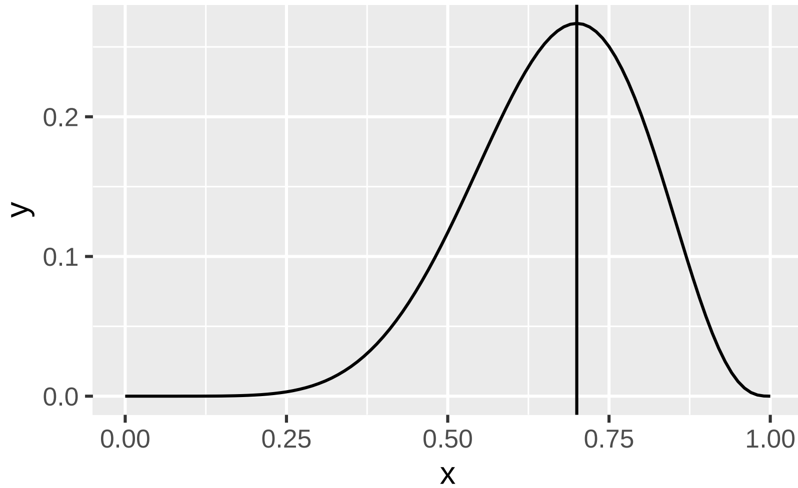
We can plot the likelihood function:

$\mathcal{L}(\theta|x) = \theta^7 (1-\theta)^3$

```
x <- seq(0, 1, by = 0.01)
y <- (factorial(10)/(factorial(3) * factorial(7))) * x^7 * (1 - x)^3

dat <- tibble(x, y)

ggplot(dat, aes(x, y)) + geom_line() + geom_vline(xintercept = 1/(3/7 + 1))
```



to re-normalize such that the maximum likelihood is 1, we need to multiply the function by some value such that when $x = ML, y = 1$. First we find what the value of y when it is at its maximum likelihood:

```
x <- 1/(3/7 + 1)

y <- x^7 * (1 - x)^3
y
```

```
## [1] 0.002223566
```

And then we divide 1 by this value to solve for the number by which we need to scale:

```
1/y
```

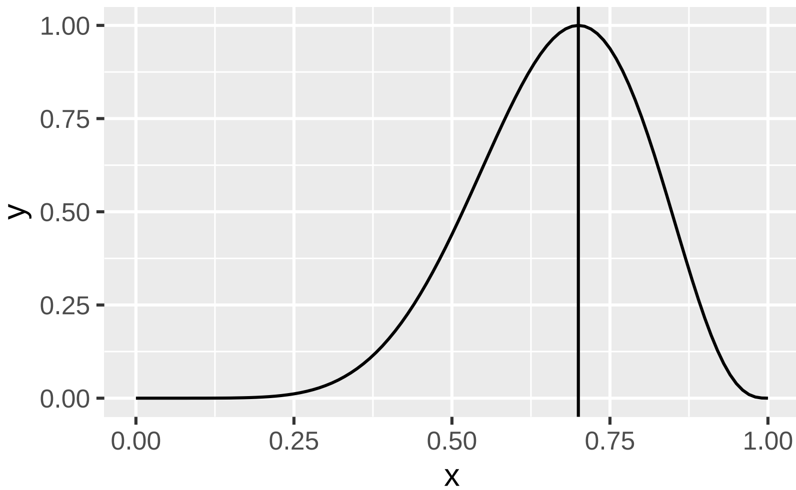
```
## [1] 449.728
```

so,

```
x <- seq(0, 1, by = 0.01)
y <- 449.728 * x^7 * (1 - x)^3

dat <- tibble(x, y)

ggplot(dat, aes(x, y)) + geom_line() + geom_vline(xintercept = 1/(3/7 + 1))
```



We see that a theta (here written x) of 0.5 (a fair coin) has a likelihood of roughly half that of our maximum likelihood - but this still seems to be within the realm of possibility.

c

The probability of seeing 7 or more heads given the coin is essentially the integral of the likelihood function from 0.7 to 1 given $\theta = 0.5$. Because this function has a discrete input, this is simply the sum of the values of the PMF from 7 to 10. The PMF is represented by the binomial PMF:

$\sum_{k=7}^{10} \binom{n}{k} 0.5^k 0.5^{10-k} = \frac{10!}{7!3!} 0.5^7 0.5^3 + \frac{10!}{8!2!} 0.5^8 0.5^2 + \frac{10!}{9!1!} 0.5^9 0.5^1 + \frac{10!}{10!0!} 0.5^1 00.5^0$

```
my_sum <- 0
for (k in 7:10) {
    y <- (factorial(10)/(factorial(k) * factorial(10 - k))) * (0.5^k) * (0.5^(10 -
        k))
    my_sum <- sum(my_sum + y)
}
my_sum
```

```
## [1] 0.171875
```

Thus, given a fair coin there is a ~17% probability that we would get 7/10 heads (not out of the realm of possibility)

d

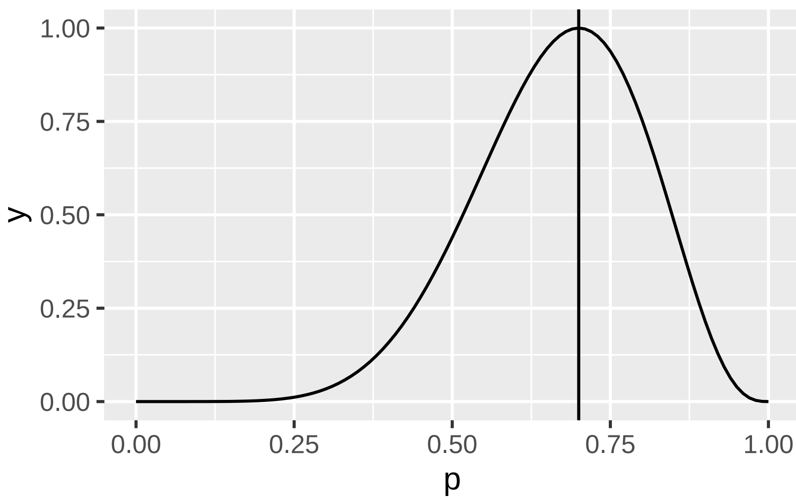
In this particular instance, $y = 10$, so
$\mathcal{L}(p|y) = \frac{9!}{3!7!}(1-p)^3 p^7$

We can see that though the constant preceding the function is different, the overall equation remains the same. Therefore the maximum likelihood will be at the same value of x (this constant can be effectively removed upon taking the derivative of this equation and setting it equal to zero)

```
p <- seq(0, 1, by = 0.01)
y <- 449.728 * (1 - p)^3 * p^7

dat <- tibble(p, y)

ggplot(dat, aes(p, y)) + geom_line() + geom_vline(xintercept = 1/(3/7 + 1))
```




```
my_sum <- 0
for (y in 3:10) {
    y <- (factorial(y - 1)/(factorial(2) * factorial(y - 2))) * (0.5^y) * (0.5^(y -
        3))
    my_sum <- sum(my_sum + y)
}
my_sum

## [1] 0.1944313
```