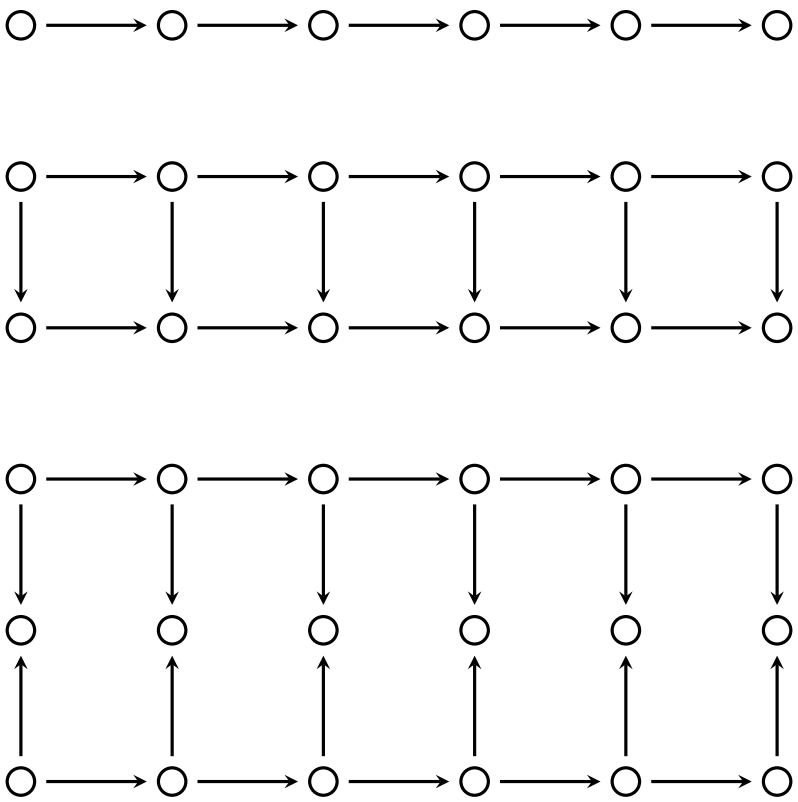

Compiling Swift Generics

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Preface

This is a book about the implementation of generic programming—also known as parametric polymorphism—in the Swift compiler. You won't learn how to *write* generic code in Swift here; the best reference for that is, of course, the official language guide [1]. This book is intended mainly for Swift compiler developers who interact with the generics implementation, other language designers who want to understand how Swift evolved, Swift programmers curious to peek under the hood, and finally, mathematicians interested in a practical application of string rewriting and the Knuth-Bendix completion procedure.

From the compiler developer's point of view, the *user* is the developer writing the code being compiled. The declarations, types, statements and expressions written in the user's program become *data structures* the compiler must analyze and manipulate. I assume some basic familiarity with these concepts, and compiler construction in general. For background reading, I recommend [2], [3], [4], and [5].

This book is divided into five parts. Part I gives a high-level overview of the Swift compiler architecture, and describes how types and declarations, and specifically, generic types and declarations, are modeled by the compiler.

- Chapter 1 summarizes every key concept in the generics implementation with a series of worked examples, and surveys capabilities for generic programming found in other programming languages.
- Chapter 2 covers Swift's compilation model and module system as well as the *request evaluator*, which adds an element of lazy evaluation to the typical "compilation pipeline" of parsing, type checking and code generation.
- Chapter 3 describes how the compiler models the *types* of values declared by the source program. Types form a miniature language of their own, and we often find ourselves taking them apart and re-assembling them in new ways. Generic parameter types, dependent member types, and generic nominal types are the three fundamental kinds; others are also summarized.
- Chapter 4 is about *declarations*, the building blocks of Swift code. Functions, structs, and protocols are examples of declarations. Various kinds of declarations can be *generic*. There is a common syntax for declaring generic parameters and stating *requirements*. Protocols can declare associated types and impose *associated requirements* on their associated types in a similar manner.

Part II focuses on the core *semantic* objects in the generics implementation. To grasp the mathematical asides, it helps to have had some practice working with definitions and proofs, at the level of an introductory course in calculus, linear algebra or combinatorics. A summary of basic math appears in Appendix A.

- Chapter 5 defines the *generic signature*, which collects the generic parameters and explicit requirements of a generic declaration. The explicit requirements of a generate signature generate a set of *derived requirements* and *valid type parameters*, which explains how we type check code *inside* a generic declaration. This formalism is realized in the implementation via *generic signature queries*.
- Chapter 6 defines the *substitution map*, a mapping from generic parameter types to replacement types. The *type substitution algebra* will explain the operations of type substitution and substitution map composition. This explains how we type check a *reference* to (sometimes called a *specialization* of) a generic declaration.
- Chapter 7 defines the *conformance*, a description of how a concrete type fulfills the requirements of a protocol, in particular its associated types. In the type substitution algebra, conformances are to protocols what substitution maps are to generic signatures.
- Chapter 8 defines *archetypes* and *generic environments*, two abstractions used throughout the compiler. Also describes the *type parameter graph* that gives us an intuitive visualization of a generic signature.

Part III explores the implementation of various compiler subsystems, while further developing the derived requirements formalism and type substitution:

- Chapter 9 details *type resolution*, which uses name lookup and substitution to resolve syntactic representations to semantic types. Checking if a substitution map satisfies the requirements of its generic signature links our two formalisms.
- Chapter 10 discusses extension declarations, which add members and conformances to existing types. Extensions can also declare *conditional conformances*, which have some interesting behaviors.
- Chapter 11 explains how we build a generic signature from the syntax written in source, via the process of *requirement minimization*, which eliminates requirements proved redundant. This chapter also shows how invalid requirements are diagnosed. A *well-formed generic signature* is one that satisfies the validity condition, and its theoretical properties have nice practical consequences.
- Chapter 12 fills out the remainder of the type substitution algebra, explaining *conformance paths*. The concept of a *recursive conformance* is explored, and finally, the type substitution algebra is shown to be Turing-complete.

Part IV is currently unfinished. It will describe opaque return types (Chapter 13), existential types (Chapter 14), and subclassing (Chapter 15).

Part V describes the Requirement Machine, a *decision procedure* for the derived requirements formalism. The original contribution here is that generic signature queries and requirement minimization are problems in the theory of *string rewriting*:

- Chapter 16 describes the entry points for generic signature queries and requirement minimization, and how they recursively build a *requirement machine* for a generic signature from the requirement machines of its *protocol components*.
- Chapter 17 introduces *finitely-presented monoids* and the *word problem*, and then presents the theoretical result that a finitely-presented monoid can be encoded as a generic signature, such that word problems become generic signature queries. Therefore, derived requirements are *at least* as expressive as the word problem; that is, undecidable in the general case.
- Chapter 18 goes in the other direction and shows that a generic signature can be encoded in the form of a finitely-presented monoid, such that generic signature queries become word problems. Therefore, derived requirements are *at most* as expressive as the word problem—which can be solved in many cases using known techniques. This is the heart of our decision procedure.
- Chapter 19 describes the Knuth-Bendix algorithm, which attempts to solve the word problem by constructing a *convergent rewriting system*. Fundamental generic signature queries can then be answered via the *normal form algorithm*. This is the brain of our decision procedure.
- Chapter 20 is unfinished. It will describe the construction of a *property map* from a convergent rewriting system; the property map answers trickier generic signature queries.
- Chapter 21 is unfinished. It will describe the handling of concrete types in the Requirement Machine.
- Chapter 22 is unfinished. It will present the algorithm for rewrite rule minimization, which is the final step in building a new generic signature.

The Swift compiler is written in C++. To avoid incidental complexity, concepts are described without direct reference to the source code. Instead, some chapters end with a **Source Code Reference** section, structured like an API reference. You can skip this material if you're only interested in the theory. No knowledge of C++ is assumed outside of these sections.

Occasional historical asides talk about how things came to be. Starting with Swift 2.2, the design of the Swift language has been guided by the Swift evolution process, where

language changes are pitched, debated, and formalized in the open [6]. I will cite Swift evolution proposals when describing various language features. You will find a lot of other interesting material in the bibliography as well, not just evolution proposals.

This book does not say much about the runtime side of the separate compilation of generics, except for a brief overview of the model in relation to the type checker in Chapter 1. To learn more, I recommend watching a pair of LLVM Developer's Conference talks: [7] which gives a summary of the whole design, and [8] which describes some recent optimizations.

Also, while most of the material should be current as of Swift 6, two recent language extensions are not covered. These features are mostly additive and can be understood by reading the evolution proposals:

1. Parameter packs, also known as variadic generics ([9], [10], [11]).
2. Noncopyable types ([12], [13]).

Source Code

The TEX code for this book lives in the Swift source repository:

`https://github.com/swiftlang/swift/tree/main/docs/Generics`

A periodically-updated typeset PDF is available from the Swift website:

`https://download.swift.org/docs/assets/generics.pdf`

Acknowledgments

I'd like to thank everyone who read earlier versions of the text, pointed out typos, and asked clarifying questions. Also, the Swift generics system itself is the result of over a decade of collaborative effort by countless people. This includes compiler developers, Swift evolution proposal authors, members of the evolution community, and all the users who reported bugs. This book attempts to give an overview of the sum total of all these contributions.

Contents

List of Algorithms

Part I.

Syntax

1. Introduction

Swift's generics implementation is best understood by first considering various design constraints faced by the compiler:

1. Generic functions should be independently type checked, without knowledge of all possible generic arguments that they are invoked with.

2. Shared libraries that export generic types and functions should be able to evolve resiliently without requiring recompilation of clients.

3. Layouts of generic types should be determined by their concrete substitutions, with fields of generic parameter type stored inline.

4. Abstraction over concrete types with generic parameters should only impose a cost across module boundaries, or in other situations where type information is not available at compile time.

The high-level design can be summarized as follows:

1. The interface between a generic function and its caller is mediated by **generic requirements**. The generic requirements describe the behavior of the generic parameter types inside the function body, and the generic arguments at the call site are checked against the function's generic requirements at compile time.

2. Generic functions receive **runtime type metadata** for each generic argument from the caller. Type metadata defines operations to abstractly manipulate values of their type without knowledge of their concrete layout.

3. Runtime type metadata is constructed for each type in the language. The **runtime type layout** of a generic type is computed recursively from the type metadata of the generic arguments. Generic types always store their contents directly, without indirection through a heap-allocated box.

4. The optimizer can generate a **specialization** of a generic function in the case where the definition is visible at the call site. This eliminates the overhead of runtime type metadata and abstract value manipulation.

The author's philosophy is that *a compiler is a library for modeling the concepts of the target language*. The Swift generics implementation defines four fundamental kinds of semantic objects: *generic signatures*, *substitution maps*, *requirement signatures*, and *conformances*. As we will see, they are understood as much by their inherent structure, as their relationships with each other. Subsequent chapters will dive into all the details, but first, we're going to present a series of worked examples.

1.1. Functions

Consider these two rather contrived function declarations:

```
func identity(_ x: Int) -> Int { return x }
func identity(_ x: String) -> String { return x }
```

Both have the same exact definition apart from the parameter and return type, and indeed you can write the same function for any concrete type. Our aesthetic sense might lead us to replace both with a single generic function:

```
func identity<T>(_ x: T) -> T { return x }
```

While this function declaration is trivial, it illustrates some important concepts and allows us to introduce terminology. We'll see a full description of the compilation pipeline in the next chapter, but for now, let's consider a simplified view where we begin with parsing, then type checking, and finally code generation.

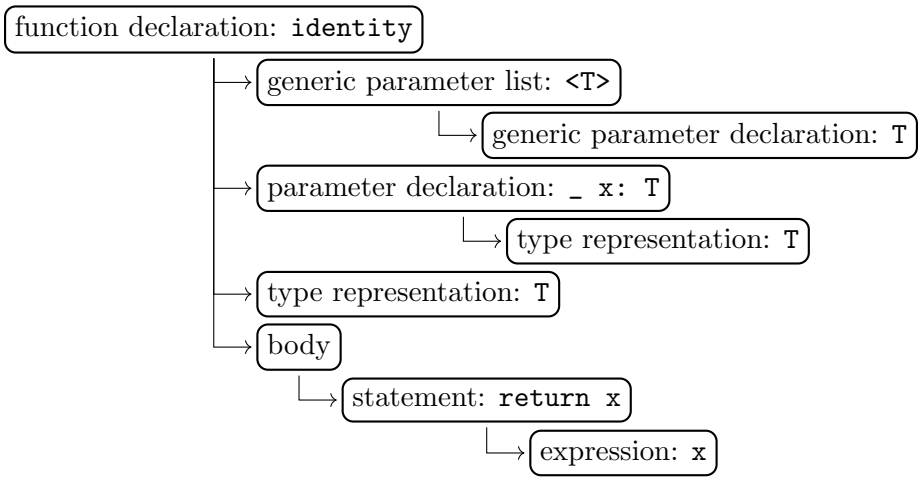
Parsing. Figure 1.1 shows the abstract syntax tree produced by the parser before type checking. The key elements:

1. The *generic parameter list* `<T>` introduces a single *generic parameter declaration* named `T`. As its name suggests, this declares the generic parameter type `T`, scoped to the entire source range of this function.

2. The *type representation* `T` appears twice, first in the declaration of the parameter "`_ x: T`" and then again, as the return type. A type representation is the purely syntactic form of a type. The parser does not perform name lookup, so the type representation stores the identifier `T` and does not refer to the generic parameter declaration of `T` in any way.

3. The function body contains an expression referencing `x`. Again, the parser does not perform name lookup, so this is just the identifier `x` and is not associated with the parameter declaration "`_ x: T`".

Figure 1.1.: The abstract syntax tree for `identity(_:)`



Type checking. The type checker constructs the *interface type* of the function declaration from the following:

1. A *generic signature*, to introduce the generic parameter type `T`. In our case, the generic signature has the printed representation `<T>`. This is the simplest generic signature, apart from the empty generic signature of a non-generic declaration. We'll see more interesting generic signatures soon.

2. The *interface type* of the parameter "`_ x: T`", which is declared to be the type representation `T` in source. The compiler component responsible for resolving this to a semantic type is called *type resolution*. In this case, we perform name lookup inside the lexical scope defined by the function, which finds the generic parameter type `T`.

3. The function's return type, which also resolves to the generic parameter type `T`.

All of this is packaged up into a *generic function type* which completely describes the type checking behavior of a reference to this function:

```
<T> (T) -> T
```

The name "`T`" is of no semantic consequence beyond name lookup. We will learn that the *canonical type* for the above erases the generic parameter type `T` to the notation `τ_0_0`. More generally, each generic parameter type is uniquely identified within its lexical scope by its *depth* and *index*.

Having computed the interface type of the function, the type checker moves on to the function's body. The type of the return statement's expression must match the return type of the function declaration. When we type check the return expression `x`, we use name lookup to find the parameter declaration "`_ x: T`". The interface type of this parameter declaration is the generic parameter type `T`, which is understood relative to the function's generic signature. The type assigned to the expression, however, is the *archetype* corresponding to `T`, denoted $[\![\texttt{T}]\!]$. We will learn later that an archetype is a self-describing form of a type parameter which behaves like a concrete type.

Code generation. We've now successfully type checked our function declaration. How might we generate code for it? Recall the two concrete implementations that we folded into our single generic function:

```
func identity(_ x: Int) -> Int { return x }
func identity(_ x: String) -> String { return x }
```

The calling conventions of these functions differ significantly:

1. The first function receives and returns the `Int` value in a machine register. The `Int` type is *trivial*,[1] meaning it can be copied and moved at will.

2. The second function is trickier. A `String` is stored as a 16-byte value in memory, and contains a pointer to a reference-counted buffer. When manipulating values of a non-trivial type like `String`, memory ownership comes into play.

The standard ownership semantics for a Swift function call are defined such that the caller retains ownership over the parameter values passed into the callee, while the callee transfers ownership of the return value to the caller. Thus the implementation of `identity(_:)` must create a logical copy of `x` and then move this copy back to the caller, and do this in a manner that abstracts over all possible concrete types.

The calling convention for a generic function passes *runtime type metadata* for each generic parameter in the function's generic signature. Runtime type metadata describes the size and alignment of a concrete type, and provides implementations of the *move*, *copy* and *destroy* operations.

We move and copy values of trivial type by copying bytes; destroy does nothing in this case. With a reference type, the value is a reference-counted pointer; copy and destroy operations update the reference count, while a move leaves the reference count unchanged. The value operations for structs and enums are defined recursively from their members. Finally, weak references and existential types also have non-trivial value operations.

[1] Or POD, for you C++ folks.

For copyable types, a move is semantically equivalent to a copy followed by a destroy, only more efficient. Traditionally, all types in the language were copyable. Swift 5.9 introduced *noncopyable types* [12], and Swift 6 extended generics to work with noncopyable types [13]. We will not discuss noncopyable types in this book.

More details

- Types: Chapter 3
- Generic parameter lists: Section 4.1
- Function declarations: Section 4.4
- Archetypes: Chapter 8
- Type resolution: Chapter 9

Substitution maps. Let us now turn our attention to the callers of generic functions. A *call expression* brings together a *callee* and a list of argument expressions. A callee is just an expression of function type. This function type's parameters must match the argument expressions, and its return type is then the type of the call expression. Some possible callees include an expression that names an existing function declaration, type expressions (which is sugar for invoking a constructor member of the type), function parameters and local variables of function type, and even other calls whose result has function type. In our example, we might call the `identity(_:)` function as follows:

```
identity(3)
identity("Hello, Swift")
```

In Swift, calls to generic functions do not specify their generic arguments explicitly; the type checker infers them from the types of call argument expressions. Generic arguments are encoded in a *substitution map*, which assigns a *replacement type* to each generic parameter type of the callee's generic signature.

The generic signature of `identity(_:)` has a single generic parameter type. Thus each of its substitution maps holds a single concrete type. We now introduce some notation. Here are two possible substitution maps, corresponding to the two calls shown above:

$$\Sigma_1 := \{\texttt{T} \mapsto \texttt{Int}\} \qquad \Sigma_2 := \{\texttt{T} \mapsto \texttt{String}\}$$

We can form the return type of a call by taking the function declaration's return type, and applying the corresponding substitution map. We denote this application using the "$\otimes$" operator. For now, this simply extracts the concrete type stored therein:

$$\texttt{T} \otimes \Sigma_1 = \texttt{Int} \qquad \texttt{T} \otimes \Sigma_2 = \texttt{String}$$

Substitution maps play a role in code generation. When calling a generic function, the compiler must realize the runtime type metadata for each replacement type in the substitution map of the call. In our example, the types `Int` and `String` are *nominal types* defined in the standard library. They are non-generic and have a fixed layout, so their runtime type metadata is accessed by calling a function, exported by the standard library, that returns the address of a constant symbol.

More details

- Substitution maps: Chapter 6

Specialization. Reification of runtime type metadata and the subsequent indirect manipulation of values incurs a performance penalty. As an alternative, if the definition of a generic function is visible at the call site, the optimizer can generate a *specialization* of the generic function by cloning the definition and applying the substitution map to all types appearing in the function's body. Definitions of generic functions are always visible to the specializer within their defining module. Shared library developers can also opt-in to exporting the body of a function across module boundaries with the `@inlinable` attribute.

More details

- `@inlinable` attribute: Section 2.5

1.2. Nominal Types

For our next example we consider a simple generic struct declaration:

```
struct Pair<T> {
  let first: T
  let second: T

  init(first: T, second: T) {
    self.first = first
    self.second = second
  }
}
```

A struct declaration is an example of a *nominal type declaration*. A *generic nominal type*, like `Pair<Int>` or `Pair<String>` is a reference to a generic nominal type declaration together with a list of generic argument types.

The in-memory layout of a struct value is determined by the interface types of its stored properties. Our `Pair` struct declares two stored properties, `first` and `second`, both with interface type `T`. Thus the layout of a `Pair` depends on the layout of the generic parameter type `T`.

The generic nominal type `Pair<T>`, formed by taking the generic parameter type as the argument, is called the *declared interface type* of `Pair`. The type `Pair<Int>` is a *specialized type* of the declared interface type, `Pair<T>`. We can obtain `Pair<Int>` from `Pair<T>` by applying a substitution map:

$$\texttt{Pair<T>} \otimes \{\texttt{T} \mapsto \texttt{Int}\} = \texttt{Pair<Int>}$$

This "factorization" is our first example of what is in fact an algebraic identity. The *context substitution map* of a generic nominal type is the substitution map formed from its generic arguments. This has the property that applying the context substitution map to the declared interface type recovers the original generic nominal type. Suppose we declare a local variable of type `Pair<Int>`:

```
let twoIntegers: Pair<Int> = ...
```

The compiler must allocate storage on the stack for this value. We take the context substitution map, and apply it to the interface type of each stored property:

$$\texttt{T} \otimes \{\texttt{T} \mapsto \texttt{Int}\} = \texttt{Int}$$

We see that a value of type `Pair<Int>` stores two consecutive values of type `Int`, which gives `Pair<Int>` a size of 16 bytes and alignment of 8 bytes. Since `Pair<Int>` is trivial, the stack allocation does not require any special cleanup once we exit its scope.

Now, we complete our local variable declaration by writing down an initial value expression which calls the constructor:

```
let twoIntegers: Pair<Int> = Pair(first: 1, second: 2)
```

While the value has the type `Pair<Int>` at the call site, inside the constructor the value being initialized is of type `Pair<T>`. We construct the runtime type metadata for `Pair<Int>` by calling the *metadata access function* for `Pair` with the runtime type metadata for `Int` as an argument. The metadata for `Pair<Int>` has two parts:

1. A common prefix present in all runtime type metadata, which includes the total size and alignment of a value, and implementations of the move, copy and destroy operations.
2. A private area specific to the declaration of `Pair` itself, which stores the runtime type metadata for `T`, followed by the *field offset vector*, storing the offset of each stored property of this specialization of `Pair<T>`.

The metadata access function for a generic type takes the metadata for each generic argument, and calculates the offset of each stored property, also obtaining the size and alignment of the entire value. The move, copy and destroy operations of the aggregate type delegate to the corresponding operations in the generic argument metadata. The constructor of `Pair` then uses the runtime type metadata for both `Pair<T>` and `T` to correctly initialize the aggregate value from its two constituent parts.

Structural types, such as function types, tuple types and metatypes, are similar to generic nominal types in that we call a metadata access function to obtain runtime type metadata for them, but this time, the metadata access function is part of the Swift runtime. For example, to construct metadata for the tuple type `(Int, Pair<String>)`, we first call the metadata access function for `Pair` to get `Pair<String>`, then call an entry point in the runtime to obtain `(Int, Pair<String>)`.

More details

- Declarations: Chapter 4
- Context substitution map: Section 6.1
- Structural types: Section 3.2

1.3. Protocols

Our `identity(_:)` and `Pair` declarations both abstract over arbitrary concrete types, but in turn, this limits their generic parameter `T` to the common capabilities shared by all types—the move, copy and destroy operations. By stating *generic requirements*, a generic declaration can impose various restrictions on the concrete types used as generic arguments, which in turn endows its generic parameter types with new capabilities provided by those concrete types.

A *protocol* abstracts over the capabilities of a concrete type. By stating a *conformance requirement* between a generic parameter type and protocol, a generic declaration can require that its generic argument is a concrete type that *conforms* to this protocol:

```
protocol Shape {
  func draw()
}

func drawShapes<S: Shape>(_ shapes: Array<S>) {
  for shape in shapes {
    shape.draw()
  }
}
```

The `drawShapes(_:)` function takes an array of values, all of the same type, which must conform to `Shape`. So far we only encountered the generic signature `<T>`. More generally, a generic signature lists one or more generic parameter types, together with their requirements. The generic signature of `drawShapes(_:)` has the single requirement [`S: Shape`]. We will use the following notation for generic signatures with requirements:

```
<S where S: Shape>
```

The interface type of `drawShapes(_:)` incorporates this generic signature into a generic function type:

```
<S where S: Shape> (Array<S>) -> ()
```

We can also write the `drawShapes(_:)` function to state the conformance requirement with a trailing `where` clause, or we can avoid naming the generic parameter `S` by using an *opaque parameter type* instead:

```
func drawShapes<S>(_ shapes: Array<S>) where S: Shape
func drawShapes(_ shapes: Array<some Shape>)
```

All three forms of `drawShapes(_:)` are ultimately equivalent, because they define the same generic signature, up to the choice of generic parameter name. In general, when there is more than one way to spell the same underlying language construct due to syntax sugar, the semantic objects "desugar" these differences into the same uniform representation.

More details

- Protocols: Section 4.3
- Requirements: Section 4.2
- Generic signatures: Chapter 5

Qualified lookup. Once we have a generic signature, we can type check the body of `drawShapes(_:)`. The `for` loop introduces a local variable "`shape`" of type ⟦`S`⟧ (we re-iterate that the generic parameter type `S` is represented as the archetype ⟦`S`⟧ inside a function body, but the distinction doesn't matter right now). This variable is referenced inside the `for` loop by the *member expression* "`shape.draw`":

```
for shape in shapes {
  shape.draw()
}
```

Our generic signature has the conformance requirement [`S: Shape`], so the caller must provide a replacement type for `S` conforming to `Shape`. We'll return to the caller's side of the equation shortly, but inside the callee, the requirement also tells us that the archetype ⟦`S`⟧ conforms to `Shape`. To resolve the member expression, the type checker performs a *qualified lookup* of the identifier `draw` with a base type of ⟦`S`⟧. A qualified lookup into an archetype checks each protocol the archetype conforms to, so we find and return the `draw()` method of the `Shape` protocol.

How does the compiler generate code for the call `shape.draw()`? Together with the runtime type metadata for `S`, the calling convention for `drawShapes(_:)` passes an additional argument, corresponding to the conformance requirement [`S: Shape`]. This argument is the *witness table* for the conformance. The layout of a witness table is determined by the protocol's members; a witness table for a conformance to `Shape` has a single entry, the implementation of the `draw()` method. To call `shape.draw()`, we load the function pointer from the witness table, and invoke it with the value of `shape`.

More details

- Name lookup: Section 2.1

Conformances. This `Circle` type states a *conformance* to the `Shape` protocol:

```
struct Circle: Shape {
  let radius: Double
  func draw() {...}
}
```

The *conformance checker* ensures that the conforming type `Circle` declares a *witness* for the `draw()` method of `Shape`, and records this fact in a *normal conformance*. We denote this normal conformance by [`Circle: Shape`]. When generating code for the declaration of `Circle`, we also emit the witness table for the normal conformance [`Circle: Shape`]. This witness table contains a pointer to the implementation of `Circle.draw()`.

Now, let's call `drawShapes(_:)` with an array of circles and look at the substitution map for the call:

```
drawShapes([Circle(radius: 1), Circle(radius: 2)])
```

When the callee's generic signature has conformance requirements, the substitution map must store a conformance for each conformance requirement. This is the "proof" that the concrete replacement type actually conforms to the protocol, as required. We denote a substitution map with conformances as follows:

$$\{\texttt{S} \mapsto \texttt{Circle};\ [\texttt{S: Shape}] \mapsto [\texttt{Circle: Shape}]\}$$

To find the normal conformance, the type checker performs a *global conformance lookup* with the concrete type and protocol:

$$[\texttt{Shape}] \otimes \texttt{Circle} = [\texttt{Circle: Shape}]$$

When generating code for the call to `drawShapes(_:)`, we visit each entry in the substitution map, emitting a reference to runtime type metadata for each replacement type, and a reference to the witness table for each conformance. In our case, we pass the runtime type metadata for `Circle` and witness table for [`Circle: Shape`].

More details

- Conformances: Chapter 7
- Conformance lookup: Section 7.1

Existential types. Note that `drawShapes(_:)` operates on a *homogeneous* array of shapes. While the array contains an arbitrary number of elements, `drawShapes(_:)` only receives a single runtime type metadata for `S`, and one witness table for the conformance requirement [`S: Shape`], which together describe the behavior of each element in the array. If we instead want a function taking a *heterogeneous* array of shapes, we can use an *existential type* as the element type of our array:

```
func drawShapes(_ shapes: Array<any Shape>) {
  for shape in shapes {
    shape.draw()
  }
}
```

This function uses the `Shape` protocol in a new way. The existential type `any Shape` is a container for a value of some concrete type, together with runtime type metadata, and a witness table describing the conformance. This container stores small values inline, otherwise it points to a heap-allocated box. Observe the difference between the type `Array<S>` from the previous variant of `drawShapes(_:)`, and `Array<any Shape>` here. Every element of the latter has its own runtime type metadata and witness table, so we can mix multiple types of shapes in one array. In the implementation, existential types are built on top of the core primitives of the generics system.

More details

- Existential types: Chapter 14

1.4. Associated Types

The standard library's `IteratorProtocol` declares an associated type. This allows us to abstract over iterators whose element type depends on the conformance:

```
protocol IteratorProtocol {
  associatedtype Element
  mutating func next() -> Element?
}
```

A conforming type must declare a member type named `Element`, and a `next()` method returning an optional value of this type. This member type, which can be a type alias or nominal type, is the *type witness* for the associated type `Element`.

We declare a `Nat` type conforming to `IteratorProtocol` with an `Element` type of `Int`, for generating an infinite stream of consecutive natural numbers:

```
struct Nat: IteratorProtocol {
  typealias Element = Int
  var x = 0

  mutating func next() -> Int? {
    defer { x += 1 }
    return x
  }
}
```

We say that `Int` is the *type witness* for `[IteratorProtocol]Element` in the conformance `[Nat: IteratorProtocol]`. We can express this in our type substitution algebra, using the type witness *projection* operation on normal conformances:

$$\pi(\texttt{[IteratorProtocol]Element}) \otimes \texttt{[Nat: IteratorProtocol]} = \texttt{Int}$$

In this case, we could have omitted the declaration of the type alias, relying on *associated type inference* to deduce the type witness. Also, more accurately, the type witness here is the `Element` *type alias type*, whose underlying type is `Int`. A type alias type is an example of a *sugared type*, equivalent semantically to its underlying type. The *optional type* `Int?` is another sugared type, equivalent to `Optional<Int>`.

More details

- Type witnesses: Section 7.3

Dependent member types. This function reads a pair of elements from an iterator:

```
func readTwo<I: IteratorProtocol>(_ iter: inout I) -> Pair<I.Element> {
  return Pair(first: iter.next()!, second: iter.next()!)
}
```

The return type is the generic nominal type `Pair<I.Element>`, obtained by applying the declaration of `Pair` to the generic argument `I.Element`. The generic argument is a *dependent member type*, built from the base type `I` together with the associated type declaration `[IteratorProtocol]Element`. This dependent member type represents the type witness in the conformance `[I: IteratorProtocol]`.

Suppose we call `readTwo(_:)` with a value of type `Nat`:

```
var iter = Nat()
print(readTwo(&iter))
```

The substitution map for the call stores the replacement type `Nat` and the conformance of `Nat` to `IteratorProtocol`. We'll call this substitution map Σ:

$$\Sigma := \left\{ \begin{array}{l} \texttt{I} \mapsto \texttt{Nat}; \\ [\texttt{I: IteratorProtocol}] \mapsto [\texttt{Nat: IteratorProtocol}] \end{array} \right\}$$

The type of the call expression is then `Pair<I.Element>` $\otimes \Sigma$. Applying a substitution map to a generic nominal type recursively substitutes the generic arguments. Since the dependent member type `I.Element` abstracts over the type witness in the conformance, we could guess `I.Element` $\otimes \Sigma =$ `Int`. We will eventually understand the next equation, to relate dependent member type substitution with type witness projection:

$$\begin{aligned} &\texttt{I.Element} \otimes \Sigma \\ &\quad = \pi(\texttt{[IteratorProtocol]Element}) \otimes [\texttt{I: IteratorProtocol}] \otimes \Sigma \\ &\quad = \pi(\texttt{[IteratorProtocol]Element}) \otimes [\texttt{Nat: IteratorProtocol}] \\ &\quad = \texttt{Int} \end{aligned}$$

We can finally say that `Pair<I.Element>` $\otimes \Sigma =$ `Pair<Int>` is the return type of our call to `readTwo(_:)`.

More details

- Dependent member type substitution: Section 7.4, Chapter 12

Bound and unbound. We now briefly introduce a concept that will later become important in our study of type substitution. If we need to make the associated type declaration explicit, we use the notation `I.[IteratorProtocol]Element`, despite this not being valid language syntax. This is the *bound* form of a dependent member type. To transform the syntactic type representation `I.Element` into a bound dependent member type, type resolution queries qualified lookup on the base type `I`, which is known to conform to `IteratorProtocol`; thus, the lookup finds the associated type declaration `Element`. For our current purposes, it is more convenient to use the *unbound* notation for a dependent member type, written in the source language style `I.Element`.

More details

- Member type representations: Section 9.1

Type parameters. Generic parameter types and dependent member types are the two kinds of *type parameters*. The generic signature of `readTwo(_:)` defines two type parameters, `I` and `I.Element`.

As with generic parameter types, dependent member types map to archetypes in the body of a generic function. We can reveal a little more about the structure of archetypes now, and say that an archetype packages a type parameter together with a generic signature. While a type parameter is just a *name* which can only be understood in relation to some generic signature, an archetype inherently "knows" what requirements it is subject to.

More details

- Type parameters: Section 3.1
- Primary archetypes: Section 8.2

Code generation. Inside the body of `readTwo(_:)`, the call expression `iter.next()!` has the type ⟦`I.Element`⟧`?`, which is force-unwrapped to yield the type ⟦`I.Element`⟧. To manipulate a value of this type abstractly, we need its runtime type metadata.

We recover the runtime type metadata for an associated type from the witness table at run time, in the same way that dependent member type substitution projects a type witness from a conformance at compile time.

A witness table for a conformance to `IteratorProtocol` consists of a metadata access function to witness the `Element` associated type, and a function pointer to witness the `next()` method. The witness table for [`Nat: IteratorProtocol`] references the runtime type metadata for `Int`, defined by the standard library.

Same-type requirements. To introduce another fundamental requirement kind, we compose `Pair` and `IteratorProtocol` in a new way, writing a function that takes two iterators and reads an element from each one:

```
func readTwoParallel<I, J>(_ i: I, _ j: J) -> Pair<I.Element>
    where I: IteratorProtocol, J: IteratorProtocol,
          I.Element == J.Element {
  return Pair(first: i.next()!, second: j.next()!)
}
```

The generic signature of our `readTwoParallel(_:)` states the *same-type requirement* [`I.Element == J.Element`]:

```
<I, J where I: IteratorProtocol, J: IteratorProtocol,
            I.Element == J.Element>
```

This generic signature defines four type parameters, `I`, `J`, `I.Element`, and `J.Element`, where the final two abstract over the same concrete type, forming an *equivalence class*. It is often convenient to refer to an entire equivalence class by a representative type parameter. To make this choice in a deterministic fashion, we *order* type parameters and say the smallest type parameter inside an equivalence class is a *reduced type*. In our example, `I.Element` is the reduced type of `J.Element`.

In the presence of same-type requirements, an archetype represents a reduced type parameter (and thus an entire equivalence class of type parameters). In the body of `readTwoParallel(_:)`, the expressions `i.next()!` and `j.next()!` both return the type ⟦`I.Element`⟧, and the call to the `Pair` constructor is made with this substitution map:

$$\{\texttt{T} \mapsto ⟦\texttt{I.Element}⟧\}$$

More details

- Reduced types: Section 5.4
- The type parameter graph: Section 8.3

Checking generic arguments. When type checking a call to `readTwoParallel(_:)`, we must ensure the same-type requirement is satisfied. Suppose we define two new iterator types, `BabyNames` and `CatNames`, both witnessing the `Element` associated type with `String`, and we call `readTwoParallel(_:)` with these types:

```
var i = BabyNames()
var j = CatNames()
print(readTwoParallel(&i, &j))
```

We're making the call with this substitution map:

$$\Sigma := \left\{ \begin{array}{l} \texttt{I} \mapsto \texttt{BabyNames} \\ \texttt{J} \mapsto \texttt{CatNames}; \\ [\texttt{I: IteratorProtocol}] \mapsto [\texttt{BabyNames: IteratorProtocol}] \\ [\texttt{J: IteratorProtocol}] \mapsto [\texttt{CatNames: IteratorProtocol}] \end{array} \right\}$$

The type checker applies Σ to both sides of the same-type requirement, which gives us $\texttt{I.Element} \otimes \Sigma = \texttt{String}$ and $\texttt{J.Element} \otimes \Sigma = \texttt{String}$. The result is `String` in both cases, so we get the following *substituted requirement*:

$$[\texttt{I.Element == J.Element}] \otimes \Sigma = [\texttt{String == String}]$$

The substituted requirement is satisfied so the code is well-typed, and we see that the call returns `Pair<String>`. Suppose instead we had substituted `I` with `Nat`:

```
var i = Nat()
var j = CatNames()
print(readTwoParallel(&i, &j))  // error
```

In this case, we get the substituted requirement [`Int == String`] which is unsatisfied, so the call is malformed and the type checker must diagnose an error.

More details

- Checking generic arguments: Section 9.3

1.5. Associated Requirements

Protocols can impose *associated requirements* on their associated types. A conforming type must then satisfy these requirements. This capability is what gives Swift generics much of their distinctive flavor. Perhaps the simplest example is the `Sequence` protocol in the standard library, which abstracts over types that can produce a fresh iterator on demand:

```
protocol Sequence {
  associatedtype Element
  associatedtype Iterator: IteratorProtocol
    where Element == Iterator.Element

  func makeIterator() -> Iterator
}
```

This protocol states two associated requirements:

- The conformance requirement [`Self.Iterator: IteratorProtocol`], stated using the sugared form, as a constraint type in the inheritance clause of the `Iterator` associated type.
- The same-type requirement [`Self.Element == Self.Iterator.Element`], which we state in a trailing `where` clause attached to the associated type.

Associated requirements are like the requirements in a generic signature, except they are rooted in the protocol `Self` type. Once again, there are multiple equivalent syntactic forms for stating them. For example, we could write out the conformance requirement explicitly, and the above `where` clause could be attached to the protocol itself for the same semantic effect.

The associated requirements of a protocol are recorded by the protocol's *requirement signature*. The `Sequence` protocol has the following requirement signature:

```
<Self where Self.Element == Self.Iterator.Element,
            Self.Iterator: IteratorProtocol>
```

Now consider this generic signature, and call it G:

```
<T, U where T: Sequence, U: Sequence,
            T.Element == U.Element>
```

We can informally describe G by looking at its equivalence classes:

- `T`, which conforms to `Sequence`.
- `U`, which conforms to `Sequence`.
- `T.Element`, `U.Element`, `T.Iterator.Element` and `U.Iterator.Element` are all in one equivalence class.
- `T.Iterator`, which conforms to `IteratorProtocol`.
- `U.Iterator`, which conforms to `IteratorProtocol`.

To make this kind of analysis precise, we will develop a theory of *derived requirements* to reason about requirements that are not explicitly stated, but are logical consequences of other requirements. The following are some interesting derived requirements of G:

[`T.Iterator: IteratorProtocol`]

[`U.Iterator: IteratorProtocol`]

[`T.Element == T.Iterator.Element`]

[`U.Element == U.Iterator.Element`]

[`T.Iterator.Element == U.Iterator.Element`]

At this point, it's worth clarifying that type parameters have a recursive structure; the base type of `U.Iterator.Element` is another dependent member type, `U.Iterator`. The theory of derived requirements will also describe the *valid type parameters* of a generic signature.

Conformances. A normal conformance stores the *associated conformance* for each associated conformance requirement of its protocol. In the runtime representation, there is a corresponding entry in the witness table for each associated conformance requirement. A witness table for `Sequence` has four entries:

1. A metadata access function to witness the `Element` associated type.
2. A metadata access function to witness the `Iterator` associated type.
3. A *witness table access function* to witness the associated conformance requirement [`Self.Iterator: IteratorProtocol`].
4. A function pointer to witness the `makeIterator()` protocol method.

Protocol inheritance is the special case of an associated conformance requirement with a subject type of `Self`. The standard library `Collection` protocol inherits from `Sequence`, so the associated conformance requirement [`Self: Sequence`] appears in the requirement signature of `Collection`. Starting from a conformance [`Array<Int>: Collection`], we can get at the conformance to `Sequence` via *associated conformance projection*:

$$\pi(\texttt{Self: Sequence}) \otimes [\texttt{Array<Int>: Collection}] = [\texttt{Array<Int>: Sequence}]$$

Similarly, at runtime, starting from a witness table for a conformance to `Collection`, we can recover a witness table for a conformance to `Sequence`, and thus all of the metadata described above. We will take a closer look at the other associated requirements of the `Collection` protocol in due time. This will lead us into the topic of *recursive* associated conformance requirements, which as we show, enable the type substitution algebra to encode any computable function.

More details

- Requirement signatures: Section 5.1
- Derived requirements: Section 5.2
- Associated conformances: Section 7.5
- Recursive conformances: Section 12.3

1.6. Related Work

A calling convention centered on a runtime representation of types was explored in a 1996 paper [14]. Swift protocols are similar in spirit to Haskell type classes, described in [15], and subsequently [16] and [17]. Swift witness tables follow the "dictionary passing" implementation strategy for type classes. Two papers subsequently introduced type classes with associated types, but without associated requirements. Using our Swift lingo, the first paper defined a formal model for associated types witnessed by nested nominal types [18]. In this model, every type witness is a distinct concrete type. To translate their example into Swift:

```
protocol ArrayElem {
  associatedtype Array
  func index(_: Array, _: Int) -> Self
}
```

The second paper described *associated type synonyms* [19]. This is the special case of an associated type witnessed by a type alias in Swift. Again, we translate their example:

```
protocol Collects {
  associatedtype Elem
  static var empty: Self { get }
  func insert(_: Elem) -> Self
  func toList() -> [Elem]
}
```

Other relevant papers from the Haskell world include [20] and [21].

C++. While C++ templates are synonymous with "generic programming" to many programmers, C++ is somewhat unusual compared to most languages with parametric polymorphism, because templates are fundamentally syntactic in nature. Compiling a template declaration only does some minimal amount of semantic analysis, with most type checking deferred to until *after* template expansion. There is no formal notion of requirements on template parameters, so whether template expansion succeeds or fails at a given expansion point depends on how the template's body uses the substituted template parameters.

The inherent flexibility of C++ templates enables some advanced metaprogramming techniques [22]. On the other hand, a template declaration's body must be visible from each expansion point, so this model is fundamentally at odds with separate compilation. Undesirable consequences include libraries where large parts must be implemented in header files, and cryptic error messages on template expansion failure.

Swift's generics model was in many ways inspired by "C++0x concepts," a proposal to extend the C++ template metaprogramming model with *concepts*, a form of type classes with associated types ([23], [24]). Concepts could declare their own associated requirements, but the full ramifications were perhaps not yet apparent to the authors when they wrote this remark:

> *"Concepts often include requirements on associated types. For example, a container's associated iterator* `A` *would be required to model the* `Iterator` *concept. This form of concept composition is slightly different from refinement but close enough that we do not wish to clutter our presentation [. . .]"*

Rust. Rust generics are separately type checked, but Rust does not define a calling convention for unspecialized generic code, so there is no separate compilation. Instead, the implementation of a generic function is *specialized*, or *monomorphized*, for every unique set of generic arguments.

Rust's *traits* are similar to Swift's protocols; traits can declare associated types and associated conformance requirements. Rust generics also allow some kinds of abstraction not supported by Swift, such as lifetime variables, generic associated types [25], and const generics [26]. On the flip side, Rust does not allow same-type requirements to be stated in full generality [27]. Instead, trait bounds can constrain associated types with a syntactic form resembling Swift's parameterized protocol types (Section 4.3), but we will show in Example 19.10 that Swift's same-type requirements are more general.

Rust's "`where` clause elaboration" is more limited than Swift's derived requirements formalism, and associated requirements sometimes need to be re-stated in the `where` clause of a generic declaration [28]. An early attempt at formalizing Rust traits appears in a PhD dissertation from 2015 [29]. A more recent effort is "Chalk," an implementation of a Prolog-like solver based on Horn clauses [30].

Java. Java generics are separately type checked and compiled. In Java, only reference types can be used as generic arguments; primitive value types must be boxed first. Generic argument types are not reified at runtime, and values of generic parameter type are always represented as object pointers. This avoids the complexity of dependent layout, but comes at the cost of more runtime type checks and heap allocation. Java generics are based on existential types and also support *variance*, a subtyping relation between different instantiations of the same generic type, defined in terms of the subtype relation on corresponding generic arguments [31].

Hylo. Hylo is a research language with a focus on mutable value semantics [32]. Hylo's generic programming capabilities are similar to Swift and Rust. The Hylo compiler implementation incorporates some ideas from this book, using the theory of string rewriting to reason about generic requirements [33].

2. Compilation Model

Most developers interact with the Swift compiler through the Xcode build system or the Swift package manager, but for simplicity's sake we're just going to consider direct invocation of `swiftc` from the command line. The `swiftc` command runs the *Swift driver*, which invokes the *Swift frontend* program to actually compile each source file; then, depending on the usage mode, the driver runs additional tools, such as the linker, to produce the final build artifact. Most of this book concerns the frontend, but we will briefly review the operation of the driver now.

In the Swift module system, all source files in a module must be built together. The Swift driver takes a list of source files on the command line, which become the *main module* being built. By default, the Swift driver generates an executable from the main module:

```
$ swiftc main.swift other.swift stuff.swift
```

Executables must define a *main function*, which is the entry point invoked when the executable is run. There are three mechanisms for doing so:

1. If only a single source file was provided, this file becomes the *main source file* of the module. If there are multiple source files and one of them is named `main.swift`, then this file becomes the main source file. The main source file is special, in that it can contain statements at the top level, outside of a function body. Top-level statements are collected into *top-level code declarations*, and the frontend generates a main function which executes each top-level code declaration in source order. Source files other than the main source file cannot contain statements at the top level.

2. In the absence of a main source file, a struct, enum or class declaration can instead be annotated with the `@main` attribute, in which case the declaration must contain a static method named `main()`. This method becomes the main entry point. This attribute was introduced in Swift 5.3 [34].

3. The older `@NSApplicationMain` and `@UIApplicationMain` attributes, deprecated since Swift 5.10 [35], provided a similar mechanism specific to Apple platforms. Attaching one of these attributes to a class conforming to `NSApplicationMain` or `UIApplicationMain`, respectively, will generate a main entry point which calls the `NSApplicationMain()` or `UIApplicationMain()` system framework function.

Invoking the driver with the `-emit-library` and `-emit-module` flags instructs it to generate a shared library, together with the serialized module file consumed by the compiler when importing the library (Section 2.5):

```
$ swiftc algorithm.swift utils.swift -module-name SudokuSolver
      -emit-library -emit-module
```

Frontend jobs. The Swift frontend itself is single-threaded, but the driver can benefit from multi-core concurrency by running multiple frontend jobs in parallel. Each frontend job compiles one or more source files; these are the *primary source files* of the frontend job. All non-primary source files are the *secondary source files* of the frontend job. The assignment of primary source files to each frontend job is determined by the *compilation mode*:

- The `-wmo` driver flag selects *whole module mode*, typically used for release builds. In this mode, the driver schedules a single frontend job. The primary files of this job are all the source files in the main module, and there are no secondary files. In whole module mode, the frontend is able to perform more aggressive optimization across source file boundaries, hence its usage for release builds.

- The `-disable-batch-mode` driver flag selects *single file mode*, with one frontend job per source file. In this mode, each frontend job has a single primary file, with all other files being secondary files. Single file mode was the default for debug builds until Swift 4.1, however these days it is only used for testing the compiler.

  Single file mode incurs inexorable overhead in the form of duplicated work between frontend jobs; if two source files reference the same declaration in a third source file, the two frontend jobs will both need to parse and type check this declaration as there is no caching across frontend jobs (the next two sections detail how the frontend deals with secondary files, with delayed parsing and the request evaluator respectively).

- The `-enable-batch-mode` driver flag selects *batch mode*, which is a happy medium between whole module and single file mode. In batch mode, the list of source files is partitioned into fixed-size batches, up to the maximum batch size. The source files in each batch become the primary files of each frontend job.

  By compiling multiple primary files in a single frontend job, batch mode amortizes the cost of parsing and type checking work performed on secondary files. At the same time, it still schedules multiple frontend jobs for parallelism on multi-core systems. Batch mode was first introduced in Swift 4.2, and is now the default for debug builds.

Note that each source file is a primary source file of exactly one frontend job, and within a single frontend job, the primary files and secondary files together form the full list of source files in the module. A single source file is therefore the minimum unit of parallelism. By default, the number of concurrent frontend jobs is determined by the number of CPU cores; this can be overridden with the `-j` driver flag. If there are more frontend jobs than can be run simultaneously, the driver queues them and kicks them off as other frontend jobs complete. In batch mode and single file mode, the driver can also perform an *incremental build* by re-using the result of previous compilations, providing an additional compile-time speedup. Incremental builds are described in Section 2.3.

The `-###` driver flag performs a "dry run" which prints all commands to run without actually doing anything. In the below example, the driver schedules three frontend jobs, with each job having a single primary source file and two secondary files. The final command is the linker invocation, which combines the output of each frontend job into our binary executable.

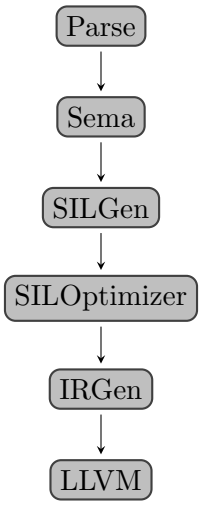
```
$ swiftc m.swift v.swift c.swift -###
swift-frontend -frontend -c -primary-file m.swift v.swift c.swift ...
swift-frontend -frontend -c m.swift -primary-file v.swift c.swift ...
swift-frontend -frontend -c m.swift v.swift -primary-file c.swift ...
ld m.o v.o c.o -o main
```

Compilation pipeline. Figure 2.1 shows a high-level view of the Swift frontend; this resembles the classic multi-pass compiler design, described in [2] or [3] for example:

- **Parse:** Source files are parsed, building the abstract syntax tree.
- **Sema:** Semantic analysis is performed, producing a type-checked syntax tree. (We'll see shortly the first two stages are not completely sequential.)
- **SILGen:** The type-checked syntax tree is lowered to "raw SIL." SIL is the Swift Intermediate Language, described in [36] and [37].
- **SILOptimizer:** The raw SIL is transformed into "canonical SIL" by a series of *mandatory passes*, which analyze the control flow graph and emit diagnostics; for example, *definite initialization* ensures that all storage locations are initialized.

  When the `-O` command line flag is specified, the canonical SIL is optimized by a series of *performance passes* to improve run-time performance and code size.
- **IRGen:** The optimized SIL is then transformed into LLVM IR.
- **LLVM:** Finally, the LLVM IR is handed off to LLVM, which performs various lower level optimizations before generating machine code. (LLVM is, of course, the project formerly known as the "Low Level Virtual Machine [38].")

Figure 2.1.: The compilation pipeline



Debugging flags. Various command-line flags are provided to run the pipeline until a certain phase, and dump the output of that phase to the terminal (or some other file, in conjunction with the `-o` flag). These are useful for debugging the compiler:

- `-dump-parse` runs only the parser, and prints the syntax tree as an s-expression.[1]
- `-dump-ast` runs only the parser and Sema, and prints the type-checked syntax tree as an s-expression.
- `-print-ast` prints the type-checked syntax tree in a form that approximates what was written in source code. This is useful for getting a sense of what declarations the compiler synthesized, for example for derived conformances to protocols like `Equatable`.
- `-emit-silgen` runs only Sema and SILGen, and prints the raw SIL output by SILGen.
- `-emit-sil` prints the canonical SIL output by the SIL optimizer. To see the output of the performance pipeline, also pass `-O`.
- `-emit-ir` prints the LLVM IR output by IRGen.
- `-S` prints the assembly output by LLVM.

[1] The term comes from Lisp. An s-expression represents a tree structure as nested parenthesized lists; e.g. `(a (b c) d)` is a node with three children `a`, `(b c)` and `d`, and `(b c)` has two children `b` and `c`.

Each pipeline phase can emit warnings and errors, collectively known as *diagnostics*. The parser attempts to recover from errors; the presence of parse errors does not prevent Sema from running. On the other hand, if Sema emits errors, compilation stops; SILGen does not attempt to lower an invalid abstract syntax tree to SIL (but SILGen can emit its own diagnostics, including those that result from lazy type checking of declarations in secondary files).

The compilation pipeline will vary slightly depending on what the driver and frontend were asked to produce. When the frontend is instructed to emit a serialized module file only, and not an object file, compilation stops after the SIL optimizer. When generating a textual interface file or TBD file, compilation stops after Sema. (Textual interfaces are discussed in Section 2.5. A TBD file is a list of symbols in a shared library, which can be consumed by the linker and is faster to generate than the shared library itself; we're not going to talk about them here.)

Frontend flags. The flags we listed above for dumping various stages of compiler output are understood by both the driver and the frontend; the driver passes them down to the frontend. Various other flags used for compiler development and debugging and only known to the frontend. If the driver is invoked with the `-frontend` flag as the first command line flag, then instead of scheduling frontend jobs, the driver spawns a single frontend job, passing it the rest of the command line without further processing:

```
$ swiftc -frontend -typecheck -primary-file a.swift b.swift
```

Another mechanism for passing flags to the frontend is the `-Xfrontend` flag. When this flag appears in a command-line invocation of the driver, the driver schedules job as usual, but the command line argument that comes immediately after is passed directly to each frontend job:

```
$ swiftc a.swift b.swift -Xfrontend -dump-requirement-machine
```

2.1. Name Lookup

Name lookup is the process of resolving identifiers to declarations. The Swift compiler does not have a distinct "name binding" phase; instead, name lookup is queried from various points in the frontend process. Broadly speaking, there are two kinds of name lookup: *unqualified lookup* and *qualified lookup*. An unqualified lookup resolves a single identifier "`foo`", while qualified lookup resolves an identifier "`bar`" relative to a base, such as a member reference expression "`foo.bar`". There are also three important variants of these two fundamental kinds, for looking up top-level declarations in other modules, resolving operators, and performing dynamic lookups of Objective-C methods.

Unqualified lookup. An unqualified lookup is always performed relative to the source location where the identifier actually appears. The source location may either be in a primary or secondary file.

Unqualified lookup consults the source file's *scope tree*, which is constructed by walking the source file's abstract syntax tree. The root scope is the source file itself. Each scope has an associated source range, and zero or more child scopes; each child scope's source range must be a subrange of the source range of its parent, and the source ranges of sibling scopes are disjoint. Each scope introduces zero or more *variable bindings*.

Unqualified lookup first finds the innermost scope containing the source location, and proceeds to walk the scope tree up to the root, searching each parent node for bindings named by the given identifier. If the lookup reaches the root node, a *top-level lookup* is performed next. This will look for top-level declarations named by the given identifier, first in all source files of the main module, followed by all imported modules.

The `-dump-scope-maps` frontend flag dumps the scope map for each source file in the main module. For example, with this program:

```
func id<T>(_ t: T) -> T {
  return t
}
```

We get the scope map below:

```
ASTSourceFileScope 0x14c131908, [1:1 - 5:1] 'id.swift'
`-AbstractFunctionDeclScope 0x14c1392c0, [1:1 - 4:1] 'id(_:)'
  `-GenericParamScope 0x14c139118, [1:1 - 4:1] param 0 'T'
    |-ParameterListScope 0x14c139238, [1:11 - 1:18]
    `-FunctionBodyScope 0x14c1392c0, [1:25 - 4:1]
      `-BraceStmtScope 0x14c139510, [1:25 - 4:1]
        `-PatternEntryDeclScope 0x14c139450, [2:7 - 4:1] entry 0 'x'
          `-PatternEntryInitializerScope 0x14c139450, [2:11 - 2:11] entry 0 'x'
```

Unqualified lookup is an important part of type resolution (Section 9.1).

Qualified lookup. A qualified lookup searches in a list of type declarations for a member with the given name. Qualified lookup recursively visits the conformed protocols of each struct, enum and class declaration, and the superclass of each class declaration. If the found member was from a protocol or superclass, we apply a substitution map, which will be described in Section 9.2. The primitive operation, which searches inside a single type declaration and its extensions only, is called *direct lookup* (Section 10.2).

Module lookup. A qualified lookup where the base is a module declaration searches for a top-level declaration in the given module and any other modules that it re-exports via "`@_exported import`".

Dynamic lookup. A qualified lookup whose base is the `AnyObject` type implements the legacy Objective-C behavior of a message send to `id`, which can invoke any method defined in any Objective-C class or protocol. In Swift, the so-called *dynamic lookup* searches a global lookup table constructed to contain all `@objc` members of all classes and protocols:

- Any class can contain `@objc` members, and the attribute can either be explicitly stated, or inferred if the method overrides an `@objc` method from the superclass.
- Protocol members are `@objc` only if the protocol itself is `@objc`.

Operator lookup. Operator symbols are declared at the top level of a module by *operator declarations*. Operator declarations have a fixity (prefix, infix, or postfix), and infix operators also have a *precedence group*. Precedence groups are partially ordered with respect to other precedence groups. Standard operators like `+` and `*` and their precedence groups are thus defined in the standard library, rather than being built-in to the language itself.

The parser parses an arithmetic expression like `2 + 3 * 6` into a flat list of nodes and operator symbols, called a *sequence expression*. The parser does not know the precedence, fixity or associativity of the `+` and `*` operators. Indeed, it does not know that they exist at all. The *pre-check* pass of the expression type checker looks up operator symbols, in this case `+` and `*`, and transforms sequence expressions into the more familiar nested tree form, according to the operator's fixity, precedence and associativity.

Operator symbols do not themselves have an implementation; they are just names. An operator symbol can be used as the name of a function implementing the operator on a specific type (for prefix and postfix operators) or a specific pair of types (for infix operators). Operator functions can be declared either at the top level, or as a member of a type. As far as a name lookup is concerned, the interesting thing about operator functions is that they are visible globally, even when declared inside of a type. Operator functions are found by consulting the operator lookup table, which contains top-level operator functions as well as member operator functions of all declared types.

When the compiler type checks the expression `2 + 3 * 6`, it must pick two specific operator functions for `+` and `*` among all the possibilities in order to make this expression type check. In this case, the overloads for `Int` are chosen, because `Int` is the default literal type for the literals `2`, `3` and `6`.

Listing 2.1 shows the definition of some custom operators and precedence groups. Note that the overload of `++` inside struct `Chicken` returns `Int`, and the overload of `++` inside struct `Sausage` returns `Bool`. The closure value stored in `fn` applies `++` to two anonymous closure parameters, `$0` and `$1`. While they do not have declared types, by simply coercing the *return type* to `Bool`, we are able to unambiguously pick the overload of `++` declared in `Sausage`. (Whether this is good style is left to the reader to judge.)

Listing 2.1.: Operator lookup in action

```
prefix operator <&>
infix operator ++: MyPrecedence
infix operator **: MyPrecedence

precedencegroup MyPrecedence {
  associativity: right
  higherThan: AdditionPrecedence
}

// Member operator examples
struct Chicken {
  static prefix func <&>(x: Chicken) {}
  static func ++(lhs: Chicken, rhs: Chicken) -> Int {}
}

struct Sausage {
  static func ++(lhs: Sausage, rhs: Sausage) -> Bool {}
}

// Top-level operator example
func **(lhs: Sausage, rhs: Sausage) -> Sausage {}

// Global operator lookup finds Sausage.++
// 'fn' has type (Sausage, Sausage) -> Bool
let fn = { ($0 ++ $1) as Bool }
```

Initially, infix operators defined their precedence as an integer value; Swift 3 introduced named precedence groups [39]. The global lookup for operator functions dates back to when all operator functions were declared at the top level. Swift 3 also introduced the ability to declare operator functions as members of types, but the global lookup behavior was retained [40].

2.2. Delayed Parsing

The "compilation pipeline" model as described is an over-simplification of the actual state of affairs. Ultimately, each frontend job only needs to generate machine code from the declarations in its primary files, so all stages from SILGen onward operate on the frontend job's primary files only. The situation while parsing and type checking is more subtle, because name lookup must find declarations in other source files, even secondary files. This requires having the abstract syntax tree for secondary files as well. However, it would be inefficient if every frontend job was required to fully parse all secondary files, because the time spent in the parser would be proportional to the number of frontend jobs multiplied by the number of source files, negating the benefits of parallelism.

The *delayed parsing* optimization solves this dilemma. When parsing a secondary file for the first time, the parser does not construct syntax tree nodes for the bodies of top-level types, extensions and functions. Instead, it operates in a high-speed mode where comments are skipped and pairs of braces are matched, but very little other work is performed. This outputs a "skeleton" representation of each secondary file. (In whole module mode, there is no delayed parsing. There are no secondary files, and delayed parsing of declarations in primary files is pointless, since they are always needed for type checking and code generation anyway.) If the body of a type or extension declaration from a secondary file is needed later—for example, if type checking of an expression in a primary file performs a name lookup into this declaration—the source range of the declaration is parsed again, this time building the full syntax tree. While it is possible to construct a pathological program where every source file triggers delayed parsing of all declarations in every other file, this does not occur in practice.

For delayed parsing to work, the skipped members of types and extensions must have no observable effect on compilation. This is always true with two exceptions: operator lookup, and dynamic lookup.

Operator lookup. As explained in the previous section, operator functions are visible globally, even when declared as a method of a type. To deal with this, the parser looks for the keyword "`func`" followed by an operator symbol when skipping a type or extension body in a secondary file. The first time an operator lookup is performed, the bodies of all types and extensions that contain operator functions are parsed again. Most types and extensions do not define operator functions, so this occurs rarely in practice.

Dynamic lookup. The situation with dynamic lookup is similar, since a method call on a value of type `AnyObject` must consult a global lookup table constructed from `@objc` members of classes, and the (implicitly `@objc`) members of `@objc` protocols. Unlike operator functions, classes and `@objc` protocols are quite common in Swift programs, however `AnyObject` lookup itself is rarely used. The first time a frontend job encounters a dynamic `AnyObject` method call, all class bodies flagged as potentially containing `@objc` methods are eagerly parsed.

There's actually one more complication here. Classes can be nested inside of other types, whose bodies are skipped if they appear in a secondary file. This is resolved with the same trick as operator lookup. When skipping the body of a type, the parser looks for occurrences of the "`class`" keyword. If the body contains this keyword, this type is parsed and its members visited recursively when building the `AnyObject` global lookup table.

Most Swift programs, even those making heavy use of Objective-C interoperability, do not contain a dynamic `AnyObject` method call in every source file, so delayed parsing remains effective.

Example 2.1. Listing 2.2 shows an example of this behavior. This program consists of three files. Suppose that the driver kicks off three frontend jobs, with a single primary file for each frontend job.

Listing 2.2.: Delayed parsing with `AnyObject` lookup

```
// a.swift
func f(x: AnyObject) {
  x.foo()
}
```

```
// b.swift
func g() {
  f()
}
```

```
// c.swift
struct Outer {
  class Inner {
    @objc func foo() {}
  }
}
```

The frontend jobs each do the following:

- The frontend job with the primary file `a.swift` will parse `b.swift` and `c.swift` as secondary files. The body of `g()` in `b.swift` is skipped. The parser also skips the body of `Outer`, but records that it contains the `class` keyword. The function `f()` in `a.swift` contains a dynamic `AnyObject` call, so this frontend job will construct the global lookup table, triggering parsing of `Outer` and `Inner` in `c.swift`.
- The frontend job with the primary file `b.swift` will parse `a.swift` and `c.swift` as secondary files. This primary file does not reference anything from `c.swift` at all, so `Outer` remains unparsed in this frontend job. Type checking the call to `f()` from `g()` also does not require parsing the *body* of `f()`.
- The frontend job with the primary file `c.swift` will parse `a.swift` and `b.swift` as secondary files, skipping the body of `f()` and `g()`.

2.3. Request Evaluator

The *request evaluator* generalizes the idea behind delayed parsing to all of type checking. As with parsing, the classic compiler design, where a single semantic analysis pass walks declarations in source order, is not well-suited for Swift:

- Declarations may be written in any order within a Swift source file, without being forward declared (unlike Pascal or C). Expressions and type annotations can also reference declarations in other source files without restriction. Finally, certain kinds of circular references are permitted.

  In particular, this means that within a single frontend job, an entity in a primary file may reference a declaration that has not yet been type checked, or is in the process of being type checked.

- Ordering issues aside, there is also the potential overhead of duplicated work across frontend jobs. Every time a frontend job type checks a declaration in a secondary file, some of the benefit of parallelism is lost, since this secondary file is necessarily the primary file of some other frontend job, and the same declaration must therefore be type checked again in the other job.

  For this reason, we want to minimize time spent type checking declarations in secondary files.

Thus, the work of type checking is split up into small, fine-grained *requests* which are evaluated on demand, instead of sequentially. There is still a semantic analysis pass that visits the declarations of each primary file in source order, but it merely kicks off requests and emits diagnostics.

Concretely, a *request* packages a list of input parameters together with an *evaluation function*. With the exception of emitting diagnostics, the request function's result should only depend on those inputs, and the results of other requests. The request evaluator directly invokes the evaluation function, and caches the result. Clients only evaluate the request via the request evaluator framework, which returns the cached value if present, detects request cycles automatically, and tracks dependency information for incremental builds.

The Swift frontend defines hundreds of request kinds; for our purposes, the most important ones are:

- The **type-check source file request** visits each declaration in a primary source file. It is responsible for kicking off enough requests to ensure that SILGen can proceed if all requests succeeded without emitting diagnostics.
- The **AST lowering request** is the entry point into SILGen, generating SIL from the abstract syntax tree for a source file.
- The **unqualified lookup request** and **qualified lookup request** perform the two kinds of name lookup described in the previous section.
- The **interface type request** is explained in Chapter 4.
- The **generic signature request** is explained in Chapter 11.

Example 2.2. Consider what happens when we type check this program:

```
let food = cook()
func cook() -> Food {}
struct Food {}
```

Notice how the initial value expression of the variable references the function, and the function's return type is the struct declared immediately after, so the inferred type of the variable is then this struct. This plays out with the request evaluator:

1. The **type-check source file request** begins by visiting the declaration of `food` and performing various semantic checks.
2. One of these checks evaluates the **interface type request** with the declaration of `food`. This is a variable declaration, so the evaluation function will type check the initial value expression and return the type of the result.
   a) In order to type check the expression `cook()`, the **interface type request** is evaluated again, this time with the declaration of `cook` as its input parameter.
   b) The interface type of `cook()` has not been computed yet, so the request evaluator calls the evaluation function for this request.

3. After computing the interface type of `food` and performing other semantic checks, the **type-check source file request** moves on to the declaration of `cook`:
   a) The **interface type request** is evaluated once again, with the input parameter being the declaration of `cook`.
   b) The result was already cached, so the request evaluator immediately returns the cached result without computing it again.

The **type-check source file request** is special, because it does not return a value; it is evaluated for the side effect of emitting diagnostics, whereas most other requests return a value. The implementation of the **type-check source file request** guarantees that if no diagnostics were emitted, then SILGen can generate valid SIL for all declarations in a primary file. However, the next example shows that SILGen can encounter invalid declarations, and diagnose errors in secondary files.

Example 2.3. Suppose we run a frontend job with the below primary file:

```
// a.swift
func open(_: Box) {}
```

We're going to look at what happens when `Box` is defined in a secondary file with a semantic error:

```
// b.swift
struct Box {
  let contents: DoesNotExist
}
```

Our frontend job does not emit any diagnostics in the semantic analysis pass, because the `contents` stored property of `Box` is not actually referenced while type checking the primary file `a.swift`. However when SILGen runs, it needs to determine whether the parameter of type `Box` to the `open()` function needs to be passed directly in registers, or via an address by computing the *type lowering* for the `Box` type. The type lowering procedure recursively computes the type lowering of each stored property of `Box`; this evaluates the **interface type request** for the `contents` property of `Box`, which emits a diagnostic because the identifier "`DoesNotExist`" does not resolve to a valid type. The interface type of the stored property then becomes the error type.

The request evaluator framework was first introduced in Swift 4.2 [41]. In subsequent releases, various ad-hoc mechanisms were gradually converted into request evaluator requests, with resulting gains to compiler performance, stability, and implementation maintainability.

Cycles. In a language supporting forward references, it is possible to write a program that is syntactically well-formed, and where all identifiers resolve to valid declarations, but is nonetheless invalid because of circularity. The classic example of this is a pair of classes where each class inherits from the other:

```
class A: B {}
class B: A {}
```

Implementing bespoke logic to detect circularity is error-prone and tedious, and a missing circularity check can result in a crash or infinite loop when the compiler encounters an invalid input program. The request evaluator centralizes cycle detection by maintaining a stack of *active requests*. Before evaluating a request, the request evaluator first checks if the active request stack already contains an equal request. In this case, calling the evaluation function would result in infinite recursion, so instead the request evaluator diagnoses an error and returns a request-specific sentinel value.

```
$ swiftc cycle.swift
cycle.swift:1:7: error: ‘A’ inherits from itself
class A: B {}
      ^
cycle.swift:2:7: note: class ‘B’ declared here
class B: A {}
      ^
```

The circularity diagnostic can be customized for each request kind; the default just says “circular reference.” If the compiler is invoked with the `-debug-cycles` frontend flag, the active request stack is also printed:

```
$ swiftc cycle.swift -Xfrontend -debug-cycles
===CYCLE DETECTED===
 ‘--TypeCheckSourceFileRequest(source_file "cycle.swift")
     ‘--SuperclassDeclRequest(cycle.(file).A@cycle.swift:1:7)
         ‘--SuperclassDeclRequest(cycle.(file).B@cycle.swift:2:7)
             ‘--SuperclassDeclRequest(cycle.(file).A@cycle.swift:1:7)
```

Debugging. A couple of command-line flags are useful for debugging compile-time performance issues. The `-stats-output-dir` flag is followed by the name of a directory, which must already exist. Each frontend job writes a new JSON file to this directory, with various counters and timers. For each kind of request, there is a counter for the number of unique requests of this kind that were evaluated, not counting requests whose results were cached. The timer records the time spent in the request’s evaluation function.

The output can be sliced and diced in various ways; one can actually make pretty effective use of `awk`, despite the JSON format:

```
$ mkdir /tmp/stats
$ swiftc ... -stats-output-dir /tmp/stats
$ awk '/InterfaceTypeRequest.wall/ { x += $2 } END { print x }' \
    /tmp/stats/*.json
```

The second command-line flag is `-trace-stats-events`. It must be passed in conjunction with `-stats-output-dir`, and enables output of a trace file to the statistics directory. The trace file records a time-stamped event for the start and end of each request evaluation function, in CSV format.

2.4. Incremental Builds

The request evaluator also records dependencies for incremental compilation, enabled by the `-incremental` driver flag. The goal of incremental compilation is to prove which files do not need to be rebuilt, in the least conservative way possible. The quality of an incremental compilation implementation can be judged as follows:[2]

1. Perform a clean build of all source files in the program, and collect the object files.

2. Make a change to one or more source files in the input program.

3. Do an incremental build, which rebuilds some subset of source files in the input program. If a source file was rebuilt but the resulting object file is identical to the one saved in Step 1, the incremental build performed *wasted work*.

4. Finally, do another clean build, which yet again rebuilds all source files in the input program. If a source file was rebuilt and the resulting object file is different to the one saved in Step 1, the incremental build was *incorrect*.

This highlights the difficulty of the incremental compilation problem. Rebuilding *too many* files is an annoyance; rebuilding *too few* files is an error. A correct but ineffective implementation would rebuild all source files every time. The opposite approach of only rebuilding the subset of source files that have changed since the last compiler invocation is also too aggressive. To see why it is incorrect, consider the program shown in Listing 2.3. Let's say the programmer builds the program, adds the overload `f: (Int) -> ()`, then builds it again. The new overload is more specific, so the call `f(123)` in `b.swift` now refers to the new overload; therefore, `b.swift` must also be rebuilt.

[2] Credit for this idea goes to David Ungar.

Listing 2.3.: Rebuilding a file after adding a new overload

```
// a.swift
func f<T>(_: T) {}

// new overload added in second version of file
func f(_: Int) {}
```

```
// b.swift
func g() {
  // new overload is selected after a.swift is updated
  f(123)
}
```

The approach taken by the Swift compiler is to construct a *dependency graph*. The frontend outputs a *dependency file* for each source file, recording all names the source file *provides*, and all names the type checker *requires* while compiling the source file. Dependency files use a binary format with the "`.swiftdeps`" file name extension. The list of provided names in the dependency file is generated by walking the abstract syntax tree, collecting all visible declarations in each source file. The list of required names is generated by the request evaluator, using the stack of active requests. Every cached request has a list of required names, and a request can optionally be either a dependency sink, or dependency source:

- A *dependency sink* is a name lookup request which records a required name. When a dependency sink request is evaluated, the request evaluator walks the stack of active requests, adding the identifier to each active request's list of required names. Thus, for every request, we track the name lookups that took place from the evaluation function.

  An important caveat is that when a request with a cached value is evaluated again, the request's cached list of required names must again be "replayed," adding them to each active request that depends on the cached value.

- A *dependency source* is a request which appears at the top of the request stack, such as the **type-check source file request** or the **AST lowering request**. A dependency source scopes some amount of work to a source file.

  After the evaluation of a dependency source request completes, all required names attributed to the request are added to the source file's list of required names.

The driver makes use of the dependency files generated by the frontend to actually perform an incremental build. This happens in two phases:

1. The first phase rebuilds all source files which have changed since the last compilation. This is the minimum set that must be rebuilt.

2. The second phase reads the dependency files, and collects all names provided by the source files rebuilt in the first phase. The source files which depend on those names are then rebuilt.

Example 2.4. To understand how request caching interacts with dependency recording, consider the program shown in Listing 2.4. Suppose the driver decides to compile *both* `a.swift` and `b.swift` in the same frontend job (in fact, the issue at hand can only appear in batch mode, when a frontend job has more than one primary file). First, the **type-check source file request** runs with the source file `a.swift`.

1. While type checking the body of `breakfast()`, the type checker evaluates the **unqualified lookup request** with the identifier "`soup`."

2. This records the identifier "`soup`" in the requires list of each active request. There is one active request, the **type-check source file request** for `a.swift`.

3. The lookup finds the declaration of `soup()` in `c.swift`.

Listing 2.4.: Recording incremental dependencies

```
// a.swift
func breakfast() {
  soup(nil)
}
```

```
// b.swift
func lunch() {
  soup(nil)
}
```

```
// c.swift
func soup(_: Pumpkin?) {}
struct Pumpkin {}
```

4. The type checker evaluates the **interface type request** with the declaration of `soup()`.
   a) The **interface type request** evaluates the **unqualified lookup request** with the identifier "`Pumpkin`."
   b) This records the identifier "`Pumpkin`" in the requires list of each active request, of which there are now two: the **interface type request** for `soup()`, and the **type-check source file request** for `a.swift`.
5. The **type-check source file request** for `a.swift` has now finished. The requires list for this request contains two identifiers, "`soup`" and "`Pumpkin`"; both are added to the requires list of the source file `a.swift`.

Next, the **type-check source file request** runs with the source file `b.swift`.

1. While type checking the body of `lunch()`, the type checker evaluates the **unqualified lookup request** with the identifier "`soup`."
2. This records the identifier "`soup`" in the requires list of each active request. There is one active request, the **type-check source file request** for `b.swift`.
3. The lookup finds the declaration of `soup()` in `c.swift`.
4. The type checker evaluates the **interface type request** with the declaration of `soup()`.
5. This request has already been evaluated, and the cached result is returned. The requires list for this request is the single identifier "`Pumpkin`." This requires list is replayed, as if the request was being evaluated for the first time. This adds the identifier "`Pumpkin`" to the requires list of each active request, of which there is just one: the **type-check source file request** for `b.swift`.
6. The **type-check source file request** for `b.swift` has now finished. The requires list for this request contains two identifiers, "`soup`" and "`Pumpkin`"; both are added to the requires list of the source file `b.swift`.

The frontend job writes out the dependency files for `a.swift` and `b.swift` upon completion. Both source files require the names "`soup`" and "`Pumpkin`." The dependency of `b.swift` on "`Pumpkin`" is correctly recorded because evaluating a request with a cached value replays the request's requires list in Step (2) above.

There's a bit more to the incremental build story than this; in particular, we haven't talked about the "interface hash" mechanism, meant to avoid rebuilding of dependent source files for changes were limited to comments, whitespace or function bodies. We're already far afield from the goal of describing Swift generics though, so the curious reader can refer to [41] and [42] for details.

2.5. Module System

The frontend represents a module by a *module declaration* containing one or more *file units*. The list of source files in a compiler invocation form the *main module*. The main module is special, because its abstract syntax tree is constructed directly by parsing source code; the file units are *source files*. There are three other kinds of modules:

- **Serialized modules** containing one or more *serialized AST file units*. When the main module imports another module written in Swift, the frontend reads a serialized module that was previously built.
- **Imported modules** consisting of one or more *Clang file units*. These are the modules implemented in C, Objective-C, or C++.
- **The builtin module** with exactly one file unit, containing types and intrinsics implemented by the compiler itself.

The main module depends on other modules via the `import` keyword, which parses as an *import declaration*. After parsing, one of the first stages in semantic analysis loads all modules imported the main module. The standard library is defined in the `Swift` module, which is imported automatically unless the frontend was invoked with the `-parse-stdlib` flag, used when building the standard library itself. As for the builtin module, it is ordinarily not visible, but the `-parse-stdlib` flag also causes it to be implicitly imported (Section 3.3).

Serialized modules. The `-emit-module` flag instructs the compiler to generate a serialized module. Serialized module files use the "`.swiftmodule`" file name extension. Serialized modules are stored in a binary format, closely tied to the specific version of the Swift compiler (when building a shared library for distribution, it is better to publish a textual interface instead, as described at the end of this section).

Name lookup into a serialized module lazily constructs declarations by deserializing records from this binary format as needed. Deserialized declarations generally look like parsed and fully type-checked declarations, but they sometimes contain less information. For example, in Section 4.2, we will describe various syntactic representations of requirements, such as `where` clauses. Since this information is only used when type checking the declaration, it is not serialized. Instead, deserialized declarations only need to store a generic signature, described in Chapter 5.

Another key difference between parsed declarations and deserialized declarations is that parsed function declarations have a body, consisting of statements and expressions. This body is never serialized, so deserialized function declarations never have a body. The one case where the body of a function is made available across module boundaries is when the function is annotated with the `@inlinable` attribute; this is implemented by serializing the SIL representation of the function instead.

Imported modules. An imported module is implemented in C, Objective-C, or C++. The Swift compiler embeds a copy of Clang and uses it to parse module maps, header files, and binary precompiled headers. Name lookup into an imported module lazily constructs Swift declarations from their corresponding Clang declarations. The Swift compiler component responsible for this is known as the "Clang importer."

Imported function declarations generally do not have bodies if the entry point was previously emitted by Clang and is available externally. Occasionally the Clang importer synthesizes accessor methods and other such trivia, which do have bodies represented as Swift statements and expressions. C functions not available externally, such as `static inline` functions declared in header files, are emitted by having Swift IRGen call into Clang.

Invoking the compiler with the `-import-objc-header` flag followed by a header file name specifies a *bridging header*. This is a shortcut for making C declarations in the bridging header visible to all other source files in the main module, without having to define a separate Clang module first. This is implemented by adding a Clang file unit corresponding to the bridging header to the main module. For this reason, compiler code should not assume that all file units in the main module are necessarily source files.

Textual interfaces. The binary module format depends on compiler internals and no attempt is made to preserve compatibility across compiler releases. When building a shared library for distribution, it is better to generate a *textual interface*:

```
$ swiftc Horse.swift -enable-library-evolution -emit-module-interface
```

Unlike the serialized module format, textual interfaces only describe the public declarations of a module. The `-enable-library-evolution` flag enables *resilience*, which is a prerequisite for emitting a textual interface. Resilience instructs clients to use more abstract access patterns which are guaranteed to only depend on the public declarations of a module. For example, it allows new stored properties to be added to a public struct. Resilience is documented in [43].

Textual interface files use the "`.swiftinterface`" file name extension. They are generated by the AST printer, which prints declarations in a format that looks very much like Swift source code, with a few exceptions:

1. Non-`@inlinable` function bodies are skipped. Bodies of `@inlinable` functions are printed verbatim, including comments, except that `#if` conditions are evaluated.

2. Various synthesized declarations, such as type alias declarations from associated type inference, witnesses for derived conformances such as `Equatable`, and so on, are written out explicitly.

3. Opaque return types also require special handling (Section 13.2).

Note that (1) above means the textual interface format is target-specific; a separate textual interface needs to be generated for each target platform, alongside the shared library itself.

When a module defined by a textual interface is imported for the first time, a frontend job parses and type checks the textual interface, and generates a serialized module file which is then consumed by the original frontend job. Serialized module files generated in this manner are cached, and can be reused between invocations of the same compiler version.

The `@inlinable` attribute was introduced in Swift 4.2 [44]. The Swift ABI was formally stabilized in Swift 5, when the standard library became part of the operating system on Apple platforms. Library evolution support and textual interfaces became user-visible features in Swift 5.1 [45].

2.6. Source Code Reference

The Swift driver is now implemented in Swift, and lives in a separate repository from the rest of the compiler:

```
https://github.com/swiftlang/swift-driver
```

The Swift frontend, standard library and runtime are found in the main repository:

```
https://github.com/swiftlang/swift
```

The major components of the Swift frontend live in their own subdirectories of the main repository. The entities modeling the abstract syntax tree are defined in `lib/AST/` and `include/swift/AST/`; among these, types and declarations are important for the purposes of this book, and will be covered in Chapter 3 and Chapter 4. The core of the SIL intermediate language is implemented in `lib/SIL/` and `include/swift/SIL/`.

Each stage of the compilation pipeline has its own subdirectory:

- `lib/Parse/`
- `lib/Sema/`
- `lib/SILGen/`
- `lib/SILOptimizer/`
- `lib/IRGen/`

The AST Context

Key source files:

- include/swift/AST/ASTContext.h
- lib/AST/ASTContext.cpp

ASTContext	*class*

The global singleton for a single frontend instance. An AST context provides a memory allocation arena, unique allocation for various immutable data types used throughout the compiler, and storage for various other global singletons.

Request Evaluator

Key source files:

- include/swift/AST/Evaluator.h
- lib/AST/Evaluator.cpp

SimpleRequest	*template class*

Each request kind is a subclass of `SimpleRequest`. The evaluation function is implemented by overriding the `evaluate()` method of `SimpleRequest`.

RequestFlags	*enum class*

One of the template parameters to `SimpleRequest` is a set of flags:

- `RequestFlags::Uncached`: indicates that the result of the evaluation function should not be cached.
- `RequestFlags::Cached`: indicates that the result of the evaluation function should be cached by the request evaluator, which uses a per-request kind `DenseMap` for this purpose.
- `RequestFlags::SeparatelyCached`: the result of the evaluation function should be cached by the request implementation itself, as described below.
- `RequestFlags::DependencySource`, `DependencySink`: if one of these is set, the request kind becomes a dependency source or sink, as described in Section 2.4.

Separate caching can be more performant if it allows the cached value to be stored directly inside of an AST node, instead of requiring the request evaluator to consult a side table. For example, many requests taking a declaration as input store the result directly inside of the `Decl` instance or some subclass thereof.

Due to expressivity limitations in C++, a bit of boilerplate is involved in the definition of a new request kind. For example, consider the `InterfaceTypeRequest`, which takes a `ValueDecl` as input and returns a `Type` as output:

- The request type ID is declared in `include/swift/AST/TypeCheckerTypeIDZone.def`.
- The `InterfaceTypeRequest` class is declared in `include/swift/AST/TypeCheckRequests.h`.
- The `InterfaceTypeRequest::evaluate()` method is defined in `lib/Sema/TypeCheckDecl.cpp`.
- The request is separately cached. The `InterfaceTypeRequest` class overrides the `isCached()`, `getCachedResult()` and `cacheResult()` methods to store the declaration's interface type inside the `ValueDecl` instance itself. These methods are implemented in `lib/AST/TypeCheckRequestFunctions.cpp`.

`Evaluator` *class*

Request evaluation is performed by calling the `evaluateOrDefault()` top-level function, passing it an instance of the request evaluator, the request to evaluate, and a sentinel value to return in case of circularity. The `Evaluator` class is a singleton, stored in the `evaluator` instance variable of the global `ASTContext` singleton. The request evaluator will either return a cached value, or invoke the evaluation function and cache the result. For example, the `getInterfaceType()` method of `ValueDecl` is implemented as follows:

```
Type ValueDecl::getInterfaceType() const {
  auto &ctx = getASTContext();
  return evaluateOrDefault(
    ctx.evaluator,
    InterfaceTypeRequest{const_cast<ValueDecl *>(this)},
    ErrorType::get(ctx)));
}
```

Name Lookup

Key source files:

- include/swift/AST/NameLookup.h
- include/swift/AST/NameLookupRequests.h
- lib/AST/NameLookup.cpp
- lib/AST/UnqualifiedLookup.cpp

The "AST scope" subsystem implements unqualified lookup for local bindings. Outside of the name lookup implementation itself, the rest of the compiler does not generally interact with it directly:

- include/swift/AST/ASTScope.h
- lib/AST/ASTScope.cpp
- lib/AST/ASTScopeCreation.cpp
- lib/AST/ASTScopeLookup.cpp
- lib/AST/ASTScopePrinting.cpp
- lib/AST/ASTScopeSourceRange.cpp

`UnqualifiedLookupRequest` *class*

Unqualified lookups are performed by evaluating an instance of this request kind. The request takes an `UnqualifiedLookupDescriptor` as input.

`UnqualifiedLookupDescriptor` *class*

Encapsulates the input parameters for an unqualified lookup:

- The name to look up.
- The declaration context where the lookup starts.
- The source location where the name was written in source. If not specified, this becomes a top-level lookup.
- Various flags, described below.

`UnqualifiedLookupFlags` *enum class*

Flags passed as part of an `UnqualifiedLookupDescriptor`.

- `UnqualifiedLookupFlags::TypeLookup`: if set, lookup ignores declarations other than type declarations. This is used in type resolution.

- `UnqualifiedLookupFlags::AllowProtocolMembers`: if set, lookup finds members of protocols and protocol extensions. Generally should always be set, except to avoid request cycles in cases where it is known the result of the lookup cannot appear in a protocol or protocol extensions.
- `UnqualifiedLookupFlags::IgnoreAccessControl` if set, lookup ignores access control. Generally should never be set, except when recovering from errors in diagnostics.
- `UnqualifiedLookupFlags::IncludeOuterResults` if set, lookup stops after finding results in an innermost scope, or to always proceed to a top-level lookup.

`DeclContext` *class*

Declaration contexts will be introduced in Chapter 4, and the `DeclContext` class in Section 4.6.

- `lookupQualified()` has various overloads, which perform a qualified name lookup into one of various combinations of types or declarations. The "`this`" parameter—the `DeclContext *` which the method is called on determines the visibility of declarations found via lookup through imports and access control; it is not the base type of the lookup.

`NLOptions` *enum*

Similar to `UnqualifiedLookupFlags`, but for `DeclContext::lookupQualified()`.

- `NL_OnlyTypes`: if set, lookup ignores declarations other than type declarations. This is used in type resolution.
- `NL_ProtocolMembers`: if set, lookup finds members of protocols and protocol extensions. Generally should always be set, except to avoid request cycles in cases where it is known the result of the lookup cannot appear in a protocol or protocol extension.
- `NL_IgnoreAccessControl`: if set, lookup ignores access control. Generally should never be set, except when recovering from errors in diagnostics.

`NominalTypeDecl` *class*

Nominal type declarations will be introduced in Chapter 4, and the `NominalTypeDecl` class in Section 4.6. The implementation of direct lookup and lazy member loading is discussed in Section 10.5.

- `lookupDirect()` performs a direct lookup, which only searches the nominal type declaration itself and its extensions, ignoring access control.

`lookupInModule()` *function*

Searches for top-level declarations within a module. Operates in one of two modes, depending on the arguments given:

- Qualified lookup into a specific module. Looks inside the given module and all of its `@_exported` imports.
- Unqualified lookup from the top-level of a source file. Looks inside all modules imported from this source file, as well as any `@_exported` imports from other source files in the main module.

Primary File Type Checking

Key source files:

- `lib/Sema/TypeCheckDeclPrimary.cpp`

The `TypeCheckSourceFileRequest` calls the `typeCheckDecl()` global function, which uses the visitor pattern to switch on the declaration kind. For each declaration kind, it performs various semantic checks and kicks off requests which may emit diagnostics.

Module System

`ModuleDecl` *class*

A module.

- `getName()` returns the module's name.
- `getFiles()` returns an array of `FileUnit`.
- `isMainModule()` answers if this is the main module.

See Section 7.6 and Section 10.5 for the global conformance lookup operations defined on `ModuleDecl`.

`FileUnit` *class*

Abstract base class representing a file unit.

`SourceFile` *class*

Represents a parsed source file from disk. Inherits from `FileUnit`.

- `getTopLevelItems()` returns an array of all top-level items in this source file.
- `isPrimary()` returns `true` if this is a primary file, `false` if this is a secondary file.
- `isScriptMode()` answers if this is the main file of a module.
- `getScope()` returns the root of the scope tree for unqualified lookup.

Imported and serialized modules get a subdirectory each:

- `lib/ClangImporter/`
- `lib/Serialization/`

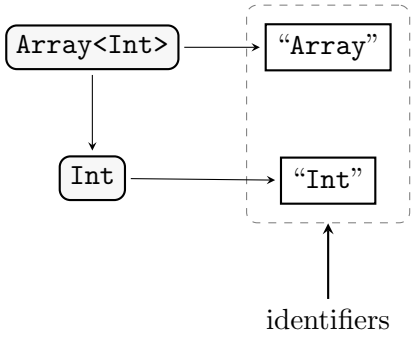
The AST printer for generating textual interfaces is implemented in a pair of files:

- `include/swift/AST/ASTPrinter.h`
- `lib/AST/ASTPrinter.cpp`

3. Types

Reasoning about types is a central concern in the implementation of a statically typed language. In Swift, various syntactic forms such as `Int`, `Array<String>` and `(Bool) -> ()` denote references to types. A *type representation* is the syntactic form of a type annotation written in source, as constructed by the parser. A *type* is a higher-level semantic object. Types are constructed from type representations by *type resolution*. They can also be built and taken apart directly.

A type representation:



Suppose the text "`Array<Int>`" appears in a valid position of the language grammar. First, the lexer splits the input text into the tokens "`Array`", "`<`", "`Int`", and "`>`". The parser inspects each token and builds a type representation.

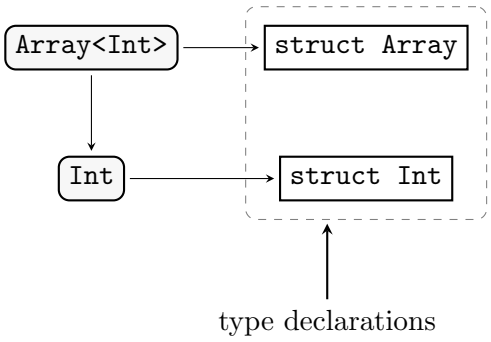
The parsed type representation `Array<Int>` has a tree structure, combining the identifier "`Array`" with the type representation `Int`. The latter is a leaf node just storing an identifier. As syntactic objects, type representations only store identifiers, which bear no relation to actual type declarations. The "shape" of a tree node is determined by the type representation's *kind*, corresponding to syntactic productions.

Type resolution calls upon name lookup to find type declarations, here `Array` and `Int`, and validates the generic argument, to produce a semantic type object.

The generic nominal type `Array<Int>` points at the `Array` type declaration, and contains a child node for its generic argument, which is the type `Int`. The latter type also points at the declaration of `Int`, and not just an identifier.

Types are categorized by *kind*, just like type representations. Each kind has a constructor operation to form a new type from *structural components*, and corresponding operations to take an existing type apart.

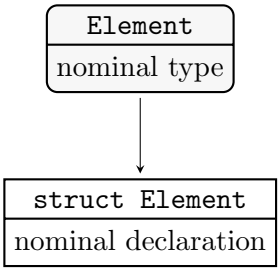
A type:



Categorization by kind. Type representation kinds are given by various productions in the grammar, such as identifiers, function type representations "`(Int) -> ()`", tuple type representations "`(Bool, String)`", and so on. The latter two kinds resolve to function types and tuple types, but type kinds are finer grained than type representation kinds in general, as we can see if we consider identifier type representations. An identifier type representation "`Element`" can resolve to one of several kinds of types; we consider a few now.

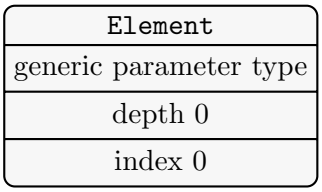
One possibility is that the identifier names a nominal type declaration. In this case, the resolved type is a **nominal type** pointing at this declaration.

```
struct Element {...}
var x: Element
```



The type checker can then perform further qualified lookups to find members of "`x`", query the stored properties of the struct to determine its layout, and so on.

The identifier might instead name a generic parameter from an outer scope, in which case the resolved type is a **generic parameter type**.

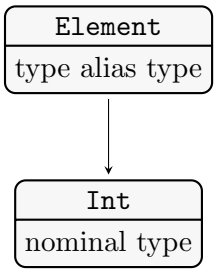



```
struct Container<Element> {
  var x: Element
}
```

Generic parameters are declared in a generic parameter list attached to a declaration. They are scoped to the body of this declaration, and can be uniquely identified by a pair of integers, the *depth* and *index*. Generic parameter types can also store a name, but the name only has significance when printed in diagnostics.

Yet another possibility is that the identifier names a type alias declaration. In this case, a special **type alias type** is returned, which wraps the *underlying type* `Int`. It behaves like `Int` in all ways, except that it can be printed back as "`Element`".

```
typealias Element = Int
var x: Element
```



Outside of type resolution, type representations do not play a big role in the compiler, so we punt on the topic of type representations until Chapter 9 and just focus on types for now. For our current purposes, it suffices to say that type resolution is really just one possible mechanism by which types are constructed. The expression checker builds types by solving a constraint system, and the generics system builds types via substitution, to give two examples.

Structural components. A type is constructed from structural components, which may either be other types, or non-type information. Common examples include: nominal types, which consist of a pointer to a declaration, together with a list of generic argument types; tuple types, which have element types and labels; and function types, which contain parameter types, return types, and various additional bits like `@escaping` and `inout`. We will give a full accounting of all type kinds and their structural components in the second half of this chapter.

Once created, types are immutable. To say that a type *contains* another type means that the latter appears as a structural component of the former, perhaps nested several levels deep. We will often talk about *replacing* a type contained by another type. This is understood as constructing a new type with the same kind as the original type, preserving all structural components except for the one being replaced. The original type is never mutated directly.

More generally, types can be transformed by taking the type apart by kind, recursively transforming each structural component, and forming a new type of the same kind from the new components. To preview Chapter 6, if `Element` is a generic parameter type, the type `Array<Int>` can be formed from `Array<Element>` by replacing `Element` with `Int`; this is called *type substitution*. The compiler provides various utilities to simplify the task of implementing recursive walks and transformations over kinds of types; type substitution is one example of such a transformation.

Canonical types. It is possible for two types to differ in their spelling, and yet be equivalent semantically:

- The Swift language defines some shorthands for common types, such as `T?` for `Optional<T>`, `[T]` for `Array<T>`, and `[K: V]` for `Dictionary<K, V>`.
- Type alias declarations introduce a new name for some existing underlying type, equivalent to writing out the underlying type in place of the type alias. The standard library, for example, declares a type alias `Void` with underlying type `()`.
- Another form of fiction along these lines is the preservation of generic parameter names. Generic parameter types written in source have a name, like "`Element`," and should be printed back as such in diagnostics, but internally they are uniquely identified in their generic signature by a pair of integers, the depth and the index. This is detailed in Section 4.1.

These constructions are the so-called *sugared types*. A sugared type has a desugaring into a more primitive form in terms of its structural components. The compiler constructs type sugar in type resolution, and attempts to preserve it as much as possible when transforming types. Preserving sugar in diagnostics can be especially helpful with more complex type aliases and such.

A *canonical type* is a type that does not recursively contain any sugared types. Type sugar has no semantic effect, for the most part. For example, it would not make sense to define two overloads of the same function that only differ by sugared types. For this reason, many operations on types compute the canonical type first to avoid having to consider sugared types in case analysis. After type checking, compiler passes such as SILGen and IRGen only deal with canonical types. The Swift runtime reifies canonical types as runtime type metadata.

The compiler can transform an arbitrary type into a canonical type by the process of *canonicalization*, which recursively replaces sugared types with their desugared form; in this way, `[(Int?, Void)]` becomes `Array<(Optional<Int>, ())>`. This operation is very cheap; each type caches a pointer to its canonical type, which is computed as needed (so types are not completely immutable, as we said previously; but the mutability cannot be observed from outside).

One notable exception where the type checker does depend on type sugar is the rule for default initialization of variables: if the variable's type is declared as the sugared optional type `T?` for some `T`, the variable's initial value expression is assumed to be `nil` if none was provided. Spelling the type as `Optional<T>` avoids the default initialization behavior:

```
var x: Int?
print(x)  // prints 'nil'

var y: Optional<Int>
print(y)  // error: use of uninitialized variable 'y'
```

Another exception is requirement inference with a generic type alias (Section 11.1).

Type equality. Types are uniquely allocated, which is made possible by them being immutable. A type `(Int) -> ()` has a unique pointer identity within a compilation; inside the tuple type `((Int) -> (), (Int) -> ())`, the two element types have the same pointer value in memory. From this, three levels of equality are defined on types:

1. **Type pointer equality** checks if two types are exactly equal as trees.
2. **Canonical type equality** checks if two types are equal after sugar is removed.
3. **Reduced type equality** checks if two types have the same reduced type with respect to the same-type requirements of a generic signature.

Each level of equality implies the next, so (1) ⇒ (2) and (2) ⇒ (3). If both types are canonical, then (1) and (2) coincide; if both are reduced, (1), (2) and (3) all coincide. Type pointer equality is only infrequently used, precisely because it is too strict; we usually do not want to consider two types to be distinct if they only differ by sugar.

Figure 3.1.: Some examples of type equality

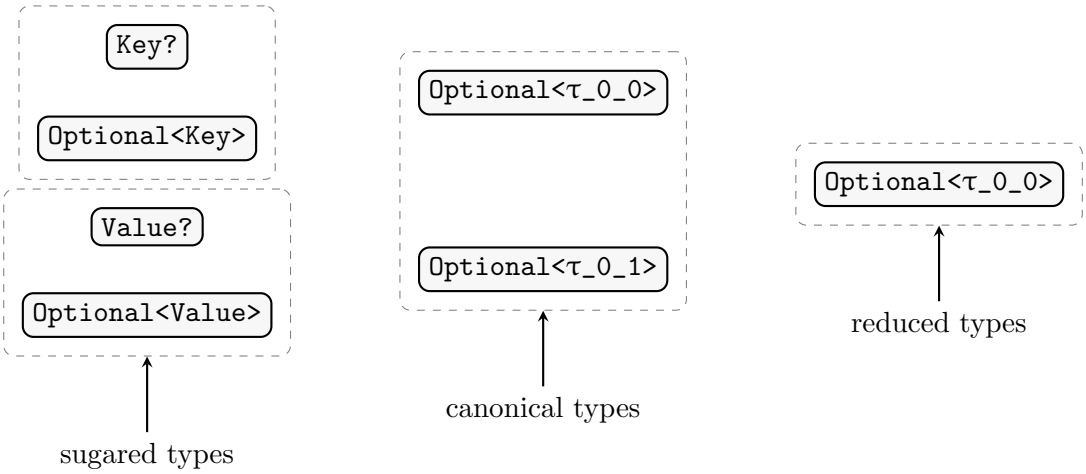


Figure 3.1 demonstrates type equality in the following extension declaration, where the `Key` (`τ_0_0`) and `Value` (`τ_0_1`) generic parameters of `Dictionary` are declared equivalent with a same-type requirement:

```
extension Dictionary where Key == Value {
  func foo(a: Key?, b: Optional<Key>, c: Value?, d: Optional<Value>) {}
}
```

Consider the four types `Key?`, `Optional<Key>`, `Value?`, and `Optional<Value>`:

- All four are distinct under type pointer equality.
- The first two have the canonical type `Optional<τ_0_0>`. Thus, the first two are equal under canonical type equality.
- The last two have the canonical type `Optional<τ_0_1>`. Again, they are equal to each other under canonical type equality (but distinct from the first two).
- When we consider the generic signature of the extension, we're left with just one reduced type, because both canonical types reduce to `Optional<τ_0_0>` via the same-type requirement. Thus, all four of the original types are equal under reduced type equality.

Reduced type equality means "equivalent as a consequence of one or more same-type requirements." We will define this equivalence of type parameters in Section 5.3 using the derived requirements formalism, and then generalize to all interface types in Section 5.5. Presenting a computable algorithm for reduced type equality is one of our main results in this book; key developments take place in Section 17.5 and Chapter 18.

3.1. Fundamental Types

We've looked at some behaviors of types in general, and informally introduced a few kinds. We now give a full accounting of all kinds of types, starting from those most important from the point of view of the generics implementation.

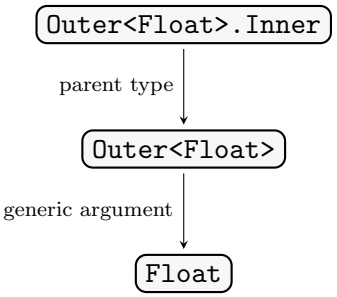
Nominal types. A *nominal type* is declared by a non-generic struct, enum or class declaration, such as `Int`. A *generic nominal type* is declared by a generic struct, enum or class declaration. They have these structural components:

- A pointer to the nominal type declaration.
- A parent type, if the nominal type declaration is nested inside of another nominal type declaration.
- A list of generic arguments, if the nominal type declaration is generic.

The parent type records the generic arguments of outer nominal type declarations. For example, with the below declarations, we can form the nominal type `Outer<Float>.Inner`:

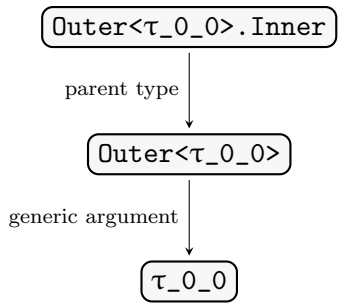
```
struct Outer<T> {
  struct Inner {}
}
```

Nominal type nesting



A nominal type declaration defines a family of nominal types, all sharing the same declaration and parent type structure, but having different generic arguments. Each of these types is a *specialized type* of the nominal type declaration.

Declared interface type



The specialized type where each generic argument is set to the corresponding generic parameter type is the *declared interface type* of the nominal type declaration. It is "universal": any specialized type is obtainable from the declared interface type by assigning a replacement type to each generic parameter type. This assignment is known as a *substitution map*, and the substitution map defined by the generic arguments of a specialized type is its *context substitution map* (Section 6.1). Finally, in the case where neither the nominal type declaration nor any of its parents are generic, the declaration defines just one *fully-concrete* specialized type, with an empty context substitution map.

Generic parameter types. A generic parameter type abstracts over a generic argument provided by the caller. Generic parameter types are declared by generic parameter declarations, described in Section 4.1. The sugared form references the declaration, and prints as the declaration's name; the canonical form only stores a depth and an index. Care must be taken not to print canonical generic parameter types in diagnostics, to avoid surfacing the "`τ_1_2`" notation to the user. (We will show how to transform a canonical generic parameter type into its sugared form at the end of Section 5.6.)

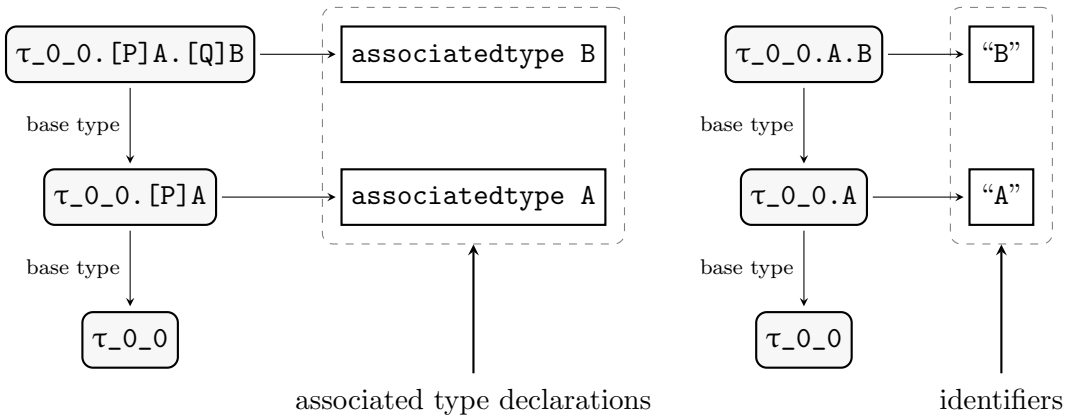
Dependent member types. A dependent member type abstracts over a concrete type that fulfills an associated type requirement. It has two structural components:

- A base type, which is a generic parameter type or another dependent member type.
- An identifier (in which case this is an *unbound* dependent member type), or an associated type declaration (in which case it is *bound*).

Unbound dependent member types fit in somewhere in between type representations and "true" types, in that they are a syntactic construct. They appear in the structural resolution stage when building a generic signature. Most type resolution happens in the interface resolution stage, after a generic signature is available, and thus dependent member types appearing in the interface types of declarations are always bound.

If `T` is the base type, `P` is a protocol and `A` is an associated type declared inside this protocol, we denote the bound dependent member type by `T.[P]A`, and the unbound dependent member type by `T.A`. The base type `T` may be another dependent member type, so we can have a series of member type accesses, like `τ_0_0.[P]A.[Q]B`. Figure 3.2 shows the recursive structure of two dependent member types, bound and unbound.

Figure 3.2.: Bound and unbound dependent member types



A dependent member type is "dependent" in the C++ sense, *not* a type dependent on a value in the "lambda cube" sense. Generic parameter types and dependent member types are together known as *type parameters*. A type that might contain type parameters but is not necessarily a type parameter itself is called an *interface type*. The above summary necessarily leaves many questions unanswered, because a significant portion of the rest of the book is devoted to type parameters. Key topics include:

- Semantic validity of type parameters (Section 5.2, Section 5.3).
- Generic signature queries (Section 5.5).
- Dependent member type substitution (Section 7.4, Chapter 12).
- Type resolution with bound and unbound type parameters (Chapter 9).

Archetype types. Type parameters derive their meaning from the requirements of a generic signature; they are only "names" of external entities, in a sense. Archetypes are an alternate "self-describing" representation. Archetypes are instantiated from a *generic environment*, which stores a generic signature together with other information (Chapter 8).

Archetypes occur inside expressions and SIL instructions. Archetypes also represent references to opaque return types (Section 13.1) and the type of the payload inside of an existential (Section 14.1). In diagnostics, an archetype is printed as the type parameter it represents. We will denote by $[\![\texttt{T}]\!]$ the archetype for the type parameter `T` in some generic environment understood from context. A type that might contain archetypes but is not necessarily an archetype itself is called a *contextual type*.

The fundamental type kinds we surveyed above—nominal types, type parameters, and archetypes—are *the Swift types that can conform to protocols*. In other words, they can satisfy the left-hand side of a conformance requirement, with the details for each type given later in Section 7.1. Also important are *constraint types*; these are *the types appearing on the right hand side of conformance requirements*. Constraint types themselves are never the types of value-producing expressions. (Type-erased values are represented by existential types, wrapping a constraint type in a level of indirection.)

Protocol types. A protocol type is the most fundamental kind of constraint type; a conformance requirement involving any other kind of constraint type can always be split up into simpler conformance requirements. A protocol type is a kind of nominal type, so it will have a parent type if the protocol declaration is nested inside of another nominal type declaration. Unlike other nominal types, protocols cannot be nested in generic contexts (Section 6.4), so neither the protocol type itself nor any of its parents can have generic arguments. Thus, there is exactly one protocol type corresponding to each protocol declaration.

Protocol composition types. A protocol composition type is a constraint type with a list of members. On the right hand side of a conformance requirement, protocol compositions *decompose* into a series of requirements for each member of the composition (Section 11.2). The members can include protocol types, a class type (at most one), and the `AnyObject` layout constraint:

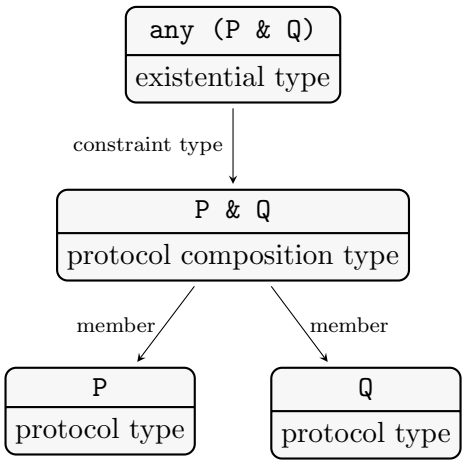
```
P & Q
P & AnyObject
SomeClass<Int> & P
```

Parameterized protocol types. A parameterized protocol type stores a protocol type together with a list of generic arguments. As a constraint type, it expands into a conformance requirement together with one or more same-type requirements, for each of the protocol's *primary associated types* (Section 4.3). The written representation looks just like a generic nominal type, except the named declaration is a protocol, for example, `Sequence<Int>`. Parameterized protocol types were introduced in Swift 5.7 [46] (the evolution proposal calls them "constrained protocol types").

3.2. More Types

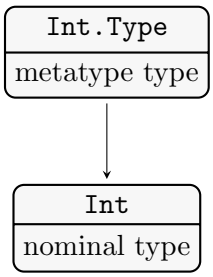
Now we will look at the various *structural types* which are part of the language. (Not to be confused with the types produced by the *structural resolution stage*, which is discussed in Chapter 9.)

Existential types. An existential type has one structural component, the *constraint type*. An existential value is a container holding a value of some unknown dynamic type that is known to satisfy the constraint; to the right we show the existential type `any (P & Q)`, which stores a value conforming to both `P` and `Q`.



The `any` keyword was added in Swift 5.6 [47]; in Swift releases prior, existential types and constraint types were the same concept in the language and implementation. (For the sake of source compatibility, a constraint type without the `any` keyword still resolves to an existential type except when it appears on the right-hand side of a conformance requirement.)

Existential types are covered in Chapter 14.

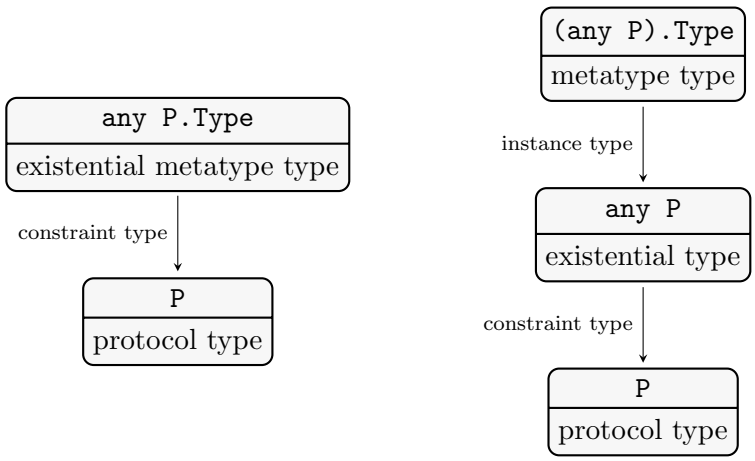


Metatype types. A type `T` can be the callee in a call expression, `T(...)`; this is shorthand for a constructor call `T.init(...)`. It can serve as the base of a static method call, `T.foo(...)`, where the type is passed as the `self` parameter. Finally, it can be directly referenced by the expression `T.self`. In all cases, the type becomes a *value*, and this value must itself be assigned a type; this type is called a *metatype*. The metatype of a type `T` is written as `T.Type`. The type `T` is the *instance type* of the metatype. For example, the type of the expression "`Int.self`" is the metatype `Int.Type`, whose instance type is `Int`. Metatypes are sometimes referred to as *concrete metatypes*, to distinguish them from existential metatypes, which we introduce below. Most concrete metatypes are singleton types with one value, the instance type itself. One exception is that the class metatype for a non-final class also has all subclasses of the class as values.

Existential metatype types. An existential metatype is a container for an unknown concrete metatype, whose instance type is known to satisfy the constraint type.

Existential metatypes are not the same as metatypes whose instance type is existential, as we can demonstrate by considering language semantics. If `P` is a protocol and `S` is a type conforming to `P`, then the existential metatype `any P.Type` can store the value `S.self`. As the existential type `any P` does not conform to `P`, the value `(any P).self` cannot be stored inside of the existential metatype `any P.Type`. In fact, the type of this value is the *concrete* metatype `(any P).Type`. Figure 3.3 compares the recursive structure of `any P.Type` against `(any P).Type`.

Figure 3.3.: Existential metatype and metatype of existential

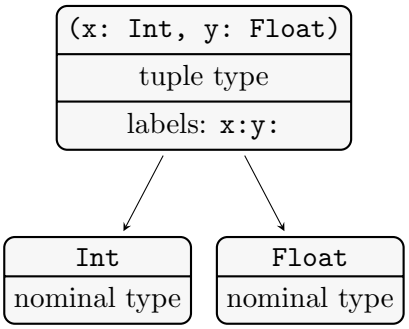


Prior to the introduction of the `any` keyword, existential metatypes were written as `P.Type`, and the concrete metatype of an existential as `P.Protocol`. This was a source of confusion, because when `T` is a non-protocol type, `T.Type` is otherwise always a concrete metatype. The below table compares the old and new syntax:

Old syntax	New syntax	Type kind
`(T).Type`	`(T).Type`	Concrete metatype (unchanged)
`P.Protocol`	`(any P).Type`	Concrete metatype of existential
`P.Type`	`any P.Type`	Existential metatype

Tuple types. A tuple type is a list of element types together with optional labels. If the list of element types is empty, we get the unique empty tuple type `()`.

An unlabeled one-element tuple type cannot be formed at all; `(T)` resolves to the same type as `T`. Labeled one-element tuple types `(foo: T)` are valid in the grammar, but are rejected by type resolution. SILGen creates them internally when it materializes the payload of an enum case (for instance, "`case person(name: String)`"), but they do not appear as the types of expressions.



Function types. A function type is the type of the callee in a call expression. It contains a parameter list, a return type, and non-type attributes. The attributes include the function's effect, lifetime, and calling convention. The effects are `throws` and `async` (part of the Swift 5.5 concurrency model [48]). Function values with non-escaping lifetime are second-class; they can only be passed to another function, captured by a non-escaping closure, or called. Only escaping functions can be returned or stored inside other values. The four calling conventions are:

- The default "thick" convention, where the function is passed as a function pointer together with a reference-counted closure context.
- `@convention(thin)`: the function is passed as a single function pointer, without a closure context. Thin functions cannot capture values from outer scopes.
- `@convention(c)`: passed as a single function pointer, and also the parameter and return types must be representable in C.
- `@convention(block)`: passed as an Objective-C block, which allow captures but must have parameter and return types representable in Objective-C.

Each entry in the parameter list contains a parameter type and some non-type bits:

- The **value ownership kind**, which can be the default, `inout`, `borrowing` or `consuming`.

  The `inout` kind is key to Swift's mutable value type model; the interested reader can consult [49] for details. The last two were introduced in Swift 5.9 [50].

- The **variadic** flag, in which case the parameter type must be an array type.

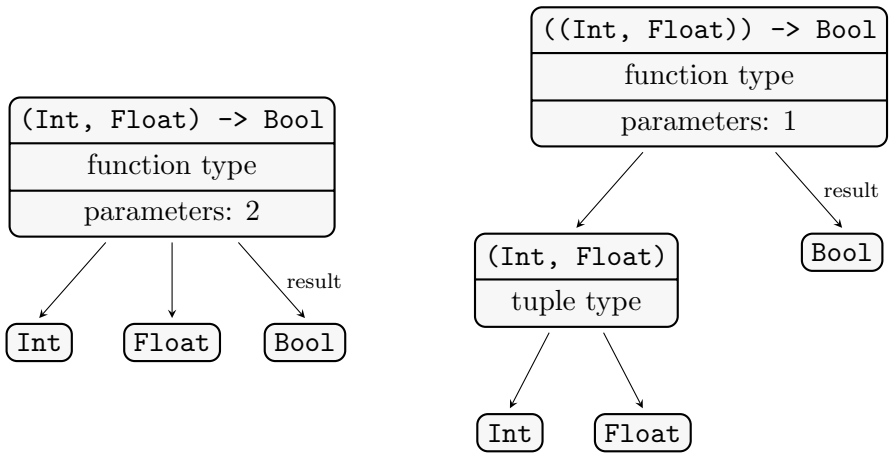
  When type checking a call to function value with a variadic parameter, the type checker collects multiple expressions from the call argument list into an implicit array expression. Otherwise, variadic parameters behave exactly like arrays once we get to SILGen and below.

- The `@autoclosure` attribute, in which case the parameter type must be another function type of the form `() -> T` for some type `T`.

  This instructs the type checker to treat the corresponding argument in the caller as if it was a value of type `T`, rather than a function type `() -> T`. The argument is then wrapped inside an implicit closure expression. In the body of the callee, an `@autoclosure` parameter behaves exactly like an ordinary function value, and can be called to evaluate the expression provided by the caller.

In functional languages such as ML and Haskell, all function types conceptually take a single parameter type. This is not the case in Swift, however. Figure 3.4 illustrates the distinction between `(Int, Float) -> Bool` and `((Int, Float)) -> Bool`; the first has two parameters, and the second has a single parameter that is of tuple type.

Figure 3.4.: Function type with two parameters, or a single tuple parameter



For convenience, the type checker defines an implicit conversion, called a "tuple splat," between function types taking a single tuple and function types of multiple arguments. This implicit conversion is only available when passing a function value as an argument to a call, and only when the function type's parameter list can be represented as a tuple (hence, it has no parameter attributes). An example:

```
func apply<T, U>(fn: (T) -> U, arg: T) -> U {
  return fn(arg)
}

// Tuple splat conversion:
// - the type of (+) is (Int, Int) -> (),
// - but apply() expects ((Int, Int)) -> ().
print(apply(fn: (+), arg: (1, 2)))
```

Another subtle point with function types is that argument labels are part of a function declaration's *name*, not a function declaration's *type*. A closure, being an anonymous function, is always called without argument labels. This includes a closure formed from an unapplied reference to a function declaration—even if the function declaration *does* have argument labels:

```
func subtract(minuend x: Int, subtrahend y: Int) -> Int {
  return x - y
}

print(subtract(minuend: 3, subtrahend: 1))  // prints 2

let fn1 = subtract  // declaration name can omit argument labels
print(fn1(3, 1))  // prints 2

let fn2 = subtract(minuend:subtrahend:)  // full declaration name
print(fn2(3, 1))  // prints 2
```

The history of Swift function types is an interesting case study in language evolution. Originally, Swift followed the classical functional language model, where a function type always had a *single* parameter type, which could be a tuple type to simulate a function of multiple parameters. Tuple types used to be able to contain `inout` and variadic elements, and furthermore, the argument labels of a function declaration were part of the function declaration's type. The existence of such "non-materializable" tuple types introduced complications throughout the type system, and argument labels had inconsistent behavior in different contexts.

The syntax for referencing a declaration name with argument labels was adopted in Swift 2.2 [51]. Subsequently, argument labels were dropped from function types in Swift 3 [52]. The distinction between a function taking multiple arguments and a function taking a single tuple argument was first hinted at in Swift 3 with [53] and [54], and became explicit in Swift 4 [55]. At the same time, Swift 4 also introduced the "tuple splat" function conversion to simulate the Swift 3 model for the cases where the old behavior was convenient. For example, the element type of `Dictionary` is a key/value pair, but often it is more convenient to call `Collection.map()` with a closure taking two arguments, instead of a single tuple argument.

Once the above proposals were implemented, the compiler continued to model function types as having a single input type for quite some time, despite this being completely hidden from the user. After Swift 5, the function type representation fully converged with the semantic model of the language.

It is worth noting that metatypes, tuple types and function types play an important role in the core language model, but they are not essential to the formal analysis of generics; one can construct a toy implementation of Swift generics from nominal types and type parameters alone. Indeed, all structural types can be seen as special kinds of type constructors, which have no special behavior other than possibly containing type parameters which can be substituted.

We finish this section by turning to sugared types. Sugared generic parameter types were already described in Section 3.1. Of the remaining kinds of sugared types, type alias types are defined by the user, and the other three are built-in to the language.

Type alias types. A type alias type represents a reference to a type alias declaration. It contains an optional parent type, a substitution map, and the substituted underlying type. The canonical type of a type alias type is the substituted underlying type. The substitution map is formed in type resolution, from any generic arguments applied to the type alias type declaration itself, together with the generic arguments of the parent type (Section 9.1). Type resolution applies this substitution map to the original underlying type of the type alias declaration to compute the substituted underlying type. The substitution map is preserved for printing, and for requirement inference (Section 11.1).

Optional types. The optional type is written as `T?` for some object type `T`; its canonical type is `Optional<T>`.

Array types. The array type is written as `[E]` for some element type `E`; its canonical type is `Array<E>`.

Dictionary types. The dictionary type is written as `[K: V]` for some key type `K` and value type `V`; its canonical type is `Dictionary<K, V>`.

3.3. Special Types

We now discuss a few of the remaining types, which are all weird in their own unique ways. They tend to only be valid in specific contexts, and some do not represent actual types of values at all. Their unexpected appearance can be a source of counter-examples and failed assertions. They all play important roles in the expression type checker, but again, do not really give us anything new if we consider the generics model from a purely formal viewpoint.

Generic function types. A generic function type has the same structural components as a function type, except it also stores a generic signature:

```
<S where S: Sequence> (S) -> S.Element
```

Generic function types are the interface types of generic function and subscript declarations. A reference to a generic function declaration from an expression always applies substitutions first, so generic function types do not appear as the types of expressions. In particular, an unsubstituted generic function value cannot be a parameter to another function, thus the Swift type system does not support higher-rank polymorphism. Type inference with higher-rank types is known to be undecidable; see [56] and [57].

Generic function types have a special behavior when their canonical type is computed. Since generic function types carry a generic signature, the parameter types and return type of a *canonical* generic function type are actually *reduced* types with respect to this generic signature (Section 5.4).

Reference storage types. A reference storage type is the type of a variable declaration adorned with the `weak`, `unowned` or `unowned(unsafe)` attribute. The wrapped type must be a class type, a class-constrained archetype, or class-constrained existential type. Reference storage types arise as the interface types of variable declarations, and as the types of SIL instructions. The types of expressions never contain reference storage types.

Placeholder types. A placeholder type represents a generic argument to be inferred by the type checker. The written representation is the underscore "`_`". They can only appear in a handful of restricted contexts, and do not appear in the types of expressions or the interface types of declarations after type checking. The constraint solver replaces placeholder types with type variables when solving the constraint system. For example, here, the interface type of the `myPets` local variable is inferred as `Array<String>`:

```
let myPets: Array<_> = ["Zelda", "Giblet"]
```

Placeholder types were introduced in Swift 5.6 [58].

Unbound generic types. Unbound generic types predate placeholder types, and can be seen as a special case. An unbound generic type is written as a named reference to a generic type declaration, without generic arguments applied. An unbound generic type behaves like a generic nominal type where all generic arguments are placeholder types. In previous example, we wrote the generic nominal type `Array<_>` with a placeholder type. The unbound generic type `Array` could have been used instead:

```
let myPets: Array = ["Zelda", "Giblet"]
```

Unbound generic types are also occasionally useful in diagnostics, to print the name of a type declaration only (like `Outer.Inner`) without the generic parameters of its declared interface type (`Outer<T>.Inner<U>`, for example). Unbound generic types are discussed in the context of type resolution in Section 9.4.

Dynamic Self types. The dynamic `Self` type appears when a class method declares a return type of `Self`. In this case, the object is known to have the same dynamic type as the base of the method call, which might be a subclass of the method's class. The dynamic `Self` type has one structural component, a class type, which is the static upper bound for the type. This concept comes from Objective-C, where it is called `instancetype`. The dynamic `Self` type in many ways behaves like a generic parameter, but it is not represented as one; the type checker and SILGen implement support for it directly. Note that the identifier "`Self`" only has this interpretation inside a class. In a protocol declaration, `Self` is the implicit generic parameter (Section 4.3). In a struct or enum declaration, `Self` is the declared interface type (Section 9.1).

Listing 3.1 demonstrates some of the behaviors of the dynamic `Self` type. Two invalid cases are shown; `invalid1()` is rejected because the type checker cannot prove that the return type is always an instance of the dynamic type of `self`, and `invalid2()` is rejected because `Self` appears in contravariant position.

Type variable types. A type variable represents the future inferred type of an expression in the expression type checker's constraint system. The expression type checker builds the constraint system by walking an expression recursively, assigning new type variables to the types of sub-expressions and recording constraints between these type variables. Solving the constraint system can have three possible outcomes:

- **One solution**—every type variable has exactly one fixed type assignment; the expression is well-typed.
- **No solutions**—some constraints could not be solved, indicating erroneous input.
- **Multiple solutions**—the constraint system is underspecified and some type variables can have multiple valid fixed type assignments.

Listing 3.1.: Dynamic `Self` type example

```
class Base {
  required init() {}

  func dynamicSelf() -> Self {
    // the type of ‘self’ in a method returning ‘Self’ is
    // the dynamic Self type.
    return self
  }

  func clone() -> Self {
    return Self()
  }

  func invalid1() -> Self {
    return Base()
  }

  func invalid2(_: Self) {}
}

class Derived: Base {}

let y = Derived().dynamicSelf()  // y has type ‘Derived’
let z = Derived().clone()  // z has type ‘Derived’
```

In the case of multiple solutions, the type checker uses heuristics to pick the "best" solution for the entire expression; if none of the solutions are clearly better than the others, an ambiguity error is diagnosed. Otherwise, we take the best solution and apply it to the expression, replacing type variables with their assigned fixed types.

The utmost care must be taken when working with type variables, because unlike all other types, they are not allocated with indefinite lifetime. Type variables live in the constraint solver arena, which grows and shrinks as the solver explores branches of the solution space. Types that *contain* type variables also need to be allocated in the constraint solver arena. This is also true of structures that contain types, such as substitution maps. Type variables "escaping" from the constraint solver can crash the compiler in odd ways. Assertions should be used to rule out type variables from appearing in the wrong places.

The printed representation of a type variable is `$Tn`, where `n` is an incrementing integer local to the constraint system. One way to see type variables in action is to pass the `-Xfrontend -debug-constraints` compiler flag.

L-value types. An l-value type represents the type of an expression appearing on the left hand side of an assignment operator (hence the "l" in l-value), or as an argument to an `inout` parameter in a function call. L-value types wrap an *object type* which is the type of the stored value; they print out as `@lvalue T` where `T` is the object type, but this is not valid syntax in the language.

L-value types appear in type-checked expressions. The reader familiar with C++ might think of an l-value type as somewhat analogous to a C++ mutable reference type "`T &`"—unlike C++ though, they are not directly visible in the source language. L-value types do not appear in the types of SIL instructions; SILGen lowers l-value accesses into accessor calls or direct manipulation of memory.

Error types. Error types are returned when type substitution encounters an invalid or missing conformance (Chapter 6). In this case, the error type wraps the original type, and prints as the original type to make types coming from malformed conformances more readable in diagnostics.

Error types are also returned by type resolution if the type representation is invalid in some way. This uses the singleton form of the error type, which prints as `<<error type>>`. To avoid user confusion, diagnostics containing the singleton error type should not be emitted. Generally, any expression whose type contains an error type does not need to be diagnosed, because a diagnostic should have been emitted elsewhere.

Built-in types. What users think of as fundamental types, such as `Int` and `Bool`, are defined as structs in the standard library. These structs wrap the various *built-in types*

which are understood directly by the compiler. Built-in types are not nominal types, so they cannot contain members, cannot have new members added via extensions, and cannot conform to protocols. Values of built-in types are manipulated using special *compiler intrinsics*. The standard library wraps built-in types in nominal types, and defines methods and operators on those nominal types which call the intrinsic functions, thereby presenting the actual interface expected by users.

For example, the `Int` struct defines a single stored property named `_value` with type `Builtin.Int`. The `+` operator on `Int` is implemented by extracting the `_value` stored property from a pair of `Int` values, calling the `Builtin.sadd_with_overflow_Int64` compiler intrinsic to add them together, and finally, wrapping the resulting `Builtin.Int` in a new instance of `Int`.

Built-in types and their intrinsics are defined in the special `Builtin` module, which is a module constructed by the compiler itself, and not built from source code. The `Builtin` module is only visible when the compiler is invoked with the `-parse-stdlib` frontend flag; the standard library is built with this flag, but user code never interacts with the `Builtin` module directly.

3.4. Source Code Reference

Key source files:

- include/swift/AST/Type.h
- include/swift/AST/Types.h
- lib/AST/Type.cpp

Other source files:

- include/swift/AST/TypeNodes.def
- include/swift/AST/TypeVisitor.h
- include/swift/AST/CanTypeVisitor.h

`Type` *class*

Represents an immutable, uniqued type. Meant to be passed as a value, it stores a single instance variable, a `TypeBase *` pointer.

The `getPointer()` method returns this pointer. The pointer is not `const`, however neither `TypeBase` nor any of its subclasses define any mutating methods. The pointer may be `nullptr`; the default constructor `Type()` constructs an instance of a null type. Most methods will crash if called on a null type; only the implicit `bool` conversion and `getPointer()` are safe.

The `getPointer()` method is only used occasionally, because types are usually passed as `Type` and not `TypeBase *`, and `Type` overloads `operator->` to forward method calls to the `TypeBase *` pointer. While most operations on types are actually methods on `TypeBase`, a few methods are also defined on `Type` itself (these are called with "`.`" instead of "`->`").

- **Various traversals:** `walk()` is a general pre-order traversal where the callback returns a tri-state value—continue, stop, or skip a sub-tree. Built on top of this are two simpler variants; `findIf()` takes a boolean predicate, and `visit()` takes a void-returning callback which offers no way to terminate the traversal.
- **Transformations:** `transformWithPosition()` and `transformRec()`. As with the traversals, the first is the most general, and the other one changes the signature of the callback to ignore a `TypePosition` parameter. The callback is invoked on all types contained within a type, recursively. It can either elect to replace a type with a new type, or leave a type unchanged and instead try to transform any of its child types.
- **Substitution:** `subst()` implements type substitution, which is a particularly common kind of transform which replaces generic parameters or archetypes with concrete types (Section 6.5).
- **Printing:** `print()` outputs the string form of a type, with many customization options; `dump()` prints the tree structure of a type in an s-expression form. The latter is extremely useful for invoking from inside a debugger, or ad-hoc print debug statements.

The `Type` class explicitly deletes the overloads of `operator==` and `operator!=` to make the choice between pointer and canonical equality explicit. To check type pointer equality of possibly-sugared types, first unwrap both sides with a `getPointer()` call:

```
if (lhsType.getPointer() == rhsType.getPointer())
  ...;
```

The more common canonical type equality check is implemented by the `isEqual()` method on `TypeBase`:

```
if (lhsType->isEqual(rhsType))
  ...;
```

`TypeBase` *class*

The root of the type kind hierarchy. Its instances are always uniqued and allocated by the AST context, either the permanent arena or constraint solver arena. Instances are usually wrapped in `Type`. The various subclasses correspond to the different kinds of types:

- `NominalType` and its four subclasses:
  - `StructType`,
  - `EnumType`,
  - `ClassType`,
  - `ProtocolType`.
- `BoundGenericNominalType` and its three subclasses:
  - `BoundGenericStructType`,
  - `BoundGenericEnumType`,
  - `BoundGenericClassType`.
- The structural types `TupleType`, `MetatypeType`.
- `AnyFunctionType` and its two subclasses:
  - `FunctionType`,
  - `GenericFunctionType`.
- `GenericTypeParamType`, `DependentMemberType`, the two type parameter types.
- `ArchetypeType`, and its three subclasses:
  - `PrimaryArchetypeType`,
  - `OpenedArchetypeType`,
  - `OpaqueArchetypeType`.
- The abstract types:
  - `ProtocolCompositionType`,
  - `ParameterizedProtocolType`,
  - `ExistentialType`,
  - `ExistentialMetatypeType`,
  - `DynamicSelfType`.
- `SugarType` and its four subclasses:
  - `TypeAliasType`,

  - `OptionalType`,
  - `ArrayType`,
  - `DictionaryType`.

- `BuiltinType` and its subclasses (there are a bunch of esoteric ones; only a few are shown below):
  - `BuiltinRawPointerType`,
  - `BuiltinVectorType`,
  - `BuiltinIntegerType`,
  - `BuiltinIntegerLiteralType`,
  - `BuiltinNativeObjectType`,
  - `BuiltinBridgeObjectType`.

- `ReferenceStorageType` and its two subclasses:
  - `WeakStorageType`,
  - `UnownedStorageType`.

- Miscellaneous types:
  - `UnboundGenericType`,
  - `PlaceholderType`,
  - `TypeVariableType`,
  - `LValueType`,
  - `ErrorType`.

Each concrete subclass defines some set of static factory methods, usually named `get()` or similar, which take the structural components and construct a new, uniqued type of this kind. There are also getter methods, prefixed with `get`, which project the structural components of each kind of type. It would be needlessly duplicative to list all of the getter methods for each subclass of `TypeBase`; they can all be found in `include/swift/AST/Types.h`.

Dynamic casts. Subclasses of `TypeBase *` are identifiable at runtime via the `is<>`, `castTo<>` and `getAs<>` template methods. To check if a type has a specific kind, use `is<>`:

```
Type type = ...;
```

```
if (type->is<FunctionType>())
  ...;
```

To conditionally cast a type to a specific kind, use `getAs<>`, which returns `nullptr` if the cast fails:

```
if (FunctionType *funcTy = type->getAs<FunctionType>())
  ...;
```

Finally, `castTo<>` is an unconditional cast which asserts that the type has the required kind:

```
FunctionType *funcTy = type->castTo<FunctionType>();
```

These template methods desugar the type if it is a sugared type, and the casted type can never itself be a sugared type. This is usually correct; for example, if `type` is the `Swift.Void` type alias type, then `type->is<TupleType>()` returns true, because it is for all intents and purposes a tuple (an empty tuple), except when printed in diagnostics.

There are also top-level template functions `isa<>`, `dyn_cast<>` and `cast<>` that operate on `TypeBase *`. Using these with `Type` is an error; the pointer must be explicitly unwrapped with `getPointer()` first. These casts do not desugar, and permit casting to sugared types. This is the mechanism used when sugared types must be distinguished from canonical types for some reason:

```
Type type = ...;

if (isa<OptionalType>(type.getPointer()))
  ...;
```

Visitors. The simplest way to exhaustively handle each kind of type is to switch over the kind, which is an instance of the `TypeKind` enum, like this:

```
Type ty = ...;
switch (ty->getKind()) {
case TypeKind::Struct: {
  auto *structTy = ty->castTo<StructType>();
  ...
}
case TypeKind::Enum:
  ...
case TypeKind::Class:
```

```
  ...
}
```

However, in most cases it is more convenient to use the *visitor pattern* instead, by subclassing `TypeVisitor` and overriding various `visit`*`Kind`*`Type()` methods. Then, invoking the visitor's `visit()` method with a type performs the same switch and dynamic cast dance above:

```
class MyVisitor: public TypeVisitor<MyVisitor> {
public:
  void visitStructType(StructType *ty) {
    ...
  }
};

MyVisitor visitor;

Type ty = ...;
visitor.visit(ty);
```

The `TypeVisitor` also defines various methods corresponding to abstract base classes in the `TypeBase` hierarchy, so, for example, you can override `visitNominalType()` to handle all nominal types at once.

The `TypeVisitor` preserves information if it receives a sugared type; for example, visiting `Int?` will call `visitOptionalType()`, while visiting `Optional<Int>` will call `visitBoundGenericEnumType()`. In the common situation where the semantics of your operation do not depend on type sugar, you can use the `CanTypeVisitor` template class instead. Here, the `visit()` method takes a `CanType`, so `Int?` will need to be canonicalized to `Optional<Int>` before being passed in.

Canonical types. The `getCanonicalType()` method outputs a `CanType` wrapping the canonical form of this `TypeBase *`. The `isCanonical()` method checks if a type is canonical.

Nominal types. A handful of methods on `TypeBase` exist which perform a desugaring cast to a nominal type (so they will also accept a type alias type or other sugared type), and return the nominal type declaration, or `nullptr` if the type isn't of a nominal kind:

- `getAnyNominal()` returns the nominal type declaration of `UnboundGenericType`, `NominalType` or `BoundGenericNominalType`.

- `getNominalOrBoundGenericNominal()` returns the nominal type declaration of a `NominalType` or `BoundGenericNominalType`.

- `getStructOrBoundGenericStruct()` returns the type declaration of a `StructType` or `BoundGenericStructType`.

- `getEnumOrBoundGenericEnum()` returns the type declaration of an `EnumType` or `BoundGenericEnumType`.

- `getClassOrBoundGenericClass()` returns the class declaration of a `ClassType` or `BoundGenericClassType`.

- `getNominalParent()` returns the parent type stored by an `UnboundGenericType`, `NominalType` or `BoundGenericNominalType`.

Recursive properties. Various predicates are computed when a type is constructed and are therefore cheap to check:

- `hasTypeVariable()` determines whether the type was allocated in the permanent arena or the constraint solver arena.

- `hasArchetype()`, `hasOpaqueArchetype()`, `hasOpenedExistential()`.

- `hasTypeParameter()`.

- `hasUnboundGenericType()`, `hasDynamicSelf()`, `hasPlaceholder()`.

- `hasLValueType()` determines whether the type contains an l-value type.

Utility operations. These encapsulate frequently-useful patterns.

- `getOptionalObjectType()` returns the type `T` if the type is some `Optional<T>`, otherwise it returns the null type.

- `getMetatypeInstanceType()` returns the type `T` if the type is some `T.Type`, otherwise it returns `T`.

- `mayHaveMembers()` answers if this is a nominal type, archetype, existential type or dynamic Self type.

Recovering the AST context. All non-canonical types point at their canonical type, and canonical types point at the AST context.

- `getASTContext()` returns the singleton AST context from a type.

`CanType` *class*

The `CanType` class wraps a `TypeBase *` pointer which is known to be canonical. The pointer can be recovered with the `getPointer()` method. It forwards various methods to either `Type` or `TypeBase *`. There is an implicit conversion from `CanType` to `Type`. In the other direction, the explicit one-argument constructor `CanType(Type)` asserts that the type is canonical; however, most of the time the `getCanonicalType()` method on `TypeBase` is used instead.

The `operator==` and `operator!=` operators are used to test `CanType` for type pointer equality. The `isEqual()` method described earlier implements canonical equality on sugared types by first canonicalizing both sides, and then checking the resulting canonical types for type pointer equality. Therefore, the following are equivalent:

```
if (lhsType->isEqual(rhsType)) ...;
if (lhsType->getCanonicalType() == rhsType->getCanonicalType()) ...;
```

The `CanType` class can be used with the `isa<>`, `cast<>` and `dyn_cast<>` templates. Instead of returning the actual `TypeBase` subclass, the latter two return a *canonical type wrapper* for that subclass. Every subclass of `TypeBase` has a corresponding canonical type wrapper; if the subclass is named `FooType`, the canonical wrapper is named `CanFooType`. Canonical type wrappers forward `operator->` to the specific `TypeBase` subclass, and define methods of their own (called with ".") which project the known-canonical components of the type.

For example, `FunctionType` has a `getResult()` method returning `Type`, so the canonical type wrapper `CanFunctionType` has a `getResult()` method returning a `CanType`. The wrapper methods are not exhaustive, and their use is not required because you can instead make explicit calls to `CanType(Type)` or `getCanonicalType()` after projecting a type that is known to be canonical.

```
CanType canTy = ...;
CanFunctionType canFuncTy = cast<FunctionType>(canTy);

// method on CanFunctionType: returns CanType(canFuncTy->getResult())
CanType canResultTy = canFuncTy.getResult();

// operator-> forwards to method on FunctionType: returns Type
CanType resultTy = CanType(canFuncTy->getResult());
```

`AnyFunctionType` *class*

This is the base class of `FunctionType` and `GenericFunctionType`.

- `getParams()` returns an array of `AnyFunctionType::Param`.
- `getResult()` returns the result type.
- `getExtInfo()` returns an instance of `AnyFunctionType::ExtInfo` storing the additional non-type attributes.

`AnyFunctionType::Param` *class*

This represents a parameter in a function type's parameter list.

- `getPlainType()` returns the type of the parameter. If the parameter is variadic (`T...`), this is the element type `T`.
- `getParameterType()` same as above, but if the parameter is variadic, returns the type `Array<T>`.
- `isVariadic()`, `isAutoClosure()` are the special behaviors.
- `getValueOwnership()` returns an instance of the `ValueOwnership` enum.

`ValueOwnership` *enum class*

The possible ownership attributes on a function parameter.

- `ValueOwnership::Default`
- `ValueOwnership::InOut`
- `ValueOwnership::Shared`
- `ValueOwnership::Owned`

`AnyFunctionType::ExtInfo` *class*

This represents the non-type attributes of a function type.

4. Declarations

Declarations are the building blocks of Swift programs. In Chapter 2, we started by viewing the user's program as a series of module declarations, where a module declaration holds file units. A file unit further holds a list of top-level declarations, which correspond to the main divisions in a source file. The different kinds of declarations are categorized into a taxonomy, and we will survey this taxonomy, as we did with types in Chapter 3. Our principal goal will be describing the syntactic representations for declaring generic parameters and stating requirements, which are common to all generic declarations; once we have that, we can proceed to Part II.

We begin with two major divisions in the declaration taxonomy:

1. A *value declaration* is a one that can be referenced by name from an expression; this includes variables, functions, and such. Every value declaration has an *interface type*, which is the type assigned to an expression that names this declaration.

2. A *type declaration* is one that can be referenced by name from within a type representation. This includes structs, type aliases, and so on. A type declaration declares a type, called the *declared interface type* of the type declaration.

Not all declarations are value declarations. An extension declaration adds members to an existing nominal type declaration, as we'll see in Chapter 10, but an extension does not itself have a name. A *top-level code declaration* holds the statements and expressions written at the top level of a source file, and again, it does not have a name, semantically.

Declaration contexts. Every declaration is contained in a *declaration context*, and a declaration context is anything that *contains* declarations. Consider this program:

```
func squares(_ nums: [Int]) -> [Int] {
  return nums.map { x in x * x }
}
```

The parameter declaration "`x`" is a child of the closure expression "`{ x in x * x }`", and not a direct child of the enclosing function declaration. So a closure expression is a declaration context, but not a declaration. On the other hand, a parameter declaration is a declaration, but not a declaration context. Finally, the `squares()` function itself is both a declaration, and a declaration context.

Type declarations. Types can be written inside expressions, so every type declaration is also a value declaration. We can understand the relationship between the interface type and declared interface type of a type declaration by looking at this example:

```
struct Horse {}
let myHorse: Horse = Horse()
```

The struct declaration `Horse` is referenced twice, first in the type representation on the left-hand side of "`=`" and then again in the initial value expression on the right. On the left-hand side, it's referenced as a type declaration; we want the *declared interface type*, which is the nominal type `Horse`, because this type's values are stored inside the `myHorse` variable. The second reference to `Horse`, within the call expression, refers to the *type itself* as a value declaration, so we want the *interface type*, which is the metatype `Horse.Type`. (Recall the diagram from Section 3.2.) When a metatype is the callee in a call expression, we interpret it as looking up the member named `init`:

```
struct Horse {}
let myHorseType: Horse.Type = Horse.self
let myHorse: Horse = myHorseType.init()
```

The interface type of a type declaration always wraps its declared interface type in a metatype type. (It sounds like a mouthful, but the idea is simple.)

Nominal type declarations. Introduced with the `struct`, `enum` and `class` keywords; Swift 5.5 also added `actor`, which to us is just a class [59]. Nominal type declarations are declaration contexts, and the declarations they contain are called their *member declarations*. If a member declaration is a function, we call it a *method*, a member variable is a *property*, and a *member type declaration* is exactly that.

Structs and classes can contain a special kind of property declaration called a *stored property declaration*. Struct values directly store their stored properties, while a class value is a reference to a heap allocated box. Enum values store exactly one element among several; enum declarations instead contain *enum element declarations*, introduced with `case`.

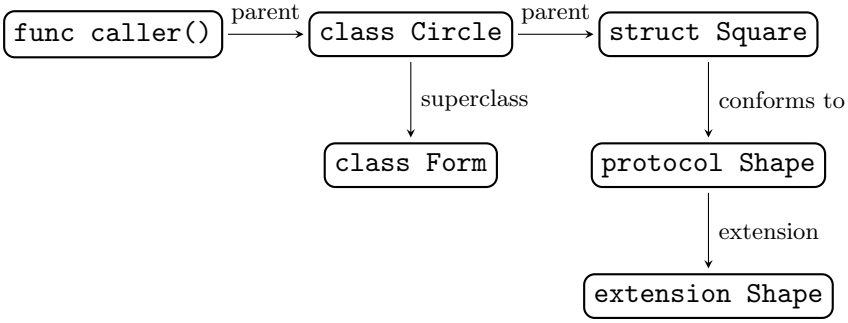
The members of a nominal type declaration are visible to name lookup (Section 2.1), both in the nominal type declaration's scope (unqualified lookup) and outside (qualified lookup). Listing 4.1 shows three features we will cover in detail later:

- Nominal type declarations can conform to protocols (Chapter 7).
- Extensions add members to existing nominal type declarations (Chapter 10).
- A class can inherit from a superclass type, and members of the superclass are also visible from the subclass (Chapter 15).

Listing 4.1.: Some behaviors of name lookup

```
class Form { static func callee1() {} }
protocol Shape { static func callee2() }
extension Shape { static func callee3() {} }

struct Square: Shape {
  class Circle: Form {
    static func caller() {
      ...  // unqualified lookup from here
    }
  }
}
```

The body of `caller()` can reference `callee1()`, `callee2()` or `callee3()` with a single identifier. To resolve the identifier, unqualified lookup must search through these declaration contexts, starting in the top-left corner:



We will say more about name lookup in Chapter 9 and Section 10.2.

A nominal type declaration declares a new type with its own name and identity (hence "nominal"). The declared interface type of a nominal type declaration is called a nominal type, which we talked about in Section 3.1:

```
struct Universe {     // declared interface type: Universe
  struct Galaxy {}    // declared interface type: Universe.Galaxy

  func solarSystem() {
    struct Planet()   // declared interface type: Planet
  }
}
```

The declared interface type of the `Galaxy` struct is `Universe.Galaxy`, while the declared interface type of `Planet` is just `Planet`, with no parent type. This reflects the semantic difference; `Galaxy` is visible to qualified lookup as a member of `Universe`, while `Planet` is only visible to unqualified lookup within the scope of `solarSystem()`; we call it a *local type declaration*. Section 6.4 gives more detail about nominal type nesting.

Type alias declarations. These are introduced by the `typealias` keyword. The underlying type is written on the right-hand side of "`=`":

```
typealias Hands = Int  // one hand is four inches
func measure(horse: Horse) -> Hands {...}
let metatype = Hands.self
```

The declared interface type of a type alias declaration is a type alias type. The canonical type of this type alias type is just the underlying type. Therefore, if we print the return type of `measure()` in a diagnostic message, we will print it as "`Hands`", but otherwise it behaves as if it were an `Int`.

As with all type declarations, the interface type of a type alias declaration is the metatype of its declared interface type. In the above, the expression "`Hands.self`" has the metatype type `Hands.Type`. This is a sugared type, canonically equal to `Int.Type`.

While type aliases are declaration contexts, the only declarations a type alias can contain are generic parameter declarations, in the event the type alias is generic.

Other type declarations. We've now seen the first two kinds of type declarations. In the next two sections, we expand on this by looking at the declarations of generic parameters, protocols and associated types. At this point, our foray into Swift generics can begin in earnest. Here are all the type declaration kinds and their declared interface types, with a representative specimen of each:

Type declaration	**Declared interface type**
Nominal type declaration: `struct Horse {...}`	Nominal type: `Horse`
Type alias declaration: `typealias Hands = Int`	Type alias type: `Hands`
Generic parameter declaration: `<T: Sequence>`	Generic parameter type: `T` (or `τ_0_0`)
Protocol declaration: `protocol Sequence {...}`	Protocol type: `Sequence`
Associated type declaration: `associatedtype Element`	Dependent member type: `Self.Element`

4.1. Generic Parameters

Various kinds of declarations can have a generic parameter list. We call them *generic declarations*. We start with those where the generic parameter list is written in source: structs, enums, classes, type aliases, functions and constructors, and subscripts. In all cases, a generic parameter list is denoted in source with the `<...>` syntax following the name of the declaration:

```
struct Outer<T> {...}
```

Each comma-separated element in this list is a *generic parameter declaration*; this is a type declaration that declares a generic parameter type. Generic parameter declarations are visible to unqualified lookup in the entire source range of the parent declaration, the one that has the generic parameter list. When generic declarations nest, each inner generic declaration is effectively parameterized by the generic parameters of its outer declarations.

Any declaration kind that can have a generic parameter list is also a declaration context in our taxonomy, because it contains other declarations; namely, its generic parameter declarations. We say that a declaration context is a *generic context* if at least one parent context has a generic parameter list.

The name of a generic parameter declaration plays no role after unqualified lookup. Instead, to each generic parameter declaration, we assign a pair of integers (or more accurately, natural numbers; they're non-negative), the *depth* and the *index*:

- The depth selects a generic parameter list; the generic parameters declared by the outermost generic parameter list are at depth zero, and we increment the depth by one for each nested generic parameter list.

- The index selects a generic parameter within a generic parameter list; we number sibling generic parameter declarations consecutively starting from zero.

Let's write some nested generic declarations inside the `Outer` struct above. In the following, `two()` is generic over `T` and `U`, while `four()` is generic over `T`, `V`, `W` and `X`:

```
struct Outer<T> {
  func two<U>(u: U) -> T {...}

  struct Both<V, W> {
    func four<X>() -> X {...}
  }
}
```

When type resolution resolves the type representation "`T`" in the return type of `two()`, it outputs a generic parameter type that prints as "`T`", if it appears in a diagnostic for example. This is a sugared type. Every generic parameter type also has a canonical form which only records the depth and index; we denote a canonical generic parameter type by "`τ_d_i`", where `d` is the depth and `i` is the index. Two generic parameter types are canonically equal if they have the same depth and index. This is sound, because the depth and index unambiguously identify a generic parameter within its lexical scope.

Let's enumerate all generic parameters visible within `two()`,

Name:	`T`	`U`
Depth:	0	1
Index:	0	0
Type:	`τ_0_0`	`τ_1_0`

and `four()`,

Name:	`T`	`V`	`W`	`X`
Depth:	0	1	1	2
Index:	0	0	1	0
Type:	`τ_0_0`	`τ_1_0`	`τ_1_1`	`τ_2_0`

The generic parameter `U` of `two()` has the same declared interface type, `τ_1_0`, as the generic parameter `V` of `four()`. This is not a problem because the source ranges of their parent declarations, `two()` and `Both`, do not intersect.

The numbering by depth can be seen in the declared interface type of a nested generic nominal type declaration. For example, the declared interface type of `Outer.Both` is the generic nominal type `Outer<τ_0_0>.Both<τ_1_0, τ_1_1>`.

Implicit generic parameters. Sometimes the generic parameter list is not written in source. Every protocol declaration has a generic parameter list with a single generic parameter named `Self` (Section 4.3), and every extension declaration has a generic parameter list cloned from that of the extended type (Chapter 10). These implicit generic parameters can be referenced by name within their scope, just like the generic parameter declarations in a parsed generic parameter list (Section 9.1).

Function, constructor and subscript declarations can also declare *opaque parameters* with the `some` keyword, possibly in combination with a generic parameter list:

```
func pickElement<E>(_ elts: some Sequence<E>) -> E {...}
```

An opaque parameter simultaneously declares a parameter value, a generic parameter type that is the type of the value, and a requirement this type must satisfy. Here, we can

refer to "`elts`" from an expression inside the function body, but we cannot name the *type* of "`elts`" in a type representation. From expression context however, the type of an opaque parameter can be obtained via the `type(of:)` special form, which produces a metatype value. This allows for invoking static methods on these types.

The **generic parameter list request** appends the opaque parameters to the parsed generic parameter list, so they follow the parsed generic parameters in index order. In `pickElement()`, the generic parameter `E` has canonical type `τ_0_0`, while the opaque parameter associated with "`elts`" has canonical type `τ_0_1`. Opaque parameter declarations also state a constraint type, which imposes a requirement on this unnamed generic parameter. We will discuss this in the next section. Note that when `some` appears in the return type of a function, it declares an *opaque return type*, which is a related but different feature (Chapter 13).

In Chapter 5, we define the generic signature, which records all visible generic parameters of a declaration, independent of surface syntax.

4.2. Requirements

The requirements of a generic declaration constrain the generic argument types that can be provided by the caller. This endows the generic declaration's type parameters with new capabilities, so they abstract over the concrete types that satisfy those requirements. We use the following encoding for requirements in the theory and implementation.

Definition 4.1. A *requirement* is a triple consisting of a *requirement kind*, a subject type `T` (usually a type parameter), and one final piece of information that depends on the requirement kind:

- A **conformance requirement** [`T: P`] states that the replacement type for `T` must conform to `P`, which must be a protocol, protocol composition, or parameterized protocol type.
- A **superclass requirement** [`T: C`] states that the replacement type for `T` must be a subclass of some class type `C`.
- A **layout requirement** [`T: AnyObject`] states that the replacement type for `T` must be represented as a single reference-counted pointer at runtime.
- A **same-type requirement** [`T == U`] states that the replacement types for `T` and `U` must be canonically equal.

When looking at concrete instances of requirements in a self-contained snippet of code, there is no ambiguity in using the same notation for the first three kinds, because the type referenced by the right-hand side determines the requirement kind. When talking about requirements in the abstract, we will explicitly state that `P` is some protocol, or `C` is some class, before talking about [`T: P`] or [`T: C`].

Constraint types. Before we introduce the trailing `where` clause syntax for stating requirements in a fully general way, let's look at the shorthand of stating a *constraint type* in the inheritance clause of a generic parameter declaration:

```
func allEqual<E: Equatable>(_ elements: [E]) {...}
```

The generic parameter declaration `E` declares the generic parameter type `τ_0_0`, and it also states the constraint type `Equatable`. This is a protocol declared in the standard library, so the stated requirement is the conformance requirement [`τ_0_0: Equatable`]. More generally, the constraint type is one of the following:

1. A protocol type, like `Equatable`.
2. A parameterized protocol type, like `Sequence<String>`.
3. A protocol composition type, like `Sequence & MyClass`.
4. A class type, like `NSObject`.
5. The `AnyObject` *layout constraint*, which restricts the possible concrete types to those represented as a single reference-counted pointer.

In the first three cases, the stated requirement becomes a conformance requirement. Otherwise, it is a superclass or layout requirement. In all cases, the subject type of the requirement is the declared interface type of the generic parameter.

Example 4.2. Notice how the generic parameter `B` of `open()` states the constraint type `Box<C>`, and this refers to the second generic parameter, `C`:

```
func open<B: Box<C>, C>(box: B) -> C {
  return box.contents!
}

class Box<Contents> {
  var contents: Contents? = nil
}
```

This illustrates a property of the scope tree: generic parameters are visible in the entire source range of a generic declaration, including in the generic parameter list itself. The declaration of `open()` thus states the superclass requirement [`τ_0_0: Box<τ_0_1>`]. Here is a possible usage of `open()`, which we leave unexplained:

```
struct Vegetable {}
class FarmBox: Box<Vegetable> {}
let vegetable: Vegetable = open(box: FarmBox())
```

Opaque parameters. Recall from Section 4.1 that an opaque parameter declares a parameter value, a generic parameter type, and a constraint type. The constraint type defines a conformance, superclass or layout requirement on the generic parameter type. Both of the following declarations state the requirement [τ_0_1: Sequence<τ_0_0>]:

```
func pickElement<E>(_ elts: some Sequence<E>) -> E {...}
func pickElement<E, S: Sequence<E>>(_ elts: S) -> E {...}
```

We will see later that constraint types also appear in various other positions, and in all cases, they state a requirement with some distinguished subject type:

1. In the inheritance clause of a protocol or associated type (Section 4.3).
2. Following the `some` keyword in return position, where it declares an opaque return type (Chapter 13).
3. Following the `any` keyword that references an existential type (Chapter 14), with the exception that the constraint type cannot be a class by itself (for example, we allow "`any NSObject & Equatable`", but "`any NSObject`" is just "`NSObject`").

Trailing where clauses. Requirements can also be stated in a `where` clause attached to the generic declaration. This allows generality that cannot be expressed using the inheritance clause of a generic parameter alone.

A `where` clause entry defines a requirement whose subject type is written explicitly, so that dependent member types can be be subject to requirements; here, we state two requirements, [τ_0_0: Sequence] and [τ_0_0.Element: Comparable]:

```
func isSorted<S>(_: S) where S: Sequence, S.Element: Comparable {...}
```

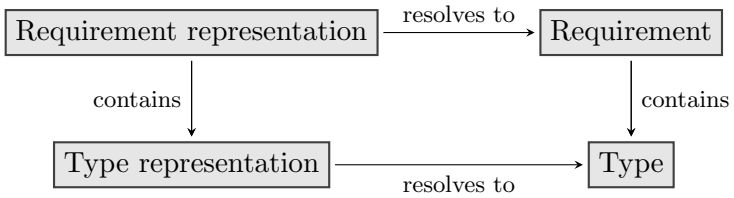
A `where` clause can also state a same-type requirement. In the next example, we state two conformance requirements using the inheritance clause syntax, another conformance requirement, and the same-type requirement [τ_0_0.Element == τ_0_1.Element]:

```
func merge<S1: Sequence, S2: Sequence>(_: S1, _: S2) -> [S1.Element]
    where S1: Comparable, S1.Element == S2.Element {...}
```

Note that there is no way to refer to an opaque parameter type within the function's `where` clause, but every declaration using opaque parameters can always be rewritten into an equivalent one using named generic parameters, so no generality is lost.

We saw in Chapter 3 that when the parser reads a type annotation in the source, it constructs a type representation, a lower-level syntactic object which must be resolved to obtain a type. Similarly, requirements have a syntactic form, called a *requirement*

representation. The parser constructs requirement representations while reading a `where` clause. The relationship between the syntactic and semantic entities is shown in this diagram:



There are only two kinds of requirement representations, because the "`:`" form cannot distinguish conformance, superclass and layout requirements until we resolve the type representation on the right-hand side:

1. A **constraint requirement representation** "`T: C`", where `T` and `C` are type representations.
2. A **same-type requirement representation** "`T == U`", where `T` and `U` are type representations.

Recall that a conformance requirement has a protocol type, protocol composition type, or parameterized protocol type on the right-hand side (Section 3.1). In the case of a protocol composition, we decompose the requirement into simpler requirements, one for each member of the composition. For example, if `MyClass` is a class, the requirement [`τ_0_0: Sequence & MyClass`] decomposes as [`τ_0_0: Sequence`] and [`τ_0_0: MyClass`], the latter being a superclass requirement. The empty protocol composition, written `Any`, is a trivial case; stating a conformance requirement to `Any` does nothing in a `where` clause, but it is allowed. Parameterized protocol types also decompose, as we'll see in Section 4.3.

The next chapter will introduce the derived requirements formalism. We will assume that only conformance requirements to protocol types remain, and similarly, that the subject types of requirements are type parameters, and not arbitrary types. Section 11.2 will show how we eliminate these unnecessary forms of generality.

Contextually-generic declarations. A generic declaration nested inside of another generic declaration can state a `where` clause, without introducing new generic parameters of its own. This is called a *contextually-generic declaration*:

```
enum LinkedList<Element> {...}

extension LinkedList {
  func sum() -> Element where Element: AdditiveArithmetic {...}
}
```

There is no semantic distinction between attaching a `where` clause to a member of a type, or moving the member to a constrained extension (Section 10.3), so the above is equivalent to the following:

```
extension LinkedList where Element: AdditiveArithmetic {
  func sum() -> Element {...}
}
```

However, for historical reasons, these two declarations have distinct mangled symbol names, so the above is not an ABI-compatible transformation.

In Chapter 5, we will see that the generic signature of a declaration records all of its requirements, regardless of they were stated in source.

History. The syntax described in this section has evolved over time:

- The `where` clause used to be written within the "`<`" and "`>`", but was moved to the current "trailing" position in Swift 3 [60].
- Generic type aliases were introduced in Swift 3 [61].
- Protocol compositions involving class types were introduced in Swift 4 [62].
- Generic subscripts were introduced in Swift 4 [63].
- Implementation limitations prevented the `where` clause from stating requirements that constrain outer generic parameters until Swift 3, and contextually-generic declarations were not allowed until Swift 5.3 [64].
- Opaque parameter declarations were introduced in Swift 5.7 [65].

4.3. Protocols

The `protocol` keyword introduces a *protocol declaration*, which is a special kind of nominal type declaration. The members of a protocol, with the exception of type aliases, are requirements that must be witnessed by corresponding members in each conforming type. In particular, the property, subscript and method declarations inside a protocol don't have bodies, but are otherwise represented in a similar way to the concrete case. A protocol declaration's declared interface type is a protocol type.

Every protocol has an implicit generic parameter list with a single generic parameter named `Self`, which abstracts over the conforming type. The declared interface type of `Self` is always `τ_0_0`; protocols cannot be nested in other generic contexts (Section 6.4), nor can they declare any other generic parameters.

The `associatedtype` keyword introduces an *associated type declaration*, which can only appear inside of a protocol. The declared interface type is a dependent member type (Section 3.1). Specifically, the declared interface type of an associated type `A` in a protocol `P` is the bound dependent member type denoted `Self.[P]A`, formed from the base type of `Self` together with `A`. A nominal type conforming to this protocol must declare a type witness for each associated type (Section 7.3).

Protocols can also state *associated requirements* on their `Self` type and its dependent member types. The conforming type and its type witnesses must satisfy the protocol's associated requirements. We will review all the ways of stating associated requirements now.

Protocol inheritance clauses. A protocol can have an inheritance clause with a list of one or more comma-separated constraint types. Each inheritance clause entry states an associated requirement with a subject type of `Self`. These are additional requirements the conforming type itself must satisfy in order to conform.

An associated conformance requirement with a subject type of `Self` establishes a *protocol inheritance* relationship. The protocol stating the requirement is the *derived protocol*, and the protocol on the right-hand side is the *base protocol*. The derived protocol is said to *inherit* from (or sometimes, *refine*) the base protocol. A qualified lookup will search through all base protocols, when the lookup begins at a derived protocol or one of its concrete conforming types.

For example, the standard library's `Collection` protocol inherits from `Sequence` by stating the associated requirement [`Self: Sequence`]:

```
protocol Collection: Sequence {...}
```

Protocols can restrict their conforming types to those with a reference-counted pointer representation by stating an `AnyObject` layout constraint in the inheritance clause:

```
protocol BoxProtocol: AnyObject {...}
```

Protocols can also limit their conforming types to subclasses of some superclass:

```
class Plant {}
class Animal {}
protocol Duck: Animal {}
class MockDuck: Plant, Duck {}  // error: not a subclass of Animal
```

A protocol is *class-constrained* if the [`Self: AnyObject`] associated requirement is either explicitly stated, or a consequence of some other associated requirement. We'll say more about the semantics of protocol inheritance clauses and name lookup in Section 5.1, Section 9.1, and Chapter 11.

Primary associated types. A protocol can declare a list of *primary associated types* with a syntax resembling that of a generic parameter list:

```
protocol IteratorProtocol<Element> {
  associatedtype Element
  mutating func next() -> Element?
}
```

While generic parameter lists introduce new generic parameter declarations, the entries in the primary associated type list reference an *existing* associated type declaration, either in the protocol itself, or some base protocol.

A *parameterized protocol type* can be formed from a reference to a protocol with primary associated types, by taking a list of generic argument types, one for each primary associated type. On the right-hand side of a conformance requirement, a parameterized protocol type decomposes into a conformance requirement to the protocol, followed by a series of same-type requirements. The following are equivalent:

```
func sumOfSquares<I>(_: I) -> Int
    where I: IteratorProtocol<Int> {...}
func sumOfSquares<I>(_: I) -> Int
    where I: IteratorProtocol, I.Element == Int {...}
```

More details appear in Section 11.2. Parameterized protocol types and primary associated types were added to the language in Swift 5.7 [46].

Associated requirements. An associated type declaration can have an inheritance clause, consisting of one or more comma-separated constraint types. Each entry defines a requirement on the declared interface type of the associated type declaration, so we get [`Self.Data: Codable`] and [`Self.Data: Hashable`] below:

```
associatedtype Data: Codable, Hashable
```

An associated type declaration can also have a trailing `where` clause, which allows associated requirements to be stated in full generality. The standard library `Sequence` protocol showcases both primary associated types, and associated requirements:

```
protocol Sequence<Element> {
  associatedtype Iterator: IteratorProtocol
  associatedtype Element where Element == Iterator.Element

  func makeIterator() -> Iterator
}
```

The associated conformance requirement on `Self.Iterator` could have been stated using a `where` clause instead:

```
associatedtype Iterator where Iterator: IteratorProtocol
```

A `where` clause can also be attached to the protocol itself; there is no semantic difference between that and attaching it to an associated type declaration:

```
protocol Sequence where Iterator: IteratorProtocol,
                        Element == Iterator.Element {...}
```

Finally, we can explicitly qualify the member types with `Self`:

```
protocol Sequence where Self.Iterator: IteratorProtocol,
                        Self.Element == Self.Iterator.Element {...}
```

In all cases, we state the same two associated requirements. Our notation is to append a subscript with the protocol name declaring the requirement:

$$[\texttt{Self.Iterator: IteratorProtocol}]_{\texttt{Sequence}}$$
$$[\texttt{Self.Element == Self.Iterator.Element}]_{\texttt{Sequence}}$$

Let's summarize all the ways of stating associated requirements in a protocol `P`:

- The protocol can state an inheritance clause. Each entry defines a conformance, superclass or layout requirement with a subject type of `Self`.
- An associated type declaration `A` can state an inheritance clause. Each entry defines a conformance, superclass or layout requirement with a subject type of `Self.[P]A`.
- Arbitrary associated requirements can be stated in trailing `where` clauses, attached to the protocol or any of its associated types, in any combination.

A protocol's associated requirements are collected in its requirement signature, which we will see is dual to a generic signature in some sense (Section 5.1). How concrete types satisfy the requirement signature will be discussed in Chapter 7.

Self requirements. The `where` clause of a protocol method or subscript requirement cannot constrain `Self` or its associated types. For example, the following protocol is rejected, because there would be no way to implement the `minElement()` requirement in a concrete conforming type whose `Element` type is *not* `Comparable`:

```
protocol SetProtocol {
  associatedtype Element  // we want 'where Element: Comparable' here
  func minElement() -> Element where Element: Comparable  // error
}
```

History. Older releases of Swift permitted protocols and associated types to state constraint types in their inheritance clauses, but more general associated requirements did not exist. Associated requirements were introduced when the trailing `where` clause syntax was generalized to associated types and protocols in Swift 4 [66].

4.4. Functions

A function declaration can appear at the top level of a source file, as a member of a nominal type or extension (which we recall is called a method declaration), or as a *local function* nested inside of another function. In this section, we will describe the computation of the interface type of a function declaration, and then conclude with a discussion how closure expressions and local functions capture values.

The **interface type request** computes the interface type of a function declaration. This is a function type or generic function type, constructed from the interface types of the function's parameter declarations, together with its return type, and generic signature, if any. If no return type is given, it becomes the empty tuple type `()`:

```
func f(x: Int, y: String) -> Bool {...}
// Interface type: (Int, String) -> Bool

func g() {...}
// Interface type: () -> ()
```

Method declarations. In addition to the formal parameters declared in its parameter list, a method declaration also has an implicit `self` parameter, to receive the value on the left-hand side of the "`.`" in the method call expression. The interface type of a method declaration is a function type which receives the `self` parameter, and returns another function which then takes the method's formal parameters. The "`->`" syntax for a function type associates to the right, so `A -> B -> C` means `A -> (B -> C)`:

```
struct Universe {
  func wormhole(x: Int, y: String) -> Bool {...}
  // Interface type: (Universe) -> (Int, String) -> Bool

  static func bigBang() {}
  // Interface type: (Universe.Type) -> () -> ()

  mutating func teleport() {}
  // Interface type: (inout Universe) -> () -> ()
}
```

The interface type of the `self` parameter is derived as follows:

- We start with the *self interface type* of the method's parent declaration context. In a struct, enum or class, this is the same as the declared interface type. In a protocol, this is the protocol `Self` type (Section 4.3). In an extension, the self interface type is that of the extended type.
- If the method is declared inside a class, and if it returns the dynamic `Self` type, we wrap the type in the dynamic `Self` type (Section 3.3).
- If the method is `static`, we wrap the type in a metatype.
- If the method is `mutating`, we pass the `self` parameter `inout`.

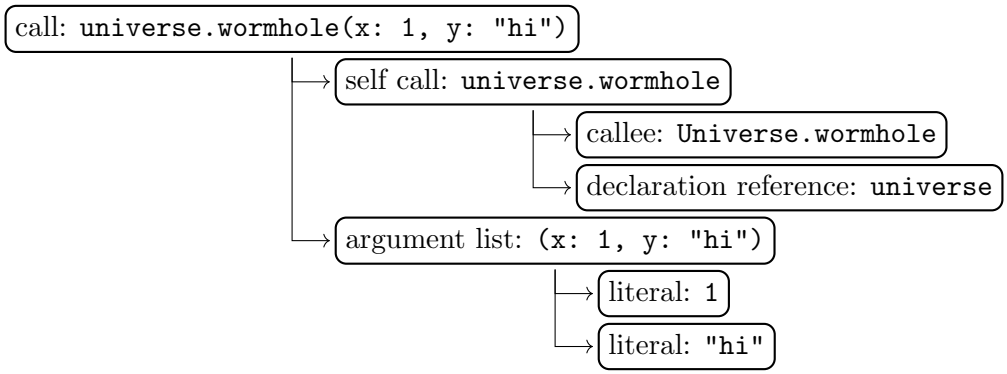
Let's compare the interface type of a method declaration with the structure of a method call expression in the abstract syntax tree, shown in Figure 4.1:

- The outer expression's callee, `universe.wormhole`, is itself a call expression, so we must evaluate this inner call expression first.

  The inner call expression applies the argument `universe` to the `self` parameter of `Universe.wormhole()`. This represents the method lookup. The return type of the inner call expression is `(Int, String) -> Bool`.
- The outer call expression applies the argument list `(x: 1, y: "hi")` to the result of the inner call expression. This represents the method call itself, with a return type of `Bool`.

The extra function call disappears in SILGen, where we lower a method to a SIL function that receives all formal parameters and the `self` parameter at once.

Figure 4.1.: The method call `universe.wormhole(x: 1, y: "hi")`



A partially-applied method reference, such as `universe.wormhole`, can also be used as a value of type `(Int, String) -> Bool`. This binds the `self` parameter but does not call the method. We handle this by wrapping the method reference in a closure that invokes the method; this closure has the same type as the method reference. In the lambda calculus formalism, this is what's known as *η-expansion*:

```
{ x, y in universe.wormhole(x: x, y: y) }
```

The unapplied form, `Universe.wormhole`, desugars into a closure returning a closure:

```
{ mySelf in { x y in mySelf.wormhole(x: x, y: y) } }
```

This desugaring simplifies SILGen, because we only need to implement a lowering for fully-applied method calls that pass all formal parameters and `self` at once.

```
struct Example {
  func instanceMethod() {}
  static func staticMethod() {}

  struct Lookup {
    func innerMethod() {}

    func test() {
      instanceMethod()  // bad
      staticMethod()    // ok
      innerMethod()     // ok
    }
  }

  func anotherMethod(x: Int) {
    struct Local {
      func test() {
        print(x)        // bad
      }
    }
  }
}
```

All function declarations must be followed by a body in the source language, except for protocol requirements. A function body can contain statements, expressions, and other declarations. (Unlike types and declarations, we will not exhaustively cover all statements and expressions in this book.) The example on the left shows some call expressions.

In a method body, an unqualified reference to a member of the innermost nominal type declaration is interpreted as having an implicit "`self.`" qualification. Thus, instance methods can refer to other instance methods this way, and static methods can refer to other static methods.

An unqualified reference to a member of an outer nominal type can only be made if the member is static, because there is no "outer `self` value" to invoke the method with; a *value* of the nested type does not contain a *value* of its parent type.

For the same reason, methods inside local types cannot refer to local variables declared outside of the local type. (Contrast this with Java inner classes for example, which can be declared as `static` or instance members of their outer class; a non-`static` inner class captures a "`this`" reference from the outer class. Inner classes nested in methods can also capture local variables in Java.)

Constructor declarations. Constructor declarations are introduced with the `init` keyword. The parent context of a constructor must be a nominal type or extension.

From the outside, the interface type of a constructor looks like a static method that returns a new instance of the type, but inside the constructor, `self` is the instance being initialized, so the interface type of `self` is the nominal type, and not its metatype. In a struct or enum, `self` is also `inout`. Constructors can delegate to other constructors in various ways. To model the delegation with a call expression, the *initializer interface type* describes the type of an in-place initialization at a location provided by the caller:

```
struct Universe {
  init(age: Int) {...}
  // Interface type: (Universe.Type) -> (Int) -> Universe
  // Initializer interface type: (inout Universe) -> (Int) -> Universe
}
```

Destructor declarations. Destructor declarations are introduced with the `deinit` keyword. They can only appear inside classes. They have no formal parameters, no generic parameter list, no `where` clause, and no return type.

Local contexts. A *local context* is any declaration context that is not a module, source file, type declaration or extension. Swift allows variable, function and type declarations to appear in local context. The following are local contexts:

- Top-level code declarations.
- Function declarations.
- Closure expressions.
- If a variable is not itself in local context (for example, it's a member of a nominal type declaration), then its initial value expression defines a new local context.
- Subscript and enum element declarations declarations are local contexts, because they can contain parameter declarations (and also a generic parameter list, in the case of a subscript).

Local functions and closures can *capture* references to other local declarations from outer scopes. We use the standard technique of *closure conversion* to lower functions with captured values into ones without. We can understand this process as introducing an additional parameter for each captured value, followed by a walk to replace references to captured values with references to the corresponding parameters in the function body. In Swift, this is part of SILGen's lowering process, and not a separate transformation on the abstract syntax tree.

The **capture info request** computes the list of declarations captured by the given function and all of its nested local functions and closure expressions.

```
func f() {
  let x = 0, y = 0

  func g() {
    var z = 0
    print(x)

    func h() {
      print(y, z)
    }
  }
}
```

Consider the three nested functions shown on the left. We proceed to compute their captures from the inside out.

The innermost function `h()` captures `y` and `z`. The middle function `g()` captures `x`. It also captures `y`, because `h()` captures `y`, but it does not capture `z`, because `z` is declared by `g()` itself. Finally, `f()` is declared at the top level, so it does not have any captures.

We can summarize this as follows (see Appendix A for a summary of set notation):

Function	Captures
`f()`	$\varnothing$
`g()`	{`x`, `y`}
`h()`	{`y`, `z`}

Algorithm 4A (Compute closure captures)**.** As input, takes the type-checked body of a closure expression or local function F. Outputs the set of captures of F.

1. Initialize the return value with an empty set, $C \leftarrow \varnothing$.
2. Recursively walk the type-checked body of F and handle each element:
3. (Declaration references) If F contains an expression that references some local variable or local function d by name, let `PARENT`(d) denote the declaration context containing d. This is either F itself, or some outer local context, because we found d by unqualified lookup from F.

   If `PARENT`(d) $\neq F$, set $C \leftarrow C \cup \{d\}$.
4. (Nested closures) If F contains a nested closure expression or local function F', then any captures of F' not declared by F are also captures of F.

   Recursively compute the captures of F'. For each d captured by F' such that `PARENT`(d) $\neq F$, set $C \leftarrow C \cup \{d\}$.
5. (Local types) If F contains a local type, do not walk into the children of the local type. Local types do not capture values; we enforce this below.
6. (Diagnose) After the recursive walk, consider each element $d \in C$. If the path of parent declaration contexts from F to d contains a nominal type declaration, we have an unsupported capture inside a local type. Diagnose an error.
7. Return C.

Local functions can also reference each other recursively. Consider the functions shown on the right and notice how `f()` and `g()` are mutually recursive. At runtime, we cannot represent this by forming two closure contexts where each one retains the other, because then neither context will ever be released.

```
func f() {
  let x = 0, y = 0, z = 0

  func g() { print(x); h() }
  func h() { print(y); g() }
  func i() { print(z); h() }
}
```

We use a second algorithm to obtain the list of *lowered captures*, by replacing any captured local functions with their corresponding capture lists, repeating this until fixed point. The final list contains variable declarations only. With our example, the captures and lowered captures of each function are as follows:

Function	Captures	Lowered
`f()`	∅	∅
`g()`	{`x`, `h()`}	{`x`, `y`}
`h()`	{`y`, `g()`}	{`x`, `y`}
`i()`	{`z`, `h()`}	{`x`, `y`, `z`}

(As a special case, if a set of local functions reference each other but capture no other state from the outer declaration context, their lowered captures will be empty, so no runtime context allocation is necessary.)

Algorithm 4B (Compute lowered closure captures)**.** As input takes the type-checked body of a closure expression or local function F. Outputs the set of variable declarations transitively captured by F.

1. Initialize the set $C \leftarrow \varnothing$; this will be the return value. Initialize an empty worklist. Initialize an empty visited set. Add F to the worklist.
2. If the worklist is empty, return C. Otherwise, remove the next function F from the worklist.
3. If F is in the visited set, go back to Step 2. Otherwise, add F to the visited set.
4. Compute the captures of F using Algorithm 4A and consider each capture d. If d is a local variable declaration, set $C \leftarrow C \cup \{d\}$. If d is a local function declaration, add d to the worklist.
5. Go back to Step 2.

This completely explains captures of `let` variables, but mutable `var` variables and `inout` parameters merit further explanation.

A *non-escaping* closure can capture a `var` or `inout` by simply capturing the memory address of the storage location. This is safe, because a non-escaping closure cannot outlive the dynamic extent of the storage location.

An `@escaping` closure can also capture a `var`, which requires promoting the `var` to a heap-allocated box with a reference count, with all variable accesses indirecting through the box. The below example can be found in every Lisp textbook. Each invocation of `counter()` allocates a new counter value on the heap, and returns three closures that reference the box; the box itself is completely hidden by the abstraction:

```
func counter() -> (read: () -> Int, inc: () -> (), dec: () -> ()) {
  var count = 0  // promoted to a box
  return ({ count }, { count += 1 }, { count -= 1 })
}
```

Before Swift 3.0, `@escaping` closures were permitted to capture `inout` parameters as well. To make this safe, the contents of the `inout` parameter were first copied into a heap-allocated box, which was captured by the closure. The contents of this box were then copied back before the function returned to its caller. This was essentially equivalent to doing the following transform, where we introduced `_n` by hand:

```
func changeValue(_ n: inout Int) {
  var _n = n  // copy the value

  let escapingFn = {
    _n += 1   // capture the box
  }

  n = _n      // write it back
}
```

In this scheme, if the closure outlives the dynamic extent of the `inout` parameter, any subsequent writes from within the closure are silently dropped. This was a source of user confusion, so Swift 3.0 banned `inout` captures from escaping closures instead [67].

In SIL, a closure is represented abstractly, as the result of this partial application operation. The mechanics of how the partially-applied function value actually stores its captures—the partially-applied arguments—are left up to IRGen. In IRGen, we allocate space for storing the captures (either on the stack for a non-escaping function type, otherwise it's on the heap for `@escaping`), and then we emit a thunk, which takes a pointer to the context as an argument, unpacks the captured values from the context, and passes them as individual arguments to the original function. This thunk together with the context forms a *thick function* value which can then be passed around.

If nothing is captured (or if all captured values are zero bytes in size), we can pass a null pointer as the context, without performing a heap allocation. If there is exactly one captured value and this value can be represented as a reference-counted pointer, we can also elide the allocation by passing the captured value as the context pointer instead. For example, if a closure's single capture is an instance of a class type, nothing is allocated. If the single capture is the heap-allocated box that wraps a `var`, we must still allocate the box for the `var`, but we avoid a second context allocation.

4.5. Storage

Storage declarations represent locations that can be read and written.

Parameter declarations. Functions, enum elements and subscripts can have parameter lists; each parameter is represented by a parameter declaration. Parameter declarations are a kind of variable declaration.

Variable declarations. Variables that are not parameters are introduced with `var` and `let`. A variable might either be *stored* or *computed*; the behavior of a computed variable is given by its accessor implementations. The interface type of a variable is the stored value type, possibly wrapped in a reference storage type if the variable is declared as `weak` or `unowned`. The *value interface type* of a variable is the storage type without any wrapping.

Variable declarations are always created alongside a *pattern binding declaration* which represents the various ways in which variables can be bound to values in Swift. A pattern binding declaration consists of one or more *pattern binding entries*. Each pattern binding entry has a *pattern* and an optional *initial value expression*. A pattern declares zero or more variables.

Here is a pattern binding declaration with a single entry that does not declare any variables:

```
let _ = ignored()
```

Here is a pattern binding declaration with a single entry, where the pattern declares a single variable:

```
let x = 123
```

We can write a more complex pattern, for example storing the first element of a tuple while discarding the second element:

```
let (x, _) = (123, "hello")
```

Here is a pattern binding declaration with a single entry, whose pattern delares two variables `x` and `y`:

```
let (x, y) = (123, "hello")
```

Here is a pattern binding declaration with two entries, declaring `x` and `y` respectively:

```
let x = 123, y = "hello"
```

And finally, here we have two pattern binding declarations, where each pattern binding declaration has a single entry declaring a single variable:

```
let x = 123
let y = "hello"
```

When a pattern binding declaration appears outside of a local context, each entry must declare at least one variable, so we reject both of the following:

```
let _ = 123

struct S {
  let _ = "hello"
}
```

A funny quirk of the pattern grammar is that typed patterns and tuple patterns do not compose in the way one might think. If "`let x: Int`" is a typed pattern declaring a variable `x` type with annotation `Int`, and "`let (x, y)`" is a tuple pattern declaring two variables `x` and `y`, we might expect "`let (x: Int, y: String)`" to declare two variables `x` and `y` with type annotations `Int` and `String` respectively; what actually happens is we get a tuple pattern declaring two variables named `Int` and `String` that binds a two-element tuple with *labels* `x` and `y`:

```
let (x: Int, y: String) = (x: 123, y: "hello")
print(Int)  // huh? prints 123
print(String)  // weird! prints "hello"
```

Subscript declarations. Subscripts are introduced with the `subscript` keyword. They can only appear as members of nominal types and extensions. The interface type of a subscript is a function type taking the index parameters and returning the storage type. The value interface type of a subscript is just the storage type. For historical reasons, the interface type of a subscript does not include the `Self` clause, the way that method declarations do. Subscripts can either be instance or static members; static subscripts were introduced in Swift 5.1 [68].

Accessor declarations. Each storage declaration has a set of accessor declarations, which are a special kind of function declaration. The accessor declarations are siblings of the storage declaration in the declaration context hierarchy. The interface type of an accessor depends the accessor kind. For example, getters return the value, and setters take the new value as a parameter. Property accessors do not take any other parameters; subscript accessors also take the subscript's index parameters. We will not need any more details about accessor and storage declarations in this book.

4.6. Source Code Reference

Key source files:

- include/swift/AST/Decl.h
- include/swift/AST/DeclContext.h
- lib/AST/Decl.cpp
- lib/AST/DeclContext.cpp

Other source files:

- include/swift/AST/DeclNodes.def
- include/swift/AST/ASTVisitor.h
- include/swift/AST/ASTWalker.h

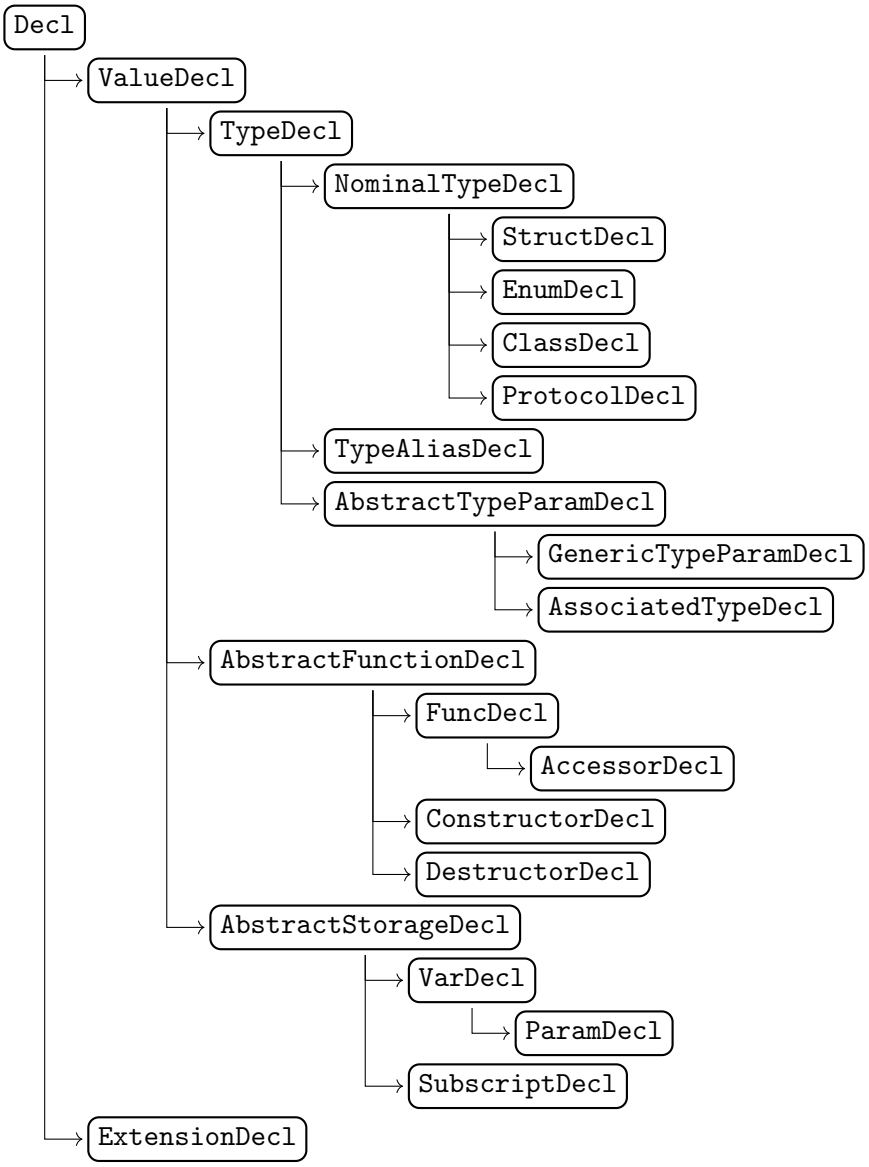
`Decl` *class*

Base class of declarations. Figure 4.2 shows various subclasses, which correspond to the different kinds of declarations defined previously in this chapter.

Instances are always allocated in the permanent arena of the `ASTContext`, either when the declaration is parsed or synthesized. The top-level `isa<>`, `cast<>` and `dyn_cast<>` template functions support dynamic casting from `Decl *` to any of its subclasses.

- `getDeclContext()` returns the parent `DeclContext` of this declaration.
- `getInnermostDeclContext()` if this declaration is also a declaration context, returns the declaration as a `DeclContext`, otherwise returns the parent `DeclContext`.
- `getASTContext()` returns the singleton AST context from a declaration.

Figure 4.2.: The `Decl` class hierarchy



Visitors. To exhaustively handle each kind of declaration, the simplest way is to switch over the kind, which is an instance of the `DeclKind` enum, like this:

```
Decl *decl = ...;
switch (decl->getKind()) {
case DeclKind::Struct: {
  auto *structDecl = decl->castTo<StructDecl>();
  ...
}
case DeclKind::Enum:
  ...
case DeclKind::Class:
  ...
}
```

However, just as with types, is can be more convenient to use the visitor pattern, by subclassing `ASTVisitor` and overriding various `visit`*Kind*`Decl()` methods. The visitor's `visit()` method performs the switch and dynamic cast dance above, and calls each method:

```
class MyVisitor: public ASTVisitor<MyVisitor> {
public:
  void visitStructDecl(StructType *decl) {
    ...
  }
};

MyVisitor visitor;

Decl *decl = ...;
visitor.visit(decl);
```

The `ASTVisitor` also defines various methods corresponding to abstract base classes in the `Decl` hierarchy, so for example an override of `visitNominalTypeDecl()` will handle all nominal type declarations at once. The `ASTVisitor` is more general than just visiting declarations; it also supports visiting statements, expressions, and type representations.

A more elaborate form is implemented by the `ASTWalker`. While the visitor visits a single declaration, the walker traverses nested declarations, statements and expressions in a pre-order walk.

`ValueDecl` *class*

Base class of named declarations.

- `getDeclName()` returns the declaration's name.
- `getInterfaceType()` returns the declaration's interface type.

Type Declarations

`TypeDecl` *class*

Base class of type declarations.

- `getDeclaredInterfaceType()` returns the type of an instance of this declaration.

`NominalTypeDecl` *class*

Base class of nominal type declarations. Also a `DeclContext`.

- `getSelfInterfaceType()` returns the self interface type of the declaration context (Section 4.4). Different from the declared interface type for protocols, where the declared interface type is a nominal but the declared self type is the generic parameter `Self`.
- `getDeclaredType()` returns the type of an instance of this declaration, without generic arguments. If the declaration is generic, this is an unbound generic type. If this declaration is not generic, this is a nominal type. This is occasionally used in diagnostics instead of the declared interface type, when the generic parameter types are irrelevant.

`TypeAliasDecl` *class*

A type alias declaration. Also a `DeclContext`.

- `getDeclaredInterfaceType()` returns the underlying type of the type alias declaration, wrapped in type alias type sugar.
- `getUnderlyingType()` returns the underlying type of the type alias declaration, without wrapping it in type alias type sugar.

Declaration Contexts

`DeclContext` *class*

Base class for declaration contexts. The top-level `isa<>`, `cast<>` and `dyn_cast<>` template functions also support dynamic casting from a `DeclContext *` to any of its subclasses. See also Section 5.6.

There are a handful of subclasses which are not also subclasses of `Decl *`:

- `ClosureExpr`.
- `FileUnit` and its various subclasses, such as `SourceFile`.
- A few other less interesting ones found in the source.

Utilities for understanding the nesting of declaration contexts:

- `getAsDecl()` if declaration context is also a declaration, returns the declaration, otherwise returns `nullptr`.
- `getParent()` returns the parent declaration context.
- `isModuleScopeContext()` returns true if this is a `ModuleDecl` or `FileUnit`.
- `isTypeContext()` returns true if this is a nominal type declaration or an extension.
- `isLocalContext()` returns true if this is not a module scope context or type context.
- `getParentModule()` returns the module declaration at the root of the hierarchy.
- `getModuleScopeContext()` returns the innermost parent which is a `ModuleDecl` or `FileUnit`.
- `getParentSourceFile()` returns the innermost parent which is a source file, or `nullptr` if this declaration context was not parsed from source.
- `getInnermostDeclarationDeclContext()` returns the innermost parent which is also a declaration, or `nullptr`.
- `getInnermostDeclarationTypeContext()` returns the innermost parent which is also a nominal type or extension, or `nullptr`.

Operations on type contexts:

- `getSelfNominalDecl()` returns the nominal type declaration if this is a type context, or `nullptr`.
- `getSelfStructDecl()` as above but result is a `StructDecl *` or `nullptr`.
- `getSelfEnumDecl()` as above but result is a `EnumDecl *` or `nullptr`.
- `getSelfClassDecl()` as above but result is a `ClassDecl *` or `nullptr`.
- `getSelfProtocolDecl()` as above but result is a `ProtocolDecl *` or `nullptr`.

- `getDeclaredInterfaceType()` delegates to the method on `NominalTypeDecl` or `ExtensionDecl` as appropriate.
- `getSelfInterfaceType()` is similar.

Generic parameters and requirements:

- `isGenericContext()` answers true if either this generic context or one of its parents has a generic parameter list.
- `isInnermostContextGeneric()` answers if this declaration context itself has a generic parameter list. Compare with `isGenericContext()`.

Generic Contexts

Key source files:

- include/swift/AST/GenericParamList.h
- include/swift/AST/Requirement.h
- lib/AST/GenericParamList.cpp
- lib/AST/NameLookup.cpp
- lib/AST/Requirement.cpp

`GenericContext` *class*

Subclass of `DeclContext`. Base class for declaration kinds which can have a generic parameter list. See also Section 5.6.

- `getParsedGenericParams()` returns the declaration's parsed generic parameter list, or `nullptr`.
- `getGenericParams()` returns the declaration's full generic parameter list, which includes any implicit generic parameters. Evaluates a `GenericParamListRequest`.
- `isGeneric()` answers if this declaration has a generic parameter list. This is equivalent to calling `DeclContext::isInnermostContextGeneric()`. Compare with `DeclContext::isGenericContext()`.
- `getGenericContextDepth()` returns the depth of the declaration's generic parameter list, or `(unsigned)-1` if neither this declaration nor any outer declaration is generic.

- `getTrailingWhereClause()` returns the declaration's trailing `where` clause, or `nullptr`.

Trailing `where` clauses are not preserved in serialized generic contexts. Most code should look at `GenericContext::getGenericSignature()` instead (Section 5.6), except when actually building the generic signature.

`GenericParamList` *class*

A generic parameter list.

- `getParams()` returns an array of generic parameter declarations.
- `getOuterParameters()` returns the outer generic parameter list, linking multiple generic parameter lists for the same generic context. Only used for extensions of nested generic types.

`GenericParamListRequest` *class*

This request creates the full generic parameter list for a declaration. Kicked off from `GenericContext::getGenericParams()`.

- For protocols, this creates the implicit `Self` parameter.
- For functions and subscripts, calls `createOpaqueParameterGenericParams()` to walk the formal parameter list and look for `OpaqueTypeRepr`s.
- For extensions, calls `createExtensionGenericParams()` which clones the generic parameter lists of the extended nominal itself and all of its outer generic contexts, and links them together via `GenericParamList::getOuterParameters()`.

`GenericTypeParamDecl` *class*

A generic parameter declaration.

- `getDepth()` returns the depth of the generic parameter declaration.
- `getIndex()` returns the index of the generic parameter declaration.
- `getName()` returns the name of the generic parameter declaration.
- `getDeclaredInterfaceType()` returns the sugared generic parameter type for this declaration, which prints as the generic parameter's name.
- `isOpaque()` answers if this generic parameter is associated with an opaque parameter.

- `getOpaqueTypeRepr()` returns the associated `OpaqueReturnTypeRepr` if this is an opaque parameter, otherwise `nullptr`.
- `getInherited()` returns the generic parameter declaration's inheritance clause.

Inheritance clauses are not preserved in serialized generic parameter declarations. Requirements stated on generic parameter declarations are part of the corresponding generic context's generic signature, so except when actually building the generic signature, most code uses `GenericContext::getGenericSignature()` instead (Section 5.6).

`GenericTypeParamType` *class*

A generic parameter type.

- `getDepth()` returns the depth of the generic parameter declaration.
- `getIndex()` returns the index of the generic parameter declaration.
- `getName()` returns the name of the generic parameter declaration if this is the sugared form, otherwise returns a string of the form "`τ_d_i`".

`TrailingWhereClause` *class*

The syntactic representation of a trailing `where` clause.

- `getRequirements()` returns an array of `RequirementRepr`.

`RequirementRepr` *class*

The syntactic representation of a requirement in a trailing `where` clause.

- `getKind()` returns a `RequirementReprKind`.
- `getFirstTypeRepr()` returns the first `TypeRepr` of a same-type requirement.
- `getSecondTypeRepr()` returns the second `TypeRepr` of a same-type requirement.
- `getSubjectTypeRepr()` returns the first `TypeRepr` of a constraint or layout requirement.
- `getConstraintTypeRepr()` returns the second `TypeRepr` of a constraint requirement.
- `getLayoutConstraint()` returns the layout constraint of a layout requirement.

`RequirementReprKind` *enum class*

- `RequirementRepr::TypeConstraint`
- `RequirementRepr::SameType`
- `RequirementRepr::LayoutConstraint`

`WhereClauseOwner` *class*

Represents a reference to some set of requirement representations which can be resolved to requirements, for example a trailing `where` clause. This is used by various requests, such as the `RequirementRequest` below, and the `InferredGenericSignatureRequest` in Section 11.5.

- `getRequirements()` returns an array of `RequirementRepr`.
- `visitRequirements()` resolves each requirement representation and invokes a callback with the `RequirementRepr` and resolved `Requirement`.

`RequirementRequest` *class*

Request which can be evaluated to resolve a single requirement representation in a `WhereClauseOwner`. Used by `WhereClauseOwner::visitRequirements()`.

`ProtocolDecl` *class*

A protocol declaration.

- `getTrailingWhereClause()` returns the protocol `where` clause, or `nullptr`.
- `getAssociatedTypes()` returns an array of all associated type declarations in the protocol.
- `getPrimaryAssociatedTypes()` returns an array of all primary associated type declarations in the protocol.
- `getInherited()` returns the parsed inheritance clause.

Trailing `where` clauses and inheritance clauses are not preserved in serialized protocol declarations. Except when actually building the requirement signature, most code uses `ProtocolDecl::getRequirementSignature()` instead (Section 5.6).

The last four utility methods operate on the requirement signature, so are safe to use on deserialized protocols:

- `getInheritedProtocols()` returns an array of all protocols directly inherited by this protocol, computed from the inheritance clause.

- `inheritsFrom()` determines if this protocol inherits from the given protocol, possibly transitively.
- `getSuperclassDecl()` returns the protocol's superclass declaration.

`AssociatedTypeDecl` *class*

An associated type declaration.

- `getTrailingWhereClause()` returns the associated type's trailing `where` clause, or `nullptr`.
- `getInherited()` returns the associated type's inheritance clause.

Trailing `where` clauses and inheritance clauses are not preserved in serialized associated type declarations. Requirements on associated types are part of a protocol's requirement signature, so except when actually building the requirement signature, most code uses `ProtocolDecl::getRequirementSignature()` instead (Section 5.6).

Function Declarations

`AbstractFunctionDecl` *class*

Base class of function-like declarations. Also a `DeclContext`.

- `getImplicitSelfDecl()` returns the implicit `self` parameter, if there is one.
- `getParameters()` returns the function's parameter list.
- `getMethodInterfaceType()` returns the type of a method without the `Self` clause.
- `getResultInterfaceType()` returns the return type of this function or method.

`ParameterList` *class*

The parameter list of `AbstractFunctionDecl`, `EnumElementDecl` or `SubscriptDecl`.

- `size()` returns the number of parameters.
- `get()` returns the `ParamDecl` at the given index.

`ConstructorDecl` *class*

Constructor declarations.

- `getInitializerInterfaceType()` returns the initializer interface type, used when type checking `super.init()` delegation.

Closure Conversion

Key source files:

- include/swift/AST/CaptureInfo.h
- include/swift/SIL/TypeLowering.h
- lib/AST/CaptureInfo.cpp
- lib/Sema/TypeCheckCaptures.cpp
- lib/SIL/IR/TypeLowering.cpp

`CaptureInfo`	*class*

An immutable list of captured values.

`CaptureInfoRequest::evaluate`	*method*

Computes the `CaptureInfo`. This is Algorithm 4A.

`TypeLowering::getLoweredLocalCaptures()`	*method*

Computes the lowered `CaptureInfo`. This is Algorithm 4B.

Storage Declarations

`AbstractStorageDecl`	*class*

Base class for storage declarations.

- `getValueInterfaceType()` returns the type of the stored value, without `weak` or `unowned` storage qualifiers.

`VarDecl`	*class*

Subclass of `AbstractStorageDecl`.

`SubscriptDecl`	*class*

Subclass of `AbstractStorageDecl` and `DeclContext`.

Part II.

Semantics

5. Generic Signatures

GENERIC SIGNATURES are semantic objects that describe the type checking behavior of generic declarations. Declarations can nest, and outer generic parameters are visible inside inner declarations, so a generic signature is a "flat" representation that collects all *generic parameter types* and *requirements* that apply in the declaration's scope. This abstracts away the concrete syntax described in the previous chapter:

- The list of generic parameter types begins with all outer generic parameters, which are followed by generic parameters from the explicit generic parameter list `<...>`, together with any generic parameter types implicitly introduced by opaque parameter declarations, like the "`some P`" in "`func f(_: some P)`".
- The list of requirements in a generic signature includes the requirements stated by outer generic declarations, as well as any requirements from generic parameter inheritance clauses, the trailing `where` clause, and opaque parameters. A fourth mechanism, called requirement inference, will be described in Section 11.1.

The following notation for generic signatures is used throughout this book, as well as debugging output from the compiler:

$$\underbrace{\texttt{<A, B, C, ...}}_{\text{generic parameters}}\ \texttt{where}\ \underbrace{\texttt{A: Sequence, B: Equatable, A.Element == Int, ...>}}_{\text{requirements}}$$

The requirements in a generic signature use the same semantic representation as the requirements in a trailing `where` clause, given by Definition 4.1, with a few additional properties. A generic signature always omits any redundant requirements from the list, and the remaining ones are written to be "as simple as possible." We will describe this in Chapter 11, when we see how the generic signature of a declaration is built from the syntactic forms described in the previous chapter, but for now, we're just going to assume we're working with an existing generic signature that was given to us by this black box.

After some preliminaries, we will go on to introduce a formal system for reasoning about requirements and type parameters in Section 5.2. This makes precise the earlier concept of the *interface type* of a declaration—it contains valid type parameters of the declaration's generic signature. Section 5.5 describes *generic signature queries*, which are fundamental primitives in the implementation, used by the rest of the compiler to answer questions about generic signatures. These questions will be statements in our formal system.

Debugging. The `-debug-generic-signatures` frontend flag gives us a glimpse into the generics implementation by printing the generic signature of each declaration being type checked. Here is a simple program with three nested generic declarations:

```
struct Outer<T: Sequence> {
  struct Inner<U> {
    func transform() where T.Element == U {
      ...
    }
  }
}
```

If we run the compiler with this flag, we see that the generic signature at each level of nesting incorporates all information from the outer declaration's generic signature:

```
debug.(file).Outer@debug.swift:1:8
Generic signature: <T where T : Sequence>

debug.(file).Outer.Inner@debug.swift:2:10
Generic signature: <T, U where T : Sequence>

debug.(file).Outer.Inner.transform()@debug.swift:3:10
Generic signature: <T, U where T : Sequence, U == T.[Sequence]Element>
```

This flag also allows us to observe *requirement minimization*. Here are three functions, each one generic over a pair of types conforming to `Sequence`:

```
func sameElt<S1: Sequence, S2: Sequence>(_ s1: S1, _ s2: S2)
    where S1.Element == S2.Element {...}

func sameIter<S1: Sequence, S2: Sequence>(_ s1: S1, _ s2: S2)
    where S1.Iterator == S2.Iterator {...}

func sameEltAndIter<S1: Sequence, S2: Sequence>(_ s1: S1, _ s2: S2)
    where S1.Element == S2.Element,
          S1.Iterator == S2.Iterator {...}
```

The first function expects two sequences with the same `Element` associated type, so "`sameElt(Array<Int>(), Set<Int>())`" for example. The second requires both have the same `Iterator` type. The third requires both, but a consequence of how `Sequence` is declared in the standard library is that the second is a stronger condition; if two

sequences have the same `Iterator` type, they will also have the same `Element` type, but not vice versa. In other words, the same-type requirement [`S1.Element == S2.Element`] is *redundant* in the trailing `where` clause of `sameEltAndIter()`. (We will be able to *prove* it when we revisit this generic signature in Example 5.11.) When we compile this program with `-debug-generic-signatures`, we observe that the generic signature of `sameElt()` is distinct from the other two, while `sameIter()` and `sameEltAndIter()` have a generic signature that looks like this:

```
<S1, S2 where S1: Sequence, S2: Sequence,
              S1.[Sequence]Iterator == S2.[Sequence]Iterator>
```

Requirement minimization is described in Section 11.4.

Canonical signatures. Generic signatures are immutable and uniqued, so two generic signatures with the same structure and pointer-equal types are always pointer-equal generic signatures. While requirements in a generic signature are minimal and sorted in a certain order, they can still differ by type sugar. A generic signature is *canonical* if all listed generic parameter types are canonical, and any types appearing in requirements are canonical. A canonical signature is computed from an arbitrary generic signature by replacing any sugared types appearing in the signature with canonical types, and this gives us the notion of canonical equality of generic signatures; two generic signatures are canonically equal if their canonical signatures are pointer-equal.

The generic signatures of `sameIter()` and `sameEltAndIter()` above are canonically equal, but not pointer-equal; while their generic parameters have the same *names*, the sugared generic parameter types refer to actual *declarations*, which are distinct. Their canonical signature looks like this:

```
<τ_0_0, τ_0_1 where τ_0_0: Sequence, τ_0_1: Sequence,
           τ_0_0.[Sequence]Iterator == τ_0_1.[Sequence]Iterator>
```

Empty generic signature. A generic declaration without a generic parameter list or trailing `where` clause inherits the generic signature from the parent context. If no outer parent context is generic, we get the *empty generic signature* with no generic parameters or requirements. The interface types described by the empty generic signature are the fully concrete types, that is, types that do not contain any type parameters.

Protocol generic signature. In Section 4.3 we saw that every protocol declaration has a generic parameter named `Self` that conforms to the protocol itself. The protocol `Self` type is always canonically equal to `τ_0_0`. We will denote the generic signature of a protocol `P` by $G_{\texttt{P}}$:

$G_{\texttt{P}}$:= `<Self where Self: P>`

5.1. Requirement Signatures

Each protocol has a *requirement signature*, which collects the protocol's associated type declarations, and the associated requirements imposed upon them. This abstracts away the concrete syntax from Section 4.3. There is a duality between generic signatures and requirement signatures, illustrated with the following diagram that shows each kind of entity with a concrete specimen to remind us of the notation:

Generic signature	**Requirement signature**
Generic parameters: `τ_0_1`	Associated types: `associatedtype Iterator`
Generic requirements: `[τ_0_1: Sequence]`	Associated requirements: $[\texttt{Self.Iterator: IteratorProtocol}]_{\texttt{Sequence}}$

(Requirement signatures also describe the type alias members of the protocol; these are called *protocol type aliases*. They're not covered by the formalism described in the next section, but we will discuss them in Section 18.4.)

If a generic signature G states a conformance requirement `[T: P]`, the requirement signature of `P` generates additional structure, which we describe informally at first:

- For every associated type `A` of `P`, we can talk about the dependent member type `T.[P]A`, which abstracts over the type witness in the conformance.
- Because the associated requirements are satisfied by every concrete conforming type, they also hold "in relation" to the abstract type parameter `T`.

To interpret a generic signature, we must also consult the requirement signatures of some set of protocols that the generic signature "depends on." The question of *which* protocols exactly is settled in Section 16.1, but for now, we can take our generic signatures to exist in the universe of all protocols visible to name lookup, declared in the source program and all serialized modules. This set of protocols is always finite.

Remember how the generic signature of a protocol `P` just has a single conformance requirement `[Self: P]`; thus, it is the simplest generic signature that allows us to "look inside" the requirement signature of `P`.

The `-debug-generic-signatures` frontend flag also prints the requirement signature of each protocol as its type checked. While we print a requirement signature as a generic signature with the single generic parameter `Self`, requirement signatures and generic signatures are distinct in theory and implementation.

Our examples often use the fact that the `Sequence` protocol states two associated requirements:

$$[\texttt{Self.Iterator: IteratorProtocol}]_{\texttt{Sequence}}$$
$$[\texttt{Self.Element == Self.Iterator.Element}]_{\texttt{Sequence}}$$

Example 5.1. To motivate the formal system in the next section, we consider how the associated requirements of a protocol can manifest in a generic signature. Here is a generic function that states a conformance requirement to `Sequence`—two, in fact:

```
func firstTwoEqual<S1: Sequence, S2: Sequence>(_ s1: S1, _ s2: S2)
    where S1.Element == S2.Element, S1.Element: Equatable {
  var iter1 = s1.makeIterator()
  var iter2 = s2.makeIterator()
  return iter1.next()! == iter2.next()!
}
```

Instead of the sugared generic parameter types "`S1`" and "`S2`", we will use canonical types to emphasize that type sugar has no semantic effect. Here is the generic signature of `firstTwoEqual()`:

```
<τ_0_0, τ_0_1 where τ_0_0: Sequence, τ_0_1: Sequence,
                    τ_0_0.Element: Equatable,
                    τ_0_0.Element == τ_0_1.Element>
```

We expect roughly the following to happen while type checking the function body:

1. "`s1`" has type `τ_0_0`, which conforms to `Sequence`, so it has a `makeIterator()` method that we can call.

2. This method returns `Self.Iterator`. We substitute `Self` with `τ_0_0`, and conclude that "`iter1`" has type `τ_0_0.Iterator`.

3. Similarly, "`iter2`" has type `τ_0_1.Iterator`.

Then, consider the sub-expression "`iter1.next()!`":

1. The requirement signature of `Sequence` tells us that `τ_0_0.Iterator` conforms to `IteratorProtocol`, so we know it has a `next()` method.

2. This method returns `Optional<Self.Element>`, and we substitute `Self` with `τ_0_0.Iterator` to get `Optional<τ_0_0.Iterator.Element>`.

3. The forced unwrap expression has type `τ_0_0.Iterator.Element`.

A similar analysis shows that "`iter2.next()!`" has type `τ_0_1.Iterator.Element`. We now look at the interface type of the `==` operator in the `Equatable` protocol:

```
<Self where Self: Equatable> (Self, Self) -> Bool
```

We see that for the expression "`iter1.next()! == iter2.next()!`" to type check, the compiler must convince itself that these two requirements below are implied by the explicit requirements of our generic signature (together with the requirement signature of `Sequence`):

[`τ_0_0.Iterator.Element == τ_0_1.Iterator.Element`]

[`τ_0_0.Iterator.Element: Equatable`]

We will formalize what this actually means first, and eventually describe a *decision procedure*. It will answer affirmatively for either of the above two requirements, but report a negative answer for the below requirement for example, because it's not a consequence of our signature:

[`τ_0_0.Iterator: Equatable`]

Our decision procedure will also tell us that `τ_0_0.Iterator.Element` is a valid type parameter, but something like `τ_0_0.Element.Iterator` is not.

5.2. Derived Requirements

This section and the one that follows will define the *derived requirements* and *valid type parameters* of a generic signature. This will allow us to precisely state those questions the compiler must be able to answer about generic signatures while type checking generic declarations.

We do this by working in a system of deductive reasoning, or a *formal system*, of the sort studied in mathematical logic [69]. We define our formal system in relation to a fixed generic signature G, so each generic signature has its own corresponding formal system. A formal system has three constituent parts, which we describe somewhat informally:

1. A description of the possible *statements* in the formal system, generated by finite combination from a fixed set of symbols.

   Our statements are requirements, such as [`τ_0_0.Element: Equatable`], and type parameters, such as `τ_0_1.Iterator`; their syntactic structure is quite simple, as we've already seen.

2. A finite set of *elementary statements* which are assumed to be true.

   Our elementary statements are the generic parameters and explicit requirements of our generic signature G; all generic parameters are valid type parameters, and all explicit requirements are trivially "derived."

3. A specification of the *inference rules* for deriving new statements from previous statements, in a manner that depends only on their formal syntactic structure.

   We will describe inference rules for requirements and type parameters shortly.

The *theory* of a formal system is the set of all statements we can prove within the system. This always includes the elementary statements, but it should not include *all* possible statements. (A theory where *everything* is true explains nothing.)

The theory of a generic signature is the set union of its derived requirements and valid type parameters. To demonstrate membership in this set, we write down a *derivation*—a finite sequence of steps that give a constructive proof of our desired statement as a consequence of the formal system:

1. A derivation necessarily begins by deriving one or more elementary statements.
2. This is followed by zero or more steps that derive new statements from previous statements via inference rules.
3. The derivation ends with the final step proving the conclusion. (However, we can also think of a derivation as proving multiple things, if we take *all* conclusions from each intermediate step.)

There can be many ways to prove the same thing, and we allow "useless" steps whose conclusions are unused, so derivations are not unique in general. The point of writing down a derivation is that we can *check* the reasoning at each step, and convince ourselves the conclusion is a true statement in the theory.

A remark for practitioners. We use our formal system to *specify* various behaviors of the implementation, but the compiler itself does not directly encode derivations as data structures. (We learn in Chapter 18 that we reason about derived requirements and valid type parameters by translating into a *string rewriting* problem.)

Notation. A derivation step will always be written on a single line, with the conclusion first; then on the right-hand side we write the "kind" of derivation step in small caps, followed by the list of assumptions. The conclusion and assumptions are statements:

conclusion (Kind *assumption*)

An *elementary derivation step* has no assumptions, so it proves an elementary statement. Any other kind of derivation step applies an inference rule to one or more assumptions, which are themselves conclusions of previous steps. When discussing a specific derivation step, we write the assumptions in place. When listing a derivation, we prefer to be concise so we *number* each step and refer to the conclusion of a prior step by number:

1. *an elementary statement of "Pooh" sort* (Pooh)
2. *a consequence of the above by the "Piglet" principle* (Piglet 1)

We use the "turnsile operator" $\vdash$ as a predicate. If G is a generic signature and D is a requirement or type parameter, $G \vdash D$ means D is an element of the theory of G, as in "if $G \vdash D$, then ...".

Our formal system has 6 kinds of elementary statements and 17 inference rules; this section together with the next will explain them all, with examples in-between:

1. We start with conformance requirements, and same-type requirements between type parameters. The theory completely describes the behavior of this subset of the language. We will also sketch out the remaining requirement kinds.
2. The next section defines some more inference rules so that the derived same-type requirements define an equivalence relation on type parameters. We will study the equivalence classes generated by this relation.
3. We initially only consider type parameters formed from unbound dependent member types. The next section will also extend our theory to explain bound dependent member types, but this won't reveal anything fundamentally new.

Elementary statements. Let G be a generic signature. We can derive each generic parameter `τ_d_i` of G:

`τ_d_i` (Generic)

Let [`T: P`] be an explicit conformance requirement of G, so `T` is some type parameter and `P` is a protocol. We can derive this requirement:

[`T: P`] (Conf)

Let [`T == U`] be an explicit same-type requirement of G, with `T` and `U` type parameters. We can derive this requirement:

[`T == U`] (Same)

Requirement signatures. We now assume we have derived a specific conformance requirement [`T: P`] for some type parameter `T` and protocol `P`. Each element of the requirement signature of `P` defines an inference rule that generates a new statement:

[`T: P`] + requirement signature of `P` = more statements

The three kinds of inference rules are AssocName, AssocConf, and AssocSame. They correspond to requirement signature elements in the same way that the Generic, Conf and Same elementary statements correspond to generic signature elements.

From each associated type declaration `A` of `P`, the AssocName inference rule derives the unbound dependent member type `T.A`, with base type `T` and identifier `A`. Notice how this step is a consequence of its assumption, the assumption being the original conformance requirement:

`T.A` (AssocName [`T: P`])

From each associated conformance requirement of `P`, the AssocConf inference rule derives the conformance requirement obtained by substituting `T` in place of `Self`. An arbitrary associated conformance requirement takes the form $[\texttt{Self.U: Q}]_{\texttt{P}}$ for some type parameter `Self.U` and protocol `Q`. The substituted type parameter we described above is denoted by "`T.U`":

$$[\texttt{T.U: Q}] \qquad (\textsc{AssocConf}\ [\texttt{Self.U: Q}]_{\texttt{P}}\ [\texttt{T: P}])$$

From each associated same-type requirement of `P`, the AssocSame inference rule derives a new same-type requirement by substituting `T` in place of `Self`. The most general form of an associated same-type requirement is $[\texttt{Self.U == Self.V}]_{\texttt{P}}$ for type parameters `Self.U` and `Self.V`, so we're forming a same-type requirement from the two substituted type parameters "`T.U`" and "`T.V`":

$$[\texttt{T.U == T.V}] \qquad (\textsc{AssocSame}\ [\texttt{Self.U == Self.V}]_{\texttt{P}}\ [\texttt{T: P}])$$

We will omit the associated requirement from the list of assumptions when listing a derivation in a fixed generic signature, because there is no ambiguity in doing so.

A few words about the meta-syntactic variables used above, such as "`T`" and "`U`" and so on. In the conclusion of an elementary derivation step, they specify the most general form of an explicit requirement of G; we're saying that we have a fixed set of elementary derivation steps, and each one can be obtained from the schema by suitable substitution of concrete entities in place of "`T`", "`U`" and "`P`". When meta-syntactic variables appear in the assumptions of a derivation step as above, they mean something else—we're defining the inference rule by pattern matching, and we can make use of the rule as long as a suitable substitution of the meta-syntactic variables matches each assumption to the conclusion of a previous step.

The general form of a type parameter inside an associated requirement is denoted by "`Self.U`" because it's a type parameter for the protocol generic signature $G_{\texttt{P}}$, so it must be `Self` recursively wrapped in dependent member types, *zero* or more times. This means that `Self` itself is a valid substitution for "`Self.U`". This comes up with protocol inheritance. If `Derived` inherits from `Base`, we have the associated requirement $[\texttt{Self: Base}]_{\texttt{Derived}}$. Given a derivation of `[T: Derived]`, we get a derivation of `[T: Base]` by adding an AssocConf derivation step for our associated requirement.

Example 5.2. The inference rules presented so far can already prove some interesting statements, but we they're not sufficient to explain everything in the informal overview from Example 5.1. Let's revisit the generic signature from that example:

```
<τ_0_0, τ_0_1 where τ_0_0: Sequence, τ_0_1: Sequence,
                    τ_0_0.Element: Equatable,
                    τ_0_0.Element == τ_0_1.Element>
```

The `Sequence` protocol declares two associated types `Iterator` and `Element`, and since `τ_0_0` conforms to `Sequence`, we can derive `τ_0_0.Element`, for example:

1. [`τ_0_0: Sequence`] (Conf)
2. `τ_0_0.Element` (AssocName 1)

We can derive `τ_0_0.Iterator` similarly. Now consider `τ_0_0.Iterator.Element`. We recall that `Sequence` declares two associated requirements:

[`Self.Iterator: IteratorProtocol`]$_{\texttt{Sequence}}$

[`Self.Element == Self.Iterator.Element`]$_{\texttt{Sequence}}$

The first associated requirement allows us to derive that `τ_0_0.Iterator` conforms to `IteratorProtocol`, and from there we can derive `τ_0_0.Iterator.Element`:

1. [`τ_0_0: Sequence`] (Conf)
2. [`τ_0_0.Iterator: IteratorProtocol`] (AssocConf 1)
3. `τ_0_0.Iterator.Element` (AssocName 2)

We can also derive `τ_0_1.Iterator.Element` if we start from [`τ_0_1: Sequence`] instead:

1. [`τ_0_1: Sequence`] (Conf)
2. [`τ_0_1.Iterator: IteratorProtocol`] (AssocConf 1)
3. `τ_0_1.Iterator.Element` (AssocName 2)

Remember our original goal in Example 5.1 was to establish a same-type requirement relating `τ_0_0.Iterator.Element` with `τ_0_1.Iterator.Element`, and not just to show that these two type parameters exist. We're going to attempt to write down a derivation. The associated same-type requirement of `Sequence` gives us a same-type requirement between `τ_0_0.Element` and `τ_0_0.Iterator.Element`:

1. [`τ_0_0: Sequence`] (Conf)
2. [`τ_0_0.Element == τ_0_0.Iterator.Element`] (AssocSame 1)

We can also derive a similar statement about `τ_0_1`:

3. [`τ_0_1: Sequence`] (Conf)
4. [`τ_0_1.Element == τ_0_1.Iterator.Element`] (AssocSame 1)

Now remember we have an explicit same-type requirement also:

5. [`τ_0_0.Element == τ_0_1.Element`] (Same)

At this point, we have derived the three same-type requirements (2), (4) and (5), but no apparent way to conclude anything else:

[τ_0_0.Element == τ_0_0.Iterator.Element]

[τ_0_1.Element == τ_0_1.Iterator.Element]

[τ_0_0.Element == τ_0_1.Element]

We put this example aside until the next section, when we introduce more inference rules for working with same-type requirements between type parameters.

Other requirements. We will now extend our formal system to cover concrete same-type requirements, superclass requirements and layout requirements, with the caveat that this extended theory is incomplete. All inferences made by these rules are correct, but they do not describe all implemented behaviors of these other requirement kinds. We will see examples in Section 5.5 and Section 11.4.

We add some more elementary statements. Let G be a generic signature. If [`T == X`] is an explicit same-type requirement of G, where `T` is a type parameter and `X` is a concrete type (so `X` is not a type parameter but it may contain type parameters):

[`T == X`] (Concrete)

If [`T: C`] is an explicit superclass requirement of G, so `C` is a class type:

[`T: C`] (Super)

If [`T: AnyObject`] is an explicit layout requirement of G:

[`T: AnyObject`] (Layout)

We also have inference rules for the associated requirement forms of the above, so in what follows, we assume we can derive [`T: P`] for some `T` and `P`.

Suppose that `P` declares an associated same-type requirement $[\texttt{Self.U == X}]_{\texttt{P}}$, for some `Self.U` and concrete type `X`. We substitute `T` for `Self` inside `Self.U` to get a type parameter `T.U`, and inside `X` to get a concrete type `X′`. We can then derive:

[`T.U == X′`] (AssocConcrete $[\texttt{Self.U == X}]_{\texttt{P}}$ [`T: P`])

For an associated superclass requirement $[\texttt{Self.U: C}]_{\texttt{P}}$, we substitute `T` for `Self` in `C` to get `C′`:

[`T.U: C′`] (AssocSuper $[\texttt{Self.U: C}]_{\texttt{P}}$ [`T: P`])

Finally, an associated layout requirement does not have anything to substitute on the right-hand side, so we derive the same statement about `T.U`:

[`T.U: AnyObject`] (AssocLayout $[\texttt{Self.U: AnyObject}]_{\texttt{P}}$ [`T: P`])

At this point, we've seen all of the kinds of elementary statements that exist, but a few more inference rules remain to be defined. Here is a quick summary of all the elementary statements and inference rules we've seen so far:

Basic model:		
Generic	Conf	Same
AssocName	AssocConf	AssocSame
Other requirements:		
Concrete	Super	Layout
AssocConcrete	AssocSuper	AssocLayout

5.3. Valid Type Parameters

In Chapter 3, we discussed how types containing type parameters have two kinds of equality. Canonical type equality tells us if the type parameters are spelled in the same way, and *reduced type equality*, relative to a generic signature, also takes same-type requirements into account. We are now in a position to define reduced type equality on type parameters. Section 5.5 will generalize it to all interface types.

Reduced type equality is an *equivalence relation*. We will talk about other equivalence relations later, so we introduce the abstraction now. Given a fixed set that we call the *domain*, a relation models some property that a pair of elements might have. In programming languages, "relational operators" are typically functions taking a pair of values to a true or false result. In mathematics, we instead imagine that a relation is the set of those ordered pairs (x, y) such that the property is true of x and y.

Definition 5.3. Let S be a set. A *relation* with domain S is a subset of the Cartesian product $S \times S$.

Definition 5.4. An *equivalence relation* $R \subseteq S \times S$ is reflexive, symmetric and transitive:

- R is *reflexive* if $(x, x) \in R$ for all $x \in S$.
- R is *symmetric* if $(x, y) \in R$ implies that $(y, x) \in R$ for all x, $y \in S$.
- R is *transitive* if (x, y), $(y, z) \in R$ implies that $(x, z) \in R$ for all x, y, $z \in S$.

Imagine a fraction a/b as a purely formal object, a pair of integers a, b with $b \neq 0$. While $1/2$, $2/4$, and $(-3)/(-6)$ are distinct fractions, they measure the same proportion, as we see by cross-multiplying denominators. We have two equivalence relations:

- a/b and c/d are "identical" if $a = b$ and $c = d$.
- a/b and c/d are "equivalent" if $ad = bc$.

Definition 5.5. Let R be an equivalence relation with domain S, and let $x \in S$. The *equivalence class* of x, denoted $[\![x]\!]$, is the set of all $y \in S$ such that $(x, y) \in R$.

Under our "equivalence of proportion" relation, the equivalence class of $1/2$ is the set of all fractions of the form $n/2n$, as n ranges over all non-zero integers. Every fraction belongs to exactly one equivalence class, so our equivalence relation partitions the set of fractions into disjoint equivalence classes. We now increase the level of abstraction, and consider the set whose *elements* are equivalence classes of fractions. This is $\mathbb{Q}$, the set of *rational numbers*, where two equivalent fractions represent the same rational number. We can repeat this construction with any equivalence relation.

Proposition 5.6. Let R be an equivalence relation with some domain S. Every element of S belongs to exactly one equivalence class of R.

Proof. The phrase "exactly one" is actually a shorthand for these two statements:

1. Every $x \in S$ belongs to *at least* one equivalence class.
2. Every $x \in S$ belongs to *at most* one equivalence class, so if $x \in [\![y]\!]$ and $x \in [\![z]\!]$ for some y, $z \in S$, then in fact $[\![y]\!] = [\![z]\!]$ (this is equality of sets, meaning they have the same elements).

For the first part, the assumption that R is reflexive says that $(x, x) \in R$ for all $x \in S$. By definition of $[\![x]\!]$, this means that $x \in [\![x]\!]$, so every x is, at the very least, an element of its *own* equivalence class $[\![x]\!]$.

For the second part, assume that for some y, $z \in S$, there exists an $x \in [\![y]\!] \cap [\![z]\!]$. To show that $[\![y]\!] = [\![z]\!]$, we prove that $[\![y]\!] \subseteq [\![z]\!]$ and $[\![z]\!] \subseteq [\![y]\!]$. To see that $[\![y]\!] \subseteq [\![z]\!]$, we take an arbitrary element $t \in [\![y]\!]$, and chase a series of equivalences to derive that $t \in [\![z]\!]$:

1. $t \in [\![y]\!]$ and $x \in [\![y]\!]$, so (t, y), $(x, y) \in R$.
2. $(x, y) \in R$, but R is symmetric, so $(y, x) \in R$.
3. (t, y), $(y, x) \in R$, but R is transitive, so $(t, x) \in R$.
4. (t, x), $(x, z) \in R$, but R is transitive, so $(t, z) \in R$; that is, $t \in [\![z]\!]$.

This gives us $[\![y]\!] \subseteq [\![z]\!]$. To prove $[\![z]\!] \subseteq [\![y]\!]$, we pretend to copy and paste the above, and swap y and z everywhere that they appear. (We tend to avoid writing out the other direction of the proof in a situation like this where it is completely mechanical.)

We've shown that if two equivalence classes of S have at least one element in common, they must in fact coincide. The only other possibility is that the two equivalence classes are disjoint, that is, their intersection is the empty set. Note that we made use of all three defining properties of an equivalence relation; the result no longer holds if we relax either assumption of reflexivity, symmetry or transitivity. □

Reduced type equality. We now define an equivalence relation on type parameters. We want to say that two type parameters are equivalent if we can derive a same-type requirement between them:

Definition 5.7. Let G be a generic signature. The *reduced type equality* relation on the valid type parameters of G is the set of all pairs `T` and `U` such that $G \vdash$ `[T == U]`.

For this to work, we must extend our formal system with three new inference rules, one for each defining axiom of an equivalence relation. The first rule says that if we can derive a valid type parameter `T`, we can derive the trivial same-type requirement `[T == T]`. Note that we're deriving a *requirement* from a *type parameter*; this is the only time we can do that:

`[T == T]` (Reflex `T`)

Next, if we can derive a same-type requirement `[T == U]` for type parameters `T` and `U`, we can derive the opposite same-type requirement:

`[U == T]` (Sym `[T == U]`)

Finally, if we can derive two same-type requirements `[T == U]` and `[U == V]` where the right-hand side of the first is identical to the left-hand side of the second, we can derive a same-type requirement that gets us from one end to the other in one jump:

`[T == V]` (Trans `[T == U]` `[U == V]`)

Given these new rules, we immediately see:

Proposition 5.8. Reduced type equality is an equivalence relation.

Proof. Each axiom follows from the corresponding inference rule:

- (Reflexivity) Suppose that `T` is a valid type parameter of G. Given a derivation $G \vdash$ `T`, we derive $G \vdash$ `[T == T]` via Reflex. Thus, `T` is equivalent to itself.
- (Symmetry) Suppose that `T` is equivalent to `U`. Given a derivation $G \vdash$ `[T == U]`, we derive $G \vdash$ `[U == T]` via Sym. Thus, `U` is equivalent to `T`.
- (Transitivity) Suppose that `T` is equivalent to `U`, and `U` is equivalent to `V`. Given the two derivations $G \vdash$ `[T == U]` and $G \vdash$ `[U == V]`, we concatenate them together, and derive $G \vdash$ `[T == V]` via Trans. Thus, `T` is equivalent to `V`.

We are careful to limit our relation's domain to the valid type parameters of G, instead of considering all type parameters that can be formed syntactically. If `T` is some invalid type parameter that cannot be derived from G, we cannot use Reflex to derive `[T == T]`, so we would no longer have an equivalence relation. □

So far, nothing prevents us from deriving a requirement [`T == U`] where `T` or `U` is not itself derivable. If this happens, we exclude the ordered pair from our relation. We will not worry about this for now, because Section 11.3 will show that such situations can be ruled out by diagnosing invalid requirements written by the user.

Example 5.9. The proof that reduced type equality is an equivalence relation may seem useless or circular, but the point is we can use these inference rules to derive more requirements. Returning to Example 5.2, we had derived (2), (4) and (5) below:

1. [`τ_0_0: Sequence`] (Conf)
2. [`τ_0_0.Element == τ_0_0.Iterator.Element`] (AssocSame 1)
3. [`τ_0_1: Sequence`] (Conf)
4. [`τ_0_1.Element == τ_0_1.Iterator.Element`] (AssocSame 3)
5. [`τ_0_0.Element == τ_0_1.Element`] (Same)

We apply Sym to (2), and then observe that (6), (5) and (4) now form a chain where the right-hand side of each one is identical to the left-hand side of the next. Two applications of Trans give us what we're looking for:

6. [`τ_0_0.Iterator.Element == τ_0_0.Element`] (Sym 2)
7. [`τ_0_0.Iterator.Element == τ_0_1.Element`] (Trans 6 5)
8. [`τ_0_0.Iterator.Element == τ_0_1.Iterator.Element`] (Trans 7 4)

We managed to derive the first requirement from Example 5.1. Along the way, we've shown that our generic signature has four "singleton" equivalence classes,

{`τ_0_0`}, {`τ_0_1`}, {`τ_0_0.Iterator`}, {`τ_0_1.Iterator`},

and one final equivalence class formed from the other four type parameters,

{`τ_0_0.Element`, `τ_0_1.Element`,
`τ_0_0.Iterator.Element`, `τ_0_1.Iterator.Element`}.

To prove that `τ_0_0.Iterator.Element` conforms to `Equatable`, we need another rule.

Compatibility. Our equivalence relation on fractions had an interesting property: if we pick any two fractions from a pair of equivalence classes and add them together (or multiply, etc), the equivalence class of the result does not depend on our choice of representatives. In Swift generics, we have an analogous goal: any observable behavior of a type parameter, be it conforming to a protocol, having a superclass bound, being fixed to a concrete type, or having a dependent member type with a given name, should only depend on its equivalence class, and not on the "spelling."

Once again, we wave our magic wand, and decree that reduced type equality is to have this property, by adding new inference rules. These rules relate same-type requirements with the other requirement kinds by replacing the other requirement's subject type. That is, if we can derive a same-type requirement [`T == U`] and a conformance requirement [`U: P`], we can derive [`T: P`]:

[`T: P`] (SameConf [`U: P`] [`T == U`])

A same-type requirement [`T == U`] also composes with the other requirement kinds:

[`T == X`] (SameConcrete [`U == X`] [`T == U`])
[`T: C`] (SameSuper [`U: C`] [`T == U`])
[`T: AnyObject`] (SameLayout [`U: AnyObject`] [`T == U`])

There is a second compatibility condition. We want equivalent type parameters to have equivalent member types, as follows. Suppose we can derive a conformance requirement [`U: P`] and a same-type requirement [`T == U`]. For every associated type `A` of `P`, we can derive a same-type requirement [`T.A == U.A`]:

[`T.A == U.A`] (SameName [`U: P`] [`T == U`])

We have yet to describe the inference rules for bound dependent member types, so that's coming up next. Apart from that, our formal system is complete, so let's look at a few more examples to justify these rules we just added.

Example 5.10. To continue Example 5.9, we can use the SameConf inference rule to derive [`τ_0_0.Iterator.Element: Equatable`] from [`τ_0_0.Element: Equatable`], because as we saw, `τ_0_0.Element` and `τ_0_0.Iterator.Element` are equivalent:

1. [`τ_0_0.Element: Equatable`] (Conf)
2. [`τ_0_0.Element == τ_0_0.Iterator.Element`] (AssocSame 1)
3. [`τ_0_0.Iterator.Element == τ_0_0.Element`] (Sym 2)
3. [`τ_0_0.Iterator.Element: Equatable`] (SameConf 1 3)

In our generic signature, this *entire* equivalence class conforms to `Equatable`:

{`τ_0_0.Element`, `τ_0_1.Element`,
`τ_0_0.Iterator.Element`, `τ_0_1.Iterator.Element`}.

Example 5.11. To see SameName in action, we now look recall the generic signature of `sameIter()` from the beginning of the chapter:

```
<S1, S2 where S1: Sequence, S2: Sequence,
              S1.Iterator == S2.Iterator>
```

We also saw another declaration, `sameIterAndElt()`, which states both requirements:

[τ_0_0.Iterator == τ_0_1.Iterator]
[τ_0_0.Element == τ_0_1.Element]

We claimed that requirement minimization will drop the second requirement because it is redundant, giving `sameIterAndElt()` the same generic signature as `sameIter()`. This means we can derive the second requirement without invoking *itself* as an elementary statement. We do this by first taking [τ_0_0.Iterator == τ_0_1.Iterator], and applying SameName to derive the same-type requirement (3):

1. [τ_0_1: Sequence] (Conf)
2. [τ_0_0.Iterator == τ_0_1.Iterator] (Same)
3. [τ_0_0.Iterator.Element == τ_0_1.Iterator.Element] (SameName 1 2)

We then rewrite both sides of (3) into their short form by making use of the associated same-type requirement of `Sequence`:

4. [τ_0_0: Sequence] (Conf)
5. [τ_0_0.Element == τ_0_0.Iterator.Element] (AssocSame 4)
6. [τ_0_1.Element == τ_0_1.Iterator.Element] (AssocSame 1)
7. [τ_0_1.Iterator.Element == τ_0_1.Element] (Sym 6)
8. [τ_0_0.Element == τ_0_1.Iterator.Element] (Trans 5 3)
9. [τ_0_0.Element == τ_0_1.Element] (Trans 8 7)

We could not have derived this requirement without SameName.

Example 5.12. We can conjure up a generic signature with infinitely many equivalence classes. We start with this protocol:

```
protocol N {
  associatedtype A: N
}
```

Now, consider the protocol generic signature $G_{\texttt{N}}$. We can derive an infinite sequence of conformance requirements from [τ_0_0: N], by repeated application of the AssocConf inference rule with the associated conformance requirement $[\texttt{Self.A: N}]_{\texttt{N}}$:

1. [τ_0_0: N] (Conf)
2. [τ_0_0.A: N] (AssocConf 1)
3. [τ_0_0.A.A: N] (AssocConf 2)
4. ...

We can also derive infinitely many valid type parameters:

5. τ_0_0 (Generic)
6. $\tau\_0\_0.A$ (AssocName 2)
7. $\tau\_0\_0.A.A$ (AssocName 3)
8. ...

Each one of these type parameters is in its own equivalence class, because we cannot derive any non-trivial same-type requirements. We've shown that $G_{\mathtt{N}}$ defines an infinite set of equivalence classes.

Example 5.13. Here is a simplified form of the standard library `Collection` protocol:

```
protocol Collection: Sequence {
  associatedtype SubSequence: Collection
      where Element == SubSequence.Element
            SubSequence == SubSequence.SubSequence
}
```

Our protocol inherits the `Element` and `Iterator` associated types from `Sequence`, then declares a new associated type and states these four associated requirements:

$$[\texttt{Self: Sequence}]_{\texttt{Collection}}$$
$$[\texttt{Self.SubSequence: Collection}]_{\texttt{Collection}}$$
$$[\texttt{Self.Element == Self.SubSequence.Element}]_{\texttt{Collection}}$$
$$[\texttt{Self.SubSequence == Self.SubSequence.SubSequence}]_{\texttt{Collection}}$$

We can read the associated requirements as follows:

1. The first conformance requirement states that all collections are sequences.
2. The second conformance requirement states that a subsequence of a collection is another collection, of a possibly different concrete type.
3. The first same-type requirement states that a subsequence of an arbitrary collection always has the same element type as the original.
4. The second same-type requirement states that a subsequence is just a subsequence of the original collection, so subsequences don't stack more than once.

For example, the `SubSequence` of `Array<Int>` is `ArraySlice<Int>`, and `SubSequence` of `ArraySlice<Int>` is again `ArraySlice<Int>`. (Type witnesses of concrete conformances are discussed in Section 7.3).

We're going to look at the protocol generic signature $G_{\texttt{Collection}}$ and attempt to understand its equivalence classes. We can use the protocol inheritance relationship to derive `τ_0_0.Iterator`:

1. [`τ_0_0: Collection`] (Conf)
2. [`τ_0_0: Sequence`] (AssocConf 1)
3. `τ_0_0.Iterator` (AssocName 2)

The first thing we notice is that `τ_0_0` and `τ_0_0.Iterator` are not equivalent to each other or to any other type parameter. Each one is in an equivalence class by itself, so we have our first two equivalence classes.

To understand how the remaining equivalence classes are formed, we notice that $[\texttt{Self.SubSequence: Collection}]_{\texttt{Collection}}$ generates an infinite family of conformance requirements, like $[\texttt{Self.A: N}]_{\texttt{N}}$ did in the previous example:

1. [`τ_0_0: Collection`] (Conf)
2. [`τ_0_0.SubSequence: Collection`] (AssocConf 1)
3. [`τ_0_0.SubSequence.SubSequence: Collection`] (AssocConf 2)
4. ...

To talk about this phenomenon, we introduce the following notation:

$$\texttt{τ\_0\_0.SubSequence}^n = \begin{cases} \texttt{τ\_0\_0} & n = 0 \\ \texttt{τ\_0\_0.SubSequence} & n = 1 \\ \texttt{τ\_0\_0.SubSequence}^{n-1}\texttt{.SubSequence} & n > 1 \end{cases}$$

So the theory of $G_{\texttt{Collection}}$ contains $[\texttt{τ\_0\_0.SubSequence}^n\texttt{: Collection}]$ for all $n \geq 0$. Furthermore, for each $n \geq 0$, we can derive the following from (1) below:

1. $[\texttt{τ\_0\_0.SubSequence}^n\texttt{: Collection}]$ (...)
2. $[\texttt{τ\_0\_0.SubSequence}^n\texttt{: Sequence}]$ (AssocConf 1)
3. $[\texttt{τ\_0\_0.SubSequence}^n\texttt{.Element == τ\_0\_0.SubSequence}^{n+1}\texttt{.Element}]$ (AssocSame 1)
4. $[\texttt{τ\_0\_0.SubSequence}^n\texttt{.Element == τ\_0\_0.SubSequence}^n\texttt{.Iterator.Element}]$ (AssocSame 2)

The infinite families (3) and (4) define a single equivalence class. This equivalence class contains $\texttt{τ\_0\_0.SubSequence}^n\texttt{.Element}$ and $\texttt{τ\_0\_0.SubSequence}^n\texttt{.Iterator.Element}$, for all $n \geq 0$. (In particular, it contains `τ_0_0.Element`.)

We can also generate another infinite family of same-type requirements:

5. $[\texttt{τ\_0\_0.SubSequence}^{n+1}\texttt{ == τ\_0\_0.SubSequence}^{n+2}]$ (AssocSame 1)

These form an equivalence class from all $\texttt{τ\_0\_0.SubSequence}^n\texttt{.SubSequence}$ for $n \geq 0$. Finally, applying SameName gives us one more infinite family:

6. $[\texttt{τ\_0\_0.SubSequence}^{n+1}\texttt{.Iterator == τ\_0\_0.SubSequence}^{n+2}\texttt{.Iterator}]$ (SameName 5 2)

These form the last equivalence class, because the `SubSequence` may have a distinct `Iterator` from the original sequence.

We see that $G_{\texttt{Collection}}$ defines five equivalence classes, and three of those contain infinitely many representative type parameters each:

$$\{\texttt{τ\_0\_0}\}, \tag{1}$$

$$\{\texttt{τ\_0\_0.Element}, \texttt{τ\_0\_0.SubSequence.Element}, \ldots\} \tag{2}$$
$$\cup\, \{\texttt{τ\_0\_0.Iterator.Element}, \texttt{τ\_0\_0.SubSequence.Iterator.Element}, \ldots\},$$

$$\{\texttt{τ\_0\_0.Iterator}\}, \tag{3}$$

$$\{\texttt{τ\_0\_0.SubSequence}, \texttt{τ\_0\_0.SubSequence.SubSequence}, \ldots\}, \tag{4}$$

$$\{\texttt{τ\_0\_0.SubSequence.Iterator}, \tag{5}$$
$$\texttt{τ\_0\_0.SubSequence.SubSequence.Iterator},$$
$$\ldots\}$$

To describe the conformance requirements that apply to each equivalence class, it is sufficient to pick a single representative conformance requirement for each combination of type parameter and protocol, because the SameConf inference rule allows us to derive the rest. This table summarizes our "pen and paper" investigation:

Representative:	**Conforms to:**
`τ_0_0`	`Collection` and `Sequence`
`τ_0_0.Element`	none
`τ_0_0.Iterator`	`IteratorProtocol`
`τ_0_0.SubSequence`	`Collection` and `Sequence`
`τ_0_0.SubSequence.Iterator`	`IteratorProtocol`

That's pretty much all there is to say about the generic signature $G_{\texttt{Collection}}$, however our above description is somewhat lacking because it doesn't make the member type relationships apparent. We will revisit all of the generic signatures we saw here in Section 8.3, when we introduce the *type parameter graph*. We will define an edge relation on equivalence classes, which become the vertices of a graph. This will give us a visual aid for understanding member type relationships.

Notice how our formal system can generate an infinite theory! We saw that $G_{\texttt{N}}$ has an infinite set of finite equivalence classes, while in $G_{\texttt{Collection}}$, the equivalence classes themselves were infinite. Of course both can happen simultaneously. A generic signature with an infinite set of infinite equivalence classes will appear in Section 17.3.

Bound dependent member types. We saw in Section 3.1 that dependent member types come in two varieties:

- Unbound dependent member types refer to an identifier. Denoted `T.A` for some base type `T` and identifier `A`.
- Bound dependent member types refer to an associated type declaration. Denoted `T.[P]A` for some base type `T` and associated type declaration `A` of some protocol `P`.

Definition 5.14. We define the following for convenience:

- An *unbound type parameter* is a generic parameter type, or an unbound dependent member type whose base type is another unbound type parameter.
- A *bound type parameter* is a generic parameter type, or a bound dependent member type whose base type is another bound type parameter.

The bound and unbound type parameters do not exhaustively partition the set of all type parameters. A generic parameter type is *both* bound and unbound per the above, while a type parameter like `τ_0_0.[Sequence]Iterator.Element` is neither bound nor unbound, because it contains a mix of bound and unbound dependent member types.

The fundamental tension here is the following:

- Unbound type parameters types appear when type resolution resolves the requirements in a `where` clause; they directly represent what was written by the user.
- Type substitution only operates on bound dependent member types, because as we will see in Section 7.4, we need all three pieces of information that describe one as a valid type parameter: the base type, the protocol, and the associated type declaration.

We often apply substitution maps to requirements of generic signatures and requirement signatures, and the interface types of declarations. Indeed, all of these semantic objects must only contain bound type parameters. We resolve this as follows:

- Requirement minimization (Section 11.4) converts unbound type parameters in the `where` clause into bound type parameters when building a generic signature.
- Queries against an existing generic signature (Section 5.5) allow unbound type parameters. We will see that one can obtain a bound type parameter by asking the generic signature for an unbound type parameter's *reduced type*.
- Type resolution makes use of generic signature queries to form bound dependent member types when resolving the interface type of a declaration (Chapter 9).

We now extend our formal system to describe these behaviors. As always, let G be a generic signature. We assume that $G \vdash$ [`T: P`] for some `T` and `P`.

We first add an inference rule that is analogous to AssocName except that it derives a bound dependent member type. From each associated type declaration `A` of `P`, the AssocDecl inference rule derives the bound dependent member type `T.[P]A`, with base type `T`, referencing the associated type declaration `A`:

`T.[P]A` (AssocDecl [`T: P`])

To encode the equivalence of bound and unbound dependent member types, we also add an inference rule that derives a same-type requirement between the two. That is, from each associated type declaration `A` of `P`, the AssocBind inference rule derives a same-type requirement between the two dependent member types that AssocDecl and AssocName would derive under the same assumptions:

[`T.[P]A == T.A`] (AssocBind [`T: P`])

The SameName inference rule, which gave us [`T.A == U.A`] from [`U: P`] and [`T == U`], also has a bound type parameter equivalent. Under the same assumptions, SameDecl derives [`T.[P]A == U.[P]A`]:

[`T.[P]A == U.[P]A`] (SameDecl [`U: P`] [`T == U`])

The SameDecl inference rule is fundamentally different from all of the others, because its introduction does not contribute any novel statements to our theory:

Proposition 5.15. Any derivation containing a SameDecl step can be transformed into one without.

Proof. Given $G \vdash$ [`U: P`] and $G \vdash$ [`T == U`], we can derive [`T.[P]A == U.[P]A`] without SameDecl by adding the below steps, for any suitable choice of `T`, `U`, `P` and `A`:

1. [`U: P`] (...)
2. [`T == U`] (...)
3. [`T.A == U.A`] (SameName 1 2)
4. [`U.[P]A == U.A`] (AssocBind 1)
5. [`U.A == U.[P]A`] (Sym 4)
6. [`T: P`] (SameConf 1 2)
7. [`T.[P]A == T.A`] (AssocBind 6)
8. [`T.[P]A == U.A`] (Trans 7 3)
9. [`T.[P]A == U.[P]A`] (Trans 8 5)

However, it is certainly more convenient to write 1 derivation step instead of 7 every time we need this equivalence. Thus, SameDecl is syntax sugar for our formal system. □

Example 5.16. Consider the protocol generic signature $G_{\texttt{Sequence}}$, and recall that `Sequence` states two associated requirements:

$$[\texttt{Self.Iterator: IteratorProtocol}]_{\texttt{Sequence}}$$
$$[\texttt{Self.Element == Self.Iterator.Element}]_{\texttt{Sequence}}$$

Let's continue to pretend that those associated requirements are written with unbound type parameters at first, but allow use of the new inference rules. We can derive the bound type parameter `τ_0_0.[Sequence]Iterator.[IteratorProtocol]Element`:

1. [`τ_0_0: Sequence`] (Conf)
2. [`τ_0_0.Iterator: IteratorProtocol`] (AssocConf 1)
3. [`τ_0_0.[Sequence]Iterator == τ_0_0.Iterator`] (AssocBind 1)
4. [`τ_0_0.[Sequence]Iterator: IteratorProtocol`] (SameConf 2 3)
5. `τ_0_0.[Sequence]Iterator.[IteratorProtocol]Element` (AssocDecl 4)

We used the new inference rules in (3) and (5). Notice how (4) looks a lot like our associated conformance requirement, but with a bound type parameter.

Now, we can also define our formal system so that the explicit requirements of our generic signature and the associated requirements of protocols are written in terms of bound type parameters, as they actually would be in the implementation:

$$[\texttt{Self.[Sequence]Iterator: IteratorProtocol}]_{\texttt{Sequence}}$$
$$[\texttt{Self.[Sequence]Element == Self.[Sequence]Iterator.[IteratorProtocol]Element}]_{\texttt{Sequence}}$$

We see that `τ_0_0.[Sequence]Iterator.[IteratorProtocol]Element` can be derived, now with fewer steps:

1. [`τ_0_0: Sequence`] (Conf)
2. [`τ_0_0.[Sequence]Iterator: IteratorProtocol`] (AssocConf 1)
3. `τ_0_0.[Sequence]Iterator.[IteratorProtocol]Element` (AssocDecl 2)

And there's a symmetry here—we can just as easily derive the unbound type parameter `τ_0_0.Iterator.Element` from our associated requirements that contain bound type parameters:

1. [`τ_0_0: Sequence`] (Conf)
2. [`τ_0_0.[Sequence]Iterator: IteratorProtocol`] (AssocConf 1)
3. [`τ_0_0.[Sequence]Iterator == τ_0_0.Iterator`] (AssocBind 1)
4. [`τ_0_0.Iterator == τ_0_0.[Sequence]Iterator`] (Sym 3)
5. [`τ_0_0.Iterator: IteratorProtocol`] (SameConf 2 4)
6. `τ_0_0.Iterator.Element` (AssocName 5)

In fact, we will prove the following in Chapter 11:

- Theorem 11.8 will show that every equivalence class of valid type parameters always contains at least one bound and at least one unbound type parameter.
- Proposition 11.9 will tell us that we can start with a list of explicit requirements written with bound or unbound type parameters without changing the theory.

Suppose a generic parameter `τ_0_0` conforms to two protocols `P1` and `P2`, and both declare an associated type named `A`. The same unbound dependent type `τ_0_0.A` can be derived from either conformance requirement, and this effectively merges the associated requirements imposed on `A` in the language semantics. This remains true once we add bound dependent member types, because we can derive:

1. [`τ_0_0: P1`] (Conf)
2. [`τ_0_0.[P1]A == τ_0_0.A`] (AssocBind 1)
3. [`τ_0_0: P2`] (Conf)
4. [`τ_0_0.[P2]A == τ_0_0.A`] (AssocBind 3)
5. [`τ_0_0.A == τ_0_0.[P2]A`] (Sym 4)
6. [`τ_0_0.[P1]A == τ_0_0.[P2]A`] (Trans 2 5)

A special case is when the first protocol inherits from the other. Re-stating an associated type declaration with the same name as an associated type of an inherited protocol has no effect. This is explored further in Section 19.4.

Bound type parameters don't seem to yield anything new. Why bother then? Type substitution must issue generic signature queries in the general case anyway, and we could just consult the generic signature every time we decompose a dependent member type. Instead, our representation basically pre-computes certain information and encodes it in the structure of the dependent member type itself. This avoids generic signature queries in simple cases, and our formal system proves this is equivalent.

Summary. A complete summary of all elementary statements and inference rules in the derived requirements formalism appears in Appendix B. Bound type parameters are trivial from a theoretical standpoint, and our theory of concrete type requirements is incomplete, so we will mostly work with this subset of elementary statements and inference rules:

Elementary statements:	Generic	Conf	Same
Inference rules:	AssocName Reflex	AssocConf Sym SameConf	AssocSame Trans SameName

So far, we've only used our formal system to derive concrete statements *in* a fixed generic signature. In later chapters, we prove results *about* generic signatures:

- In Chapter 11, we describe how we diagnose invalid requirements, and show that if no diagnostics are emitted, our formal system has a particularly nice theory.
- In Chapter 12, we take a closer look at derived conformance requirements, to describe substitution of dependent member types; in particular, we prove that a certain algorithm must terminate.
- In Chapter 17, we use derived requirements to show that a Swift protocol can encode an arbitrary finitely-presented monoid, which demonstrates that a generic signature can have an undecidable theory.
- In Chapter 18, we will translate the explicit requirements of a generic signature into rewrite rules, and then show that derived requirements correspond to *rewrite paths* under this mapping. This provides a correctness proof for the implementation.

5.4. Reduced Type Parameters

An equivalence class of type parameters might be infinite, or finite but large. For this reason, we cannot model an equivalence class as a set in the implementation. Instead, we define a way to consistently select a unique representative from each equivalence class, called the *reduced type parameter* of the equivalence class. The reduced type parameter can "stand in" for its entire equivalence class, and the set of all reduced type parameters gives another description of the equivalence class structure of a generic signature.

To continue the analogy with fractions from the previous section, we don't usually think of a single rational number as an infinite set of fractions. Instead, after performing a series of arithmetic operations, we *reduce* the result to lowest terms by eliminating common factors from the numerator and denominator. For example, $2/4$ and $(-3)/(-6)$ reduce to $1/2$, while $1/2$ is already reduced. This gives us a new way to check if two fractions are equivalent: we reduce both, and then check if the reduced fractions are identical. A key fact is that we can find a reduced fraction by *ordering* the elements of its equivalence class: the reduced fraction has the smallest positive denominator.

Definition 5.17. A *partial order* $R \subseteq S \times S$ is anti-reflexive and transitive:

- R is *anti-reflexive* if $(x, x) \notin R$ for all $x \in S$.
- R is *transitive* if (x, y), $(y, z) \in R$ implies that $(x, z) \in R$.

If R is clear from context, we write $x < y$ instead of $(x, y) \in R$ and $x \not< y$ instead of $(x, y) \notin R$. We also define $>$, $\leq$ and $\geq$ in the usual way in terms of $<$.

Note that $x < y$ and $y < x$ cannot both be true at once, because then we'd have $x < x$ by transitivity, which contradicts the requirement that $x \not< x$ for all $x \in S$. Therefore in a partial order, *at most* one of the below is true for a pair of elements x, $y \in S$:

1. $x = y$,
2. $x < y$,
3. $x > y$.

If none of these are true, we say x and y are *incomparable*. We will consider general partial orders later, but for now we're going to restrict our attention to those orders where *exactly* one of the three conditions above is true, for any pair of elements:

Definition 5.18. A *linear order* is a partial order without incomparable elements. In a linear order, $x \not< y$ is another way of saying $x \geq y$.

We will define a linear order on type parameters, denoted by $<$. This will express the idea that if $G \vdash$ `[T == U]` and `T` $<$ `U`, then `T` is "more reduced" than `U`, within this equivalence class that contains both. The minimum out of all representatives is then the reduced type parameter itself. We will start by measuring the "complexity" of a type parameter by counting the number of dependent member types involved in its construction:

Definition 5.19. The *length* of a type parameter `T`, denoted by $|$`T`$|$, is a natural number. The length of a generic parameter type is 1, while the length of a dependent member type `U.A` or `U.[P]A` is recursively defined as $|$`U`$| + 1$.

We want the reduced type parameter to be one of minimum possible length, and we also want to be able to apply substitution maps to it, so it ought to be a bound type parameter. This gives us two conditions our type parameter order must satisfy:

1. If $|$`T`$| < |$`U`$|$, then `T` $<$ `U`.
2. Bound dependent member types precede unbound dependent member types.

Example 5.20. In Example 5.13, we studied a simplified `Collection` protocol. We saw that the equivalence class of `τ_0_0.SubSequence` in $G_{\texttt{Collection}}$ contains an infinite sequence of type parameters of increasing length,

$$\{\texttt{τ\_0\_0.SubSequence}^n\}_{n \geq 1}.$$

After adding inference rules for bound dependent member types, our equivalence class will also contain some of those:

$$\{\texttt{τ\_0\_0.[Collection]SubSequence}^n\}_{n \geq 1}.$$

(It also contains "mixed" bound and unbound type parameters.) The reduced type parameter of this equivalence class must be `τ_0_0.[Collection]SubSequence`.

Each equivalence class of $G_{\texttt{Collection}}$ has a unique representative of minimum length, which isn't true of all generic signatures in general. This means that if two equivalent type parameters have the same length, they only differ in being bound or unbound, so our two conditions completely determine the reduced type parameter of each equivalence class:

```
τ_0_0
τ_0_0.[Sequence]Element
τ_0_0.[Sequence]Iterator
τ_0_0.[Collection]SubSequence
τ_0_0.[Collection]SubSequence.[Sequence]Iterator
```

(The real `Collection` protocol defines these reduced types together with a few more equivalence classes related to the `Index` and `Indices` associated types.)

In general, we must also order type parameters of equal length, so we show how we do this now. All of the below algorithms take a pair of values x and y and then compute whether $x < y$, $x > y$ or $x = y$ simultaneously. We start with the generic parameters, which are the type parameters of length 1.

Algorithm 5A (Generic parameter order)**.** Takes two generic parameter types `τ_d_i` and `τ_D_I` as input. Returns one of "<", ">" or "=" as output.

1. If `d` < `D`, return "<".
2. If `d` > `D`, return ">".
3. If `d` = `D` and `i` < `I`, return "<".
4. If `d` = `D` and `i` > `I`, return ">".
5. If `d` = `D` and `i` = `I`, return "=".

To order dependent member types, we must order associated type declarations, and to do that we must order protocol declarations.

Algorithm 5B (Protocol order)**.** Takes protocols `P` and `Q` as input, and returns one of "<", ">" or "=" as output.

1. Compare the parent module names of `P` and `Q` with the usual lexicographic order on identifiers. Return the result if it is "<" or ">". Otherwise, `P` and `Q` are declared in the same module, so keep going.
2. Compare the names of `P` and `Q` and return the result if it is "<" or ">". If `P` and `Q` are actually the same protocol, return "=". Otherwise, the program is invalid because it declares two protocols with the same name. Any tie-breaker can be used, such as source location.

Suppose the `Barn` module declares a `Horse` protocol, and the `Swift` module declares `Collection`. Then `Horse` precedes `Collection`, because they are declared in different modules, and `Barn` < `Swift`. If the `Barn` module also declares a `Saddle` protocol, then `Horse` < `Saddle`, because both are from the same module, so we compare their protocol names.

In the previous section, we showed that re-stating an inherited associated type has no effect in our formal system. We also want this to have no effect on reduced types, so our type parameter order must prefer certain associated type declarations over others:

Definition 5.21. A *root associated type* is an associated type whose parent protocol does not inherit from any protocol having an associated type with the same name.

In the following, `Derived.Foo` is not a root, because `Derived` inherits `Base` and `Base` also declares an associated type named `Foo`. However, `Derived.Bar` is a root:

```
protocol Base {
  associatedtype Foo  // root
}

protocol Derived: Base {
  associatedtype Foo  // not a root
  associatedtype Bar  // root
}
```

We define the associated type order in terms of the protocol order.

Algorithm 5C (Associated type order)**.** Takes associated type declarations $\texttt{A}_1$ and $\texttt{A}_2$ as input, and returns one of "<", ">" or "=" as output.

1. First, compare their names lexicographically. Return the result if it is "<" or ">". Otherwise, both associated types have the same name, so keep going.
2. If $\texttt{A}_1$ is a root associated type and $\texttt{A}_2$ is not, return "<".
3. If $\texttt{A}_2$ is a root associated type and $\texttt{A}_1$ is not, return ">".
4. Otherwise, $\texttt{A}_1$ and $\texttt{A}_2$ are both root associated types. Compare the parent protocols of $\texttt{A}_1$ and $\texttt{A}_2$ using Algorithm 5B and return the result if it is "<" or ">".
5. Otherwise, we have two associated types with the same name in the same protocol. If $\texttt{A}_1$ and $\texttt{A}_2$ are the same associated type declaration, return "=".
6. Otherwise, the program is invalid. As with the protocol order, any tie-breaker can be used, such as source location.

We now have enough to order all type parameters. The type parameter order does not distinguish type sugar, so it outputs "=" if and only if `T` and `U` are canonically equal.

Algorithm 5D (Type parameter order)**.** Takes type parameters `T` and `U` as input, and returns one of "<", ">" or "=" as output.

1. If `T` is a generic parameter type and `U` is a dependent member type, then $|\texttt{T}| < |\texttt{U}|$, so return "<".
2. If `T` is a dependent member type and `U` is a generic parameter type, then $|\texttt{T}| > |\texttt{U}|$, so return ">".
3. If `T` and `U` are both generic parameter types, compare them using Algorithm 5A and return the result.
4. Otherwise, both are dependent member types. Recursively compare the base type of `T` with the base type of `U`, and return the result it is "<" or ">".
5. Otherwise, `T` and `U` have canonically equal base types.
6. If `T` is bound and `U` is unbound, return "<".
7. If `T` is unbound and `U` is bound, return ">".
8. If `T` and `U` are both unbound dependent member types, compare their names lexicographically and return the result.
9. If `T` and `U` are both bound dependent member types, compare their associated type declarations using Algorithm 5C and return the result.

Definition 5.22. Let G be a generic signature. A valid type parameter `T` of G is a *reduced type parameter* if $G \vdash [\texttt{T == U}]$ implies that $\texttt{T} \leq \texttt{U}$ for all `U`.

The type parameter order is a linear order, so if we can identify a reduced type parameter in some equivalence class, we know that it must be unique. However, we haven't seen how to calculate the reduced type parameter yet. Indeed, we haven't even established that an equivalence class of type parameters *has* a minimum element. For example, the set of integers $\mathbb{Z}$ certainly does not, under the usual linear order:

$$\cdots < -2 < -1 < 0 < 1 < 2 < \cdots$$

We wish to rule out this possibility:

Definition 5.23. A linear order with domain S is *well-founded* if every non-empty subset of S contains a minimum element.

In the more general case of a partial order, this is not the right definition, because a set consisting of two incomparable elements does not have a minimum element. For a partial order, the correct condition is that the domain S must not contain an *infinite descending chain*, which is an infinite sequence of elements $x_i \in S$ such that:

$$x_1 > x_2 > x_3 > \ldots$$

For a linear order the two conditions are equivalent.

Any partial order on a finite set is always well-founded. The usual linear order on the natural numbers $\mathbb{N}$ is also well-founded (essentially by *definition* of $\mathbb{N}$).

Proposition 5.24. The type parameter order of Algorithm 5D is well-founded.

Proof. Let S be any non-empty set of type parameters, possibly infinite. We must prove that S contains a minimum element.

First, define the set $L(S) \subseteq \mathbb{N}$ by taking the length of each type parameter in S. While $L(S)$ might be an infinite set, it must have a minimum element under the linear order on $\mathbb{N}$, say $n \in L(S)$. Let $S_n \subseteq S$ be the subset of S containing only type parameters of length n. If S_n has a minimum element, it will also be the minimum element of S.

On the other hand, a set of type parameters of fixed length must be finite. If our entire program and all imported modules declare a total of g distinct generic parameters and a associated types, then the number of type parameters of length n, generated by all generic signatures, cannot exceed $g(2a)^n$. (We count each associated type twice, to account for bound and unbound type parameters, but this exact number is irrelevant; the important fact is that it's finite.)

Thus, the finite set S_n contains a minimum element, and so does S, and since S was chosen arbitrarily, we conclude that the type parameter order is well-founded. □

The type parameter order plays an important role in the Swift ABI. For example, the mangled symbol name of a generic function incorporates the function's parameter types and return type, and the mangler takes reduced types to ensure that cosmetic changes to the declaration have no effect on binary compatibility.

To define reduced type parameters, we compared elements in a *single* equivalence class. We also use the type parameter order to establish an order relation *between* equivalence classes, by comparing their reduced types. We will extend this to a partial order on *requirements* in Algorithm 11D. This requirement order surfaces in the Swift ABI:

1. The calling convention for a generic function passes a witness table for each explicit conformance requirement in the function's generic signature, sorted in this order.

2. The in-memory layout of a witness table stores an associated witness table for each associated conformance requirement in the protocol's requirement signature, sorted in this order.

In Chapter 8, we introduce *archetypes*, a self-describing representation of a reduced type parameter packaged up with a generic signature, which behaves more like a concrete type inside the compiler. An archetype thus represents an entire equivalence class of type parameters, and for this reason we will also use the equivalence class notation $[\![\texttt{T}]\!]$ to denote the archetype corresponding to the type parameter `T`.

A more complete treatment of equivalence relations and partial orders can be found in a discrete mathematics textbook, such as [70]. Something to keep in mind is that some authors say that a partial order is reflexive instead of anti-reflexive, so $\leq$ is the basic operation. This necessitates the additional condition that if both $x \leq y$ and $y \geq x$ are true, then $x = y$; without this assumption, we have a *preorder*. Sometimes, a linear order is also called a *total order*, and a set with a well-founded order is said to be *well-ordered*. The type parameter order is actually a special case of a *shortlex order*. Under reasonable assumptions, a shortlex order is always well-founded. We will see another instance of a shortlex order in Section 12.2, and then generalize the concept when we introduce the theory of string rewriting in Section 17.5.

5.5. Generic Signature Queries

In the implementation, we answer questions about the derived requirements and valid type parameters of a generic signature by invoking *generic signature queries*. These are methods on the `GenericSignature` class, used throughout the compiler. We can formalize the behavior of these queries using the notation we developed earlier in this chapter. Each query defines a mathematical function that takes a generic signature and at least one other piece of data, and evaluates to some property of this signature.

Basic queries. We saw that the equivalence class structure of a generic signature is generated by conformance requirements, and same-type requirements between type parameters. The first four queries allow us to answer questions about this structure, and their behavior is completely defined by the derived requirements formalism.

- **Query:** `requiresProtocol`$(G, \texttt{T}, \texttt{P})$

  Input: type parameter `T`, protocol `P`.

  Output: true or false: $G \vdash [\texttt{T: P}]$?

  Decide if a type parameter conforms to a given protocol.

- **Query:** `areReducedTypeParametersEqual`$(G, \texttt{T}, \texttt{U})$

  Input: type parameter `T`, type parameter `U`.

  Output: true or false: $G \vdash [\texttt{T == U}]$?

  Decide if two type parameters are in the same equivalence class.

- **Query:** `isValidTypeParameter`(G, `T`)

  Input: type parameter `T`.

  Output: true or false: $G \vdash$ `T`?

  Decide if a type parameter is valid.

- **Query:** `getRequiredProtocols`(G, `T`)

  Input: type parameter `T`.

  Output: All protocols `P` such that $G \vdash$ [`T: P`].

  Produce a list of all protocols this type parameter is known to conform to.

From a theoretical point of view, the two fundamental queries are `requiresProtocol`() and `areReducedTypeParametersEqual`(). The other two, `isValidTypeParameter`() and `getRequiredProtocols`(), while primitive in the implementation, can be formalized in terms of the others. Here is `isValidTypeParameter`(G, `T`):

- A generic parameter type is valid if it appears in G.
- An unbound dependent member type `U.A` is valid if there exists a protocol `P` in `getRequiredProtocols`(G, `U`) such that `P` declares an associated type named `A`.
- A bound dependent member type `U.[P]A` is valid if `requiresProtocol`(G, `U`, `P`).

As for `getRequiredProtocols`(), because the "for all" quantifier is selecting from a finite universe of protocols, a correct but inefficient implementation would repeatedly check if `requiresProtocol`(G, `T`, `P`) for each `P` in turn. We will describe the data structure that allows this to be implemented efficiently in Chapter 20.

The `getRequiredProtocols`() query is used when type checking a member reference expression like "`foo.bar`" where the type of "`foo`" is a type parameter, because we resolve "`bar`" by a qualified name lookup into this list of protocols. Qualified lookup recursively visits inherited protocols, so the list is minimal in the sense that no protocol inherits from any other. The protocols are also sorted using Algorithm 5B.

Example 5.25. Let G be the generic signature from Example 5.1:

```
<τ_0_0, τ_0_1 where τ_0_0: Sequence, τ_0_1: Sequence,
                    τ_0_0.Element: Equatable,
                    τ_0_0.Element == τ_0_1.Element>
```

1. `requiresProtocol`(G, `τ_0_0.Element`, `Equatable`) is true.
2. `requiresProtocol`(G, `τ_0_0.Iterator.Element`, `Equatable`) is true.
3. `requiresProtocol`(G, `τ_0_0.Iterator`, `Equatable`) is false.

4. `areReducedTypeParametersEqual`(G, `τ_0_0.Element`, `τ_0_1.Element`) is true.
5. `areReducedTypeParametersEqual`(G, `τ_0_0.Iterator`, `τ_0_1.Iterator`) is false.
6. `isValidTypeParameter`(G, `τ_0_0.Element`) is true.
7. `isValidTypeParameter`(G, `τ_0_0.Iterator.Element`) is true.
8. `isValidTypeParameter`(G, `τ_0_0.Element.Iterator`) is false.
9. `getRequiredProtocols`(G, `τ_0_0.Iterator`) is {`IteratorProtocol`}.

Concrete types. The next two queries concern concrete same-type requirements. We can ask if a type parameter is fixed to a concrete type, and then we can ask for this concrete type. (We use "$\vdash$" below, but recall from Section 5.2 that we don't have a complete set of inference rules to describe the implemented behavior of concrete same-type requirements.)

- **Query:** `isConcreteType`(G, `T`)

  Input: type parameter `T`.

  Output: true or false: exists some concrete type `X` such that $G \vdash$ [`T == X`]?

  Decide if a type parameter is fixed to a concrete type.

- **Query:** `getConcreteType`(G, `T`)

  Input: type parameter `T`.

  Output: some concrete type `X` such that $G \vdash$ [`T == X`].

  Output the fixed concrete type of a type parameter.

Example 5.26. Let G be the generic signature,

```
<τ_0_0 where τ_0_0: Foo, τ_0_0.[Foo]B == Int>
```

with protocol `Foo` as below:

```
protocol Foo {
  associatedtype A where A == Array<B>
  associatedtype B
}
```

1. `isConcreteType`(G, `τ_0_0.[Foo]A`) is true.
2. `getConcreteType`(G, `τ_0_0.[Foo]A`) is `Array<τ_0_0.[Foo]B>`.
3. `getConcreteType`(G, `τ_0_0.[Foo]B`) is `Int`.

Reduced types. We now generalize reduced type equality from Section 5.3 to an equivalence relation on all interface types. In the previous example, the first call to `getConcreteType()` returns a concrete type containing `τ_0_0.[Foo]B`, which is itself fixed to the concrete type `Int`. We can write down three interface types that all abstract over the same concrete replacement type in our generic signature:

```
τ_0_0.A
Array<τ_0_0.[Foo]B>
Array<Int>
```

This gives us the first principle: if one interface type can be obtained from another by replacement of type parameters that are fixed to concrete types, then both interface types ought to be equivalent. Now, consider this signature, with the same protocol `Foo`:

```
<τ_0_0, τ_0_1 where τ_0_0: Foo, τ_0_1 == τ_0_0.[Foo]B>
```

Here, instead of completely fixing it to `Int`, we just ask that the concrete replacement type for `τ_0_0.[Foo]B` is canonically equal to the concrete replacement type for `τ_0_1`. In this generic signature, the following three interface types abstract over the same concrete replacement type:

```
τ_0_0.A
Array<τ_0_0.[Foo]B>
Array<τ_0_1>
```

This gives us the second principle: we replace each type parameter within an interface type with its reduced type parameter. We assume we have a subroutine to compute the reduced type parameter of an equivalence class already. We then iterate our two simplifying transformations until fixed point:

Algorithm 5E (Reduce interface type)**.** Takes a generic signature G, and an interface type `T`, as input. Outputs the reduced type of `T`.

1. If `T` is a type parameter:
   a) If `isConcreteType`(G, `T`): call `getConcreteType`(G, `T`), apply the algorithm recursively, and return the result.
   b) Otherwise, find the reduced type parameter `T′` in the equivalence class of `T` and return `T′`.
2. Otherwise, `T` is a concrete type. If `T` does not contain any child types, return `T`.
3. Otherwise, recursively reduce each child type of `T`, and construct a new type from these reduced child types together with any non-type attributes of `T`.

This algorithm outputs a special kind of interface type:

Definition 5.27. An interface type is a *reduced type* in case the following two conditions hold for each type parameter contained in the interface type:

1. Every such type parameter is a reduced type parameter.
2. No such type parameter is fixed to a concrete type.

To guarantee termination, we must rule out self-referential same-type requirements like [`τ_0_0 == Array<τ_0_0>`], otherwise we'll get stuck reducing `τ_0_0`, `Array<τ_0_0>`, `Array<Array<τ_0_0>>`, and so on. We'll describe how in Section 20.1, but for now we just assume they don't appear.

A fully-concrete type (one without type parameters) is trivially a reduced type. In the implementation, we say that a reduced type must also be canonical, so it cannot contain sugared types. This gives us the reduced type equality relation on interface types: two interface types are equivalent under this relation if they have canonically equal reduced types. The following two generic signature queries pertain to reduced types:

- **Query:** `isReducedType`(G, `T`)

  Input: interface type `T`

  Output: true or false: is `T` canonically equal to its reduced type?

  Decide if an interface type is already a reduced type.

- **Query:** `getReducedType`(G, `T`)

  Input: interface type `T`

  Output: the reduced type of `T`.

  Compute the reduced type of an interface type using Algorithm 5E. This will output a canonical type, so it will not contain type sugar.

Example 5.28. Equivalence of interface types is a *coarser* relation than equivalence of type parameters, because if two type parameters are equivalent as type parameters, they must also be equivalent as interface types. The converse does not hold. Let G be the generic signature:

```
<τ_0_0, τ_0_1 where τ_0_0 == Int, τ_0_1 == Int>
```

1. `areReducedTypeParametersEqual`(G, `τ_0_0`, `τ_0_1`) is false.
2. `getReducedType`(G, `τ_0_0`) is `Int`.
3. `getReducedType`(G, `τ_0_1`) is `Int`.

Thus, `τ_0_0` and `τ_0_1` are equivalent as interface types, but not as type parameters.

Other requirements. The final set of queries concern superclass and layout requirements. Once again, we note that "$\vdash$" is aspirational, because not all implemented behaviors of these requirement kinds are described by our formal system.

- **Query:** `getSuperclassBound`$(G, \texttt{T})$

  Input: type parameter `T`.

  Output: some concrete class type `C` such that $G \vdash$ [`T: C`].

  Output the superclass bound, if the type parameter has one.

- **Query:** `requiresClass`$(G, \texttt{T})$

  Input: type parameter `T`.

  Output: true or false: $G \vdash$ [`T: AnyObject`]?

  Decide if `T` has a single retainable pointer representation.

- **Query:** `getLayoutConstraint`$(G, \texttt{T})$

  Input: type parameter `T`.

  Output: some layout constraint `L` such that `T` $\vdash$ [`T: L`].

  Output the layout constraint, if the type parameter has one.

  The `AnyObject` layout constraint is the only one that can be explicitly written in source. A second kind of layout constraint, `_NativeClass`, is implied by a superclass bound being a native Swift class, meaning a class not inheriting from `NSObject`. The `_NativeClass` layout constraint in turn implies the `AnyObject` layout constraint.

  The two differ in how reference counting operations are lowered in IRGen. Classes of unknown ancestry use the Objective-C runtime entry points, whereas native class instances use different entry points in the Swift runtime.

Example 5.29. Let G be the generic signature,

```
<τ_0_0, τ_0_1, τ_0_2 where τ_0_0: Form, τ_0_1: Shape, τ_0_2: Entity>
```

together with the three declarations:

```
class Shape {}
protocol Form: AnyObject {}
protocol Entity: Shape, Form {}
```

1. `getSuperclassBound`(G, `τ_0_0`) is null.
2. `getSuperclassBound`(G, `τ_0_1`) is `Shape`.

3. `getSuperclassBound`(G, `τ_0_2`) is `Shape`.
4. `requiresClass`(G, `τ_0_0`) is true.
5. `requiresClass`(G, `τ_0_1`) is true.
6. `requiresClass`(G, `τ_0_2`) is true.

We can write down a derivation of `requiresClass`(G, `τ_0_0`):

1. [`τ_0_0: Form`] (Conf)
2. [`τ_0_0: AnyObject`] (AssocLayout 1)

However, we cannot write down a derivation of `requiresClass`(G, `τ_0_1`), even though we *should* be able to. The implementation says this derived requirement holds, because any concrete replacement type for `τ_0_1` that satisfies the superclass requirement must also satisfy the layout requirement. We are unable to make this inference within our formal system, so it is *incomplete*. In this case, the missing rule is one that allows us to derive [`T: AnyObject`] from a superclass requirement [`T: C`]. A formal description of the missing rules remains a work in progress.

Sugared types. To avoid printing canonical generic parameter types like `τ_d_i` in diagnostics, a special query can be used to transform a canonical type into a sugared type, using the generic parameter names of a given generic signature.

- **Query:** `getSugaredType`(G, `T`)

  Input: interface type `T`.

  Output: a canonically equal interface type, written using the sugared types of G.

Combined queries. To simplify archetype construction inside a generic environment (Section 8.1), we use a special entry point to perform multiple lookups at once. Its behavior is otherwise completely determined by the other queries.

- **Query:** `getLocalRequirements`(G, `T`)

  Input: type parameter `T`.

  Output: a single structure with the result of several queries.

  Output all known data points about the equivalence class of `T`:

  `getReducedType`(G, `T`)
  `getRequiredProtocols`(G, `T`)
  `getSuperclassBound`(G, `T`)
  `getLayoutConstraint`(G, `T`)

5.6. Source Code Reference

Key source files:

- include/swift/AST/GenericSignature.h
- include/swift/AST/Requirement.h
- include/swift/AST/RequirementSignature.h
- lib/AST/GenericSignature.cpp

Other source files:

- include/swift/AST/Decl.h
- include/swift/AST/DeclContext.h
- lib/AST/Decl.cpp
- lib/AST/DeclContext.cpp

`DeclContext` *class*

See also Section 4.6.

- `getGenericSignatureOfContext()` returns the generic signature of the innermost generic context, or the empty generic signature if there isn't one.

`GenericContext` *class*

See also Section 4.6.

- `getGenericSignature()` returns the declaration's generic signature, computing it first if necessary. If the declaration does not have a generic parameter list or trailing `where` clause, returns the generic signature of the parent context.

`GenericSignature` *class*

Represents an immutable, uniqued generic signature. Meant to be passed as a value, it stores a single instance variable, a `GenericSignatureImpl *` pointer.

The `getPointer()` method returns this pointer. The pointer is not `const`, however `GenericSignatureImpl` does not define any mutating methods.

The pointer may be `nullptr`, representing an empty generic signature; the default constructor `GenericSignature()` constructs this value. There is an implicit `bool` conversion which tests for the empty generic signature.

The `getPointer()` method is only used occasionally, because the `GenericSignature` class overloads `operator->` to forward method calls to the `GenericSignatureImpl *` pointer. Some operations on generic signatures are methods on `GenericSignature` (called with "`.`") and some on `GenericSignatureImpl` (called with "`->`").

Methods of `GenericSignature` are safe to call with an empty generic signature, which is presented as having no generic parameters or requirements. Methods forwarded to `GenericSignatureImpl` can only be invoked if the signature is non-empty.

The `GenericSignature` class explicitly deletes `operator==` and `operator!=` to make the choice between pointer and canonical equality explicit. To check pointer equality of generic signatures, first unwrap both sides with a `getPointer()` call:

```
if (lhsSig.getPointer() == rhsSig.getPointer())
  ...;
```

The more common canonical signature equality check is implemented by the `isEqual()` method on `GenericSignatureImpl`:

```
if (lhsSig->isEqual(rhsSig))
  ...;
```

Various accessor methods:

- `getGenericParams()` returns an array of `GenericTypeParamType`. If the generic signature is empty, this is the empty array, otherwise it contains at least one generic parameter.
- `getInnermostGenericParams()` returns an array of `GenericTypeParamType` with the innermost generic parameters only, that is, those with the highest depth. If the generic signature is empty, this is the empty array, otherwise it contains at least one generic parameter.
- `getRequirements()` returns an array of `Requirement`. If the generic signature is empty, this is the empty array.
- `getCanonicalSignature()` returns the canonical signature. If the generic signature is empty, returns the canonical empty generic signature.
- `getPointer()` returns the underlying `GenericSignatureImpl *`.

Computing reduced types:

- `getReducedType()` returns the reduced type of an interface type for this generic signature. If the generic signature is empty, the type must be fully concrete, and is returned unchanged.

Other:

- `print()` prints the generic signature, with various options to control the output.
- `dump()` prints the generic signature, meant for use from the debugger or ad-hoc print debug statements.

Also see Section 11.5.

`GenericSignatureImpl` *class*

The backing storage of a generic signature. Instances of this class are allocated in the AST context, and are always passed by pointer.

- `isEqual()` checks if two generic signatures are canonically equal.
- `getSugaredType()` given a type containing canonical type parameters that is understood to be written with respect to this generic signature, replaces the generic parameter types with their "sugared" forms, so that the name is preserved when the type is printed out to a string.
- `forEachParam()` invokes a callback on each generic parameter of the signature; the callback also receives a boolean indicating if the generic parameter type is reduced or not—a generic parameter on the left hand side of a same-type requirement is not reduced.
- `areAllParamsConcrete()` answers if all generic parameters are fixed to concrete types via same-type requirements, which makes the generic signature somewhat like an empty generic signature. Fully-concrete generic signatures are lowered away at the SIL level.

The generic signature queries from Section 5.5 are methods on `GenericSignatureImpl`:

- Predicate queries:
  - `isValidTypeParameter()`
  - `requiresProtocol()`
  - `requiresClass()`
  - `isConcreteType()`
- Property queries:
  - `getRequiredProtocols()`
  - `getSuperclassBound()`
  - `getConcreteType()`

  - `getLayoutConstraint()`

- Reduced type queries:
  - `areReducedTypeParametersEqual()`
  - `isReducedType()`
  - `getReducedType()`

`CanGenericSignature` *class*

The `CanGenericSignature` class wraps a `GenericSignatureImpl *` pointer which is known to be canonical. The pointer can be recovered with the `getPointer()` method. There is an implicit conversion from `CanGenericSignature` to `GenericSignature`. The `operator->` forwards method calls to the underlying `GenericSignatureImpl`.

The `operator==` and `operator!=` operators are used to test `CanGenericSignature` for pointer equality. The `isEqual()` method of `GenericSignatureImpl` implements canonical equality on arbitrary generic signatures by first canonicalizing both sides, then checking the resulting canonical signatures for pointer equality. Therefore, the following are equivalent:

```
if (lhsSig->isEqual(rhsSig))
  ...;

if (lhsSig.getCanonicalSignature() == rhsSig.getCanonicalSignature())
  ...;
```

The `CanGenericSignature` class inherits from `GenericSignature`, and so inherits all of the same methods. Additionally, it overrides `getGenericParams()` to return an array of `CanGenericTypeParamType`.

`Requirement` *class*

A generic requirement. See also Section 9.5 and Section 11.5.

- `getKind()` returns the `RequirementKind`.
- `getSubjectType()` returns the subject type.
- `getConstraintType()` returns the constraint type if the requirement kind is not `RequirementKind::Layout`, otherwise asserts.
- `getProtocolDecl()` returns the protocol declaration of the constraint type if this is a conformance requirement with a protocol type as the constraint type.

- `getLayoutConstraint()` returns the layout constraint if the requirement kind is `RequirementKind::Layout`, otherwise asserts.

`RequirementKind` *enum class*

An enum encoding the four kinds of requirements.

- `RequirementKind::Conformance`
- `RequirementKind::Superclass`
- `RequirementKind::Layout`
- `RequirementKind::SameType`

`ProtocolDecl` *class*

See also Section 4.6 and Section 11.5.

- `getRequirementSignature()` returns the protocol's requirement signature, first computing it, if necessary.
- `requiresClass()` answers if the protocol is a class-constrained protocol.

`RequirementSignature` *class*

A protocol requirement signature.

- `getRequirements()` returns an array of `Requirement`.
- `getTypeAliases()` returns an array of `ProtocolTypeAlias`.

Also see Section 11.5.

`ProtocolTypeAlias` *class*

A protocol type alias descriptor.

- `getName()` returns the name of the alias.
- `getUnderlyingType()` returns the underlying type of the type alias. This is a type written in terms of the type parameters of the requirement signature.

`TypeBase` *class*

See also Section 3.4.

- `isTypeParameter()` answers if this type is a type parameter; that is, a generic parameter type, or a `DependentMemberType` whose base is another type parameter.
- `hasTypeParameter()` answers if this type is itself a type parameter, or if it contains a type parameter in structural position. For example, `Array<τ_0_0>` will answer `false` to `isTypeParameter()`, but `true` to `hasTypeParameter()`.

`DependentMemberType` *class*

A type abstracting over a type witness in a conformance.

- `getBase()` returns the base type; for example, given `τ_0_0.Foo.Bar`, will answer `τ_0_0.Foo`.
- `getName()` returns the identifier naming the associated type.
- `getAssocType()` if this is a bound `DependentMemberType`, returns the associated type declaration, otherwise if it is unbound, returns `nullptr`.

`TypeDecl` *class*

See also Section 4.6.

- `compare()` compares two protocols by the protocol order (Definition 5B), returning one of the following:
  - -1 if this protocol precedes the given protocol,
  - 0 if both protocol declarations are equal,
  - 1 if this protocol follows the given protocol.

`compareDependentTypes()` *function*

Implements the type parameter order (Algorithm 5D), returning one of the following:

- -1 if the left hand side precedes the right hand side,
- 0 if the two type parameters are equal as canonical types,
- 1 if the left hand side follows the right hand side.

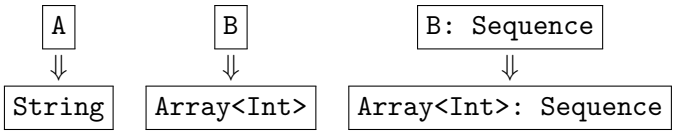
6. Substitution Maps

Substitution maps arise when the type checker needs to reason about a reference to a generic declaration, specialized with list of generic arguments. Abstractly, a substitution map defines a replacement type corresponding to each type parameter of a generic signature; applying a substitution map to the interface type of a generic declaration recursively replaces the type parameters therein, producing the type of the specialized reference.

The generic signature of a substitution map is called the *input generic signature*. A substitution map stores its input generic signature, and the generic signature's list of generic parameters and conformance requirements determine the substitution map's shape:

```
<A, B where B: Sequence, B.[Sequence]Element == Int>
```

Formally, a substitution map consists of a replacement type for each generic parameter, and a conformance for each conformance requirement:



We can collect all of the above information in a table:

Generic parameters		**Replacement types**
`A`	$\mapsto$	`String`
`B`	$\mapsto$	`Array<Int>`
Conformance requirements		**Conformances**
`[B: Sequence]`	$\mapsto$	`[Array<Int>: Sequence]`

Or more concisely,

$$\left\{ \begin{array}{l} \texttt{A} \mapsto \texttt{String} \\ \texttt{B} \mapsto \texttt{Array<Int>}; \\ \\ \texttt{[B: Sequence]} \mapsto \texttt{[Array<Int>: Sequence]} \end{array} \right\}$$

Listing 6.1.: Substitution maps in type checking

```
func genericFunction<A, B: Sequence>(_: A, _: B)
    where B.Element == Int {}

struct GenericType<A, B: Sequence> where B.Element == Int {
  func nonGenericMethod() {}
}

// substitution map for the call is {A := String, B := Array<Int>}.
genericFunction("hello", [1, 2, 3])

// the type of 'value' is GenericType<String, Array<Int>>.
let value = GenericType<String, Array<Int>>()

// the context substitution map for the type of 'value' is
// {A := String, B := Array<Int>}.
value.nonGenericMethod()
```

Example 6.1. We often use the Greek letter Σ, in various forms, to denote substitution maps. For example, Σ, Σ_1, Σ_2, Σ', etc. For now we'll just work with the single substitution map defined above, so let's denote it by Σ:

$$\Sigma := \left\{ \begin{array}{l} \texttt{A} \mapsto \texttt{String} \\ \texttt{B} \mapsto \texttt{Array<Int>}; \\ \\ [\texttt{B: Sequence}] \mapsto [\texttt{Array<Int>: Sequence}] \end{array} \right\}$$

Our substitution map appears while type checking the program shown in Listing 6.1. Here, all three of `genericFunction()`, `GenericType` and `nonGenericMethod()` have the same generic signature, `<A, B where B: Sequence, B.Element == Int>`. When type checking a generic function call, the expression type checker infers the generic arguments from the types of the argument expressions. When referencing a generic type, the generic arguments can be written explicitly. In fact, all three declarations are also referenced with the same substitution map. (In the case of a generic type, this substitution map is called the *context substitution map*, as you will see in Section 6.1.)

Type substitution. Substitution maps operate on interface types. Recall that an interface type is a type *containing* valid type parameters for some generic signature, which may itself not be a type parameter; for example, one possible interface type is

`Array<T>`, if `T` is a generic parameter type. Let's introduce the formal notation $\textsc{Type}(G)$ to mean the set of interface types for a generic signature G. Then, if $\texttt{T} \in \textsc{Type}(G)$ and Σ is a substitution map with input generic signature G, we can *apply* Σ to `T` to get a new type. This operation is called *type substitution*. The interface type here is called the *original type*, and the result of the substitution is the *substituted type*. We will think of applying a substitution map to an interface type as an binary operation:

$$\texttt{T} \otimes \Sigma$$

The $\otimes$ binary operation is a *right action* of substitution maps on types. (We could have instead defined a left action, but later we will see the right action formulation is more natural for expressing certain identities. Indeed, we'll develop this notation further throughout this book.)

Type substitution recursively replaces any type parameters appearing in the original type with new types derived from the substitution map, while preserving the "concrete structure" of the original type. Thus the behavior of type substitution is ultimately defined by how substitution maps act the two kinds of type parameters: generic parameters and dependent member types:

- Applying a substitution map to a generic parameter type returns the corresponding replacement type from the substitution map.

  Type substitution does not care about generic parameter sugar in the original type; replacement types for generic parameters are always looked up by depth and index in the substitution map.

- Applying a substitution map to a dependent member type derives the replacement type from one of the substitution map's conformances.

  Now, we haven't talked about conformances yet. There is a circularity between substitution maps and conformances—substitution maps can store conformances, and conformances can store substitution maps. We will look at conformances in great detail in Chapter 7. The derivation of replacement types for dependent member types is discussed in Section 7.4.

Example 6.2. Applying the substitution map from our running example to sugared and canonical generic parameter types produces the same results:

$$\left\{ \begin{array}{l} \texttt{A} \\ \texttt{τ\_0\_0} \\ \texttt{B} \\ \texttt{τ\_0\_1} \end{array} \right\} \otimes \Sigma = \left\{ \begin{array}{l} \texttt{String} \\ \texttt{String} \\ \texttt{Array<Int>} \\ \texttt{Array<Int>} \end{array} \right\}$$

Listing 6.2.: Applying a substitution map to four interface types

```
struct GenericType<A, B: Sequence> where B.Element == Int {
  typealias T1 = A
  typealias T2 = B
  typealias T3 = (A.Type, Float)
  typealias T4 = (Optional<A>) -> B
}

let t1: GenericType<String, Array<Int>>.T1 = ...
let t2: GenericType<String, Array<Int>>.T2 = ...
let t3: GenericType<String, Array<Int>>.T3 = ...
let t4: GenericType<String, Array<Int>>.T4 = ...
```

Example 6.3. Listing 6.2 shows a generic type with four member type alias declarations. There are four global variables, and the type of each global variable is written as a member type alias reference with the same type base type, `GenericType<String, Array<Int>>`.

Type resolution resolves a member type alias reference by applying a substitution map to the underlying type of the type alias declaration. Here, the underlying type of each type alias declaration is an interface type for the generic signature of `GenericType`, and the substitution map is the substitution map Σ of Example 6.1.

The type of each global variable `t1`, `t2`, `t3` and `t4` is determined by applying Σ to the underlying type of each type alias declaration:

	Original type	**Substituted type**
t1	A	String
t2	B	Array<Int>
t3	(A.Type, Float)	(String.Type, Float)
t4	(Optional<A>) -> B	(Optional<String>) -> Array<Int>

The first two original types are generic parameters, and substitution directly projects the corresponding replacement type from the substitution map; the second two original types are substituted by recursively replacing generic parameters they contain.

References to generic type alias declarations are more complex because in addition to the generic parameters of the base type, the generic type alias will have generic parameters of its own. Section 9.1 describes how the substitution map is computed in this case.

Substitution failure. Substitution of an interface type containing dependent member types can *fail* if any of the conformances in the substitution map are invalid. In this case, an error type is returned instead of signaling an assertion. Invalid conformances can appear in substitution maps when the user's own code is invalid; it is not an invariant violation as long as other errors are diagnosed elsewhere and the compiler does not proceed to SILGen with error types in the abstract syntax tree.

Output generic signature. If the replacement types in the substitution map are fully concrete—that is, they do not contain any type parameters—then all possible substituted types produced by this substitution map will also be fully concrete. If the replacement types are interface types for some *output* generic signature, the substitution map will produce interface types for this generic signature. The output generic signature might be a different from the *input* generic signature of the substitution map.

The output generic signature is not stored in the substitution map; it is implicit from context. Also, fully-concrete types can be seen as valid interface types for *any* generic signature, because they do not contain type parameters at all. Keeping these caveats in mind, we have this essential principle:

> **A substitution map defines a transformation from the interface types of its input generic signature to the interface types of its output generic signature.**

Recall our notation $\textsc{Type}(G)$ for the set of interface types of G. We also use the notation $\textsc{Sub}(G, H)$ for the set of substitution maps with input generic signature G and output generic signature H. We make use of this notation to formalize our principle. If $\mathtt{T} \in \textsc{Type}(G)$ and $\Sigma \in \textsc{Sub}(G, H)$, then $\mathtt{T} \otimes \Sigma \in \textsc{Type}(H)$, and thus the $\otimes$ binary operation is a function between the following sets:

$$\textsc{Type}(G) \otimes \textsc{Sub}(G, H) \longrightarrow \textsc{Type}(H)$$

Canonical substitution maps. Substitution maps are immutable and uniqued, just like types and generic signatures. A substitution map is canonical if all replacement types are canonical types and all conformances are canonical conformances. A substitution map is canonicalized by constructing a new substitution map from the original substitution map's canonicalized replacement types and conformances.

As with types, canonicalization gives substitution maps two levels of equality; two substitution maps are equal pointers if their replacement types and conformances are equal pointers. Two substitution maps are canonically equal if their canonical substitution maps are equal pointers; or equivalently, if their replacement types and conformances are canonically equal.

Applying a canonical substitution map to a canonical original type is not guaranteed to produce a canonical substituted type. However, there are two important invariants that do hold:

1. Given two canonically equal original types, applying the same substitution map to both will produce two canonically equal substituted types.
2. Given an original type and two canonically equal substitution maps, applying the two substitution maps to this type will also produce two canonically equal substituted types.

6.1. Generic Arguments

A nominal type is *specialized* if the type itself or one of its parent types is a generic nominal type. That is, `Array<Int>` and `Array<Int>.Iterator` are both specialized types, but `Int` and `String.UTF8View` are not. Equivalently, a nominal type is specialized if its nominal type declaration is a generic context—that is, if the type declaration itself has a generic parameter list, or an outer declaration context has one.

Every specialized type determines a unique substitution map for the generic signature of its declaration, called the *context substitution map*. The context substitution map replaces the generic parameters of the type declaration with the corresponding generic arguments of the specialized type.

Let's say that d is a nominal type declaration with generic signature G. The declared interface type of d, which we will denote by $\mathtt{T}_d$, is an element of $\textsc{Type}(G)$. Suppose that $\mathtt{T}$ is some specialized type of d whose generic arguments are interface types for a generic signature H, so that $\mathtt{T} \in \textsc{Type}(H)$. The context substitution map of $\mathtt{T}$ is a substitution map $\Sigma \in \textsc{Sub}(G, H)$, such that applying it to the declared interface type of d gives us back $\mathtt{T}$. That is,

$$\mathtt{T} = \mathtt{T}_d \otimes \Sigma.$$

To demonstrate the above identity, consider the generic signature of the `Dictionary` type declaration in the standard library:

```
<Key, Value where Key: Hashable>
```

One possible specialized type for `Dictionary` is the type `Dictionary<Int, String>`; this type is related to the declared interface type of `Dictionary` by this substitution map:

$$\texttt{Dictionary<τ\_0\_0, τ\_0\_1>} \otimes \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{Int} \\ \texttt{τ\_0\_1} \mapsto \texttt{String}; \\ \\ \texttt{[τ\_0\_0: Hashable]} \mapsto \texttt{[Int: Hashable]} \end{array} \right\} = \texttt{Dictionary<Int, String>}$$

The identity substitution map. What is the context substitution map of a type declaration's declared interface type? By definition, if Σ is the context substitution map of $\mathtt{T}_d$, then $\mathtt{T}_d \otimes \Sigma = \mathtt{T}_d$; it leaves the declared interface type unchanged. That is, this substitution map maps every generic parameter of the type declaration's generic signature to itself. If we look at the `Dictionary` type again, we can write down this substitution map:

$$\texttt{Dictionary<τ\_0\_0, τ\_0\_1>} \otimes \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{τ\_0\_0} \\ \texttt{τ\_0\_1} \mapsto \texttt{τ\_0\_1}; \\ \\ \texttt{[τ\_0\_0: Hashable]} \mapsto \texttt{[τ\_0\_0: Hashable]} \end{array} \right\} = \texttt{Dictionary<τ\_0\_0, τ\_0\_1>}$$

This is called the *identity substitution map* for this generic signature; every generic signature has one. We denote the identity substitution map of a generic signature G by 1_G. Then, $1_g \in \textsc{Sub}(G, G)$, and if $\mathtt{T} \in \textsc{Type}(G)$, we have

$$\mathtt{T} \otimes 1_G = \mathtt{T}.$$

Applying the identity substitution map to any interface type leaves it unchanged, with three caveats:

1. The interface type must only contain type parameters which are valid in the input generic signature G of this identity substitution map 1_G.
2. Substitution might change type sugar, because generic parameters appearing in the original interface type might be sugared differently than the input generic signature of this identity substitution map. Therefore, canonical equality of types is preserved, not necessarily pointer equality.
3. We won't talk about archetypes until Chapter 8, but you may have met them already. Applying the identity substitution map to a contextual type containing archetypes replaces the archetypes with equivalent type parameters. There is a corresponding *forwarding substitution map* which maps all generic parameters to archetypes; the forwarding substitution map acts as the identity in the world of contextual types.

The empty substitution map. The empty generic signature only has a single unique substitution map, the *empty substitution map*, so the context substitution map of a non-specialized nominal type is the empty substitution map. In our notation, the empty substitution map is denoted $\{\}$. The only valid interface types of the empty generic signature are the fully-concrete types. The action of the empty substitution map leaves fully-concrete types unchanged, so for example, $\texttt{Int} \otimes \{\} = \texttt{Int}$.

The empty substitution map {} is almost never the same as the identity substitution map 1_G. In fact, they only coincide if G is the empty generic signature. Applying the empty substitution map to an interface type containing type parameters is a substitution failure and returns an error type.

$$\texttt{τ\_0\_0.[Sequence]Element} \otimes \{\} = \texttt{<<error type>>}$$

6.2. Composing Substitution Maps

Suppose that we have three generic signatures, F, G and H, and a pair of substitution maps: $\Sigma_1 \in \textsc{Sub}(F, G)$, and $\Sigma_2 \in \textsc{Sub}(G, H)$. If we start with an interface type $\texttt{T} \in \textsc{Type}(F)$, then $\texttt{T} \otimes \Sigma_1 \in \textsc{Type}(G)$. If we then apply Σ_2 to $\texttt{T} \otimes \Sigma_1$, we get an interface type in $\textsc{Type}(H)$:

$$(\texttt{T} \otimes \Sigma_1) \otimes \Sigma_2$$

The *composition* of the substitution maps Σ_1 and Σ_2, denoted by $\Sigma_1 \otimes \Sigma_2$, is the unique substitution map which satisfies the following equation for all $\texttt{T} \in \textsc{Type}(F)$:

$$\texttt{T} \otimes (\Sigma_1 \otimes \Sigma_2) = (\texttt{T} \otimes \Sigma_1) \otimes \Sigma_2$$

That is, applying the composition of two substitution maps is the same as applying the first substitution map followed by the second. Since $(\texttt{T} \otimes \Sigma_1) \otimes \Sigma_2 \in \textsc{Type}(H)$, we see that $\Sigma_1 \otimes \Sigma_2 \in \textsc{Sub}(F, H)$; the input generic signature of the composition is the input generic signature of the first substitution map, and the output generic signature of the composition is the output generic signature of the second. Substitution map composition can thus be understood as a function between sets:

$$\textsc{Sub}(F, G) \otimes \textsc{Sub}(G, H) \longrightarrow \textsc{Sub}(F, H)$$

To understand how the composition $\Sigma_1 \otimes \Sigma_2$ is actually constructed from Σ_1 and Σ_2 in the implementation, we decompose Σ_1 by applying it to each generic parameter and conformance requirement of the generic signature F:

$$\Sigma_1 := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{τ\_0\_0} \otimes \Sigma_1 \\ \ldots; \\ [\texttt{τ\_0\_0: P}] \mapsto [\texttt{τ\_0\_0: P}] \otimes \Sigma_1 \\ \ldots \end{array} \right\}$$

This looks like a circular definition, but what it really says is that the behavior of Σ_1 is completely determined by these primitive elements of its input generic signature. Now,

Listing 6.3.: Motivating substitution map composition

```
struct Outer<A> {
  var inner: Inner<Array<A>, A>
}

struct Inner<T, U> {
  var value: (T) -> U
}

let outer: Outer<Int> = ...
let x = outer.inner.value
```

we define $\Sigma_1 \otimes \Sigma_2$ by applying Σ_2 to each element of Σ_1:

$$\Sigma_1 \otimes \Sigma_2 := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \big((\texttt{τ\_0\_0} \otimes \Sigma_1) \otimes \Sigma_2\big) \\ \ldots; \\ [\texttt{τ\_0\_0: P}] \mapsto \big(([\texttt{τ\_0\_0: P}] \otimes \Sigma_1) \otimes \Sigma_2\big) \\ \ldots \end{array} \right\}$$

Under this definition, if we take a generic parameter $\texttt{τ\_d\_i}$ or conformance requirement $[\texttt{τ\_d\_i: P}]$ of F, we see that $\Sigma_1 \otimes \Sigma_2$ satisfies the necessary identity on these primitive elements of F:

$$\texttt{τ\_d\_i} \otimes (\Sigma_1 \otimes \Sigma_2) = (\texttt{τ\_d\_i} \otimes \Sigma_1) \otimes \Sigma_2$$
$$[\texttt{τ\_d\_i: P}] \otimes (\Sigma_1 \otimes \Sigma_2) = ([\texttt{τ\_d\_i: P}] \otimes \Sigma_1) \otimes \Sigma_2$$

Since the behavior of $\Sigma_1 \otimes \Sigma_2$ is completely determined by these primitive elements of its input generic signature, this is actually true for any interface type $\texttt{T} \in \textsc{Type}(F)$:

$$\texttt{T} \otimes (\Sigma_1 \otimes \Sigma_2) = (\texttt{T} \otimes \Sigma_1) \otimes \Sigma_2$$

Example 6.4. Listing 6.3 shows an example where substitution map composition can help reason about the types of chained member reference expressions. The `inner` stored property of `Outer` has type `Inner<Array<A>, A>`. Here is the context substitution map of this type, which we will refer to as Σ_1:

$$\Sigma_1 := \{\texttt{T} \mapsto \texttt{Array<A>}, \texttt{U} \mapsto \texttt{A}\}$$

The interface type of the `inner` stored property is a specialization of the nominal type `Inner` with generic signature `<T, U>`, so the input generic signature of Σ_1 is `<T, U>`. The

interface type of `inner` is declared inside the nominal type `Outer` with generic signature `<A>`, so the output generic signature of Σ_1 is `<A>`.

Now, let's look at the `outer` global variable. It has the type `Outer<Int>`, with the following context substitution map, which we denote as Σ_2:

$$\Sigma_2 := \{\texttt{A} \mapsto \texttt{Int}\}$$

The input generic signature of Σ_2 is `<A>`, the generic signature of `Outer`. The output generic signature of Σ_2 is the empty generic signature, because its replacement types are fully concrete. We can compose Σ_1 and Σ_2, because the output generic signature of Σ_1 is the same as the input generic signature of Σ_2:

$$\Sigma_1 \otimes \Sigma_2 = \{\texttt{T} \mapsto \texttt{Array<A>}, \texttt{U} \mapsto \texttt{A}\} \otimes \{\texttt{A} \mapsto \texttt{Int}\} = \{\texttt{T} \mapsto \texttt{Array<Int>}, \texttt{U} \mapsto \texttt{Int}\}$$

Now, the substituted type of `outer.inner.value` can be derived from the interface type of `value` in two equivalent ways:

1. By applying Σ_1 to `(T) -> U` and then applying Σ_2 to the result:

$$\begin{aligned}&(\texttt{(T) -> U} \otimes \Sigma_1) \otimes \Sigma_2 \\ &\quad = \texttt{(Array<A>) -> A} \otimes \Sigma_2 \\ &\quad = \texttt{(Array<Int>) -> Int}\end{aligned}$$

2. By applying the composition $\Sigma_1 \otimes \Sigma_2$ to `(T) -> U`:

$$\begin{aligned}&\texttt{(T) -> U} \otimes (\Sigma_1 \otimes \Sigma_2) \\ &\quad = \texttt{(T) -> U} \otimes \{\texttt{T} \mapsto \texttt{Array<Int>}, \texttt{U} \mapsto \texttt{Int}\} \\ &\quad = \texttt{(Array<Int>) -> Int}\end{aligned}$$

The final substituted type, `(Array<Int>) -> Int`, is the same in both cases.

If $\Sigma \in \textsc{Sub}(F, G)$, then the identity substitution maps 1_F and 1_G have a natural behavior under substitution map composition:

$$1_F \otimes \Sigma = \Sigma \qquad \Sigma \otimes 1_G = \Sigma$$

The second identity carries the same caveat as the identity $\texttt{T} \otimes 1_G = \texttt{T}$ for types; it is only true if the replacement types of Σ are interface types. If they are contextual types, the archetypes will be replaced with equivalent type parameters, as we will explain in Section 8.2.

Example 6.5. Recall the generic signatures F and G, and the substitution map $\Sigma_1 := \{\texttt{T} \mapsto \texttt{Array<A>}, \texttt{U} \mapsto \texttt{A}\} \in \textsc{Sub}(F, G)$ from Example 6.4. We can write down the identity substitution maps 1_F and 1_G:

$$1_F := \{\texttt{T} \mapsto \texttt{T}, \texttt{U} \mapsto \texttt{U}\}$$
$$1_G := \{\texttt{A} \mapsto \texttt{A}\}$$

Now, one can verify that both of these hold:

$$\{\texttt{T} \mapsto \texttt{T}, \texttt{U} \mapsto \texttt{U}\} \otimes \{\texttt{T} \mapsto \texttt{Array<A>}, \texttt{U} \mapsto \texttt{A}\} = \{\texttt{T} \mapsto \texttt{Array<A>}, \texttt{U} \mapsto \texttt{A}\}$$
$$\{\texttt{T} \mapsto \texttt{Array<A>}, \texttt{U} \mapsto \texttt{A}\} \otimes \{\texttt{A} \mapsto \texttt{A}\} = \{\texttt{T} \mapsto \texttt{Array<A>}, \texttt{U} \mapsto \texttt{A}\}$$

Thus $1_F \otimes \Sigma_1 = \Sigma_1 \otimes 1_G = \Sigma_1$. Note that the left and right identity substitution maps are different in this case, because the input and output generic signatures of Σ_1 are different.

One final rule here is that substitution map composition is *associative*. This means that both possible ways of composing three substitution maps will output the same result:

$$(\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3 = \Sigma_1 \otimes (\Sigma_2 \otimes \Sigma_3)$$

Putting everything together, if $\texttt{T}$ is some type, all of the following are equivalent when defined (and by our compatibility conditions, if one is defined, all are):

$$((\texttt{T} \otimes \Sigma_1) \otimes \Sigma_2) \otimes \Sigma_3$$
$$(\texttt{T} \otimes \Sigma_1) \otimes (\Sigma_2 \otimes \Sigma_3)$$
$$(\texttt{T} \otimes (\Sigma_1 \otimes \Sigma_2)) \otimes \Sigma_3$$
$$\texttt{T} \otimes ((\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3)$$
$$\texttt{T} \otimes (\Sigma_1 \otimes (\Sigma_2 \otimes \Sigma_3))$$

Thus, our type substitution algebra allows us to omit grouping parentheses without introducing ambiguity:

$$\texttt{T} \otimes \Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3$$

Categorically speaking. A *category* is a collection of *objects* and *morphisms*. (Very often the morphisms are functions of some sort, but they might also be completely abstract.) Each morphism is associated with a pair of objects, the *source* and *destination*. The collection of morphisms with source A and destination B is denoted $\text{Hom}(A, B)$. The morphisms of a category must obey certain properties:

1. For every object A, there is an *identity morphism* $1_A \in \text{Hom}(A, A)$.

2. If $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(B, C)$ are a pair of morphisms, there is a third morphism $g \circ f \in \text{Hom}(A, C)$, called the *composition* of g with f.

3. Composition respects the identity: if $f \in \text{Hom}(A, B)$, then $f \circ 1_A = 1_B \circ f = f$.

4. Composition is associative: if $f \in \text{Hom}(A, B)$, $g \in \text{Hom}(B, C)$ and $h \in \text{Hom}(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

We define *the category of generic signatures* as follows:

- The objects are generic signatures.
- The morphisms are substitution maps (a technicality here is that their replacement types must not contain archetypes).
- The source of a morphism (substitution map) is the input generic signature of the substitution map.
- The destination of a morphism (substitution map) is the output generic signature of the substitution map.
- The identity morphism is the identity substitution map (you will see later it does not act as the identity on archetypes, which is why we rule them out above).
- The composition of morphisms $g \circ f$ is the composition of substitution maps $f \otimes g$ (note that we must reverse the order here for the definition to work).

Category theory often comes up in programming when working with data structures and higher-order functions; an excellent introduction to the topic is [71]. While we don't need to deal with categories in the abstract here, but we will encounter another idea from category theory, the commutative diagram, in Section 7.3.

6.3. Building Substitution Maps

Now that we've seen how to get substitution maps from types, and how to compose existing substitution maps, it's time to talk about building substitution maps from scratch using the two variants of the **get substitution map** operation.

The first variant constructs a substitution map directly from its three constituent parts: a generic signature, an array of replacement types, and an array of conformances. The arrays must have the correct length—equal to the number of generic parameters and conformance requirements, respectively. Conformances satisfy an additional validity condition where they must match the conformance requirements of the generic signature:

1. The conforming type of a conformance must be canonically equal to the result of applying the substitution map to the subject type of the corresponding conformance requirement.

2. The protocol of a conformance must be the same as the protocol on the right hand side of the corresponding conformance requirement.

This variant of **get substitution map** is used when constructing a substitution map from a deserialized representation, because a serialized substitution map is guaranteed to satisfy the above invariants. It is also used when building a protocol substitution map, because the shape is sufficiently simple—just a single replacement type and a single conformance.

The second variant takes the input generic signature and a pair of callbacks:

1. The **replacement type callback** maps a generic parameter type to a replacement type. It is invoked with each generic parameter type to populate the replacement types array.

2. The **conformance lookup callback** maps a protocol conformance requirement to a conformance. It is invoked with each conformance requirement to populate the conformances array.

The conformance lookup callback takes three parameters:

1. The *original type*; this is the subject type of the conformance requirement.

2. The *substituted type*; this is the result of applying the substitution map to the original type, which should be canonically equal to the conforming type of the conformance that will be returned.

3. The protocol declaration named by the conformance requirement.

The callbacks can be arbitrarily defined by the caller. Several pre-existing callbacks also implement common behaviors. For the replacement type callback,

1. The **query substitution map** callback looks up a generic parameter in an existing substitution map.

2. The **query type map** callback looks up a generic parameter in a hashtable.

For the conformance lookup callback,

1. The **global conformance lookup** callback performs a global conformance lookup (Section 7.1).

2. The **local conformance lookup** callback performs a local conformance lookup into another substitution map (Section 7.4).

3. The **make abstract conformance** callback asserts that the substituted type is a type variable, type parameter or archetype, and returns an abstract conformance (also in Section 7.4). It is used when it is known that the substitution map can be constructed without performing any conformance lookups, as is the case with the identity substitution map.

Specialized types only store their generic arguments, not conformances, so the context substitution map of a specialized type is constructed by first populating a `DenseMap` with the generic arguments of the specialized type and all of its parent types, and then invoking the **get substitution map** operation with the **query type map** and **global conformance lookup** callbacks.

The identity substitution map of a generic signature is constructed from a replacement type callback which just returns the input generic parameter together with the **make abstract conformance** callback.

Example 6.6. A substitution map which does *not* satisfy the invariants specified above, and thus cannot be constructed. First, the generic signature:

```
<T where T: Sequence, T.[Sequence]Element: Comparable>:
```

And the substitution map:

$$\Sigma := \left\{ \begin{array}{l} \texttt{T} \mapsto \texttt{Array<Int>}; \\ \\ [\texttt{T: Sequence}] \mapsto [\texttt{Array<Int>: Sequence}] \\ [\texttt{T.[Sequence]Element: Comparable}] \mapsto [\texttt{String: Comparable}] \end{array} \right\}$$

The generic signature has two conformance requirements:

$[\texttt{T: Sequence}]$

$[\texttt{T.[Sequence]Element: Comparable}]$

Applying the substitution map to the subject type of each requirement produces the expected conforming types:

$\texttt{T} \otimes \Sigma = \texttt{Array<Int>}$

$\texttt{T.[Sequence]Element} \otimes \Sigma = \texttt{Int}$

The substitution map violates the invariant, because the conforming type of the second conformance is `String`, and not `Int` as was expected.

6.4. Nested Nominal Types

Nominal type declarations can appear inside other declaration contexts, subject to the following restrictions:

Listing 6.4.: A nominal type declaration in a generic local context

```
func f<T>(t: T) {
  struct Nested {  // error
    let t: T

    func printT() {
      print(t)
    }
  }

  Nested(t: t).printT()
}

func g() {
  f(t: 123)
  f(t: "hello")
}
```

1. Structs, enums and classes cannot be nested in generic local contexts.
2. Structs, enums and classes cannot be nested in protocols or protocol extensions.
3. Protocols cannot be nested in generic contexts.

We're going to explore the implementation limitations behind these restrictions, and possible future directions for lifting them. (The rest of the book talks about what the compiler does, but this section is about what the compiler *doesn't* do.)

Types in generic local contexts. This restriction is a consequence of a shortcoming in the representation of a nominal type. Recall from Chapter 3 that nominal types and generic nominal types store a parent type, and generic nominal types additionally store a list of generic arguments, corresponding to the generic parameter list of the nominal type declaration. This essentially means there is no place to store the generic arguments from outer local contexts, such as functions.

Listing 6.4 shows a nominal type nested inside of a generic function. The generic signature of `Nested` contains the generic parameter `T` from the outer generic function `algorithm()`. However, under our rules, the declared interface type of `Nested` is a singleton nominal type, because `Nested` does not have its own generic parameter list, and its parent context is not a nominal type declaration. This means there is no way to

recover a context substitution map for this type because the generic argument for `T` is not actually stored anywhere.

In the source language, there is no way to specialize `Nested`; the reference to `T` inside `f()` is always understood to be the generic parameter `T` of the outer function. However, inside the compiler, different generic specializations can still arise. If the two calls to `f()` from inside `g()` are specialized and inlined by the SIL optimizer for example, the two temporary instances of `Nested` must have different in-memory layouts, because in one call `T` is `Int`, and in the other `T` is `String`.

A better representation for the specializations of nominal types would replace the parent type and list of generic arguments with a single "flat" list that includes all outer generic arguments as well. This approach could represent generic arguments coming from outer local contexts without loss of information.

Luckily, this "flat" representation is already implemented in the Swift runtime. The runtime type metadata for a nominal type includes all the generic parameters from the nominal type declaration's generic signature, not just the generic parameters of the nominal type declaration itself. So while lifting this restriction would require some engineering effort on the compiler side, it would be a backward-deployable and ABI-compatible change.

Types in protocol contexts. Allowing struct, enum and class declarations to appear inside protocols and protocol extensions would come down to deciding if the protocol `Self` type should be "captured" by the nested type.

If the nested type captures `Self`, the code shown in Listing 6.4 would become valid. With this model, the `Nested` struct depends on `Self`, so it would not make sense to reference it as a member of the protocol itself, like `P.Nested`. Instead, `Nested` would behave as if it was a member of every conforming type, like `S.Nested` above (or even `T.Nested`, if `T` is a generic parameter conforming to `P`). At the implementation level, the generic signature of a nominal type nested inside of a protocol context would include the protocol `Self` type, and the *entire* parent type, for example `S` in `S.Nested`, would become the replacement type for `Self` in the context substitution map.

The alternative is to prohibit the nested type from referencing the protocol `Self` type. The nested type's generic signature would *not* include the protocol `Self` type, and `P.Nested` would be a valid member type reference. The protocol would effectively act as a namespace for the nominal types it contains, with the nested type not depending on the conformance to the protocol in any way.

Protocols in other declaration contexts. The final possibility is the nesting of protocols inside other declaration contexts, such as functions or nominal types. This breaks down into two cases, illustrated in Listing 6.6:

1. Protocols inside non-generic declaration contexts.

Listing 6.5.: A nominal type declaration nested in a protocol context

```
protocol P {}

extension P {
  typealias Me = Self

  struct Nested {  // error
    let value: Me  // because what would this mean?

    func method() {
      print(value)
    }
  }

  func f() {
    Nested(value: self).method()
  }
}

struct S1: P {}
struct S2: P {}  // are S1.Nested and S2.Nested distinct?
```

Listing 6.6.: Protocol declaration nested inside other declaration contexts

```
struct Outer {
  protocol P {}  // allowed as of SE-0404
}

struct S: Outer.P {}

func generic<T>(_: T) {
  protocol P {  // error
    func f(_: T)  // because what would this mean?
  }
}
```

2. Protocols inside generic declaration contexts.

The first case was originally prohibited, but is now permitted as of Swift 5.10 [72]; the non-generic declaration context acts as a namespace to which the protocol declaration is scoped, but apart from the interaction with name lookup this has no other semantic consequences. The second case is more subtle. If we were to allow a "generic protocol" to be parameterized by its outer generic parameters in addition to just the protocol `Self` type, we would get what Haskell calls a "multi-parameter type class." Multi-parameter type classes introduce some complications, for example undecidable type inference [73].

6.5. Source Code Reference

Key source files:

- `include/swift/AST/SubstitutionMap.h`
- `lib/AST/SubstitutionMap.cpp`
- `lib/AST/TypeSubstitution.cpp`

Other source files:

- `include/swift/AST/GenericSignature.h`
- `include/swift/AST/Type.h`
- `include/swift/AST/Types.h`

`Type`	*class*

See also Section 3.4.

- `subst()` applies a substitution map to this type and returns the substituted type.

`TypeBase`	*class*

See also Section 3.4 and Section 5.6.

- `getContextSubstitutionMap()` returns this type's context substitution map with respect to the given `DeclContext`.

`SubstitutionMap`	*class*

Represents an immutable, uniqued substitution map.

As with `Type` and `GenericSignature`, this class stores a single pointer, so substitution maps are cheap to pass around as values. The default constructor `SubstitutionMap()` constructs an empty substitution map. The implicit `bool` conversion tests for a non-empty substitution map.

The overload of `operator==` implements substitution map pointer equality. Canonical equality can be tested by first canonicalizing both sides:

```
if (subMap1.getCanonical() == subMap2.getCanonical())
  ...;
```

Accessor methods:

- `empty()` answers if this is the empty substitution map; this is the logical negation of the `bool` implicit conversion.
- `getGenericSignature()` returns the substitution map's input generic signature.
- `getReplacementTypes()` returns an array of `Type`.
- `hasAnySubstitutableParams()` answers if the input generic signature contains at least one generic parameter not fixed to a concrete type; that is, it must be non-empty and not fully concrete (see the `areAllParamsConcrete()` method of `GenericSignatureImpl` from Section 5.6).

Recursive properties computed from replacement types:

- `hasArchetypes()` answers if any of the replacement types contain a primary archetype or opened existential archetype.
- `hasOpenedExistential()` answers if any of the replacement types contain an opened existential archetype.
- `hasDynamicSelf()` answers if any of the replacement types contain the dynamic Self type.

Canonical substitution maps:

- `isCanonical()` answers if the replacement types and conformances stored in this substitution map are canonical.
- `getCanonical()` constructs a new substitution map by canonicalizing the replacement types and conformances of this substitution map.

Composing substitution maps (Section 6.2):

- `subst()` applies another substitution map to this substitution map, producing a new substitution map.

Two overloads of the `get()` static method are defined for constructing substitution maps (Section 6.3).

`get(GenericSignature, ArrayRef<Type>, ArrayRef<ProtocolConformanceRef>)` builds a new substitution map from an input generic signature, an array of replacement types, and array of conformances.

`get(GenericSignature, TypeSubstitutionFn, LookupConformanceFn)` builds a new substitution map by invoking a pair of callbacks to produce each replacement type and conformance.

A static method for constructing protocol substitution maps:

- `getProtocolSubstitutions()` builds a new substitution map from a conforming type and a conformance of this type to a protocol.

`TypeSubstitutionFn` *type alias*

The type signature of a replacement type callback for `SubstitutionMap::get()`.

```
using TypeSubstitutionFn
  = llvm::function_ref<Type(SubstitutableType *dependentType)>;
```

The parameter type is always a `GenericTypeParamType *` when the callback is used with `SubstitutionMap::get()`.

`QuerySubstitutionMap` *struct*

A callback intended to be used with `SubstitutionMap::get()` as a replacement type callback. Overloads `operator()` with the signature of `TypeSubstitutionFn`.

Constructed from a `SubstitutionMap`:

```
QuerySubstitutionMap{subMap}
```

`QueryTypeSubstitutionMap` *struct*

A callback intended to be used with `SubstitutionMap::get()` as a replacement type callback. Overloads `operator()` with the signature of `TypeSubstitutionFn`.

Constructed from an LLVM `DenseMap`:

```
DenseMap<SubstitutableType *, Type> typeMap;

QueryTypeSubstitutionMap{typeMap}
```

`LookupConformanceFn` *type alias*

The type signature of a conformance lookup callback for `SubstitutionMap::get()`.

```
using LookupConformanceFn = llvm::function_ref<
    ProtocolConformanceRef(CanType origType,
                           Type substType,
                           ProtocolDecl *conformedProtocol)>;
```

`LookUpConformanceInModule` *struct*

A callback intended to be used with `SubstitutionMap::get()` as a conformance lookup callback. Overloads `operator()` with the signature of `LookupConformanceFn`.

Performs a global conformance lookup. Constructed without arguments:

```
LookUpConformanceInModule()
```

`LookUpConformanceInSubstitutionMap` *struct*

A callback intended to be used with `SubstitutionMap::get()` as a conformance lookup callback. Overloads `operator()` with the signature of `LookupConformanceFn`.

Constructed with a `SubstitutionMap`:

```
LookUpConformanceInSubstitutionMap{subMap}
```

`MakeAbstractConformanceForGenericType` *struct*

A callback intended to be used with `SubstitutionMap::get()` as a conformance lookup callback. Overloads `operator()` with the signature of `LookupConformanceFn`.

Constructed without arguments:

```
MakeAbstractConformanceForGenericType()
```

`GenericSignature` *class*

See also Section 5.6.

- `getIdentitySubstitutionMap()` returns the substitution map that replaces each generic parameter with itself.

7. Conformances

Conformances relate concrete types with protocols. The last chapter informally introduced conformances in substitution maps, which record the fulfillment of a conformance requirement in a generic signature. We will resume this discussion shortly, and you will see that conformances to protocols with associated types play an important role in type substitution. But first, let's understand the representation and lookup of conformances.

Formally, a conformance describes how a concrete type *witnesses* each requirement of a protocol that it conforms to. There are three kinds of conformance:

1. An **invalid conformance** denotes that a type does not actually conform to the protocol.
2. An **abstract conformance** denotes that a type conforms to the protocol, but it is not known where this conformance was declared (Section 7.4).
3. A **concrete conformance** represents a conformance with a known definition.

Concrete conformances break down further into four sub-kinds, the first two being our primary focus for now:

1. A **normal conformance** declares a conformance on a nominal type or extension.
2. A **specialized conformance** results from type substitution (Section 7.2).
3. A **self conformance** describes a self-conforming protocol, which is only possible in a few very special cases (Section 14.3).
4. An **inherited conformance** describes the conformance of a subclass when the conformance was declared on the superclass (Section 15.1).

Normal conformances. Structs, enums and classes can conform to protocols. A normal conformance represents the *declaration* of a conformance. Normal conformances are declared in the inheritance clause of a nominal type or extension:

```
struct Horse: Animal {...}

struct Cow {...}
extension Cow: Animal {...}
```

The above code declares two normal conformances, [`Horse: Animal`] and [`Cow: Animal`]. Each type or extension declaration has a list of *local conformances*, which are the normal conformances declared on that type or extension. In the above, the struct declaration `Horse` has a single local conformance. The struct declaration `Cow` does not have any local conformances itself, but the *extension* of `Cow` has one.

Nominal type declarations have a *conformance lookup table*, which stores the local conformances of the type and any of its extensions, together with conformances inherited from the superclass, if the type declaration is a class declaration. Extension declarations do not have a conformance lookup table of their own; their local conformances are part of the extended type's conformance lookup table. The conformance lookup table is used to implement global conformance lookup. The rest of the compiler does not interact directly with conformance lookup tables.

Broken down into constituent parts, a normal conformance stores the following:

- **The type:** the declared interface type of the conforming context.
- **The protocol:** this is the protocol being conformed to.
- **The conforming context:** either a nominal type declaration (if the conformance is stated on the type) or an extension thereof (if the conformance is stated on an extension).
- **The generic signature:** the generic signature of the conforming context. If the conformance context is a nominal type declaration or an unconstrained extension, this is the generic signature of the nominal type. If the conformance context is a constrained extension, this generic signature will have additional requirements, and the conformance becomes a conditional conformance. Conditional conformances are described in Section 10.4.
- **Type witnesses:** a mapping from each associated type of the protocol to the concrete type witnessing the associated type requirement. This is an interface type written in terms of the generic signature of the conformance. Section 7.3 will talk about type witnesses.
- **Associated conformances:** a mapping from the conformance requirements of the requirement signature to a conformance of the substituted subject type to the requirement's protocol. Section 7.5 will talk about associated conformances.
- **Value witnesses:** for each value requirement of the protocol, the declaration witnessing the requirement. This declaration is either a member of the conforming nominal type, an extension of the conforming nominal type, or it is a default witness from a protocol extension. The mapping is more elaborate than just storing the witness declaration; Chapter 15.4 goes into the details.

7.1. Conformance Lookup

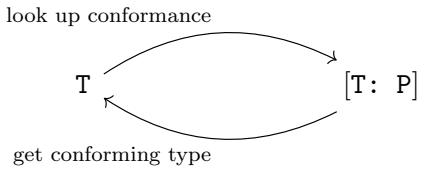
A conformance can be obtained via *global conformance lookup*, an operation defined as taking a protocol declaration and a conforming type. Global conformance lookup can be understood as the action of a protocol declaration `P` on the left of a type `T`. We denote the protocol as [`P`] to avoid confusion with the protocol *type*, and use the notation [`T: P`] to represent the found conformance of `T` to `P`, whatever kind of conformance it may be:

$$[\texttt{P}] \otimes \texttt{T} = [\texttt{T: P}]$$

Global conformance lookup performs a case analysis on the given conforming type, returning one of several kinds of conformance:

- For the declared interface type of a nominal type declaration, it consults the conformance lookup table, and returns a normal conformance to the protocol, if one exists.
- For another specialized type of a nominal type declaration, it returns a specialized conformance, as described in the next section.
- For a type parameter, it returns an abstract conformance (Section 7.4).
- In any case not covered by the above, it returns an invalid conformance.

From a conformance [`T: P`], we can obtain the conformed protocol `P` and conforming type `T`. The conforming type of a valid conformance found by global conformance lookup is canonically equal to the type that was handed to the lookup operation (the two types might differ by type sugar, so are not required to be equal as type pointers). We can exhibit this with a *commutative diagram*:



A commutative diagram is a diagram where every path with the same start and end leads to the same result. The above commutative diagram shows two pairs of paths:

1. Starting from a type, we look up its conformance to a fixed protocol, and get the conforming type of this conformance. This takes us back to the original type.
2. Starting from a conformance, we get its conforming type, and perform a global conformance lookup with this type and our protocol. This gives us the original conformance.

So far, we've only introduced normal conformances. If d is a nominal type declaration and $\mathtt{T}_d$ is the declared interface type of d, then a normal conformance of $\mathtt{T}_d$ to `P` is written as $[\mathtt{T}_d\mathtt{:}\ \mathtt{P}]$ in our notation, as its conforming type is $\mathtt{T}_d$:

$$[\mathtt{P}] \otimes \mathtt{T}_d = [\mathtt{T}_d\mathtt{:}\ \mathtt{P}]$$

We will look at more equations and commutative diagrams in the next section, after a brief interlude to discuss a conceptual difficulty.

Coherence. If a conformance is declared on an extension, the conforming type and the conformed protocol might be in different modules. We consider the three possibilities. First, we can conform our own type to a protocol from another module:

```
struct MyType {...}
extension MyType: Hashable {...}
```

We can also conform a type from another module to our own protocol:

```
protocol MyProtocol {...}
extension Int: MyProtocol {...}
```

Finally, in the most general case, both the conforming type and conformed protocol are declared in modules separate from ours. We call this a *retroactive conformance*:

```
extension String.UTF8View: Hashable {...}
// warning: extension declares a conformance of imported type
// 'UTF8View' to imported protocol 'Hashable'; this will not
// behave correctly if the owners of 'Swift' introduce this
// conformance in the future
```

As of Swift 6, the compiler diagnoses this situation, enforcing what is sometimes known as the *orphan rule* in other languages with similar capabilities. For source compatibility reasons, this is just a warning, but it ought to be an error. The diagnostic can be silenced by explicitly stating the intent with the `@retroactive` attribute to the conformance, offering an escape hatch for this situations that warrant it (and our example certainly does not):

```
extension String.UTF8View: @retroactive Hashable {...}
```

Retroactive conformances are problematic because they allow one to declare *overlapping conformances*.

The compiler rejects a conformance declaration if some existing conformance is statically visible, so this scenario cannot occur with `Int` and `Hashable` for instance, because any attempt to define a overlapping conformance will be caught. With retroactive conformances, this is no longer the case, so we cannot guarantee the *coherence* of global conformance lookup. Suppose a concrete type `K` is defined in some common module, and two unrelated modules each declare a conformance of `K` to `Hashable` unbeknownst to the other. If a module then imports all three, it becomes possible to construct two values of type `Set<K>` with incompatible hash functions; passing such a value across module boundaries will result in undefined behavior.

A similar scenario can occur with library evolution. Suppose a library author publishes the concrete type `K`, and a third party defines a retroactive conformance of `K` to `Hashable`. If the library vendor then adds their own conformance of `K` to `Hashable`, the previously-compiled client might encounter incorrect behavior at runtime.

Unfortunately, the orphan rule does not completely close the hole, because of class inheritance and library evolution. Consider a framework which defines an open class and a protocol:

```
public protocol MyProtocol {}
open class BaseClass {}
```

A client might declare a subclass of `BaseClass` and conform it to `MyProtocol`, concluding this it is safe to do so because the conforming type, `DerivedClass`, is owned by the client, and thus this is not a retroactive conformance:

```
import OtherLibrary

class DerivedClass: BaseClass {}
extension DerivedClass: MyProtocol {}
```

However, in the next version of the framework, the framework author might decide to conform `BaseClass` to `MyProtocol`. At this point, `DerivedClass` has two overlapping conformances to `MyProtocol`; the inherited conformance from `BaseClass`, and the local conformance of `DerivedClass`.

Future directions. The theoretical model could one day be extended to fully support overlapping conformances. The global conformance lookup operation would need to be changed to disambiguate in the case of multiple lookup results, possibly by receiving some kind of lexical context from the caller, in addition to the type and protocol.

The runtime equivalent of a global conformance lookup is a *dynamic cast* from a concrete type to an existential type. Dynamic casts would also need to be extended to deal with overlapping conformances.

7.2. Conformance Substitution

Now, we will define global conformance lookup with a specialized nominal type $\texttt{T}$. We know from Section 6.1 that $\texttt{T}$ splits up into the declared interface type $\texttt{T}_d$ and context substitution map, Σ. We begin by expanding out $\texttt{T}$ in the global conformance lookup $[\texttt{P}] \otimes \texttt{T}$:

$$[\texttt{P}] \otimes \texttt{T} = [\texttt{P}] \otimes (\texttt{T}_d \otimes \Sigma)$$

So far, we don't have any means to simplify the above expression further. However, recall that when we first introduced type substitution and substitution map composition, we defined them to not depend on order of operations. Thus, if $\texttt{T}$ is a type and Σ_1, Σ_2, and Σ_3 are substitution maps, we showed that $(\texttt{T} \otimes \Sigma_1) \otimes \Sigma_2 = \texttt{T} \otimes (\Sigma_1 \otimes \Sigma_2)$, and also $(\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3 = \Sigma_1 \otimes (\Sigma_2 \otimes \Sigma_3)$. Analogously, we would like for the following to hold:

$$[\texttt{P}] \otimes (\texttt{T}_d \otimes \Sigma) = ([\texttt{P}] \otimes \texttt{T}_d) \otimes \Sigma$$

Now, we can simplify the right hand side, at least partially. We have $[\texttt{P}] \otimes \texttt{T}_d = [\texttt{T}_d\texttt{:}\ \texttt{P}]$. This leaves us with the new kind of expression $[\texttt{T}_d\texttt{:}\ \texttt{P}] \otimes \Sigma$ that we have not seen before; we need to apply a substitution map to a conformance. Just as applying Σ to $\texttt{T}_d$ outputs the specialized type $\texttt{T}$, applying Σ to the normal conformance $[\texttt{T}_d\texttt{:}\ \texttt{P}]$ outputs a *specialized conformance*, denoted $[\texttt{T}\texttt{:}\ \texttt{P}]$, which describes how $\texttt{T}$ meets the requirements of $\texttt{P}$:

$$[\texttt{P}] \otimes \texttt{T} = [\texttt{T}\texttt{:}\ \texttt{P}] = [\texttt{T}_d\texttt{:}\ \texttt{P}] \otimes \Sigma$$

The behavior of a specialized conformance $[\texttt{T}\texttt{:}\ \texttt{P}]$ can be specified entirely in terms of the underlying normal conformance $[\texttt{T}_d\texttt{:}\ \texttt{P}]$, together with the *conformance substitution map* Σ. When $\texttt{T}$ is not the declared interface type of some nominal type, we write $[\texttt{T}\texttt{:}\ \texttt{P}]$ as a notational convenience, if it is clear that $\texttt{T} = \texttt{T}_d \otimes \Sigma$ for some $\texttt{T}_d$ and Σ. In the implementation, the "formal" pair $[\texttt{T}_d\texttt{:}\ \texttt{P}] \otimes \Sigma$ is the more primitive notion.

There is also a special case involving the identity substitution map. Suppose that the generic signature of d is G. If the conformance substitution map is 1_G, a specialized conformance is not constructed; substitution returns the original normal conformance:

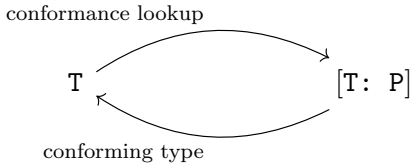
$$[\texttt{T}_d\texttt{:}\ \texttt{P}] \otimes 1_G = [\texttt{T}_d\texttt{:}\ \texttt{P}]$$

To make this concrete, consider the specialized type `Array<Int>`. The normal confor-

mance [Array<τ_0_0>: Sequence] is defined in the standard library. So, we have:

$$\begin{aligned}
&[\texttt{Sequence}] \otimes \texttt{Array<Int>} \\
&\quad = [\texttt{Sequence}] \otimes (\texttt{Array<}\tau\texttt{_0_0>} \otimes \{\tau\texttt{_0_0} \mapsto \texttt{Int}\}) \\
&\quad = ([\texttt{Sequence}] \otimes \texttt{Array<}\tau\texttt{_0_0>}) \otimes \{\tau\texttt{_0_0} \mapsto \texttt{Int}\}) \\
&\quad = [\texttt{Array<}\tau\texttt{_0_0>: Sequence}] \otimes \{\tau\texttt{_0_0} \mapsto \texttt{Int}\} \\
&\quad = [\texttt{Array<Int>: Sequence}]
\end{aligned}$$

Going backwards, the conforming type of a specialized conformance [T: P] is defined as the conforming type of the normal conformance with the conformance substitution map Σ applied. This is just $\texttt{T} = \texttt{T}_d \otimes \Sigma$. This fact is not *completely* trivial; it makes our diagram commute:



For example, the conforming type of [Array<Int>: Sequence] is Array<Int>.

Canonical conformances. Just like types and substitution maps, specialized conformances are immutable and uniquely-allocated. A specialized conformance is *canonical* if the substitution map is canonical. We compute the canonical specialized conformance from an arbitrary specialized conformance by replacing its substitution map with a canonical substitution map. Normal and abstract conformances are always canonical.

More about substitution. As with substitution maps, we define the output generic signature of a conformance. For a normal conformance, this is the generic signature of the conforming context (the nominal itself, or an extension). For a specialized conformance, we take it to be the output generic signature of the conformance substitution map. We can denote the set of all conformances with output generic signature G by $\textsc{Conf}(G)$. Now, if we have a normal conformance $[\texttt{T}_d\texttt{: P}] \in \textsc{Conf}(G)$ and a substitution map $\Sigma \in \textsc{Sub}(G, G')$, then $[\texttt{T}_d\texttt{: P}] \otimes \Sigma \in \textsc{Conf}(G')$, by definition.

With this notation, we can understand how type substitution applies a substitution map to a *specialized* conformance. Suppose $\Sigma' \in \textsc{Sub}(G', G'')$. Then, applying Σ' to the specialized conformance $[\texttt{T}_d\texttt{: P}] \otimes \Sigma$ outputs a new specialized conformance with the composed substitution map:

$$([\texttt{T}_d\texttt{: P}] \otimes \Sigma) \otimes \Sigma' = [\texttt{T}_d\texttt{: P}] \otimes (\Sigma \otimes \Sigma')$$

This means that, for example, the expression $[\texttt{P}] \otimes \texttt{T}_d \otimes \Sigma \otimes \Sigma'$ can be evaluated in one of several ways, corresponding to each possible placement of parentheses, but all evaluations produce the same result—a conformance to `P` with conforming type $\texttt{T}_d \otimes \Sigma \otimes \Sigma'$.

Thus, we can apply substitution maps to conformances, just like we can apply them to types, and compose them with other substitution maps. To recap, we now have three "overloads" of $\otimes$ in our type substitution algebra:

$$\textsc{Type}(G) \otimes \textsc{Sub}(G, G') \longrightarrow \textsc{Type}(G')$$
$$\textsc{Conf}(G) \otimes \textsc{Sub}(G, G') \longrightarrow \textsc{Conf}(G')$$
$$\textsc{Sub}(G, G') \otimes \textsc{Sub}(G', G'') \longrightarrow \textsc{Sub}(G, G'')$$

7.3. Type Witnesses

A concrete type fulfills the associated type requirements of a protocol by declaring a *type witness* for each associated type. Type witnesses are declared in one of four ways:

1. Via a **member type declaration** having the same name as the associated type. Usually this member type is a type alias, but it is legal to use a nested nominal type declaration as well. If the conforming type is a class, the member type may also be defined in a superclass.

2. Via a **generic parameter** having the same name as the associated type. If the conforming type is not generic but is nested inside of a generic context, a generic parameter of the innermost generic context can be used.[1]

3. Via **associated type inference**, where it is implicitly derived from the declaration of a witness to a value requirement. (This is actually an extremely complex topic which warrants an entire chapter in itself.)

4. Via a **default type witness** on the associated type declaration, which is used if all else fails.

The conformance checker is responsible for resolving type witnesses and ensuring they satisfy the requirements of the protocol's requirement signature, as described earlier in Section 5.1. The problem of checking whether concrete types satisfy generic requirements is covered in Section 9.3.

Example 7.1. Listing 7.1 illustrates all four possibilities. In `WithMemberType`, the type witness is the nested struct type named `T`. In the three remaining cases, the conformance checker actually synthesizes a type alias named `T` as a member of the conforming type.

[1] The latter being allowed was probably an oversight, but it's the behavior implemented today.

Listing 7.1.: Different ways of declaring a type witness in a conformance

```
protocol P {
  associatedtype T = Int
  func f(_: T)
}

extension P {
  func f(_: T) {}
}

// Four conforming types:

struct WithMemberType: P {
  struct T {}
}

struct WithGenericParam<T>: P {
  // synthesized: typealias T = T
}

struct WithInferredType: P {
  // synthesized: typealias T = String
  func f(_: String) {}
}

struct WithDefault: P {
  // synthesized: typealias T = Int
}
```

In the case of `WithGenericParam`, the type alias type `T` has as its underlying type the generic parameter, also named `T`. Thus it might seem that the generic parameter `T` is a member type of `WithGenericParam<T>`:

```
func squared(_ x: WithGenericParam<Int>.T) -> Int {
  return x * x
}
```

However, the member type `T` does not refer to the generic parameter declaration itself, but to the synthesized type alias declaration. If `WithGenericParam` did not declare a conformance to `P`, there would be no member type named `T`, and the type representation `WithGenericParam<Int>.T` would diagnose an error. Type resolution of member types is discussed in detail in Section 9.2.

Projection. Given a conformance and an associated type of the conformed protocol, we can ask the conformance for the corresponding type witness. The next section will explain how type substitution of dependent member types uses the type witnesses of a conformance, but first we need to develop the "algebra" of type witnesses.

We denote a type witness projection by $\pi(\texttt{[P]A})$, where `A` is some associated type declared in a protocol `P`. (Of course, π here has nothing to do with the circle constant $3.1415926\ldots$; rather, it means "projection", or perhaps "protocol.")

A type witness projection $\pi(\texttt{[P]A})$ can be applied to a conformance, as long as the conformance is to the same protocol `P`; so if `W` is the type witness (where "`W`" is understood as a placeholder for an arbitrary interface type, and not literally a referenced to some type declaration named "`W`"), we define:

$$\pi(\texttt{[P]A}) \otimes [\texttt{T: P}] := \texttt{W}$$

The type witnesses of a normal conformance are interface types which may contain type parameters from the generic signature of the conforming context. In other words, if $[\texttt{T}_d\texttt{: P}] \in \textsc{Conf}(G)$, then $\pi(\texttt{[P]A}) \otimes [\texttt{T}_d\texttt{: P}] \in \textsc{Type}(G)$.

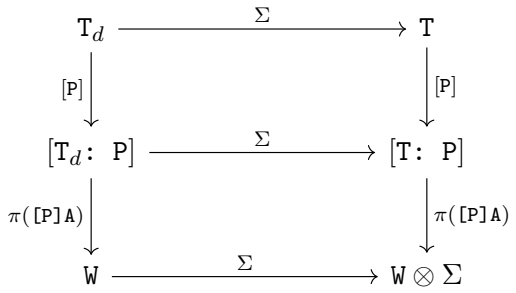
In Listing 7.1, we have the following:

$$\pi(\texttt{[P]T}) \otimes [\texttt{WithMemberType: P}] = \texttt{WithMemberType.T}$$
$$\pi(\texttt{[P]T}) \otimes [\texttt{WithGenericParam: P}] = \texttt{τ\_0\_0}$$
$$\pi(\texttt{[P]T}) \otimes [\texttt{WithInferredType: P}] = \texttt{String}$$
$$\pi(\texttt{[P]T}) \otimes [\texttt{WithDefault: P}] = \texttt{Int}$$

The type witnesses of a specialized conformance are completely determined by the underlying normal conformance and conformance substitution map. Once again, only one possible definition leads to a self-consistent algebra. If $[\texttt{T: P}] = [\texttt{T}_d\texttt{: P}] \otimes \Sigma$:

$$\pi(\texttt{[P]A}) \otimes [\texttt{T: P}] = \pi(\texttt{[P]A}) \otimes \big([\texttt{T}_d\texttt{: P}] \otimes \Sigma\big) = \big(\pi(\texttt{[P]A}) \otimes [\texttt{T}_d\texttt{: P}]\big) \otimes \Sigma$$

Figure 7.1.: Type witnesses of normal and specialized conformances



Thus, if $\mathtt{T} = \mathtt{T}_d \otimes \Sigma$ and $\pi(\mathtt{[P]A}) \otimes [\mathtt{T}_d\mathtt{:\ P}] = \mathtt{W}$, then $\pi(\mathtt{[P]A}) \otimes [\mathtt{T:\ P}] = \mathtt{W} \otimes \Sigma$. Also, note that if $\Sigma \in \textsc{Sub}(G, G')$, then $\pi(\mathtt{[P]A}) \otimes [\mathtt{T:\ P}] \in \textsc{Type}(G')$. So the type witnesses of a normal or specialized conformance are understood in relation to the output generic signature of the conformance, just as the replacement types of a substitution map contain type parameters for the output generic signature of the substitution map.

We now study the relationship between global conformance lookup and type witness projection. Consider the "general type witness expression":

$$\pi(\mathtt{[P]A}) \otimes [\mathtt{P}] \otimes \mathtt{T}_d \otimes \Sigma$$

There are three ways to place parentheses in this expression, and the key property is that the three different evaluation orders produce the same result:

1. We can look up the conformance of $\mathtt{T}_d$ to $\mathtt{P}$ to get the normal conformance $[\mathtt{T}_d\mathtt{:\ P}]$, project the type witness $\pi(\mathtt{[P]A})$ to get $\mathtt{W}$, and apply Σ to get $\mathtt{W} \otimes \Sigma$.

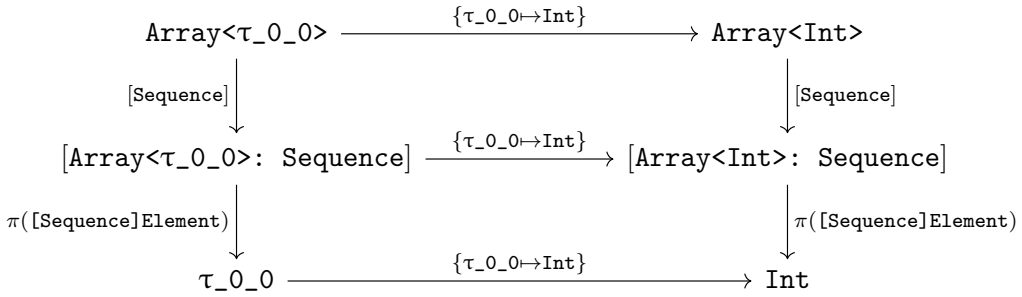
2. We can look up the conformance of $\mathtt{T}_d$ to $\mathtt{P}$ to get the normal conformance $[\mathtt{T}_d\mathtt{:\ P}]$, apply Σ to get the specialized conformance $[\mathtt{T:\ P}]$, and project the type witness $\pi(\mathtt{[P]A})$ to get $\mathtt{W} \otimes \Sigma$.

3. We can apply Σ to $\mathtt{T}_d$ to get $\mathtt{T}$, look up the conformance of $\mathtt{T}$ to $\mathtt{P}$ to get the specialized conformance $[\mathtt{T:\ P}]$, and project the type witness $\pi(\mathtt{[P]A})$ to get $\mathtt{W} \otimes \Sigma$.

Figure 7.1 exhibits the above in the form of a commutative diagram; the three paths from $\mathtt{T}_d$ to $\mathtt{W} \otimes \Sigma$ correspond to the three possible evaluations above.

Example 7.2. To make this concrete, say we look up the conformance of `Array<Int>` to `Sequence`, and then get the type witness for the `Element` associated type. The declared interface type of `Array` is `Array<τ_0_0>`, and the type witness of the `Element` associated type in [`Array<τ_0_0>: Sequence`] is the `τ_0_0` generic parameter type.

Figure 7.2.: Type witnesses of the conformances of `Array<τ_0_0>` and `Array<Int>` to `Sequence`



Our specialized type is `Array<Int>` with context substitution map $\{\texttt{τ\_0\_0} \mapsto \texttt{Int}\}$, so the type witness for `Element` in `[Array<Int>: Sequence]` is $\texttt{τ\_0\_0} \otimes \{\texttt{τ\_0\_0} \mapsto \texttt{Int}\}$, which is `Int`. Figure 7.2 shows the commutative diagram for this case. We can start at the top left and always end up at the bottom right, independent of which of the three paths we take.

If we take $\textsc{AssocType}_{\texttt{P}}$ to mean the set of all associated type declarations of `P`, and $\textsc{Conf}_{\texttt{P}}(G)$ as the subset of $\textsc{Conf}(G)$ containing only conformances to `P`, then type witness projection is the following new "overload" of the $\otimes$ operation:

$$\textsc{AssocType}_{\texttt{P}} \otimes \textsc{Conf}_{\texttt{P}}(G) \longrightarrow \textsc{Type}(G)$$

7.4. Abstract Conformances

An *abstract conformance* represents the conformance of a type parameter of some generic signature to a protocol. Unlike concrete conformances, abstract conformances do not refer to a specific declaration, nor do they store their type witnesses directly. They are to concrete conformances, what type parameters are to concrete types. If it is understood that `T` is a type parameter, we use the same notation for abstract conformances as for concrete conformances:

$$[\texttt{P}] \otimes \texttt{T} = [\texttt{T: P}]$$

If the conformance *requirement* `[T: P]` can be derived in a generic signature G, then there is an abstract conformance $[\texttt{T: P}] \in \textsc{Conf}(G)$. (To put it another way, an abstract conformance `[T: P]` exists if the `requiresProtocol`$(G, \texttt{T}, \texttt{P})$ is true.)

To complete our type substitution algebra, we now define what it means to project a type witness from, or apply a substitution map to, an abstract conformance. Recall that

a bound dependent member type `T.[P]A` is a valid type parameter if it can be derived by applying the AssocDecl inference rule to some derived conformance requirement `[T: P]`:

$$\texttt{T.[P]A} \qquad (\textsc{AssocDecl}\ [\texttt{T: P}])$$

The derived conformance requirement `[T: P]` gives us the abstract conformance `[T: P]`, and its type witness for `A` must be the dependent member type `T.[P]A`:

$$\pi(\texttt{[P]A}) \otimes [\texttt{T: P}] = \texttt{T.[P]A}$$

As every valid dependent member type is the type witness of an abstract conformance, we can define the application of a substitution map to a dependent member type by factoring the dependent member type in this manner:

$$\texttt{T.[P]A} \otimes \Sigma = \big(\pi(\texttt{[P]A}) \otimes [\texttt{T: P}]\big) \otimes \Sigma = \pi(\texttt{[P]A}) \otimes \big([\texttt{T: P}] \otimes \Sigma\big)$$

We saw how normal and specialized conformances are substituted in Section 7.2. Now, it appears we need the ability to apply a substitution map to an abstract conformance. This operation is named *local conformance lookup* by analogy with global conformance lookup. Local conformance lookup is *compatible* with global conformance lookup, in this sense: a local conformance lookup with some substitution map Σ, type parameter `T` and protocol `P` must return the same conformance as performing a global conformance lookup of $\texttt{T} \otimes \Sigma$ to `P`. This can be expressed as an equation:

$$([\texttt{P}] \otimes \texttt{T}) \otimes \Sigma = [\texttt{P}] \otimes (\texttt{T} \otimes \Sigma)$$

Local conformance lookup is not actually implemented in terms of global conformance lookup, though. In the simple case, when the abstract conformance corresponds to an explicit conformance requirement in the substitution map's input generic signature, local conformance lookup returns one of the conformances stored by the substitution map. In the general case, local conformance lookup derives the conformance via a *conformance path*. This will be revealed in Chapter 12.

Example 7.3. Listing 7.2 shows an example of dependent member type substitution. We're going to work through how the compiler derives the type of the `iter` variable. The type annotation references the `InnerIterator` member type alias with a base type of `Concatenation<Array<Array<Int>>`, so we need to apply the context substitution map of this base type to the underlying type of the type alias declaration.

The generic signature of `Concatenation` is the following:

```
<Elements where Elements: Sequence,
                Elements.[Sequence]Element: Sequence>
```

Listing 7.2.: Applying a substitution map to a dependent member type

```
struct Concatenation<Elements: Sequence>
    where Elements.Element: Sequence {
  typealias InnerIterator = Elements.Element.Iterator
}

// What is the type of 'iter'?
let iter: Concatenation<Array<Array<Int>>.InnerIterator = ...
```

The context substitution map of `Concatenation<Array<Array<Int>>` is a substitution map for the above input generic signature:

$$\left\{ \begin{array}{l} \texttt{Elements} \mapsto \texttt{Array<Array<Int>>}; \\ \\ [\texttt{Elements: Sequence}] \mapsto [\texttt{Array<Array<Int>>: Sequence}] \\ [\texttt{Elements.Element: Sequence}] \mapsto [\texttt{Array<Int>: Sequence}] \end{array} \right\}$$

The underlying type of the `InnerIterator` type alias is the bound dependent member type `Elements.[Sequence]Element.[Sequence]Iterator`. The compiler applies our substitution map to this dependent member type in three steps:

1. The base type of the dependent member type is `Elements.[Sequence]Element`, and the associated type `Iterator` is defined in the `Sequence` protocol. Therefore there is an abstract conformance

   $[\texttt{Elements.[Sequence]Element: Sequence}]$

2. Applying the substitution map to this abstract conformance performs a local conformance lookup into the substitution map. The conforming type and protocol of the abstract conformance is exactly equal to the second conformance requirement in the generic signature, so the local conformance lookup returns the conformance $[\texttt{Array<Array<Int>>: Sequence}]$.

3. The final step projects the type witness for `Iterator` from this conformance. This is a specialized conformance, with the conformance substitution map:

   $\{\texttt{τ\_0\_0} \mapsto \texttt{Array<Int>}\}$

   In the normal conformance, `Iterator` is `IndexingIterator<Array<τ_0_0>>`, so:

   $$\begin{aligned} &\pi(\texttt{[Sequence]Iterator}) \otimes [\texttt{Array<Array<Int>>: Sequence}] \\ &\quad = \texttt{IndexingIterator<Array<τ\_0\_0>>} \otimes \{\texttt{τ\_0\_0} \mapsto \texttt{Array<Int>}\} \\ &\quad = \texttt{IndexingIterator<Array<Array<Int>>>} \end{aligned}$$

So our final substituted type is `IndexingIterator<Array<Array<Int>>>`.

Example 7.4. An attentive reader might remember from Section 6.3 that the construction of the context substitution map of a specialized type is a little tricky, because we have to recursively compute the substituted subject type of each conformance requirement in the generic signature and then perform a global conformance lookup. In the previous example, the generic signature of `Concatenation` has two conformance requirements, and their original and substituted subject types are as follows:

$$\texttt{Elements} \Rightarrow \texttt{Array<Array<Int>>}$$

$$\texttt{Elements.[Sequence]Element} \Rightarrow \texttt{Array<Int>}$$

The computation of each substituted subject type can be understood as applying the *partially-constructed* context substitution map that has been built so far to each original subject type. For the first subject type, the substitution projects the replacement type of the `Elements` generic parameter:

$$\texttt{Elements} \otimes \left\{ \begin{array}{l} \texttt{Elements} \mapsto \texttt{Array<Array<Int>>}; \\ \_\_ \\ \_\_ \end{array} \right\} = \texttt{Array<Array<Int>>}$$

The second time around, the original subject type is itself a dependent member type, so type substitution recursively performs the same dance with a local conformance lookup and type witness projection:

$$\texttt{Elements.[Sequence]Element} \otimes \left\{ \begin{array}{l} \texttt{Elements} \mapsto \texttt{Array<Array<Int>>}; \\ \texttt{[Elements: Sequence]} \mapsto \texttt{[Array<Array<Int>>: Sequence]} \\ \_\_ \end{array} \right\} = \texttt{Array<Int>}$$

Protocol substitution maps. If `T` is any interface type (not just a type parameter) conforming to some protocol `P`, the normal, specialized or abstract conformance `[T: P]` defines a substitution map for the protocol generic signature `<Self where Self: P>` in a natural way. This is the *protocol substitution map*, denoted by $\Sigma_{\texttt{[T: P]}}$:

$$\Sigma_{\texttt{[T: P]}} := \{\texttt{Self} \mapsto \texttt{T}; \texttt{[Self: P]} \mapsto \texttt{[T: P]}\}$$

Suppose that our protocol `P` declares an associated type `A`. Note that `Self.[P]A` is the declared interface type of the associated type `A`, and $\texttt{Self.[P]A} = \pi(\texttt{[P]A}) \otimes \texttt{[Self: P]}$. Then applying the protocol substitution map to this dependent member type gives the same result as projecting the type witness from the conformance:

$$\texttt{Self.[P]A} \otimes \Sigma_{\texttt{[T: P]}} = \pi(\texttt{[P]A}) \otimes \big(\texttt{[Self: P]} \otimes \Sigma\big) = \pi(\texttt{[P]A}) \otimes \texttt{[T: P]}$$

7.5. Associated Conformances

The conformance requirements stated in a protocol's requirement signature are known as *associated conformance requirements* and the concrete conformance corresponding to one is an *associated conformance*. There is an interesting duality between substitution maps and conformances:

Generic signatures:		**Protocols:**
Generic parameter	$\Leftrightarrow$	Associated type
Requirement in signature	$\Leftrightarrow$	Associated requirement
Substitution maps:		**Conformances:**
Replacement type	$\Leftrightarrow$	Type witness
Replacement conformance	$\Leftrightarrow$	Associated conformance

Generic signatures and requirement signatures are similar except that the former constrains generic parameters, while the latter constrains associated types. A substitution map records a replacement type for each generic parameter and a replacement conformance for each conformance requirement. A normal conformance records a type witness for each associated type and an associated conformance for each associated conformance requirement.

Recall from Section 5.1 that the printed representation of a requirement signature looks like a generic signature with a single `Self` generic parameter. For example, here is the abridged requirement signature of the standard library's `Collection` protocol:

```
<Self where Self: Sequence, Self.Index: Comparable, ...>
```

The special case of an associated conformance requirement with a subject type of `Self` represents a protocol inheritance relationship, as you already saw in Section 5.1. Other associated conformance requirements constrain the protocol's associated types.

The associated conformances of a normal conformance are populated lazily by the **associated conformance request**. The request computes the substituted subject type of an associated conformance requirement by applying the protocol substitution map for the conformance, and then performs a global conformance lookup with this subject type. This is the same idea as when we do conformance lookups to populate a substitution map (Section 6.3).

The substituted subject type is obtained by applying the protocol substitution map to the subject type of each associated conformance requirement. For example, in the conformance of `Array<τ_0_0>` to `Collection`, the substituted subject type of the requirement [`Self: Sequence`] is just the conforming type:

$$\texttt{Self} \otimes \Sigma_{[\texttt{Array<τ\_0\_0>: Collection}]} = \texttt{Array<τ\_0\_0>}$$

The substituted subject type of `Self.Index` is the type witness for `Index`, which is `Int`:

$$\begin{aligned}&\texttt{Self.Index} \otimes \Sigma_{\texttt{[Array<τ\_0\_0>: Collection]}}\\ &\quad = \pi(\texttt{[Collection]Index}) \otimes \texttt{[Array<τ\_0\_0>: Collection]}\\ &\quad = \texttt{Int}\end{aligned}$$

With the substituted subject types on hand, the conformance checker then performs a global conformance lookup to find each associated conformance:

$$\texttt{[Sequence]} \otimes \texttt{Array<τ\_0\_0>} = \texttt{[Array<τ\_0\_0>: Sequence]}$$
$$\texttt{[Comparable]} \otimes \texttt{Int} = \texttt{[Int: Comparable]}$$

Notation. In the type substitution algebra, we will write an associated conformance requirement as $\pi(\texttt{Self.Index: Comparable})$. Associated conformance requirements are quite different from abstract conformances, which use the notation $\texttt{[T: P]}$. An abstract conformance states that a type parameter is known to conform to a protocol in some *generic* signature (possibly as a non-trivial consequence of other requirements), whereas an associated conformance requirement is a *specific* requirement directly appearing in a protocol's *requirement* signature.

Projection. Projecting an associated conformance from a normal conformance can be understood as the action of an associated conformance requirement (from a protocol's requirement signature) on the left of a normal conformance (to this protocol):

$$\pi(\texttt{Self.U: Q}) \otimes \texttt{[}\mathtt{T}_d\texttt{: P]}$$

With a specialized conformance, we do the same thing as when getting a type witness; first, we get the associated conformance from the underlying normal conformance, and then we apply the conformance substitution map:

$$\begin{aligned}&\pi(\texttt{Self.U: Q}) \otimes \texttt{[T: P]}\\ &\quad = \pi(\texttt{Self.U: Q}) \otimes \big(\texttt{[}\mathtt{T}_d\texttt{: P]} \otimes \Sigma\big)\\ &\quad = \big(\pi(\texttt{Self.U: Q}) \otimes \texttt{[}\mathtt{T}_d\texttt{: P]}\big) \otimes \Sigma\end{aligned}$$

Now we can project associated conformances from normal conformances and specialized conformances. Last but not least, we need to define associated conformance projection from an abstract conformance. Just as the type witnesses of an abstract conformance are dependent member types, associated conformances of abstract conformances are other abstract conformances. In Section 12.1, we will show that *all* abstract conformances can be defined this way:

$$\pi(\texttt{Self.U: Q}) \otimes \texttt{[T: P]} = \texttt{[T.U: Q]}$$

Listing 7.3.: Different kinds of associated conformances

```
protocol P {
  associatedtype A: Equatable
  associatedtype B: Equatable
  associatedtype C: Equatable
}

struct S<T: Equatable>: P {
  typealias A = Int
  typealias B = Array<Int>
  typealias C = T
}
```

If we define $\textsc{AssocConf}_{\texttt{P}}$ as the set of all associated conformance requirements in a protocol `P`, then we get a new "overload" of the $\otimes$ binary operation:

$$\textsc{AssocConf}_{\texttt{P}} \otimes \textsc{Conf}_{\texttt{P}}(G) \longrightarrow \textsc{Conf}(G)$$

One more overload of $\otimes$ is described in Section 9.3. A complete summary of the type substitution algebra appears in Appendix C.

Example 7.5. The associated conformances of a normal conformance can themselves be any kind of conformance, including normal, specialized or abstract. Listing 7.3 shows these possibilities. The protocol `P` states three associated conformance requirements, and each of the associated conformances of the normal conformance `S<T>: P` are a different kind of conformance:

Requirement	**Associated conformance**	**Kind**
π(`A: Equatable`)	[`Int: Equatable`]	Normal
π(`B: Equatable`)	[`Array<Int>: Equatable`]	Specialized
π(`C: Equatable`)	[`T: Equatable`]	Abstract

The case where the associated conformance is abstract is important, because it arises when the type witness is a type parameter of the conforming type's generic signature.

Now, let $\Sigma := \{\texttt{τ\_0\_0} \mapsto \texttt{String}; [\texttt{τ\_0\_0: Equatable}] \mapsto [\texttt{String: Equatable}]\}$. Consider what happens when we apply the projection π(`Self.C: Equatable`) to the specialized conformance $[\texttt{S<String>: P}] = [\texttt{S<τ\_0\_0>: P}] \otimes \Sigma$:

$$\begin{aligned}
&\pi(\texttt{Self.[P]C: Equatable}) \otimes [\texttt{S<String>: P}] \\
&\quad = \big(\pi(\texttt{Self.[P]C: Equatable}) \otimes [\texttt{S<τ_0_0>: P}]\big) \otimes \Sigma \\
&\quad = [\texttt{τ_0_0: Equatable}] \otimes \Sigma
\end{aligned}$$

The associated conformance projection operation actually turns around and reduces to a local conformance lookup into the substitution map, which gives us the final result:

$$[\tau\_0\_0\texttt{: Equatable}] \otimes \Sigma = [\texttt{String: Equatable}]$$

This has some unexpected consequences, which are explored in Section 12.3.

7.6. Source Code Reference

Global Conformance Lookup

Key source files:

- `include/swift/AST/ConformanceLookup.h`
- `lib/AST/ConformanceLookupTable.h`
- `lib/AST/ConformanceLookup.cpp`
- `lib/AST/ConformanceLookupTable.cpp`

Other source files:

- `include/swift/AST/DeclContext.h`

`lookupConformance()` *function*

Performs global conformance lookup of a type to a protocol. Does not check conditional requirements. To check conditional requirements, use `checkConformance()` described in Section 10.5.

`ConformanceLookupTable` *class*

A conformance lookup table for a nominal type. Every `NominalTypeDecl` has a private instance of this class, but it is not exposed outside of the global conformance lookup implementation.

`IterableDeclContext` *class*

Base class inherited by `NominalTypeDecl` and `ExtensionDecl`.

- `getLocalConformances()` returns a list of conformances directly declared on this nominal type or extension.

`NominalTypeDecl` *class*

See also Section 4.6.

- `getAllConformances()` returns a list of all conformances declared on this nominal type, its extensions, and inherited from its superclass, if any.

Operations on Conformances

Key source files:

- include/swift/AST/ProtocolConformanceRef.h
- include/swift/AST/ProtocolConformance.h
- lib/AST/ProtocolConformanceRef.cpp
- lib/AST/ProtocolConformance.cpp

`ProtocolConformanceRef` *class*

A protocol conformance. Stores a single pointer, and is cheap to pass around by value.

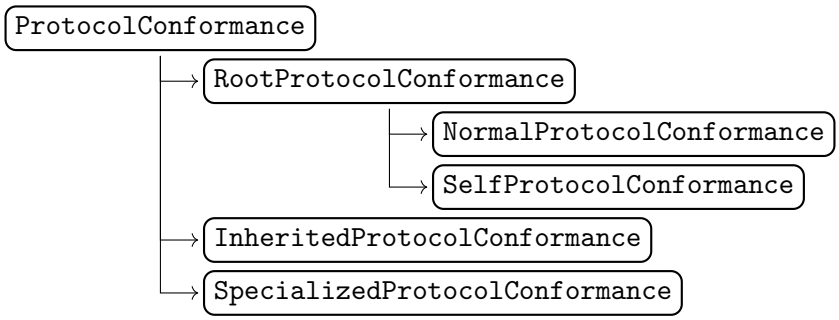
- `isInvalid()` answers if this is an invalid conformance reference, meaning the type did not actually conform to the protocol.
- `isAbstract()` answers if this is an abstract conformance reference.
- `isConcrete()` answers if this is a concrete conformance reference.
- `getConcrete()` returns the `ProtocolConformance` instance if this is a concrete conformance.
- `getRequirement()` returns the `ProtocolDecl` instance if this is an abstract or concrete conformance.
- `subst()` returns the protocol conformance obtained by applying a substitution map to this conformance.

`ProtocolConformance` *class*

A concrete protocol conformance. This class is the root of a class hierarchy shown in Figure 7.3. Concrete protocol conformances are allocated in the AST context, and are always passed by pointer. See Section 10.5 for documentation about conditional conformance.

- `getType()` returns the conforming type.
- `getProtocol()` returns the conformed protocol.
- `getTypeWitness()` returns the type witness for an associated type.
- `getAssociatedConformance()` returns the associated conformance for a conformance requirement in the protocol's requirement signature.

Figure 7.3.: The `ProtocolConformance` class hierarchy



- `subst()` returns the protocol conformance obtained by applying a substitution map to this conformance.

`RootProtocolConformance` *class*

Abstract base class for `NormalProtocolConformance` and `SelfProtocolConformance`. Inherits from `ProtocolConformance`.

`NormalProtocolConformance` *class*

A normal protocol conformance. Inherits from `RootProtocolConformance`.

- `getDeclContext()` returns the conforming declaration context, either a nominal type declaration or extension.
- `getGenericSignature()` returns the generic signature of the conforming context.

`SpecializedProtocolConformance` *class*

A specialized protocol conformance. Inherits from `ProtocolConformance`.

- `getGenericConformance()` returns the underlying normal conformance.
- `getSubstitutionMap()` returns the conformance substitution map.

Lazy Loading

The interface between conformances and the module system is mediated by an abstract base classes defined in the below header file:

- `include/swift/AST/LazyResolver.h`

`LazyConformanceLoader` *class*

Abstract base class implemented by different kinds of modules to fill out conformances. For serialized modules, this populates the mapping from requirements to witnesses by deserializing records. For imported modules, this populates the mapping by inspecting Clang declarations.

`AssociatedConformanceRequest` *class*

Request evaluator request for lazily looking up an associated conformance of a normal conformance that was parsed from source. Computes the substituted subject type of the requirement and calls global conformance lookup.

8. Archetypes

An archetype encapsulates a reduced type parameter and a generic signature, two semantic objects we met in Chapter 5. This representation is self-describing, so it can answer questions about protocol conformance and such without further context. An archetype thus behaves like a concrete type in many ways; it is the "most general" concrete type that satisfies the requirements of its type parameter. (Recall that a type parameter is essentially a *name* that requires a separate generic signature to interpret.) Archetypes are created by mapping type parameters into a *generic environment*, an object derived from a generic signature. Multiple environments can be instantiated from a single generic signature, each one generating a distinct family of archetypes. Exactly one of those is the *primary* generic environment, whose archetypes are called *primary archetypes*. We will discuss these first.

A primary generic environment defines a correspondence between type parameters and primary archetypes:

- **Mapping into an environment** recursively replaces type parameters in the given interface type with their archetypes from the given primary generic environment.
- **Mapping out of an environment** recursively replaces primary archetypes with their reduced type parameters.

Mapping into an environment produces a type which no longer contains type parameters, but instead may contain primary archetypes. We call this a *contextual type*.

Contextual types and primary archetypes are used in place of interface types inside the expressions that appear in generic function bodies, but archetypes don't surface as a distinct concept in the language model; it's a representation change. SILGen lowers expressions to SIL instructions which consume and produce SIL values, so the types of SIL values are also contextual types. Finally, when IRGen generates code for a generic function, archetypes become *values* representing the runtime type metadata provided by the caller of the generic function.

We will see that various operations on interface types, such as generic signature queries and type substitution, have analogs in the world of contextual types. As a practical matter, understanding these is important for compiler developers. The representation of archetypes will also motivate the *type parameter graph* in Section 8.3, which will give us a new way to think about generic signatures. Finally, in Section 8.4 we give a historical sketch of the Swift 3 generics implementation, based on the construction of a finite type parameter graph where archetypes were the source of truth.

Nesting. A nested generic declaration with a distinct generic signature (because it introduced new generic parameters or requirements) also introduces a fresh primary generic environment; we do not "inherit" primary archetypes from the outer scope. All primary archetypes, including those that represent outer generic parameters, are instantiated from the innermost environment. (If the inner declaration does not declare new generic parameters or requirements, it has the same generic signature, and thus the same generic environment, as its parent context.)

This allows correct modeling of nested declarations that impose new requirements on outer generic parameters, which we discussed in Section 4.2. This was not supported prior to Swift 3 because outer primary archetypes were re-used. This simple program demonstrates this behavior:

```
struct Box<T> {
  var contents: T?

  mutating func take() -> T {
    let value: T = self.contents!
    self.contents = nil
    return value
  }

  func compare(_ other: T) -> Bool where T: Equatable {
    guard let value: T = self.contents else { return false }
    return value == other
  }
}
```

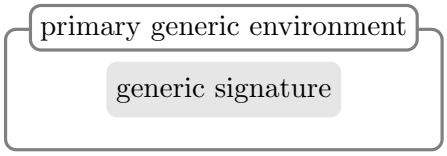
The `take()` and `compare()` methods have different generic signatures, with `take()` inheriting the generic signature of struct `Box`, and `compare()` adding a conformance requirement. The type representation "`T`" resolves to the *same* generic parameter type `T` (or `τ_0_0`) in all source locations where it appears. However, this generic parameter maps to two different archetypes in the generic environment of each function. Thus, the type of the expression "`self.contents`" is actually different in `take()` and `compare()`. Let's use the notation $[\![\texttt{T}]\!]_d$ for the archetype obtained by mapping the type parameter `T` into the generic environment of some declaration d. Here, we have two archetypes:

1. $[\![\texttt{T}]\!]_{\text{take}}$, which does not conform to any protocols.
2. $[\![\texttt{T}]\!]_{\text{compare}}$, which conforms to `Equatable`.

By working with archetypes instead of type parameters, the constraint solver knows this, without having to plumb the generic signature of the current context through everywhere that types of expressions must be reasoned about.

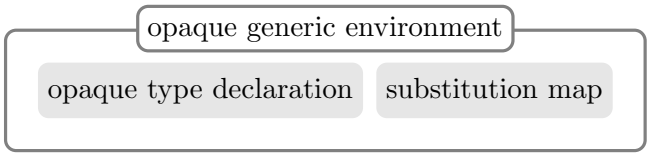
Generic environment kinds. Formally, a generic environment is a uniquing key, which together with a reduced type parameter, identifies an archetype. There are three kinds of generic environment, each with its own uniquing key:

- As described above, every generic signature has exactly one **primary generic environment** of primary archetypes.



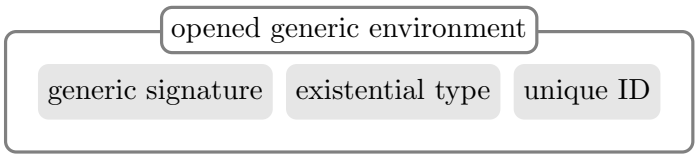
  Primary generic environments preserve the sugared names of generic parameters for the printed representation of an archetype, so two generic signatures which are not equal pointers will instantiate distinct primary generic environments, even if those generic signatures are canonically equal.

- When a declaration has an opaque return type, an **opaque generic environment** is created for each unique substitution map of the owner declaration's generic signature.



  Unlike primary archetypes, opaque archetypes are visible outside of any lexical scope; using the substitution map, they can represent a reference to a (substituted) return type of the owner declaration. Opaque archetypes are also valid inside interface types, in particular they also represent the return type in the interface type of the owner declaration itself. Details are in Chapter 13.

- An **opened generic environment** is created when an existential value is opened at a call site. Formally, opened archetypes represent the concrete payload stored inside an existential type.



  Every opening expression gets a fresh unique ID, and therefore a new opened generic environment. In the AST, opened archetypes cannot "escape" from their opening expression; in SIL, they are introduced by an opening instruction and are similarly scoped by the dominance relation on the control flow graph. Details in Chapter 14.

Archetype equality. Some notes about contextual types:

1. Every primary archetype is a contextual type. Otherwise, a contextual type is a concrete type that contains a primary archetype, like `Array<⟦T⟧>` formed from the primary archetype ⟦T⟧.
2. A fully-concrete type like `Array<Int>` is both an interface type and a contextual type.
3. An interface type like `Array<T>` is *not* a contextual type.
4. A contextual type will only ever contain primary archetypes from a single generic environment.
5. Archetypes from non-primary generic environments may appear in both interface types and contextual types, but we won't worry about those for now.

In the following, let G be a generic signature, and write $\textsc{Type}(G)$ for the set of all interface types of G, as before. Now, let's say that $\textsc{Type}(⟦G⟧)$ is the set of all contextual types that contain primary archetypes of G. We can understand mapping into and out of an environment as giving us a pair of functions:

$$\mathsf{in}_G \colon \textsc{Type}(G) \longrightarrow \textsc{Type}(⟦G⟧)$$
$$\mathsf{out}_G \colon \textsc{Type}(⟦G⟧) \longrightarrow \textsc{Type}(G)$$

In Section 5.3, we defined the reduced type equality relation on the valid type parameters of G, and in Section 5.5, we extended this to all of $\textsc{Type}(G)$ via the recursive definition of a reduced type. The fact that an archetype's type parameter is a reduced type means the following when it comes to equivalence of types:

1. If $G \vdash$ `[T == U]` for type parameters `T` and `U`, then in_G(`T`) and in_G(`U`) give two equal pointers to the same archetype. The three relations of reduced type equality, canonical type equality and type pointer equality coincide on archetypes.
2. If $G \vdash$ `[T == X]` for some type parameter `T` and concrete type `X`, then we define in_G(`T`) := in_G(`X`). A type parameter fixed to a concrete type is not represented by an archetype; instead, we map it into the environment by recursively mapping the concrete type.
3. If `T` is some interface type, then $\mathsf{out}_G(\mathsf{in}_G$(`T`)) is canonically equal to the reduced type of `T` when computed with Algorithm 5E.
4. If two interface types are equivalent under reduced type equality, the corresponding contextual types are equivalent under canonical type equality.

Taken together, the operations of in_G and canonical type equality faithfully represent the equivalence class structure of a generic signature. We will see how archetypes subsume the remaining generic signature queries next.

8.1. Local Requirements

If we have a type parameter `T` in some generic signature G, we can understand the behavior of its equivalence class by looking at the list of all conformance, superclass and layout requirements with a subject type of `T` that can be derived from G. When working with type parameters, we can recover this information by making use of generic signature queries (Section 5.5). When working with an archetype $[\![\texttt{T}]\!]$, we can instead ask the archetype for its *local requirements*, which are stored directly within the data structure for the archetype itself:

- **Required protocols:** A list of all protocols `P` such that $G \vdash$ [`T: P`]. This is the result of the `getRequiredProtocols`() generic signature query.
- **Superclass bound:** A class type `C` such that $G \vdash$ [`T: C`]. This is the result of the `getSuperclassBound`() generic signature query.
- **Requires class flag:** True if $G \vdash$ [`T: AnyObject`]. This is the result of the `requiresClass`() generic signature query.
- **Layout constraint:** A more fine-grained view than the previous, to differentiate between Objective-C and Swift-native reference counting. This is the result of the `getLayoutConstraint`() generic signature query.

When we create an archetype, we issue the `getLocalRequirements`() generic signature query to gather all of the above information in one shot. Local requirements can be inspected directly by looking at an archetype. Even more importantly, they also allow archetypes to take on the behaviors of concrete types.

Qualified name lookup. An archetype can serve as the base type for a qualified name lookup (Section 2.1). The visible members are those of the archetype's required protocols, superclass declaration, and finally, the concrete protocol conformances of the superclass. This explains how we type check a member reference expression where the base type is an archetype.

Global conformance lookup. Consider the two declarations below, together with the generic signature `<Box, Elt where Box: Base<Elt>, Box: Proto>`:

```
protocol Proto { func requirement() }
class Base<Elt>: OtherProto { func otherRequirement() }
```

If we have a value of type $[\![\texttt{Box}]\!]$, a call to `requirement()` must dispatch through the witness table; whereas a call to `otherRequirement()` can directly invoke the concrete witness, even though when we don't have the concrete replacement type for $[\![\texttt{Box}]\!]$.

To model this, global conformance lookup returns an abstract conformance when given the archetype $[\![\texttt{Box}]\!]$ of G and `Proto`, but a concrete conformance when given `OtherProto`:

$$[\texttt{Proto}] \otimes [\![\texttt{Box}]\!] = [[\![\texttt{Box}]\!]\texttt{: Proto}]$$
$$[\texttt{OtherProto}] \otimes [\![\texttt{Box}]\!] = [\texttt{Base<}[\![\texttt{Elt}]\!]\texttt{>: OtherProto}]$$

The concrete conformance is a specialized conformance, formed from the normal conformance `[Base<τ_0_0>: OtherProto]` with the conformance substitution map:

$$[\texttt{Base<τ\_0\_0>: OtherProto}] \otimes \{\texttt{τ\_0\_0} \mapsto [\![\texttt{Elt}]\!]\}$$
$$= [\texttt{Base<}[\![\texttt{Elt}]\!]\texttt{>: OtherProto}]$$

We will resume discussing substitution maps with contextual replacement types shortly, but first let's formalize the above behavior of global conformance lookup.

Recall that with a type parameter, global conformance lookup returns an abstract conformance without being able to check if the conformance actually holds (Section 7.4). With archetypes, we implement a more useful behavior that distinguishes several cases. Let $[\![\texttt{T}]\!]$ be an archetype with generic signature G, and let `P` be a protocol:

1. If $G \vdash [\texttt{T: C}]$ for some class type `C`, and `C` conforms to `P`, then the archetype *conforms concretely* to `P`. The class type `C` may contain type parameters, so global conformance lookup recursively calls itself on $\mathsf{in}_G(\texttt{C})$:

   $$[\texttt{P}] \otimes [\![\texttt{T}]\!] = [\texttt{P}] \otimes \mathsf{in}_G(\texttt{C}) = [\mathsf{in}_G(\texttt{C})\texttt{: P}]$$

   The result is actually wrapped in an *inherited conformance*, which doesn't do much except report the conforming type as $[\![\texttt{T}]\!]$ rather than $\mathsf{in}_G(\texttt{C})$ (Section 15.1).

2. If $G \vdash [\texttt{T: P}]$, the archetype *conforms abstractly*, and thus global conformance lookup returns an abstract conformance:

   $$[\texttt{P}] \otimes [\![\texttt{T}]\!] = [[\![\texttt{T}]\!]\texttt{: P}]$$

3. Otherwise, $[\![\texttt{T}]\!]$ does not conform to `P`, and global conformance lookup returns an invalid conformance.

In (2), we return an abstract conformance whose subject type is an archetype. We need to explain what this means. To recap, when the original type of an abstract conformance is a type parameter `T`, the type witness for `[P]A` is a dependent member type `T.[P]A`. Similarly, the associated conformance for $[\texttt{Self.U: Q}]_{\texttt{P}}$ is the abstract conformance `[T.U: Q]`. So, when the original type is an archetype $[\![\texttt{T}]\!]$, we must also map this type witness or associated conformance into the environment:

$$\pi(\texttt{[P]A}) \otimes [[\![\texttt{T}]\!]\texttt{: P}] = [\![\texttt{T.[P]A}]\!] = \mathsf{in}_G(\texttt{T.[P]A})$$
$$\pi(\texttt{Self.U: Q}) \otimes [[\![\texttt{T}]\!]\texttt{: P}] = [[\![\texttt{T.U}]\!]\texttt{: Q}] = [\texttt{Q}] \otimes \mathsf{in}_G(\texttt{T.U})$$

8.2. Archetype Substitution

Recall that $\textsc{Sub}(G, H)$ is set of substitution maps with input generic signature G and output generic signature H, that is, their replacement types are interface types. To understand how type substitution behaves when the original type is a contextual type, we define a new form of our $\otimes$ operator:

$$\textsc{Type}(\llbracket G \rrbracket) \otimes \textsc{Sub}(G, H) \longrightarrow \textsc{Type}(H)$$

To apply a substitution map Σ to a contextual type $\mathtt{T}' \in \textsc{Type}(\llbracket G \rrbracket)$, we first map the contextual type out of its environment, and then apply the substitution map to this interface type:

$$\mathtt{T}' \otimes \Sigma = \mathsf{out}_G(\mathtt{T}') \otimes \Sigma$$

If the replacement types of Σ are interface types, we always get an interface type back, regardless of whether the original type was an interface type or a contextual type.

Substitution maps can also have contextual replacement types. This will be important in Section 9.3, so we will introduce the notation now. We write $\textsc{Sub}(G, \llbracket H \rrbracket)$ for the set of substitution maps whose replacement types are drawn from $\textsc{Type}(\llbracket H \rrbracket)$. Now, if Σ has contextual replacement types, we always get a contextual type back, regardless of whether the original type was an interface type or contextual type:

$$\textsc{Type}(G) \otimes \textsc{Sub}(G, \llbracket H \rrbracket) \longrightarrow \textsc{Type}(\llbracket H \rrbracket)$$
$$\textsc{Type}(\llbracket G \rrbracket) \otimes \textsc{Sub}(G, \llbracket H \rrbracket) \longrightarrow \textsc{Type}(\llbracket H \rrbracket)$$

Substitution map composition generalizes like so:

$$\textsc{Sub}(F, G) \otimes \textsc{Sub}(G, \llbracket H \rrbracket) \longrightarrow \textsc{Sub}(F, \llbracket H \rrbracket)$$
$$\textsc{Sub}(F, \llbracket G \rrbracket) \otimes \textsc{Sub}(G, \llbracket H \rrbracket) \longrightarrow \textsc{Sub}(F, \llbracket H \rrbracket)$$

Forwarding substitution map. When working in the constraint solver, SILGen, or anywhere else that deals with both interface types and contextual types, a special substitution map often appears. If G is a generic signature, the *forwarding substitution map* of G, denoted $1_{\llbracket G \rrbracket}$, sends each generic parameter $\tau\_\mathtt{d}\_\mathtt{i}$ of G to the corresponding contextual type $\mathsf{in}_G(\tau\_\mathtt{d}\_\mathtt{i})$ in the primary generic environment of G:

$$1_{\llbracket G \rrbracket} := \{\tau\_0\_0 \mapsto \mathsf{in}_G(\tau\_0\_0), \ldots, \tau\_\mathtt{m}\_\mathtt{n} \mapsto \mathsf{in}_G(\tau\_\mathtt{m}\_\mathtt{n}); \ldots\}$$

Note that $1_{\llbracket G \rrbracket} \in \textsc{Sub}(G, \llbracket G \rrbracket)$. The forwarding substitution map looks similar to the identity substitution map $1_G \in \textsc{Sub}(G, G)$ which sends every generic parameter to itself, from Section 6.2:

$$1_G := \{\tau\_0\_0 \mapsto \tau\_0\_0, \ldots, \tau\_\mathtt{m}\_\mathtt{n} \mapsto \tau\_\mathtt{m}\_\mathtt{n}; \ldots\}$$

If $\mathtt{T} \in \textsc{Type}(G)$ and $\mathtt{T}' \in \textsc{Type}(\llbracket G \rrbracket)$, we can apply 1_G and $1_{\llbracket G \rrbracket}$ to each one of $\mathtt{T}$ and $\mathtt{T}'$:

$$\mathtt{T} \otimes 1_G = \mathtt{T}$$
$$\mathtt{T} \otimes 1_{\llbracket G \rrbracket} = \mathsf{in}_G(\mathtt{T})$$
$$\mathtt{T}' \otimes 1_G = \mathsf{out}_G(\mathtt{T}') \otimes 1_G = \mathsf{out}_G(\mathtt{T}')$$
$$\mathtt{T}' \otimes 1_{\llbracket G \rrbracket} = \mathsf{out}_G(\mathtt{T}') \otimes 1_{\llbracket G \rrbracket} = \mathsf{in}_G(\mathsf{out}_G(\mathtt{T}')) = \mathtt{T}'$$

Or in words,

- Applying the identity substitution map leaves an interface type unchanged, while a contextual type is mapped out of the environment.
- Applying the forwarding substitution map leaves a contextual type unchanged, while an interface type is mapped into the environment.

We can use this to convert between substitution maps with contextual and interface replacement types, by composing the substitution map with the identity or forwarding substitution map on the right. If $\Sigma \in \textsc{Sub}(G, H)$ and $\Sigma' \in \textsc{Sub}(G, \llbracket H \rrbracket)$, we have:

$$\Sigma \otimes 1_{\llbracket H \rrbracket} \in \textsc{Sub}(G, \llbracket H \rrbracket)$$
$$\Sigma' \otimes 1_H \in \textsc{Sub}(G, H)$$

Furthermore, if $T \in \textsc{Type}(G)$,

$$T \otimes (\Sigma \otimes 1_{\llbracket G \rrbracket}) = \mathsf{in}_H(T \otimes \Sigma)$$
$$T \otimes (\Sigma' \otimes 1_G) = \mathsf{out}_H(T \otimes \Sigma')$$

Finally, substitution maps support a **map replacement types out of environment** operation which is more direct than composing with the identity substitution map:

$$\mathsf{out}_H \colon \textsc{Sub}(G, \llbracket H \rrbracket) \longrightarrow \textsc{Sub}(G, H)$$

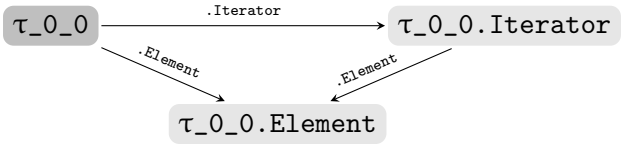
Invariants. In the implementation, a pair of predicates distinguish interface types from contextual types:

- `hasTypeParameter()` tests if the type contains type parameters.
- `hasPrimaryArchetype()` tests if the type contains primary archetypes.

These predicates are used to check preconditions in cases where a piece of compiler logic only expects to be called with an interface type or a contextual type, but not vice versa. Generally, the predicate asserted is the negation of the *opposite* condition: logic that expects only interface types should assert that the given type does not contain archetypes, while logic that expects only contextual types should assert that the type does not contain a type parameter.

8.3. The Type Parameter Graph

Let G be the generic signature `<τ_0_0 where τ_0_0: Sequence>`, and consider the type parameter `τ_0_0.Element`. One way to think about the operation $\mathsf{in}_G(\texttt{τ\_0\_0.Element})$ is that we start from the archetype $[\![\texttt{τ\_0\_0}]\!]$, and then "jump" to $[\![\texttt{τ\_0\_0.Element}]\!]$ in one step, labeled "`.Element`". Similarly, to find $\mathsf{in}_G(\texttt{τ\_0\_0.Iterator.Element})$, we start from $[\![\texttt{τ\_0\_0}]\!]$, take a step to $[\![\texttt{τ\_0\_0.Iterator}]\!]$, and then take one more step, at which point we again end up at $[\![\texttt{τ\_0\_0.Element}]\!]$, because `τ_0_0.Element` is the reduced type of `τ_0_0.Iterator.Element`:



We will define an object called the *type parameter graph* of a generic signature, such that valid type parameters are *paths* in this graph, and equivalence classes of type parameters, or archetypes, are the *vertices*. Two equivalent type parameters define a pair of paths with the same destination vertex. We will have occasion to study other directed graphs later, so as usual we begin with the abstract definitions.

Definition 8.1. A *directed graph* is a pair (V, E) consisting of a set of vertices V together with a set of edges E, where every edge $e \in E$ has an associated *source* vertex and a *destination* vertex, denoted $\mathrm{src}(e)$ and $\mathrm{dst}(e)$, respectively.

Some books (for example, [70]) define directed graphs such that an edge is exactly an ordered pair of vertices, $(\mathrm{src}(e), \mathrm{dst}(e))$. This disallows graphs where two edges share the same source and destination. Our formulation is more general because the source and destination are *properties* of an edge, which may have inherent identity of its own. This is called a *directed multi-graph* in [74].

A finite directed graph can be visualized by plotting each vertex as a point, and each edge as an arrow pointing from the source towards the destination. If either V or E is infinite, then we say that (V, E) is an *infinite graph*. We can still visualize some finite *subgraph* (V', E'), where $V' \subseteq V$ and $E' \subseteq E$, if the remaining structure is understood somehow.

Definition 8.2. Let (V, E) be a directed graph. A *path* $p := (v, (e_1, \ldots, e_n))$ starts at a source vertex $v \in V$ and follows a sequence of zero or more edges $(e_1, \ldots, e_n)$, not necessarily distinct, which must satisfy the following conditions:

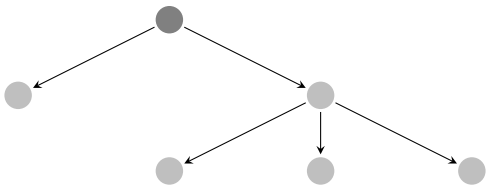
1. If the path contains at least one edge, the source of the first edge must equal the source vertex: $\mathrm{src}(e_1) = v$.
2. If the path contains at least two edges, the source of each subsequent edge must equal the destination of the preceding edge: $\mathrm{src}(e_{i+1}) = \mathrm{dst}(e_i)$ for $0 < i \leq n$.

Each path also has a *source* and *destination*. The source of a path p is the source vertex v, and the destination is either the source vertex v again (if the sequence of edges is empty), or the destination of the final edge:

$$\mathrm{src}(p) := v$$
$$\mathrm{dst}(p) := \begin{cases} v & (\text{if } n = 0) \\ \mathrm{dst}(e_n) & (\text{if } n > 0) \end{cases}$$

The *length* of a path is the number of edges in the path. Every vertex $v \in V$ defines an empty path (of length zero), denoted 1_v, with source vertex v followed by an empty sequence of edges; by the above definitions, we have $\mathrm{src}(1_v) = \mathrm{dst}(1_v) = v$. Every edge e also defines a one-element path (of length one), with source vertex $\mathrm{src}(e)$ and destination vertex $\mathrm{dst}(e)$. We can also denote the one-element path as e, because $\mathrm{src}(e)$ and $\mathrm{dst}(e)$ have the same meaning whether we interpret e as an edge or a path.

Definition 8.3. A *tree* is a directed graph with a distinguished root vertex, and the property that there is one unique path from the root to every other vertex. This describes various forms of tree data structures, familiar to all programmers:



We now have enough graph theory to define our main object of interest.

Definition 8.4. The *type parameter graph* of a generic signature G is defined as follows:

- The vertex set contains a distinguished root vertex. (But we can sometimes omit it, as described later.)
- For each equivalence class of type parameters of G, the vertex set also contains a vertex labeled with the reduced type of the equivalence class. If the reduced type of some equivalence class is a concrete type, the corresponding vertex is labeled with this concrete type.
- For each generic parameter `τ_d_i` of G, the edge set contains an edge with the root vertex as the source, and the equivalence class of this generic parameter, $[\![$`τ_d_i`$]\!]$, as the destination. We label this edge with the generic parameter type "`τ_d_i`".
- If $G \vdash$ `U.A` and $G \vdash$ [`U.A == V`] for some pair of type parameters `T` and `U` and some identifier `A`, the edge set contains an edge with source $[\![$`U`$]\!]$ and destination $[\![$`V`$]\!]$, labeled "`.A`".

Proposition 8.5. The edge relation described above is *well-defined*.

Proof. That is, if we're given two equivalence classes, the question of whether there is an edge connecting one to the other does not depend on which representative type parameter we pick from each one. To see why, suppose that $G \vdash [\texttt{U == U}']$, $G \vdash [\texttt{V == V}']$, $G \vdash \texttt{U.A}$, and $G \vdash [\texttt{U.A == V}]$, so there is an edge labeled "`.A`" with source `U` and destination `V`. We note that the condition $G \vdash \texttt{U.A}$ means $G \vdash [\texttt{U: P}]$ for some protocol `P` that declares an associated type named `A`. Using all this, we can then derive $G \vdash [\texttt{U}'\texttt{.A == V}']$:

1. $[\texttt{U == U}']$ (...)
2. $[\texttt{V == V}']$ (...)
3. $[\texttt{U: P}]$ (...)
4. $[\texttt{U.A == V}]$ (...)
5. $[\texttt{U.A == U}'\texttt{.A}]$ (SameName 3 4)
6. $[\texttt{U}'\texttt{.A == U.A}]$ (Sym 5)
7. $[\texttt{U}'\texttt{.A == V}]$ (Trans 6 4)
8. $[\texttt{U}'\texttt{.A == V}']$ (Trans 7 2)

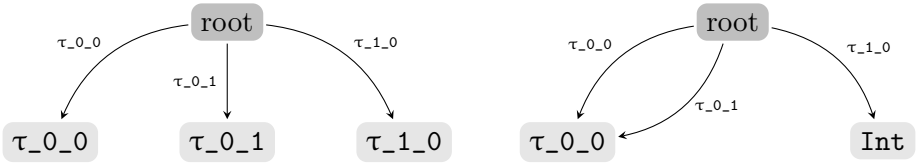
This result would no longer hold if we didn't have the SameName inference rule, so this proposition gives us another justification for the existence of this inference rule. □

Every unbound type parameter $\texttt{τ\_d\_i.A}_1\texttt{...A}_n$ defines a path from the root vertex to the type parameter's equivalence class. Two type parameters belong to the same equivalence class if and only if their paths end at the same destination. If a type parameter is equivalent to a concrete type, the type parameter's path ends at a vertex labeled by this concrete type. (We could also model bound type parameters as paths by adding more edges, but we will not pursue this direction, because as we recall from Section 5.3, bound type parameters don't really give us anything new.)

Example 8.6. Consider these two generic signatures:

```
<τ_0_0, τ_0_1, τ_1_0>
<τ_0_0, τ_0_1, τ_1_0 where τ_0_0 == τ_0_1, τ_1_0 == Int>
```

The first signature has no requirements at all, and each generic parameter is in its own equivalence class. Its type parameter graph is a tree, shown on the left. The second signature collapses the first two and fixes the third to a concrete type. This time, the graph is not a tree, because we have two paths from the root that end at the same vertex.

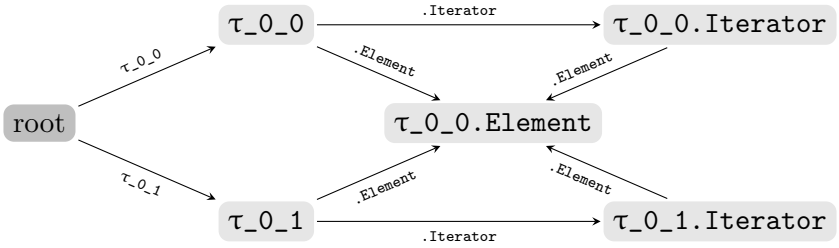


Without conformance requirements, there isn't much more we can do, as our supply of vertices is limited. We can generate more vertices by adding conformance requirements to protocols with associated types.

Example 8.7. We're going to revisit the generic signature from Example 5.2, but let's drop the [τ_0_0.Element: Equatable] requirement (the Equatable protocol does not declare any associated types, so this requirement does not generate any new vertices or edges in our graph):

```
<τ_0_0, τ_0_1 where τ_0_0: Sequence, τ_0_1: Sequence,
                    τ_0_0.Element == τ_0_1.Element>
```

We already studied the equivalence class structure of this generic signature extensively. We can now present what we already know in the form of a type parameter graph, which allows us to look at the member type relationships in a new way:



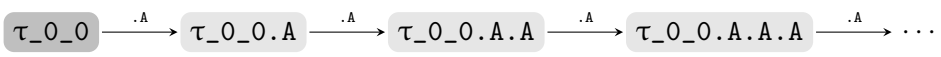
Notice how the paths corresponding to each one of the type parameters τ_0_0.Element, τ_0_1.Element, τ_0_0.Iterator.Element, and τ_0_1.Iterator.Element all have the same destination, because they belong to the same equivalence class ⟦τ_0_0.Element⟧.

When our generic signature only has a single generic parameter τ_0_0, we can omit the root vertex. This changes our formulation slightly, because now a type parameter maps to a path that starts at τ_0_0, so it has one fewer step.

Example 8.8. Recall this protocol from Example 5.12:

```
protocol N {
  associatedtype A: N
}
```

We saw that the protocol generic signature $G_{\texttt{N}}$ (that is, <τ_0_0 where τ_0_0: N>) defines an infinite set of equivalence classes. Its type parameter graph is called a *ray*, or an infinite path:

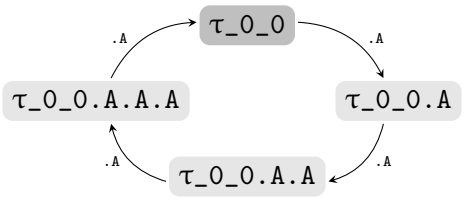


In general, the type parameter graph may be infinite. If we think of archetypes as the vertices in this graph, then we see that the lazy construction of archetypes means that in any given compilation session, the compiler will explore an arbitrarily large but finite subgraph of the type parameter graph.

Example 8.9. Now let's change `N` by adding an associated same-type requirement:

```
protocol Z4 {
  associatedtype A: Z4 where Self == Self.A.A.A.A
}
```

Just like $G_{\texttt{N}}$, the protocol generic signature $G_{\texttt{Z4}}$ still defines an infinite set of valid type parameters, but the set of equivalence classes is now finite. The type parameter graph is the *cyclic graph* of order 4:

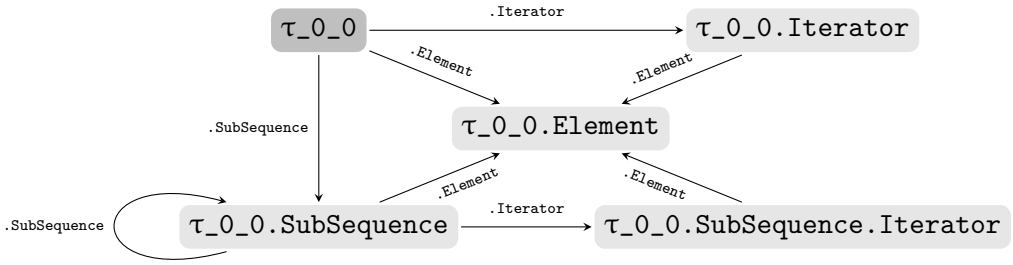


For example, `τ_0_0.A` and `τ_0_0.A.A.A.A.A` are equivalent. Note that a generic signature with an infinite theory but only finitely many equivalence classes must have at least one infinite equivalence class. In $G_{\texttt{Z4}}$, *every* equivalence class is infinite.

Example 8.10. Recall the simplified `Collection` protocol from Example 5.13:

```
protocol Collection: Sequence {
  associatedtype SubSequence: Collection
      where Element == SubSequence.Element
      where SubSequence == SubSequence.SubSequence
}
```

We studied the protocol generic signature $G_{\texttt{Collection}}$ and again we saw that it defines an infinite theory, but only a finite set of equivalence classes. Here is the type parameter graph:



The equivalence class $[\![\texttt{τ\_0\_0.SubSequence}]\!]$ contains infinitely many type parameters, and just like $G_{\texttt{Z4}}$, the type parameter graph contains a *cycle*, or a path with the same source and destination, which we can take any number of times to generate more type parameters in this equivalence class:

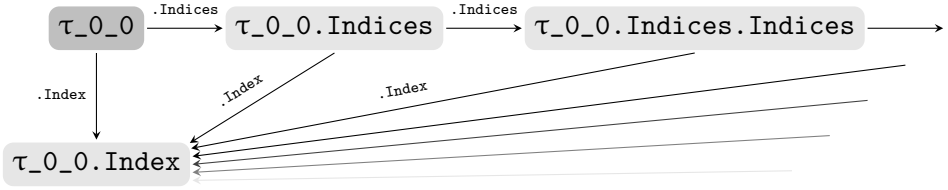
```
τ_0_0.SubSequence
τ_0_0.SubSequence.SubSequence
τ_0_0.SubSequence.SubSequence.SubSequence
...
```

The other infinite equivalence classes are formed by taking the cycle zero or more times, followed by some other edge. In general, if a generic signature has an infinite theory but a finite set of equivalence classes, the type parameter graph will contain at least one cycle.

Example 8.11. The real `Collection` protocol declares two more associated types, `Index` and `Indices`, and imposes some associated requirements on them. We can subset this out into a new protocol to see what it looks like:

```
protocol Indexable {
  associatedtype Index
  associatedtype Indices: Indexable
    where Index == Indices.Index
}
```

The protocol generic signature $G_{\texttt{Indexable}}$ defines an infinite set of equivalence classes, like $G_{\texttt{N}}$. The equivalence class of `τ_0_0.Index` also has infinitely many representatives:



If v and w are vertices in some directed graph (V, E), we say that v is a *successor* of w if they are joined by a edge $e \in E$, that is $\text{src}(e) = v$ and $\text{dst}(e) = w$. Similarly, w is then a *predecessor* of v. The type parameter graph of any generic signature has the property that a vertex can only have a finite set of successors, because the successors are defined by associated type declarations. (We will investigate directed graphs where every vertex has a finite set of successors in Section 12.3.) The above example shows that the analogous statement does not hold for the *predecessor* set of a vertex though, because the vertex $[\![\texttt{τ\_0\_0.Index}]\!]$ has an infinite set of predecessors in $G_{\texttt{Indexable}}$.

In Chapter 12, we will construct a second directed graph for a generic signature, called the *conformance path graph*.

8.4. The Archetype Builder

A generic environment gives us a faithful representation of a generic signature: the in_G operation encodes the reduced type equality relation, and the local requirements of its archetypes encode everything else. Today, a generic environment is a *derived* representation, in the sense that it sits on top of generic signature queries. This was not always the case, and back in Swift 3, archetypes were the fundamental primitives. In this section, we will describe the `ArchetypeBuilder` algorithm that was used to build them. While now obsolete and no longer part of the implementation, the algorithm has a certain elegance to it, and understanding it will motivate the material in Part V.

We can group generic signatures into three families of increasing complexity:

1. Generic signatures with a *finite theory*, that is a finite set of derived requirements and valid type parameters. Example 8.7 is a member of this family.
2. Generic signatures with an infinite theory, but a *finite set of equivalence classes* of type parameters. Example 8.9 and Example 8.10 belong to this family.
3. Generic signatures with an *infinite set of equivalence classes*. This family includes Example 8.8 and Example 8.11, and we will later see it is the most difficult.

Swift 3 could only accept the first family, because protocols were prohibited from stating recursive conformance requirements on their associated types (so today's `Collection` protocol could not be expressed). In general, the associated requirements of a protocol took on a much simpler form; protocols could state inheritance relationships, and they could impose conformance requirements on their associated types, but associated same-type requirements were not supported. Also, while Swift 3 did allow superclass and concrete same-type requirements, we're going to ignore them because they only involve a little bit of additional bookkeeping and are not central to the algorithm.

Potential archetypes. The algorithm represents a partially-formed equivalence class of type parameters using a data structure called a *potential archetype*. In memory, a potential archetype t stores these four fields:

$\texttt{FORWARD}(t)$	An optional pointer to another potential archetype.
$\texttt{TYPE}(t)$	A type parameter. This field is immutable after creation.
$\texttt{PROTO}(t)$	A set of protocol types.
$\texttt{MEMBERS}(t)$	A set of (identifier, pointer to potential archetype) pairs.

A potential archetype t always represents a fixed type parameter `T`, so that $\texttt{TYPE}(t) = \texttt{T}$ for the lifetime of t. In the initial state, we have $\texttt{FORWARD}(t) = \text{null}$, $\texttt{PROTO}(t) = \varnothing$, and $\texttt{MEMBERS}(t) = \varnothing$. Potential archetypes are allocated sequentially while the algorithm runs, and then freed all at once when the entire archetype builder instance is freed.

The archetype builder establishes the following invariant. Any potential archetype with a null forwarding pointer represents the reduced type parameter of its equivalence class. In this case, $\texttt{PROTO}(t)$ is the set of all protocols this equivalence class conforms to, and $\texttt{MEMBERS}(t)$ lists all of the successors of the equivalence class in the type parameter graph. Otherwise, if a potential archetype t has a non-null forwarding pointer, it points to some other potential archetype in the same equivalence class, and the other potential archetype precedes it in type parameter order; that is, $\texttt{TYPE}(\texttt{FORWARD}(t)) < \texttt{TYPE}(t)$. By following this chain, we eventually end up at the reduced type parameter.

We now describe how this invariant is established, then show how the resulting data structure can implement generic signature queries.

Eager expansion. The archetype builder begins by creating a potential archetype for each declared generic parameter; these are called *root* potential archetypes. Potential archetypes that correspond to dependent member types come into existence from the *expansion* of conformance requirements. The expansion of a conformance requirement is the first of the two principal operations on potential archetypes. A protocol can impose conformance requirements on its associated types, so this expansion process is recursive. Notice how in Step 6b, we must catch and diagnose recursive conformance requirements to protocols we've already seen, to guarantee termination; this is of course the major limitation with the "eager expansion" approach.

Algorithm 8A (Expand conformance requirement)**.** Takes a potential archetype t and a protocol `P` as input. Has side effects.

1. If $\texttt{FORWARD}(t)$ is non-null, load $t \leftarrow \texttt{FORWARD}(t)$, and repeat if necessary.
2. If $\texttt{P} \in \texttt{PROTO}(t)$, return.
3. Otherwise, set $\texttt{PROTO}(t) \leftarrow \texttt{PROTO}(t) \cup \{\texttt{P}\}$.
4. For each protocol `Q` appearing in the inheritance clause of the declaration of `P`, recursively apply this algorithm to t and `Q`.
5. Let $\texttt{T} := \texttt{TYPE}(t)$.
6. For each associated type declaration `A` of `P`:
   a) If $\texttt{MEMBERS}(t)$ does not have an entry for `A`, create a new potential archetype v with $\texttt{TYPE}(v) = \texttt{T.A}$. Add the ordered pair $(\texttt{A}, v)$ to $\texttt{MEMBERS}(t)$.

   Otherwise, $\texttt{MEMBERS}(t)$ contains an existing entry $(\texttt{A}, v)$ with $\texttt{TYPE}(v) = \texttt{T.A}$.

   b) For each protocol `Q` listed in the inheritance clause of the declaration of `A`, first check if `Q` already appears among the set of all protocols we've visited in prior recursive calls; if so, emit a diagnostic. Otherwise, add `Q` to the visited set, and recursively apply this algorithm to v and `Q`.

The second principal operation is that of merging two potential archetypes, to form a larger equivalence class. This implements same-type requirements.

Algorithm 8B (Merge potential archetypes)**.** Receives a pair of potential archetypes t and u as input. Has side effects.

1. If either $\texttt{FORWARD}(t)$ or $\texttt{FORWARD}(u)$ is non-null, follow the forwarding pointers.
2. If t and u are equal pointers to the same potential archetype, return.
3. If $\texttt{TYPE}(u) < \texttt{TYPE}(t)$ under Algorithm 5D, first swap t and u. (This step is what eventually guarantees that $\texttt{FORWARD}(t) = \text{null}$ if and only if $\texttt{TYPE}(t)$ is a reduced type parameter.)
4. Set $\texttt{FORWARD}(u) \leftarrow t$.
5. Set $\texttt{PROTO}(t) \leftarrow \texttt{PROTO}(t) \cup \texttt{PROTO}(u)$.
6. Let $\texttt{T} := \texttt{TYPE}(t)$.
7. For each entry $(\texttt{A}, w) \in \texttt{MEMBERS}(u)$,
   a) If $\texttt{MEMBERS}(t)$ does not contain a potential archetype for $\texttt{A}$, create a new potential archetype v such that $\texttt{TYPE}(v) = \texttt{T.A}$. Add $(\texttt{A}, v)$ to $\texttt{MEMBERS}(t)$.
   b) Otherwise, $\texttt{MEMBERS}(t)$ contains an existing entry $(\texttt{A}, v)$ with $\texttt{TYPE}(v) = \texttt{T.A}$.
   c) In either case, invoke this algorithm recursively to merge v and w.

We need one more utility subroutine to actually resolve an arbitrary type parameter to a potential archetype, starting from the roots:

Algorithm 8C (Resolve potential archetype)**.** Receives the array of root potential archetypes, and a type parameter $\texttt{T}$. Returns the potential archetype for $\texttt{T}$, or null if one has not been created.

1. If $\texttt{T}$ is a generic parameter, let t be the corresponding root potential archetype. If $\texttt{FORWARD}(t)$ is non-null, load $t \leftarrow \texttt{FORWARD}(t)$, and repeat if necessary. Return t.
2. Otherwise, $\texttt{T}$ must be a dependent member type $\texttt{U.A}$ with some base type $\texttt{T}$ and identifier $\texttt{A}$. First, invoke the algorithm recursively to resolve $\texttt{U}$ to a potential archetype u.
3. If $\texttt{MEMBERS}(u)$ contains a potential archetype for $\texttt{A}$, let t be this potential archetype. If $\texttt{FORWARD}(t)$ is non-null, load $t \leftarrow \texttt{FORWARD}(t)$, and repeat if necessary. Return t.
4. Otherwise, return null. The type parameter $\texttt{T}$ is either invalid, or we haven't yet expanded a conformance requirement involved in its derivation.

We're now ready to look at the main algorithm. We visit each requirement written in source, expanding conformance requirements and merging pairs of potential archetypes corresponding to same-type requirements. There is a minor complication: requirements can be written in any order, so for example, if we process [`T.Element: Equatable`] before [`T: Sequence`], the potential archetype for `T.Element` may not have been created yet. This is allowed, and if Algorithm 8C returns null, we add the requirement to the "delayed requirements" list and re-process it again after attempting all other requirements.

Algorithm 8D (Historical `ArchetypeBuilder` algorithm)**.** Receives a list of generic parameters, requirements, and protocol declarations as input. Outputs an array of root potential archetypes. There are three intermediate data structures: a list of pending requirements, a list of delayed requirements, and a flag to record forward progress.

1. (Initialize) Add all requirements to the pending list. Set the flag to false. Create a root potential archetype for each generic parameter.
2. (Reprocess) If the pending list is empty and the flag is set, move any requirements from the delayed list to the pending list, and clear the flag.
3. (Check) If the pending list is still empty, go to Step 8.
4. (Resolve) Remove a requirement from the pending list.
5. (Conformance) For a conformance requirement [`T: P`], invoke Algorithm 8C to resolve `T` to a potential archetype. If this potential archetype does not exist, add [`T: P`] to the delayed list and set the flag. Otherwise, invoke Algorithm 8A to expand the conformance requirement.
6. (Same-type) For a same-type requirement [`T == U`], invoke Algorithm 8C to resolve each one of `T` and `U` to an potential archetype. If either potential archetype does not exist, add [`T == U`] to the delayed list and set the flag. Otherwise, invoke Algorithm 8B to merge the two potential archetypes.
7. (Repeat) Go back to Step 2.
8. (Diagnose) If we end up here, there are no more pending requirements. Any requirements remaining on the delayed list reference type parameters which cannot be resolved; diagnose them as invalid.

Once the algorithm returns, the potential archetype structure now encodes the finite type parameter graph built from the input requirements, assuming we didn't diagnose an invalid recursive conformance. At this point, no more expansion or merging takes place; the remaining potential archetypes, those without forwarding pointers, are "frozen" into immutable archetype *types* that describe this generic signature to the rest of the compiler. While generic signature queries did not exist back then, we can reconcile this algorithm with our present model by defining each generic signature query as follows:

- `isValidTypeParameter`(G, `T`): Resolve `T` to a potential archetype t, and check if t is non-null.
- `getReducedType`(G, `T`): Resolve `T` to a potential archetype t, and return `TYPE`(t).
- `areReducedTypeParametersEqual`(G, `T`, `U`): Resolve both `T` and `U` to potential archetypes t and u, and check if t and u are equal pointers to the same potential archetype.
- `requiresProtocol`(G, `T`, `P`): Resolve `T` to a potential archetype t, and check if $\texttt{P} \in \texttt{PROTO}(t)$.
- `getRequiredProtocols`(G, `T`): Resolve `T` to a potential archetype t, and return `PROTO`(t).

The archetype builder is an example of a *union-find* or *disjoint set* data structure (see, for example, Section 2.3.3 of [75]). An instance of this data structure defines an equivalence relation. Starting from a collection of singleton equivalence classes, the two fundamental operations are *union*, to merge two equivalence classes, and *find*, to resolve an element to its equivalence class. In our case, Algorithm 8B is *union*, and Algorithm 8C is *find* (whereas the expansion in Algorithm 8A is our own thing).

Our description of these algorithms omits some standard techniques for optimizing these operations; for example, when *find* follows a forwarding pointer, we usually want to write the pointer back to its original storage location as well, to eliminate repeated traversals on subsequent lookups.

This kind of data structure first appeared in the early history of compilers in the implementation of "`EQUIV` statements" for declaring that two symbols ought to share a storage location. These statements thus define an equivalence relation on symbols, and the performance gains from a careful implementation were noted in [76]. A survey of later techniques appears in [77].

Lazy expansion. From a theoretical point of view, the archetype builder's approach amounts to exhausive enumeration of all derived requirements and valid type parameters of a generic signature, made slightly more efficient by the choice of data structure (the asymmetry in the handling of member types in Algorithm 8B means we skip parts of the search space that would yield nothing new).

The eager expansion model survived the introduction of protocol `where` clauses in Swift 4 [66], and thus associated requirements, with only relatively minor changes. The introduction of recursive conformances in Swift 4.1 [78] necessitated a larger overhaul. Once the type parameter graph becomes infinite, the eager conformance requirement expansion of Algorithm 8A no longer makes sense. The `ArchetypeBuilder` was renamed to the `GenericSignatureBuilder` as part of a re-design where the recursive expansion was now performed as needed, within the lookup of Algorithm 8C itself [79].

In the lazy expansion model, the potential archetype structure was highly mutable, as generic signature queries would create new potential archetypes and merge existing potential archetypes while exploring new subgraphs of the type parameter graph. The on-going mutation prevented sharing of structure between `GenericSignatureBuilder` instances, and every generic signature that depends on a complicated standard library protocol, such as `RangeReplaceableCollection`, would eventually construct its own copy of a large graph of potential archetypes. This would significantly impact compiler memory usage and performance.

Lazy expansion also suffered from correctness problems. Even the second family of generic signatures, those with an infinite theory but a finite set of equivalence classes, did not always work properly. Example 8.9 happened to be one such instance, and we will later see another in Example 19.10. An even more fundamental problem was discovered later. As we will see in Section 17.4, the third family of generic signatures, those having an infinite set of equivalence classes, actually contains generic signatures where the question of reduced type equality is *undecidable*. Such generic signatures must be rejected by any correct implementation of Swift. By virtue of its design, the `GenericSignatureBuilder` pretended to be able to accept any generic signature and answer queries about it. Lazy expansion was a dead end! This seemed surprising at the time, because in the absence of recursive conformances, eager expansion completely "solved" Swift generics in a straightforward way.

These problems motivated the search for a sound and decidable foundation on top of which Swift generics should be built, which became the Requirement Machine. Instead of incrementally constructing finite subgraphs of the type parameter graph, the correct approach is to construct a *convergent rewriting system*. While more abstract, this is actually much *simpler* than lazy expansion of the type parameter graph. Just like with the original eager expansion design, the rewriting system is a finite data structure that is built once and then remains immutable after construction. Unlike eager expansion, a rewriting system can describe an infinite set of equivalence classes, and in many cases, it can also encode a *finite* set without exhaustive enumeration. We will describe the contemporary approach and present a correctness proof in Part V.

8.5. Source Code Reference

`GenericEnvironment` *class*

A generic environment. Instances are allocated in the AST context, and passed by pointer. A flat array maps generic parameter types to archetypes, and a separate side table is allocated to store the mapping for dependent member types.

- `getGenericSignature()` returns this generic environment's generic signature.

- `mapTypeIntoContext()` returns the contextual type obtained by mapping an interface type into this generic environment.
- `getForwardingSubstitutionMap()` returns a substitution map for mapping each generic parameter to its contextual type—an archetype, or a concrete type if the generic parameter is fixed to a concrete type via a same-type requirement.
- `getOrCreateArchetypeFromInterfaceType()` is the private entry point which maps a single type parameter to an archetype, creating the archetype the first time. This method uses the `getLocalRequirements`() generic signature query.

`GenericSignature` *class*

See also Section 5.6.

- `getGenericEnvironment()` returns the primary generic environment associated with this generic signature.

`TypeBase` *class*

See also Section 3.4.

- `mapTypeOutOfContext()` returns the interface type obtained by mapping this contextual type out of its generic environment.

`SubstitutionMap` *class*

See also Section 6.5.

- `mapReplacementTypesOutOfContext()` returns the substitution map obtained by mapping this substitution map's replacement types and conformances out of their generic environment.

`ProtocolConformanceRef` *class*

See also Section 7.6.

- `mapConformanceOutOfContext()` returns the protocol conformance obtained by mapping this protocol conformance out of its generic environment.

`DeclContext` *class*

See also Section 4.6.

- `getGenericEnvironmentOfContext()` returns the generic environment of the innermost generic declaration containing this declaration context.

- `mapTypeIntoContext()` Maps an interface type into the primary generic environment for the innermost generic declaration. If at least one outer declaration context is generic, this is equivalent to:

```
dc->getGenericEnvironmentOfContext()->mapTypeIntoContext(type);
```

  For convenience, the `DeclContext` version of `mapTypeIntoContext()` also handles the case where no outer declaration is generic. In this case, it returns the input type unchanged, after asserting that it does not contain any type parameters (since type parameters appearing outside of a generic declaration are nonsensical).

`ArchetypeType` *class*

An archetype.

- `getName()` returns the name of this archetype, which is either the name of its generic parameter type or associated type declaration.
- `getFullName()` returns the "dotted" name of this archetype, which looks like a string representation of its type parameter.

Taking an archetype apart:

- `getInterfaceType()` returns the reduced type parameter of this archetype.
- `getGenericEnvironment()` returns the archetype's generic environment.
- `isRoot()` answers if the reduced type parameter is a generic parameter type.

Local requirements (Section 8.1):

- `getConformsTo()` returns the archetype's required protocols. This set does not include inherited protocols. To actually check if an archetype conforms to a specific protocol, use global conformance lookup (Section 7.6) instead of looking through this array.
- `getSuperclass()` returns the archetype's superclass bound, or the empty `Type` if there isn't one.
- `requiresClass()` answers with the requires class flag.
- `getLayoutConstraint()` returns the layout constraint, or the empty layout constraint if there isn't one.

`PrimaryArchetypeType` *class*

Subclass of `ArchetypeType` representing a primary archetype.

Part III.

Specialties

9. Type Resolution

Type resolution transforms the syntactic type representations produced by the parser into the semantic types of Chapter 3. Type representations have a tree structure. The leaf nodes are *identifier type representations* without generic arguments, such as `Int`. Nodes with children include *member type representations* which recursively store a base type representation, such as `T.Element`. There are also type representations for function types, metatypes, tuples, and existentials; they have children and follow the same shape as the corresponding kind of type. Finally, identifier and member type representations may have children in the form of generic arguments, such as `Array<Int>`. One of our main goals in this chapter is to understand how type resolution forms a generic nominal type from a reference to a type declaration and a list of generic arguments.

Type resolution builds the *resolved type* by consulting the type representation itself, as well as contextual information describing where the type representation appears:

1. Identifier type representations are resolved by unqualified lookup of their identifier; this depends on the type representation's source location. For example, a type representation might name a generic parameter declared in the current scope.

2. Certain type representations are also resolved based on their semantic position. For example, a function type representation appearing in parameter position resolves to a non-escaping function type unless annotated with the `@escaping` attribute; in any other position, a function type representation resolves to an escaping function type. This behavior was introduced in Swift 3 [80].

We encode contextual information in the *type resolution context*, consisting of the below:

1. The *declaration context* where the type representation is written. This is passed to unqualified lookup for identifier type representations. Fully resolving dependent member types also requires the generic signature of this declaration context.

2. A *type resolution context* and a set of *type resolution flags* together encode the semantic position of the type representation inside its declaration context.

3. A *type resolution stage* specifies if type resolution may query the generic signature of this declaration context. We use type resolution to build the generic signature, but type resolution uses the generic signature to resolve dependent member types and check generic arguments. The staged resolution breaks the request cycle by delaying certain semantic checks until the generic signature is available.

The two type resolution stages are *structural* resolution stage and *interface* resolution stage; we say we resolve a type *in* the given stage:

1. Structural resolution stage does not use the current declaration context's generic signature, and so it doesn't validate type parameters or check generic arguments.

2. Interface resolution stage requests the current context's generic signature first, and issues generic signature queries against this signature to perform semantic checks.

The **generic signature request** resolves various type representations so that it can build a generic signature from user-written requirements, as we will see in Chapter 11. This must be done in the structural resolution stage.

The **interface type request** resolves type representations in the interface resolution stage, to form a value declaration's interface type from semantically well-formed types. By resolving types in the interface stage, the **interface type request** depends on the **generic signature request**.

Structural resolution stage differs from interface resolution stage in two respects:

- References to associated types resolve to unbound dependent member types in the structural resolution stage, and references to invalid member types are not diagnosed. We describe this in Section 9.2.

- Generic arguments are not checked to satisfy the requirements of a generic nominal type in the structural resolution stage. We'll describe checking generic arguments in Section 9.3.

All invocations of type resolution "downstream" of the generics implementation must use the interface resolution stage, to not admit invalid types. To emit the full suite of diagnostics for type representations resolved in the structural stage, in particular inheritance clauses and trailing `where` clauses of generic declarations, the **type check source file request** revisits these type representations and resolves them again in the interface resolution stage.

While the structural resolution stage skips some semantic checks, it can still produce diagnostics; name lookup can fail to resolve an identifier, and certain simpler semantic invariants are still enforced, such as checking that the *number* of generic arguments is correct. For these reasons, care must be taken to not emit the same diagnostics twice if the same invalid type representation is resolved in both stages.

The general mechanism for this is to have each type representation store an "invalid" flag; after diagnosing an error, type resolution sets the invalid flag and returns an error type as the resolved type. If an invalid type representation is resolved again, type resolution immediately returns another error type without visiting the type representation or emitting any new diagnostics. A type representation can only transition from valid to invalid, and never back again.

9.1. Identifier Type Representations

An *identifier type representation* is a single identifier that names a type declaration in some outer scope. We find the *resolved type declaration* via unqualified lookup, starting from the source location of the type representation (Section 2.1). We then form the resolved type, which will be a nominal type, type alias type, generic parameter type, or dependent member type, depending on the type declaration's kind. We show some examples before describing the general principle.

Nominal types. A top-level non-generic nominal type declaration declares a single type, which we referred to as the declared interface type in Chapter 4. Here, the type representation resolves to the struct type `Int` declared by the standard library:

```
var x: Int = ...
```

If a nominal type declaration has a generic parameter list, it instead declares a new type for every possible list of generic arguments. Both identifier and member type representations may contain a list of generic arguments; we discuss how generic arguments are applied and checked in Section 9.3.

We allow the generic arguments to be omitted when the resolved type declaration's generic parameters are visible from the type representation's lexical scope; the generic *arguments* of the resolved type are taken to be the generic *parameters* of the resolved type declaration. In other words, the resolved type becomes the nominal type declaration's declared interface type, without a substitution map applied. For example:

```
struct Outer<T> {
  // Interface type of ‘x’ is ‘Optional<Outer<T>.Inner>’
  var x: Inner?

  class Inner {
    // Interface type of ‘y’ is ‘Outer<T>’
    var y: Outer
  }

  struct GenericInner<U> {
    // Return type of ‘f’ is ‘Outer<T>.GenericInner<U>’
    func f() -> GenericInner {}
  }
}
```

Section 9.4 describes another special case where generic arguments can be omitted when referencing a generic nominal type.

The next series of examples will show that a substitution map may need to be applied to the declared interface type. Recall the discussion from Section 2.1 and Chapter 4. As unqualified lookup visits each outer scope, it considers the scope kind and takes appropriate action to resolve identifiers for that kind of scope. If the scope is a nominal type declaration or extension, unqualified lookup performs a qualified lookup into this nominal type to search for member types of the nominal type.

Unqualified lookup might find a type declaration that is not a direct member of an outer scope. Instead, the resolved type declaration might be a member of a superclass or conformed protocol of some outer type declaration. In this case, the declared interface type is written in terms of type parameters that are not in our current scope, so we must apply a substitution map. The substitution map's input generic signature is that of the resolved type declaration, and its output generic signature is the generic signature of the innermost generic declaration that contains our type representation.

If the resolved type declaration was found in a superclass, this substitution map is the *superclass substitution map*, which we will describe completely in Chapter 15. In the below example, unqualified lookup of `Inner` inside `Derived` finds the member of `Base`. The generic parameter `T` of `Base` is always `Int` in `Derived`, so the superclass substitution map we apply to members of `Base` when seen from `Derived`, is $\{\texttt{T} \mapsto \texttt{Int}\}$:

```
class Base<T> {
  struct Inner {}
}

class Derived: Base<Int> {
  // Interface type of ‘x’ is ‘Base<Int>.Inner’
  var x: Inner = ...
}
```

One edge case here is that if a protocol or protocol extension imposes a superclass requirement on `Self`, we might find a member of this superclass when resolving an identifier type representation inside our protocol or protocol extension. Swift does not support recursive superclass requirements, so the superclass bound of `Self` in this case must always be a fully-concrete type. To continue our example, we can reference `Inner` from a protocol extension of `Proto`, which requires that `Self` inherit from `Derived`:

```
protocol Proto: Derived {}

extension Proto {
  // Return type of ‘f’ is ‘Base<Int>.Inner’
  func f() -> Inner {}
}
```

Generic parameters. Unqualified lookup will find generic parameter declarations if any of the outer declaration contexts have generic parameter lists.

```
struct G<T> {
  func f<U>(...) {
    var x: T = ...    // Canonical type of 'x' is τ_0_0
    var y: U = ...    // Canonical type of 'y' is τ_1_0
  }
}
```

The resolved type is the declared interface type of the generic parameter declaration, which is the corresponding generic parameter type.

The identifier Self. Speaking of `Self`, inside a protocol or protocol extension this refers to the implicit generic parameter named `Self`, also known as `τ_0_0` (Section 4.3), which we resolve like a reference to any other named generic parameter:

```
protocol Proto {
  // Return type of 'f' is the 'Self' generic parameter of 'Proto'
  func f() -> Self
}
```

Inside the source range of a struct or enum declaration or an extension thereof, `Self` is shorthand for the declared interface type of this nominal type declaration. This is not a generic parameter type at all, but rather a nominal type, generic or non-generic:

```
struct Outer<T> {
  // Return type of 'f' is 'Outer<T>'
  func f() -> Self {}
}
```

Inside the source range of a class declaration or an extension of one, `Self` stands for the dynamic `Self` type, wrapping the declared interface type of the class:

```
extension Outer {
  class InnerClass {
    // Return type of 'f' is the dynamic Self type of
    // 'Outer<T>.InnerClass'
    func f() -> Self {}
  }
}
```

The dynamic `Self` type was previously described in Section 3.3. Historically, Swift only had `Self` in protocols and dynamic `Self` in classes, and the latter could only appear in the return type of a method. Swift 5.1 introduced the ability to state dynamic `Self` in more positions, and also refer to "static" `Self` inside struct and enum declarations [81].

Type aliases. Identifier type representations can also refer to type alias declarations, generalizing the behavior described for nominal type declarations above. Once again, we take the declared interface type of the type alias declaration, and possibly apply a substitution map. While the declared interface type of a nominal type declaration is a nominal type, the declared interface type of a type alias declaration is a type alias type. This is a sugared type, canonically equal to the underlying type of the type alias declaration.

If the named type alias declaration is in a local context, the resolved type is the declared interface type, with no substitution map applied:

```
func f<T>() {
  typealias A = (Int, T)

  // Interface type of ‘a’ is canonically equal to ‘(Int, T)’
  let a: A = ...
}
```

If the resolved type alias declaration is a member of a superclass of some outer type declaration, we must apply a substitution map, just like we do when resolving a nominal type found inside a superclass:

```
class Base<T> {
  typealias InnerAlias = T?
}

class Derived: Base<Int> {
  // Return type of ‘f’ is canonically equal to ‘Optional<Int>’
  func f() -> InnerAlias {}
}
```

As explained in Section 6.4, a nominal type declaration cannot be a member of a protocol or protocol extension, but a type alias declaration can. We say it's a *protocol type alias*. Unqualified lookup will find such a type alias declaration from within the scope of any nominal type declaration that conforms to this protocol. We discuss protocol type aliases in the next section when we talk about member type representations.

Associated types. If the identifier type representation is located within the source range of a protocol or protocol extension, the protocol's associated type declarations are visible to unqualified lookup. The resolved type is the declared interface type of the associated type declaration, which is a dependent member type around "`Self`":

```
protocol Pair {
  associatedtype A
  associatedtype B

  // Interface type of ‘a’ is ‘Self.[P]A’
  var a: A { get }

  // Interface type of ‘b’ is ‘Self.[P]B’
  var b: B { get }
}
```

Associated type declarations are also visible from within the protocol's conforming types. Recall that associated types can be witnessed by generic parameters, member type declarations, or inference (Section 7.3). When the type witness is a generic parameter or member type, unqualified lookup will always find the witness *before* the associated type declaration. However, if the type witness is inferred, unqualified lookup will find the associated type declaration:

```
struct S: Pair {
  // Explicit type witness:
  typealias A = Int

  // Inferred type witness:
  // typealias B = String

  var a: A  // resolved type is canonically equal to ‘Int’
  var b: String  // ‘B == String’ is inferred from ‘b’

  // Return type of ‘f’ and ‘g’ is canonically equal to ‘String’
  func f() -> B {}
  func g() -> B {}
}
```

Note that `f()` and `g()` have the same return type, but the two type representations are resolved in a slightly different manner. Let's assume that we compute the interface type of `f()` first (but the other order might also arise, depending on how the remainder of the program is structured).

Resolving the return type of `f()`, we find the associated type declaration `B` of `Pair`. Type resolution performs a global conformance lookup to find the concrete conformance [`S: Pair`], then projects the type witness for `B` from this conformance. This evaluates the **type witness request**, which *synthesizes* the type alias `B` inside `S`. The declared interface type of `S.B`, canonically equal to `String`, is returned as the type witness.

Resolving the return type of `g()`, we now find the type alias `S.B` we just synthesized. This shows that associated type inference has a side effect on the member lookup table of `S`, but it effectively acts as a form of lazy caching; the first lookup triggers the synthesis of this member.

Modules. A module declaration is a special kind of type declaration that can only be used as the base of a member type representation. Otherwise a bare module is an error:

```
_ = Swift.self  // error: expected module member name after module name
```

Of course `import` declarations are usually followed by a bare module name, but they are resolved by different means than type resolution.

Summary. If the resolved type declaration is in local context, or at the top level of a source file, the resolved type is just its declared interface type. Otherwise, we found the resolved type declaration by performing a qualified lookup into some outer nominal type or extension. In this case, the resolved type declaration might be a *direct* member of the outer nominal, or a member of a superclass or protocol. In the direct case, or if we started from a protocol and found a member of another protocol, we again return the member's declared interface type. Otherwise, we build a substitution map whose output generic signature is the generic signature of the outer nominal type or extension. We apply it to the member's declared interface type to get the resolved type:

- In the **superclass case**, we take the outer nominal's superclass bound and the member's parent class declaration, and build the superclass substitution map.
- In the **protocol case**, we take the outer nominal's declared interface type and the member's parent protocol, and build the protocol substitution map from the conformance. (We described this as projecting a type witness from a conformance instead of applying a substitution map; we'll say more about the protocol case in the next section, when we generalize these concepts further.)

Source ranges. In the scope tree, a nominal type or extension declaration actually defines *two* scopes, one nested within the other. The smaller source range contains the *body* only, from "`{`" to "`}`". The larger source range starts with the declaration's opening keyword, such as "`class`" or "`extension`", and continues until the "`}`". In particular,

the `where` clause is inside the larger source range, but outside of the smaller source range. Within a protocol or protocol extension, the protocol's associated type members (and type alias members, too) are always visible in both scopes:

```
// This is OK; 'Element' resolves to 'Self.[Collection]Element'
extension Collection where Element == Int {...}
```

Within a nominal type declaration, generic parameter declarations are visible in the larger scope, but member types can only be seen in the body:

```
// 'A' cannot be referenced here:
struct G<T> where A: Equatable {
  typealias A = T

  // 'A' can be referenced here:
  var a: A = ...
}
```

Prior discussion of name lookup from protocol contexts appeared in Section 4.3.

9.2. Member Type Representations

A *member type representation* consists of a *base* type representation together with an identifier, joined by "." in the concrete syntax. The base might be an identifier type representation, or recursively, another member type representation. The base may also have generic arguments. The general procedure for resolving a member type representation is the following. We start by recursively resolving the base type reprensetation; then, we issue an qualified lookup (Section 2.1) to look for a member type declaration with the given name, inside the resolved base type. This finds the resolved type declaration, from which we compute the resolved type by applying a substitution map.

The resolved types obtained this way include nominal types, dependent member types, and type alias types. We classify the various behaviors by considering each kind of base type in turn, and describing its member types.

Module base. When the base is an identifier naming a module, the member type representation refers to a top-level type declaration within this module:

```
var x: Swift.Int = ...
```

The type declarations referenced this way are those that can appear at the top level of a module, namely nominal types and type aliases. The resolved type is the declared interface type of the declaration.

Type parameter base. As we alluded at the beginning of this chapter, the behavior of member type resolution with a type parameter base depends on the type resolution stage. We will begin with a description of the interface resolution stage, even though it happens last, because it implements the "complete" behavior.

Interface resolution stage. In this stage we interpret the type parameter relative to the current declaration context's generic signature. The requirements imposed on this type parameter give a list of protocols, and possibly a superclass bound, or a concrete type. The member type declarations of these types are the member type declarations of our type parameter. This includes:

1. Associated type declarations from all conformed protocols.
2. Type alias declarations from all conformed protocols and their extensions.
3. Member types of the concrete superclass type, if any.
4. If the type parameter is fixed to a concrete type via a same-type requirement, the members of this concrete type.

We get the list of types that qualified lookup must look inside by collecting the results of the `getRequiredProtocols`(), `getSuperclassBound`(), and `getConcreteType`() generic signature queries (Section 5.5).

The first case, where qualified lookup finds an associated type declaration, is extremely important; this is how we form type parameters recursively in type resolution. If some type parameter `T` conforms to a protocol, say `Provider`, and this protocol declares an associated type `Entity`, then type resolution of "`T.Entity`" will find the associated type declaration `Entity` by performing a qualified lookup into `Provider`.

```
protocol Provider {
  associatedtype Entity
}

struct G<T: Provider> {
  var x: T.Entity = ...
}
```

To obtain the resolved type `T.[Provider]Entity` from `Self.[Provider]Entity` (the declared interface type of the `Entity` associated type), we must replace `Self` with `T`. We do this by applying the protocol substitution map, which satisfies the protocol generic signature $G_{\texttt{Provider}}$ with the abstract conformance [`T: Provider`]:

$$\Sigma_{[\texttt{T: Provider}]} := \{\texttt{Self} \mapsto \texttt{T};\ [\texttt{Self: Provider}] \mapsto [\texttt{T: Provider}]\}$$

Applying $\Sigma_{\texttt{[T: Provider]}}$ to `Self.[Provider]Entity` projects the type witness from this conformance, with an equation we've seen a few times now, for example in Section 7.4:

$$\begin{aligned}\texttt{Self.[Provider]Entity} \otimes \Sigma_{\texttt{[T: Provider]}} &\\ &= \pi(\texttt{[Provider]Entity}) \otimes [\texttt{Self: Provider}] \otimes \Sigma_{\texttt{[T: Provider]}} \\ &= \pi(\texttt{[Provider]Entity}) \otimes [\texttt{T: Provider}] \\ &= \texttt{T.[Provider]Entity}\end{aligned}$$

The second case, where we found a protocol type alias as a member of the base, is closely related. The resolved type is again obtained by replacing all occurrences of `Self` in the underlying type of the type alias with the resolved base type of the member type representation. We're also going to make things slightly more interesting by using a dependent member type `T.Element` as the base type, rather than the generic parameter `T`:

```
protocol Subscriber {
  associatedtype Parent: Provider
  typealias Content = Self.Parent.Entity  // or just Parent.Entity
}

func process<T: Sequence>(_: T) where T.Element: Subscriber {
  // Interface type of 'x' is canonically equal to
  // 'T.Element.Parent.Entity'
  let x: T.Element.Content = ...
}
```

The above behavior can be explained by applying the protocol substitution map to the declared interface type of the type alias declaration:

$$\begin{aligned}\texttt{Self.[Subscriber]Parent.[Provider]Entity} \otimes \Sigma_{\texttt{[T.Element: Subscriber]}} &\\ &= \texttt{T.[Sequence]Element.[Subscriber]Parent.[Provider]Entity}\end{aligned}$$

In this example, the underlying type of the protocol type alias was a type parameter, but it can also be a concrete type that recursively contains type parameters, too.

References to associated type declarations and protocol type aliases are resolved by the same rule: we apply a protocol substitution map to the declared interface type of the resolved type declaration. We assumed the base type is a type parameter, so we form a protocol substitution map from an abstract conformance. We'll see shortly that when the base type is concrete, we can still reference associated type declarations and protocol type aliases, and we form a protocol substitution map from a concrete conformance instead.

Structural resolution stage. Before we talk about type parameter bases subject to superclass or concrete type requirements, let's take a moment to describe the structural resolution stage. In the structural resolution stage, we don't have a generic signature, so we cannot perform generic signature queries or qualified lookup. Instead, a member type representation with a type parameter base always resolves to an unbound dependent member type formed from the base type and identifier. First, we need an example where structural resolution actually takes place. We can take the `Provider` protocol declared prior, and write a function with a trailing `where` clause:

```
func f<T: Provider>(_: T) where T.Entity: Equatable {...}
```

The **generic signature request** resolves the type representation `T.Entity` in the structural resolution stage, to obtain the unbound dependent member type `T.Entity`. We have two user-written requirements, from which we build our generic signature:

{[`T: Provider`], [`T.Entity: Equatable`]}

Requirement minimization rewrites the second requirement into one that contains the bound dependent member type `T.[Provider]Entity`, and we get the following generic signature:

```
<T where T: Provider, T.[Provider]Entity: Equatable>
```

At this point, `f()` has a generic signature, so type representations appearing inside the function can be resolved in the interface resolution stage. The rewriting of requirements that use unbound dependent member types into requirements that use bound dependent member type swill be completely justified in Section 11.4.

We discussed bound and unbound dependent member types in Section 5.3. An unbound dependent member type from the structural resolution stage can be converted into a bound dependent member type by first checking the `isValidTypeParameter`() generic signature query, followed by `getReducedType`(). The first check is needed since an unbound dependent member type, being a syntactic construct, might not name a valid member type at all. However, the usual situation is that the entire type representation is resolved again in the interface resolution stage, at which point an invalid type parameter is resolved to an error type and a diagnostic is emitted. In particular, type representations in the trailing `where` clause of each declaration are re-visited by the **type-check source file request**, which walks all top-level declarations in source order and emits further diagnostics.

Suppose now we change `f()`, to add the invalid requirement [`T.Foo: Equatable`]:

```
func f<T: Provider>(_: T)
    where T.Entity: Equatable, T.Foo: Equatable {...}
```

The invalid requirement will be dropped by requirement minimization, and the **generic signature request** does not emit any diagnostics. Instead, the invalid member type representation `T.Foo` will be diagnosed by **type-check source file request**, because qualified lookup would fail to find a member type named `Foo` in `Provider` when we revisit the `where` clause in interface resolution stage. We will resume the discussion of how invalid requirements are diagnosed in Section 11.3.

Next, we look at this function `g()`, which references our previously-seen protocol type alias `Content` from its trailing `where` clause.

```
func g<T: Sequence>(_: T)
    where T.Element: Subscriber, T.Element.Content: Equatable {...}
```

Here, we form the requirement [`T.Element.Content: Equatable`] whose subject type is the unbound dependent member type `T.Element.Content`. Indeed, this is not a type alias type at all. As we will learn in Section 18.4, protocol type aliases introduce rewrite rules, and the computation of reduced types replaces the unbound dependent member type with the underlying type of the protocol type alias. In this case, we get a generic signature with a requirement having quite the long subject type,

[`T.[Sequence]Element.[Subscriber]Parent.[Provider]Entity: Equatable`].

Type aliases can also appear in protocol extensions. However, such aliases cannot be referenced from positions resolved in the structural resolution stage, and in particular, trailing `where` clauses. We will see later that type aliases from protocol extensions do not define rewrite rules, so the reduction described above cannot be performed. This situation is detected after the fact, when the type representation is revisited in the interface resolution stage:

```
extension Provider {
  typealias Object = Entity
}

// error: 'Object' was defined in extension of protocol 'Provider'
// and cannot be referenced from a 'where' clause
struct G<T: Provider> where T.Object: Equatable {}
```

Remaining cases. We now wrap up the case where we have a type parameter base, the type parameter is subject to a concrete same-type or superclass requirement, and the resolved type declaration is a member of this concrete type. We compute the resolved type from this type declaration by proceeding as if the base type was this concrete type or superclass bound instead of the type parameter spelled by the user. Once again, various limitations surface if this type representation is also resolved in structural resolution stage. The "concrete contraction" pass allows certain cases to work (Section 21.3).

Concrete base. Now, consider a member type representation where the base resolves to something that's not a type parameter. The only interesting possibilities are when the base type is a nominal type or dynamic `Self` type, because function types and such do not have member types. (The dynamic `Self` type is simply unwrapped to its underlying nominal type, allowing member types of the innermost class declaration to be referenced by "`Self.Foo`", which is perhaps slightly more explicit than the almost equivalent "`Foo`".)

Unlike the type parameter case, here we immediately perform a qualified lookup into the base type without consulting the generic signature of the current context. When both the base and the member type declaration are non-generic nominal types, and the member type is a direct member of the base, the resolved type is just the declared interface type of the member. The member might also be found in a superclass, and not as a direct member of the base. We saw an almost identical example in the previous section, where the reference to `Inner` was instead an identifier type representation inside the body of `Derived`:

```
class Base<T> {
  struct Inner {}
}

class Derived: Base<Int> {}

// Interface type of 'x' is Base<Int>.Inner
var x: Derived.Inner = ...
```

The superclass is generic, so we apply the superclass substitution map to the declared interface type of `Inner` to get the resolved type for `Inner` as a member of `Derived`:

$$\texttt{Base<T>.Inner} \otimes \{\texttt{T} \mapsto \texttt{Int}\} = \texttt{Base<Int>.Inner}$$

Next, suppose we wish to reference `Inner` directly, as a member of `Base`. The `Base` class is generic, and we haven't seen how generic arguments are applied when resolving a type representation yet, but for now assume we know how to resolve the base type here:

```
var y: Base<String>.Inner = ...
```

Given the generic nominal type `Base<String>`, we resolve the member type representation written above by applying the context substitution map of `Base<String>` to the declared interface type of `Inner`, which we can do because `Inner` has the same generic signature as `Base`:

$$\texttt{Base<T>.Inner} \otimes \{\texttt{T} \mapsto \texttt{String}\} = \texttt{Base<String>.Inner}$$

The substitutions above are trivial in a sense, because the referenced member is a nominal type declaration, and to compute the resolved type, we simply transfer over the generic arguments from the base type. However, when the member is a type alias declaration, we can actually encode an arbitrary substitution. Each choice of base type defines a possible substitution map, which is then applied to the underlying type of the type alias, which can be any valid interface type for its generic signature. The reader may recall that when substitution maps were introduced in Chapter 6, one of the motivating examples was Example 6.3, showing substitutions performed when resolving member type representations with type alias members. We're going to look at a few interesting examples of type alias members now.

The type alias might be declared in a constrained extension of the base. If the constrained extension introduces new conformance requirements, the underlying type of the type alias declaration may reference new type parameters not present in the base type's generic signature. Recall that `Optional` type has a single generic parameter `Wrapped`:

```
extension Optional where Wrapped: Sequence {
  typealias OptionalElement = Optional<Wrapped.Element>
}

// Interface type of ‘x’ is ‘Optional<Int>’
var x: Optional<Array<Int>>.OptionalElement = ...
```

The context substitution map of `Optional<Array<Int>>` is $\{\texttt{Wrapped} \mapsto \texttt{Array<Int>}\}$, which is not a valid substitution map for the constrained extension's generic signature, because there's no conformance for the requirement `[Wrapped: Sequence]`. Applying this substitution map to the underlying type of our type alias would signal substitution failure and return the error type, because the substituted type for the type parameter `Wrapped.[Sequence]Element` cannot be resolved without this conformance:

$$\texttt{Optional<Wrapped.[Sequence]Element>} \otimes \{\texttt{Wrapped} \mapsto \texttt{Array<Int>}\} = \texttt{<<error type>>}$$

Instead we build the context substitution map of our base type, but using the generic signature of our constrained extension. This uses global conformance lookup to resolve the conformance:

$$\Sigma := \left\{ \begin{array}{l} \texttt{Wrapped} \mapsto \texttt{Array<Int>}; \\ [\texttt{Wrapped: Sequence}] \mapsto [\texttt{Array<Int>: Sequence}] \end{array} \right\}$$

Now, applying Σ to the underlying type of our type alias outputs the expected result by projecting the type witness from the conformance:

$$\texttt{Optional<Wrapped.[Sequence]Element>} \otimes \Sigma = \texttt{Optional<Int>}$$

If instead we had a `Wrapped` type that does not conform to `Sequence`, for example as in `Optional<Float>.OptionalElement`, we again get a substitution failure so the resolved type becomes an error type. We will see in the next section how such type representations are diagnosed.

To complete the discussion of a member type representation with a concrete base, we finally consider the possibility that the named member is an associated type or protocol type alias declared in some conformed protocol. When resolving a protocol member with a type parameter base, we used the protocol substitution map constructed from the abstract conformance of the type parameter to the protocol. With a concrete base, we do the same thing, except we form the protocol substitution map from a *concrete* conformance.

In the below, `Tomato` conforms to `Plant`, so we can access the protocol type alias `Food` as a member of `Tomato`:

```
struct Ketchup {}
struct Pasta<Flavor> {}

protocol Plant {
  associatedtype Sauce
  typealias Food = Pasta<Sauce>
  consuming func process() -> Sauce
}

struct Tomato: Plant {
  consuming func process() -> Ketchup {...}
}

// Interface type of ‘x’ is canonically equal to ‘Ketchup’
var x: Tomato.Sauce = ...

// Interface type of ‘y’ is canonically equal to ‘Pasta<Ketchup>’
var y: Tomato.Food = ...
```

When we resolve the interface type of “`x`” and then “`y`”, we see that in both cases the resolved type declaration is a member of `Plant`, so we build the protocol substitution map $\Sigma_{[\texttt{Tomato: Plant}]}$ from the normal conformance [`Tomato: Plant`]:

$$\Sigma_{[\texttt{Tomato: Plant}]} := \{\texttt{Self} \mapsto \texttt{Tomato};\ [\texttt{Self: Plant}] \mapsto [\texttt{Tomato: Plant}]\}$$

To resolve the interface type of “`x`”, we apply $\Sigma_{[\texttt{Tomato: Plant}]}$ to `Self.[Plant]Sauce`, the declared interface type of `Sauce`. This is equivalent to projecting the type witness

for `Sauce` from this conformance, giving us the resolved type `Ketchup`:

$$\begin{aligned}&\texttt{Self.[Plant]Sauce} \otimes \Sigma_{\texttt{[Tomato: Plant]}}\\ &\quad= \pi(\texttt{[Plant]Sauce}) \otimes [\texttt{Self: Plant}] \otimes \Sigma_{\texttt{[Tomato: Plant]}}\\ &\quad= \pi(\texttt{[Plant]Sauce}) \otimes [\texttt{Tomato: Plant}]\\ &\quad= \texttt{Ketchup}\end{aligned}$$

To resolve the interface type of "`y`", where we have the type representation `Tomato.Food`, we apply $\Sigma_{\texttt{[Tomato: Plant]}}$ to `Pasta<Self.[Plant]Sauce>`, the underlying type of `Food`, which recursively transforms the type parameter contained therein:

$$\begin{aligned}&\texttt{Pasta<Self.[Plant]Sauce>} \otimes \Sigma_{\texttt{[Tomato: Plant]}}\\ &\quad= \texttt{Pasta<Ketchup>}\end{aligned}$$

Protocol base. This is really just a funny edge case. A protocol type alias is visible as a member type of the protocol type *itself* when its underlying type does not depend on the protocol `Self` type. The `Food` protocol type alias above cannot be referenced as a member of the protocol type `Plant`, because the underlying type of `Food` contains `Self`, and `Plant` is not a valid replacement type for `Self`; `Plant` does not conform to `Plant`! Type resolution must reject the member type representation "`Plant.Food`":

```
// error: cannot access type alias 'Food' from 'Plant'; use a
// concrete type or generic parameter base instead
var x: Plant.Food = ...
```

Similarly, an associated type member can *never* be referenced with a protocol base, like "`Plant.Sauce`"; its declared interface type *always* contains `Self`. However, if the underlying type of some protocol type alias does not contain `Self`, the type alias is just a shortcut for another globally-visible type that does not depend on the conformance. We allow the reference because no substitution needs to be performed:

```
protocol Pet {
  typealias Age = Int
}

// 'Pet.Age' is just another spelling for 'Int'
func celebratePetBirthday(_ age: Pet.Age) {}
```

Due to peculiarities of type substitution, protocol type aliases that are also generic are always considered to depend on `Self`, even if their underlying type does not reference `Self`, so they cannot be referenced with a protocol base. (In structural resolution stage, a generic type alias cannot be referenced with a type parameter base, either. Perhaps it is best not to stick generic type aliases inside protocols, at all.)

General principle. Let's say that H is the generic signature of the current context, and `T` is the resolved base type of our member type representation, obtained via a recursive call to type resolution. We perform a qualified lookup after considering the base type `T`:

- If `T` is a type parameter, we look through a list of nominal type declarations discovered by generic signature queries against `T` and H.
- If `T` is a nominal type, we look into the nominal type declaration of `T`.

If qualified lookup fails or finds more than one type declaration, we diagnose an error. Otherwise, let d be the resolved type declaration with declared interface type $\texttt{T}_d$ and generic signature G. We construct a substitution map $\Sigma \in \textsc{Sub}(G, H)$, then compute $\texttt{T}_d \otimes \Sigma$ to get the final resolved type. We build Σ from the base type `T` and information about the parent context of d. There are three cases to handle:

- In the **direct case**, d is a direct member of the nominal type declaration of `T`, and Σ is the context substitution map of `T`.
- In the **superclass case**, d is a member of a superclass of `T`, and Σ is the superclass substitution map formed from `T` and the parent class of d.
- In the **protocol case**, d is a member of some protocol `P`, either via conformance (if `T` is concrete) or protocol inheritance (if `T` is a type parameter), and Σ is the protocol substitution map $\Sigma_{[\texttt{T}:\ \texttt{P}]}$ for the conformance of `T` to `P`, found via global conformance lookup.

If the base type `T` is a type parameter subject to a concrete same-type requirement or a superclass requirement, we replace `T` with the corresponding concrete type obtained by a generic signature query against H before proceeding to compute Σ above.

In all three cases above, d might be defined in a constrained extension that imposes further conformance requirements. When building Σ, we resolve any of these additional conformances via global conformance lookup (Section 6.3).

This is called a *context substitution map for a declaration context*. This concept generalizes the context substitution map of a type from Section 6.1, which was an inherent property of a type, without reference to a declaration context. If `T` is a nominal type and d is a direct member of the nominal type declaration of `T`, the context substitution map of `T` for the parent context of d is simply the context substitution map of `T`.

The context substitution map of a type for a declaration context also arises when type checking member reference expressions, like "`foo.bar`", that appear in a function body. In this case, qualified lookup will find any member named `bar`, not just a type declaration, and the type of the expression is computed by applying the corresponding context substitution map to the interface type of `bar`.

History. In pre-evolution Swift, associated type declarations in protocols were declared with the `typealias` keyword, and protocol type aliases did not exist. Swift 2.2 added the distinct `associatedtype` keyword for declaring associated types, to make space for protocol type aliases in the language [82]. Protocol type aliases were then introduced as part of Swift 3 [83].

Caching the type declaration. Having computed the resolved type of an identifier or member type representation, we stash the resolved type declaration within our type representation, as a sort of cache. If the type representation is resolved again (perhaps once in the structural stage, and then again in the interface stage), we skip name lookup and proceed directly to computing the resolved type from the stored type declaration. The optimization was more profitable in the past, when type resolution actually had *three* stages. The third stage would resolve interface types to archetypes, but it has since been subsumed by the **map type into environment** operation on generic environments. We also pre-populate this cache when parsing textual SIL, by assigning a type declaration to certain type representations. Name lookup would otherwise not find these declarations, because of SIL syntax oddities that we're not going to discuss here.

9.3. Applying Generic Arguments

Identifier and member type representations may be equipped with generic arguments, where each generic argument is recursively another type representation:

```
Array<Int>
Dictionary<String, Array<Int>>
Big<Int>.Small<String>
```

To resolve a type representation with generic arguments, we begin by finding the resolved type declaration using name lookup, and then perform a few additional steps not present in the non-generic case. Let's say that d is the resolved type declaration, G is the generic signature of d, and G' is the generic signature of the parent context of d. If the parent generic signature G' is empty, then we're referencing a top-level generic declaration, like `Array<Int>` for example. (If G is *also* empty, then d is not actually generic at all; we will diagnose below.) Notice there are several new semantic checks here, each of which can diagnose errors and return an error type:

1. We check that d has a generic parameter list, and ensure we have the correct number of generic arguments.

2. We recursively resolve all generic argument type representations, to form an array of generic argument types.

3. We form a substitution map Σ for G from these generic argument types.
4. We check that Σ satisfies all explicit requirements of G.
5. We compute the substituted type $\mathtt{T}_d \otimes \Sigma$, where $\mathtt{T}_d$ is the declared interface type of d, to get the final resolved type.

Let's say that H is the generic signature of the declaration context where the type representation appears. The check in Step 1 consults the generic parameter list of d; this is a syntactic construct describing the innermost generic parameters of G we met in Section 4.1. This does not require knowledge of either G or H, so we perform this check in both interface resolution stage and structural resolution stage. Checking requirements in Step 4 requires knowledge of both G and H, and we only do it in the interface resolution stage. How this checking is done is a topic unto itself; we will turn our attention to it shortly.

We want our resolved type to be an interface type for H, so $\mathtt{T}_d \otimes \Sigma \in \textsc{Type}(H)$. Since $\mathtt{T}_d \in \textsc{Type}(G)$, we wish to construct a $\Sigma \in \textsc{Sub}(G, H)$. In the previous section, we dealt with the case where d does not introduce any new generic parameters or requirements, but may possibly be in a generic context. In this simple case, the resolved type was $\mathtt{T}_d \otimes \Sigma'$ where Σ' is the context substitution map for `T` with respect to the declaration context of d. In the general case, $\Sigma' \in \textsc{Sub}(G', H)$ and not $\textsc{Sub}(G, H)$, but the "difference" between the two is that a substitution map for G must also fix d's innermost generic parameters and witness any new conformance requirements; so we form $\Sigma \in \textsc{Sub}(G, H)$ by adding to Σ' the generic argument types from Step 2, and resolving their conformances.

That is all a fancy way to say, the recursive nesting of type declarations gives rise to a recursive nesting of type representations. Type resolution forms a substitution map at each nesting level by adding new generic arguments to the previous substitution map.

Example 9.1. Suppose we're resolving the interface type of "`x`" below:

```
struct Big<T> {
  struct Small<U, V> {}
}

struct From<X: Sequence> {
  var x: Big<X.Iterator>.Small<X.Element, Int> = ...
}
```

To illustrate the connection between nested type declarations and generic parameter depth, we're going to work with canonical types. We are resolving a type representation inside `From`, and indeed the generic arguments contain type parameters from the generic signature of `From`, canonically `<τ_0_0 where τ_0_0: Sequence>`.

We first resolve the base type representation `Big<X.Iterator>`. The resolved type declaration is `Big`. Since the parent context of `Base` has an empty generic signature, the generic parameter of `Big` has the canonical type `τ_0_0`. We form a substitution map $\{\texttt{τ\_0\_0} \mapsto \texttt{τ\_0\_0.Iterator}\}$ from our generic argument. Substitution gives us the resolved base type:

$$\texttt{Big<τ\_0\_0>} \otimes \{\texttt{τ\_0\_0} \mapsto \texttt{τ\_0\_0.Iterator}\} = \texttt{Big<τ\_0\_0.Iterator>}$$

Next, we resolve the member type declaration `Small` via qualified lookup into the base type. The generic parameters `U` and `V` of `Small` appear at depth 1, so they declare the generic parameter types `τ_1_0` and `τ_1_1`. We take the context substitution map of the base type, which we saw is $\{\texttt{τ\_0\_0} \mapsto \texttt{τ\_0\_0.Iterator}\}$, and insert our generic arguments as the replacement types for `τ_1_0` and `τ_1_1`:

$$\Sigma := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{τ\_0\_0.Iterator} \\ \texttt{τ\_1\_0} \mapsto \texttt{τ\_0\_0.Element} \\ \texttt{τ\_1\_1} \mapsto \texttt{Int} \end{array} \right\}$$

Substitution gives us the final resolved type:

$$\begin{aligned} &\texttt{Big<τ\_0\_0>.Small<τ\_1\_0, τ\_1\_1>} \otimes \Sigma \\ &\qquad = \texttt{Big<τ\_0\_0.Iterator>.Small<τ\_0\_0.Element, Int>} \end{aligned}$$

Note that the `τ_0_0` on the left-hand side is interpreted relative to the generic signature of the resolved type declaration, while `τ_0_0` on the right is relative to the generic signature of `From`.

Checking generic arguments. Returning to Step 4 from the beginning of the present section, we're given a substitution map $\Sigma \in \textsc{Sub}(G, H)$ and we must decide if Σ satisfies the requirements of G, the generic signature of the referenced type declaration. We apply Σ to each explicit requirement of G to get a series of *substituted requirements*. A substituted requirement is a statement about concrete types that is either true or false; we proceed to check each one by global conformance lookup, canonical type comparison, and so on, as will be described below. If any substituted requirements are unsatisfied, we diagnose an error. This checking of substituted requirements is a generally useful operation, used not only by type resolution; we discuss other applications at the end of the section.

Our generic argument types can contain type parameters of H (the generic signature of the current context), and checking a substituted requirement may raise questions about H. To avoid separately passing in H, we require that a substituted requirement's types are expressed in terms of *archetypes* instead (see Section 8.2 for a refresher). Thus, we begin by mapping Σ into the primary generic environment of H, and assume henceforth that $\Sigma \in \textsc{Sub}(G, [\![H]\!])$.

Definition 9.2. We denote by $\textsc{Req}(G)$ the set of all requirements whose left-hand and right-hand side types contain type parameters of G (all explicit and derived requirements of G are also elements of $\textsc{Req}(G)$, but there are many more). Similarly, let $\textsc{Req}(\llbracket H \rrbracket)$ denote the set of requirements written using the primary archetypes of H.

We then define *requirement substitution* as a new "overload" of $\otimes$:

$$\textsc{Req}(G) \otimes \textsc{Sub}(G, \llbracket H \rrbracket) \to \textsc{Req}(\llbracket H \rrbracket)$$

Requirement substitution must apply Σ to every type parameter appearing in a given requirement R by considering the requirement kind. In all of the below, `T` is the subject type of the requirement, so $\texttt{T} \in \textsc{Type}(G)$:

- For a **conformance requirement** `[T: P]`, we apply Σ to `T`. The protocol type `P` remains unchanged because it does not contain any type parameters:

  $$[\texttt{T: P}] \otimes \Sigma := [(\texttt{T} \otimes \Sigma)\texttt{: P}]$$

- For a **superclass requirement** `[T: C]`, we apply Σ to `T` as well as the superclass bound `C`, which might be a generic class type containing type parameters of G:

  $$[\texttt{T: C}] \otimes \Sigma := [(\texttt{T} \otimes \Sigma)\texttt{: } (\texttt{C} \otimes \Sigma)]$$

- For a **same-type requirement** `[T == U]`, we apply Σ to both sides; `U` is either a type parameter or a concrete type that may contain type parameters of G:

  $$[\texttt{T == U}] \otimes \Sigma := [(\texttt{T} \otimes \Sigma)\texttt{ == } (\texttt{U} \otimes \Sigma)]$$

- For a **layout requirement** `[T: AnyObject]`, we apply Σ to `T`. The right-hand side remains invariant under substitution:

  $$[\texttt{T: AnyObject}] \otimes \Sigma := [(\texttt{T} \otimes \Sigma)\texttt{: AnyObject}]$$

Having obtained a substituted requirement, we can check that it is satisfied. Before describing how this is done, we look at a few examples to help motivate what follows.

Example 9.3. Let's begin by looking at how type resolution proceeds to resolve the underlying type of `A` and `B`:

```
struct Concat<T: Sequence, U: Sequence> where T.Element == U.Element {}

// (1) all requirements satisfied
typealias A = Concat<String, Substring>

// (2) 'T.Element == U.Element' unsatisfied
typealias B = Concat<Array<Int>, Set<String>>
```

When resolving the underlying type of type alias `A`, we form this substitution map:

$$\Sigma_a := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{String} \\ \texttt{τ\_0\_1} \mapsto \texttt{Substring}; \\ \\ [\texttt{τ\_0\_0: Sequence}] \mapsto [\texttt{String: Sequence}] \\ [\texttt{τ\_0\_1: Sequence}] \mapsto [\texttt{Substring: Sequence}] \end{array} \right\}$$

We apply Σ_a to each explicit requirement in the generic signature of `Concat`:

$$[\texttt{τ\_0\_0: Sequence}] \otimes \Sigma_a = [\texttt{String: Sequence}]$$
$$[\texttt{τ\_0\_1: Sequence}] \otimes \Sigma_a = [\texttt{Substring: Sequence}]$$
$$[\texttt{τ\_0\_0.Element == τ\_0\_1.Element}] \otimes \Sigma_a = [\texttt{Character == Character}]$$

The first two substituted requirements claim their subject types conform to `Sequence`. We can check these by performing a global conformance lookup, which returns a valid concrete conformance in both cases, so we conclude both requirements are satisfied. The final substituted requirement is a statement that `Character` and `Character` are the same type, which also appears to be true. Thus, Σ_a satisfies the generic signature of `Concat`, and we successfully form the resolved type `Concat<String, Substring>`.

When resolving the underlying type of type alias `B`, we see it is invalid:

$$\Sigma_b := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{Array<Int>} \\ \texttt{τ\_0\_1} \mapsto \texttt{Set<String>}; \\ \\ [\texttt{τ\_0\_0: Sequence}] \mapsto [\texttt{Array<Int>: Sequence}] \\ [\texttt{τ\_0\_1: Sequence}] \mapsto [\texttt{Set<String>: Sequence}] \end{array} \right\}$$

The substitution map Σ_b does not satisfy the same-type requirement because the conformances of `Array<Int>` and `Set<String>` to `Sequence` have different type witnesses for `Element`:

$$[\texttt{τ\_0\_0.Element == τ\_0\_1.Element}] \otimes \Sigma_b = [\texttt{Int == String}]$$

We diagnose the failure and reject the declaration of `B`.

Example 9.4. Let's take `Concat` as above, but reference it with generic arguments from a generic context. Let G be the generic signature of `Concat`, and H the generic signature of `OuterGeneric`. We are resolving the interface type of "`x`":

```
struct OuterGeneric<C: Collection> {
  var x: Concat<C, C.SubSequence> = ...
}
```

We build $\Sigma \in \textsc{Sub}(G, [\![H]\!])$ by mapping our generic arguments into the primary generic environment of H:

$$\Sigma := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto [\![\texttt{C}]\!] \\ \texttt{τ\_0\_1} \mapsto [\![\texttt{C.SubSequence}]\!]; \\ \\ [\texttt{τ\_0\_0: Sequence}] \mapsto [[\![\texttt{C}]\!]\texttt{: Sequence}] \\ [\texttt{τ\_0\_1: Sequence}] \mapsto [[\![\texttt{C.SubSequence}]\!]\texttt{: Sequence}] \end{array} \right\}$$

Note that our original requirements contained type parameters of G, but now involve the primary archetypes of H after substitution:

$$[\texttt{τ\_0\_0: Sequence}] \otimes \Sigma = [[\![\texttt{C}]\!]\texttt{: Sequence}]$$
$$[\texttt{τ\_0\_1: Sequence}] \otimes \Sigma = [[\![\texttt{C.SubSequence}]\!]\texttt{: Sequence}]$$
$$[\texttt{τ\_0\_0.Element == τ\_0\_1.Element}] \otimes \Sigma = [[\![\texttt{C.Element}]\!] \texttt{ == } [\![\texttt{C.Element}]\!]]$$

We check the first two requirements by performing a global conformance lookup on each archetype. Recalling Section 8.1, this is implemented as a generic signature query against the archetype's generic signature. In fact, from the associated requirements [Self: Sequence] and [Self.SubSequence: Collection] of Collection, we can derive $H \vdash$ [C: Sequence] and $H \vdash$ [C.SubSequence: Sequence]. Global conformance lookup succeeds and outputs a pair of abstract conformances:

$$[\texttt{Sequence}] \otimes [\![\texttt{C}]\!] = [[\![\texttt{C}]\!]\texttt{: Sequence}]$$
$$[\texttt{Sequence}] \otimes [\![\texttt{C.SubSequence}]\!] = [[\![\texttt{C.SubSequence}]\!]\texttt{: Sequence}]$$

(In fact, we could have also checked that Σ satisfies a conformance requirement with a local conformance lookup into Σ itself.)

The third requirement is a same-type requirement, so we apply Σ to both sides; being dependent member types, we project a type witness from each abstract conformance:

$$\texttt{τ\_0\_0.Element} \otimes \Sigma = \pi(\texttt{[Sequence]Element}) \otimes [[\![\texttt{C}]\!]\texttt{: Sequence}]$$
$$\texttt{τ\_0\_1.Element} \otimes \Sigma = \pi(\texttt{[Sequence]Element}) \otimes [[\![\texttt{C.SubSequence}]\!]\texttt{: Sequence}]$$

The two type witness projections construct the dependent member types τ_0_0.Element and τ_0_0.SubSequence.Element, and map them into the generic environment of H. Since $H \vdash$ [τ_0_0.Element == τ_0_0.SubSequence.Element], both map to the same reduced type in H, so they also define the same archetype:

$$\pi(\texttt{[Sequence]Element}) \otimes [[\![\texttt{C}]\!]\texttt{: Sequence}] = [\![\texttt{C.Element}]\!]$$
$$\pi(\texttt{[Sequence]Element}) \otimes [[\![\texttt{C.SubSequence}]\!]\texttt{: Sequence}] = [\![\texttt{C.Element}]\!]$$

Both sides of our substituted same-type requirement are canonically equal, so we can conclude that all requirements of G are satisfied, and our type representation is valid.

Algorithm 9A (Check if requirement is satisfied)**.** Takes a substituted requirement $R \in \textsc{Req}(\llbracket H \rrbracket)$ as input, where the generic signature H is not given explicitly; R may contain primary archetypes of H, but not type parameters. Returns true if R is satisfied, false otherwise. In the below, `T` is the concrete subject type of R, so $\texttt{T} \in \textsc{Type}(\llbracket H \rrbracket)$. We handle each requirement kind as follows:

- For a **conformance requirement** [`T: P`], we perform the global conformance lookup $\texttt{P} \otimes \texttt{T}$. There are three possible outcomes:
  1. If we get an abstract conformance, it must be that `T` is an archetype of H whose type parameter conforms to `P`. Return true.
  2. If we get a concrete conformance, it might be conditional (Section 10.4). These conditional requirements are also substituted requirements of $\textsc{Req}(\llbracket H \rrbracket)$, and we check them by recursively invoking this algorithm. If all conditional requirements are satisfied (or if there aren't any), return true.
  3. If we get an invalid conformance, or if the conditional requirement check failed above, return false.
- For a **superclass requirement** [`T: C`], we proceed as follows:
  1. If `T` is a class type canonically equal to `C`, return true.
  2. If `T` and `C` are two distinct generic class types for the same class declaration, return false.
  3. If `T` does not have a superclass type (Chapter 15), return false.
  4. Otherwise, let $\texttt{T}'$ be the superclass type of `T`. Recursively apply the algorithm to the superclass requirement [$\texttt{T}'$`: C`].
- For a **layout requirement** [`T: AnyObject`], we check if `T` is a class type, an archetype satisfying the `AnyObject` layout constraint, or an `@objc` existential, and if so, we return true. Otherwise, we return false. (We'll discuss representation of existentials in Chapter 14.)
- For a **same-type requirement** [`T == U`], we check if `T` and `U` are canonically equal.

Contextually-generic declarations. A type declaration with a trailing `where` clause but no generic parameter list was called a contextually-generic declaration in Section 4.2. A non-generic declaration inside a constrained extension is conceptually similar; we'll meet constrained extensions in Section 10.3. In both cases, the generic signature G of the referenced declaration and the generic signature of the parent context G' share the same generic parameters, but G has additional requirements not present in G'. While there are no generic arguments to apply, we still proceed to check that Σ satisfies the requirements of G.

Example 9.5. The `Inner` type below demonstrates the first case:

```
struct Outer<T: Sequence> {
  struct Inner where T.Element == Int {}
}

// all requirements are satisfied
typealias A = Outer<Array<Int>>.Inner

// 'T.Element == Int' is unsatisfied
typealias B = Outer<Array<String>>.Inner
```

The second type alias `B` is ultimately invalid. While the base `Outer<Array<String>>` resolves without error, the member type representation fails because this substituted requirement is unsatisfied, with Σ as the context substitution map of the base type:

$$[\texttt{τ\_0\_0.Element == Int}] \otimes \Sigma = [\texttt{String == Int}]$$

Example 9.6. There is one more detail left to discuss. Again using our `Concat` type, we now consider the following type alias:

```
// (3) 'T: Sequence' unsatisfied;
// 'T.Element == U.Element' substitution failure
typealias C = Concat<Float, Set<Int>>
```

We get the following substitution map; note that it contains an invalid conformance:

$$\Sigma_c := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{Float} \\ \texttt{τ\_0\_1} \mapsto \texttt{Set<Int>}; \\ \text{(invalid)} \\ [\texttt{τ\_0\_1: Sequence}] \mapsto [\texttt{Set<Int>: Sequence}] \end{array} \right\}$$

We apply this substitution map to each requirement of our generic signature:

$$[\texttt{τ\_0\_0: Sequence}] \otimes \Sigma_a = [\texttt{Float: Sequence}]$$
$$[\texttt{τ\_0\_1: Sequence}] \otimes \Sigma_a = [\texttt{Set<Int>: Sequence}]$$
$$[\texttt{τ\_0\_0.Element == τ\_0\_1.Element}] \otimes \Sigma_a = [\texttt{<<error type>> == Int}]$$

The first conformance requirement is unsatisfied and will be diagnosed. The same-type requirement contains error types after substitution, and does not need to be diagnosed; in fact, its failure is a consequence of the first conformance requirement being unsatisfied.

After applying our substitution map, but before checking the substituted requirement via the above algorithm, we look for error types. The presence of error types in the substituted requirement is indicative of one of several problems:

1. One of the generic argument types contained an error type, meaning an error was diagnosed earlier by type resolution.
2. The original type was a dependent member type, and the substituted base type does not conform to the member type's protocol. This means that an earlier conformance requirement was unsatisfied, and hence already diagnosed.
3. The declaration of a normal conformance may itself contain error types for any invalid or missing type witnesses, in which case projecting a type witness may output an error type; again, we will have diagnosed an error earlier, when checking the conformance.

In all cases, a diagnostic was already emitted, thus the requirement itself need not be diagnosed. We called this a *substitution failure* in Chapter 6.

Algorithm 9B (Check if substitution map satisfies requirements)**.** Takes two inputs:

1. A substitution map $\Sigma \in \textsc{Sub}(G, [\![H]\!])$.
2. Some list of elements of $\textsc{Req}(G)$. (When checking the generic arguments of a type representation, these are the explicit requirements of the generic signature G.)

As output, returns an *unsatisfied* list, and a *failed* list. If both output lists are empty, all input requirements are satisfied by Σ.

1. (Setup) Initialize the two output lists, initially empty.
2. (Done) If the input list is empty, return.
3. (Next) Remove the next requirement R from the input list and compute $R \otimes \Sigma$.
4. (Failed) If $R \otimes \Sigma$ contains error types, move R to the failed list.
5. (Check) Move R to the unsatisfied list if Algorithm 9A reports that $R \otimes \Sigma$ is unsatisfied.
6. (Loop) Go back to Step 2.

If any requirements appear in the unsatisfied list, type resolution diagnoses a series of errors at the source location of the generic type representation, one for each unsatisfied requirement. Requirements on the failed list are dropped, because another diagnostic will have been emitted, as explained previously. If at least one requirement was unsatisfied or failed, the resolved type becomes the error type; this enforces the invariant that clients of type resolution will not encounter any types that do not satisfy generic requirements—with the important exception of the error type itself!

There's more. Other places where we use Algorithm 9B:

1. When checking the declaration of a normal conformance $[\texttt{T}_d\texttt{: P}]$ where $\texttt{T}_d$ is the declared interface type of some nominal type declaration d, we must decide if the given set of type witnesses satisfy the associated requirements of `P`. In other words, we take the protocol substitution map $\Sigma_{[\texttt{T: P}]}$, and apply it to each associated requirement of `P` (Section 5.1).

2. When a concrete type `T` conforms to a protocol `P` via a conditional conformance, we check if the context substitution map of `T` satisfies the conditional requirements of `[T: P]`. We will describe this in Section 10.4.

3. The conditional requirements of a conditional conformance are also computed via the same algorithm when the declaration of the conformance is type checked. We ask which requirements in the generic signature of the constrained extension are *not* satisfied by the generic signature of the extended type.

4. Checking if a subclass method is a well-formed override of a superclass method asks whether the generic signature of the subclass method satisfies each requirement of the generic signature of the superclass method (Section 15.2).

There are also two related problems which follow different code paths but reason about requirements in the same way as above:

1. The expression type checker translates generic requirements to constraints when type checking a reference to a generic function; these constraints are then solved by the constraint solver and a substitution map is formed for the call. This is entirely analogous to what happens in type resolution when referencing a generic type declaration.

2. Requirement inference is the step in building a new generic signature where we *add* requirements to ensure that certain substituted requirements will be satisfied (Section 11.1).

9.4. Unbound Generic Types

Introduced in Section 3.3, the *unbound generic type* represents a reference to a generic type declaration without generic arguments, and a *placeholder type* represents a specific missing generic argument.

Unbound generic types and placeholder types only appear when permitted by the type resolution context. These contexts are those syntactic positions where missing generic arguments can be filled in by some other mechanism, for example by using the expression type checker to infer the type of an expression.

The following type resolution contexts allow unbound generic types or placeholder types, with each context handling them in some specific manner:

1. The type annotation of a variable declaration with an initial value expression may have either unbound generic types or placeholder types in various nested positions:

   ```
   let callback: () -> (Array) = { return [1, 2, 3] }
   let dictionary: Dictionary<Int, _> = { 0: "zero" }
   ```

   The missing generic arguments are inferred from the initial value expression when computing the variable declaration's interface type.

2. Types written in expressions, such as metatype values (`GenericType.self`), the callees of constructor invocations (`GenericType(...)`), and target types of casts (`x as GenericType`). In all cases, the generic arguments are inferred from the surrounding expression.

3. The extended type of an extension is typically written as an unbound generic type:

   ```
   struct GenericType<T: Sequence> { ... }
   extension GenericType { ... }
   ```

   We will see in Section 10.3 that writing generic arguments for the extended type also has meaning, as a shorthand for a series of same-type requirements. Placeholder types cannot appear here.

4. The underlying type of a type alias may contain an unbound generic type (but not a placeholder type). This is a shorthand for a generic type alias that forwards its generic arguments, so the below are equivalent, given `GenericType` as above:

   ```
   typealias GenericAlias = GenericType
   typealias GenericAlias<T: Sequence> = GenericType<T>
   ```

   If the underlying type of a type alias is an unbound generic type, it cannot have its own generic parameter list, and vice versa. Additionally, only the outermost type representation can resolve to an unbound generic type. Both of the following are thus invalid:

   ```
   typealias WrongGenericAlias<T> = GenericType
   typealias WrongFunctionAlias = () -> (GenericType)
   ```

Notice how the first two contexts allow any type representation to resolve to an unbound generic type even if it appears in a nested position, while in the last two, only the topmost type representation can resolve to an unbound generic type.

A limitation. An unbound generic type can refer to either a nominal type declaration, or a type alias declaration. However, an unbound generic type referring to a type alias declaration cannot be the parent type of a nominal type. It is an error to access a member type of a generic type alias without providing generic arguments, even when an unbound generic type referencing a nominal type declaration can appear in the same position:

```
struct GenericType<T> {
  struct Nested {
    let value: T
  }
}

typealias GenericAlias = GenericType

_ = GenericType.Nested(value: 123)        // OK
_ = GenericAlias<Int>.Nested(value: 123)  // OK

_ = GenericAlias.Nested(value: 123)       // error
_ = GenericAlias<Int>.Nested(value: 123)  // OK
```

A future direction. If we think of type representations as syntactic, and types as semantic, unbound generic types occupy a weird point in between. They are produced by type resolution and refer to type declarations, but they do not actually survive type checking. When all is said and done, generic arguments in expressions will either be resolved or replaced with error types. We could eliminate unbound generic types from the implementation by reworking type resolution to take a callback that fills in missing generic arguments. Each context where an unbound generic type or placeholder type can appear would supply its own callback. For example, the expression type checker would provide returns a fresh type variable type. This callback model is partially implemented today, but all existing callbacks are trivial; the callback's presence simply communicates to type resolution that an unbound generic type is permitted in this position.

9.5. Source Code Reference

Key source files:

- include/swift/AST/TypeResolutionStage.h
- lib/Sema/TypeCheckType.h

- lib/Sema/TypeCheckType.cpp

A client of type resolution must first create a `TypeResolutionOptions`, and then a `TypeResolution` instance from that. The latter defines a `resolveType()` method to resolve a `TypeRepr *` to a `Type`. We start with `TypeResolutionOptions` and its constituent parts, `TypeResolverContext` and `TypeResolutionFlags`.

`TypeResolutionOptions` *class*

Encodes the semantic behavior of type resolution, consisting of a base context, a current context, and flags. The base context and context are initially identical. When type resolution recurses on a nested type representation, it preserves the base context, possibly changes the current context, and clears the `Direct` flag. Thus, the base context encodes the overall syntactic position of the type representation, while the context encodes the role of the current type representation.

- `TypeResolutionOptions(TypeResolutionContext)` creates a new instance with the given base context, and no flags.
- `getBaseContext()` returns the base `TypeResolverContext`.
- `getContext()` returns the current `TypeResolverContext`.
- `getFlags()` returns the `TypeResolutionFlags`.

`TypeResolverContext` *enum class*

The type resolution context, which encodes the position of a type representation:

- `TypeResolverContext::None`: no special type handling is required.
- `TypeResolverContext::GenericArgument`: generic arguments of a bound generic type.
- `TypeResolverContext::ProtocolGenericArgument`: generic arguments of a parameterized protocol type.
- `TypeResolverContext::TupleElement`: elements of a tuple element type.
- `TypeResolverContext::AbstractFunctionDecl`: the base context of a function declaration's parameter list.
- `TypeResolverContext::SubscriptDecl`: the base context of a subscript declaration's parameter list.
- `TypeResolverContext::ClosureExpr`: the base context of a closure expression's parameter list.

- `TypeResolverContext::FunctionInput`: the type of a parameter in a function type or declaration.
- `TypeResolverContext::VariadicFunctionInput`: the type of a variadic parameter in a function type or declaration. This differs from the above in that nested function types become `@escaping` by default.
- `TypeResolverContext::InoutFunctionInput`: the type of an `inout` parameter in a function type or declaration.
- `TypeResolverContext::FunctionResult`: the result type of a function type or declaration.
- `TypeResolverContext::PatternBindingDecl`: the type of a pattern in a pattern binding declaration.
- `TypeResolverContext::ForEachStmt`: the type of a variable in a `for` statement.
- `TypeResolverContext::ExtensionBinding`: the extended type of an extension declaration.
- `TypeResolverContext::InExpression`: a type appearing in expression context.
- `TypeResolverContext::ExplicitCastExpr`: the target type of an `as` cast expression.
- `TypeResolverContext::EnumElementDecl`: the base context of an enum element parameter list.
- `TypeResolverContext::EnumPatternPayload`: the payload type of an enum element pattern. Tweaks the behavior of tuple element labels.
- `TypeResolverContext::TypeAliasDecl`: the underlying type of a non-generic type alias.
- `TypeResolverContext::GenericTypeAliasDecl`: the underlying type of a generic type alias.
- `TypeResolverContext::ExistentialConstraint`: the constraint type of an existential type (Chapter 14);
- `TypeResolverContext::GenericRequirement`: the constraint type of a conformance requirement in a `where` clause.
- `TypeResolverContext::SameTypeRequirement`: the subject type or constraint type of a same-type requirement in a `where` clause.

- `TypeResolverContext::ProtocolMetatypeBase`: the instance type of a protocol metatype, like `P.Protocol`.
- `TypeResolverContext::MetatypeBase`: the base type of a concrete metatype, like `T.Type`.
- `TypeResolverContext::ImmediateOptionalTypeArgument`: the argument of an optional type, like `T?`. This just tailors some diagnostics.
- `TypeResolverContext::EditorPlaceholderExpr`: the type of an editor placeholder.
- `TypeResolverContext::Inherited`: the inheritance clause of a concrete type.
- `TypeResolverContext::GenericParameterInherited`: the inheritance clause of a generic parameter.
- `TypeResolverContext::AssociatedTypeInherited`: the inheritance clause of an associated type.
- `TypeResolverContext::CustomAttr`: the type of a custom attribute referencing a property wrapper or result builder.

`TypeResolutionFlags` *enum class*

A set of flags to control various esoteric behaviors:

- `TypeResolutionFlags::AllowUnspecifiedTypes`: allows type resolution to succeed if no type representation was actually provided. Used when resolving type annotations which can be omitted, such as the types of variable declarations and the types of closure parameters.
- `TypeResolutionFlags::SILType`: used when resolving a position where lowered SIL types, such as SIL function types, can appear.
- `TypeResolutionFlags::SILMode`: used when resolving a position where ordinary AST types appear, but in textual SIL, which enables special syntax, for example `@dynamic_self` for the dynamic `Self` type.
- `TypeResolutionFlags::FromNonInferredPattern`: when resolving an explicitly-stated type annotation in a pattern binding, muffles warnings where the type of a variable binding is inferred as `()` or `Never`, which suggests a programmer mistake unless the type was explicitly stated.

- `TypeResolutionFlags::Direct`: set when resolving the top-level type representation, and cleared when type resolution recurses on a child. Allows certain type representations which cannot be nested inside other type representations, such as `inout T` for an `inout` parameter type.
- `TypeResolutionFlags::SilenceErrors`: disables diagnostics, simply returning an error type if type resolution fails. Used in pattern resolution when it is not known if a pattern refers to a type or some other member.
- `TypeResolutionFlags::AllowModule`: allows an identifier type representation with a single component to resolve to a `ModuleType`, which is usually prohibited.
- `TypeResolutionFlags::AllowUsableFromInline`: makes internal type declarations annotated with the `@usableFromInline` attribute visible, which is used in the implementation of the `@_specialize` attribute.
- `TypeResolutionFlags::DisallowOpaqueTypes`: prevents opaque parameter declarations from appearing in contexts where they would otherwise be allowed, such as the generic arguments of a parameterized protocol type `any P<some Q>`.
- `TypeResolutionFlags::Preconcurrency`: when resolving a reference to a type alias marked with the `@preconcurrency` attribute, strips off `@Sendable` and global actor attributes from function types appearing in the underlying type of the type alias.

`TypeResolution` *class*

After initializing options, a client must create an instance of this class by calling one of two factory methods, depending on the desired type resolution stage. Each method takes a `DeclContext`, `TypeResolutionOptions`, and a pair of callbacks for resolving placeholder types and unbound generic types:

- `forStructural()` creates a type resolution in structural resolution stage.
- `forInterface()` creates a type resolution in interface resolution stage.

The main thing you can do with an instance is, of course, resolve types:

- `resolveType()` takes a `TypeRepr *` and returns a `Type`.

Another way to use this class is to call a static utility method that creates a new type resolution in interface resolution stage, resolves a type representation, and maps it into a generic environment to get a contextual type. This is used by the expression type checker to resolve types appearing in expression context:

- `resolveContextualType()`

The actual implementation of `resolveType()` uses these getter methods:

- `getDeclContext()` returns the declaration context where this type representation appears.
- `getASTContext()` returns the global AST context singleton.
- `getStage()` returns the current `TypeResolutionStage`.
- `getOptions()` returns the current `TypeResolutionOptions`.
- `getGenericSignature()` returns the current `GenericSignature` in interface resolution stage.

`TypeResolutionStage` *enum class*

Encodes the type resolution stage:

- `TypeResolutionStage::Structural`
- `TypeResolutionStage::Interface`

`ResolveTypeRequest` *class*

The implementation of the `TypeResolution::resolveType()` entry point evaluates the `ResolveTypeRequest`. While this request is not cached, it uses the request evaluator infrastructure to detect and diagnose circular type resolution. The evaluation function destructures the given type representation by kind using a visitor class.

`TypeResolver` *class*

A recursive visitor, internal to the implementation of type resolution. The methods of this visitor implement type resolution for each kind of type representation.

Type Representations

Key source files:

- include/swift/AST/TypeRepr.h
- lib/Sema/TypeCheckType.cpp

`TypeRepr` *class*

Abstract base class for type representations. This class hierarchy resembles `TypeBase`, `Decl`, `ProtocolConformance`, and others.

Type representations store a source location and kind:

- `getLoc()`, `getSourceRange()` returns the source location and source range of this type representation.
- `getKind()` returns the `TypeReprKind`.

Each `TypeReprKind` corresponds to a subclass of `TypeRepr`. Instances of subclasses support safe downcasting via the `isa<>`, `cast<>` and `dyn_cast<>` template functions,

```
if (auto *indentRepr = dyn_cast<IdentTypeRepr>(typeRepr))
  ...

auto *funcRepr = cast<FunctionTypeRepr>(typeRepr);
...

assert(isa<ArrayTypeRepr>(typeRepr));
```

Identifier Type Representations

`DeclRefTypeRepr` *class*

Abstract base class for identifier and member type representations.

- `getNameRef()` returns the identifier.
- `getGenericArgs()` returns the array of generic argument type representations, if any were written.

Recall that type representations cache the type declaration found by name lookup to speed up the case where both type resolution stages resolve the same type representation:

- `getBoundDecl()` returns the cached type declaration.
- `getDeclContext()` for the first component only, returns the outer declaration context where unqualified lookup found the type declaration.
- `setValue()` records a cached type declaration and the declaration context from which it was found.

Note that `getDeclContext()` is not always the same as the declaration context of the type declaration, because the type declaration might be a member of a conformed protocol or superclass of the declaration context, having been reached indirectly. However, the declaration context is always one of the outer declaration contexts in which the type representation appears.

`IdentTypeRepr` *class*

Subclass of `DeclRefTypeRepr`. Represents something like `Int` or `Array<Int>`.

Unqualified Lookup

Key source files:

- lib/Sema/TypeCheckType.cpp
- lib/Sema/TypeCheckNameLookup.cpp

`resolveTopLevelIdentTypeComponent()` *function*

Resolves the first component of an identifier type representation. Performs an unqualified lookup, disambiguates results, handles the special case of a `Self` reference, applies generic arguments if they were provided, and diagnoses errors.

`TypeChecker::lookupUnqualifiedType()` *function*

Wrapper around unqualified lookup which only considers type declarations.

`getSelfTypeKind()` *function*

Determines if `Self` in the given context refers to the innermost nominal type declaration, the dynamic `Self` type of the innermost class declaration, or if it is invalid to utter `Self`.

`resolveTypeDecl()` *function*

Resolves an unqualified reference to a type declaration to a type. This handles the behavior where omitted generic arguments when referencing type within its own context are deduced to be the corresponding generic parameters. It also handles various edge-case behaviors where the referenced type declaration was a nested type of a different nominal type, such as a superclass or conformed protocol, and requires substitutions.

Member Type Representations

`MemberTypeRepr` *class*

Subclass of `DeclRefTypeRepr` representing a member type representation.

- `getBase()` returns the base type representation.

Qualified Lookup

Key source files:

- lib/Sema/TypeCheckType.cpp
- lib/Sema/TypeCheckNameLookup.cpp

`resolveNestedIdentTypeComponent()` *function*

Resolves subsequent components of an identifier type representation. Performs a qualified lookup, disambiguates results, applies generic arguments if they were provided, and diagnoses errors.

`TypeResolution::resolveDependentMemberType()` *function*

Implements the special behavior where a member type of a type parameter directly resolves to an unbound dependent member type in the structural resolution stage, or via name lookup to a bound dependent member type or type alias type in the interface resolution stage.

`TypeChecker::lookupMemberType()` *function*

Wrapper around qualified lookup which only considers type declarations, and resolves type witnesses of concrete types when it finds an associated type declaration with a concrete base type.

`TypeChecker::isUnsupportedMemberTypeAccess()` *function*

Determines if a member type access is invalid. This includes such oddities as accessing a member type of an unbound generic type referencing a type alias (Section 9.4), or accessing a dependent protocol type alias where the base type is the protocol itself (Section 9.2).

Applying Generic Arguments

- `include/swift/AST/Requirement.h`
- `lib/AST/Requirement.cpp`
- `lib/Sema/TypeCheckGeneric.cpp`

`applyGenericArguments()` *function*

Constructs a nominal type or type alias type, given a type declaration and a list of generic arguments. In the interface resolution stage, also checks that the generic arguments satisfy the requirements of the type declaration's generic signature.

`TypeResolution::applyUnboundGenericArguments()` *method*

Factor of `applyGenericArguments()` to build a substitution map from the base type and generic arguments and check requirements of a generic declaration.

`TypeResolution::checkContextualRequirements()` *method*

Factor of `applyGenericArguments()` to build the substitution map from the base type and check requirements of contextually-generic declarations.

`Requirement` *class*

See also Section 5.6. Requirement substitution:

- `subst()` applies a substitution map to this requirement, returning the substituted requirement.

Checking requirements:

- `checkRequirement()` answers if a single requirement is satisfied, implementing Algorithm 9A. Any conditional requirements are returned via a `SmallVector` supplied by the caller, who must then check these requirements recursively.
- `checkRequirements()` checks each requirement in an array; if any have their own conditional requirements, those are checked too. This implements Algorithm 9B for when the caller needs to check a condition without emitting diagnostics. For example, this is used by `checkConformance()` to check conditional requirements in Section 10.5.

`checkGenericArgumentsForDiagnostics()` *function*

Uses `Requirement::checkRequirement()` to implements a variant of Algorithm 9B that records additional information for diagnostics. This is used by type resolution and the conformance checker.

10. Extensions

Extensions add members to existing nominal type declarations. We refer to this nominal type declaration as the *extended type* of the extension. The extended type may have been declared in the same source file, another source file of the main module, or in some other module. Extensions themselves are declarations, but they are *not* value declarations in the sense of Chapter 4, meaning the extension itself cannot be referenced by name. Instead, the members of an extension are referenced as members of the extended type, visible to qualified name lookup.

Consider a module containing a pair of struct declarations, `Outer` and `Outer.Middle`:

```
public struct Outer<T> {
  public struct Middle<U> {}
}
```

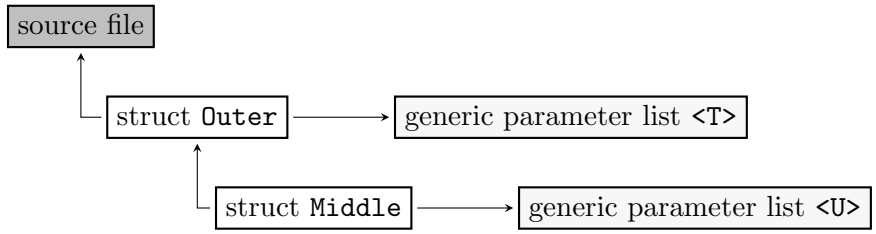
A second module can declare an extension of `Outer.Middle` to add a method named `foo()` and a nested type named `Outer.Middle.Inner`:

```
extension Outer.Middle {
  func foo() {}
  struct Inner<V> {}
}
```

If a third module subsequently imports both the first and second module, it will see the members `Outer.Middle.foo()` and `Outer.Middle.Inner` just as if they were defined inside `Outer.Middle` itself.

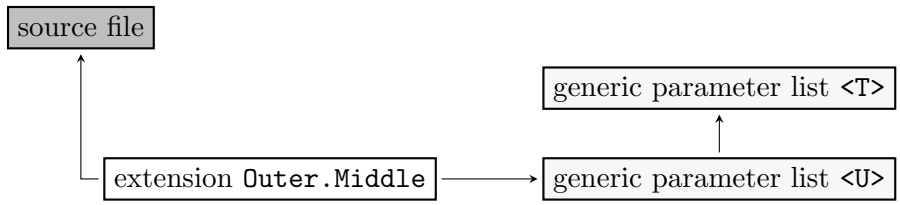
Extensions and generics. The generic parameters of the extended type are visible in the scope of the extension's body. Each extension has a generic signature, which describes the interface types of its members. The generic signature of an "unconstrained" extension is the same as that of the extended type. Extensions can impose additional requirements on their generic parameters via a `where` clause; this declares a *constrained extension* with its own generic signature (Section 10.3). An extension can also state a conformance to a protocol, which is represented as normal conformance visible to global conformance lookup. If the extension is unconstrained, this is essentially equivalent to stating a conformance on the extended type. If the extension is constrained, the conformance becomes a *conditional conformance* (Section 10.4).

Let's begin by taking a closer look at generic parameter lists of extensions, by way of our nested type `Outer.Middle` and its extension above. Recall how generic parameters of nominal types work from Section 4.1: their names are lexically scoped to the body of the type declaration, and each generic parameter uniquely identified by its depth and index. We can represent the declaration context nesting and generic parameter lists of our nominal type declaration `Outer.Middle` with a diagram like the following:



We create the fiction that each generic parameter in the scope of the extended type is also visible inside the extension by *cloning* generic parameter declarations. The cloned declarations have the same name, depth and index as the originals, but they are parented to the extension. This ensures that looking up a generic parameter inside an extension finds a generic parameter with the same depth and index as the one with the same name in the extended type. Since all generic parameter in a single generic parameter list have the same depth, an extension conceptually has multiple generic parameter lists, one for each level of depth (that is, the generic context nesting) of the extended type.

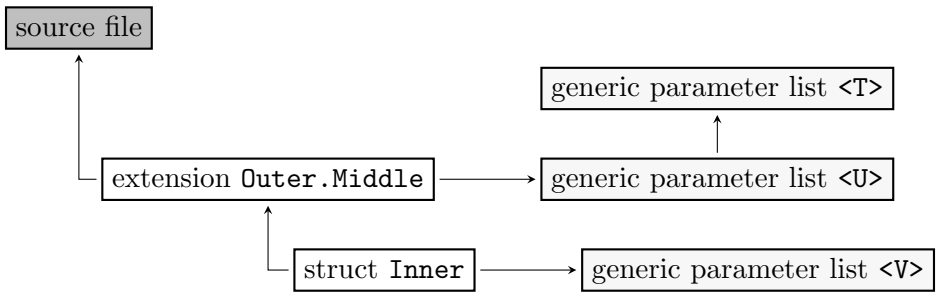
This is represented by linking the generic parameter lists together via an optional "outer list" pointer. The innermost generic parameter list is "the" generic parameter list of the extension, and the head of the list; it is cloned from the extended type. Its outer list pointer is cloned from the extended type's parent generic context, if any, and so on. The outermost generic parameter list has a null outer list pointer. In our extension of `Outer.Middle`, this looks like so:



Unqualified name lookup traverses the outer list pointer when searching for generic parameter declarations by name. Now, consider the nested type `Inner` from our example, which declares a generic parameter list of its own:

```
extension Outer.Middle {
  struct Inner<V> {}
}
```

When an extension declares a nested generic type, the depth of the nested type's generic parameters becomes one greater than the depth of the extension's innermost generic parameter list. Thus, the generic parameter `V` of `Inner` has depth 2 and index 0. All three generic parameter lists are visible inside the body of `Outer.Middle.Inner`:



The declared interface type of an extension is the declared interface type of the extended type; the self interface type of an extension is the self interface type of the extended type. The two concepts coincide except when the extended type is a protocol, in which case the declared interface type is the protocol type, whereas the self interface type is the protocol `Self` type.

Other behaviors. An extension member generally behaves as if were declared inside the extended type itself, with some differences:

- Members of *protocol extensions* are not requirements imposed on the concrete type in the way that members of a protocol declaration are; they actually have a body.

  A protocol extension member with the same name and interface type as a protocol requirement acts as a *default witness*, to be used if the conforming type does not provide its own witness for this requirement:

```
protocol P {
  func f()
}

extension P {
  func f() {
    print("default witness for P.f()")
  }
}

struct S: P {}

S().f()  // calls the default witness for P.f()
```

- Extensions of struct and class declarations cannot add stored properties, only computed properties:

```
struct S {
  var x: Int
}

extension S {
  var y: Int  // stored property: error
  var flippedX: Int { return -x }  // computed property: okay
}
```

  All stored properties must be initialized by all declared constructors; extensions being able to introduce stored properties hithero-unknown to the original module would break this invariant. Another reason is that the stored property layout of a struct or class is computed when the struct or class declaration is emitted, and there is no mechanism for extensions to alter this layout after the fact.

- Extensions cannot add new cases to enum declarations, for similar reasons as stored properties; doing so would complicate both the static exhaustiveness checking for `switch` statements and the in-memory layout computation of the enum.

- When the extended type is a class, the methods of the extension are implicitly `final`, and the extension's methods are not permitted to override methods from the superclass. Non-`final` methods declared inside a class are dispatched through a vtable of function pointers attached to the runtime metadata of the class, and there is no mechanism for extensions to add new entries or replace existing entries inherited from the superclass.

- The rules for nested types in extensions are the same as those for other nominal types (Section 6.4). An extension of a struct, enum or class may contain nested structs, enums and classes, while a protocol extension cannot just as a protocol cannot. Extensions themselves must be at the top level of a source file, but the extended type can be nested inside of another nominal type (or extension).

10.1. Extension Binding

The extended type of an extension is given as a type representation following the `extension` keyword. Extension members are made available to qualified lookup by the process of *extension binding*, which resolves the type representation of an extension to its extended type, and adds the extension's members to the extended type's name lookup table.

A complication is that the extended type of an extension can itself be declared inside of another extension. Extension binding cannot simply visit all extension declarations in a single pass in source order, because of ordering dependencies between extensions and nested types. Instead, multiple passes are performed; a failure to bind an extension is not a fatal condition, and instead failed extensions are revisited later after subsequent extensions are successfully bound. This process iterates until fixed point.

Algorithm 10A (Bind extensions)**.** Takes a list of all extensions in the main module as input, in any order.

1. Initialize the pending list, adding all extension declarations.
2. Initialize the delayed list to be empty.
3. Initialize the flag to be clear.
4. (Check) If the pending list is empty, go to Step 6.
5. (Resolve) Remove an extension from the pending list and attempt to resolve its extended type. If resolution succeeds, associate the extension with the resolved nominal type declaration and set the flag. If resolution fails, add the extension to the delayed list but do not emit any diagnostics and leave the flag unchanged. Go to Step 4.
6. (Retry) If the flag is set, move the entire contents of the delayed list to the pending list, clear the delayed list, clear the flag, and go to Step 4. Otherwise, return.

The worklist-driven extension binding algorithm was introduced in Swift 5. Older compiler releases attempted to bind extensions in a single pass, something that could either succeed or fail depending on declaration order. This incorrect behavior was one of the most frequently reported bugs of all time [84].

Invalid extensions. If extension binding fails to resolve the extended type of an extension, it simply remains on the delayed list without any diagnostics emitted. Invalid extensions are diagnosed later, when the **type-check source file request** visits all primary files and attempts to resolve the extended types of any extensions again.

Extension binding uses a more limited form of type resolution, because we only need to resolve the type representation to a *type declaration* and not a *type*. This type declaration must be a nominal type declaration, so the extended type is typically written as an *identifier* or *member* type representation (Sections 9.1 and 9.2). Extension binding runs early in the type checking process, immediately after parsing and import resolution. We cannot build generic signatures or check conformances in extension binding, because those requests assume that extension binding has already taken place, freely relying on qualified lookup to find members of arbitrary extensions.

In particular, extension binding cannot find declarations synthesized by the compiler, including type aliases created by associated type inference. Also, it cannot perform type substitution; this rules out extensions of generic type aliases whose underlying type is itself a type parameter.

If extension binding fails while the **type-check source file request** successfully resolve the extended type when it visits the extension later, we emit a special diagnostic, tailored by a couple of additional checks. If ordinary type resolution returned a type alias type `Foo` desugaring to a nominal type `Bar`, the type checker emits the "extension of type `Foo` must be declared as an extension of `Bar`" diagnostic. Even though we know what the extended type should be at this point, we must still diagnose an error; it is too late to bind the extension, because other name lookups may have already been performed, potentially missing out on finding members of this extension.

Example 10.1. An invalid extension of an inferred type alias:

```
protocol Animal {
  associatedtype FeedType
  func eat(_: FeedType)
}

struct Horse: Animal {
  func eat(_: Hay) {}
}

// error: extension of type 'Horse.FeedType' must be declared as an
// extension of 'Hay'
extension Horse.FeedType {...}
```

Extension binding fails to resolve `Horse.FeedType`, because this type alias was not stated explicitly and is inferred by the conformance checker from the witness `Horse.eat(_:)`. Therefore, the type alias does not exist at extension binding time, and we specifically do not trigger its synthesis.

Example 10.2. An invalid extension of a type alias with an underlying type that is a type parameter:

```
typealias G<T: Sequence> = T.Element

// error: extension of type 'G<Array<Int>>' must be declared as an
// extension of 'Int'
extension G<Array<Int>> {...}
```

Extension binding fails to resolve `G<Array<Int>>`, because this requires performing a type substitution that depends on the conformance [`Array: Sequence`]:

$$\texttt{T.Element} \otimes \{\texttt{T} \mapsto \texttt{Array<Int>};\ [\texttt{T: Sequence}] \mapsto [\texttt{Array<Int>: Sequence}]\}$$
$$= \texttt{Int}$$

The other case where extension binding fails but **type-check source file request** is able to resolve the extended type is when this type is not actually a nominal type. In this situation, the type checker emits the fallback "non-nominal type cannot be extended" diagnostic:

```
typealias Fn = () -> ()

// error: you wish
extension Fn {
  ...
}
```

The extension binding algorithm is quadratic in the worst case, where each pass over the pending list is only able to bind exactly one extension. However, only unrealistic code examples trigger this pathological behavior. Indeed, the first pass binds all extensions of types not nested inside of other extensions, and the second pass binds extensions of types nested inside extensions that were bound by the first pass, which covers the vast majority of cases in reasonable user programs.

Example 10.3. The following order of extensions requires four passes to bind; by adding more nested types, the number of iterations can be increased arbitrarily:

```
struct Outer {}

extension Outer.Middle.Inner {}  // bound in 3rd pass

extension Outer.Middle {  // bound in 2nd pass
  struct Inner {}
}

extension Outer {  // bound in 1st pass
  struct Middle {}
}

extension DoesNotExist {}  // remains on delayed list
```

In the first pass, the extensions of `Outer.Middle.Inner` and `Outer.Middle` both fail, but we do successfully bind the extension of `Outer`. The second pass once again fails to bind the extension of `Outer.Middle.Inner`, but it does bind the extension of `Outer.Middle`. Finally, the third pass binds the extension of `Outer.Middle.Inner`. Notice how in each of the first three passes, we are able to bind at least one extension. The invalid extension of `DoesNotExist` remains on the delayed list after each of the first three passes. Since the fourth pass does not make any forward progress, the algorithm stops. Later we end up in the diagnostic path, which resolves `DoesNotExist` via ordinary type resolution, surfacing the "unknown type" error.

Local types. Because extensions can only appear at the top level of a source file, the extended type must ultimately be visible from the top level. This allows extensions of types nested inside other top-level types, but precludes extensions of local types nested inside of functions or other local contexts, because there is ultimately no way to name a local type from the top level of a source file. (As a curious consequence, local types cannot conditionally conform to protocols, since the only way to declare a conditional conformance is with an extension!)

10.2. Direct Lookup

Now we're going to take closer look at *direct lookup*, the primitive operation that looks for a value declaration with the given name inside of a nominal type and its extensions. This was first introduced in Section 2.1 as the layer below qualified name lookup, which performs direct lookups into the base type, its conformed protocols, and superclasses.

Nominal type declarations and extensions are *iterable declaration contexts*, meaning they contain member declarations. Before discussing direct lookup, let's consider what happens if we ask an iterable declaration context to list its members. This is a lazy operation which triggers work the first time it is called:

- Iterable declaration contexts parsed from source are populated by delayed parsing; when the parser first reads a source file, the bodies of iterable declaration contexts are skipped, and only the source range is recorded. Asking for the list of members goes and parses the source range again, constructing declarations from their parsed representation (Section 2.2).
- Iterable declaration contexts from binary and imported modules are equipped with a *lazy member loader* which serves a similar purpose. Asking the lazy member loader to list all members will build the corresponding Swift declarations from deserialized records or imported Clang declarations. The lazy member loader can also find just those declarations with a *specific* name, as explained below; this is the more common operation, since it is much more efficient.

Member lookup table. Every nominal type declaration has an associated *member lookup table*, which is used for direct lookup. This table maps each identifier to a list of value declarations with that name (multiple value declarations can share a name because Swift allows type-based overloading). The declarations in a member lookup table are understood to be members of one or more iterable declaration contexts, which are exactly the type declaration itself and all of its extensions. These iterable declaration contexts might originate from a mix of different module kinds. For example, the nominal type itself might be an Objective-C class from an imported Objective-C module, with one extension declared in a binary Swift module, and another extension defined in the main module, parsed from source.

The lookup table is populated lazily, in a manner resembling a state machine. Say we're asked to perform a direct lookup for some given name. If this is the first direct lookup, we populate the member lookup table with *all* members from any *parsed* iterable declaration contexts, which might trigger delayed parsing. Each entry in the member lookup table stores a "complete" bit. The "complete" bit of these initially-populated entries is *not* set, meaning that the entry only contains those members that were parsed from source. If any iterable declaration contexts originate from binary and imported modules, direct lookup then asks each lazy member loader to selectively load only those members with the given name. (Parsed declaration contexts do not offer this level of laziness, because there is no way to parse a subset of the members only.) After the lazy member loaders do their work, the lookup table entry for this name is now complete, and the "complete" bit is set. Later when direct lookup finds a member lookup table entry with the "complete" bit set, it knows this entry is fully populated, and the stored list of declarations is returned immediately without querying the lazy member loaders.

This *lazy member loading* mechanism ensures that only those members which are actually referenced in a compilation session are loaded from serialized and imported iterable declaration contexts.

Example 10.4. Listing 10.1 shows an example of lazy member loading. Observe the following:

- The `NSHorse` class itself is declared in Objective-C in a header file. Suppose this header file is part of the `HorseKit` module, imported by the two Swift source files.
- The Swift source file `b.swift` declares an extension of `NSHorse`. Let's say that this is a secondary file in the current frontend job.
- The Swift source file `c.swift` declares a function which calls some methods on `NSHorse`. For our example this needs to be a primary file of this frontend job.
- The `trot()` method in the class has a different interface type than the `trot()` method in the extension, so they are two distinct overloaded methods with the same name.

Listing 10.1.: A class implemented in Objective-C, with an extension written in Swift

```
// a.h
@interface NSHorse: NSObject
- (void) trot: (int) x;
- (void) canter: (float) x;
@end
```

```
// b.swift
import HorseKit

extension NSHorse {
  func walk(_: Float) {}
  func trot(_: Float) {}
}
```

```
// c.swift
import HorseKit

func ride(_ horse: NSHorse) {
  horse.walk(1.0)
  horse.trot(2.0)
  horse.walk(1.0)
}
```

The compiler begins by parsing both Swift source files, skipping over the extension body with delayed parsing because it appears in a secondary file. Next, extension binding is performed, which associates the extension of `NSHorse` with the imported class declaration of `NSHorse`. Finally, we type check the body of the `ride()` function, since it appears in a primary file. Type checking the expressions in the body performs three direct lookups into `NSHorse`, first with the name `walk`, then `trot`, and finally `walk` again.

(`walk`) The table is initially empty, so it must be populated with the members of all parsed iterable declaration contexts first. This triggers delayed parsing of the extension, which adds two entries to our member lookup table:

Name	Declarations	Complete?
`walk`	`NSHorse.walk` in extension of `NSHorse`	×
`trot`	`NSHorse.trot` in extension of `NSHorse`	×

At this point, neither entry is complete, because we haven't looked for members inside the class itself, which was imported and not parsed. Direct lookup does that next, by asking the lazy member loader associated with the class declaration to load any members named `walk`. The class does not define such a member, so we simply mark the member lookup table entry as complete:

Name	Declarations	Complete?
`walk`	`NSHorse.walk` in extension of `NSHorse`	✓
`trot`	`NSHorse.trot` in extension of `NSHorse`	×

Direct lookup returns `NSHorse.walk()` to the type checker.

(`trot`) While the member lookup table already has an entry named `trot`, it is not complete, so the lazy member loader for the class declaration is consulted again. This time, the class contains a member with this name also. The `trot` method is imported from Objective-C and added to the member lookup table:

Name	Declarations	Complete?
`walk`	`NSHorse.walk` in extension of `NSHorse`	✓
`trot`	`NSHorse.trot` in extension of `NSHorse` `NSHorse.trot` in class `NSHorse`	✓

Direct lookup now returns both overloads of `NSHorse.trot()` to the type checker.

(`walk`) The third lookup finds that the corresponding member lookup table entry is already complete, so we immediately return the existing declaration stored there without consulting the lazy member loader. Notice that the `canter` method of `NSHorse` was never referenced, so it did not need to be imported at all.

History. Lazy member loading was introduced in Swift 4.1 to avoid wasted work in deserializing or importing members that are never referenced. The speedup is most pronounced when multiple frontend jobs perform direct lookup into a common set of deserialized or imported nominal types. The overhead of loading all members of this common set of types was multiplied across frontend jobs prior to the introduction of lazy member loading.

10.3. Constrained Extensions

An extension can impose its own requirements on the generic parameters of the extended type; we refer to this as a *constrained extension*. These requirements are always additive; the generic signature of a constrained extension is built from the generic signature of the extended type, together with the new requirements. The members of a constrained extension are then only available on specializations of the extended type that satisfy these requirements. There are three ways to declare a constrained extension:

1. using a `where` clause,
2. by writing a bound generic type as the extended type,
3. by writing a generic type alias type as the extended type, with some restrictions.

Case 1 is the most general form; Case 2 and Case 3 can be expressed by writing the appropriate requirements in a `where` clause. An extension that does not fall under one of these three cases is sometimes called an *unconstrained extension* when it is important to distinguish it from a constrained extension. The generic signature of an unconstrained extension is the same as the generic signature of the extended type.

Here is a constrained extension of `Set` which constrains the `Element` type to `Int`:

```
extension Set where Element == Int {...}
```

The generic signature of `Set` is `<Element where Element: Hashable>`. Adding the same-type requirement [`Element == Int`] makes the requirement [`Element: Hashable`] redundant since `Int` conforms `Hashable`, so the extension's generic signature becomes `<Element where Element == Int>`.

This example demonstrates that while the requirements of the constrained extension abstractly imply those of the extended type, they are not a "syntactic" subset—the [`Element: Hashable`] requirement does not appear in the constrained extension's generic signature, because it became redundant when [`Element == Int`] was added.

In the early days of Swift, this extension was not supported at all, because the compiler did not permit same-type requirements between generic parameters and concrete types; only dependent member types could be made concrete. This restriction was lifted in Swift 3, and many concepts described in this book, most importantly substitution maps and generic environments, were introduced as part of this effort.

Extending a generic nominal type. An extension of a generic nominal type with generic arguments is shorthand for a `where` clause that constrains each generic parameter to a concrete type. The generic argument types must be fully concrete; they cannot reference the generic parameters of the extended type declaration. The previous example can be more concisely expressed using this syntax:

```
extension Set<Int> {...}
```

A non-generic type alias type whose underlying type is a bound generic type can also be used in the same manner:

```
typealias StringMap = Dictionary<String, String>
extension StringMap {...}
```

This shorthand was introduced in Swift 5.7 [85].

Extending a generic type alias. A generic type alias is called a "pass-through type alias" if it satisfies the following three conditions:

1. the underlying type of the generic type alias must be a generic nominal type,
2. the type alias must have the same number of generic parameters as the underlying type declaration,
3. the type alias must substitute each generic parameter of the underlying type declaration with the corresponding generic parameter of the type alias (where the correspondence means they have the same depth and index).

A pass-through type alias essentially equivalent to the underlying generic nominal type, except that it may introduce additional requirements via a `where` clause. An extension of a pass-through type alias is equivalent to a constrained extension of the underlying nominal type which states these additional requirements. Note that the pass-through

type alias is not required to give its generic parameters the same names as its underlying type, which is slightly confusing, because the names used by the type alias declaration do not matter to the extension at all; the generic parameter list of the extension is always cloned from the extended nominal type, and not the type alias.

Example 10.5. While it is a generally useful feature, the ability to extend a pass-through type alias was motivated by the desire to maintain source compatibility after a specific standard library change.

The standard library defines a generic struct named `Range` with a single `Bound` generic parameter conforming to `Comparable`:

```
struct Range<Bound: Comparable> {...}
```

Prior to Swift 4.2, the standard library also had a separate `CountableRange` type with a different set of requirements:

```
struct CountableRange<Bound: Stridable>
    where Bound.Stride: SingedInteger {}
```

The `Stridable` protocol inherits from `Comparable`, so every valid generic argument of `CountableRange` was also also suitable for `Range`. Also, `CountableRange` offered a superset of the API of `Range`. The only difference between the two was that `CountableRange` conformed to more protocols than `Range`. After conditional conformances were added to the language, `CountableRange` no longer made sense as a separate type. Swift 4.2 replaced `CountableRange` with a generic type alias:

```
typealias CountableRange<Bound: Stridable> = Range<Bound>
    where Bound.Stride: SignedInteger
```

In Swift 4.1, the following were extensions of two different nominal types:

```
extension Range {...}

extension CountableRange {...}
```

In Swift 4.2, the second extension became equivalent to a constrained extension of `Range`, once the compiler understood pass-through type aliases:

```
extension Range
    where Bound: Stridable,
          Bound.Stride: SignedInteger {...}
```

Example 10.6. The following type aliases are not pass-through type aliases.

1. The underlying type of this type alias is not a nominal type:

   ```
   typealias A<T> = () -> T
   ```

2. This type alias has the wrong number of generic parameters:

   ```
   typealias B<Value> = Dictionary<AnyHashable, Value>
   ```

3. This type alias does not substitute the second generic parameter of the underlying type with the second generic parameter of the type alias:

   ```
   typealias D<Key, Value> = Dictionary<Key, (Value) -> Int>
   ```

10.4. Conditional Conformances

A conformance written on a nominal type or unconstrained extension implements the protocol's requirements for all specializations of the nominal type, and we call this an *unconditional* conformance. A conformance declared on a *constrained* extension is what's known as a *conditional conformance*, which implements the protocol requirements only for those specializations of the extended type which satisfy the requirements of the extension. Conditional conformances were introduced in Swift 4.2 [86].

For example, arrays have a natural notion of equality, defined in terms of the equality operation on the element type. However, there is no reason to restrict all arrays to only storing `Equatable` types. Instead, the standard library defines a conditional conformance of `Array` to `Equatable` where the element type is `Equatable`:

```
struct Array<Element> {...}

extension Array: Equatable where Element: Equatable {
  func ==(lhs: Self, rhs: Self) -> Bool {
    guard lhs.count == rhs.count else { return false }

    for i in 0..<lhs.count {
      // 'Element: Equatable' conditional requirement is used here
      guard lhs[i] == rhs[i] else { return false }
    }

    return true
  }
}
```

More complex conditional requirements can also be written. We previously discussed overlapping conformances and coherence in Section 7.1, and conditional conformances inherit an important restriction. A nominal type can only conform to a protocol once, so in particular, overlapping conditional conformances are not supported and we emit a diagnostic:

```
struct G<T> {}

protocol P {}

extension G: P where T == Int {}
extension G: P where T == String {}  // error
```

Computing conditional requirements. A conditional conformance stores a list of *conditional requirements*. If G is the generic signature of the constrained extension where the conformance was declared, and H is the generic signature of the conforming type, then certainly G satisfies all requirements of H. If the converse is also true, we have an unconditional conformance. Otherwise, the conditional requirements of the conformance are precisely those requirements of G not satisfied by H. A simple example is the conditional conformance of `Dictionary` to `Equatable`:

```
extension Dictionary: Equatable where Value: Equatable {...}
```

The generic signature of `Dictionary` is `<Key, Value where Key: Hashable>`. The generic signature of our constrained extension has two requirements:

```
<Key, Value where Key: Hashable, Value: Equatable>
```

The first requirement is already satisfied by the generic signature of `Dictionary` itself; the second requirement is the conditional requirement of our conformance.

We compute the conditional requirements by handing Algorithm 9B the requirements of G together with the forwarding substitution map $1_{[\![H]\!]}$. The algorithm outputs a list of failed and unsatisfied requirements. They're not really "failed" or "unsatisfied" though; instead, the concatenation of these two lists is precisely the list of conditional requirements in our normal conformance.

The "failed" case corresponds to a conditional requirement whose subject type is not even a valid type parameter of H. The [`T.Element: Hashable`] requirement below has the subject type `T.Element`, which was defined by [`T: Sequence`], which is itself a conditional requirement, so [`G: Sequence`] has two conditional requirements:

```
struct G<T> {}

extension G: Sequence where T: Sequence, T.Element: Hashable {...}
```

Specialized conditional conformances. Building upon Section 7.1, we now describe how substitution relates to conditional conformances. If $\texttt{T}_d$ is the declared interface type of some nominal type declaration d that *unconditionally* conforms to `P`, and $\texttt{T} = \texttt{T}_d \otimes \Sigma$ is a specialized type of d for some substitution map Σ, then looking up the conformance of `T` to [`P`] returns a specialized conformance with conformance substitution map Σ:

$$[\texttt{P}] \otimes \texttt{T} = [\texttt{T}_d\texttt{: P}] \otimes \Sigma$$

If $[\texttt{T}_d\texttt{: P}]$ is conditional though, we cannot take Σ to be the context substitution map of `T`. The type witnesses of $[\texttt{T}_d\texttt{: P}]$ might contain type parameters of the constrained extension, and not just the conforming type; however the context substitution map of `T` has the generic signature of the conforming type. We must set Σ to be the context substitution map of `T` for the generic signature of the constrained extension, as in Section 9.2. Indeed, this is the same problem as member type resolution when the referenced type declaration is declared in a constrained extension; we must perform some additional global conformance lookups to populate the substitution map completely.

The conditional requirements of the specialized conformance [`T: P`] are the substituted requirements obtained by applying Σ to each conditional requirement of $[\texttt{T}_d\texttt{: P}]$. This makes the following diagram commute for each conditional requirement R:

$$\begin{array}{ccc} [\texttt{T}_d\texttt{: P}] & \xrightarrow{\text{apply } \Sigma} & [\texttt{T: P}] \\ \Big\downarrow{\scriptstyle \text{get conditional requirement}} & & \Big\downarrow{\scriptstyle \text{get conditional requirement}} \\ R & \xrightarrow{\text{apply } \Sigma} & R \otimes \Sigma \end{array}$$

Consider the [`Array<τ_0_0>: Equatable`] conformance from the standard library. The generic signature of `Array` is `<τ_0_0>` with no requirements, while the conformance context is the constrained extension with signature `<τ_0_0 where τ_0_0: Equatable>`. The context substitution map of `Array<Int>` for the constrained extension is:

$$\Sigma := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{Int}; \\ [\texttt{τ\_0\_0: Equatable}] \mapsto [\texttt{Int: Equatable}] \end{array} \right\}$$

We compose this substitution map with the normal conformance, and obtain a specialized conditional conformance, denoted [`Array<Int>: Equatable`]:

$$[\texttt{Array<τ\_0\_0>: Equatable}] \otimes \Sigma = [\texttt{Array<Int>: Equatable}]$$

The normal conformance has a single conditional requirement, [`τ_0_0: Equatable`]. We apply Σ to get the conditional requirement of the specialized conformance:

$$[\texttt{τ\_0\_0: Equatable}] \otimes \Sigma = [\texttt{Int: Equatable}]$$

Global conformance lookup. Global conformance lookup purposely does not check conditional requirements, allowing callers to answer two different questions:

1. Does this specialized type with its specific list of generic arguments conform to the protocol?
2. What requirements should a list of generic arguments satisfy to make the type conform?

When the first interpretation is desired, a convenience entry point can be used that combines global conformance lookup followed by Algorithm 9A to check any conditional requirements.

Checking conditional requirements is more subtle than checking for the presence of invalid conformances in the conformance substitution map. It is true that their presence indicates a conditional requirement is unsatisfied; for example, `Array<AnyObject>` does not conditionally conform to `Equatable`, since an `AnyObject` is not `Equatable`, and we get the following specialized conformance:

$$\begin{aligned}
&[\texttt{Equatable}] \otimes \texttt{Array<AnyObject>} \\
&= [\texttt{Array<}\tau\texttt{_0_0>: Equatable}] \otimes \left\{ \begin{array}{l} \tau\texttt{_0_0} \mapsto \texttt{AnyObject}; \\ [\tau\texttt{_0_0: Equatable}] \mapsto \text{invalid} \end{array} \right\} \\
&= [\texttt{Array<AnyObject>: Equatable}]
\end{aligned}$$

However, the failure of conditional requirements of other kinds, such as same-type, superclass and layout requirements, is not immediately apparent from the structure of the substitution map. In the below, the conformance [`Pair<Int, String>: Diagonal`] might look fine at first sight because it does not contain any invalid conformances, but it does not satisfy the conditional requirement [`τ_0_0 == τ_0_0`]:

```
protocol Diagonal {}
struct Pair<T, U> {}
extension Pair: Diagonal where T == U {}
```

Thus, conditional requirements must always be checked by Algorithm 9A, and not in some "ad-hoc" way by digging through the substitution map for instance.

Protocol inheritance. Protocol inheritance is modeled as an associated conformance requirement on `Self`, so for instance `Derived` has an associated conformance requirement [`Self: Base`]:

```
protocol Base {...}
protocol Derived: Base {...}
```

When checking a conformance to `Derived`, the conformance checker ensures that the conforming type satisfies the [`Self: Base`] requirement. When the conformance to `Base` is unconditional, this always succeeds, because the conformance declaration also implies an unconditional conformance to the base protocol:

```
struct Pair<T, U> {}
extension Pair: Derived {...}
// implies 'extension Pair: Base {}'
```

The nominal type's conformance lookup table synthesizes these implied conformances and makes them available to global conformance lookup. With conditional conformances, such implied conformances are not synthesized because there is no way to guess what the conditional requirements should be. The conformance checker still checks the associated conformance requirement on `Self` though, so the user must first explicitly declare a conformance to each base protocol when writing a conditional conformance.

Suppose we wish for `Pair` to conform to `Derived` when [`τ_0_0 == Int`]:

```
extension Pair: Derived where T == Int {...}
```

The compiler will diagnose an error unless there is also an *explicit* conformance of `Pair` to `Base`. There are several possible ways to declare a conformance of `Pair` to `Base`. The simplest way is to conform unconditionally:

```
extension Pair: Base {...}
```

Things get more interesting if the conformance to `Base` is also conditional, because then the conformance to `Derived` only makes sense if the conditional requirements of [`Pair: Derived`] imply the conditional requirements of [`Pair: Base`]. We establish this condition as follows. There are three generic signatures in play here:

1. The signature of `Pair`; call it H.
2. The signature of the extension declaring conformance to `Base`; we call it G_1.
3. The signature of the extension declaring conformance to `Derived`; we call it G_2.

The conformance [`Pair: Derived`] satisfies the [`Self: Base`] associated conformance requirement of `Derived` if any specialization of `Pair` which satisfies the conditional requirements of (3) also satisfies the conditional requirements of (2). In the conformance checker, this falls out from the general case of checking associated requirements.

To each associated requirement, we apply the protocol substitution map for the normal conformance, followed by the forwarding substitution map for the generic signature of the conformance. In our example, this gives us the following substituted requirement:

$$[\texttt{Self: Base}] \otimes \Sigma_{[\texttt{Pair: Derived}]} \otimes 1_{[\![G_2]\!]} = [\texttt{Pair<}[\![\texttt{T}]\!]\texttt{,Int>: Base}]$$

The next step asks Algorithm 9A if the substituted requirement is satisfied. This issues a global conformance lookup and proceeds to check its conditional requirements:

$$[\texttt{Base}] \otimes \texttt{Pair<}[\![\texttt{T}]\!]\texttt{,Int>} = [\texttt{Pair<}[\![\texttt{T}]\!]\texttt{,Int>: Base}]$$

Since $\texttt{Pair<}[\![\texttt{T}]\!]\texttt{,Int>} \in \textsc{Type}([\![G_2]\!])$, we get $[\texttt{Base}] \otimes \texttt{Pair<}[\![\texttt{T}]\!]\texttt{,Int>} \in \textsc{Conf}([\![G_2]\!])$. The conditional requirements of this conformance are the requirements of G_1 substituted with the primary archetypes of G_2. Knowing if they are satisfied or not determines the validity of the conformance to `Derived`.

We now look at three different definitions of the conformance [`Pair: Base`], and see how each one impacts the validity of the conformance [`Pair: Derived`]:

1. The conditional conformance of `Pair` to `Base` might also require that `T` is `Int`:

   ```
   extension Pair: Base where T == Int {...}
   ```

   The conditional requirement here is [`τ_0_0 == Int`], the same as in the `Derived` conformance, and after substitution we get:

   $$[\texttt{τ\_0\_0 == Int}] \otimes \{\texttt{τ\_0\_0} \mapsto \texttt{Int}, \texttt{τ\_0\_1} \mapsto [\![\texttt{U}]\!]\} = [\texttt{Int == Int}]$$

   This requirement is satisfied by Algorithm 9A, so the conditional requirements of [`Pair: Derived`] imply the conditional requirements of [`Pair: Base`], and the conformance to `Derived` is valid.

2. Instead, we could conform `Pair` to `Base` when `T` conforms to `Equatable`:

   ```
   extension Pair: Base where T: Equatable {...}
   ```

   Now, [`Pair: Base`] has the conditional requirement [`τ_0_0: Equatable`], and after substitution we get:

   $$[\texttt{τ\_0\_0: Equatable}] \otimes \{\texttt{τ\_0\_0} \mapsto \texttt{Int}, \texttt{τ\_0\_1} \mapsto [\![\texttt{U}]\!]\} = [\texttt{Int: Equatable}]$$

   This is also satisfied, so once again our conditional conformance [`Pair: Derived`] is valid.

3. Finally consider this, which renders invalid our conformance to `Derived`:

   ```
   extension Pair: Base where U: Equatable {...}
   ```

   The conditional requirement [`τ_0_0: Equatable`] becomes [$[\![\texttt{U}]\!]$`: Equatable`] after substitution:

   $$[\texttt{τ\_0\_1: Equatable}] \otimes \{\texttt{τ\_0\_0} \mapsto \texttt{Int}, \texttt{τ\_0\_1} \mapsto [\![\texttt{U}]\!]\} = [[\![\texttt{U}]\!]\texttt{: Equatable}]$$

The archetype ⟦U⟧ does not conform to `Equatable` because [`τ_0_1: Equatable`] is not a derived requirement of G_2, so the conditional requirement [`τ_0_0 == Int`] of [`Pair: Derived`] does not imply the conditional requirement [`τ_0_1: Equatable`] of [`Pair: Base`]. For this reason, when [`Pair: Base`] is written as above, the compiler rejects our conformance [`Pair: Derived`].

Termination. Conditional conformances can express non-terminating computation at compile time. The below code is taken from a bug report which remains unfixed for the time being [87]:

```
protocol P {}

protocol Q {
  associatedtype A
}

struct G<T: Q> {}

extension G: P where T.A: P {}

struct S: Q {
  typealias A = G<S>
}

func takesP<T: P>(_: T.Type) {}
takesP(G<S>.self)
```

The `takesP()` function has the below generic signature:

```
<τ_0_0 where τ_0_0: P>
```

In the snippet, this function is called with `G<S>` as the generic argument for `τ_0_0`. This substitution map for this call also needs to store the conformance [`G<S>: P`], but this conformance cannot be constructed. The normal conformance [`G<τ_0_0>: P`] is declared on a constrained extension with the following generic signature:

```
<τ_0_0 where τ_0_0: Q, τ_0_0.[Q]A: P>
```

To construct the specialized conformance [`G<S>: P`] from this normal conformance, we must first construct the conformance substitution map. The substitution map should map `τ_0_0` to `S`, and store a pair of conformances, for the conformance requirements

$[\texttt{τ\_0\_0: Q}]$ and $[\texttt{τ\_0\_0.[Q]A: P}]$:

$$\Sigma := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{S}; \\ \\ [\texttt{τ\_0\_0: Q}] \mapsto [\texttt{S: Q}] \\ [\texttt{τ\_0\_0.[Q]A: P}] \mapsto ??? \end{array} \right\}$$

The conforming type of the first conformance should be $\texttt{τ\_0\_0} \otimes \Sigma = \texttt{S}$, so the first conformance is the normal conformance $[\texttt{S: Q}]$. The conforming type of the second conformance should be $\pi([\texttt{Q}]\texttt{A}) \otimes [\texttt{S: Q}] = \texttt{G<S>}$; so the second conformance is exactly $[\texttt{G<S>: P}]$, the conformance we're in the process of constructing. A substitution map cannot form a cycle with a specialized conformance it contains.

Now, suppose we could represent a circular substitution map of this kind:

$$\Sigma := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{S}; \\ \\ [\texttt{τ\_0\_0: Q}] \mapsto [\texttt{S: Q}] \\ [\texttt{τ\_0\_0.[Q]A: P}] \mapsto [\texttt{G<τ\_0\_0>: P}] \otimes \Sigma \end{array} \right\}$$

Unfortunately, this would still not allow our example to type check. The next problem we would encounter is checking the conditional requirement, $[\texttt{τ\_0\_0.[Q]A: P}]$. Applying Σ gives us the substituted conditional requirement:

$$[\texttt{τ\_0\_0.[Q]A: P}] \otimes \Sigma = [\texttt{G<S>: P}]$$

So `G<S>` conditionally conforms to `P` if the conditional requirement $[\texttt{G<S>: P}]$ is satisfied; this conditional requirement is satisfied if `G<S>` conditionally conforms to `P`. At this point, we are stuck in a loop, again. For now, the compiler crashes on this example while constructing the conformance substitution map; hopefully a future update to this book will describe the eventual resolution of this bug by imposing an iteration limit.

Our example only encodes an infinite loop, but conditional conformances can actually express arbitrary computation. This is shown for the Rust programming language, which also has conditional conformances, in [88]. We'll also see another example of non-terminating compile-time computation in Section 12.3.

Soundness. Substitution maps in a valid program should satisfy a certain condition.

Definition 10.7. If Σ is a substitution map with input generic signature G and fully-concrete replacement types, then Σ is *well-formed* if for each derived requirement R of G, the substituted requirement $R \otimes \Sigma$ is satisfied according to Algorithm 9A.

(The restriction to fully-concrete replacement types is not a real limitation; as we saw in Section 9.3, we can first compose Σ with the forwarding substitution map $1_{[\![H]\!]}$ for some generic signature H if necessary).

If the derived requirements of G are an infinite set, we cannot directly check if Σ is well-formed. Algorithm 9B only checks if Σ satisfies each *explicit* requirement of G, and unfortunately, this is not sufficient.

An immediate counter-example appears when the substituted requirement depends on a conformance that does not satisfy the associated requirements of its protocol. For example, we might declare a `Bad` type conforming to `Sequence` whose `Iterator` type witness does not conform to `IteratorProtocol`:

```
struct Bad: Sequence {
  typealias Iterator = Int  // error
}
```

The substitution map $\Sigma := \{\texttt{τ\_0\_0} \mapsto \texttt{Bad}; [\texttt{τ\_0\_0: Sequence}] \mapsto [\texttt{Bad: Sequence}]\}$ satisfies all explicit requirements of its generic signature, but it does not satisfy the derived requirement [`τ_0_0.[Sequence]Iterator: IteratorProtocol`]. However, *this* is not the soundness hole; we diagnose an error when we check the conformance, so the program is invalid.

We now show an example where we can write down a substitution map that is not well-formed, and yet Algorithm 9B does not produce any diagnostics at all, including from checking conformances. We start with two protocols:

```
protocol Bar {
  associatedtype Beer
  func brew() -> Beer
}

protocol Pub {
  associatedtype Beer
  func pour() -> Beer
}
```

We declare a function generic over types that conform to both `Bar` and `Pub`, so it has the generic signature `<τ_0_0 where τ_0_0: Bar, τ_0_0: Pub>`:

```
func both<T: Bar & Pub>(_ t: T) -> (T.Beer, T.Beer) {
  return (t.brew(), t.pour())
}
```

Now, we declare a `BrewPub` struct, and pass an instance of `BrewPub<Int>` to our function:

```
struct BrewPub<T> {}
let result = both(BrewPub<Int>())
```

For this call expression to type check, `BrewPub` must conform to `Bar` and `Pub`; assume for a moment these conformances have already been declared. Let Σ be the substitution map for the call:

$$\Sigma := \left\{ \begin{array}{l} \texttt{τ\_0\_0} \mapsto \texttt{BrewPub<Int>}; \\ \\ [\texttt{τ\_0\_0: Bar}] \mapsto [\texttt{BrewPub<Int>: Bar}] \\ [\texttt{τ\_0\_0: Pub}] \mapsto [\texttt{BrewPub<Int>: Pub}] \end{array} \right\}$$

Applying Σ to `τ_0_0.[Bar]Beer` and `τ_0_0.[Pub]Beer` projects the type witness from each conformance:

$$\texttt{τ\_0\_0.[Bar]Beer} \otimes \Sigma = \pi(\texttt{[Bar]Beer}) \otimes [\texttt{BrewPub<Int>: Bar}]$$
$$\texttt{τ\_0\_0.[Pub]Beer} \otimes \Sigma = \pi(\texttt{[Pub]Beer}) \otimes [\texttt{BrewPub<Int>: Pub}]$$

For Σ to be well-formed, both type witnesses must be canonically equal, because in the generic signature of `both()`, the bound dependent member types `τ_0_0.[Bar]Beer` and `τ_0_0.[Pub]Beer` are both equivalent to the unbound dependent type `τ_0_0.Beer`.

If `BrewPub` conforms to these protocols unconditionally, the redeclaration checking rules prevent us from declaring the two conformances to have distinct type witnesses:

```
extension BrewPub: Bar {
  typealias Beer = Float
  func brew() -> Float { return 0.0 }
}

extension BrewPub: Pub {
  typealias Beer = String  // error: invalid redeclaration
  func pour() -> String { return "" }
}
```

If the conformances are conditional though, neither type alias is visible from the other constrained extension, so they are not rejected as redeclarations. All that remains is to pick conditional requirements such that our generic argument type `Int` satisfies both:

```
extension BrewPub: Bar where T: Equatable {
  typealias Beer = Float
  func brew() -> Float { return 0.0 }
}

extension BrewPub: Pub where T: ExpressibleByIntegerLiteral {
  typealias Beer = String
  func pour() -> String { return "" }
}
```

When we call our `both()` function, the caller expects to receive a tuple of two elements both having the same reduced type `τ_0_0.[Bar]Beer`. Inside the function, the type checker thinks that the calls to `brew()` and `pour()` return the same type. However, what actually happens is that one returns a `Float` while the other returns a `String`. This quickly results in undefined behavior. This soundness hole remains unfixed at the time of writing. There are two possible approaches to solving this problem:

1. We can tighten the rules around redeclarations of type aliases in constrained extensions, requiring that all such type aliases have canonically equal underlying types. This would reject the conditional conformances written above as invalid.

2. We can extend Algorithm 9B to look for incompatible conformances in the given substitution map. This would accept the conformances, but reject the call to `both()` written above.

Section 19.2 sketches out a potential implementation of (2) at the end of Example 19.8.

10.5. Source Code Reference

Key source files:

- `include/swift/AST/Decl.h`
- `lib/AST/Decl.cpp`

`ExtensionDecl` *class*

Represents an extension declaration.

- `getExtendedNominal()` returns the extended type declaration. Returns `nullptr` if extension binding failed to resolve the extended type. Asserts if extension binding has not yet visited this extension.

- `computeExtendedNominal()` actually evaluates the request which resolves the extended type to a nominal type declaration. Only used by extension binding.

- `getExtendedType()` returns the written extended type, which might be a type alias type or a generic nominal type. This uses ordinary type resolution, so it only happens after extension binding. This is used to implement the syntax sugar described at the start of Section 10.3.

- `getDeclaredInterfaceType()` returns the declared interface type of the extended type declaration.

- `getSelfInterfaceType()` returns the self interface type of the extended type declaration. Different than the declared interface type for protocol extensions, where the declared interface type is the protocol type, but the self interface type is the protocol `Self` type.

Extension Binding

Key source files:

- include/swift/AST/NameLookup.h
- lib/AST/Decl.cpp
- lib/AST/NameLookup.cpp
- lib/Sema/TypeChecker.cpp

`bindExtensions()` *function*

Takes a `ModuleDecl *`, which must be the main module, and implements Algorithm 10A.

`ExtendedNominalRequest` *class*

The request evaluator request which resolves the extended type to a nominal type declaration. This calls out to a restricted form of type resolution which does not apply generic arguments or perform substitution.

`directReferencesForTypeRepr()` *function*

Takes a `TypeRepr *` and the extension's parent declaration context, which is usually a source file. Returns a vector of `TypeDecl *`. Some of the type declarations might be type alias declarations; the next entry point fully resolves everything to a list of nominal types.

`resolveTypeDeclsToNominal()` *function*

Takes a list of `TypeDecl *` and outputs a list of `NominalTypeDecl *`, by resolving any type alias declarations to nominal types, using the same restricted form of type resolution recursively.

If the output list is empty, type resolution failed and the extension cannot be bound. If the output list contains more than one entry, type resolution produced an ambiguous result. For now, we always take the first nominal type declaration from the output list in that case.

Direct Lookup and Lazy Member Loading

Key source files:

- `lib/AST/NameLookup.cpp`
- `include/swift/AST/LazyResolver.h`

`DirectLookupRequest` *class*

The request evaluator request implementing direct lookup. The entry point is the `NominalTypeDecl::lookupDirect()` method, which was introduced in Section 2.6. This request is uncached, because the member lookup table effectively implements caching outside of the request evaluator.

To understand the implementation of `DirectLookupRequest::evaluate()`, one can start with the following functions:

- `prepareLookupTable()` adds all members from extensions without lazy loaders, and members loaded so far from extensions with lazy loaders, without marking any entries as complete.
- `populateLookupTableEntryFromLazyIDCLoader()` asks a lazy member loader to load a single entry and adds it to the member lookup table.
- `populateLookupTableEntryFromExtensions()` adds all members from extensions that do not have lazy member loaders.

`MemberLookupTable` *class*

Every `NominalTypeDecl` has an instance of `MemberLookupTable`, which maps declaration names to lists of `ValueDecl`. The most important methods:

- `find()` returns an iterator pointing at an entry, which might be missing or incomplete.
- `isLazilyComplete()` answers if an entry is complete.
- `markLazilyComplete()` marks an entry as complete.

`LazyMemberLoader` *class*

Abstract base class implemented by different kinds of modules to look up top-level declarations and members of types and extensions. For the main module, this consults lookup tables built from source, for serialized modules this deserializes records and builds declarations from them, for imported modules this constructs Swift declarations from Clang declarations.

Constrained Extensions

Key source files:

- lib/Sema/TypeCheckDecl.cpp
- lib/Sema/TypeCheckGeneric.cpp

The `GenericSignatureRequest` was previously introduced in Section 11.5. It delegates to a pair of utility functions to implement special behaviors of extensions.

`collectAdditionalExtensionRequirements()` *function*

Collects an extension's requirements from the extended type, which handles extensions of pass-through type aliases (`extension CountableRange {...}`) and extensions of bound generic types (`extension Array<Int> {...}`).

`isPassthroughTypealias()` *function*

Answers if a generic type alias satisfies the conditions that allow it to be the extended type of an extension.

Conditional Conformances

Key source files:

- include/swift/AST/GenericSignature.h
- include/swift/AST/ProtocolConformance.h
- lib/AST/GenericSignature.cpp
- lib/AST/ProtocolConformance.cpp
- lib/Sema/TypeCheckProtocol.cpp

`checkConformance` *function*

A utility function that first calls `lookupConformance()` (Section 7.6), and then checks any conditional requirements using `checkRequirements()`, described in Section 9.5.

The `NormalProtocolConformance` and `SpecializedProtocolConformance` classes were previously introduced in Section 7.6.

`NormalProtocolConformance` *class*

- `getConditionalRequirements()` returns an array of conditional requirements; this is non-empty exactly when this is a conditional conformance.

`SpecializedProtocolConformance` *class*

- `getConditionalRequirements()` applies the conformance substitution map to each conditional requirement of the underlying normal conformance.

`GenericSignatureImpl` *class*

See also Section 5.6.

- `requirementsNotSatisfiedBy()` returns an array of those requirements of this generic signature not satisfied by the given generic signature. This is used for computing the conditional requirements of a `NormalProtocolConformance`.

11. Building Generic Signatures

Building a generic signature from user-written requirements is something we glossed over before, and it's time to detail it now. We're going to fill in missing steps between the syntax for declaring generic parameters and stating requirements of Sections 4.1 and 4.2, and the generic signature of Chapter 5, a semantic representation of the generic parameters and requirements of a declaration.

The requirements in a generic signature must be reduced, minimal, and ordered in a certain way (we will see the formal definitions in Section 11.4). To build a generic signature then, we must convert user-written requirements into an *equivalent* set of requirements that satisfy these additional invariants. We're going to start from the entry points for building generic signatures and peel away the layers:

- The **generic signature request** lazily builds a generic signature for a declaration written in source. This is a multi-step process; we call out to type resolution to construct the user-written requirements from syntactic representations, and then ultimately pass these requirements to the minimization algorithm. We will see how this request decomposes the declaration's syntactic forms before delegating to the **inferred generic signature request** to do most of the work.

- The **abstract generic signature request** instead builds a generic signature from a list of generic parameters and requirements. No name lookup or type resolution occurs here, because the inputs are semantic objects already.

- All generic signatures are ultimately created by the **primitive constructor**, which takes generic parameters and requirements known to already obey the necessary invariants, and simply allocates and initializes the semantic object.

  Outside of the generics implementation, the primitive constructor is used in a handful of cases when it is already known the requirements satisfy the necessary conditions. When deserializing a generic signature from a serialized module for example, we know it satisfied the invariants when it was serialized.

  We also use the primitive constructor to build the protocol generic signature `<Self where Self: P>` for a protocol `P`, because we know it satisfies the invariants by definition. Anywhere else we need to build a generic signature "from scratch," we use the **abstract generic signature request** instead.

- The **requirement signature request** lazily builds a requirement signature for a protocol written in source. The flow resembles the inferred generic signature request; we start by resolving user-written requirements, then perform minimization. There's no "abstract" equivalent of the requirement signature request; a requirement signature is always attached to a specific named protocol declaration.

Now we look at each involved request in detail.

Generic signature request. Two easy cases are handled first:

1. If the declaration is a protocol or an unconstrained protocol extension, we build the protocol generic signature `<Self where Self: P>` using the primitive constructor.
2. A declaration having neither a generic parameter list nor a trailing `where` clause simply inherits the generic signature from its parent context. If the declaration is at the top level of a source file, we return the empty generic signature. Otherwise, we recursively evaluate the **generic signature request** against the parent.

In every other case, the generic signature request kicks off the lower-level **inferred generic signature request**, handing it a list of arguments:

1. The generic signature of the parent context, if any.

   The generic signature of a nested declaration extends that of the parent with additional generic parameters and requirements.

2. The current generic context's generic parameter list, if any.

   At least one of the first two inputs must be specified; absent a parent signature or any generic parameters to add, the resulting generic signature is necessarily empty, and the caller handles this case by not evaluating the request at all.

3. The current generic context's trailing `where` clause, if any.

   The inferred generic signature request will call out to type resolution to resolve the requirement representations written here into requirements.

4. Any additional requirements to add.

   This enables a shorthand syntax for declaring a constrained extension by writing a generic nominal type or a "pass-through" generic type alias as the extended type (Section 10.3).

5. A list of types eligible for requirement inference.

   If we're building the generic signature of a function or subscript declaration, this consists of the declaration's parameter and return types; otherwise, it is empty (Section 11.1).

6. A source location for diagnostics.

Inferred generic signature request. A possibly apocryphal story says the name "inferred generic signature request" was chosen because "requirement inference" is one of the steps below, but this does not infer the generic signature in any real sense. Rather, this request transforms user-written requirements into a minimal, reduced form via a multi-step process shown in Figure 11.1:

1. **Requirement resolution** builds user-written requirements, from constraint types stated in generic parameter inheritance clauses and requirement representations in the trailing `where` clause. This happens in structural resolution stage (Chapter 9), so the resolved requirements may contain unbound dependent member types, which will reduce to bound dependent member types in requirement minimization.

   Diagnostics are emitted here if type resolution fails.

2. **Requirement inference** (Section 11.1) allows certain requirements to be omitted in source if they can be inferred. These inferred requirements are added to the list of user-written requirements.

3. **Decomposition and desugaring** (Section 11.2) transforms requirements collected by the first two stages, rewriting conformance and same-type requirements into a simpler form, and detecting trivial requirements that are always or never satisfied.

   Diagnostics are emitted here if a requirement is always unsatisfied as written.

4. **Requirement minimization** is where the real magic happens; we will talk about the invariants established by this in Section 11.4, and pick up the topic again when we get to the construction of a requirement machine from desugared requirements in Chapter 16.

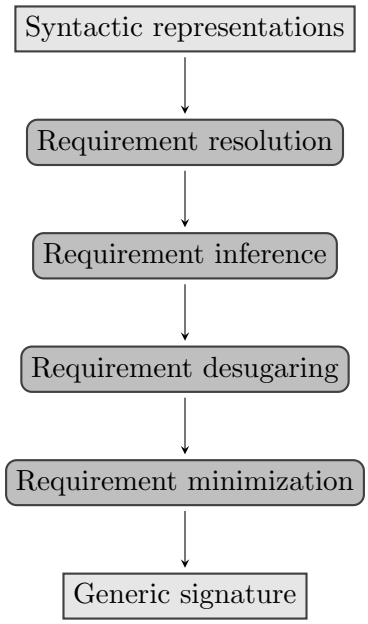
   Diagnostics are emitted here if no substitution map can satisfy the generic signature as written; the signature is said to contain *conflicting requirements* in this case.

The diagnostics mentioned above are emitted at the source location of the declaration, which is given to the request. This source location is used for one more diagnostic, an artificial restriction of sorts. Once we have a generic signature, we ensure that every innermost generic parameter is a reduced type. If a generic parameter is not reduced, it must be equivalent to a concrete type or an earlier generic parameter; it serves no purpose and should be removed, so we diagnose an error[1]:

```
// error: same-type requirement makes generic parameter 'T' non-generic
func add<T>(_ lhs: T, _ rhs: T) -> T where T == Int {
  return lhs + rhs
}
```

[1] Well, it's a warning prior to `-swift-version 6`.

Figure 11.1.: Overview of the inferred generic signature request



The restriction only concerns innermost generic parameters, so we allow this:

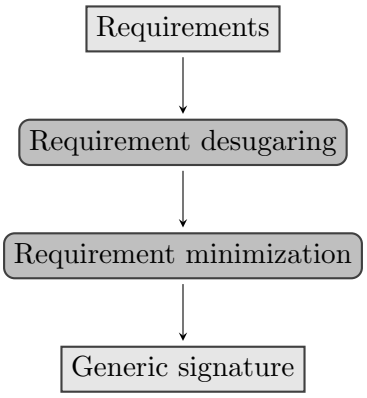
```
struct Outer<T> {
  func f() where T == Int {...}
}
```

And this:

```
extension Outer where Element == Int {
  func f() {...}
}
```

Abstract generic signature request. This request builds a generic signature from a list of generic parameters and requirements provided by the caller. The flow here is simpler, as shown in Figure 11.2. As input, it takes an optional parent generic signature, a list of generic parameter types to add, and a list of requirements to add. At least one of the first two parameters must be specified; if there is no parent generic signature and there are no generic parameters to add, the result would be the empty generic signature, which the caller is expected to handle by not evaluating this request at all.

Figure 11.2.: Overview of the abstract generic signature request



Like the inferred generic signature request, the abstract generic signature request decomposes, desugars and minimizes requirements; for this reason, it is preferred over using the primitive constructor. Often this request is invoked with a list of substituted requirements obtained by applying a substitution map to each requirement of some original generic signature. This request is used in various places:

- Computing the generic signature of an opaque type declaration (Chapter 13).
- Computing the generic signature for an opened existential type (Section 14.1).
- Checking class method overrides (Section 15.2).
- Witness thunks (Section 15.5).

This request does not perform requirement inference. No diagnostics are emitted either; instead, an error value is returned which is checked by the caller.

Requirement signature request. This request builds the requirement signature for a given *protocol component*, or a collection of one or more mutually-recursive protocols; this will be explained in Section 16.1. The evaluation function begins by evaluating two subordinate requests to collect the user-written requirements of each protocol:

- The **structural requirements request** collects associated requirements from the protocol's inheritance clause, associated type inheritance clauses, any `where` clauses on the protocol's associated types, and the `where` clause on the protocol itself. Refer to Section 4.3 for a description of the syntax.
- The **type alias requirements request** collects protocol type aliases and converts them to same-type requirements. These are discussed further in Section 18.4.

The requirement signature request takes these user-written associated requirements, and decomposes, desugars and minimizes them, much like the requirements of a generic signature. There is also a primitive constructor for requirement signatures. It is the last step of the requirement signature request. We also use it after reading a protocol from a serialized module, because the requirements were already minimal when serialized.

Protocol inheritance clauses. When searching for a member of a protocol, name lookup must also visit inherited protocols. Consider this example:

```
protocol Base {
  associatedtype Other: Base
  typealias Salary = Int
}

protocol Good: Base {
  typealias Income = Salary
}
```

The underlying type of `Income` in `Good` refers to `Salary` in `Base`. Protocol inheritance relationships are encoded in the protocol's requirement signature. A concrete type that conforms to `Good` must also conform to `Base`, so `Good` has an associated conformance requirement [`Self: Base`]. Name lookup cannot issue generic signature queries though, because building the requirement signature depends on type resolution, which depends on name lookup. To avoid request cycles, name lookup must interpret the inheritance clause of a protocol directly, without interfacing with generics. This introduces a minor limitation, which we now describe.

A general fact is that the protocol inheritance relation is transitive, so `Most` also inherits from `Base` here, because `Most` inherits from `Good`, and `Good` inherits from `Base`:

```
protocol Most: Good {}
```

We can understand this behavior by writing down a derivation for the requirement [`τ_0_0: Base`] in the protocol generic signature $G_{\texttt{Most}}$. We start with [`τ_0_0: Most`] and apply the associated conformance requirements [`Self: Good`]$_{\texttt{Most}}$ and [`Self: Base`]$_{\texttt{Good}}$:

1. [`τ_0_0: Most`] (Conf)
2. [`τ_0_0: Good`] (AssocConf 1)
3. [`τ_0_0: Base`] (AssocConf 2)

When the protocol inheritance relationship is a syntactic consequence of the inheritance clause, we can write down a derivation like the above where every step has a subject type

of `τ_0_0`. This is why a name lookup into `Good` knows to look into `Base`. On the other hand, it is possible to devise a protocol declaration where [`τ_0_0: Base`] is a non-trivial consequence of a same-type requirement between `τ_0_0` and some other type parameter. For example, here we have $G_{\texttt{Bad}} \vdash$ [`τ_0_0: Base`], which is not immediately apparent, because nothing is stated in the protocol's inheritance clause:

```
protocol Bad {
  associatedtype Tricky: Base where Self == Tricky.Other
  typealias Income = Salary  // error
}
```

We diagnose an error, because we are unable to resolve `Salary` as a member of `Bad`. To understand why, notice that the derivation $G_{\texttt{Bad}} \vdash$ [`τ_0_0: Base`] involves the associated same-type requirement $[\texttt{Self == Self.Tricky.Other}]_{\texttt{Bad}}$, so it is not a consequence of the protocol's syntactic inheritance clause:

1. [`τ_0_0: Bad`] (Conf)
2. [`τ_0_0.Tricky: Base`] (AssocConf 1)
3. [`τ_0_0.Tricky.Other: Base`] (AssocConf 2)
4. [`τ_0_0 == τ_0_0.Tricky.Other`] (AssocSame 1)
5. [`τ_0_0: Base`] (SameConf 3 4)

A special code path detects this problem and diagnoses a warning to explain what is going on. After building a protocol's requirement signature, the **type-check source file request** verifies that any conformance requirements known to be satisfied by `Self` actually appear in the protocol's inheritance clause, or `where` clause entries with a subject type of `Self`.

In our example, we discover the unexpected derived requirement [`τ_0_0: Base`] of $G_{\texttt{Bad}}$, but at this point it is too late to attempt the failed name lookup of `Salary` again. The compiler instead suggests that the user should explicitly state the inheritance from `Base` in the inheritance clause of `Bad`:

```
$ swiftc bad.swift

bad.swift:8:22: error: cannot find type ‘Salary’ in scope
  typealias Income = Salary
                     ^~~~~~
bad.swift:6:10: warning: protocol ‘Bad’ should be declared to refine
‘Base’ due to a same-type constraint on ‘Self’
protocol Bad {
         ^
```

A mistake of this sort is baked into the standard library's `SIMDScalar` protocol. The `Self` type of `SIMDScalar` must conform to all of `Equatable`, `Hashable`, `Encodable` and `Decodable`, via an associated same-type requirement:

```
public protocol SIMDScalar {
  ...
  associatedtype SIMD2Storage: SIMDStorage
    where SIMD2Storage.Scalar == Self
  ...
}

public protocol SIMDStorage {
  associatedtype Scalar: Codable, Hashable
  ...
}
```

However, these requirements are not stated in `SIMDScalar`'s inheritance clause. Since the inheritance clause of a protocol is part of the Swift ABI, this omission cannot be fixed at this point, so the type checker specifically muffles the warning when building the standard library.

11.1. Requirement Inference

Consider a function that returns the unique elements of a collection:

```
func uniqueElements<S: Sequence>(_ seq: S) -> Set<S.Element>
    where S.Element: Hashable {...}
```

When resolving the function's return type, we must establish that the generic argument `τ_0_0.Element` satisfies the requirements in the generic signature of the `Set` type declared in the standard library:

```
struct Set<Element: Hashable> {...}
```

The sole substituted requirement is satisfied (Section 9.3) because the generic signature of `uniqueElements()` contains the requirement [`τ_0_0.Element: Hashable`]. In fact, this requirement *has* to be part of our function's generic signature, for the return type, and thus the overall function declaration, to be valid at all. The *requirement inference* feature allows us to omit this requirement:

```
func uniqueElements<S: Sequence>(_ seq: S) -> Set<S.Element>
    /* where S.Element: Hashable */ {...}
```

These *inferred requirements* are not the derived requirements in the sense of Section 5.2. They appear in the generic signature, along with user-written requirements; they are not consequences of other requirements. Both declarations of `uniqueElements()` above really are identical, and in particular they have the *same* generic signature with an explicit conformance requirement [`τ_0_0.Element: Hashable`]. Consumers of generic signatures do not know or care about requirement inference.

Inferred requirements. When building the generic signature of a declaration, we collect inferred requirements by visiting type representations written in the following specific positions:

1. The parameter and return types of function and subscript declarations, if the generic declaration we're given is one of those. In our `uniqueElements()` example, we infer requirements from the function's return type.

2. Types appearing in inheritance clauses of generic parameter declarations. The only interesting case is a generic superclass bound. Here, the generic signature of `Foo` has the explicit superclass requirement [`τ_0_0: Base<τ_0_1>`] and an inferred requirement [`τ_0_1: Equatable`], the latter written in the commented-out `where` clause:

```
class Base<T: Equatable> {...}
struct Foo<T: Base<U>, U>
    /* where U: Equatable */ {...}
```

3. Types appearing within the requirements of a `where` clause; for example, the right-hand side of a same-type requirement. Here, we get the inferred requirement [`τ_0_1: Hashable`]:

```
struct Foo<T: Sequence, U>
    where T.Element == Set<U> /*, U: Hashable */ {...}
```

4. The typed throws feature, introduced with Swift 6 [89], allows specifying the type of error thrown by a function. For function, subscript and constructor declarations, we infer a conformance requirement to `Error` with the declaration's thrown error type as the subject type:

```
func f<E>(_: Int) throws(E) /* where E: Error */ {...}
```

   If requirement inference encounters a function type in any other position, it also considers the thrown error type there:

```
func f<E>(_: () throws(E) -> ()) /* where E: Error */ {...}
```

(From this list, we see that types appearing in the *body* of a declaration do not participate in requirement inference; it would be quite unfortunate if the interface type of a function, for example, could not be determined without type checking the function body.)

We resolve each of the enumerated type representations in the structural resolution stage, because the generic signature of the current declaration isn't known; we're building it right now. Let's say that H is the generic signature being built; we can still talk about H in the abstract, we just can't define anything concrete to depend on generic signature queries against H. The resolved type possibly contains type parameters of H, so it is element of $\textsc{Type}(H)$. From the resolved type, we obtain a set of inferred requirements by walking its child types recursively. We look for children that are generic nominal types or generic type alias types, and extract a substitution map from each one:

- For a generic nominal type, we take the context substitution map (Section 6.1).
- For a generic type alias type, we take the substitution map stored inside the type.

Let's call this substitution map Σ, and let G be the input generic signature of Σ. This is the generic signature of the referenced nominal type or type alias declaration. The output generic signature of Σ is H, so $\Sigma \in \textsc{Sub}(G, H)$. We proceed as if we were checking generic arguments, applying Σ to each requirement of G. This gives us a substituted requirement in $\textsc{Req}(H)$. (Recall that when checking generic arguments in the interface stage, we worked with archetypes, so we'd get something in $\textsc{Req}(\llbracket H \rrbracket)$; here, we instead want the substituted requirement to talk about type parameters).

In all examples we've seen so far, the inferred requirement's subject type was a type parameter of H, and the inferred requirement was just added to H unchanged. To preview the next section, the requirement's subject type might be fully concrete after substitution; for example, here we get the useless inferred requirement [`Int: Hashable`]:

```
func f<T>(_: T, _: Set<Int>) {}
```

This requirement is not a statement about the type parameters of `f()` at all; it is just "always true," so it does not affect the generic signature of `f()`. A more complex scenario is possible with conditional conformances. In the next example, the inferred requirement is [`τ_0_0: Hashable`]:

```
func f<T>(_: Set<Array<T>>) /* where T: Hashable */ {}
```

From the generic nominal type `Set<Array<τ_0_0>>`, we obtain the inferred conformance requirement [`Array<τ_0_0>: Hashable`]. The standard library declares a conditional conformance of `Array` to `Hashable` when the element type is `Hashable`. We replace our original requirement with the conditional requirements of this conformance, which gives us [`τ_0_0: Hashable`].

In general, when a requirement's subject type is not a type parameter, we repeatedly rewrite it into a (possibly empty) set of simpler requirements, until only requirements about type parameters remain; this *requirement desugaring* procedure is described in the next section.

After desugaring, inferred requirements are passed on to requirement minimization, where they might become redundant. For example, the user might re-state all inferred requirements explicitly, as in our original version of `uniqueElements()`; these duplicate requirements are eliminated by requirement minimization. More elaborate examples of redundant inferred requirements can also be constructed. In the below, we infer the requirement [`τ_0_0.Iterator: IteratorProtocol`], but it's not an explicit requirement of the generic signature of `f()` because it can be derived from [`τ_0_0: Sequence`]:

```
struct G<T: IteratorProtocol> {}

func f<T: Sequence>(_: T) -> G<T.Iterator> {...}
```

Of course, the user can explicitly re-state the redundant requirement in the `where` clause just for good luck, which would make it twice redundant, in a way.

Outer generic parameters. The outer generic parameters of a declaration can be subject to inferred requirements. Consider a generic struct with three methods:

```
struct G<T, U> {
  func f<V>(_: Set<U>, _: V) /* where U: Hashable */ {}
  func f(_: Set<U>) where T: Equatable /*, U: Hashable */ {}
  func f(_: Set<U>) {}  // error!
}
```

We infer requirements if the declaration has generic parameters *or* a `where` clause, so in the first two, we infer [`τ_0_1: Hashable`]. The third method inherits the generic signature of the struct, so the requirement is not satisfied and we diagnose an error.

Generic type aliases. A reference to a generic type alias resolves to a sugared type alias type (Section 3.2). This sugared type prints as written when it appears in diagnostic messages, but it is canonically equal to its substituted underlying type, behaving like it otherwise. So here, the interface type of "`x`" prints as `OptionalElement<Array<Int>>`, but it is canonically equal to its substituted underlying type, `Optional<Int>`:

```
typealias OptionalElement<T: Sequence> = Optional<T.Element>

let x: OptionalElement<Array<Int>> = ...
```

In the above, we form the substituted underlying type `Optional<Int>` by applying a substitution map that replaces `τ_0_0` with `Array<Int>` to the underlying type of the type alias declaration, `Optional<T.Element>`. This gives us the structural components of a type alias type: a reference to a type alias declaration, a substituted underlying type, and a substitution map. The substitution map is used when printing the sugared type's generic arguments. Cruicially to our topic at hand, this substitution map is also considered by requirement inference, making this one of a handful of language features where the appearance of a sugared type *does* have a semantic effect. In this example, we infer the requirement [`τ_0_0: Sequence`] from considering `OptionalElement<τ_0_0>`:

```
func maybePickElement<T>(_ sequence: T) -> OptionalElement<T>
```

Mostly of theoretical interest is the fact that the underlying type of a type alias need not mention the generic parameter types of the type alias at all. Before parameterized protocol types were added to the language, a few people discovered a funny trick to simulate something similar:

```
typealias SequenceOf<T, E> = Any
    where T: Sequence, T.Element == E
```

The underlying type of `SequenceOf` is just `Any`, and it's generic signature has the two requirements [`τ_0_0: Sequence`] and [`τ_0_1 == τ_0_0.Element`]. Now, we can state this type alias in the inheritance clause of a generic parameter declaration:

```
func sum<S: SequenceOf<S, Int>>(_: T) {...}
```

The inheritance clause entry introduces the requirement [`τ_0_0: Any`]; this has no effect, because `Any` is the empty protocol composition. However, requirement inference also visits the type alias type `SequenceOf<S, Int>`, which is the whole trick. This type alias type has the following substitution map:

$$\Sigma := \{\texttt{τ\_0\_0} \mapsto \texttt{τ\_0\_0},\ \texttt{τ\_0\_1} \mapsto \texttt{Int};\ [\texttt{τ\_0\_0: Sequence}] \mapsto [\texttt{τ\_0\_0: Sequence}]\}$$

Applying Σ to the generic signature of `SequenceOf` gives us our two inferred requirements:

$$[\texttt{τ\_0\_0: Sequence}] \otimes \Sigma = [\texttt{τ\_0\_0: Sequence}]$$
$$[\texttt{τ\_0\_1 == τ\_0\_0.Element}] \otimes \Sigma = [\texttt{τ\_0\_1 == Int}]$$

These requirements survive minimization and appear in the generic signature of `sum()`. We get the same generic signature as both of the following:

```
func sum<S: Sequence<Int>>(_: T) {...}
func sum<S: Sequence>(_: T) where S.Element == Int {...}
```

Protocols. Unlike the inferred generic signature request, the requirement signature request does not perform requirement inference; all associated requirements imposed on the concrete conforming type must be explicitly stated in source. For example, we must state the [`Self.Particle: Hashable`] requirement below, because otherwise the type `Set<Self.Particle>` appearing in the same-type requirement would not satisfy `Set`'s [`τ_0_0: Hashable`] requirement:

```
protocol Cloud {
  associatedtype Particle /* must be : Hashable */
  associatedtype Particles: Sequence
      where Particles.Element == Set<Particle>  // error
}
```

The technical reason for this restriction is the following. Before we can build the requirement signature of a protocol, we must construct the *protocol dependency graph*. This graph, which we will meet in Section 12.3, encodes the relation between each protocol, and any protocols that appear on the right-hand sides of its associated conformance requirements. A key property is that this graph can be recovered by performing name lookup operations alone, without queries against other requirement signatures or generic signatures. If we could infer requirements in protocols, this would no longer hold; we could define `Cloud` as above without explicit mention of `Hashable`, so our protocol dependency graph would have a "non-obvious" edge relation.

11.2. Decomposition and Desugaring

Before proceeding to requirement minimization, we eliminate some unneeded generality in the specification of requirements. To backtrack a little, here is where we are:

- If we're evaluating the **inferred generic signature request**, we're about half-way through Figure 11.1; we've gathered a set of user-written and inferred requirements that together describe the type parameters of a generic declaration.
- If we're evaluating the **abstract generic signature request**, we *start* here; as shown in Figure 11.2, the caller gives us a set of requirements as input.

In the derived requirements formalism, the right-hand side of a conformance requirement is always a protocol type, however in the syntax we can also write a protocol composition type, or a parameterized protocol type. Such conformance requirements must be split up—this is *requirement decomposition*. We also recall that the left-hand side of a derived requirement is always a type parameter, whereas requirement inference (or even the user) can write down a requirement with any subject type. Such requirements are also split up into zero or more simpler requirements, which gives us *requirement desugaring*.

Decomposition. This formalizes the syntax sugar from Sections 4.2 and 4.3. For example, the standard library defines the `Codable` type alias, whose underlying type is a composition of two protocols, `Decodable` and `Encodable`:

```
typealias Codable = Decodable & Encodable
```

We can state conformance to `Codable` in the inheritance clause of a generic parameter:

```
func ride<Horse: Codable>(_: Horse) {}
```

We can understand the above by decomposing [`τ_0_0: Decodable & Encodable`] into the two conformance requirements [`τ_0_0: Decodable`] and [`τ_0_0: Encodable`]. The right-hand side of a conformance requirement might also be parameterized protocol type, which is sugar for a series of same-type requirements, constraining each of the protocol's primary associated types to the corresponding generic argument:

```
func search<E, S: Sequence<E>>(...) {}
```

The conformance requirement [`τ_0_1: Sequence<τ_0_0>`] written above decomposes into two requirements, [`τ_0_1: Sequence`] and [`τ_0_1.[Sequence]Element == τ_0_0`].

We now present requirement decomposition in the form of an algorithm. Mostly this is a restatement of the above, with additional details and a couple of edge cases.

Algorithm 11A (Decompose conformance requirement)**.** Takes a list of conformance requirements as input. Outputs a new list of equivalent requirements where all conformance requirements have a protocol type on the right hand side.

1. Initialize an empty list to collect the output requirements.
2. Add all input requirements to the worklist.
3. (Check) If the worklist is empty, return the output list.
4. (Loop) Remove a conformance requirement [`T: X`] from the worklist, where `X` is a "protocol-like" type, with one of the three kinds below.
5. (Base case) If `X` is a protocol type, output the conformance requirement [`T: X`].
6. (Composition) If `X` is a protocol composition type $\texttt{M}_1$ `&` `...` `&` $\texttt{M}_n$, visit each member $\texttt{M}_i \in$ `X`. If $\texttt{M}_i$ is a class type, output the superclass requirement [`T:` $\texttt{M}_i$]. If $\texttt{M}_i$ is `AnyObject`, output the layout requirement [`T: AnyObject`]. Otherwise, $\texttt{M}_i$ is either a protocol type, or it can be decomposed further. Add the conformance requirement [`T:` $\texttt{M}_i$] to the worklist.

7. (Parameterized) If `X` is a parameterized protocol type `P<B`$_1$`, ..., B`$_n$`>` with base protocol type `P` and generic arguments `B`$_i$, decompose the requirement as follows:
   a) Output the conformance requirement [`T: P`].
   b) For each primary associated type `A`$_i$ of `P`, apply the protocol substitution map for `T` to the declared interface type of `A`$_i$:

$$\texttt{Self.[P]A}_i \otimes \left\{ \begin{array}{l} \texttt{Self} \mapsto \texttt{T}; \\ [\texttt{Self: P}] \mapsto [\texttt{T: P}] \end{array} \right\} = \texttt{T.[P]A}_i$$

   Output the same-type requirement [`T.[P]A`$_i$ `== G`$_i$].

8. (Error) If `X` is anything else, we have an invalid requirement, for example [`T: Int`]. Diagnose an error.

9. (Next) Go back to Step 3.

Usually `T` will be a type parameter in Step 7, and the substituted type becomes the dependent member type `T.[P]A`$_i$. Our algorithm relies on type substitution to also cope with `T` being a concrete type. When `T` is a concrete type, the subject type of each introduced same-type requirement has to be the type witness for `A`$_i$ in the concrete conformance [`T: P`]. For example, given [`Array<τ_0_0>: Sequence<Int>`], we output [`Array<τ_0_0>: Sequence`] and [`τ_0_0 == Int`]. Admittedly, this is a rather a silly edge case. One might think that requirement inference would exercise this scenario:

```
typealias G<T: Sequence<Int>> = T
func f<E>(_: G<Array<E>>) {}
```

However, we build the generic signature of `G` first, so we've already decomposed the requirement [`τ_0_0: Sequence<Int>`] by the time we get around to building the generic signature of `f()`; when we visit `G<Array<τ_0_0>` in requirement inference, we substitute the decomposed and desugared (and indeed, the minimal) requirements of `G`. In fact, the only way to exercise this is to write it down directly:

```
func f<E>(_: Array<E>) where Array<E>: Sequence<Int> {}
```

A more useful application of decomposition is the following. The **abstract generic signature request** performs decomposition to deal with the opaque return types of Chapter 13 and existential types of Chapter 14. We reason about these kinds of types by using this request to build an "auxiliary" generic signature describing the type. The constraint type after the `some` or `any` keyword defines a conformance requirement; this requirement must be decomposed by the same algorithm, so that we can interpret something like "`any Sequence<Int>`" or "`some Equatable & AnyObject`".

Desugaring. Requirement inference is one of several instances of a useful pattern: we apply a substitution map to the minimal requirements of a generic signature, and build a new generic signature from these substituted requirements. Section 15.2 will use the **abstract generic signature request** in this way to check class method overrides, for example. When the original requirements are minimal, the substituted requirements are already decomposed, but it is possible for such a requirement to have a left-hand side that is not a type parameter, but a concrete type, introduced by the substitution.

A thought experiment helps us understand the meaning of a requirement whose left-hand side is not a type parameter. If the subject type of some requirement R is an interface type, not necessarily a type parameter, we cannot describe it with the derived requirements formalism. However, we can still apply a substitution map Σ to R, and check if the substituted requirement $R \otimes \Sigma$ is satisfied with Algorithm 9A; nothing in the algorithm depends on the *original* requirement's subject type being a type parameter.

We will transform R into a set of simpler requirements $\{R_1, \ldots, R_n\}$, such that for *every* substitution map Σ, $R \otimes \Sigma$ is satisfied if and only if for all $i \leq n$, $R_i \otimes \Sigma$ is satisfied. Of course we're not iterating over all possible substitution maps in the implementation, this is just a thought experiment. If we believe that the below transformation upholds the desired property, we can conclude that R can be replaced with $\{R_1, \ldots, R_n\}$ without changing the meaning of our generic signature.

This rule essentially determines the implementation of requirement desugaring. Let's first consider a requirement that does not contain type parameters at all, such as [`Int: Hashable`] or [`Int == String`]. Applying a substitution map cannot ever change our requirement, so it is always true or always false; we can check it with Algorithm 9A:

- If the requirement is satisfied, we we can delete it, that is, replace it with the empty set of requirements, without violating our invariant. It doesn't contribute anything new.
- If the requirement is unsatisfied on the other hand, the only way to proceed is to replace it with something else unsatisfiable, so we must diagnose an error and give up.

The second case merits further explanation. A generic declaration whose requirements cannot be satisfied by any substitution map is essentially useless, and we will see in Section 11.4 that we try to uncover situations where two requirements are in conflict with each other and cannot be simultaneously satisfied. Here though, requirement desugaring detects the trivial case where a requirement "conflicts" with itself.

Next, we consider a conformance requirement with a left-hand side that contains type parameters, like [`Array<τ_0_0>: Sequence`] or [`Array<τ_0_0>: Hashable`]. With global conformance lookup, we can find the concrete conformance of the subject type to this protocol. If the conformance is invalid, we have a trivial conflict; if unconditional, the requirement is trivially satisfied.

If we get a conditional conformance here, our invariant *leaves us no choice* but to replace the original requirement with the conditional requirements of this conformance (an unconditional conformance has an empty set of conditional requirements, so there's really just one rule here). For example, because of the conditional conformance of `Array` to `Hashable`, we know that `Array<τ_0_0>` $\otimes \Sigma$ conforms to `Hashable` if and only if `τ_0_0` $\otimes \Sigma$ conforms to `Hashable`, for any substitution map Σ. Therefore, [`Array<τ_0_0>: Hashable`] must desugar to [`τ_0_0: Hashable`], which explains how we infer [`τ_0_0: Hashable`] from `Set<Array<τ_0_0>>`, an example we saw earlier:

```
func f<T>(_: Set<Array<T>>) /* where T: Hashable */ {}
```

Finally, to desugar a same-type requirement, we consider four possibilities:

1. Both sides are type parameters.
2. Left-hand side is a type parameter, right-hand side is a concrete type.
3. Left-hand side is a concrete type, right-hand side is a type parameter.
4. Both sides are concrete types.

The first two cases already have the correct form, so we're done. The third case reduces to the second, because we can swap the two sides. Indeed, a same-type requirement is also a statement that both sides have the same reduced type, and this relation is symmetric. In the fourth case, we have something like this,

[`Dictionary<τ_0_0, Bool> == Dictionary<Int, τ_0_1>`]

which we would like to desugar into two requirements:

[`τ_0_0 == Int`]
[`Bool == τ_0_1`]

The first output is already desugared; the second one must be flipped, and then we're done. On the other hand, say we're given [`Array<τ_0_0> == Set<τ_0_0>`] instead. No substitution map can transform the two sides into the same type, so this requirement can never be satisfied. This gives us the general rule.

For two concrete types to be equivalent under all substitutions, they can only differ in certain ways. Two types *match* if they have the same kind, same number of structural component types, and exactly equal non-type information (examples of this include the declaration of a nominal type, the labels of a tuple, value ownership kinds of function parameters, and so on). Our definition of matching is not recursive, so `Array<Array<Int>>` and `Array<Set<Int>>` still match because everything lines up at the outermost level, but their two children `Array<Int>` and `Set<Int>` do not match.

If the two types in a same-type requirement do not match, we have a conflict, so we diagnose an error and give up. Otherwise, the types have an equal number of children; we walk the children in parallel and construct a simpler set of same-type requirements equivalent to the original. This is a recursive process; some of these requirements might need further desugaring, or lead to conflicts, and so on.

Algorithm 11B (Desugar same-type requirement)**.** Takes an arbitrary same-type requirement as input. Outputs a list of desugared requirements and a list of conflicting requirements.

1. Initialize empty output and conflict lists.
2. Add the input requirement to the worklist.
3. (Next) Take the next requirement [`T == U`] from the worklist.
4. (Abstract) If `T` and `U` are both type parameters, add [`T == U`] to the output list.
5. (Concrete) If `T` is a type parameter and `U` is concrete, output [`T == U`].
6. (Flipped) If `T` is concrete and `U` is a type parameter, output [`U == T`].
7. (Redundant) If `T` and `U` are canonically equal, nothing non-trivial will be generated below, so we immediately go to Step 10.
8. (Recurse) If `T` and `U` match, let $\texttt{T}_1 \ldots \texttt{T}_n$ and $\texttt{U}_1 \ldots \texttt{U}_n$ be the children of `T` and `U`. For each $1 \leq i \leq n$, add [$\texttt{T}_i$ `==` $\texttt{U}_i$] to the worklist.
9. (Conflict) If `T` and `U` do not match, add [`T == U`] to the conflict list and diagnose.
10. (Loop) If the worklist is empty, return. Otherwise, go back to Step 3.

Next we have the algorithm for desugaring an arbitrary requirement. This runs after Algorithm 11A, so we assume conformance requirements have been decomposed. Notice how this algorithm generalizes the "requirement is satisfied" check of Algorithm 9A: if we desugar a requirement that does not contain any type parameters, the conflict list will be empty if and only if the requirement is satisfied.

Algorithm 11C (Desugar requirement)**.** Takes an arbitrary requirement as input. Outputs a list of desugared requirements and a list of conflicting requirements.

1. Initialize empty output and conflict lists.
2. Add the input requirement to the worklist.
3. (Next) Take the next requirement from the worklist. If the requirement's subject type is a type parameter, add it to the output list and go to Step 5.

4. (Desugar) Otherwise, the subject type is a concrete type. Handle each requirement kind as follows:
   a) For a **conformance requirement** [`T: P`], perform the global conformance lookup [`P`] ⊗ `T`:
      i. If we get a concrete conformance, add the conditional requirements, if any, to the worklist.
      ii. If we get an invalid conformance, add [`T: P`] to the conflict list.
   b) For a **superclass requirement** [`T: C`]:
      i. If `T` and `C` are two specializations of the same class declaration, add the same-type requirement [`T == C`] to the worklist.
      ii. If `T` does not have a superclass type (Chapter 15), then `T` cannot be a subclass of `C`; add [`T: C`] to the conflict list.
      iii. Otherwise, let `T′` be the superclass type of `T`. Add the superclass requirement [`T′: C`] to the worklist.
   c) For a **layout requirement** [`T: AnyObject`], any type parameters contained in the concrete type `T` have no bearing on the outcome. It suffices to apply Algorithm 9A. If unsatisfied, add [`T: AnyObject`] to the conflict list.
   d) For a **same-type requirement** [`T == U`], apply Algorithm 11B and add the results to the output and conflict lists.
5. (Loop) If the worklist is empty, return. Otherwise, go back to Step 3.

Requirements on the conflict list are diagnosed as errors. Requirements on the output list have the correct desugared form for minimization. To re-state:

Definition 11.1. A *desugared requirement* is a requirement where:

1. The left-hand side is a type parameter.
2. For a conformance requirement, the right-hand side is a protocol type.

11.3. Well-Formed Requirements

After desugaring and decomposition, our user-written requirements are now in a form where we can reason about them with the derived requirements formalism. So far, we have not attempted to impose any restrictions on the explicit requirements from which all else is derived, other than those requirements being *syntactically* well-formed. In the next section, we will precisely state the invariants of a generic signature, but first, we need to extend our theory with a notion of *semantically* well-formed requirements. It turns out we need this to diagnose certain malformed generic signatures.

To motivate our notion of well-formedness, let's return to the idea of a *valid type parameter* from Section 5.2. Specifically, we're going to look at prefixes of valid type parameters. Consider the type parameter `τ_0_0.Element.Element` below:

```
struct Concat<C: Collection> where C.Element: Collection {
  let x: C.Element.Element = ...
}
```

A type parameter is a *prefix* of another type parameter if it is equal to the other's base type, or the base type of the base type, at any level of nesting. So `τ_0_0.Element` and `τ_0_0` are the two prefixes of `τ_0_0.Element.Element`. From the user's point of view, if "`C.Element.Element`" is a meaningful utterance to write down in the program, "`C.Element`" certainly should be as well! This suggests a reasonable property that any "good" generic signature ought to have: *every prefix of a valid type parameter is itself a valid type parameter*. This is not our final condition though, so we generalize further.

When we say that a type parameter `T` is valid in a generic signature G, we mean that we have a derivation $G \vdash$ `T`. Only a handful of derivation steps have a type parameter as a conclusion (see Appendix B), and by inspecting these steps, we can see another way to characterize the validity of a type parameter `T`:

- If `T` is a generic parameter, then `T` is valid if and only if it appears in G.
- If `T` is an unbound dependent member type `U.A`, then `T` is valid if and only if some protocol `P` declares an associated type named `A`, and $G \vdash$ [`U: P`].
- If `T` is a bound dependent member type `U.[P]A`, then `T` is valid if and only if $G \vdash$ [`U: P`], where `P` is the parent protocol of the associated type declaration denoted by `[P]A`.

If we have one of those generic signatures where every prefix of `T` is also valid, then in the case where `T` is a dependent member type above, the subject type `U` of [`U: P`] is a prefix of `T`, so `U` must be valid. We say that [`U: P`] is a *well-formed* requirement if its subject type `U` is valid. This idea of well-formedness allows us to subsume "prefix validity" with a more general idea: *every derived conformance requirement must be well-formed*. In other words, we want $G \vdash$ [`U: P`] to imply $G \vdash$ `U` for all `U` and `P`.

We can also find motivation in type substitution. Let G be a generic signature where some derived requirement of G, not necessarily a conformance requirement, contains an invalid type parameter. By Algorithm 9A, no substitution map Σ can satisfy this requirement, because after we apply Σ, the invalid type parameter becomes an error type. Recalling our earlier notion of a well-formed *substitution map* from Definition 10.7, we conclude that G cannot have any well-formed substitution maps at all! To rule this out, we generalize our condition to cover all requirements, not just conformance requirements.

Definition 11.2. A requirement is *well-formed* with respect to a generic signature G if all type parameters contained in the requirement are valid type parameters of G:

- A **conformance requirement** [`T: P`] is well-formed if $G \vdash$ `T`.
- A **superclass requirement** [`T: C`], with $\{\texttt{C}_1, \ldots, \texttt{C}_n\}$ denoting the set of type parameters contained in `C`, is well-formed if $G \vdash$ `T` and $G \vdash \texttt{C}_i$ for all $1 \leq i \leq n$.
- A **layout requirement** [`T: AnyObject`] is well-formed if $G \vdash$ `T`.
- A **same-type requirement** [`T == U`], with $\{\texttt{U}_1, \ldots, \texttt{U}_n\}$ denoting the set of type parameters contained in `U`, is well-formed if $G \vdash$ `T` and $G \vdash \texttt{U}_i$ for all $1 \leq i \leq n$. (If the right-hand side `U` is a type parameter, the set of type parameters here is trivially {`U`}.)

We will be talking about both "derived" and "well-formed" requirements below, so before we completely lose the reader, let's make the distinction totally clear. Recall the generic signature of `Concat`:

```
<τ_0_0 where τ_0_0: Collection, τ_0_0.Element: Collection>
```

We cannot derive [`τ_0_0.Element.Element: Hashable`] from this generic signature; nothing in our signature is `Hashable`. However, this requirement is certainly well-formed, because `τ_0_0.Element.Element` is a valid type parameter in our generic signature. Thus, a derived requirement is provably *true*; a well-formed requirement makes sense as a *question*. So starting from prefix validity, we've arrived at our final statement: we want our generic signatures to only prove things that make sense!

Definition 11.3. A generic signature G is *well-formed* if all derived requirements of G are well-formed.

This definition doesn't immediately give us an algorithm for checking well-formedness. The derived requirements of a generic signature are an infinite set in general, so we cannot enumerate them all. We will resolve this dilemma shortly.

Notice how if $G \vdash$ [`T: P`] for some protocol `P`, the well-formedness of G implicitly depends on the well-formedness of the associated requirements of `P`; we interpret G with respect to its *protocol dependency set*, containing those protocols that can appear on the right-hand side of a derived conformance requirement. We will return to this topic in Section 16.1; for now, we can take this set to contain *all* protocols as a conservative approximation.

One more thing. The result we proved while motivating Definition 11.3 will be useful later, so let's re-state it for posterity:

Proposition 11.4. Let G be a well-formed generic signature, and let `T` be a valid type parameter of G. Then every prefix of `T` is also a valid type parameter of G.

Diagnostics. We now give an example of a generic signature that is not well-formed. We return to our `Concat` type, except here we "forgot" to state that `C` must conform to `Collection`:

```
struct Bad<C /* : Collection */> where C.Element: Collection {}
```

Nevertheless, starting with the explicit requirement [`τ_0_0.Element: Collection`], we can derive other requirements and valid type parameters; to pick a few at random:

1. [`τ_0_0.Element: Collection`] (Conf)
2. [`τ_0_0.Element: Sequence`] (AssocConf 1)
3. [`τ_0_0.Element.Iterator: IteratorProtocol`] (AssocConf 2)
4. `τ_0_0.Element.Iterator.Element` (AssocName 3)

The derived requirements (1) and (2) are not well-formed because their subject types are not valid type parameters, and the valid type parameter (4) has an invalid prefix `τ_0_0.Element`. Clearly, `Bad` ought to be rejected by the compiler. What actually happens when we type check `Bad`? Recall the type resolution stage from Chapter 9. We first resolve the requirement [`τ_0_0.Element: Collection`] in structural resolution stage, and we get a requirement whose subject type is an unbound dependent member type. We don't know that this requirement is not well-formed, yet.

After we build the generic signature for `Bad`, we revisit the `where` clause again, and resolve the requirement in the interface resolution stage. As the subject type is not a valid type parameter, type resolution diagnoses an error and returns an error type:

```
bad.swift:1:23: error: 'Element' is not a member type of type 'C'
struct Bad<C> where C.Element: Sequence {}
                      ^
```

Next, we define what it means for an associated requirement to be well-formed:

Definition 11.5. An associated requirement of a protocol `P` is *well-formed* if it is a well-formed requirement for the protocol generic signature $G_{\texttt{P}}$; that is, all type parameters it contains are valid type parameters in $G_{\texttt{P}}$.

A generic signature may depend on protocols written in source, or protocols from serialized modules. For protocols written in source, type resolution checks that their associated requirements are well-formed by visiting them again in the interface resolution stage, once the protocol's requirement signature has been built. Protocols from serialized modules already have well-formed associated requirements, because they were checked before serialization. Thus, if type resolution does not diagnose any errors, all user-written requirements are well-formed. The next theorem says this is a sufficient condition for all generic signatures in the main module to be well-formed.

Theorem 11.6. Suppose that a generic signature G satisfies these conditions:

- Every explicit requirement of G is well-formed.
- For every protocol `P` such that $G \vdash$ [`T: P`] for some `T`, every associated requirement of `P` is well-formed (in the protocol generic signature $G_{\texttt{P}}$).

Then every derived requirement of G is well-formed; in other words, G is well-formed.

To prove this theorem, we must expand our repertoire for reasoning about derivations. First, we recall the protocol generic signature from Section 5.1. If `P` is any protocol, then its generic signature, which we denote by $G_{\texttt{P}}$, has the single requirement [`Self: P`]. As always, the protocol `Self` type is sugar for `τ_0_0`.

The protocol generic signature describes the structure generated by the protocol's requirement signature. These are the valid type parameters and derived requirements inside the declaration of the protocol and its unconstrained extensions. These type parameters are all rooted in the protocol `Self` type, and the derived requirements talk about these `Self`-rooted type parameters. Informally, anything we can say about the protocol `Self` type in $G_{\texttt{P}}$, should also be true of an arbitrary type parameter `T` in some other generic signature G where $G \vdash$ [`T: P`]. We will now make this precise.

For example, we might first define an algorithm in a protocol extension of `Collection`, and then call our algorithm from another generic function:

```
extension Collection {
  func myComplicatedAlgorithm() {...}
}

func anotherAlgorithm<C: Collection>(_ c: C, _ index: C.Element.Index)
    where C.Element: Collection {
  c[index].myComplicatedAlgorithm()
}
```

The generic signature of `anotherAlgorithm()` is the same as `Concat` from earlier. Let's call it G. The reference to `myComplicatedAlgorithm()` has this substitution map:

$$\left\{ \begin{array}{l} \texttt{Self} \mapsto \texttt{τ\_0\_0.Element}; \\ [\texttt{Self: Collection}] \mapsto [\texttt{τ\_0\_0.Element: Collection}] \end{array} \right\}$$

The input generic signature is $G_{\texttt{Collection}}$, and output generic signature is G, so for generics to "work" we would expect that applying this substitution map to a valid type parameter or derived requirement of $G_{\texttt{Collection}}$ should give us a valid type parameter or derived requirement in G.

Suppose that `myComplicatedAlgorithm()` makes use of `Self.SubSequence.Index` conforming to `Comparable` inside the body. We can derive this requirement, together with the validity of `Self.SubSequence.Index` (so our requirement is well-formed):

1. [`Self: Collection`] (Conf)
2. [`Self.SubSequence: Collection`] (AssocConf 1)
3. `Self.SubSequence.Index` (AssocName 2)
4. [`Self.SubSequence.Index: Comparable`] (AssocConf 2)

We apply our substitution map to (3) and (4). Of course, we have yet to explain how to apply a substitution map to a dependent member type! We will get to that in the next chapter, but for now, let's make the simplifying assumption that we're performing a syntactic replacement of "`Self`" with "`τ_0_0.Element`". This gives us:

`τ_0_0.Element.SubSequence.Index`

[`τ_0_0.Element.Self.SubSequence.Index: Comparable`]

We can show that the first is a valid type parameter of G, and the second a derived requirement of G, by taking our original derivation in $G_{\texttt{Collection}}$, and replacing `Self` with `τ_0_0.Element` throughout:

1. [`τ_0_0.Element: Collection`] (Conf)
2. [`τ_0_0.Element.SubSequence: Collection`] (AssocConf 1)
3. [`τ_0_0.Element.SubSequence.Index: Comparable`] (AssocConf 2)

This works because [`τ_0_0.Element: Collection`] was an *explicit* requirement of G, so it's not as general as we'd like. Suppose we started with a more complicated *derived* conformance requirement, like $G \vdash$ [`τ_0_0.Indices: Collection`] in the same signature:

1. [`τ_0_0: Collection`] (Conf)
2. [`τ_0_0.Indices: Collection`] (AssocConf 1)

To build a derivation for $G \vdash$ [`τ_0_0.Indices.SubSequence.Index: Comparable`], we first replace the elementary step [`Self: Collection`] with the entire *derivation* of $G \vdash$ [`τ_0_0.Indices: Collection`]; then we substitute `Self` with `τ_0_0.Indices` in all remaining steps:

1. [`τ_0_0: Collection`] (Conf)
2. [`τ_0_0.Indices: Collection`] (AssocConf 1)
3. [`τ_0_0.Indices.SubSequence: Collection`] (AssocConf 2)
4. [`τ_0_0.Indices.SubSequence.Index: Comparable`] (AssocConf 3)

A protocol generic signature has two elementary derivation steps, so we might instead have a derivation that starts with the GENERIC elementary step for `Self`. For example, we can derive the requirement [`Self == Self`] in $G_{\texttt{Collection}}$:

1. `Self` (GENERIC)
2. [`Self == Self`] (REFLEX 1)

To get a derivation $G \vdash$ [`τ_0_0.Indices == τ_0_0.Indices`], we make use of the fact that `τ_0_0.Indices` is a valid type parameter of G. We replace the elementary derivation step for `Self` with the entire derivation of $G \vdash$ `τ_0_0.Indices`:

1. [`τ_0_0: Collection`] (CONF)
2. `τ_0_0.Indices` (ASSOCNAME 1)
3. [`τ_0_0.Indices == τ_0_0.Indices`] (REFLEX 2)

We now state the general result. We need this for the proof of Theorem 11.6, and also later in Section 12.1, when we show that every derived conformance requirement has a derivation of a certrain form.

Lemma 11.7 (Formal substitution)**.** Let G be an arbitrary generic signature. Suppose that $G \vdash$ `T` and $G \vdash$ [`T: P`] for some type parameter `T` and protocol `P`. Then if we take a valid type parameter or derived requirement of $G_{\texttt{P}}$ and replace `Self` with `T` throughout, we get a valid type parameter or derived requirement of G, just rooted in `T`. That is:

- If $G_{\texttt{P}} \vdash$ `Self.U`, then $G \vdash$ `T.U`.
- If $G_{\texttt{P}} \vdash R$, then $G \vdash R'$, where R' is the substituted requirement obtained by replacing `Self` with `T` in R.

Proof. Suppose we wanted to write down an *algorithm* for the above. As input, we're given a generic signature G, a pair of derivations $G \vdash$ `T` and $G \vdash$ [`T: P`], and some derivation in $G_{\texttt{P}}$. We rewrite the derivation in $G_{\texttt{P}}$ into a derivation in G by visiting each consecutive step. First, consider the elementary derivation steps of $G_{\texttt{P}}$:

`Self` (GENERIC)
[`Self: P`] (CONF)

Each one is replaced with the *entire derivation* of `T` or [`T: P`], respectively:

`T` (...)
[`T: P`] (...)

For all other derivation steps, we substitute `T` in place of `Self` anywhere it appears in the derivation step's statement. This gives us the desired derivation in G. □

A few words about the above proof. To be completely rigorous, we should go through each inference rule and argue that the substitution outputs a meaningful derivation step in each case. We're not going to do that, because we will demonstrate this form of exhaustive case analysis in the proof of Theorem 11.6 instead. Finally, note that our lemma makes the assumption that [`T: P`] is well-formed, because we need $G \vdash$ `T`. However, we are specifically not assuming that G is well-formed. In fact, we will use this lemma in the proof of Theorem 11.6, so such an assumption would be circular.

Structural induction. Suppose that $P(n)$ is some property of the natural numbers, and we wish to show that it is always true of all $n \in \mathbb{N}$. We can argue by *induction*, writing down a proof in two parts:

- The *base case*, that $P(0)$ is true.
- The *inductive step*, where we *assume* that $P(n)$ is true for a fixed but arbitrary n, and then prove that $P(n+1)$ is true as a consequence.

For example, let's say $P(n)$ is the statement that $0 + 1 + 2 + \cdots + n = n(n+1)/2$ for some given $n \in \mathbb{N}$. We can show this statement is always true it by induction:

- To show that $P(0)$, we substitute $n = 0$ into our formula. The left-hand side is zero, and the right-hand side is $n(n+1)/2 = 0(0+1)/2 = 0$.
- To show the inductive step, we assume $P(n)$, so, $0 + 1 + 2 + \cdots + n = n(n+1)/2$. We add $n+1$ to both sides. The right-hand side simplifies as follows, establishing $P(n+1)$:

$$\frac{n(n+1)}{2} + (n+1) = \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}$$

A proof by induction is sort of like a "recipe" that tells us how to derive a proof for some specific $n \in \mathbb{N}$. For example, suppose we wish to show $P(2)$. We start by writing down the base case $P(0)$. Then, the inductive step with $n = 0$ shows that $P(0)$ implies $P(1)$, but we know $P(0)$ is true, thus $P(1)$ is true. Next, we apply the inductive step with $n = 1$, and conclude that $P(2)$ is true. We can use our "recipe" to construct a proof for any n this way.

Proof by induction is an axiom of *Peano arithmetic*, a formal system for the natural numbers. One way to think of it, is that induction says we can reach any natural number by repeated application of the *successor function* "1+" to "0". Notice how this encoding of the natural numbers resembles a Swift enum:

```
enum Nat {
  case zero
  indirect case successor(Nat)
}
```

For example, $3 = 1 + (1 + (1 + 0))$, or indeed:

```
let three: Nat = .successor(.successor(.successor(.zero)))
```

Now, we return to our derived requirements formalism. Our statements are finite data structures, and while they are more complex than instances of `Nat`, they have the same essential quality:

- The elementary statements—the generic parameters and explicit requirements—play the role of the number 0.
- Every other statement is generated by applying a fixed inference rule to one or more previous statements. The inference rules are analogous to the successor function that increments a number by 1.

If P is a property of requirements and type parameters and G is some generic signature, we can establish that P is true for all derived requirements and valid type parameters of G by *structural induction*. A proof by structural induction has two parts:

- The *base case* establishes that $P(D)$ is true for every elementary statement D. Recall that these are the Generic steps for each generic parameter of G, and the steps for each explicit requirement of G.
- The *inductive step*, where we have a derivation step with assumptions $D_1, \ldots, D_n$ and conclusion D. We assume $P(D_1), \ldots, P(D_n)$ hold, then argue that $P(D)$ must hold as a consequence. We can perform a case analysis over the various inference rules to handle each kind of inference rule.

If we wish to prove a statement about derived requirements only, we slightly modify the above scheme. The base case no longer needs to consider Generic steps, but also, the Reflex step *becomes* a base case, because none of its assumptions are requirements. The following proof, which we are now in a position to state, uses the modified scheme.

Proof of Theorem 11.6. We're given a requirement R such that $G \vdash R$, and a type parameter contained in R. We must construct a derivation of this type parameter.

Base case. We start with the elementary statements:

[`T: P`]	(Conf)
[`T == U`]	(Same)
[`T == X`]	(Concrete)
[`T: C`]	(Super)
[`T: AnyObject`]	(Layout)

The first assumption in our theorem was that all explicit requirements of G are well-formed. In other words, every type parameter in R is a valid type parameter of G, so we're done. The other base case is that we have a requirement obtained by the Reflex inference rule, from some valid type parameter `T`:

$$[\texttt{T == T}] \qquad \text{(Reflex T)}$$

We must have $G \vdash \texttt{T}$, so the requirement $[\texttt{T == T}]$ is then well-formed by definition.

Inductive step. We must handle each kind of inference rule in turn. Consider an AssocBind step, for some type parameter `T`, protocol `P` and associated type `A`:

$$[\texttt{T.[P]A == T.A}] \qquad \text{(AssocBind [T: P])}$$

The conclusion is a derived same-type requirement that contains two type parameters, `T.[P]A` and `T.A`. We can derive both from the conformance requirement `[T: P]` using AssocDecl and AssocName:

1. `[T: P]` (...)
2. `T.[P]A` (AssocDecl 1)
3. `T.A` (AssocName 2)

Next, consider the SameName and SameDecl derivation steps:

$$[\texttt{T.A == U.A}] \qquad \text{(SameName [U: P] [T == U])}$$
$$[\texttt{T.[P]A == U.[P]A}] \qquad \text{(SameDecl [U: P] [T == U])}$$

We have four type parameters to derive: `T.A`, `T.[P]A`, `U.A` and `U.[P]A`. We first derive `[T: P]` from `[U: P]`, and then apply AssocDecl and AssocName to `[T: P]` and `[U: P]`:

1. `[U: P]` (...)
2. `[T == U]` (...)
3. `[T: P]` (SameConf 1 2)
4. `T.A` (AssocName 3)
5. `T.[P]A` (AssocDecl 3)
6. `U.A` (AssocName 1)
7. `U.[P]A` (AssocDecl 1)

To handle the derivation steps generated by the associated requirements of a protocol `P`, we must make use of the second assumption of our theorem, together with Lemma 11.7. The simplest case is an AssocConf or AssocLayout step:

$$[\texttt{T.U: Q}] \qquad \text{(AssocConf } [\texttt{Self.U: Q}]_{\texttt{P}} \text{ [T: P])}$$
$$[\texttt{T.U: AnyObject}] \qquad \text{(AssocLayout } [\texttt{Self.U: AnyObject}]_{\texttt{P}} \text{ [T: P])}$$

By the induction hypothesis, $G \vdash \texttt{T}$. We must show that $G \vdash \texttt{T.U}$. The requirement on the right-hand side is obtained by replacing `Self` with `T` in some associated conformance requirement $[\texttt{Self.U: Q}]_\texttt{P}$. By assumption, this associated requirement is well-formed with respect to $G_\texttt{P}$, so we have a derivation $G_\texttt{P} \vdash \texttt{Self.U}$. All the conditions of Lemma 11.7 are satisfied, thus we can construct a derivation $G \vdash \texttt{T.U}$.

For an associated same-type requirement $[\texttt{Self.U == Self.V}]_\texttt{P}$ we repeat the same construction to derive $G \vdash \texttt{T.U}$ and $G \vdash \texttt{T.V}$:

$$[\texttt{T.U == T.V}] \qquad (\textsc{AssocSame}\ [\texttt{Self.U == Self.V}]_\texttt{P}\ [\texttt{T: P}])$$

If we are looking at a concrete same-type requirement $[\texttt{Self.U == X}]_\texttt{P}$, or a superclass requirement $[\texttt{Self.U: C}]_\texttt{P}$, we again use Lemma 11.7 to derive each type parameter that appears in $\texttt{X}'$ and $\texttt{C}'$, which we use to denote the types obtained from `X` and `C`, respectively, by structural replacement of `Self` with `T`:

$$[\texttt{T.U == X}'] \qquad (\textsc{AssocConcrete}\ [\texttt{Self.U == X}]_\texttt{P}\ [\texttt{T: P}])$$
$$[\texttt{T.U: C}'] \qquad (\textsc{AssocSuper}\ [\texttt{Self.U: C}]_\texttt{P}\ [\texttt{T: P}])$$

Finally, all other derivation steps have the property that the type parameters appearing in their conclusion already appear in their assumptions, so by the induction hypothesis, the derived requirement is well-formed:

$$[\texttt{U == T}] \qquad (\textsc{Sym}\ [\texttt{T == U}])$$
$$[\texttt{T == V}] \qquad (\textsc{Trans}\ [\texttt{T == U}]\ [\texttt{U == V}])$$
$$[\texttt{T: P}] \qquad (\textsc{SameConf}\ [\texttt{U: P}]\ [\texttt{T == U}])$$
$$[\texttt{T == X}] \qquad (\textsc{SameConcrete}\ [\texttt{U == X}]\ [\texttt{T == U}])$$
$$[\texttt{T: C}] \qquad (\textsc{SameSuper}\ [\texttt{U: C}]\ [\texttt{T == U}])$$
$$[\texttt{T: AnyObject}] \qquad (\textsc{SameLayout}\ [\texttt{U: AnyObject}]\ [\texttt{T == U}])$$

This completes the induction. □

Formally, structural induction depends on a well-founded order (Section 5.4), so we would use the "containment" order on derivations. However, the "recursive algorithm" viewpoint is good enough for us. Induction over the natural numbers is covered in introductory books such as [70]; for structural induction in formal logic, see something like [90]. We will use structural induction over derivations again to study conformance paths in Section 12.1, encode finitely-presented monoids as protocols in Section 17.3, and finally present a correctness proof for the Requirement Machine in Chapter 18.

Before we end this section, we revisit bound and unbound type parameters from Section 5.3, and prove one final result. We previously claimed that every equivalence class of type parameters contains both a bound and unbound type parameter representative. This actually requires the assumption that our generic signature is well-formed.

Theorem 11.8. Let G be a well-formed generic signature, and suppose $\texttt{T}$ is a valid type parameter of G. Then, the equivalence class of $\texttt{T}$ also contains two type parameters (not necessarily unique), which we denote by $\texttt{T}^*$ and $\texttt{T}_*$, such that:

1. $\texttt{T}^*$ is a bound type parameter,
2. $\texttt{T}_*$ is an unbound type parameter,
3. $\texttt{T}^*$ and $\texttt{T}_*$ have the same length as $\texttt{T}$,
4. $\texttt{T}^* \leq \texttt{T} \leq \texttt{T}_*$ under the type parameter order.

Furthermore, if $\texttt{T}$ is a reduced type parameter, then $\texttt{T}^*$ is canonically equal to $\texttt{T}$, and thus every reduced type parameter of G is a bound type parameter.

Proof. We proceed by induction on the length of the type parameter $\texttt{T}$. In the base case, we prove that the property holds for all generic parameters; in the inductive step, we prove it holds for any dependent member type whose base type again has the property.

Base case. If $\texttt{T}$ is a generic parameter type $\texttt{τ\_d\_i}$, we can set both $\texttt{T}^*$ and $\texttt{T}_*$ to $\texttt{T}$, so $\texttt{T}^* \leq \texttt{T} \leq \texttt{T}_*$ holds. Then, we derive a trivial same-type requirement twice:

1. $\texttt{τ\_d\_i}$ (Generic)
2. $[\texttt{τ\_d\_i == τ\_d\_i}]$ (Reflex 1)
3. $[\texttt{τ\_d\_i == τ\_d\_i}]$ (Reflex 1)

Inductive step. Assume that $\texttt{T}$ is a dependent member type, either $\texttt{U.[P]A}$ (bound) or $\texttt{U.A}$ (unbound), for some $\texttt{U}$ and $\texttt{A}$ of $\texttt{P}$. We have $G \vdash [\texttt{U: P}]$ because $\texttt{T}$ is valid, and $G \vdash \texttt{U}$ because $[\texttt{U: P}]$ is well-formed. The length of $\texttt{U}$ is one less than the length of $\texttt{T}$, so the induction hypothesis then gives us a pair of same-type requirements $[\texttt{U == U}^*]$ and $[\texttt{U == U}_*]$, such that $\texttt{U}^* \leq \texttt{U} \leq \texttt{U}_*$. We set $\texttt{T}^* := \texttt{U}^*\texttt{.[P]A}$ and $\texttt{T}_* := \texttt{U}_*\texttt{.A}$, and derive three new same-type requirements via SameDecl, AssocBind and SameName:

1. $[\texttt{U.[P]A == T}^*]$ (SameDecl $[\texttt{U == U}^*]$ $[\texttt{U: P}]$)
2. $[\texttt{U.[P]A == U.A}]$ (AssocBind $[\texttt{U: P}]$)
3. $[\texttt{U.A == T}_*]$ (SameName $[\texttt{U == U}_*]$ $[\texttt{U: P}]$)

From the definition of the type parameter order in Section 5.4, we also see that:

$$\texttt{T}^* \leq \texttt{U.[P]A} < \texttt{U.A} \leq \texttt{T}_*$$

If $\texttt{T}$ is the bound dependent member type $\texttt{U.[P]A}$, we already have $[\texttt{T == T}^*]$ as (1), but we must derive $[\texttt{T == T}_*]$:

4. $[\texttt{T == T}_*]$ (Trans 2 3)

If `T` is the unbound dependent member type `U.A`, we find ourselves in the opposite situation. We already have [`T == T`$_*$] as (3), but we derive [`T == T`*] with Sym and Trans:

5. [`U.[P]A == T`] (Sym 2)
6. [`T == T`*] (Trans 5 1)

This completes the induction. To prove the second part of the theorem, we further assume that `T` is reduced. By the preceding argument, we get a bound type parameter `T`* such that `T`$^* \leq$ `T`. On the other hand, a reduced type parameter is the smallest element in its equivalence class, so `T` $\leq$ `T`*. We conclude that `T`* is canonically equal to `T` if `T` is reduced. □

11.4. Requirement Minimization

The last step shown in Figure 11.1 and 11.2 is called *requirement minimization*. To finish building our generic signature, we must transform the list of desugared requirements into a list of *minimal* requirements. These minimal requirements are then given to the primitive constructor, together with the list of generic parameter types collected at the start of the process, and we have our generic signature!

Because of the central role that generic signatures play in the Swift ABI, it is worth describing the requirement minimization problem in the abstract, which is our goal in the present section. The implementation itself will be revealed in Part V, when we build a convergent rewriting system from desugared requirements (Section 18.4) and then perform rewrite rule minimization (Chapter 22). The material in this section was based on [91], an earlier write-up about the `GenericSignatureBuilder`.

We can summarize the key behaviors of requirement minimization:

1. Type substitution only accepts bound dependent member types. To ensure that we can apply a substitution map to the requirements of a generic signature, as we do in Algorithm 9B for example, each requirement is rewritten to use bound dependent member types.

2. Generic signatures describe the calling convention of generic functions, the layout of nominal type metadata, the mangling of symbol names, and so on. To ensure that trivial syntactic changes do not affect ABI, each requirement in a generic signature is *reduced* into the simplest possible form, redundant requirements are dropped to produce a *minimal* list, and this list is sorted in canonical order.

3. We need to detect and diagnose generic signatures with *conflicting requirements* that cannot be satisfied by any well-formed substitution map. This allows us to assume that all generic signatures in the main module are satisfiable as long as no diagnostics are emitted during type checking.

Equivalence of requirements. In our derived requirements formalism, a generic signature is just a list of requirements. We saw in the previous section that the only real assumption we need from those requirements is that they are desugared requirements. Thus, when taken with the list of generic parameter types, the desugared requirements given to minimization already form a generic signature in our theory.

We will define a *minimal* generic signature as one satisfying the additional conditions we just described. We will understand requirement minimization as a mathematical function that takes a generic signature and outputs a minimal generic signature. In the implementation, all generic signatures come from requirement minimization, and so all generic signatures are actually minimal. To justify this transformation, we introduce the notion of *equivalent* generic signatures. Two generic signatures are equivalent if they generate the same *theory*; that is, they have the same set of valid type parameters and derived requirements. Note that in particular, if two generic signatures G_1 and G_2 are equivalent, then G_1 is well-formed if and only if G_2 is well-formed.

Proposition 11.9. Any two generic signatures G_1 and G_2 satisfying the below conditions are *equivalent*, meaning they generate the same theory:

1. G_1 and G_2 have the same list of generic parameter types.
2. Every explicit requirement of G_1 can be derived in G_2.
3. Every explicit requirement of G_2 can be derived in G_1.

Proof. We will show that G_1 and G_2 generate the same theory by first arguing that the theory of G_1 is a subset of the theory of G_2, and vice versa.

Suppose we are given a derivation $G_1 \vdash D$, where D is either a valid type parameter or derived requirement of G_1. We must show that D is a valid type parameter or derived requirement of G_2. The elementary derivation steps that can appear in $G_1 \vdash D$ are those defined by the generic parameters and explicit requirements of G_1. We can mechanically construct a derivation of $G_2 \vdash D$ from $G_1 \vdash D$ to arrive at our conclusion:

1. By the first assumption, every Generic step that derives a generic parameter of G_1 is already a valid Generic step for G_2, so we leave it unchanged.
2. By the second assumption, every elementary step for an explicit requirement of G_1 has a corresponding *derivation* of the same requirement in G_2. We replace each elementary step with its derivation.
3. All other derivation steps remain unchanged.

This is a derivation in G_2, and so the theory of G_1 is a subset of the theory of G_2. To get the other inclusion, we make the same argument but with G_1 and G_2 swapped, and use the third assumption in step 2 instead. □

Note that we do not attempt to delete "redundant" generic parameters. For example, `<T, U where T: Sequence, U == T.Element>` and `<T where T: Sequence>` are also "equivalent" in some broader sense, because any derivation in the former theory can be transformed into one for the latter by replacing `U` with `T.Element`. The latter is also more "minimal" because the calling convention for a function with this signature only needs to pass type metadata for `T`, rather than both `T` and `U`. Since we do not pursue this direction, our notion of equivalence bakes in the idea that both the original and minimal generic signatures have the same identical list of generic parameter types.

We want requirement minimization to output a minimal generic signature that is equivalent, but there are two important exceptions to keep in mind:

1. If any of the original requirements are not well-formed, we cannot rewrite them to use bound dependent member types, so we drop them instead, and the minimal generic signature describes a smaller theory. This is fine; type resolution will have diagnosed errors already, and we will not proceed to code generation.

2. Our derived requirements formalism does not explain all implemented behaviors of superclass, layout and concrete same-type requirements. If these requirement kinds are among the explicit requirements of the generic signature, or the associated requirements of some protocol, minimization may output a generic signature with a different theory. We will see examples of this after.

As an aside, Proposition 11.9 and Theorem 11.8 allow us to finally explain why we can take the explicit requirements in a derivation to contain bound *or* unbound dependent member types, depending on occasion. Indeed, sometimes we would omit the `[P]`'s to save space, other times we left them in to be explicit. It turns out that it doesn't matter; as long as the explicit requirements we start with are well-formed, we can derive either list from the other and arrive at the same theory.

Reduced requirements. We're now going to work towards a definition of a minimal generic signature. Consider this function:

```
func uniqueElements1<T: Sequence>(_: T) -> Int
    where T.Element: Hashable {}
```

When building the generic signature of `uniqueElements1()`, we get two user-written requirements from type resolution in the structural resolution stage:

{[`T: Sequence`], [`T.Element: Hashable`]}

The subject type of the second requirement is an unbound dependent member type formed from the generic parameter type `T` and identifier `Element`. We can replace the

second requirement with [`T.[Sequence]Element: Hashable`]; the latter's subject type is a bound dependent member type, formed from `T` and the associated type declaration `Element` of `Sequence`. To justify this, we can derive [`T.[Sequence]Element: Hashable`] from [`T.Element: Hashable`]:

1. [`T.Element: Hashable`] (Conf)
2. [`T: Sequence`] (Conf)
3. [`T.[Sequence]Element == T.Element`] (AssocBind 2)
4. [`T.[Sequence]Element: Hashable`] (SameConf 1 3)

And vice versa:

1. [`T.[Sequence]Element: Hashable`] (Conf)
2. [`T: Sequence`] (Conf)
3. [`T.[Sequence]Element == T.Element`] (AssocBind 2)
4. [`T.Element == T.[Sequence]Element`] (Sym 3)
5. [`T.Element: Hashable`] (SameConf 1 4)

This is how we arrive at the generic signature that we see if we test this example with the `-debug-generic-signatures` flag:

```
<T where T: Sequence, T.[Sequence]Element: Hashable>
```

Now, notice that the second conformance requirement's subject type is not just bound, it is *reduced*. This suggests a stronger condition. We could instead spell the conformance to `Hashable` with the subject type `T.Iterator.Element`:

```
func uniqueElements2<T: Sequence>(_: T) -> Int
    where T.Iterator.Element: Hashable {}
```

If we test this with `-debug-generic-signatures`, we see that `uniqueElements2()` has the same generic signature as `uniqueElements1()`. We start from these requirements:

{[`T: Sequence`], [`T.Iterator.Element: Hashable`]}

We can similarly show that this is an equivalent list:

{[`T: Sequence`],
 [`T.[Sequence]Iterator.[IteratorProtocol]Element: Hashable`]}

However, the reduced type of `T.Iterator.Element` is `T.[Sequence]Element`, because of the associated same-type requirement in `Sequence`, so in fact we can simplify the second conformance requirement further without changing the theory:

{[`T: Sequence`], [`T.[Sequence]Element: Hashable`]}

This is a list of *reduced* requirements, because the subject type of each is a reduced type parameter in our generic signature. We will define reduced requirements more generally now. In what follows, there is an important distinction between same-type requirements between type parameters ([`T.A == T.B`]), and same-type requirements where the right hand side is a concrete type ([`T.A == Array<T.B>`]). We will treat them as essentially two different kinds of requirements. The below definition is straightforward for every requirement kind, except for a same-type requirement between type parameters.

Definition 11.10. Let G be a generic signature, and let R be an explicit requirement of G. We say R is a *reduced requirement* if the following holds:

- For a **conformance requirement** [`T: P`]: `T` is a reduced type parameter.
- For a **layout requirement** [`T: AnyObject`]: `T` is a reduced type parameter.
- For a **superclass requirement** [`T: C`]: both `T` and `C` are reduced types.
- For a **same-type requirement** [`T == X`] where `X` is a **concrete type**: `T` is a reduced type parameter and `X` is a reduced type.
- For a **same-type requirement** [`T == U`] where `U` is a **type parameter**:
  1. `T` $<$ `U` under the type parameter order.
  2. `T` is either a reduced type parameter, or identical to the right-hand side of some other (explicit) same-type requirement in G.
  3. `T` is not identical to the left-hand side of any same-type requirement of G.
  4. `U` has the property that the only derivations $G \vdash$ [`U′ == U`] where `U′` $<$ `U` must involve the explicit requirement [`T == U`] itself; that is, `U` cannot be further reduced by any other requirement of G.

For a same-type requirement between type parameters, we cannot simply say that both sides are reduced type parameters, because the only way this can happen is if we have a trivial same-type requirement [`T == T`] for some reduced type parameter `T`. An example will clarify the situation. Recall this funny protocol from Section 5.3:

```
protocol N {
  associatedtype A: N
}
```

Now, consider these two generic structs:

```
struct Hook1<T, U> where U: N, T == T.A, T.A == U.A {}
struct Hook2<T, U> where U: N, T == T.A, U.A == T {}
```

We will look at `Hook1` first, but we will see that both in fact have the same generic signature. To build the generic signature of `Hook1`, we start with these user-written requirements:

{[`U: N`], [`T == T.A`], [`T.A == U.A`]}

As before, we derive an equivalent list involving only bound dependent member types:

{[`T: N`], [`T == T.[N]A`], [`T.[N]A == U.[N]A`]}

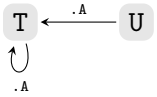
This list of requirements is already reduced and minimal, so we get this generic signature:

```
<T, U where U: N, T == T.[N]A, T.[N]A == U.[N]A>
```

To better understand this generic signature, note that we can derive [`T: N`]:

1. [`U: N`] (Conf)
2. [`U.[N]A: N`] (AssocConf 1)
3. [`T.[N]A == U.[N]A`] (Same)
4. [`T.[N]A: N`] (SameConf 1 3)
5. [`T == T.[N]A`] (Same)
6. [`T: N`] (SameConf 4 5)

Both `T` and `U` conform to [`N`], so each has a member type named `A`, but because of our same-type requirements, these member types are equivalent to `T`. We get this type parameter graph:



Next, we look at `Hook2`, which only differs in how the second same-type requirement is specified. We write the requirements of `Hook2` with bound dependent member types:

{[`U: N`], [`T == T.[N]A`], [`U.[N]A == T`]}

The last requirement is not reduced, because `U.[N]A` > `T`. However, we can derive [`T.[N]A == U.[N]A`], which is reduced, from [`U.[N]A == T`]:

1. [`U.[N]A == T`] (Same)
2. [`T == T.[N]A`] (Same)
3. [`U.[N]A == T.[N]A`] (Trans 1 2)
4. [`T.[N]A == U.[N]A`] (Sym 3)

And vice versa:

1. [`T.[N]A == U.[N]A`] (Same)
2. [`T == T.[N]A`] (Same)
3. [`T == U.[N]A`] (Trans 1 2)
4. [`U.[N]A == T`] (Sym 3)

This shows that `Hook1` and `Hook2` actually have the same minimal generic signature, which `-debug-generic-signatures` will confirm.

There's a bit of history behind this test case. Our derivation of [`T: N`] involved the same-type requirement [`T == T.A`], which contains a member type of `T`. This was too difficult for the `GenericSignatureBuilder` to reason about, so we only accepted `Hook2` but not `Hook1`, despite `Hook1` being the one whose requirements are written in reduced form according to the rules of the Swift ABI. This bug was fixed with the Requirement Machine.

Minimal requirements. Here is another variant of a function we saw earlier:

```
func uniqueElements3<T: Sequence>(_: T) -> Int
    where T.Element: Hashable,
          T.Iterator: IteratorProtocol,
          T.Element: Equatable {}
```

We get this list of reduced requirements:

{[`T: Sequence`], [`T.[Sequence]Element: Hashable`],
 [`T.[Sequence]Iterator: IteratorProtocol`],
 [`T.[Sequence]Element: Equatable`]}

The third and fourth requirements do not give us anything new, because they can be derived from the other two. Thus, `uniqueElements3()` has the same minimal generic signature as `uniqueElements1()` and `uniqueElements2()`. (In the past, we diagnosed redundant requirements as a warning, but this was removed in Swift 5.7, so now they're just dropped.)

Definition 11.11. Let G be a generic signature.

- An explicit requirement R of G is a *redundant requirement* if we can write down a derivation $G \vdash R$ using only the remaining requirements, without invoking R as an elementary statement.
- We say G is a *minimal* generic signature if every explicit requirement of G is reduced, and no explicit requirement of G is redundant.

The next example shows that a list of reduced requirements may have more than one *distinct* minimal subset. We declare three generic structs that only differ in conformance requirements; the reader may again want to try this with `-debug-generic-signatures`:

```
struct Knot1<T, U> where T: N, T == U.A,       U == T.A {}
struct Knot2<T, U> where       T == U.A, U: N, U == T.A {}
struct Knot3<T, U> where T: N, T == U.A, U: N, U == T.A {}
```

We're going to look at the generic signature of `Knot1` first. Type resolution produces the following list of user-written requirements for `Knot1`:

{[T: N], [T == U.A], [U == T.A]}

Here is the equivalent list of reduced requirements:

{[T: N], [T == U.[N]A], [U == T.[N]A]}

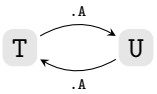
This list is minimal, so we get the following generic signature for `Knot1`:

<T, U where T: N, T == U.[N]A, U == T.[N]A>

Notice that we have an explicit requirement [T: N], and we can also derive [U: N]:

1. [T: N] (Conf)
2. [T.[N]A: N] (AssocConf 1)
3. [U == T.[N]A] (Same)
4. [U: N] (SameConf 2 3)

We will now show the type parameter graph for `Knot1`. The requirements look similar to `Hook1` because we have two equivalence classes that have a member type named `A`, but the same-type requirements act in a different way. Each member type `A` now takes us to *opposite* equivalence class:



Now, consider `Knot2`. Here is our list of reduced requirements:

{[T == U.[N]A], [U: N], [U == T.[N]A]}

This list is again minimal, so we get the following generic signature for `Knot2`:

<T, U where T == U.[N]A, U: N, U == T.[N]A>

In `Knot2`, we have an explicit requirement [`U: N`], and we can derive [`T: N`]:

1. [`U: N`] (Conf)
2. [`U.[N]A: N`] (AssocConf 1)
3. [`T == U.[N]A`] (Same)
4. [`T: N`] (SameConf 2 3)

We've shown that each one of [`T: N`] and [`U: N`] can be derived in both `Knot1` and `Knot2`. Also, the remaining explicit requirements are identical, so we now see that we have two distinct minimal generic signatures that generate the same theory.

Finally, we look at `Knot3`. Here are our reduced requirements:

{[`T: N`], [`T == U.[N]A`], [`U: N`], [`U == T.[N]A`]}

Unlike `Knot1` and `Knot2`, the reduced requirements of `Knot3` are not minimal, because each one of [`T: N`] and [`U: N`] can be derived from the other. In a situation like this, requirement minimization prefers to delete the redundant requirement with the larger subject type under the type parameter order.

In this case, that means we delete [`U: N`] first. Our derivation of [`T: N`] involves [`U: N`], so by Proposition 11.9, we can replace the explicit requirement [`U: N`] with its derivation to obtain a new derivation of [`T: N`]:

1. [`T: N`] (Conf)
2. [`T.[N]A: N`] (AssocConf 1)
3. [`U == T.[N]A`] (Same)
4. [`U: N`] (SameConf 2 3)
5. [`U.[N]A: N`] (AssocConf 4)
6. [`T == U.[N]A`] (Same)
7. [`T: N`] (SameConf 5 6)

We are no longer able to derive [`T: N`] from the *remaining* requirements, because the above derivation of [`T: N`] starts with the elementary derivation step for [`T: N`]. Thus, having deleted [`U: N`], the explicit requirement [`T: N`] is no longer redundant, and the generic signature of `Knot3` is the same as `Knot1`:

```
<T, U where T: N, T == U.[N]A, U == T.[N]A>
```

The fact that `Knot2` has a distinct generic signature from the other two was actually due to a quirk of the `GenericSignatureBuilder`, and this behavior is now part of the Swift ABI. The stronger form of requirement minimization that guarantees uniqueness would actually be *simpler* to implement, and we will explain the minor complication with the legacy behavior in Section 22.4.

There is another downside, from a theoretical standpoint. With type parameters, we are able to check equivalence by comparing their reduced types. However, we cannot check two lists of requirements for "theory equivalence" by comparing minimal generic signatures, because they are not unique. In practice though, nothing seems to call for this equivalence check; this is unlike type parameters, which are checked for reduced type equality all over the place.

The important invariant that we do maintain, to make textual interfaces work for example, is *idempotence*: meaning that when we pass from user-written requirements to minimal requirements for the first time, we are allowed to make choices amongst multiple minimal subsets, but if we take this output and build a new generic signature from those minimal requirements *again*, we must arrive at the *same* minimal generic signature. This also justifies the optimization of re-using requirement machines in Chapter 16.

Conflicting requirements. In the previous section, we related the idea of the well-formed substitution map (Definition 10.7) and the well-formed generic signature (Definition 11.3); a necessary condition for a generic signature to have any well-formed substitution maps at all, is for it to be a well-formed generic signature.

If we limit ourselves to the subset of the language with only conformance requirements and same-type requirements between type parameters, this condition is also sufficient, meaning we can mechanically construct a well-formed substitution map for a well-formed generic signature. We first declare a single concrete nominal type, call it struct `S`, and conform `S` to each protocol $\texttt{P}_i$ that our generic signature G depends on:

- Any method, variable and subscript requirements of $\texttt{P}_i$ can be witnessed by stub implementations that call `fatalError()`.
- Any associated types of $\texttt{P}_i$ can be witnessed by type alias members of `S` declared to have underlying type `S`.

We then build a substitution map for G by replacing each generic parameter type with `S` and each conformance requirement with the corresponding normal conformance $[\texttt{S: P}_i]$. Every derived requirement of G has the form $[\texttt{T: P}_i]$ for some type parameter `T` and protocol $\texttt{T}_i$, or `[T == U]` for a pair of type parameters `T` and `U`. Applying our substitution map, we always get either $[\texttt{S: P}_i]$ or `[S == S]`. Both requirements are satisfied no matter what, so our substitution map is well-formed.

With superclass, layout, and concrete same-type requirements, the picture is more complicated. When type parameters are required to have specific concrete types, we cannot "collapse" the entire generic signature down to a single point. Indeed, as we said, *conflicting requirements* might prevent our generic signature from being satisfied by *any* substitution map. There is another complication. While we can write derivations involving these "exotic" requirement kinds, certain inference rules are missing. The type substitution algebra will fill in some of these gaps.

Suppose that Σ is a substitution map where all replacement types are fully concrete, so we can apply Σ to a requirement and then apply Algorithm 9A to check if its satisfied. Further, let's say that we can derive two concrete same-type requirements, where both have the same subject type parameter $\texttt{T}$ of G:

1. $[\texttt{T} \texttt{ == } \texttt{X}_1]$ (...)
2. $[\texttt{T} \texttt{ == } \texttt{X}_2]$ (...)

(Note that with our existing inference rules, each requirement is either explicit, or an associated requirement of a protocol after substitution of `Self`; there are no other ways to "compose" concrete same-type requirements right now.) We can apply Σ to both requirements to get a pair of substituted requirements:

$$[\texttt{T} \otimes \Sigma \texttt{ == } \texttt{X}_1 \otimes \Sigma] \qquad \text{and} \qquad [\texttt{T} \otimes \Sigma \texttt{ == } \texttt{X}_2 \otimes \Sigma].$$

If Σ is a well-formed substitution map, it must satisfy both substituted requirements; that is, $\texttt{T} \otimes \Sigma$ must be canonically equal to $\texttt{X}_1 \otimes \Sigma$, and also $\texttt{T} \otimes \Sigma$ must be canonically equal to $\texttt{X}_2 \otimes \Sigma$. But canonical type equality is transitive, so it follows that $\texttt{X}_1 \otimes \Sigma$ must be canonically equal to $\texttt{X}_2 \otimes \Sigma$.

In other words, a necessary condition for Σ to be well-formed is that it must satisfy the requirement $[\texttt{X}_1 \texttt{ == } \texttt{X}_2] \otimes \Sigma$. Now, $[\texttt{X}_1 \texttt{ == } \texttt{X}_2]$ is not a derived requirement of G, because it has a concrete type on both sides. However, we know what to do if the user writes such a requirement directly; we apply Algorithm 11B. If we do the same here, each of the possible outcomes gives us further information about G:

- If $\texttt{X}_1$ does not match $\texttt{X}_2$ in the sense used by the desugaring algorithm, then no substitution map Σ can simultaneously satisfy both $[\texttt{T} \texttt{ == } \texttt{X}_1]$ and $[\texttt{T} \texttt{ == } \texttt{X}_2]$, so these two requirements are in conflict with each other, and G must be rejected.
- Otherwise, we always obtain a simpler list of requirements $\{R_1, \ldots, R_n\}$ with the property that Σ satisfies $[\texttt{X}_1 \texttt{ == } \texttt{X}_2]$ if and only if it satisfies R_i for all $1 \leq i \leq n$. If one of the original derived requirements was actually an explicit requirement of G, we can replace it with $\{R_1, \ldots, R_n\}$ without changing the "intended meaning" of the generic signature. (This might change the theory though, but only because our theory is missing some inference rules, as we already said.)

Let's revisit Example 5.26. We declare a protocol `Foo` with two associated types, together with an associated same-type requirement $[\texttt{Self.A == Array<Self.B>}]_{\texttt{Foo}}$:

```
protocol Foo {
  associatedtype A where A == Array<B>
  associatedtype B
}
```

Now, consider this function:

```
func f1<T: Foo>(_: T) where T.A == Array<Int> {}
```

We can derive two concrete same-type requirements involving `T`; the first one is explicit, the second is a consequence of the associated same-type requirement in `Foo`:

1. [`T.A == Array<Int>`] (Concrete)
2. [`T: Foo`] (Conf)
3. [`T.A == Array<T.B>`] (AssocConcrete 2)

As per our discussion above, a substitution map satisfying these requirements must also satisfy [`Array<Int> == Array<T.B>`], which desugars to [`T.B == Int`]. We can, in fact, replace our explicit same-type requirement with this desugared requirement, so the final generic signature is this:

```
<T where T: Foo, T.[Foo]B == Int>
```

Notice how these two lists of requirements are *not* equivalent by Proposition 11.9, because we cannot derive [`T.B == Int`] from the first list of requirements; once the derived requirements formalism has been fully fleshed out, these ought to become equivalent:

{[`T: Foo`], [`T.A == Array<Int>`]}
{[`T: Foo`], [`T.B == Int`]}

Next, we're going to change our example slightly to get a pair of conflicting requirements, and we see that we diagnose an error:

```
func f2<T: Foo>(_: T) where T.A == Set<Int> {}

// error: no type for 'T.A' can satisfy both 'T.A == Set<Int>' and
// 'T.A == Array<T.C>'
```

What if *both* of our same-type requirements are explicit? We now take this protocol:

```
protocol Foe {
  associatedtype X
  associatedtype Y
}
```

So that we can define this function:

```
func f3<T: Foe>(_: T) where T.X == Set<T.Y>, T.A == Set<Int> {}
```

In `f3()`, we can replace *either* requirement with [`T.X == Int`] without changing the "meaning" of our generic signature, but since the first requirement's subject type is not reduced, we replace it first. Thus, we get the following generic signature:

```
<T where T: Foe, T.[Foe]X == Set<Int>, T.[Foe]Y == Int>
```

We now state the general definition.

Definition 11.12. Let G be a well-formed generic signature. If G has a pair of derived requirements R_1 and R_2 where $R_1 \otimes \Sigma$ and $R_2 \otimes \Sigma$ cannot both be satisfied by the same substitution map Σ, then R_1 and R_2 define a pair of *conflicting requirements*. A generic signature G is *conflict-free* if it does not have any pairs of conflicting requirements. The pairs of derived requirements that can lead to conflicts are enumerated below:

1. For two concrete same-type requirements [`T == X`$_1$] and [`T == X`$_2$], we desugar the "combined" requirement [`X`$_1$ `== X`$_2$], as we already saw. Here and every remaining case below, desugaring will either detect a conflict, or produce a simpler list of requirements that can replace one of the two original requirements.

2. For a concrete same-type requirement [`T == X`] and superclass requirement [`T: C`], we desugar [`X: C`], which can be satisfied only if `X` is a class type that is also a subclass of `C`.

3. For a same-type requirement [`T == X`] and a layout requirement [`T: AnyObject`], we desugar [`X: AnyObject`], which can be satisfied only if `X` is a class type.

4. For a same-type requirement [`T == X`] and a conformance requirement [`T: P`], we desugar [`X: P`], which can be satisfied only if `X` conforms to `P`.

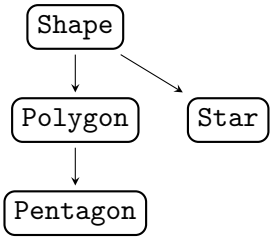
5. For two superclass requirements [`T: C`$_1$] and [`T: C`$_2$], we must consider the superclass relationship between the declarations of `C`$_1$ and `C`$_2$:
   a) If the class declaration of `C`$_1$ is a subclass of the declaration of `C`$_2$, we desugar [`C`$_1$`: C`$_2$], and [`T: C`$_2$] becomes redundant.
   b) If the class declaration of `C`$_2$ is a subclass of the declaration of `C`$_1$, we desugar [`C`$_2$`: C`$_1$], and [`T: C`$_1$] becomes redundant.
   c) If the two declarations are unrelated, the requirements conflict.

6. For a superclass requirement [`T: C`] and a layout requirement [`T: AnyObject`], we desugar [`C: AnyObject`], which is always satisfied and cannot conflict.

7. For a superclass requirement [`T: C`] and a conformance requirement [`T: P`], we desugar [`C: P`]. If `C` conforms to `P`, the conformance requirement [`T: P`] becomes redundant. However, if `C` does not conform to `P`, there is no conflict; the generic signature just requires a subclass of `C` that *also* conforms to `P`.

We will explain how the implementation deals with superclass, layout and concrete same-type requirements, sans theory, in Chapters 20 and 21, but we're going to look at two examples here.

Our next example involves superclass requirements. While a description of generic classes awaits us in Chapter 15, we only need non-generic classes to demonstrate the key ideas in requirement minimization. Since it is such a cliché at this point, we're going to follow the trend and go with the classic object oriented "shape hierarchy", shown on the right. We also introduce a `Canvas` protocol, which subjects it associated type to an associated superclass requirement $[\texttt{Self.Boundary: Polygon}]_{\texttt{Canvas}}$:




```
class Shape {}
class Polygon: Shape {}
class Pentagon: Polygon {}
class Star: Shape {}

protocol Canvas {
  associatedtype Boundary: Polygon
}
```

Our first function imposes a more general superclass bound on `C.Boundary` than what is implied by the conformance requirement `[C: Canvas]`:

```
func h1<C: Canvas>(_: C) where C.Boundary: Shape {}
```

Because every `Polygon` is also a `Shape`, the requirement `[C.Boundary: Shape]` is redundant, so we're just left with `[C: Canvas]`.

The second function tightens the superclass bound on `C.Boundary`:

```
func h2<C: Canvas>(_: C) where C.Boundary: Pentagon {}
```

Not every `Polygon` is a `Pentagon`, so the minimal generic signature of `h2()` includes the requirement `[C.[Canvas]Boundary: Pentagon]` in addition to `[C: Canvas]`.

Finally, if we attempt to impose an unrelated superclass bound, we get an error diagnosing the conflict:

```
func h3<C: Canvas>(_: C) where C.Boundary: Star {}

// error: no type for 'C.Boundary' can satisfy both 'C.Boundary : Star'
// and 'C.Boundary : Polygon'
```

Our final example looks at the interaction between conformance and concrete same-type requirements. Consider the generic signature of the extension of `Box`:

```
struct Box<Contents: Sequence> {}
extension Box where Contents == Array<Int> {}
```

To build the generic signature of the extension, we take the generic signature of `Box`, and add the requirement [`Contents == Array<Int>`], so requirement minimization starts with this list of requirements:

{[`Contents: Sequence`], [`Contents == Array<Int>`]}

By Definition 11.12, we can understand the interaction between the two requirements by desugaring the requirement [`Array<Int>: Sequence`]. We look up the conformance and see that this requirement is satisfied, so our original requirement [`Contents: Sequence`] is redundant. We're left with the same-type requirement [`Contents == Array<Int>`], so here is our extension's generic signature:

`<Contents where Contents == Array<Int>>`

This generic signature certainly generates a different, much smaller theory than the original list of requirements that includes [`Contents: Sequence`]. For example, the dependent member type `Contents.Element` is not a valid type parameter in the new generic signature, precisely because we cannot derive [`Contents: Sequence`]. This gives us another example where requirement minimization does not preserve equivalence of generic signatures, because of gaps in our theoretical understanding.

This is one of those `GenericSignatureBuilder` behaviors that proved to be slightly obnoxious in hindsight; it would have been nicer if the conformance requirement was not actually considered to be redundant here. As a practical matter, it means that a valid type parameter of the extended type's generic signature might no longer be a valid type parameter in the generic signature of the extension. The implementation of the `getReducedType`() generic signature query makes a special exception to allow such type parameters anyway, by attempting to resolve the concrete conformance (Section 20.4).

Requirement order. Once we have our minimal requirements, the last step is to sort them in a canonical way. As defined, the below algorithm is a partial order because it can return "⊥", but this can only happen if both requirements have the same subject type and same kind, and they're not conformance requirements. Consulting Definition 11.12, we see this if two minimal requirements have this property, they must conflict. Thus, the requirements of a minimal conflict-free generic signature can be linearly ordered without ambiguity.

Algorithm 11D (Requirement order)**.** Takes two requirements as input, and returns one of "<", ">", "=" or "⊥" as output.

1. (Subject) Compare the subject types of the two requirements with Algorithm 5D. Return the result if it is "<" or ">". Otherwise, both requirements have the same subject type.

2. (Kind) Compare their kinds. If they have different kinds, return "<" or ">" based on the relative order below:

   $$\text{superclass} < \text{layout} < \text{conformance} < \text{same-type}$$

   Otherwise, both requirements have the same subject type and the same kind.

3. (Protocol) If both are conformance requirements, compare their protocols with Algorithm 5B, and return one of "<", "=", or ">".

4. (Incomparable) Otherwise, both requirements have the same subject type and kind, and they're not conformance requirements. Return "⊥".

Requirement signatures. The associated requirements in a protocol's requirement signature are minimized just like the explicit requirements of a generic signature. The proof of Proposition 11.9 can be tweaked slightly to instead use a equivalence relation on requirement signatures. (The key idea is that for every associated requirement of a protocol `P`, we can derive the corresponding requirement in the protocol generic signature $G_{\texttt{P}}$.) Minimal and reduced associated requirements are defined in the same way, and the **requirement signature request** always outputs a minimal requirement signature. We can describe the conflicts among the associated requirements using Definition 11.12, and finally, sort the minimal associated requirements in a requirement signature using Algorithm 11D.

The major difference is that we must minimize all requirement signatures of a set of mutually-dependent protocols, or a *protocol component*, simultaneously. We will discuss this again in Section 16.1 and see an example in Section 22.1.

11.5. Source Code Reference

Requests

Key source files:

- `include/swift/AST/TypeCheckRequests.h`
- `lib/AST/RequirementMachine/RequirementMachineRequests.cpp`

The header file declares the requests; the evaluation functions are implemented by the Requirement Machine (Section 16.3).

`GenericSignature` *class*

See also Section 5.6.

- `get()` is the primitive constructor, which builds a generic signature directly from a list of generic parameters and minimal requirements.

`GenericSignatureRequest` *class*

The `GenericContext::getGenericSignature()` method (Section 5.6) evaluates this request, which either returns the parent declaration's generic signature, or evaluates `InferredGenericSignatureRequest` with the appropriate arguments.

`InferredGenericSignatureRequest` *class*

The request evaluator request for building a generic signature from requirements written in source. The arguments were previously discussed at the beginning of this chapter:

1. The `GenericSignature` of the parent context, if any.
2. The current generic context's `GenericParamList`, if any.
3. The current generic context's `WhereClauseOwner`, if any.
4. A vector of `Requirement` storing any additional requirements to add.
5. A vector of `TypeLoc` eligible for requirement inference.
6. A flag indicating whether generic parameters can be subject to concrete same-type requirements.

`WhereClauseOwner` *class*

A reference to a `where` clause attached to a declaration, This is a wrapper type used by requirement resolution, which is the first step in building the generic signature in `InferredGenericSignatureRequest`. Constructed with one of the following:

- A `GenericContext` representing a generic declaration.
- An `AssociatedTypeDecl`, which can also have a `where` clause despite not being a `GenericContext`.
- A `GenericParamList`, which only has a `where` clause in textual SIL.

- A `SpecializeAttr` representing a `@_specialize` attribute.
- An instance of a `TrailingWhereClause`, which is the primitive form for several of the above.

A pair of methods are defined for working with the requirements in the `where` clause:

- The `getRequirements()` method returns an array of `RequirementRepr`, which is the parsed representation of a requirement.
- The `visitRequirements()` method takes a callback and a `TypeResolutionStage`. It resolves each `RequirementRepr` to a `Requirement`, and passes both to the callback.

The `visitRequirements()` method is called with `TypeResolutionStage::Structural` by the `InferredGenericSignatureRequest` and `RequirementSignatureRequest`.

The `TypeCheckSourceFileRequest` then visits all `where` clauses in every primary file again, this time with `TypeResolutionStage::Interface`. This is how invalid dependent member types in `where` clauses get diagnosed. Recall that the structural resolution stage builds unbound dependent member types, without any knowledge of what associated type declarations are visible. The interface resolution stage actually performs a name lookup using the generic signature that was built, catching invalid dependent member types.

`AbstractGenericSignatureRequest` *class*

The request evaluator request for building a generic signature from a list of generic parameters and requirements. The result is a pair consisting of a generic signature and some error flags. Most callers don't care about the error flags, so they use the following function instead.

`buildGenericSignature()` *function*

A utility function wrapping the `AbstractGenericSignatureRequest`. It checks and discards the returned error flags. If the `CompletionFailed` error flag is set, this aborts the compiler. The other two flags are ignored.

`GenericSignatureErrorFlags` *enum class*

Error flags returned by `AbstractGenericSignatureRequest`. We will see these conditions again in Chapter 16; they prevent the requirement machine for this signature from being *installed*.

- `HasInvalidRequirements`: the original requirements were not well-formed, or were in conflict with each other. Any errors in the requirements handed to this request usually mean there was another error diagnosed elsewhere, like an invalid

conformance, so this flag being set is not really actionable to the rest of the compiler. Without source location information, this error cannot be diagnosed in a friendly manner.

- `HasConcreteConformances`: the generic signature had non-redundant concrete conformance requirements, which is an internal flag used to prevent the requirement machine from being installed. It does not indicate an error condition to the caller. See Section 21.3 for discussion.
- `CompletionFailed`: the completion procedure could not construct a convergent rewriting system within the maximum number of steps (see the discussion of termination that immediately follows Algorithm 19E). This is actually fatal, so the `buildGenericSignature()` wrapper function aborts the compiler in this case.

`RequirementSignature` *class*

See also Section 5.6.

- `get()` is the primitive constructor, which builds a requirement signature directly from a list of minimal requirements and protocol type aliases.

`RequirementSignatureRequest` *class*

The `ProtocolDecl::getRequirementSignature()` method (Section 5.6) evaluates this request, which computes the protocol's requirement signature if the protocol is in the main module, or deserializes it if the protocol is from a serialized module.

`StructuralRequirementsRequest` *class*

The `ProtocolDecl::getStructuralRequirements()` method evaluates this request to resolve the user-written requirements from which the protocol's requirement signature is formed.

`TypeAliasRequirementsRequest` *class*

The `ProtocolDecl::getTypeAliasRequirements()` method evaluates this request to collect the type alias declarations in a protocol and converts them into requirements from which the protocol's requirement signature is formed.

Requirement Resolution

Key source files:

- `include/swift/AST/Requirement.h`
- `lib/AST/RequirementMachine/RequirementLowering.cpp`

User-written requirements are wrapped in the `StructuralRequirement` type, which stores a `Requirement` together with a source location used for diagnostics. A couple of functions defined in `RequirementLowering.cpp` construct `StructuralRequirement` instances. The `InferredGenericSignatureRequest` calls these functions directly. The `RequirementSignatureRequest` delegates to `StructuralRequirementsRequest`, which uses them to resolve requirements written in a protocol declaration.

`rewriting::realizeRequirement()` *function*

Calls the `WhereClauseOwner::visitRequirements()` method to resolve requirements written in `where` clauses, and wraps the results in `StructuralRequirement` instances.

`rewriting::realizeInheritedRequirements()` *function*

Resolves the inheritance clause entries of a type declaration, then builds conformance, superclass and layout requirements with the appropriate subject type, and wraps them in `StructuralRequirement` instances.

Requirement Inference

Key source files:

- lib/AST/RequirementMachine/RequirementLowering.h
- lib/AST/RequirementMachine/RequirementLowering.cpp

The `realizeRequirement()` and `realizeInheritedRequirements()` functions take a flag indicating if requirement inference should be performed; recall that requirement inference is not performed in protocols, so `StructuralRequirementsRequest` does not pass this flag.

`rewriting::inferRequirements()` *function*

Recursively walks a `Type` and constructs `StructuralRequirement` instances from all generic nominal and generic type alias types contained therein.

Requirement Desugaring

Key source files:

- lib/AST/RequirementMachine/Diagnostics.h
- lib/AST/RequirementMachine/RequirementLowering.h
- lib/AST/RequirementMachine/RequirementLowering.cpp

The `realizeRequirement()` and `realizeInheritedRequirements()` functions also perform requirement desugaring. For `AbstractGenericSignatureRequest`, requirement desugaring is the entry point where the fun begins; it starts from a list of requirements instead of resolving user-written requirement representations.

`rewriting::desugarRequirement()` *function*

Establishes the invariants in Definition 11.1, splitting up conformance requirements and simplifying requirements where the subject type is a concrete type.

`RequirementError` *class*

Represents a redundant or conflicting requirement detected by requirement desugaring or minimization.

Requirement Minimization

Key source files:

- `lib/AST/GenericSignature.cpp`

Just as Section 11.4 only describes the invariants around minimization, here we only call out the code related to checking those invariants. For the actual implementation of minimization, see Section 22.6.

`Requirement` *class*

See also Section 5.6.

- `compare()` implements the requirement order (Algorithm 11D), returning one of the following:
  - -1 if this requirement precedes the given requirement,
  - 0 if the two requirements are equal,
  - 1 if this requirement follows the right hand side.

  This method will assert if the two requirements are incomparable; an example is when two superclass requirements have the same subject type. Incomparable requirements should not appear in a generic signature.

`GenericSignatureImpl` *class*

See also Section 5.6.

- `verify()` ensures that all explicit requirements in this signature are desugared (Definition 11.1), reduced (Definition 11.10), minimal (Definition 11.11), and ordered (Algorithm 11D). Any violations report a fatal error that crashes the compiler even in no-assert builds, since such generic signatures should not be built at all.

12. Conformance Paths

We now return to tie up a loose end in type substitution. We encountered type substitution in Chapter 6, first in its most basic form, as a structural mapping of generic parameter types to the replacement types of a substitution map. The next level of detail appeared when we considered type substitution of the other kind of type parameter: the dependent member type. A substitution map can contain conformances in addition to replacement types, and Chapter 7 showed that under type substitution, a dependent member type corresponds to a type witness of a conformance. To find this conformance, we introduced the so-called local conformance lookup operation, but only defined it in the simplest case, where the conformance is directly stored in the substitution map. In this chapter, we show that the general case works by evaluating a *conformance path*.

To motivate conformance paths, consider this generic signature, together with the protocol definitions from Listing 12.1:

```
<T, U where T: Collection, U == T.[Collection]SubSequence>
```

Let's define a substitution map for this generic signature and perform some calculations:

$$\Sigma := \left\{ \begin{array}{l} \texttt{T} \mapsto \texttt{String} \\ \texttt{U} \mapsto \texttt{Substring}; \\ \\ [\texttt{T: Collection}] \mapsto [\texttt{String: Collection}] \end{array} \right\}$$

The substitution map Σ satisfies the requirements of our generic signature. To see why, we apply Σ to both sides of the requirement $[\texttt{U == T.[Collection]SubSequence}]$:

$$\texttt{U} \otimes \Sigma$$
$$\texttt{T.[Collection]SubSequence} \otimes \Sigma$$

The first substituted type is the replacement type for `U` in Σ, which is `Substring`. The second substituted type can be computed with what we learned about dependent member type substitution from Section 7.4. Recalling that a dependent member type is a type witness of an abstract conformance, we proceed as follows:

$$\begin{aligned} &\texttt{T.[Collection]SubSequence} \otimes \Sigma \\ &\quad = \big(\pi(\texttt{[Collection]SubSequence}) \otimes [\texttt{T: Collection}]\big) \otimes \Sigma \\ &\quad = \pi(\texttt{[Collection]SubSequence}) \otimes \big([\texttt{T: Collection}] \otimes \Sigma\big) \end{aligned}$$

Listing 12.1.: Example to motivate conformance paths

```
protocol IteratorProtocol {
  associatedtype Element
}

protocol Sequence {
  associatedtype Element
  associatedtype Iterator: IteratorProtocol
    where Element == Iterator.Element
}

protocol Collection: Sequence {
  associatedtype SubSequence: Collection
    where Element == SubSequence.Element,
          SubSequence == SubSequence.SubSequence
}
```

An abstract conformance represents a derived conformance requirement in our generic signature. In fact, [`T: Collection`] is a derived requirement in the trivial sense, as it is explicitly stated. Thus, Σ directly stores a conformance for [`T: Collection`], namely [`String: Collection`]. From this conformance, we then project the type witness for `SubSequence`, and obtain the final result:

$$\begin{aligned}
&\texttt{T.[Collection]SubSequence} \otimes \Sigma \\
&\quad = \big(\pi(\texttt{[Collection]SubSequence}) \otimes [\texttt{T: Collection}]\big) \otimes \Sigma \\
&\quad = \pi(\texttt{[Collection]SubSequence}) \otimes \big([\texttt{T: Collection}] \otimes \Sigma\big) \\
&\quad = \pi(\texttt{[Collection]SubSequence}) \otimes [\texttt{String: Collection}] \\
&\quad = \texttt{Substring}
\end{aligned}$$

This shows the same-type requirement is satisfied, as applying Σ to both sides yields `Substring`.

Having established the validity of the substitution map, we now attempt to determine the substituted type $\texttt{U.[Sequence]Iterator} \otimes \Sigma$. Again, we can express the dependent member type as the type witness of an abstract conformance:

$$\texttt{U.[Sequence]Iterator} \otimes \Sigma = \pi(\texttt{[Sequence]Iterator}) \otimes ([\texttt{U: Sequence}] \otimes \Sigma)$$

However, [`U: Sequence`] does not appear in our generic signature, thus the corresponding conformance is not directly stored inside Σ. To understand what happens next, we need some terminology.

Root conformances. The *root conformances* of a substitution map are those directly stored in the substitution map. The *root abstract conformances* of a generic signature are those corresponding to the explicit conformance requirements of the generic signature. These two concepts are closely related via the identity substitution map 1_G: the root conformances of 1_G are the root abstract conformances of its input generic signature G. In our example, [`T: Collection`] is a root abstract conformance of our generic signature, but [`U: Sequence`] is not, because the latter represents a derived conformance requirement, not explicitly stated in the generic signature (we will present a derivation later).

The general case of applying a substitution map to an abstract conformance is called *local conformance lookup*. Without knowing anything about local conformance lookup, we can already deduce the result of [`U: Sequence`] $\otimes\, \Sigma$ only using concepts introduced previously. We know the abstract conformance [`U: Sequence`] can be obtained by the *global* conformance lookup [`Sequence`] $\otimes$ `U`. We also know that `U` $\otimes\, \Sigma =$ `Substring`. Therefore, from the associativity of "$\otimes$", we see that:

$$\begin{aligned}
&[\texttt{U: Sequence}] \otimes \Sigma \\
&\quad = \big([\texttt{Sequence}] \otimes \texttt{U}\big) \otimes \Sigma \\
&\quad = [\texttt{Sequence}] \otimes \big(\texttt{U} \otimes \Sigma\big) \\
&\quad = [\texttt{Sequence}] \otimes \texttt{Substring} \\
&\quad = [\texttt{Substring: Sequence}].
\end{aligned}$$

So, to be consistent with the rest of our theory, local conformance lookup must output the conformance [`Substring: Sequence`], given [`U: Sequence`] and Σ. As an operation on substitution maps, local conformance lookup is also limited in what it can do. Starting from one of the root conformances in a substitution map, the only way to derive new conformances is by associated conformance projection, possibly repeated multiple times.

Conformance paths. Our example is based on a handful of concrete conformances, simplified from their real definitions in the Swift standard library. Figure 12.1 lists the type witnesses and associated conformances of each one:

[`String: Collection`]
[`Substring: Collection`]
[`Substring: Sequence`]

Our substitution map Σ contains a single root conformance [`String: Collection`], which corresponds to the root abstract conformance [`T: Collection`]. Whatever else we do, we must first project this root conformance from the substitution map:

[`T: Collection`] $\otimes\, \Sigma =$ [`String: Collection`]

Figure 12.1.: Examples of associated conformances

Conformance of `String: Collection`		
Associated type		**Type witness**
π(`[Collection]SubSequence`)	$\mapsto$	`Substring`
Conformance requirement		**Conformance**
π(`Self: Sequence`)	$\mapsto$	[`String: Sequence`]
π(`Self.SubSequence: Collection`)	$\mapsto$	[`Substring: Collection`]

Conformance of `Substring: Collection`		
Associated type		**Type witness**
π(`[Collection]SubSequence`)	$\mapsto$	`Substring`
Conformance requirement		**Conformance**
π(`Self: Sequence`)	$\mapsto$	[`Substring: Sequence`]
π(`Self.SubSequence: Collection`)	$\mapsto$	[`Substring: Collection`]

Conformance of `Substring: Sequence`		
Associated type		**Type witness**
π(`[Sequence]Element`)	$\mapsto$	`Character`
π(`[Sequence]Iterator`)	$\mapsto$	`IndexingIterator<Substring>`

Notice how the [`String: Collection`] conformance contains the associated conformance [`Substring: Collection`], corresponding to the associated conformance requirement π(`Self.SubSequence: Collection`) of the `Collection` protocol. We project it out:

$$\pi(\texttt{Self.SubSequence: Collection}) \otimes [\texttt{String: Collection}]$$
$$= [\texttt{Substring: Collection}]$$

We're looking for [`Substring: Sequence`], not [`Substring: Collection`], but we can get there with just one more jump. The protocol inheritance relationship between `Collection` and `Sequence` is expressed via another associated conformance requirement:

$$\pi(\texttt{Self: Sequence}) \otimes [\texttt{Substring: Collection}]$$
$$= [\texttt{Substring: Sequence}]$$

Once we have the [`Substring: Sequence`] conformance, projecting the type witness for `Iterator` gives us the final substituted type:

$$\pi(\texttt{[Sequence]Iterator}) \otimes [\texttt{Substring: Sequence}]$$
$$= \texttt{IndexingIterator<Substring>}$$

If we set aside for now the question of *why* this works at all—a very important question we will study in detail—we see the substituted type of `U.[Sequence]Iterator` can be derived in four steps from Σ: we project the root conformance, then we project an associated conformance, then another, and finally we project the type witness. The expansion of the local conformance lookup operation is shown with braces:

$$\texttt{U.[Sequence]Iterator} \otimes \Sigma$$
$$= \pi(\texttt{[Sequence]Iterator}) \otimes \underbrace{([\texttt{U: Sequence}] \otimes \Sigma)}_{\text{local conformance lookup}}$$
$$= \pi(\texttt{[Sequence]Iterator}) \otimes \Big(\pi(\texttt{Self: Sequence}) \otimes \Big(\pi(\texttt{Self.SubSequence: Collection}) \otimes ([\texttt{T: Collection}] \otimes \Sigma)\Big)\Big) \Big\} \text{local conformance lookup}$$
$$= \pi(\texttt{[Sequence]Iterator}) \otimes [\texttt{Substring: Sequence}]$$
$$= \texttt{IndexingIterator<Substring>}$$

The three steps in the middle form a *conformance path* for [`U: Sequence`]:

$$\pi(\texttt{Self: Sequence}) \otimes \pi(\texttt{Self.SubSequence: Collection}) \otimes [\texttt{T: Collection}]$$

Formally, a conformance path is an ordered tuple $(s_0, s_1, \ldots, s_n)$, of length n, where $n \geq 1$, the first step s_0 is a root abstract conformance, and each subsequent step s_i is an associated conformance requirement. A "trivial" path of length exactly one is just a single root abstract conformance. Instead of using ordered tuple notation, observe we write conformance paths from *right to left*:

$$s_n \otimes \cdots \otimes s_1 \otimes s_0$$

First, let's assume we already have the means to obtain a conformance path for an abstract conformance. A conformance path only depends on the generic signature, and not the contents of the substitution map. Local conformance lookup *evaluates* this conformance path with the given substitution map Σ. This evaluation operation can be understood with our type substitution algebra; this justifies our right-to-left notation:

$$s_n \otimes \big(\cdots \otimes \big(s_1 \otimes (s_0 \otimes \Sigma)\big) \cdots\big)$$

We now show the algorithms for local conformance lookup and dependent member type substitution. In the next section, we will build up some more theory in anticipation of revealing Algorithm 12E for actually finding conformance paths.

Algorithm 12A (Local conformance lookup)**.** As input, takes a substitution map Σ, and an abstract conformance [`T: P`].

1. Let `C` be an invalid conformance. This will be the return value.
2. Find a conformance path $s_n \otimes \cdots \otimes s_1 \otimes s_0$ for [`T: P`] using Algorithm 12E.
3. (Initialize) Let $i := 1$.
4. (Root) Suppose s_0 is the root abstract conformance [$\texttt{T}_0$`: `$\texttt{P}_0$]. Project the root conformance corresponding to [$\texttt{T}_0$`: `$\texttt{P}_0$] from Σ, and assign the result to `C`.
5. (Check) If $i = n + 1$, return `C`.
6. (Step) Suppose s_i is the associated conformance projection $\pi($`Self.[`$\texttt{P}_{i-1}$`]`$\texttt{T}_i$`: `$\texttt{P}_i$$)$. Apply the projection to `C` to get a conformance to $\texttt{P}_i$, and assign the result to `C`.
7. (Next) Increment i and go back to Step 5.

In Step 6, `C` must be a conformance to $\texttt{P}_{i-1}$ for the projection to make sense; also, we expect that on the last iteration, $\texttt{P}_n$ is equal to `P`, the conformed protocol of the original abstract conformance. This gives us a validity condition on conformance paths, which we will explore in the next section. For now, we again assume that Algorithm 12E gives us such a path.

Algorithm 12B (Substitute dependent member type)**.** As input, takes a dependent member type `T.[P]A` and a substitution map Σ. The dependent member type is understood to be a valid type parameter in the substitution map's input generic signature. Outputs the substituted type `T.[P]A` $\otimes \Sigma$.

1. Let `A` be the associated type declaration referenced by the dependent member type. (This algorithm does not support unbound dependent member types.)
2. Let `P` be the protocol containing this associated type declaration.
3. Let `T` be the base type parameter of the dependent member type. (This could be another dependent member type, or a generic parameter.)
4. (Lookup) Construct the abstract conformance [`T: P`] and invoke Algorithm 12A to perform the local conformance lookup [`T: P`] $\otimes \Sigma$.
5. (Project) Apply the type witness projection π(`[P]A`) to this conformance and return the result.

12.1. Validity and Existence

Returning to our example, let's see what happens when we evaluate our conformance path with the identity substitution map 1_G, instead of our substitution map Σ. Since [`T: Collection`] $\otimes\, 1_G =$ [`T: Collection`], the first step is a no-op. Next, we perform two associated conformance projections on an abstract conformance. By the rules of our type substitution algebra, this simplifies as follows:

$$\begin{aligned}&\pi(\texttt{Self: Sequence}) \otimes \big(\pi(\texttt{Self.SubSequence: Collection}) \otimes [\texttt{T: Collection}]\big)\\ &\quad = \pi(\texttt{Self: Sequence}) \otimes [\texttt{T.SubSequence: Collection}]\\ &\quad = [\texttt{T.SubSequence: Sequence}]\end{aligned}$$

We claimed this conformance path represents [`U: Sequence`], but simplifying it in the above manner gives us [`T.SubSequence: Sequence`], which is not identical. We can justify this by noting that both conformances have the same conformed protocol, and the two subject types `T.SubSequence` and `U` belong to the same equivalence class, due to the explicit same-type requirement of our generic signature.

Thus, just like the reduced type equality relation on type parameters (Section 5.3), we can define an equivalence relation on abstract conformances. We say that two abstract conformances are equivalent if they name the same protocol, and their subject types are equivalent. A *reduced abstract conformance* is then one whose subject type is a reduced type parameter. Every equivalence class of abstract conformances contains a unique reduced abstract conformance, and two abstract conformances are equivalent if their reduced abstract conformances are identical.

In our example, [`U: Sequence`] and [`T.SubSequence: Sequence`] are two abstract conformances that belong to the same equivalence class. The former is reduced, while the latter is not (and the former is the reduced abstract conformance of the latter).

Validity. We've seen how to implement dependent member type substitution in terms of conformance paths, but so far, we've only established the legitimacy of one specific conformance path. Now, we ask three questions:

1. Does every conformance path simplify to a valid abstract conformance?
2. Does every valid abstract conformance have at least one conformance path?
3. How do we find a conformance path for a valid abstract conformance?

We will answer (1) and (2) first, and present an algorithm for (3) in the next section. We begin with an algorithm for directly evaluating a conformance path to an abstract conformance, without a substitution map. This is called *simplifying* a conformance path. Studying the preconditions of this algorithm leads to a notion of validity for conformance paths. This algorithm is also later used by Algorithm 12E.

Algorithm 12C (Simplify conformance path)**.** Takes a generic signature and a conformance path $s_n \otimes \cdots \otimes s_1 \otimes s_0$. Outputs an abstract conformance. This conformance will have the same conformed protocol as the last step of the conformance path.

1. (Initialize) Let `C` be an invalid conformance. This will be the return value. Also let $i := 1$.
2. (Root) Suppose s_0 is the root abstract conformance [$\texttt{T}_0$`: `$\texttt{P}_0$]. Assign this root abstract conformance to `C`.
3. (Check) If $i = n + 1$, return `C`.
4. (Step) Suppose s_i is the associated conformance projection $\pi($`Self.[`$\texttt{P}_{i-1}$`]`$\texttt{T}_i$`: `$\texttt{P}_i$). Apply the projection to `C` to get a conformance to $\texttt{P}_i$, and assign the result to `C`.
5. (Next) Increment i and go back to Step 4.

Every step s_i in a conformance path names a protocol on the right hand side; call this protocol $\texttt{P}_i$. A *valid conformance path* in a generic signature G satisfies two conditions:

- The first step s_0 must be a root abstract conformance of G.
- Every subsequent step s_i is an associated conformance projection defined in the protocol $\texttt{P}_{i-1}$.

Recall that an abstract conformance $[\texttt{T: P}]$ is valid in a generic signature G if the conformance requirement $[\texttt{T: P}]$ can be derived in G. To show that a valid conformance path simplifies to a valid abstract conformance, we construct a special kind of derivation, called a *primitive derivation*, from the conformance path.

The first step of the conformance path translates to a Conf derivation step (recall that a conformance path begins with a *root* abstract conformance, and not an arbitrary abstract conformance, for otherwise we could not make use of the Conf derivation step):

$$\vdash [\texttt{T}_0\texttt{: P}_0] \tag{1}$$

If the conformance path has length 1, we're done; we have our primitive derivation. Otherwise, for each remaining step in the conformance path, we add an AssocConf step for the corresponding associated conformance requirement:

$$\begin{aligned}&\ldots\\&(n),\ [\texttt{Self.A}_n\texttt{: P}_n]_{\texttt{P}_{n+1}} \vdash [\texttt{T}_0\texttt{.A}_1\texttt{...A}_n\texttt{.A}_{n+1}\texttt{: P}_{n+1}] && (n+1)\\&\ldots\end{aligned}$$

The derivation is well-formed at every step, and the subject type and conformed protocol of the final derived conformance requirement is the same as the abstract conformance output by Algorithm 12C. This shows the aforesaid algorithm outputs a valid abstract conformance, as was claimed.

Now, recall our favorite conformance path:

$$\pi(\texttt{Self: Sequence}) \otimes \pi(\texttt{Self.SubSequence: Collection}) \otimes [\texttt{T: Collection}]$$

We saw this conformance path simplifies to $[\texttt{T.SubSequence: Sequence}]$. Performing the above construction produces this primitive derivation:

$$\begin{aligned}&\vdash [\texttt{T: Collection}] && (1)\\&\vdash [\texttt{Self.SubSequence: Collection}]_{\texttt{Collection}} && (2)\\&(1),\ (2) \vdash [\texttt{T.SubSequence: Collection}] && (3)\\&\vdash [\texttt{Self: Sequence}]_{\texttt{Collection}} && (4)\\&(3),\ (4) \vdash [\texttt{T.SubSequence: Sequence}] && (5)\end{aligned}$$

We saw that $[\texttt{T.SubSequence: Sequence}]$ is equivalent to $[\texttt{U: Sequence}]$ via the same-type requirement $[\texttt{U == T.SubSequence}]$. This means we can extend the above primitive derivation with a Same derivation step to get a derivation for $[\texttt{U: Sequence}]$:

$$\begin{aligned}&\vdash [\texttt{U == T.SubSequence}] && (6)\\&(5),\ (6) \vdash [\texttt{U: Sequence}] && (7)\end{aligned}$$

Existence. The above procedure always works in general. If we receive a conformance path for [T: P], and the conformance path simplifies to [T′: P] for a possibly different type parameter T′, then we can construct a primitive derivation for [T′: P]. We also know that T and T′ must be equivalent, so there exists a derivation for the same-type requirement [T == T′]. From these two derivations, we get a derivation for [T: P].

Now, we will go in the other direction. First, observe that a primitive derivation defines a conformance path, as follows: the initial GenSig derivation step becomes the root abstract conformance at the start of the path, and each subsequent ReqSig derivation step becomes an associated conformance projection. Then, the next theorem shows a derivation of a conformance requirement [T: P] always splits into two parts: a primitive derivation [T′: P], together with a same-type requirement [T == T′]. Since a derived conformance requirement defines an abstract conformance, and a primitive derivation defines a conformance path, we can conclude that every abstract conformance has a conformance path.

We now look at more ways of building new derivations from existing ones; we will use these results in Section 12.1. Recall that we can take a same-type requirement [T == U] from [U: P] and [T.[P]A == U.[P]A], where protocol P declares an associated type A. By iterated application of the Member step, we perform a construction with *any* valid type parameter Self.V in the protocol generic signature <Self where Self: P>, to obtain a derived requirement [T.V == U.V].

Lemma 12.1. Let G be a well-formed generic signature, and suppose that $G \vDash$ [T == U] for type parameters T and U. If T′ is any valid type parameter having T as a prefix, then $G \vDash$ [T′ == U′], where U′ is the type parameter obtained by replacing T with U in T′.

Proof. We proceed by induction on the length of T′, building the desired same-type requirement by repeated application of the SameDecl or SameName derivation step.

Base case. When T′ has the same length as T, they must in fact be identical, since the latter is a prefix of the former, and thus U′ is also the same as U. We have the necessary derivation $G \vDash$ [T == U] by assumption.

Inductive step. We write T′ as a dependent member type T″.[P]A or T″.A, with base type T″ and associated type A. We have $G \vDash$ T′ by assumption, and so $G \vDash$ T″ by Proposition 11.4. By the inductive hypothesis, we have a derivation $G \vDash$ [T″ == U″].

Note that $G \vDash$ T′ also implies that $G \vDash$ [T″: P]. If T′ is bound, we apply a SameDecl step to [T″ == U″] and [T″: P]:

$$\ldots \vdash [T'' : P] \tag{1}$$
$$\ldots \vdash [T'' == U''] \tag{2}$$
$$(1), (2) \vdash [T''.[P]A == U''.[P]A] \tag{3}$$

If T′ is unbound, we just change the final step to a SameName:

$$(1), (2) \vdash [T''.A == U''.A] \tag{3}$$

In both cases, we get the desired derivation $G \vDash [\texttt{T}' \texttt{ == } \texttt{U}']$. □

Theorem 12.2. Let G be a valid generic signature, with `T` a type parameter and `P` some protocol. If $G \vDash [\texttt{T: P}]$, then there exists at least one conformance path for $[\texttt{T: P}]$. In other words, there exists a type parameter $\texttt{T}' \in \textsc{Type}(G)$, such that:

1. $G \vDash [\texttt{T}'\texttt{: P}]$, via a primitive derivation.
2. $G \vDash [\texttt{T == T}']$.

Proof. The proof relies on a few results developed in Section 11.3. Let's call the two requisite derivations D_1 and D_2. We do a structural induction on the derivation for $[\texttt{T: P}]$, building up D_1 and D_2 step by step. We only need to consider the derivation steps which produce new conformance requirements:

1. Conf steps: $\vdash [\texttt{T: P}]$.
2. SameConf steps: $[\texttt{T == U}], [\texttt{U: P}] \vdash [\texttt{T: P}]$.
3. AssocConf steps: $[\texttt{U: P}], [\texttt{Self.V: Q}] \vdash [\texttt{U.V: Q}]$.

First case. Notice that a Conf step is the base case of our structural induction, because there are no assumptions on the left-hand side of $\vdash$. We set $\texttt{T}' := \texttt{T}$. A derivation of a single explicit conformance requirement is already primitive by our definition, so D_1 is just:

$$\vdash [\texttt{T: P}] \tag{1}$$

To construct D_2, we recall that G is well-formed, thus $G \vDash \texttt{T}$, since `T` appears in $[\texttt{T: P}]$. We then derive $[\texttt{T == T}]$ via an Ident derivation step:

$$\ldots \vdash \texttt{T} \tag{1}$$
$$(2) \vdash [\texttt{T == T}] \tag{2}$$

Second case. We have a SameConf derivation step:

$$[\texttt{T == U}], [\texttt{U: P}] \vdash [\texttt{T: P}]$$

By the inductive hypothesis, we can assume the derivation of $[\texttt{U: P}]$ has already been split up into two derivations: a primitive derivation D_1' of $G \vDash [\texttt{U}'\texttt{: P}]$, and a derivation D_2' of $G \vDash [\texttt{U == U}']$. We set $D_1 := D_1'$. Then, we construct D_2 from D_2' by adding an Equiv derivation step:

$$\ldots [\texttt{T == U}] \tag{1}$$
$$\ldots [\texttt{U == U}'] \tag{2}$$
$$(1), (2) \vdash [\texttt{T == U}'] \tag{3}$$

Third case. The trickiest scenario is when we have a AssocConf derivation step:

$$[\texttt{U: P}],\ [\texttt{Self.V: Q}] \vdash [\texttt{U.V: Q}]$$

By induction, we again assume the derivation of $[\texttt{U: P}]$ has been split up into D_1' and D_2' as above. We construct D_1 from D_1' by adding an AssocConf derivation step for the same associated requirement:

$$\ldots [\texttt{U'': P}] \tag{1}$$

$$(1),\ [\texttt{Self.V: Q}]_{\texttt{P}} \vdash [\texttt{U''.V: Q}] \tag{2}$$

This is a primitive derivation by construction. We derive the same-type requirement $G \vDash [\texttt{U.V == U''.V}]$, giving us D_2. To do that, we note that G is well-formed, so $G \vDash \texttt{U.V}$. Lemma 12.1 then gives us the desired derivation, completing the induction. □

12.2. The Conformance Path Graph

Theorem 12.2 gives us a way to construct a conformance path from a derivation of a conformance requirement, but this does not immediately lead to an effective algorithm, for two reasons. First, the derived requirements formalism is a purely theoretical tool; we don't actually directly build derivations in the implementation. Second, a conformance requirement may have multiple derivations and thus multiple conformance paths. We must be able to deterministically choose the "best" conformance path in some sense, since conformance paths are part of the ABI in the form of symbol mangling.

To address these issues, we reformulate finding a conformance path as a graph theory problem. We construct a graph where the paths through the graph are the conformance paths of a generic signature. This graph might be infinite, but we can visit these paths in a certain order. We simplify each path to an abstract conformance with Algorithm 12C, and compare this with the abstract conformance whose conformance path we were asked to produce. If we have a match, we're done. Otherwise, we check the next path, and so on. Theorem 12.2 now reveals its worth, because it ensures we must eventually find a conformance path which simplifies to our abstract conformance. Thus, our search must end after a finite number of steps.

Definition 12.3. The *conformance path graph* of a generic signature G is the directed graph defined as follows:

- The vertices are the reduced abstract conformances of G.
- The edges are the associated conformance projections.

  That is, if $[\texttt{T: P}]$ is an abstract conformance and $\pi(\texttt{Self.V: Q})$ is an associated conformance requirement of $\texttt{P}$, we form the abstract conformance $\pi(\texttt{Self.V: Q}) \otimes [\texttt{T: P}] = [\texttt{T.V: Q}]$. If $\texttt{U}$ is the reduced type of $\texttt{T.V}$, there is an edge with source vertex $[\texttt{T: P}]$ and destination vertex $[\texttt{U: Q}]$.

Crucially, a conformance path is actually a path, in the graph theoretical sense, whose source vertex is a root abstract conformance. Theorem 12.2 can be interpreted as a statement about the conformance path graph, namely that every vertex is reachable by a path from a root vertex.

Let's construct the conformance path graph for our running example. We have a single conformance path of length 1:

$$p_1 = [\texttt{T: Collection}]$$

The `Collection` protocol has two associated conformance requirements; adding them onto p_1 gives us the two conformance paths of length 2:

$$p_{21} = \pi(\texttt{Self: Sequence}) \otimes p_1$$
$$p_{22} = \pi(\texttt{Self.SubSequence: Collection}) \otimes p_1$$

The conformance path p_{21} likewise has a single successor:

$$p_{31} = \pi(\texttt{Self.Iterator: IteratorProtocol}) \otimes p_{21}$$

The conformance path p_{22} has two successors:

$$p_{32} = \pi(\texttt{Self: Sequence}) \otimes p_{22}$$
$$p_{33} = \pi(\texttt{Self.SubSequence: Collection}) \otimes p_{22}$$

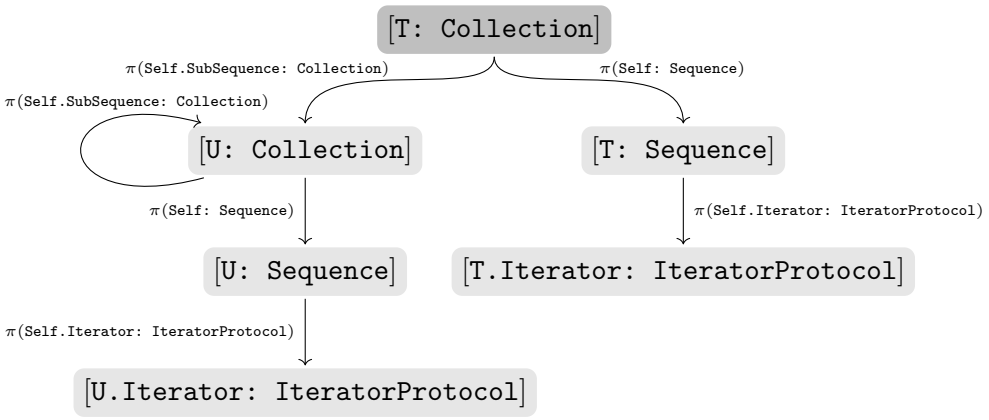
We do not need to consider the successors of p_{33}, because p_{22} and p_{33} simplify to two equivalent abstract conformances, by the same-type requirement in the `Collection` protocol, [`Self.SubSequence == Sub.SubSequence.SubSequence`]:

$$p_{22} = [\texttt{T.SubSequence: Collection}]$$
$$p_{33} = [\texttt{T.SubSequence.SubSequence: Collection}]$$

In fact, we can construct infinitely many conformance paths, which simplify to different representatives of this equivalence class. Let $s := \pi(\texttt{Self.SubSequence: Collection})$. Then, the following are all equivalent, and p_{22} is reduced:

$$p_{22}$$
$$s \otimes p_{22} \qquad (= p_{33})$$
$$s \otimes s \otimes p_{22}$$
$$s \otimes s \otimes s \otimes p_{22}$$
$$\ldots$$

Figure 12.2.: Conformance path graph for Listing 12.1



Finally, considering the successor of p_{32} gives us the sole reduced conformance path of length 4:

$$p_4 = \pi(\texttt{Self.Iterator: IteratorProtocol}) \otimes p_{32}$$

The `IteratorProtocol` protocol does not declare any conformance requirements, so we're done. We can simplify each conformance path to a reduced abstract conformance:

$$\begin{aligned}
p_1 &\Rightarrow [\texttt{T: Collection}] \\
p_{21} &\Rightarrow [\texttt{T: Sequence}] \\
p_{22} &\Rightarrow [\texttt{T.SubSequence: Collection}] \\
p_{31} &\Rightarrow [\texttt{Self.Iterator: IteratorProtocol}] \\
p_{32} &\Rightarrow [\texttt{T.SubSequence: Sequence}] \\
p_4 &\Rightarrow [\texttt{T.SubSequence.Iterator: IteratorProtocol}]
\end{aligned}$$

Figure 12.2 shows the graph; notice how [`U: Collection`] has an edge looping back to itself.

Déjà vu. If this looks familiar, recall that we already studied another directed graph associated with a generic signature, the type parameter graph of Section 8.3. Both graphs are generated by a generic signature, and describe equivalence classes of the reduced type equality relation. It is instructive to compare the two:

	Type parameter graph	**Conformance path graph**
Vertices:	Reduced type parameters	Reduced abstract conformances
Edges:	Associated types	Associated conformances
Paths:	Type parameters	Conformance paths

Conformance path equivalence. We saw that two distinct conformance paths can simplify to equivalent abstract conformances, in which case the corresponding paths through the graph end at the same vertex. We say two conformance paths are *equivalent* in this case. This gives us an equivalence relation, defined in terms of the equivalence relation on abstract conformances (which in turn was defined in terms of the reduced type equality relation on type parameters.) Just as with type parameters and abstract conformances, we can define a linear order on conformance paths. A *reduced* conformance path is one that precedes all other conformance paths in its equivalence class.

Algorithm 12D (Conformance path order)**.** Takes two conformance paths x and y as input, and returns one of "<", ">", or "=" as output.

1. (Shorter) If x has fewer elements than y, return "<".
2. (Longer) If x has more elements than y, return ">".
3. (Initialize) If x and y have the same length, we compare their elements. Let $i := 0$ and $\texttt{N} := |x|$.
4. (Equal) If $i = \texttt{N}$, we didn't find any differences, so $x = y$. Return "=".
5. (Subscript) Let x_i, y_i be the ith elements of x and y, respectively.
6. (Compare) Treating x_i and y_i as requirements, compare them with Algorithm 11D. Return the result if it is "<" or ">".
7. (Next) Increment i and go back to Step 4.

Note that this is a linear order, because in Step 6, Algorithm 11D cannot return "$\perp$", since protocol conformance requirements are always linearly ordered with respect to each other. Much like the type parameter order of Algorithm 5D, this is a special case of a shortlex order, which we generalize in Section 17.5.

Enumeration. The conformance path graph from our previous example has a finite set of vertices, but this is not true in general. However, it is true that each vertex only has finitely many successors, as there can only be finitely many associated conformance requirements. (This is also true in the type parameter graph, where successors are given by associated *type* declarations). A graph with this property is said to be *locally finite*. A locally finite graph only has finitely many paths of any fixed length $n \in \mathbb{N}$ (this can be shown by induction on path length). This implies that the conformance path order is well-founded (by the same argument as Proposition 5.24). It follows that every equivalence class of conformance paths contains a reduced conformance path. Hence, enumerating conformance paths in increasing order performs a breadth-first search which visits every path, and always visits the reduced conformance paths first.

Theorem 12.2 provides a necessary termination condition for our search. This is actually also sufficient; the only other potential source of non-termination is the reduced type computation using the `getReducedType`() generic signature query, but the theory of rewrite systems will give us that guarantee in Section 17.5. Thus, we can always find a conformance path in a finite number of steps. This exhaustive enumeration is relatively inefficient, but we can improve upon it with two modifications:

1. While searching for a conformance path, we visit all preceding conformance paths. If we cache the result of simplifying each conformance path, we can reuse these results if a subsequent lookup requests an earlier conformance path, and avoid restarting the search.

2. If we encounter a conformance path equivalent to one we've already seen, the new conformance path must not be reduced, since we visit paths in increasing order. Thus its successors are not reduced either, and do not need to be considered. This happened in our example when we encountered p_{33} after already visiting p_{22}.

Putting everything together, we finally get an algorithm to find conformance paths. This algorithm maintains the following persistent state for each generic signature:

- A table mapping reduced abstract conformances to conformance paths. Initially empty.
- A non-negative integer `N`, representing the maximum length of conformance paths stored in the table. Initially 0.
- A growable array `B` storing all conformance paths of length `N`. Initially empty.
- A growable array `B′` storing all conformance paths of length `N` + 1. Initially empty.

Algorithm 12E (Find conformance path)**.** Takes a generic signature G and an abstract conformance [`T: P`] valid in G as input. Outputs a conformance path for [`T: P`].

1. (Precondition) Check that `requiresProtocol`(G, `T`, `P`). If this is not true, signal an error. This guarantees termination below.
2. (Reduce) Set `T` ← `getReducedType`(G, `T`).
3. (Check) If the table contains an entry for [`T: P`], return the conformance path associated with this entry.
4. (Initialize) If `N` = 0, initialize `B` with a conformance path of length 1 for each of the generic signature's root abstract conformances, and set `N` = 1.
5. (Loop) For every conformance path $c \in$ `B`:

a) (Simplify) Invoke Algorithm 12C to simplify c to an abstract conformance. Denote this abstract conformance by $[\texttt{T}_c\texttt{: P}_c]$.

b) (Reduce) Set $\texttt{T}_c \leftarrow \texttt{getReducedType}(G, \texttt{T}_c)$.

c) (Check) If the table already contains an entry for $[\texttt{T}_c\texttt{: P}_c]$, continue onto the next iteration of Step 5.

d) (Record) Otherwise, add an entry with key $[\texttt{T}_c\texttt{: P}_c]$ and value c to the table.

e) (Successors) For every associated conformance requirement $\pi(\texttt{Self.A}_i\texttt{: P}_i)$ of protocol $\texttt{P}_c$, add the conformance path $\pi(\texttt{Self.A}_i\texttt{: P}_i) \otimes c$ to $\texttt{B}'$.

6. (Refill) Swap the contents of $\texttt{B}$ with $\texttt{B}'$, and clear $\texttt{B}'$. Increment $\texttt{N}$.

7. (Retry) Go back to Step 1.

Algorithmic complexity. Despite the memoization performed above, Algorithm 12E requires exponential time in the worst case. We can demonstrate this by constructing a generic signature with 2^{n-1} unique conformance paths of length n. Consider the protocol generic signature $G_{\texttt{P}}$ with the following protocol $\texttt{P}$:

```
protocol P {
  associatedtype A: P
  associatedtype B: P
}
```

A conformance path for this signature must begin with the root abstract conformance $[\texttt{Self: P}]$, followed by an arbitrary combination of $\pi(\texttt{Self.A: P})$ and $\pi(\texttt{Self.B: P})$. This generic signature has one conformance path of length 1, 2 conformance paths of length 2, 4 conformance paths of length 3, and so on, with a total of 2^{n-1} unique conformance paths of length n. Therefore, finding a conformance path for an abstract conformance whose subject type has length n, requires enumerating $1+2+4+\cdots+2^{n-1} = 2^n - 1$ possibilities. Compiling a function that forces the compiler to do just that incurs a measurable performance penalty:

```
func callee<T: P>(_: T.Type) {}

func caller<T: P>(_: T.Type) {
  callee(T.A.B.A.B.A.B.A.B.A.B.A.B.A.B.A.B.A.B.A.B.A.B.self)
}
```

In Section 22.4, we will show the algorithm for finding a minimal set of conformance requirements in a generic signature. The algorithm is based on the idea that we can

calculate a finite set of *conformance equations* which completely describe the potentially-infinite conformance path graph. While some details would need to be worked out, it should be possible to one day construct conformance paths directly from conformance equations, instead of the current approach of exhaustive enumeration.

12.3. Recursive Conformances

We saw a generic signature with an infinite type parameter graph in Section 8.3, and the previous section mentioned the possibility of an infinite conformance path graph. Now, we will show that both graphs are infinite if either one is infinite, and then attempt to better understand generic signatures where this is the case.

Proposition 12.4. For a generic signature G, the following are equivalent:

1. G has infinitely many reduced type parameters.
2. G has infinitely many reduced abstract conformances.

This allows us to define an *infinite generic signature* as one satisfying the equivalent conditions above.

Proof. For (1) $\Rightarrow$ (2), note that the set of generic parameter types is always finite, so it suffices to only consider reduced dependent member types. Suppose we're given an infinite set of reduced dependent member types; we must produce an infinite set of abstract conformances. Each dependent member type `T.[P]A` is equivalent to an ordered pair consisting of a type witness projection π(`[P]A`) and an abstract conformance [`T: P`]; the first element of the pair is drawn from a finite set, so a counting argument shows that the mapping that takes the second element of each pair must give us an infinite set of abstract conformances.

Furthermore, these abstract conformances must be reduced, meaning their subject types are reduced. To see why, note that whenever `T.[P]A` is a reduced dependent member type, its base type `T` must be reduced as well (otherwise, if $G \vDash$ [`T′ == T`] with `T′` $<$ `T`, we could construct from this a derivation of [`T′.[P]A == T.[P]A`] with `T′.[P]A` $<$ `T.[P]A`, contradicting the assumption that `T.[P]A` is reduced).

A similar argument establishes (2) $\Rightarrow$ (1). We're given an infinite set of reduced abstract conformances, and we must produce an infinite set of reduced type parameters. Each abstract conformance [`T: P`] uniquely determines an ordered pair, consisting of a protocol declaration `P` and a type parameter `T`. The set of protocol declarations is finite, so again, taking the second element of each pair gives us an infinite set of reduced type parameters. □

Next, we will characterize infinite generic signatures by way of another directed graph.

Definition 12.5. The *protocol dependency graph* is defined as follows:

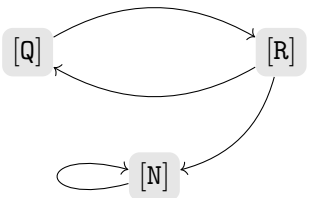
- The vertices are protocol declarations.
- The edges are again the associated conformance requirements. The source of an edge is the protocol declaring the associated conformance requirement, and the destination is the protocol on the right hand side of the requirement.

The protocol dependency graph differs from the type parameter and conformance path graphs in several respects:

- It is a global construction, and not associated with a single generic signature.
- It is finite, as a program can only define a finite set of protocols and associated conformance requirements.
- There are no root vertices or connectivity requirements, and the graph may contain a large number of disjoint components.

A conformance path defines a path in the protocol dependency graph: we map each abstract conformance to its conformed protocol, observing that the edge relation in both graphs is associated conformance projection. Also, an infinite generic signature must have conformance paths of arbitrary length, as there are only finitely many conformance paths of any *fixed* length. The protocol dependency graph is finite, so the protocol dependency path induced by a sufficiently-long conformance path then has a cycle.

A *recursive conformance requirement* is an associated conformance requirement that is part of a cycle; hence, a necessary condition for writing down an infinite generic signature is that the protocol dependency graph must contain a cycle. Thus, prior to Swift 4.1 introducing recursive conformance requirements [78], the protocol dependency graph was required to be acyclic, and every generic signature was necessarily finite.



[O]

The protocol dependency graph for Listing 12.2 is shown on the right. It has two distinct cycles; the first joins [Q] with [R], and the second is a loop at [N]. This gives us three recursive associated conformance requirements, by the above definition:

π(`Self.[N]A: N`)
π(`Self.[Q]B: R`)
π(`Self.[R]D: Q`)

Note that π(`Self.[R]C: N`) is *not* recursive, and [O] is not connected with the other protocols at all; in fact, it does not declare any associated conformance requirements.

Listing 12.2.: Protocol dependency graph example

```
protocol N {
  associatedtype A: N
}

protocol Q {
  associatedtype B: R
}

protocol R {
  associatedtype C: N
  associatedtype D: Q
}

protocol O {
  associatedtype E
}
```

Now consider the protocol generic signature $G_{\texttt{Q}}$ from Listing 12.2. Figure 12.3 shows the conformance path graph for this generic signature, an infinite tree with the abstract conformance [`Self: Q`] as the root. One possible path from the root is the conformance path for [`Self.B.D.B.C: N`] in $G_{\texttt{Q}}$:

$$\pi(\texttt{Self.C: N}) \otimes \pi(\texttt{Self.B: R}) \otimes \pi(\texttt{Self.D: Q}) \otimes \pi(\texttt{Self.B: R}) \otimes [\texttt{Self: Q}]$$

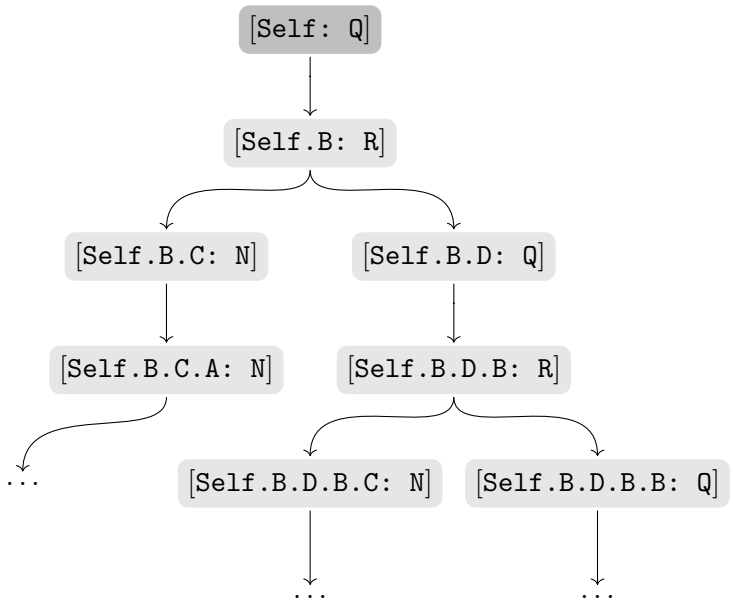
Writing down the list of protocols from each step (remember that conformance paths are read from right to left!) we get a path in the protocol dependency graph; the path visits [`Q`] and [`R`] twice, exhibiting the existence of a cycle:

$$[\texttt{Q}] \longrightarrow [\texttt{R}] \longrightarrow [\texttt{Q}] \longrightarrow [\texttt{R}] \longrightarrow [\texttt{N}]$$

We will encounter protocol dependency graphs again in Chapter 16, when we describe the construction of the rewrite system for a generic signature.

We know that infinite generic signatures have conformance paths of arbitrary length; the same argument applies to type parameters as well. We will now strengthen this to show that there exists an infinite sequence of type parameters or conformance paths of increasing length, where each one is a proper prefix of the next. Recall that the type parameter graph and conformance path graph both have finitely many root vertices, and each vertex only has finitely many successors. This allows us to make use of a classic theorem Dénes Kőnig proved in 1927 (see [92], or Section 2.3.4.3 of [75]).

Figure 12.3.: Conformance path graph for $G_{\texttt{Q}}$ from Listing 12.2



Theorem 12.6 (Kőnig's infinity lemma)**.** Assume an infinite directed graph satisfies two conditions:

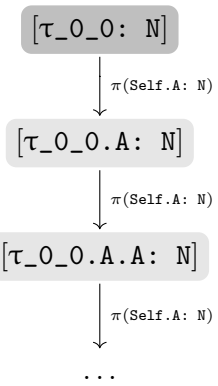
1. Every vertex has finitely many successors (such a graph is said to be locally finite).
2. Every vertex is reachable by a path from a fixed finite set of root vertices.

Then the graph must contain an infinite path: this is an infinite sequence of distinct vertices, where each element is joined to the next by an edge.

Proof. To choose the first element of the sequence, note that there are only finitely many roots, but every vertex is reachable from at least one root by assumption, so at least one root must be the origin of infinitely many distinct paths. This root serves as the first element of our sequence. We can then always add another vertex to our sequence, as long as the last vertex is the origin of infinitely many distinct paths, not involving any vertex already in our sequence. However, as each vertex has finitely many successors, so at least one successor of the last vertex has the necessary property. We can iterate this construction arbitrarily in this manner, giving us the desired infinite sequence of vertices. □

The protocol generic signature $G_{\texttt{N}}$ with the protocol `N` from Listing 12.2 is then the quintessential infinite generic signature, in a sense, because its conformance path graph

Figure 12.4.: Conformance path graph for protocol `N` from Listing 12.2



is just one ray, shown in Figure 12.4. Let $\Sigma_{[\texttt{T: N}]}$ be a protocol substitution map for a conformance to `N`. This is a substitution map with input generic signature $G_{\texttt{N}}$, which we can apply to each abstract conformance of $G_{\texttt{N}}$, obtaining a sequence of concrete conformances:

$$[\texttt{τ\_0\_0: N}] \otimes \Sigma_{[\texttt{T: N}]} = [\texttt{T: N}]$$
$$[\texttt{τ\_0\_0.A: N}] \otimes \Sigma_{[\texttt{T: N}]} = \pi(\texttt{Self.A: N}) \otimes [\texttt{T: N}]$$
$$[\texttt{τ\_0\_0.A.A: N}] \otimes \Sigma_{[\texttt{T: N}]} = \pi(\texttt{Self.A: N}) \otimes \pi(\texttt{Self.A: N}) \otimes [\texttt{T: N}]$$
$$\ldots$$

Local conformance lookup associates a substituted conformance with each abstract conformance. The substituted conformances are not necessarily distinct, but the mapping is compatible with associated conformance projection, as we saw in Section 7.5. To understand the structure we obtain here, we define yet another directed graph.

Definition 12.7. The *conformance evaluation graph* of a substitution map is the following directed graph:

- The vertices are the substituted conformances obtained by applying the substitution map to each abstract conformance of its input generic signature.
- The edge relation is given by associated conformance projection.

The conformance path graph encodes all conformance paths of a *generic signature*, while the conformance evaluation graph encodes the substituted conformances one can obtain from a *substitution map*.

In fact, this mapping from the conformance path graph to the conformance evaluation graph, given by perform a local conformance lookup on each abstract conformance, is a special kind of mapping between directed graphs.

Definition 12.8. Let $G := (V_1, E_1)$ and $H := (V_2, E_2)$ be directed graphs. A *graph homomorphism* $f\colon G \to H$ is a pair of functions, sending vertices to vertices and edges to edges, satisfying a compatibility condition:

$$
\begin{aligned}
&f_V\colon V_1 \to V_2\\
&f_E\colon E_1 \to E_2\\
&f_V(\mathrm{src}(e)) = \mathrm{src}(f_E(e))\\
&f_V(\mathrm{dst}(e)) = \mathrm{dst}(f_E(e)).
\end{aligned}
$$

That is, if two vertices are joined by an edge in E_1, their image must be joined by an edge in E_2 (and if both vertices map to the same vertex, then E_2 contains a loop at this vertex). An immediate consequence of this definition is that a graph homomorphism also maps paths in G to paths in H.

Let's now reconsider the application of a protocol substitution map $\Sigma_{[\texttt{T: N}]}$, where `T` is an arbitrary concrete type, to each abstract conformance of $G_{\texttt{N}}$. In light of the above, that we're actually looking at is a graph homomorphism from the conformance path graph of $G_{\texttt{N}}$, to the conformance evaluation graph of $\Sigma_{[\texttt{T: N}]}$. We saw that the conformance path graph of $G_{\texttt{N}}$ is a ray, and local conformance lookup maps each successive vertex to the result of applying $\pi(\texttt{Self.A: N})$ some number of times to the root conformance `[T: N]`. There are three possibilities, each one corresponding to different outcomes of the repeated application of $\pi(\texttt{Self.A: N})$ to `[T: N]`:

1. We eventually end up back at `[T: N]`; the ray is mapped to a cycle, and the conformance evaluation graph is finite.

2. Every application of $\pi(\texttt{Self.A: N})$ gives us a new conformance we have not seen before; the ray is mapped to another ray, and we have an infinite conformance evaluation graph.

3. We eventually end up back at some conformance we've already seen, but not `[T: N]`; the ray is mapped to a finite path leading to a cycle. The conformance evaluation graph is again finite in this case.

We will now construct an example of each of the three situations above, by defining different concrete types that conform to `N`.

Recursive normal conformances. Consider a pair of types conforming to `N`:

```
struct X: N {
  typealias A = Y
}

struct Y: N {
  typealias A = X
}
```

The normal conformance [`X: N`] records the type witness `Y` for `A`, and also the associated conformance [`Y: N`] for π(`Self.A: N`):

`X: N`		
Associated type		**Type witness**
π(`[N]A`)	$\mapsto$	`Y`
Conformance requirement		**Conformance**
π(`Self.[N]A: N`)	$\mapsto$	[`Y: N`]

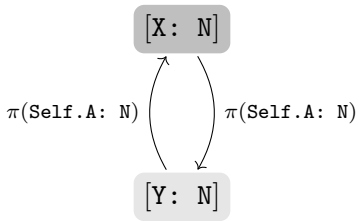
The conformance [`Y: N`] is likewise a mirror image that points back to `X` and [`X: N`]. We can express the relationship between the two conformances via associated conformance projection:

$$\pi(\texttt{Self.A: N}) \otimes [\texttt{X: N}] = [\texttt{Y: N}]$$
$$\pi(\texttt{Self.A: N}) \otimes [\texttt{Y: N}] = [\texttt{X: N}]$$

We can construct a substitution map for our generic signature `<τ_0_0 where τ_0_0: P>` which replaces `τ_0_0` with the concrete type `X`:

$$\{\texttt{τ\_0\_0} \mapsto \texttt{X};\ [\texttt{τ\_0\_0: N}] \mapsto [\texttt{X: N}]\}$$

The conformance evaluation graph of this substitution map only has two vertices, the root conformance [`X: N`] and the other normal conformance [`Y: N`], and the vertices form a cycle. The recursive conformance requirement allows us to jump from one conformance to the other and back again:



Recursive specialized conformances. When a recursive conformance requirement is witnessed by a specialized conformance, we can observe an infinite conformance evaluation graph:

```
struct G<T>: N {
  typealias A = G<G<T>>
}
```

Writing the canonical type `τ_0_0` in place of `T`, we can exhibit the information stored in the normal conformance [`G<τ_0_0>: N`] with the below table.

`G<τ_0_0>: N`		
Associated type		**Type witness**
π(`[N]A`)	$\mapsto$	`G<G<τ_0_0>>`
Conformance requirement		**Conformance**
π(`Sef.[N]A: N`)	$\mapsto$	[`G<G<τ_0_0>>: N`]

Let $\Sigma :=$ {`τ_0_0` $\mapsto$ `G<τ_0_0>`}. Then, the associated conformance for π(`Self.A: N`) is a specialized conformance having Σ as the conformance substitution map. Thus:

$$\pi(\texttt{[N]A}) \otimes [\texttt{G<τ\_0\_0>: N}] = \texttt{G<τ\_0\_0>} \otimes \Sigma$$
$$\pi(\texttt{Self.A: N}) \otimes [\texttt{G<τ\_0\_0>: N}] = [\texttt{G<τ\_0\_0>: N}] \otimes \Sigma$$

In particular, the normal conformance points back to itself via the associated conformance. Now consider this specialized conformance:

$$[\texttt{G<Int>: N}] = [\texttt{G<τ\_0\_0>: N}] \otimes \{\texttt{τ\_0\_0} \mapsto \texttt{Int}\}$$

The choice of `Int` is arbitrary—any type can be used as the generic argument, since the generic parameter `τ_0_0` is not constrained. We see that:

$$\pi(\texttt{Self.A: N}) \otimes [\texttt{G<Int>: N}] = [\texttt{G<G<Int>>: N}]$$
$$\pi(\texttt{Self.A: N}) \otimes \pi(\texttt{Self.A: N}) \otimes [\texttt{G<Int>: N}] = [\texttt{G<G<G<Int>>>: N}]$$
$$\dots$$

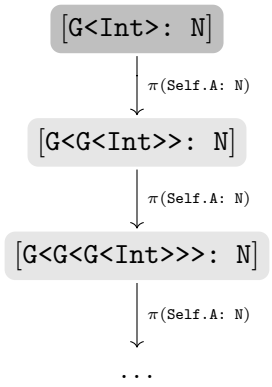
More generally, projecting the associated conformance any number of times on the left is equivalent to applying Σ the same number of times on the right, which adds new levels of "`G<>`":

$$\underbrace{\pi(\texttt{Self.A: N}) \otimes \cdots \otimes \pi(\texttt{Self.A: N})}_{n \text{ times}} \otimes [\texttt{G<τ\_0\_0>: N}] \otimes \{\texttt{τ\_0\_0} \mapsto \texttt{Int}\}$$
$$= [\texttt{G<τ\_0\_0>: N}] \otimes \underbrace{\Sigma \otimes \cdots \otimes \Sigma}_{n \text{ times}} \otimes \{\texttt{τ\_0\_0} \mapsto \texttt{Int}\}$$

Now suppose we take the substitution map for `<τ_0_0 where τ_0_0: P>` which replaces `τ_0_0` with `G<Int>`:

$$\{\texttt{τ\_0\_0} \mapsto \texttt{G<Int>};\ [\texttt{τ\_0\_0: N}] \mapsto [\texttt{G<Int>: N}]\}$$

We've shown that this substitution map has an infinite conformance evaluation graph. Starting from the root conformance `[G<Int>: N]`, we can construct arbitrarily "large" specialized conformances; the conformance evaluation graph is a ray, just like the original conformance path graph:



Recursive abstract conformances. We saw a recursive conformance requirement being fulfilled by a normal conformance, and then a specialized conformance. The third possibility, where this conformance is abstract, reveals another interesting phenomenon:

```
struct H<T: N>: N {
  typealias A = T
}
```

This time, the normal conformance `[H<τ_0_0>: N]` is not circular at all. The associated conformance is the abstract conformance `[τ_0_0: N]`:

`G<τ_0_0>: N`		
Associated type		**Type witness**
π(`[N]A`)	$\mapsto$	`τ_0_0`
Conformance requirement		**Conformance**
π(`Self.[N]A: N`)	$\mapsto$	`[τ_0_0: N]`

Let's define two substitution maps since they will appear multiple times below:

$$\Sigma_{\texttt{H}} := \{\texttt{τ\_0\_0} \mapsto \texttt{H<τ\_0\_0>};\ [\texttt{τ\_0\_0: N}] \mapsto [\texttt{H<τ\_0\_0>: N}]\}$$
$$\Sigma_{\texttt{X}} := \{\texttt{τ\_0\_0} \mapsto \texttt{X};\ [\texttt{τ\_0\_0: N}] \mapsto [\texttt{X: N}]\}$$

Consider the specialized conformance [H<H<X>>: N]. The conformance substitution map is the composition $\Sigma_{\texttt{H}} \otimes \Sigma_{\texttt{X}}$:

$$[\texttt{H<H<X>>: N}] = [\texttt{H<}\tau\texttt{\_0\_0>: N}] \otimes \Sigma_{\texttt{H}} \otimes \Sigma_{\texttt{X}}$$

Also we have these two identities:

$$\pi(\texttt{Self.A: N}) \otimes [\texttt{H<}\tau\texttt{\_0\_0>: N}] = [\tau\texttt{\_0\_0: N}]$$
$$[\tau\texttt{\_0\_0: N}] \otimes \Sigma_{\texttt{H}} = [\texttt{H<}\tau\texttt{\_0\_0>: N}]$$

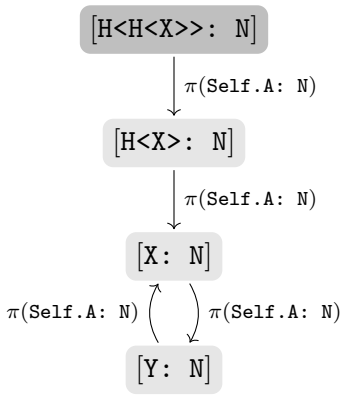
Now we can calculate $\pi(\texttt{Self.A: N}) \otimes [\texttt{H<H<X>>: N}]$:

$$\begin{aligned}
&\pi(\texttt{Self.A: N}) \otimes [\texttt{H<H<X>>: N}] \\
&= \pi(\texttt{Self.A: N}) \otimes [\texttt{H<}\tau\texttt{_0_0>: N}] \otimes \Sigma_{\texttt{H}} \otimes \Sigma_{\texttt{X}} \\
&= [\tau\texttt{_0_0: N}] \otimes \Sigma_{\texttt{H}} \otimes \Sigma_{\texttt{X}} \\
&= [\texttt{H<}\tau\texttt{_0_0>: N}] \otimes \Sigma_{\texttt{X}} \\
&= [\texttt{H<X>: N}]
\end{aligned}$$

We jumped from [H<H<X>>: N] to [H<X>: N] by performing the associated conformance projection. If we project the associated conformance again, we get:

$$\begin{aligned}
&\pi(\texttt{Self.A: N}) \otimes [\texttt{H<X>: N}] \\
&= \pi(\texttt{Self.A: N}) \otimes [\texttt{H<}\tau\texttt{_0_0>: N}] \otimes \Sigma_{\texttt{X}} \\
&= [\tau\texttt{_0_0: N}] \otimes \Sigma_{\texttt{X}} \\
&= [\texttt{X: N}]
\end{aligned}$$

We see that at each step, the projection "peels off" an H<>, until we finally reach the leaf type X, at which point the projection cycles between [X: N] and [Y: N], as we've already seen. We can now draw the conformance evaluation graph for [H<H<X>>: N], which consists of a path leading to a cycle, the cycle being the conformance evaluation graph of [X: N]:



Non-terminating substitutions. We gave a termination proof for Algorithm 12E; we can always find a conformance path in a finite number of steps. However, we cannot make the same guarantee about Algorithm 12B for evaluating of a conformance path. This example demonstrates non-terminating substitution:

```
struct S: N {
  typealias A = F<S>
}

struct F<T: N>: N {
  typealias A = T.A.A
}
```

We have a pair of types conforming to `N`, and thus two normal conformances. Here is the first normal conformance:

`S: N`		
Associated type		**Type witness**
π(`[N]A`)	$\mapsto$	`F<S>`
Conformance requirement		**Conformance**
π(`Self.[N]A: N`)	$\mapsto$	[`F<S>: N`]

And the second:

`F<τ_0_0>: N`		
Associated type		**Type witness**
π(`[N]A`)	$\mapsto$	`τ_0_0.[N]A.[N]A`
Conformance requirement		**Conformance**
π(`Self.[N]A: N`)	$\mapsto$	[`τ_0_0.[N]A.[N]A: N`]

We now claim that there is no possible substituted type which can be ascribed to the declaration of `value` in the below code:

```
func f<T: N>(_: T.Type) -> T.A.A.Type { ... }

let value = f(F<S>.self)
```

Define these two substitution maps:

$$\Sigma_S := \{\texttt{τ\_0\_0} \mapsto \texttt{S};\ [\texttt{τ\_0\_0: N}] \mapsto [\texttt{S: N}]\}$$
$$\Sigma_F := \{\texttt{τ\_0\_0} \mapsto \texttt{F<τ\_0\_0>};\ [\texttt{τ\_0\_0: N}] \mapsto [\texttt{F<τ\_0\_0>: N}]\}$$

In the first conformance, we can expand both the type witness (as a declared interface type and context substitution map) and the associated conformance (as a normal conformance and conformance substitution map) in terms of Σ_S:

$$\pi(\texttt{[N]A}) \otimes [\texttt{S: N}] = \texttt{F<}\tau\texttt{\_0\_0>} \otimes \Sigma_S$$
$$\pi(\texttt{Self.A: N}) \otimes [\texttt{S: N}] = [\texttt{F<}\tau\texttt{\_0\_0>: N}] \otimes \Sigma_S$$

In the second conformance, the type witness is a type parameter, and the associated conformance is abstract, so we can expand both using a conformance path:

$$\pi(\texttt{[N]A}) \otimes [\texttt{F<}\tau\texttt{\_0\_0>: N}] = \pi(\texttt{[N]A}) \otimes \pi(\texttt{Self.A: N}) \otimes [\tau\texttt{\_0\_0: N}]$$
$$\pi(\texttt{Self.A: N}) \otimes [\texttt{F<}\tau\texttt{\_0\_0>: N}] = \pi(\texttt{Self.A: N}) \otimes \pi(\texttt{Self.A: N}) \otimes [\tau\texttt{\_0\_0: N}]$$

We will attempt to compute $\tau\texttt{\_0\_0.[N]A} \otimes \Sigma_F \otimes \Sigma_S$ using the conformances $[\texttt{S: N}]$ and $[\texttt{F<}\tau\texttt{\_0\_0>: N}]$. Since $[\tau\texttt{\_0\_0: N}] \otimes \Sigma_F = [\texttt{F<}\tau\texttt{\_0\_0>: N}]$, we have:

$$\begin{aligned}
&\tau\texttt{_0_0.[N]A} \otimes \Sigma_F \otimes \Sigma_S \\
&\quad = \big(\pi(\texttt{[N]A}) \otimes [\tau\texttt{_0_0: N}]\big) \otimes \Sigma_F \otimes \Sigma_S \\
&\quad = \pi(\texttt{[N]A}) \otimes \big([\tau\texttt{_0_0: N}] \otimes \Sigma_F\big) \otimes \Sigma_S \\
&\quad = \pi(\texttt{[N]A}) \otimes [\texttt{F<}\tau\texttt{_0_0>: N}] \otimes \Sigma_S
\end{aligned}$$

Next, we project the type witness $\pi(\texttt{[N]A}) \otimes [\texttt{F<}\tau\texttt{\_0\_0>: N}]$ and use $[\tau\texttt{\_0\_0: N}] \otimes \Sigma_S = [\texttt{S: N}]$:

$$\begin{aligned}
&\big(\pi(\texttt{[N]A}) \otimes [\texttt{F<}\tau\texttt{_0_0>: N}]\big) \otimes \Sigma_S \\
&\quad = \big(\pi(\texttt{[N]A}) \otimes \pi(\texttt{Self.A: N}) \otimes [\tau\texttt{_0_0: N}]\big) \otimes \Sigma_S \\
&\quad = \pi(\texttt{[N]A}) \otimes \pi(\texttt{Self.A: N}) \otimes \big([\tau\texttt{_0_0: N}] \otimes \Sigma_S\big) \\
&\quad = \pi(\texttt{[N]A}) \otimes \pi(\texttt{Self.A: N}) \otimes [\texttt{S: N}]
\end{aligned}$$

But, the only further simplification we can apply here is $\pi(\texttt{Self.A: N}) \otimes [\texttt{S: N}] = [\texttt{F<}\tau\texttt{\_0\_0>: N}] \otimes \Sigma_S$, which brings us back to the left hand side of the equation prior:

$$\begin{aligned}
&\pi(\texttt{[N]A}) \otimes \big(\pi(\texttt{Self.A: N}) \otimes [\texttt{S: N}]\big) \\
&\quad = \pi(\texttt{[N]A}) \otimes \big([\texttt{F<}\tau\texttt{_0_0>: N}] \otimes \Sigma_S\big)
\end{aligned}$$

It appears that our attempt to evaluate $\tau\texttt{\_0\_0.[N]A} \otimes \Sigma_F \otimes \Sigma_S$ gets stuck in a loop. This exposes a hole in our theory, because it shows that local conformance lookup—and our type substitution operator "$\otimes$"—are actually *partial* functions, which do not always output a result. Similarly, the substitution maps Σ_S and Σ_F do not have conformance evaluation graphs. (While for the most part this theoretical hole doesn't matter, in Section 17.5, we will sketch out an alternative way of formalizing "$\otimes$" which avoids this difficulty.)

Indeed, at the time of writing, the compiler actually terminates with a stack overflow while attempting to perform this type substitution. In the future, the compiler should detect this situation and diagnose it, instead of crashing. A savvy compiler engineer will immediately suggest at least two possible approaches for doing so:

1. We can impose a maximum iteration count in type substitution, terminating with an error if this count is exceeded.

2. We might hope to detect the circularity ahead of time, by carefully analyzing all normal conformances and their type witnesses.

The first approach seems less elegant, because it also necessarily rejects *valid* programs that happen to exceed the maximum iteration count.[1] The second approach could certainly be programmed to detect this particular example "ahead of time", without evaluation. However, the next section will show that we can't actually do this in general; there is no way to detect *all* non-terminating substitutions without resorting to the first approach.

12.4. The Halting Problem

This section will show that the type substitution algebra can express any computable function, and explore the consequences of this fact. First, we will review some standard results in computability theory; details can be found in any book on the subject, such as [93].

In 1937, Alan Turing published a paper where he set out to make precise what is meant by a "computable number," in the intuitive sense that $2 + \frac{3}{7}$, $\sqrt{5}$ and π are all computable [94]. In Turing's view, for a number to be computable, one must be able to write down a finite series of steps which give an effective procedure for computing an approximate representation of this number to any arbitrary precision. Turing formalized these computational steps in the form of an abstract machine which reads and writes symbols on an infinite tape—a *Turing machine* (the precise definition can be found in the literature, but it doesn't matter for our purposes).

It suffices to only consider the base-2 representation of numbers, with the two symbols "0" and "1". Furthermore, an integer is always computable, since the effective procedure there is to list out the finite sequence of binary digits; thus, the problem reduces to defining a notion of computability for real numbers in the interval $[0, 1)$. For a rational number, the binary expansion always either terminates, or repeats; we can consider a terminating expansion as one that ends with an infinite repeating sequence of "0". With an irrational number, the sequence of digits carries on forever, without repetition.

[1] However, in practice, it is difficult to imagine a real-world use of generics where any single type substitution would ever need to perform more than, say, 1000 evaluation steps, when expressed with our type substitution algebra.

Thus, a computable number is one for which we can define a Turing machine which continues to output a stream of "0" and "1" while it is running; we can obtain a binary approximation with any desired number of digits by letting it run long enough. It is important that even for a number with a terminating binary expansion, the machine outputs digits forever; so a Turing machine representing $\frac{1}{2}$ must produce 1, 0, 0, ..., instead of simply emitting 1 and then stopping. To understand why, suppose we are given a Turing machine representing some computable number, and we require a binary approximation with 100 digits; if our machine outputs 99 digits and then appears to get stuck, do we simply assume all remaining digits are 0, or do we wait a little while longer, hoping it will output another digit, which might be 1?

Turing considered a machine to be well-behaved if it continued to output binary digits forever; he called such machines "circle-free." A "circular" machine is one that is not circle-free; it gets "stuck" after outputting some finite sequence of binary digits. Turing wanted to know if, given a Turing machine, there was an effective procedure to determine if it was circle-free.

In order to construct machines to reason about other machines, Turing defined the "standard description" of a machine, encoding its internal state as a finite sequence of digits, and a "satisfactory" integer as one whose binary representation corresponds to the standard description of a circle-free machine. One might then ask, can we define a circle-free Turing machine, which takes an arbitrary integer as input, and outputs "1", "1", ... if this integer is satisfactory, and "0", "0", ... otherwise? As it turns out, this is not possible.

Theorem 12.9. No circle-free Turing machine can determine if an arbitrary integer is satisfactory.

Today, we formulate Turing's work in terms of the *halting problem*, which asks if the machine reaches some final "good" state in a finite number of steps, rather than asking if a machine continues to output digits forever. After all, these Turing machines can perform more varied tasks than approximation of computable numbers. Each question can be reduced to the other though, showing that both are equivalent:

- Suppose we could determine if a Turing machine is circle-free. Then, we could determine if an arbitrary machine halts, if we modify it to output an infinite sequence of "0" when it reaches the halting state instead. The modified machine is circle-free if and only if the original machine halts.

- Now, suppose we could determine if a Turing machine halts on every possible input. Then, we could determine if an arbitrary machine is circle-free, by constructing a new machine which simulates the execution of the original, taking an arbitrary initial state as input, halting as soon as it outputs a single digit. Then the modified machine always halts if and only if the original machine is circle-free.

Before we look at an informal proof of the undecidability of the halting problem, we make a philosophical remark. Turing's results have value because they are not narrowly just about the Turing machine formalism itself, but any form of computation capable of simulating a Turing machine. Concurrently with Turing's work, Alonzo Church invented a formal system called the *lambda calculus*, and demonstrated a similar undecidability result [95]; Turing actually went on to become Church's student. It was then shown that every lambda-computable function can be expressed by Turing machine and vice versa, showing the two formalisms have equal expressive power. Indeed, every computational formalism discovered since has been shown to be equivalent to the Turing machine. The *Church-Turing thesis* is the statement that all forms of universal computation are equivalent in this sense.

We know Swift is Turing-complete, so we can prove the halting problem with an imagined Swift dialect having an introspection facility for function values: either giving their source code, syntax tree, machine code, or anything else. (This is not a real requirement though; with a bit of setup, the function type `() -> ()` appearing below could be replaced with `String`, or `Array<Int>`, or any other possible representation of a computable function.) We also require our functions to be deterministic, in the sense that every execution carries out the same series of steps, and they do not depend on any external input or output, other than the arguments given to the function itself.

Now, we assume we have a magical function, that in a finite number of steps, can decide if some other function always halts—a *termination checker*:

```
func halts(_ f: () -> ()) -> Bool {...}
```

In other words, `halts(f)` returns true if a call to `f()` eventually returns, or false if `f()` runs forever. Furthermore, we're assuming that `halts()` itself computes a result in a finite number of steps; this rules out a trivial implementation of `halts()` which "decides" if `f()` halts by running it:

```
func halts(_ f: () -> ()) -> Bool {
  f()  // This is cheating and doesn't count!
  return true
}
```

Thus, under our assumption, `halts({ halts(g) })` itself always halts, for every other function `g`. Now, keeping this in mind, we proceed to define a "naughty" function:

```
func naughty() {
  if halts(naughty) {  // Do the opposite of what halts() says we do!
    while true {}
  }
}
```

This function is naughty, because it leads to a contradiction. If `halts(naughty)` were to return true, then `naughty()` would run forever, thus `halts(naughty)` cannot be true. Correspondingly, if `halts(naughty)` were to return false, then `naughty()` would terminate, so `halts(naughty)` cannot be false. The only other possibility is that `halts(naughty)` runs forever without returning anything. But this cannot happen either, since we assumed that `halts()` itself makes its determination in a finite number of steps. Thus, no function `halts()` can exist with the behavior we specified—we say the halting problem is *undecidable*.

To make this argument precise, some care must be taken in the formal construction of the "fixed point" `naughty()` function, on whose existence the whole argument hinges. The intuition here, though, is that in any sufficiently expressive formal system, we can always "outwit" any purported termination checking algorithm, no matter how sophisticated it may appear to be, by encoding the termination checker itself within our formalism, and acting on what it might do.

Note that while termination checking is undecidable in full generality, we can still solve restricted forms of the problem:

1. By carefully studying a *specific* program, we might be able to write down a proof showing that it terminates. Such a proof, however, cannot generalize to arbitrary programs.

2. If the input language is sufficiently restricted, for example if general recursion and loops are not permitted, we might be able to construct a termination checker. In this case, our language cannot express an arbitrary Turing machine, but it might still be sufficient for our purposes.

3. We solve certain "simple" cases of non-termination, such as an unconditional infinite loop. Compilers frequently implement this as a best-effort diagnostic. In this case, we can't detect non-terminating programs if the control flow is too complex for the checker to reason about.

4. Finally, we might let the input program run for a finite period of time, or a finite number of steps, and reject it if it does not halt before then. In this case, we might reject a valid terminating program if it still takes too long to run.

Tag systems. To see that Swift type substitution has an undecidable halting problem, we must be able to encode an arbitrary Turing machine in terms of substitution maps and conformances, but we do this in a somewhat circuitous manner. We will define *tag systems*, which are another universal computational formalism, and show how to encode one particular tag system in the type substitution algebra. The same encoding readily generalizes to any tag system, which establishes that type substitution has an undecidable halting problem.

Emil L. Post introduced tag systems in a 1943 paper [96]. A tag system consists of the following three parts:

- A finite alphabet of symbols.
- A fixed $n \in \mathbb{N}$, called the *deletion number*.
- A table associating a *production*—a string of zero or more symbols—to each symbol of the alphabet.

A tag system operates on an input string, also written with the symbols of the alphabet. The evaluation rule transforms the input string into an output string by repeated application of the following steps:

1. If the input string has fewer than n symbols, stop evaluation.
2. Otherwise, fetch the symbol at the start of the input string, and look up the production associated with this symbol in the table.
3. Append the production at the end of the input string.
4. Remove the first n symbols from the beginning of the input string.
5. Go back to Step 1.

It is not immediately apparent from the definition that tag systems are a universal form of computation in the Church-Turing sense. Post himself wondered if there is a computable algorithm to determine if, for an arbitrary input string, a given tag system will halt. This remained an open question until 1961, when it was discovered that indeed, it is possible to encode a Turing machine as a tag system, in such a manner that the tag system halts if and only if the Turing machine halts. Thus, tag systems have equal expressive power to Turing machines, the lambda calculus, and everything else, and furthermore, the halting problem for tag systems is also undecidable [97]. Tag systems constructed from Turing machines require a large alphabet, so we merely note that it is possible to do so, without showing an explicit example. Instead, we will focus on a much simpler tag system, which nonetheless demonstrates the expressivity of the formalism.

The Collatz conjecture. The *Collatz conjecture*, named after Lothar Collatz, is a famous unsolved problem in number theory. Consider the following function, $f\colon \mathbb{N} \to \mathbb{N}$:

$$f(n) = \begin{cases} n/2 & n \text{ even} \\ 3n+1 & n \text{ odd} \end{cases}$$

For any $n \in \mathbb{N}$, we can construct an infinite sequence of natural numbers, by repeated application of f to n, called the *Collatz sequence* for n:

$$n,\ f(n),\ f(f(n)),\ f(f(f(n))),\ \ldots$$

For example, if we take $n = 3$, then we get the Collatz sequence

$$3,\ 10,\ 5,\ 16,\ 8,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ \ldots$$

Notice how after a few steps, the Collatz sequence for 3 reaches the value 1; upon reaching 1, any Collatz sequence enters a loop consisting of 1, 4, 2, 1, and so on. While this has been observed with every Collatz sequence computed to date, it is not known if this is true in general.

We will now perform the construction from [98], and define a tag system which computes the Collatz sequence, for a given natural number represented as a string. This tag system halts if the sequence reaches 1, and thus, it halts on all inputs if and only if the Collatz conjecture is true. We take the three-symbol alphabet $\{a, b, c\}$, a deletion number of 2, and the following three production rules:

$$a \mapsto bc$$
$$b \mapsto a$$
$$c \mapsto aaa$$

To evaluate the Collatz sequence for $n \in \mathbb{N}$, we take the input string to be n repetitions of the symbol "a". Since the deletion number is 2, the evaluation rule stops when the input string has fewer than 2 symbols; in particular, it stops if the input string is just "a", corresponding to the number 1. Let's look at the evaluation of this tag system with the input string "$aaaa$"; at each step, we look at the first symbol, add the corresponding production at the end, and finally delete the first two symbols:

1. $\underline{a}aaa \Rightarrow aaaa\underline{bc} \Rightarrow aabc$
2. $\underline{a}abc \Rightarrow aabc\underline{bc} \Rightarrow bcbc$
3. $\underline{b}cbc \Rightarrow bcbc\underline{a} \Rightarrow bca$
4. $\underline{b}ca \Rightarrow bca\underline{a} \Rightarrow aa$
5. $\underline{a}a \Rightarrow aa\underline{bc} \Rightarrow bc$
6. $\underline{b}c \Rightarrow bc\underline{a} \Rightarrow a$
7. a

Evaluation now stops; the length of "a" is smaller than the deletion number of 2. Now consider the evaluation steps where the input string consists entirely of "a"; in order, we have "$aaaa$", "aa", and "a". This is the Collatz sequence for 4, which is 4, 2, 1. If we had instead started with the input string "$aaaaa$", we would get the Collatz sequence for 5, and so on.

Swift type substitution. To encode this tag system in Swift, we begin with a protocol declaration, together with the five concrete types conforming to `P` shown in Listing 12.3:

```
protocol P {
  associatedtype A: P
  associatedtype B: P
  associatedtype C: P
  associatedtype Del: P
  associatedtype Next
}
```

We only need one generic parameter type for the below, so let's write τ in place of the generic parameter type `τ_0_0`. We're going to define a series of substitution maps for the protocol generic signature $G_{\texttt{P}}$ (that is, `<τ where τ: P>`). The simplest is the identity substitution map, which we will denote in the below by the symbol 1:

$$1 := \{\tau \mapsto \tau;\ [\tau\texttt{: P}] \mapsto [\tau\texttt{: P}]\}$$

We will build up our tag system by considering how to perform each of these in turn:

1. Encoding the input string.
2. Deleting symbols from the start of the input string.
3. Recognition of the halting state, when the length of the input string is smaller than the deletion number.
4. Appending a production at the end of the input string.
5. The evaluation rule, which iterates all the above repeatedly until reaching the halting state.

For (1), we encode the input string as a composition of substitution maps defined in terms of the concrete types `AA`, `BB`, `CC`, and `End`:

$$\Sigma_{\texttt{AA}} := \{\tau \mapsto \texttt{AA<}\tau\texttt{>};\ [\tau\texttt{: P}] \mapsto [\texttt{AA<}\tau\texttt{>: P}]\}$$
$$\Sigma_{\texttt{BB}} := \{\tau \mapsto \texttt{BB<}\tau\texttt{>};\ [\tau\texttt{: P}] \mapsto [\texttt{BB<}\tau\texttt{>: P}]\}$$
$$\Sigma_{\texttt{CC}} := \{\tau \mapsto \texttt{CC<}\tau\texttt{>};\ [\tau\texttt{: P}] \mapsto [\texttt{CC<}\tau\texttt{>: P}]\}$$
$$\Sigma_{\texttt{End}} := \{\tau \mapsto \texttt{End};\ [\tau\texttt{: P}] \mapsto [\texttt{End: P}]\}$$

Listing 12.3.: Concrete types for the Collatz tag system

```
struct Halt: P {
  typealias A = Halt
  typealias B = Halt
  typealias C = Halt
  typealias Del = Halt
  typealias Next = Halt
}

struct End: P {
  typealias A = AA<End>
  typealias B = BB<End>
  typealias C = CC<End>
  typealias Del = Halt
  typealias Next = Halt
}

struct AA<T: P>: P {
  typealias A = AA<T.A>
  typealias B = AA<T.B>
  typealias C = AA<T.C>
  typealias Del = T
  typealias Next = Del.Del.B.C.Next
}

struct BB<T: P>: P {
  typealias A = BB<T.A>
  typealias B = BB<T.B>
  typealias C = BB<T.C>
  typealias Del = T
  typealias Next = Del.Del.A.Next
}

struct CC<T: P>: P {
  typealias A = CC<T.A>
  typealias B = CC<T.B>
  typealias C = CC<T.C>
  typealias Del = T
  typealias Next = Del.Del.A.A.A.Next
}
```

We represent the symbol "a" as $\Sigma_{\texttt{AA}}$, "b" as $\Sigma_{\texttt{BB}}$ and "c" as $\Sigma_{\texttt{CC}}$. We also compose with $\Sigma_{\texttt{End}}$ on the right. For example, the empty string "" maps to $\Sigma_{\texttt{End}}$, the string "b" maps to $\Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{End}}$, and the string "$abc$" maps to $\Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}}$.

Applying these substitution maps to the generic parameter type τ produces various concrete types built up from `End`, `AA<>`, `BB<>`, and `CC<>`:

$$\tau \otimes \Sigma_{\texttt{End}} = \texttt{End}$$
$$\tau \otimes \Sigma_{\texttt{BB}} = \texttt{BB<End>}$$
$$\tau \otimes \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}} = \texttt{AA<BB<CC<End>>>}$$

For (2), we use the `Del` associated type of `P` to delete symbols from the beginning of the input string. Specifically, we let:

$$\Sigma_{\texttt{Del}} := \{\tau \mapsto \tau\texttt{.[P]Del}; [\tau\texttt{: P}] \mapsto [\tau\texttt{.[P]Del: P}]\}.$$

Now, consider the composition $\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{AA}}$. From the declaration of `AA`, we see that the type witness $\pi(\texttt{[P]Del})$ and associated conformance $\pi(\texttt{Self.[P]Del: P})$ of $[\texttt{AA<}\tau\texttt{>: P}]$ are the following:

$$\pi(\texttt{[P]Del}) \otimes [\texttt{AA<}\tau\texttt{>: P}] = \tau$$
$$\pi(\texttt{Self.[P]Del: P}) \otimes [\texttt{AA<}\tau\texttt{>: P}] = [\tau\texttt{: P}]$$

Now, we apply $\Sigma_{\texttt{AA}}$ to the constituents of $\Sigma_{\texttt{Del}}$, making use of the above:

$$\begin{aligned}\tau\texttt{.[P]Del} &\otimes \Sigma_{\texttt{AA}} \\ &= \pi(\texttt{[P]Del}) \otimes [\tau\texttt{: P}] \otimes \Sigma_{\texttt{AA}} \\ &= \pi(\texttt{[P]Del}) \otimes [\texttt{AA<}\tau\texttt{>: P}] \\ &= \tau\end{aligned}$$

$$\begin{aligned}[\tau\texttt{.[P]Del: P}] &\otimes \Sigma_{\texttt{AA}} \\ &= \pi(\texttt{Self.[P]Del: P}) \otimes [\tau\texttt{: P}] \otimes \Sigma_{\texttt{AA}} \\ &= \pi(\texttt{Self.[P]Del: P}) \otimes [\texttt{AA<}\tau\texttt{>: P}] \\ &= [\tau\texttt{: P}]\end{aligned}$$

Notice that τ and $[\tau\texttt{: P}]$ are the constituents of the identity substitution map of $G_{\texttt{P}}$, which we said we denote by 1; so we've shown that $\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{AA}} = 1$. The computation of $\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{BB}}$ and $\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{CC}}$ proceeds in an entirely analogous fashion. Hence, we have the following relationships:

$$\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{AA}} = 1$$
$$\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{BB}} = 1$$
$$\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{CC}} = 1$$

For example, deleting the first symbol from "*abc*" gives us "*bc*", or, in type substitution,

$$\begin{aligned}
&\Sigma_{\texttt{Del}} \otimes (\Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}}) \\
&(\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{AA}}) \otimes \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}} \\
&\qquad = (1 \otimes \Sigma_{\texttt{BB}}) \otimes \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}} \\
&\qquad = \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}}.
\end{aligned}$$

For (3), the halting state is expressed with a substitution map having the `Halt` concrete type as the replacement type:

$$\Sigma_{\texttt{Halt}} := \{\tau \mapsto \texttt{Halt};\ [\tau\texttt{: P}] \mapsto [\texttt{Halt: P}]\}$$

From the declaration of `End`, we see that applying $\pi(\texttt{[P]Del})$ and $\pi(\texttt{Self.[P]Del: P})$ to the conformance $[\texttt{End: P}]$ gives us the halting state:

$$\begin{aligned}
&\pi(\texttt{[P]Del}) \otimes [\texttt{End: P}] = \texttt{Halt} \\
&\pi(\texttt{Self.[P]Del: P}) \otimes [\texttt{End: P}] = [\texttt{Halt: P}]
\end{aligned}$$

Thus, we've shown that deleting a symbol from the start of the empty string gives us the halting state:

$$\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{End}} = \Sigma_{\texttt{Halt}}$$

It can also be seen that:

$$\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{Halt}} = \Sigma_{\texttt{Halt}}$$

For (4), appending symbols at the end of the string is accomplished by the `A`, `B` and `C` associated types of `P`. We define three more substitution maps:

$$\begin{aligned}
&\Sigma_{\texttt{A}} := \{\tau \mapsto \tau\texttt{.[P]A};\ [\tau\texttt{: P}] \mapsto [\tau\texttt{.[P]A: P}]\} \\
&\Sigma_{\texttt{B}} := \{\tau \mapsto \tau\texttt{.[P]B};\ [\tau\texttt{: P}] \mapsto [\tau\texttt{.[P]B: P}]\} \\
&\Sigma_{\texttt{C}} := \{\tau \mapsto \tau\texttt{.[P]C};\ [\tau\texttt{: P}] \mapsto [\tau\texttt{.[P]C: P}]\}
\end{aligned}$$

Consider the composition $\Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{End}}$; we're adding the symbol "*a*" to the empty string "". We apply $\Sigma_{\texttt{End}}$ to the constituents of $\Sigma_{\texttt{A}}$:

$$\begin{aligned}
&\tau\texttt{.[P]A} \otimes \Sigma_{\texttt{End}} \\
&\qquad = \pi(\texttt{[P]A}) \otimes [\tau\texttt{: P}] \otimes \Sigma_{\texttt{End}} \\
&\qquad = \pi(\texttt{[P]A}) \otimes [\texttt{End: P}] \\
&\qquad = \texttt{AA<}\tau\texttt{>} \otimes \Sigma_{\texttt{End}}
\end{aligned}$$

$$\begin{aligned}
&[\tau\texttt{.[P]A: P}] \otimes \Sigma_{\texttt{End}} \\
&\qquad = \pi(\texttt{Self.[P]A: P}) \otimes [\tau\texttt{: P}] \otimes \Sigma_{\texttt{End}} \\
&\qquad = \pi(\texttt{Self.[P]A: P}) \otimes [\texttt{End: P}] \\
&\qquad = [\texttt{AA<}\tau\texttt{>: P}] \otimes \Sigma_{\texttt{End}}
\end{aligned}$$

Again, the calculation of $\Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{End}}$ and $\Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{End}}$ is similar, and we get:

$$\Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{End}} = \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{End}}$$
$$\Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{End}} = \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{End}}$$
$$\Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{End}} = \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}}$$

Adding symbols at the end of a non-empty string is more interesting, because applying the substitution maps $\Sigma_{\texttt{A}}$, $\Sigma_{\texttt{B}}$ and $\Sigma_{\texttt{C}}$ on the *left* has the effect of applying $\Sigma_{\texttt{AA}}$, $\Sigma_{\texttt{BB}}$, $\Sigma_{\texttt{CC}}$ on the right. For example, we can compute $\Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{AA}}$, by applying $\Sigma_{\texttt{AA}}$ to the constituents of $\Sigma_{\texttt{B}}$:

$$\begin{aligned}
&\tau\texttt{.[P]B} \otimes \Sigma_{\texttt{AA}} \\
&\quad = \pi(\texttt{[P]B}) \otimes [\tau\texttt{: P}] \otimes \Sigma_{\texttt{AA}} \\
&\quad = \pi(\texttt{[P]B}) \otimes [\texttt{AA<}\tau\texttt{>: P}] \\
&\quad = \texttt{AA<}\tau\texttt{>} \otimes \Sigma_{\texttt{B}}
\end{aligned}$$

$$\begin{aligned}
&[\tau\texttt{.[P]B: P}] \otimes \Sigma_{\texttt{AA}} \\
&\quad = \pi(\texttt{Self.[P]B: P}) \otimes [\tau\texttt{: P}] \otimes \Sigma_{\texttt{AA}} \\
&\quad = \pi(\texttt{Self.[P]B: P}) \otimes [\texttt{AA<}\tau\texttt{>: P}] \\
&\quad = [\texttt{AA<}\tau\texttt{>: P}] \otimes \Sigma_{\texttt{B}}
\end{aligned}$$

Thus, we have $\Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{AA}} = \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{B}}$; the two substitution maps commute. The same argument shows that:

$$\begin{array}{lll}
\Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{AA}} = \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{A}}, & \Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{AA}} = \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{B}}, & \Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{AA}} = \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{C}}, \\
\Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{BB}} = \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{A}}, & \Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{BB}} = \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{B}}, & \Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{BB}} = \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{C}}, \\
\Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{CC}} = \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{A}}, & \Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{CC}} = \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{B}}, & \Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{CC}} = \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{C}}.
\end{array}$$

For example, adding "*c*" at the end of "*ab*" gives us "*abc*"; or, in type substitution:

$$\begin{aligned}
&\Sigma_{\texttt{C}} \otimes (\Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{End}}) \\
&\quad = \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{End}} \\
&\quad = \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{End}} \\
&\quad = \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}}
\end{aligned}$$

Observe that $\Sigma_{\texttt{C}}$ commutes with $\Sigma_{\texttt{AA}}$, $\Sigma_{\texttt{BB}}$ and $\Sigma_{\texttt{CC}}$ until it abuts $\Sigma_{\texttt{End}}$ on the right, before becoming $\Sigma_{\texttt{CC}}$, which represents the symbol "*c*". Here we see the difference between `End` and `Halt`; we can add new symbols to `End`, representing the empty string, but once we reach the halting state, we can't add any more symbols to our string. The type witneses for `A`, `B` and `C` in the conformance [`Halt: P`] are all `Halt` again, for this reason:

$$\Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{Halt}} = \Sigma_{\texttt{Halt}}$$
$$\Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{Halt}} = \Sigma_{\texttt{Halt}}$$
$$\Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{Halt}} = \Sigma_{\texttt{Halt}}$$

Finally, we get to (5), the evaluation rule itself. The evaluation rule depends on the first symbol in the string. The first symbol is the outermost generic type—for example, "abc" is represented as the type `AA<BB<CC<End>>>`—so we encode the evaluation rule with the `Next` associated type in each conformance to `P`. Notice how each one begins by deleting the first two symbols from the beginning of the string, then appends the production, and finally, repeats the next evaluation step by projecting `Next` again:

$$\pi(\texttt{[P]Next}) \otimes [\texttt{AA<}\tau\texttt{>: P}] = \tau\texttt{.Del.Del.B.C.Next}$$
$$\pi(\texttt{[P]Next}) \otimes [\texttt{BB<}\tau\texttt{>: P}] = \tau\texttt{.Del.Del.A.Next}$$
$$\pi(\texttt{[P]Next}) \otimes [\texttt{CC<}\tau\texttt{>: P}] = \tau\texttt{.Del.Del.A.A.A.Next}$$

In contrast, evaluation stops if we start with an empty string, or if we're already in the halting state:

$$\pi(\texttt{[P]Next}) \otimes [\texttt{End: P}] = \texttt{Halt}$$
$$\pi(\texttt{[P]Next}) \otimes [\texttt{Halt: P}] = \texttt{Halt}$$

It is more convenient for what follows to express the above using substitution map composition instead of type witness projection:

$$\tau\texttt{.[P]Next} \otimes \Sigma_{\texttt{AA}} = \tau\texttt{.[P]Next} \otimes \Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{Del}}$$
$$\tau\texttt{.[P]Next} \otimes \Sigma_{\texttt{BB}} = \tau\texttt{.[P]Next} \otimes \Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{Del}}$$
$$\tau\texttt{.[P]Next} \otimes \Sigma_{\texttt{CC}} = \tau\texttt{.[P]Next} \otimes \Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{Del}}$$

$$\tau\texttt{.[P]Next} \otimes \Sigma_{\texttt{End}} = \texttt{Halt}$$
$$\tau\texttt{.[P]Next} \otimes \Sigma_{\texttt{Halt}} = \texttt{Halt}$$

An immediate consequence is that there can be only three possible outcomes when we apply a substitution map composition representing some input string, to the type parameter τ`.[P]Next`:

1. Evaluation eventually reaches the halting state, where the substituted type is `Halt`.
2. Evaluation gets stuck in a loop, and never reaches the halting state.
3. Evaluation grows the input string indefinitely, and never reaches the halting state.

So the substituted type is always known to be `Halt`, as long as substitution halts—but the substitution halts if and only if the Collatz conjecture is true. Of course, termination checking for type substitution doesn't hinge on resolving the Collatz conjecture, because we can also encode a tag system for a Turing machine. We enlarge the alphabet by adding associated types to `P` and declaring more generic nominal types conforming to `P`, and we encode the deletion number and production rules as type witnesses for π(`[P]Next`).

Finally, an example. Type checking the following code verifies the Collatz conjecture for $n = 2$. The substitution map for the call to `collatz()` is $\Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{End}}$, which corresponds to the input string "*aa*":

```
func collatz<T: P>(_: T.Type) -> T.Next.Type {
  return T.Next.self
}

collatz(AA<AA<End>>.self)
```

The substituted return type can be computed with our identities. The parentheses serve to draw attention to each simplification step, but other evaluation orders are possible, and all are equivalent:

$$
\begin{aligned}
&(\tau.\texttt{[P]Next} \otimes \Sigma_{\texttt{AA}}) \otimes \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{End}} \\
&\quad = \tau.\texttt{[P]Next} \otimes \Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{B}} \otimes (\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{AA}} \otimes \Sigma_{\texttt{AA}}) \otimes \Sigma_{\texttt{End}} \\
&\quad = \tau.\texttt{[P]Next} \otimes (\Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{End}}) \\
&\quad = \tau.\texttt{[P]Next} \otimes (\Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}}) \\
&\quad = (\tau.\texttt{[P]Next} \otimes \Sigma_{\texttt{BB}}) \otimes \Sigma_{\texttt{CC}} \otimes \Sigma_{\texttt{End}} \\
&\quad = \tau.\texttt{[P]Next} \otimes \Sigma_{\texttt{A}} \otimes (\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{BB}} \otimes \Sigma_{\texttt{CC}}) \otimes \Sigma_{\texttt{End}} \\
&\quad = \tau.\texttt{[P]Next} \otimes (\Sigma_{\texttt{A}} \otimes \Sigma_{\texttt{End}}) \\
&\quad = (\tau.\texttt{[P]Next} \otimes \Sigma_{\texttt{AA}}) \otimes \Sigma_{\texttt{End}} \\
&\quad = \tau.\texttt{[P]Next} \otimes \Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{Del}} \otimes (\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{End}}) \\
&\quad = \tau.\texttt{[P]Next} \otimes \Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{B}} \otimes (\Sigma_{\texttt{Del}} \otimes \Sigma_{\texttt{Halt}}) \\
&\quad = \tau.\texttt{[P]Next} \otimes \Sigma_{\texttt{C}} \otimes (\Sigma_{\texttt{B}} \otimes \Sigma_{\texttt{Halt}}) \\
&\quad = \tau.\texttt{[P]Next} \otimes (\Sigma_{\texttt{C}} \otimes \Sigma_{\texttt{Halt}}) \\
&\quad = \tau.\texttt{[P]Next} \otimes \Sigma_{\texttt{Halt}} \\
&\quad = \texttt{Halt}
\end{aligned}
$$

We reach the halting state, because the Collatz sequence for $n = 2$ reaches 1.

Further discussion. The discussion of generic function types in Section 3.3 alluded to the undecidability of type inference in the System F formalism. In Section 10.4, we looked at an example of a non-terminating conditional conformance check in Swift, and cited a similar example in Rust. Later, Section 17.4 will show that Swift reduced type equality is also undecidable in the general case, but we will provide a termination guarantee by restricting the problem. Countless other instances of undecidable type checking problems are described in the literature, exploiting clever and varied tricks

to encode arbitrary computation in terms of types. We will cite just two more: Java generics were shown to be Turing-complete in [99], and TypeScript in [100]; the latter example also encodes the Collatz sequence, but in a completely different manner than we did in this section. Finally, the Collatz conjecture, which fundamentally has no relation to type systems or programming languages, is discussed in [101] and [102].

12.5. Source Code Reference

Key source files:

- `include/swift/AST/GenericSignature.h`
- `include/swift/AST/SubstitutionMap.h`
- `lib/AST/GenericSignature.cpp`
- `lib/AST/SubstitutionMap.cpp`
- `lib/AST/TypeSubstitution.cpp`

`ConformancePath::Entry`	*type alias*

An entry in a conformance path; `std::pair<CanType, ProtocolDecl *>`.

`ConformancePath`	*class*

Represents a conformance path as an array of one or more entries. The first entry corresponds to a conformance requirement in the generic signature; each subsequent entry is an associated conformance requirement.

`GenericSignatureImpl`	*class*

The `getConformancePath()` method returns the conformance path for a type parameter and protocol declaration. For other methods, see Section 5.6.

`SubstitutionMap`	*class*

The `lookupConformance()` method implements Algorithm 12A for performing a local conformance lookup. For other methods, see Section 6.5.

`getMemberForBaseType()`	*function*

A static helper function in `Type.cpp`, used by `Type::subst()`. Implements Algorithm 12B.

Finding Conformance Paths

Key source files:

- `lib/AST/RequirementMachine/GenericSignatureQueries.cpp`

The `getConformancePath()` method on `GenericSignature` calls the method with the same name in `RequirementMachine`. The latter implements Algorithm 12E. A pair of instance variables model the algorithm's persistent state:

- `ConformancePaths` is the table of known conformance paths.
- `CurrentConformancePaths` is the buffer of conformance paths at the currently enumerated length.

Algorithm 12E traffics in reduced type parameters, while the actual implementation deals with instances of `Term`. A term is the internal Requirement Machine representation of a type parameter, as you will see in Chapter 18. This avoids round-trip conversions between `Term` and `Type` when computing reduced types that are only used for comparing against other reduced types. A `Term` can function as a hash table key and is otherwise equivalent to a `Type` here.

Part IV.

Supplements

13. Opaque Return Types

14. Existential Types

15. Class Inheritance

Part V.

The Requirement Machine

16. Basic Operation

The final part of this book is devoted to the implementation of generic signature queries (Section 5.5) and requirement minimization (Section 11.4). More precisely, we're looking at the *decision procedure* for the derived requirements formalism. We achieve this by translating the requirements of each generic signature and protocol into *rewrite rules*, and then analyzing the relationships between these rewrite rules.

We use the proper noun "the Requirement Machine" to mean the compiler component responsible for this reasoning, while *a* requirement machine—a common noun—is an *instance* of a data structure storing a list of rewrite rules that describe a generic signature or protocol.

In this chapter, we give a high-level overview of how requirement machines operate, without saying anything about what goes on inside. Understanding their inner workings will take two more chapters. Chapter 17 introduces the theory of finitely-presented monoids and string rewriting, and Chapter 18 details the translation of requirements into rewrite rules.

Let's go. There are two entry points into the Requirement Machine:

- To answer a **generic signature query**, we construct a requirement machine from the requirements of the given generic signature; we call this a *query machine*. We then consult the machine's *property map*: a description of all conformance, superclass, layout and concrete type requirements imposed on each type parameter.
- To **build a new generic signature**, we construct a requirement machine from a list of user-written requirements; we call this a *minimization machine*. We compute the minimal requirements as a side-effect of the construction process.

We have two more kinds of requirement machine for reasoning about protocols:

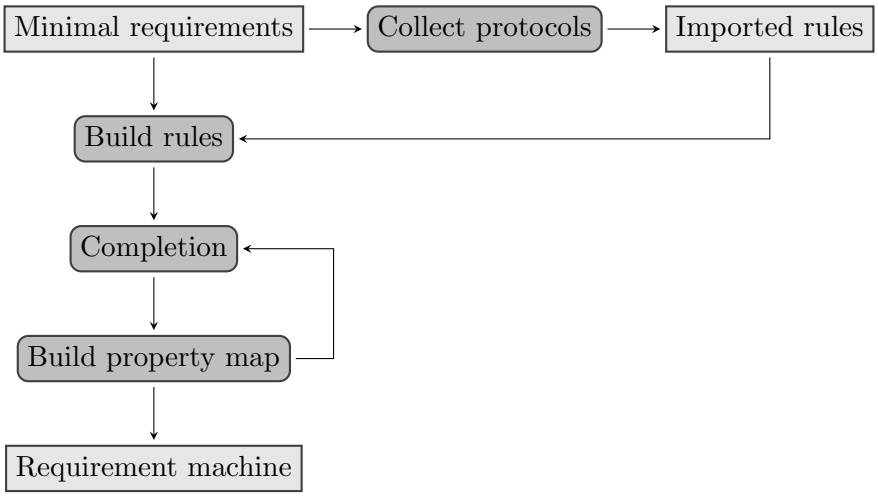
- To answer questions about **a protocol's requirement signature**, we construct a requirement machine from the protocol's requirement signature; we call this a *protocol machine*.
- To **build a new requirement signature** for a protocol written in source, we construct a requirement machine from the list of user-written requirements; we call this a *protocol minimization machine*. We compute the minimal associated requirements as a side-effect of the construction process.

An important fact about requirement machines is that the list of rewrite rules is divided into *local rules*, corresponding to the initial requirements the machine was built from, and *imported rules*, which as we see shortly, describe associated requirements of protocols referenced by the local rules. We now look at each of the four requirement machine kinds in detail.

Query machines. We maintain a table of *query machine* instances inside a singleton object, called the *rewrite context*. We can use *canonical* generic signatures as keys, because the translation of requirements into rewrite rules ignores type sugar. We build a query machine from a generic signature like so:

1. We translate the generic signature's explicit requirements into rewrite rules using Algorithm 18J of Section 18.4 to get the list of local rules for our query machine.
2. We lazily construct a *protocol machine* for each protocol appearing on the right-hand side of a conformance requirement in our signature.
3. We use Algorithm 16B to collect local rules from each of these protocol machines as well as all protocol machines they transitively reference, to get the list of imported rules for our query machine.
4. We run the *completion procedure* (Chapter 19) to add new local rules which are "consequences" of other rules. This gives us a *convergent rewriting system*.
5. We build the property map data structure (Chapter 20).
6. If property map construction added new local rules, we return to Step 3 and repeat the process until no more local rules are added.

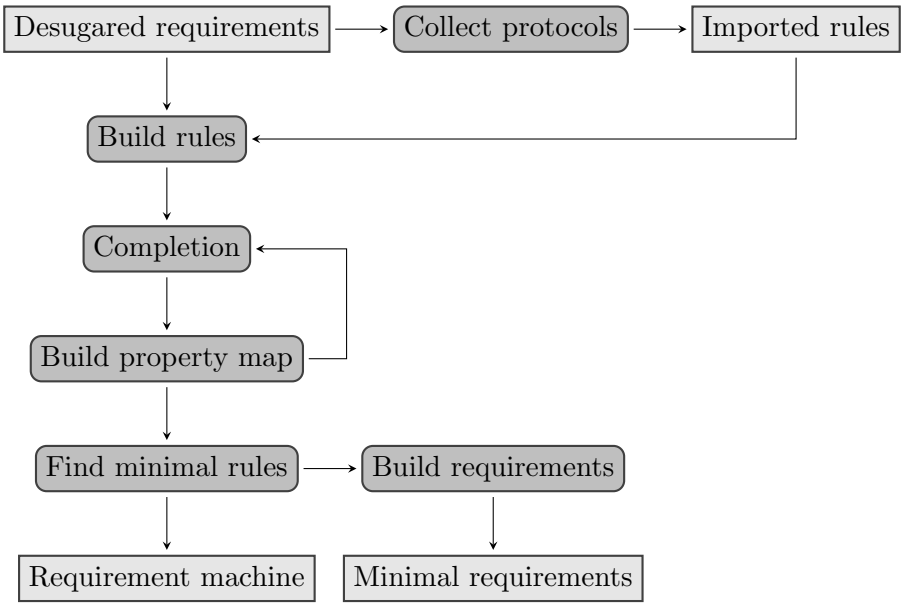
Once built, a query machine remains live for the duration of the compilation session.



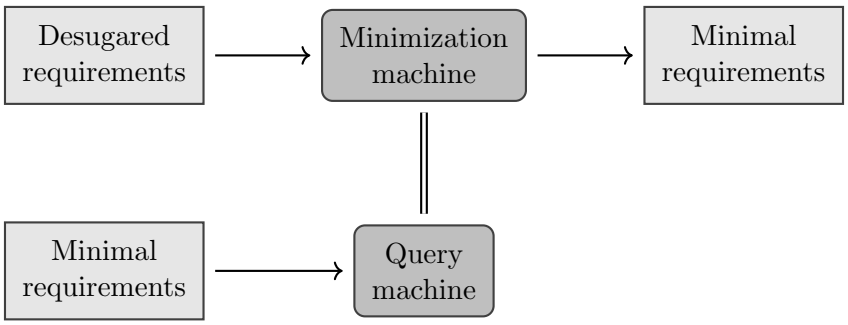
Minimization machines. To compute the list of minimal requirements for a new generic signature, we build a *minimization machine*. This is the final step in Figure 11.1 and Figure 11.2 of Chapter 11. The flow is similar to building a query machine, except now the inputs are user-written requirements after desugaring (Section 11.2). The other main differences are:

1. Completion must do additional work if the input requirements contain unbound type parameters, as they do when they come from the structural resolution stage.
2. Completion also records *rewrite loops*, which describe relations between rewrite rules—in particular, this describes which rules are redundant because they are a consequence of existing rules.
3. When constructing the property map, we note any conflicting requirements, to be diagnosed if this generic signature has a source location.
4. After completion and property map construction, we process the rewrite loops to find a minimal subset of local rules which imply the rest. This is the topic of Chapter 22.
5. Finally, we translate the minimal rules into requirements to get the list of minimal requirements that we output. This is explained in Section 22.5.

Minimization machines have a temporary lifetime that is scoped to the **inferred generic signature request** or **abstract generic signature request**.



An optimization. When building a new generic signature, we get the list of minimal requirements as a *side effect* of constructing the minimization machine. While we could just discard the minimization machine once we're done, we first check the rewrite context for the presence of a query machine for the same generic signature. If we have not built one yet, we *install* the new minimization machine, transferring ownership of it to the rewrite context. This saves effort in the case where we build a new generic signature for a declaration written in the main module and then proceed to issue queries against the new generic signature while type checking the declaration's body.



For this optimization to work, the requirements as the user wrote them must generate the same theory, in the sense of Proposition 11.9, as the minimal requirements *output* by the minimization machine. This equivalence almost always holds true, except for four situations we describe below. The first three only occur with invalid code, and are accompanied by diagnostics. The fourth is not an error, just a rare edge case where the optimization cannot be performed:

1. Minimization always outputs a list of well-formed requirements, meaning they only refer to valid type parameters (Section 11.3). If any of the user-written requirements are not well-formed, we drop them, so we output a list of minimal requirements that generates a different theory.

2. When two requirements are in conflict such that both cannot be simultaneously satisfied (Definition 11.12), we sometimes drop one or the other. A new query machine will, again, not include the conflicting requirements.

3. As we will see in Section 17.4, completion may fail if the user-written requirements are too complex to reason about. In this case the minimization machine outputs an empty list of minimal requirements.

4. If a conformance requirement is made redundant by a same-type requirement that fixes a type parameter to a concrete type (such as [`T: P`] and [`T == X`] where `X` is a concrete type conforming to `P`), we drop the conformance requirement, but this changes the theory; we will talk about this in Chapter 21.

If any of these conditions hold, we record this fact while building the minimization machine. This prevents the minimization machine from being installed into the rewrite context, forcing us to discard it instead. The `-disable-requirement-machine-reuse` frontend flag is intended for debugging. It disables this optimization entirely, forcing us to discard all minimization machines immediately without attempting to install them.

Protocol machines. A *protocol machine* collects the associated requirements of a *protocol component*, or a set of mutually-dependent protocol declarations. For now, we proceed as if each component consists of a single protocol, which is the typical scenario; the next section describes the general case.

Protocol machines have a global lifetime and are owned by the rewrite context. The procedure for building a protocol machine is similar to building a query machine:

1. We translate the associated requirements of each protocol into rewrite rules, using Algorithm 18K of Section 18.4. These become the protocol machine's local rules.
2. We lazily construct a protocol machine for each protocol that appears on the right-hand side of an associated conformance requirement.
3. We use Algorithm 16B to collect local rules from each of these protocol machines as well as all protocol machines they transitively reference, to get our imported rules. Thus, protocol machines recursively import rules from other protocol machines.
4. Completion and property map construction proceed as in the query machine case.

Usually the user's program will declare an assortment of protocols, with various generic types and functions then stating conformance requirements to those protocols, so that each protocol is mentioned in more than one generic signature. Therefore, requirement machines for different generic signatures often have many rewrite rules in common, because they depend on the same protocols.

The role of protocol machines is to serve as containers for these shared rules. This eliminates the repeated work that would result from repeatedly translating the associated requirements of a protocol, and processing them with the completion procedure, in every requirement machine that contains a conformance requirement to this protocol. Instead, we look up the protocol machine and *import* its rewrite rules when building a requirement machine that depends on a protocol.

Protocol minimization machines. To actually build the requirement signature of a protocol written in source, we construct a *protocol minimization machine*. A protocol minimization machine has a temporary lifetime, scoped to the **requirement signature request**. The minimal requirements of the requirement signature are computed as a side-effect of constructing the protocol minimization machine.

In the absence of error conditions, we install the protocol minimization machine in the rewrite context, so that it becomes a long-lived protocol machine for the same protocol component. In fact, we only ever directly construct a protocol machine for a protocol written in source if we failed to install the protocol minimization machine. Otherwise, protocol machines are usually only built when the user's program references a protocol from a serialized module, such as the standard library.

Circularity. There is a potential source of circularity in the construction of a query machine. If we encounter a type parameter subject to both a conformance and a concrete same-type requirement, say [`T: P`] and [`T == X`] where `X` is a concrete type conforming to `P`, we may look up one or more type witnesses in this conformance. This might perform type substitution, and recursively invoke a generic signature query. If we issue a query against the same generic signature whose query machine we are constructing now, we report a fatal error and give up.

Summary. The four machine kinds correspond to the four combinations of "domain" (generic signature or requirement signature) and "purpose" (querying an existing entity, or building a new entity of that kind):

Query machine	Minimization machine
Protocol machine	**Protocol minimization machine**

Notice that a protocol machine is the protocol version of a query machine, and a protocol minimization machine is the protocol version of a minimization machine. Machines in the first column are owned by the rewrite context; those in the second have a temporary lifetime, but can be installed in the rewrite context so they become an instance of the first.

Example 16.1. We saw these three declarations at the start of Chapter 5:

```
func sameElt<S1: Sequence, S2: Sequence>(_ s1: S1, _ s2: S2)
    where S1.Element == S2.Element {...}

func sameIter<S1: Sequence, S2: Sequence>(_ s1: S1, _ s2: S2)
    where S1.Iterator == S2.Iterator {...}

func sameEltAndIter<S1: Sequence, S2: Sequence>(_ s1: S1, _ s2: S2)
    where S1.Element == S2.Element,
          S1.Iterator == S2.Iterator {...}
```

We remarked that `sameElt()` and `sameEltAndIter()` have the same generic signature, while `sameIter()` is distinct from the other two. Suppose we type check the declarations in order as written above:

1. We construct a minimization machine from the requirements of `sameElt()`. We have two conformance requirements to `Sequence`, so we must first construct the protocol machine for `Sequence`. Since `Sequence` states an associated conformance requirement to `IteratorProtocol`, we also construct the protocol machine for `IteratorProtocol`.

2. Once we get the generic signature of `sameElt()`, we associate the generic signature with this minimization machine in the rewrite context. While type checking the body of `sameElt()`, we will issue queries against this generic signature, reusing the minimization machine we already built.

3. Next, we construct a minimization machine from the requirements of `sameIter()`. We import rules from the protocol machines for `Sequence` and `IteratorProtocol`, which we already built, saving some work. We get our new generic signature and install it in the rewrite context.

4. Finally, we construct a minimization machine to build the generic signature of `sameEltAndIter()`. We get the same generic signature as `sameIter()`, and we already have a query machine for it in the rewrite context. We discard the new minimization machine.

If we pass the `-debug-requirement-machine=timers` frontend flag, we can watch this happen in real time:

```
$ swiftc three.swift -Xfrontend -debug-requirement-machine=timers
+ started InferredGenericSignatureRequest @ three.swift:1:6
| + started getRequirementMachine() [ Sequence ]
| | + started getRequirementMachine() [ IteratorProtocol ]
| | + finished getRequirementMachine() in 21us: [ IteratorProtocol ]
| + finished getRequirementMachine() in 61us: [ Sequence ]
+ finished InferredGenericSignatureRequest in 204us:
<S1, S2 where S1 : Sequence, S2 : Sequence, S1.Element == S2.Element>
+ started InferredGenericSignatureRequest @ three.swift:4:6
+ finished InferredGenericSignatureRequest in 88us:
<S1, S2 where S1 : Sequence, S2 : Sequence, S1.Iterator == S2.Iterator>
+ started InferredGenericSignatureRequest @ three.swift:7:6
+ finished InferredGenericSignatureRequest in 78us:
<S1, S2 where S1 : Sequence, S2 : Sequence, S1.Iterator == S2.Iterator>
```

We will see more Requirement Machine debugging flags in Section 16.2.

Example 16.2. If `P` is any protocol, a query machine for the protocol generic signature `<Self where Self: P>` is particularly easy to build; we take the protocol machine for `P` and all protocols `P` depends on, collect all of their local rules, and then add a single local rule for [`Self: P`].

History. To continue the historical sketch of Section 8.4: the Requirement Machine shipped in Swift 5.6, where it was used for generic signature queries only. Requirement minimization was re-implemented in Swift 5.7, with the old logic available behind a flag. The `GenericSignatureBuilder` was then fully removed in Swift 5.8.

16.1. Protocol Components

We now take a closer look at how generic signatures and protocols depend on other protocols, and fully explain how protocol machines work. We begin by revisiting the protocol dependency graph, from Definition 12.5 of Section 12.3. Recall that the vertices are protocol declarations, and the edges are associated conformance requirements. That is, if $[\texttt{Self.U: P}]_{\texttt{Q}}$ is some associated conformance requirement, we define:

$$\mathrm{src}([\texttt{Self.U: Q}]_{\texttt{P}}) := \texttt{P},$$
$$\mathrm{dst}([\texttt{Self.U: Q}]_{\texttt{P}}) := \texttt{Q}.$$

There is an alternative definition of this concept using derived requirements.

Definition 16.3. A protocol `P` *depends on* a protocol `Q` if we can derive a conformance to `Q` from the protocol generic signature $G_{\texttt{P}}$; that is, if $G_{\texttt{P}} \vdash [\texttt{Self.U: Q}]$ for some type parameter `Self.U`. We write $\texttt{P} \prec \texttt{Q}$ if this relationship holds. The set of *protocol dependencies* of `P` is the set of all protocols `Q` such that $\texttt{P} \prec \texttt{Q}$.

Proposition 16.4. The $\prec$ relation is reflexive and transitive.

Proof. For the first part, let `P` be any protocol. We can always derive $G_{\texttt{P}} \vdash [\texttt{Self: P}]$ via the explicit requirement $[\texttt{Self: P}]$ of $G_{\texttt{P}}$, therefore $\texttt{P} \prec \texttt{P}$, so $\prec$ is reflexive.

For the second part, let `P`, `Q` and `R` be protocols such that $\texttt{P} \prec \texttt{Q}$ and $\texttt{Q} \prec \texttt{R}$. By definition, $G_{\texttt{P}} \vdash [\texttt{Self.U: Q}]$ and $G_{\texttt{Q}} \vdash [\texttt{Self.V: R}]$ for some type parameters `Self.U` and `Self.V`. By Lemma 11.7, $G_{\texttt{P}} \vdash [\texttt{Self.U.V: R}]$, where `Self.U.V` is the type parameter formed by substituting `Self` with `Self.U` in `Self.V`. Therefore, $\texttt{P} \prec \texttt{R}$. □

In fact, $\prec$ is exactly the reachability relation in the protocol dependency graph.

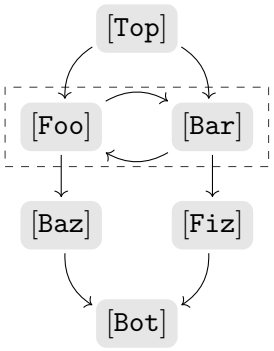
Proposition 16.5. Let `P` and `Q` be protocols. Then, $\texttt{P} \prec \texttt{Q}$ if and only if `Q` is reachable from `P` by a path in the protocol dependency graph.

Proof. A non-empty path in the protocol dependency graph is determined by the edges visited, that is, by a sequence of associated conformance requirements. This sequence of requirements has the property that each consecutive associated requirement has to be declared in the protocol named by the previous requirement. Also, when a derivation contains a sequence of consecutive AssocConf steps, the sequence of requirements referenced by each step obeys the same compatibility condition.

($\Leftarrow$) Suppose that `P` $\prec$ `Q`. By definition, there exists a type parameter `Self.U` such that $G_{\texttt{P}} \vdash$ [`Self.U: Q`]. By Theorem 12.2, we can find a type parameter `Self.U'` such that $G_{\texttt{P}} \vdash$ [`Self.U == Self.U'`], and the derivation of [`Self.U': Q`] takes a particularly simple form: it begins with a Conf step for the explicit requirement [`Self: P`], followed by a series of AssocConf derivation steps. By the previous remark, this sequence of steps give us a path from `P` to `Q` in the protocol dependency graph.

($\Rightarrow$) Suppose there is a path from `P` to `Q` in the protocol dependency graph. Consider the sequence of visited edges. We construct a derivation of a conformance requirement in $G_{\texttt{P}}$ by starting with a Conf step for the explicit requirement [`Self: P`], and adding a series of AssocConf steps for each edge visited in our path. By the previous remark, this is a valid derivation. We see that $G_{\texttt{P}} \vdash$ [`Self.U: Q`] for some type parameter `Self.U`, showing that `P` $\prec$ `Q`. □

Recursive conformances. In Section 12.3 we saw that recursive conformance requirements introduce cycles in the protocol dependency graph. Listing 16.1 shows some protocol declarations. The protocol dependency graph for this example, drawn on the right, has a cycle.



Each one of `Foo` and `Bar` points at the other via a pair of mutually-recursive associated conformance requirements. Based on our description so far, we cannot build one protocol machine without first building the other: a circular dependency.

We solve this by grouping protocols into *protocol components*, so in our example, `Foo` and `Bar` belong to the same component. A protocol machine describes an entire protocol component, and the local rules of a protocol machine include the associated requirements of all protocols in the component. If we consider dependencies between protocol *components* as opposed to *protocols*, we get a directed *acyclic* graph. To make this all precise, we step back to consider directed graphs in the abstract.

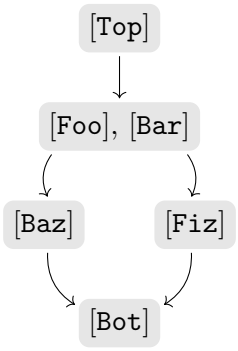
Strongly connected components. Suppose (V, E) is any directed graph, and $\prec$ is the reachability relation, so $x \prec y$ if there is a path with source x and destination y. We say that x and y are *strongly connected* if both $x \prec y$ and $y \prec x$ are true; we denote this relation by $x \equiv y$ in the following. This is an equivalence relation:

- (Reflexivity) If $x \in V$, then $x \prec x$ via the empty path at x, and thus $x \equiv x$.
- (Symmetry) If $x \equiv y$, then $y \equiv x$, by definition of $\equiv$.
- (Transitivity) If $x \equiv y$ and $y \equiv z$, then in particular $x \prec y$ and $y \prec z$, so $x \prec z$, because $\prec$ is transitive. By the same argument, we also have $y \prec x$ and $z \prec y$, hence $z \prec x$. Therefore, $x \equiv z$.

Listing 16.1.: Protocol component demonstration

```
protocol Top {
  associatedtype A: Foo
  associatedtype B: Bar
}

protocol Foo {
  associatedtype A: Bar
  associatedtype B: Baz
}

protocol Bar {
  associatedtype A: Foo
  associatedtype B: Fiz
}

protocol Baz {
  associatedtype A: Bot
}

protocol Fiz {
  associatedtype A: Bot
}

protocol Bot {}
```

The equivalence classes of $\equiv$ are called the *strongly connected components* of (V, E). We can define a graph where each *vertex* is a strongly connected component of the original graph; two components are joined by an edge if and only if some vertex in the first component is joined by a path with another vertex in the second component in the original graph. (This is well-defined, because if $x_1 \equiv x_2$ and $y_1 \equiv y_2$, then $x_1 \prec y_1$ if and only if $x_2 \prec y_2$.) The graph of strongly connected components is a *directed acyclic graph* (or DAG); that is, it does not have any non-empty paths with the same source and destination. Furthermore, if the original graph was already acyclic, then each vertex will be in a strongly connected component by itself.



Let's define the *protocol component graph* to be the graph of strongly connected components of the protocol dependency graph. The protocol component graph of Listing 16.1 is shown on the left. This looks almost the same as the original protocol dependency graph, except that we've collapsed `Foo` and `Bar` down to a single vertex. While the protocol component graph is always acyclic, it is not a tree or a forest in the general case; as we see in our example, we have two distinct paths with source `Top` and destination `Bot`, so components can share "children." To ensure we do not import duplicate rules, we must visit every downstream protocol machine once, and take only the *local* rules from each.

A debugging flag prints out each strongly connected component as it is formed. Let's try with our example:

```
$ swiftc protocols.swift -Xfrontend -debug-requirement-machine=protocol-dependencies
Connected component: [Bot]
Connected component: [Fiz]
Connected component: [Baz]
Connected component: [Bar, Foo]
Connected component: [Top]
```

The `-debug-requirement-machine=timers` frontend flag also logs the construction of protocol components. The indentation follows the recursive construction of protocol machines:

```
+ started RequirementSignatureRequest [ Top ]
| + started RequirementSignatureRequest [ Bar Foo ]
| | + started RequirementSignatureRequest [ Fiz ]
| | | + started RequirementSignatureRequest [ Bot ]
| | | + finished RequirementSignatureRequest in 46us: [ Bot ]
| | + finished RequirementSignatureRequest in 81us: [ Fiz ]
| | + started RequirementSignatureRequest [ Baz ]
| | + finished RequirementSignatureRequest in 21us: [ Baz ]
| + finished RequirementSignatureRequest in 179us: [ Bar Foo ]
+ finished RequirementSignatureRequest in 211us: [ Top ]
```

Tarjan's algorithm. We use an algorithm invented by Robert Tarjan [103] to discover the strongly connected components of the protocol dependency graph. Specifically, we number the strongly connected components, assign a component ID to each vertex, and build a table mapping each component ID to the list of vertices it contains.

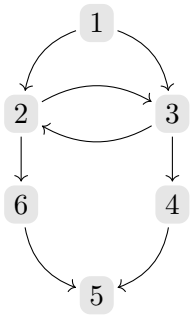
Tarjan's algorithm is incremental. We do not need to build the entire input graph upfront, which would force us to deserialize every protocol in every imported module first. When asked to produce the component ID for a given vertex, we check if the vertex has already been assigned a component ID, in which case we return it immediately. If one has not yet been assigned, the vertex must belong to a component distinct from all others discovered so far; we visit all vertices reachable from the original vertex and form connected components along the way.

Recall that if $v \in V$ is a vertex, then $w \in V$ is a *successor* of v if there exists an edge $e \in E$ with $\mathrm{src}(e) = v$ and $\mathrm{dst}(e) = w$. To find the strongly connected component of v, Tarjan's algorithm performs a *depth-first search*: given v, we recursively visit the first successor of v we haven't visited yet, and its successors, and so on, before moving on to the next successor of v. (Contrast this with the *breadth-first search* for finding a conformance path in Section 12.2, where we visit all successors of v before digging deeper into their successors.)

Tarjan's algorithm requires us to specify the input graph in the form of a callback mapping a given vertex to its list of successors. This allows the graph to be explored incrementally. In our case, the successors of a protocol are obtained by evaluating the **protocol dependencies request**, which is implemented as follows:

- If the protocol is declared inside the main module, we evaluate the **structural requirements request** and collect the protocols appearing on the right-hand side of the protocol's associated conformance requirements written in source.
- If the protocol is part of a serialized module, we evaluate the **requirement signature request** and collect the protocols appearing on the right-hand side of the protocol's associated requirements.

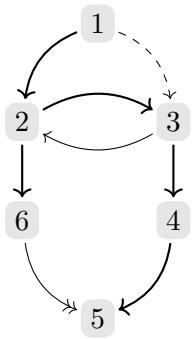
We number vertices in the order they are visited, by assigning the current value of a counter to each visited vertex v, prior to visiting the successors of v. We denote the value assigned to a vertex v by `NUMBER`(v). Our traversal can determine if a vertex has been previously visited by checking if `NUMBER`(v) is set.



We return to the protocol dependency graph from Listing 16.1. A depth-first search originating from `Top` produces the numbering of vertices shown on the right, where the first vertex is assigned the value 1. Here, the entire graph was ultimately reachable from 1; more generally, we get a numbering of the subgraph reachable from the initial vertex.

When we visit a vertex v, we look at each edge $e \in E$ with $\text{src}(e) = v$. Suppose that $\text{dst}(e)$ is some other vertex w; we consider the state of $\texttt{NUMBER}(w)$, and classify the edge e as a tree edge, ignored edge, frond, or cross-link.

If $\texttt{NUMBER}(w)$ has not yet been set, we say that e is a *tree edge*; the tree edges define a *spanning tree* of the subgraph explored by the search. Otherwise, we saw w earlier in our search, and the edge e has one of the three remaining kinds. If $\texttt{NUMBER}(w) \geq \texttt{NUMBER}(v)$ (with equality corresponding to an edge from v to itself, which we allow), any path from v to w must pass through a common ancestor of v and w in the spanning tree. We say that e is an *ignored edge*, because e cannot generate any new strongly connected components. The final two kinds arise when $\texttt{NUMBER}(w) < \texttt{NUMBER}(v)$. We say that e is a *frond* if w is also an ancestor of v in the spanning tree; otherwise, e is a *cross-link*. (While the distinction between the final two kinds plays a role in Tarjan's correctness proof, we'll see that algorithm handles fronds and cross-links uniformly.)



If the input graph is a forest, then every edge is a tree edge; we visit a subgraph that is a spanning tree of itself. If the input graph is a directed *acyclic* graph, then all edges are either tree edges or ignored edges, and once again, no two distinct vertices can be strongly connected. Tarjan's insight is that we can understand the non-trivial strongly connected components of our graph by looking at the fronds and cross-links.

The taxonomy of edges in our running example is illustrated on the left. The tree edges are drawn with a thick arrow ($\longrightarrow$), and we also have one edge of each of the remaining kinds: an ignored edge ($\dashrightarrow$), a frond ($\longrightarrow$), and a cross-link ($\twoheadrightarrow$).

The correctness of the algorithm hinges on the following property of depth-first search: any two strongly connected vertices have a common ancestor in the spanning tree, so each strongly connected component is a *subtree* of the spanning tree. We define *root* of a strongly connected component as the root of this subtree. To identify the root of each strongly connected component, we associate another integer with each vertex v, denoted by $\texttt{LOWLINK}(v)$. This value is defined such that if $\texttt{LOWLINK}(v) = \texttt{NUMBER}(v)$, then v is the root of a strongly connected component. In our example, $\texttt{LOWLINK}(v) = \texttt{NUMBER}(v)$ for all v with the exception of "3"; we have $\texttt{LOWLINK}(3) = \texttt{NUMBER}(2)$, since "2" and "3" are in the same strongly connected component having "2" as the root.

We repeatedly update $\texttt{LOWLINK}(v)$ while visiting the successors of v, and check if $\texttt{LOWLINK}(v) = \texttt{NUMBER}(v)$ afterwards. If so, we've discovered a new strongly connected component! We maintain a pushdown stack in such a way that at this point, the strongly connected component consists of some number of vertices from the top of the stack.

We push each vertex onto the stack before visiting its successors, but we only pop vertices from the stack when forming a new strongly connected component; a vertex remains on the stack until we visit its component's root. Thus, while the pushes and pops appear unbalanced, every vertex that is pushed is eventually popped.

The algorithm also needs a cheap way to check if a vertex is currently on the stack. We associate one more piece of state with each vertex v, a single bit denoted $\texttt{ONSTACK}(v)$, to be set when v is pushed and cleared when v is popped.

Algorithm 16A (Tarjan's algorithm)**.** As input, takes a vertex v, and a callback to output the successor vertices of an arbitrary vertex reachable from v. Upon returning, v will have a component ID assigned. The algorithm updates two maps: to each vertex, we assign a component ID, and to each component ID, we assign a list of member vertices. Also uses a stack, and two global incrementing counters: the next unused vertex ID, and the next component ID.

1. (Memoize) If v already has a component ID assigned, return this component ID, which can be used to look up its members.
2. (Invariant) Otherwise, ensure that $\texttt{NUMBER}(v)$ has not been set. If it is set but v does not have a component ID, we performed an invalid re-entrant call (see below).
3. (Visit) Set $\texttt{NUMBER}(v)$ to the next vertex ID. Set $\texttt{LOWLINK}(v) \leftarrow \texttt{NUMBER}(v)$. Set $\texttt{ONSTACK}(v) \leftarrow$ true. Push $\texttt{NUMBER}(v)$ on the stack.
4. (Successors) For each successor w of v:
   a) If $\texttt{NUMBER}(w)$ is not set, we have a tree edge. Recursively invoke the algorithm on w, and set $\texttt{LOWLINK}(v) \leftarrow \min(\texttt{LOWLINK}(v), \texttt{LOWLINK}(w))$.
   b) If $\texttt{NUMBER}(w) \geq \texttt{NUMBER}(v)$, we have an ignored edge; do nothing.
   c) Otherwise, $\texttt{NUMBER}(w) < \texttt{NUMBER}(v)$, so we have a frond or cross-link. If $\texttt{ONSTACK}(w)$ is true, set $\texttt{LOWLINK}(v) \leftarrow \min(\texttt{LOWLINK}(v), \texttt{NUMBER}(w))$.
5. (Not root?) If $\texttt{LOWLINK}(v) \neq \texttt{NUMBER}(v)$, return.
6. (Root) Assign the next unused component ID to v, and add v to this component ID.
7. (Add vertex) Pop a vertex v' from the stack (which must be non-empty at this point) and clear $\texttt{ONSTACK}(v')$. Add v' to the component of v, and associate the component ID with v'.
8. (Repeat) If the stack is not empty, go back to Step 7. Otherwise, return.

After the outermost recursive call returns, the stack will always be empty. Note that while this algorithm is recursive, it is not re-entrant; in particular, the act of getting the successors of a vertex must not trigger the computation of the same strongly connected components. This is enforced in Step 2. In our case, this can happen because getting the successors of a protocol performs type resolution; in practice this should be extremely difficult to hit, so for simplicity we report a fatal error and exit the compiler instead of attempting to recover.

Protocol components. We maintain two tables in rewrite context:

1. A map from protocol declarations to *protocol nodes*.

   The protocol node for `P` stores the state associated with `P` by Tarjan's algorithm: `NUMBER(P)`, `LOWLINK(P)`, `ONSTACK(P)` and a component ID for `P`.

2. A map from component IDs to *protocol components*.

   A protocol component is a list of protocol declarations together with a requirement machine. The latter is either a protocol minimization machine constructed from user-written requirements, or a protocol machine constructed from the requirement signatures of those protocols. This requirement machine is lazily constructed when first needed, and not when the component itself is formed by Algorithm 16A.

Generic signatures. The idea of a protocol having a dependency relationship on another protocol generalizes to a generic signature depending on a protocol. The protocol dependencies of a generic signature are those that appear on the right-hand side of a derived conformance requirement.

Definition 16.6. A generic signature G has a *protocol dependency* on a protocol `P`, denoted $G \prec \texttt{P}$, if we can derive a conformance to `P` from G; that is, $G \vdash [\texttt{T: P}]$ for some type parameter `T`. The *protocol dependency set* of a generic signature G is the set of all protocols `P` such that $G \prec \texttt{P}$.

This new form of $\prec$ is not a binary relation in the sense of Definition 5.3, because the operands are in different sets. However, it is still transitive in the sense that $G \prec \texttt{P}$ and $\texttt{P} \prec \texttt{Q}$ together imply $G \prec \texttt{Q}$. Note that the two definitions of $\prec$ are related via the protocol generic signature: $G_{\texttt{P}} \prec \texttt{Q}$ if and only if $\texttt{P} \prec \texttt{Q}$.

Now, consider the protocols appearing on the right-hand side of the *explicit* conformance requirements of our generic signature G; call these the *successors* of G, by analogy with the successors of a protocol in the protocol dependency graph. If we consider all protocols reachable via paths originating from the successors of G, we arrive at the complete set of protocol dependencies of G.

We end this section with the algorithm to collect imported rules for a new requirement machine. We find all protocols reachable from some initial set, compute their strongly connected components, lazily construct a protocol machine for each component, and finally, collect the local rules from each requirement machine.

Algorithm 16B (Import rules from protocol components)**.** Takes a list of protocols that are the immediate successors of a generic signature or protocol. Outputs a list of imported rules.

1. (Initialize) Initialize a worklist and add all input protocols to the worklist. Initialize an empty set S of visited protocols. Initialize an empty set M of requirement machines (uniqued by pointer equality).

2. (Check) If the worklist is empty, go to Step 8.
3. (Next) Otherwise, remove the next protocol `P` from the worklist. If `P` $\in S$, go back to Step 2, otherwise set $S \leftarrow S \cup \{\texttt{P}\}$.
4. (Component) Use Algorithm 16A to compute the component ID for `P`.
5. (Machine) Let m be the requirement machine for this protocol component, creating it first if necessary. If $m \notin M$, set $M \leftarrow M \cup \{m\}$.
6. (Successors) Add each protocol dependency of `P` to the worklist.
7. (Loop) Go back to Step 1.
8. (Collect) Collect the local rules from each $m \in M$ and return.

We will encounter the protocol dependency graph again when we talk about the Knuth-Bendix completion procedure in Chapter 19. Completion looks for *overlaps* between pairs of rules, and we will use the protocol dependency graph to cut down on the work performed by showing that pairs of rules where both are imported need not be considered.

A protocol component is always operated on as an indivisible unit; for example, in Chapter 22 we will see that requirement minimization must consider all protocols in a component simultaneously to get correct results.

16.2. Debugging Flags

We've already seen two examples of the `-debug-requirement-machine` flag:

```
-debug-requirement-machine=timers
-debug-requirement-machine=protocol-dependencies
```

More generally, the argument to this flag is a comma-separated list of options, where the possible options are the following:

- `timers`: Chapter 16.
- `protocol-dependencies`: Section 16.1.
- `simplify`: Section 18.5.
- `add`, `completion`: Chapter 19.
- `concrete-unification`, `conflicting-rules`, `property-map`: Chapter 20.
- `concretize-nested-types`, `conditional-requirements`: Section 21.1.

- `concrete-contraction`: Section 21.3.
- `homotopy-reduction`, `homotopy-reduction-detail`, `propagate-requirement-ids`: Section 22.1.
- `minimal-conformances`, `minimal-conformances-detail`: Section 22.4.
- `minimization`, `redundant-rules`, `redundant-rules-detail`, `split-concrete-equiv-class`: Chapter 22.5.

There are two final debugging flags. We maintain some histograms in the rewrite context. The `-analyze-requirement-machine` flag dumps them at the end of the compilation session. They can be helpful when working on performance optimizations:

- A count of the unique symbols allocated, by kind (Section 18.2).
- A count of the number of terms allocated, by length (Section 18.3).
- Statistics about the rule trie (Section 18.5) and property map trie (Chapter 20).
- Statistics about the minimal conformances algorithm (Section 22.4).

The `-dump-requirement-machine` flag prints each requirement machine before and after the completion procedure runs. The printed representation includes a list of rewrite rules, the property map, and all rewrite loops. The output will begin to make sense after Chapter 18.

16.3. Source Code Reference

Key source files:

- `lib/AST/RequirementMachine/`

The Requirement Machine implementation is private to `lib/AST/`. The remainder of the compiler interacts with it indirectly, through the generic signature query methods on `GenericSignature` (Section 5.6) and the various requests for building new generic signatures (Section 11.5).

The Rewrite Context

Key source files:

- `lib/AST/RequirementMachine/RewriteContext.h`
- `lib/AST/RequirementMachine/RewriteContext.cpp`

`ASTContext` *class*

The global singleton for a single frontend instance. See also Section 2.6.

- `getRewriteContext()` returns the global singleton `RewriteContext` for this frontend instance.

`rewriting::RewriteContext` *class*

A singleton object to manage construction of requirement machines, building of the protocol component graph, and unique allocation of symbols and terms.

- `getRequirementMachine(CanGenericSignature)` returns a query machine for the given generic signature, creating one first if necessary.
- `getRequirementMachine(ProtocolDecl *)` returns a for the protocol component which contains the given protocol, creating one first if necessary.
- `getProtocolComponentImpl()` is a private helper which returns the protocol component containing the given protocol.
- `getProtocolComponentRec()` implements Tarjan's algorithm (Algorithm 16A) for computing strongly connected components.
- `isRecursivelyConstructingRequirementMachine(CanGenericSignature)` returns true if we are currently constructing a query machine for this generic signature.
- `isRecursivelyConstructingRequirementMachine(ProtocolDecl *)` returns true if we are currently constructing a protocol machine for the given protocol's component. These two methods are used to break cycles in associated type inference, since otherwise re-entrant construction triggers an assertion in the compiler.
- `installRequirementMachine(CanGenericSignature, std::unique_ptr<RequirementMachine>)` takes a minimization machine and associates it with the given signature.
- `installRequirementMachine(ProtocolDecl *, std::unique_ptr<RequirementMachine>)` takes a protocol minimization machine and associates it with the given protocol's component.

`GenericSignatureImpl` *class*

See also Section 5.6.

- `getRequirementMachine()` returns the query machine for this generic signature, by asking the rewrite context to produce one and then caching the result in an instance variable of the `GenericSignatureImpl` instance itself.

  This method is used by the implementation of generic signature queries; apart from that, there should be no reason to reach inside the requirement machine instance from elsewhere in the compiler.

`ProtocolDependenciesRequest` *class*

A request evaluator request which computes the all protocols referenced from a given protocol's associated conformance requirements. These are the successors of the protocol in the .

`ProtocolDecl` *class*

See also Section 4.6.

- `getProtocolDependencies()` evaluates the `ProtocolDependenciesRequest`.

`rewriting::RequirementMachine` *class*

Represents a list of generic requirements in a manner amenable to automated reasoning.

- `initWithGenericSignature()` initializes a new query machine from the requirements of an existing generic signature.
- `initWithWrittenRequirements()` initializes a new minimization machine from user-written requirements when building a new generic signature.
- `initWithProtocolSignatureRequirements()` initializes a new protocol machine from existing requirement signatures of each protocol in a protocol component.
- `initWithProtocolWrittenRequirements()` initializes a new protocol minimization machine for computing new requirement signatures for each protocol in a protocol component.
- `verify()` checks various invariants, called after construction.
- `dump()` dumps the requirement machine's rewrite rules and property map.

Requests

Key source files:

- `lib/AST/RequirementMachine/RequirementMachineRequests.cpp`

The evaluation functions for the below requests are implemented in the Requirement Machine.

`InferredGenericSignatureRequest::evaluate` *method*

Evaluation function for building a new generic signature from requirements written in source. Constructs a minimization machine.

This ultimately implements the `GenericContext::getGenericSignature()` method; see Section 5.6.

`AbstractGenericSignatureRequest::evaluate` *method*

Evaluation function for building a new generic signature from a list of generic parameters and requirements. Constructs a minimization machine. This ultimately implements the `buildGenericSignature()` function; see Section 11.5.

`RequirementSignatureRequest::evaluate` *method*

Evaluation function for obtaining the requirement signature of a protocol. This either deserializes the requirement signature, or constructs a protocol minimization machine from user-written requirements and uses it to build a new requirement signature. This ultimately implements the `ProtocolDecl::getRequirementSignature()` method; see Section 5.6.

Debugging

Key source files:

- `lib/AST/RequirementMachine/Debug.h`
- `lib/AST/RequirementMachine/Histogram.h`

`rewriting::RewriteContext` *class*

- `RewriteContext()` the rewrite context constructor splits the string value of the `-debug-requirement-machine` flag into comma-separated tokens, and maps each token to an element of the `DebugFlags` enum.
- `getDebugFlags()` returns the `DebugFlags` enum.
- `beginTimer()` starts a timer identified by the given string, and logs a message.
- `endTimer()` ends a timer identified by the given string, and logs a message. Must be paired with the call to `beginTimer()`.

`rewriting::DebugFlags` *enum class*

An `OptionSet` of debugging flags to control various forms of debug output printed by the Requirement Machine.

`rewriting::Histogram` *class*

A utility class for collating points into a histogram. Each histogram has a fixed number of buckets, together with an initial value, and an "overflow" bucket. Points are mapped to buckets by subtracting the initial value and comparing against the total number of buckets. If the result exceeds the total number of buckets, it is recorded in an "overflow" bucket.

If the compiler is invoked with the `-analyze-requirement-machine` frontend flag, various histograms in the `RequirementMachine` instance are printed when the compiler exits.

- `Histogram(unsigned size, unsigned start)` creates a new histogram.
- `add(unsigned)` records a point in the histogram.
- `dump()` prints the histogram as ASCII art.

17. Monoids

Monoids are one of the fundamental objects that we study in abstract algebra. Their simplicity allows for a great deal of generality, so we will quickly narrow our focus to *finitely-presented* monoids. We will see that every finitely-presented monoid can be encoded by a Swift generic signature. After a brief detour into computability theory, we will introduce the *word problem*, and relate the word problem to our derived requirements formalism of Section 5.2. The word problem turns out to be undecidable in general, but we will see it can be solved in those finitely-presented monoids that admit a *convergent* presentation. This prepares us for for the next chapter, where we attempt to solve derived requirements by encoding a generic signature as a finitely-presented monoid. We begin with standard definitions, found in texts such as [104] or [105].

Definition 17.1. A *monoid* $(M, \cdot, \varepsilon)$ is a structure consisting of a set M, a binary operation "$\cdot$", and an identity element $\varepsilon \in M$, which together satisfy the following three axioms:

- The set M is *closed* under the binary operation: for all $x, y \in M$, $x \cdot y \in M$.
- The binary operation is *associative*: for all $x, y, z \in M$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
- The element ε acts as the *identity*: for all $x \in M$, $x \cdot \varepsilon = \varepsilon \cdot x = x$.

When the binary operation and identity element are clear from context, we can write M instead of $(M, \cdot, \varepsilon)$. We also sometimes omit the symbol "$\cdot$", so if $x, y \in M$, then we can write xy instead of $x \cdot y$. Finally, associativity allows us to omit parentheses without ambiguity in an expression like $x \cdot y \cdot z$ or xyz.

Specific instances of monoids can be quite complicated and abstract, but our first example is easy to describe.

Example 17.2. The set of natural numbers $\mathbb{N}$ satisfies the monoid axioms if we take addition as our operation and zero as the identity element:

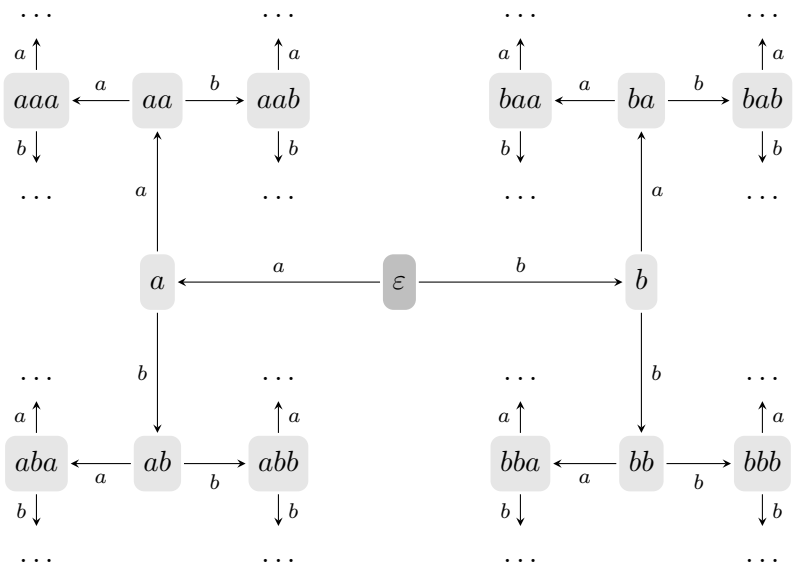
- The sum of two natural numbers is again a natural number.
- Addition is associative: for all $x, y, z \in \mathbb{N}$, $(x + y) + z = x + (y + z)$.
- Zero is the identity: for all $x \in \mathbb{N}$, $x + 0 = 0 + x = x$.

We will see shortly that $(\mathbb{N}, +, 0)$ is an instance of the following construction.

Definition 17.3. Let A be a set. The *free monoid* generated by A, denoted A^*, is the set of all finite strings of elements of A. (This is the same notation as the " $*$ " operator in a *regular expression*.) The set A is called the *alphabet* or *generating set* of A^*. The generating set may be finite or infinite, but we assume it is finite unless stated otherwise. The binary operation on A^* is *string concatenation*, and the identity element ε is the empty string. The elements of A^* are also called *terms*. We say that u is a *subterm* of t if $t = xuy$ for some $x, y \in A^*$. The *length* of a term $t \in A^*$ is denoted $|t| \in \mathbb{N}$.

The reader might wish to review the discussion of the type parameter graph from Section 8.3. In abstract algebra, the *Cayley graph* of a monoid plays a similar role, and we will discover many parallels between the two. We will describe the general construction in the next section, but for now we'll just look the Cayley graph of the free monoid A^*. The vertices in this graph are the terms of A^*, and the vertex for the identity element is the distinguished root vertex. Then, for each vertex t and each generator $g \in A$, we also add an edge with source t and destination tg, and label this edge "g". (This is sometimes called the *right* Cayley graph, and the left Cayley graph can then be defined in the opposite way by joining t with gt.)

Example 17.4. The free monoid with two generators $\{a, b\}^*$ consists of all finite strings made up of a and b. Two typical elements are $abba$ and bab, and their concatenation is $abba \cdot bab = abbabab$. Unlike $(\mathbb{N}, +, 0)$, this monoid operation is not *commutative*, so for example, $abba \cdot bab \neq bab \cdot abba$. The Cayley graph of $\{a, b\}^*$ is an infinite binary tree. Every vertex has two successors, corresponding to multiplication on the right by a and b, respectively:



Every term defines a path in the Cayley graph; we start from the root ε, and follow a series of successive "a" or "b" edges until we arrive at the vertex representing our term. In general, every vertex in the Cayley graph has the same number of successors, equal to the number of elements in our generating set A.

Before we continue, we need a shorthand where xxx becomes x^3, and so on:

Definition 17.5. Let M be a monoid. If $x \in M$ and $n \in \mathbb{N}$, we can define x^n as the "nth power" of x:

$$x^n = \begin{cases} \varepsilon & n = 0 \\ x & n = 1 \\ x^{n-1} \cdot x & n > 1 \end{cases}$$

The choice of notation is justified by the following "law of exponents."

Proposition 17.6. Let M be a monoid. If $x \in M$ is some element, then $x^m \cdot x^n = x^{m+n}$ for all m, $n \in \mathbb{N}$.

Proof. We start with a fixed but arbitrary $m \in \mathbb{N}$, and then prove the result by induction on $n \in \mathbb{N}$ (Section 11.3).

Base case. We need to show that $x^m \cdot x^0 = x^{m+0}$. We make use of the definition of x^0, the identity element axiom of M, and the fact that $m = m + 0$ in $\mathbb{N}$:

$$x^m \cdot x^0 = x^m \cdot \varepsilon = x^m = x^{m+0}$$

Inductive step. First, we assume the induction hypothesis:

$$x^m \cdot x^{n-1} = x^{m+n-1}$$

Then, we use the monoid operation of M to multiply both sides by x:

$$(x^m \cdot x^{n-1}) \cdot x = x^{m+n-1} \cdot x$$

We use the associativity of M and the definition of x^n to rewrite the left-hand side:

$$(x^m \cdot x^{n-1}) \cdot x = x^m \cdot (x^{n-1} \cdot x) = x^m \cdot x^{n-1+1} = x^m \cdot x^n$$

Finally, we use the definition of x^n to rewrite the right-hand side:

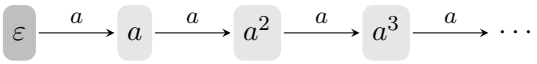
$$x^{m+n-1} \cdot x = x^{m+n-1+1} = x^{m+n}$$

Putting everything together, we get $x^m \cdot x^n = x^{m+n}$. By induction, this is actually true for all $n \in \mathbb{N}$, and since we also started with an arbitrary $m \in \mathbb{N}$, we're done. □

Example 17.7. The free monoid over one generator $\{a\}^*$ is *isomorphic* to $(\mathbb{N}, +, 0)$. We will talk about isomorphisms later, but essentially this means that both define the same object, up to a choice of notation.

Indeed, every non-zero element of $\mathbb{N}$ can be uniquely expressed as a sum of 1's. For example, $3 = 1 + 1 + 1$ and $5 = 1 + 1 + 1 + 1 + 1$. This means we can write ε instead of 0, a instead of 1, and $\cdot$ instead of $+$. Now, the equation $3 + 5 = 8$ translates into $aaa \cdot aaaaa = aaaaaaaa$, or using exponent notation, $a^3 \cdot a^5 = a^8$.

Next, consider the Cayley graph of $\{a\}^*$. Here, every vertex has one successor, because the generating set has a single element a. This looks like the type parameter graph of protocol `N` from Example 8.8, and we will see why in Section 17.3:



Example 17.8. The free monoid over an empty set $\varnothing^*$ is $\{\varepsilon\}$, the one-element set that consists of the empty string. This is called the *trivial monoid*. Its Cayley graph is a single vertex.

17.1. Finitely-Presented Monoids

Every element of a free monoid has a *unique* expression as a product of generators. Finitely-presented monoids are more general, because multiple distinct combinations can name the same element. To model this phenomenon, we start from a finite set of *rewrite rules*, which then generate an equivalence relation on terms. We first turned to equivalence relations in Section 5.3, to understand the same-type requirements of a generic signature; the below construction is similar.

A *monoid presentation* is a list of generators and rewrite rules:

$$\langle \underbrace{a_1, \ldots, a_m}_{\text{generators}} \mid \underbrace{u_1 \sim v_1, \ldots, u_n \sim v_n}_{\text{rewrite rules}} \rangle$$

Also, if $A := \{a_1, \ldots, a_m\}$ and $R := \{(u_1, v_1), \ldots, (u_n, v_n)\} \subseteq A^* \times A^*$ are finite sets, we can form the monoid presentation $\langle A \mid R \rangle$. The equivalence relation that R generates on the terms of A^* is denoted by $x \sim_R y$, or $x \sim y$ when R is clear from context. Note that when we write $x = y$, we always mean that x and y are *identical* in A^*, not just equivalent in $\langle A \mid R \rangle$.

We will describe the intuitive model behind the term equivalence relation first, and then give a rigorous definition in the next section. Syntactically, a rewrite rule $(u, v) \in R$ is a pair of terms. Semantically, a rewrite rule tells us that if we find u anywhere within a term, we can replace this subterm with v, and obtain another equivalent term. Term equivalence is symmetric, so we can also replace v with u, too. Finally, it is transitive, so we can iterate these rewrite steps any number of times to prove an equivalence.

A monoid presentation $\langle A \,|\, R \rangle$ then defines a *finitely-presented monoid*. The elements of a finitely-presented monoid are the equivalence classes of $\sim_R$, the binary operation is concatenation of terms, and the identity element is the equivalence class of ε.

Example 17.9. Finitely-presented monoids generalize the free monoids, because every free monoid (with a finite generating set) is also finitely presented if we start from an empty set of rewrite rules. In this case $x \sim y$ if and only if $x = y$.

Example 17.10. Consider the finite set $\{0, 1, 2, 3\}$ with the binary operation $+$ given by the following table. The operation is addition modulo 4. We call this monoid $\mathbb{Z}_4$. This method of specifying a finite monoid is called a *Cayley table*:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

If we write a instead of 1, ε instead of 0, and $\cdot$ instead of $+$, we can also describe $\mathbb{Z}_4$ as a finitely-presented monoid with one generator and one rewrite rule:

$$\mathbb{Z}_4 := \langle a \,|\, a^4 \sim \varepsilon \rangle$$

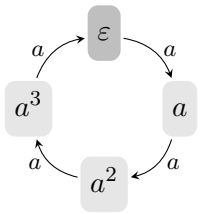
The rule $a^4 \sim \varepsilon$ allows us to insert or delete $aaaa$, which gives us this general principle:

$$a^m \sim a^n \qquad \text{if and only if} \qquad m \equiv n \pmod 4$$

Therefore, the rule partitions the terms of $\{a^*\}$ into four infinite equivalence classes:

$$\{\varepsilon, a^4, a^8, \ldots, a^{4k}, \ldots\}$$
$$\{a, a^5, a^9, \ldots, a^{4k+1}, \ldots\}$$
$$\{a^2, a^6, a^{10}, \ldots, a^{4k+2}, \ldots\}$$
$$\{a^3, a^7, a^{11}, \ldots, a^{4k+3}, \ldots\}$$

In the Cayley graph of a finitely-presented monoid, the vertices are equivalence classes of terms. The edge set is now defined on these equivalence classes, so for each equivalence class $[\![t]\!]$ and generator $g \in A$, we have an edge with source $[\![t]\!]$ and destination $[\![tg]\!]$. We will see later this does not depend on our choice of labels. With $\mathbb{Z}_4$ we can label each vertex with the shortest term in each equivalence class. The Cayley graph for $\mathbb{Z}_4$ is the same as the graph we associated with the `Z4` protocol from Example 8.9:



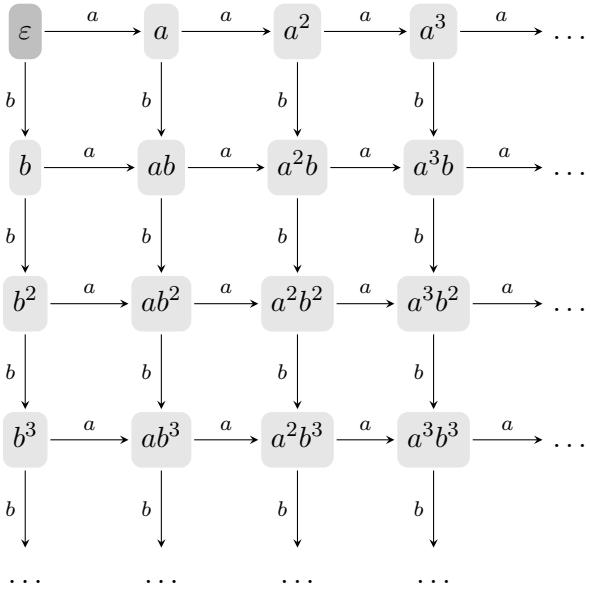
Example 17.11. We saw that the free monoid $\{a, b\}^*$ is not commutative. If we wish to make it commutative in the "most general" way, we can consider the *free commutative monoid* on two generators:

$$\langle a, b \,|\, ba \sim ab \rangle$$

The rule says we can permute the a's and b's within a term, but nothing else, so in particular the terms in an equivalence class must all have the same length. For this reason, every equivalence class is finite, and furthermore, we can transform an arbitrary term into a "canonical form" where the a's come first. This allows us to calculate by summing exponents (in fact, this monoid is isomorphic to the Cartesian product $\mathbb{N} \times \mathbb{N}$, whose elements are ordered pairs added component-wise). For example:

$$a^2b^3 \cdot a^7b = a^9b^4$$

The existence of this canonical form means that every equivalence class contains a unique term of the form $a^m b^n$ for m, $n \in \mathbb{N}$, so we will use these as the labels in our Cayley graph. Now, multiplication by a increments the first exponent, and multiplication by b increments the second. If we arrange the vertices on a grid, then we see that each vertex is joined with the vertex immediately below and to the right:



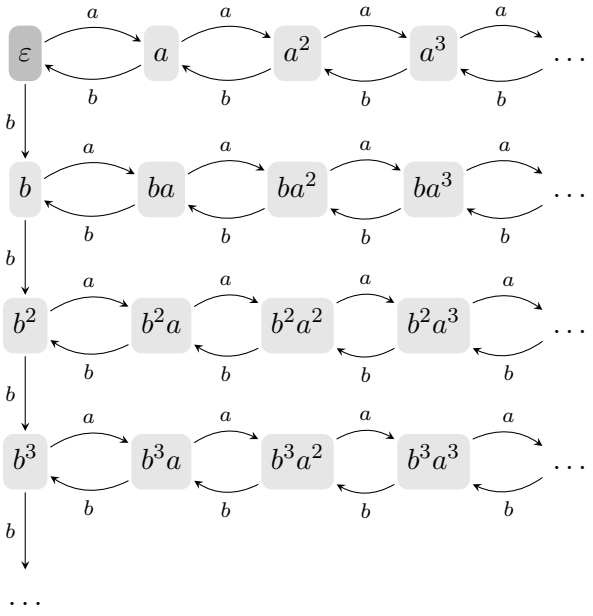
This Cayley graph (as well as the one for $\mathbb{Z}_4$) has something we didn't observe in free monoids: multiple paths with the same source and destination. For example, because $ababab \sim aaabbb$, we can start from ε and then follow the edges by reading either term from left to right. In either case, we end up at a^3b^3. The equivalence class of a^3b^3 describes the "walking routes" that take us to the bottom right corner of the grid.

Example 17.12. For the next example, we also start with two generators and a single rewrite rule, but the resulting monoid looks rather different.

If we're given a string of "`(`" and "`)`" and we wish to determine if the parentheses are balanced, we can repeatedly delete occurrences of the substring "`()`" until none remain. If we end up with the empty string, then we know that every opening parenthesis was matched by a closing parenthesis in the original string. Now, write a instead of "`(`" and b instead of "`)`". This is the *bicyclic monoid*, where balanced strings of parentheses are precisely those terms equivalent to ε:

$$\langle a, b \,|\, ab \sim \varepsilon \rangle$$

Consider the term $baaba$, which contains ab as a subterm. Our rule allows us to replace this subterm with ε, thus $baaba \sim baa$. We can then insert ab in a different position, so for example $baa \sim babaa$. Equivalence is transitive, so we also have $baaba \sim babaa$. Note that this monoid is not commutative, for instance $ab \not\sim ba$. In this monoid, every equivalence class has a unique term of minimum length, the one without any ab's. This term has the form $b^m a^n$ for $m, n \in \mathbb{N}$, so we will label our vertices with those. As in the previous example, we arrange the vertices of the Cayley graph on a grid:



To understand the edge relation better, observe that if we multiply any term by a on the right, we can then multiply by b to "cancel out" the a and end up back where we started. On the other hand, if there are no more a's to cancel on the right, adding a new b takes us into a new "state" from which we cannot return. There is a unique shortest path from ε to every other vertex; it is the one that does not contain ab.

Example 17.13. Not every "elementary" monoid is finitely presented, one example being the natural numbers under *multiplication*. This is a consequence of these four properties of the natural numbers:

1. Every non-zero natural number has a unique representation as a product of prime numbers (*The Fundamental Theorem of Arithmetic*).
2. There are infinitely many prime numbers (*Euclid's Theorem*).
3. Multiplication of natural numbers is commutative.
4. Multiplication by zero always yields zero.

These give us an *infinite* presentation of $(\mathbb{N}, \cdot, 1)$. This monoid is generated by zero and all the prime numbers: $\{0, 2, 3, 5, 7, 11, \ldots\}$. Then, we add the rule $0 \cdot 0 \sim 0$ because zero times zero is zero, followed by two infinite sets of rewrite rules:

- For every pair of distinct prime numbers p and q, we have $p \cdot q \sim q \cdot p$, so our binary operation is commutative.
- For every prime number p, we have $0 \cdot p \sim 0$ and $p \cdot 0 \sim 0$, so that zero behaves like the zero element.

The unique prime factorization of a natural number is really a term written using these generators, for example $84 = 2^2 \cdot 3 \cdot 7$. This monoid is a far more intricate object than $(\mathbb{N}, +, 0)$, which we recall was just the free monoid with one generator.

Mathematical aside. A typical course of abstract algebra often starts with the study of *groups*, which are monoids where each element x has an inverse element x^{-1}, such that $x \cdot x^{-1} = x^{-1} \cdot x = \varepsilon$. Groups have a lot more structure than monoids, all because of this one additional axiom. The natural numbers under addition are not a group, but if we instead consider the set of all integers $\mathbb{Z}$, then $\mathbb{Z}$ is a group under addition, because each $n \in \mathbb{Z}$ has an inverse element: $n + (-n) = (-n) + n = 0$. Addition is commutative, and a group with a commutative binary operation is also called an *Abelian group*.

The next level of structure is the *ring*. A ring is a set R with *two* binary operations, denoted $+$ and $\cdot$, and two special elements, 0 and 1. The ring axioms state that $(R, +, 0)$ is an Abelian group, $(R, \cdot, 1)$ is a monoid, and $+$ and $\cdot$ are related via the *distributive law*, so $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$. With the usual addition and multiplication operations, $\mathbb{Z}$ becomes a ring. If every non-zero element has a multiplicative inverse, we have a *division ring*. If we also say that multiplication is commutative, we have a *field*. In a field, the non-zero elements define an Abelian group under multiplication. The set of rational numbers $\mathbb{Q}$ is a field.

This tower of abstractions also describes the complex numbers, quaternions, matrices, polynomials, modular arithmetic, and other "number-like" objects which commonly arise in various applications of mathematics.

17.2. Equivalence of Terms

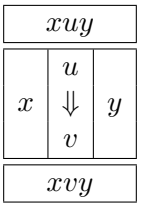
The Cayley graph of a monoid presentation helped us understand the relationships between equivalence classes. To better see the structure within a single equivalence class, we follow [106] and introduce another directed graph, called the *rewrite graph* of a monoid presentation $\langle A \mid R \rangle$. We begin by describing the edges of our graph.

Definition 17.14. A monoid presentation $\langle A \mid R \rangle$ generates a set of *rewrite steps*:

1. A rewrite step is an ordered tuple of terms, written $x(u \Rightarrow v)y$, where x, $y \in A^*$, and one of (u, v) or $(v, u) \in R$.
2. If $(u, v) \in R$, we say that $x(u \Rightarrow v)y$ is a *positive* rewrite step. If $(v, u) \in R$, then $x(u \Rightarrow v)y$ is a *negative* rewrite step.
3. The terms x and y are called the left and right *whiskers*, respectively. If x or y is identically ε, we omit them in the notation.
4. For rewrite steps to serve as edges in our graph, we define the *source* and *destination* of a rewrite step $s := x(u \Rightarrow v)y$ as the following terms of A^*:

$$\mathrm{src}(s) := xuy$$
$$\mathrm{dst}(s) := xvy$$

The rewrite step $x(u \Rightarrow v)y$ represents the transformation of xuy into xvy via the replacement of u with v, while leaving the whiskers x and y unchanged. (We can assume without loss of generality that $u \neq v$, ruling out redundant steps where $\mathrm{src}(s) = \mathrm{dst}(s)$.) We can draw a picture:



Definition 17.15. The *rewrite graph* of a monoid presentation $\langle A \mid R \rangle$ has the terms of A^* as vertices, and rewrite steps as edges. Except in the trivial case where $A = \varnothing$, this graph is infinite, but we will only look at small subgraphs at any one time. For example, every rewrite step determines a subgraph with two vertices and one edge:

$$xuy \xRightarrow{x(u \Rightarrow v)y} xvy$$

From the rewrite graph, we will then define $x \sim_R y$ to mean that this graph has a *path* with source x and destination y. The following definition of a rewrite path is identical to the general case of a path in a directed graph from Section 8.3.

Definition 17.16. A *rewrite path* consists of an initial term $t \in A^*$ together with a sequence of zero or more rewrite steps $(s_1, \ldots, s_n)$ which satisfy the following conditions:

1. If the rewrite path contains at least one rewrite step, the source of the first rewrite step must equal the initial term: $\mathrm{src}(s_1) = t$.

2. If the rewrite path contains at least two steps, the source of each subsequent step must equal the destination of the preceding step: $\mathrm{src}(s_{i+1}) = \mathrm{dst}(s_i)$ for $0 < i \leq n$.

Like a rewrite step, each rewrite path also has a source and destination. We define the source and destination of a rewrite path p as follows, where n is the length of p:

$$\mathrm{src}(p) := t$$
$$\mathrm{dst}(p) := \begin{cases} \mathrm{dst}(s_n) & \text{if } n > 0 \\ t & \text{if } n = 0 \end{cases}$$

To each term $t \in A^*$ we can associate a rewrite path with an empty sequence of rewrite steps, called the *empty rewrite path* and denoted 1_t. We have $\mathrm{src}(1_t) = \mathrm{dst}(1_t) = t$. (This is why a rewrite path must store its initial term, to specify the source and destination in the empty case.) A non-empty rewrite path p can be written as a composition of steps: $p = s_1 \circ \cdots \circ s_n$. We can also visualize a rewrite path as a path in the rewrite graph:

$$\mathrm{src}(p) \xRightarrow{s_1} \cdots \xRightarrow{\cdots} \cdots \xRightarrow{s_n} \mathrm{dst}(p)$$

Example 17.17. Recall the bicyclic monoid $\langle a, b \,|\, ab \sim \varepsilon \rangle$ from Example 17.12. We saw that $baaba \sim baa$ because we obtain baa by deleting the subterm ab from $baaba$. We can encode this with a rewrite step $ba(ab \Rightarrow \varepsilon)a$:

$$baaba \xRightarrow{ba(ab \Rightarrow \varepsilon)a} baa$$

We also saw that $baa \sim babaa$. We can encode the transformation of baa into $babaa$ with a rewrite step $b(\varepsilon \Rightarrow ab)aa$:

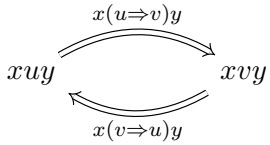
$$baa \xRightarrow{b(\varepsilon \Rightarrow ab)aa} babaa$$

The destination of the first rewrite step is also the source of the second rewrite step. We can compose them to form the rewrite path $ba(ab \Rightarrow \varepsilon)a \circ b(\varepsilon \Rightarrow ab)aa$. This rewrite path encodes the fact that $baaba \sim babaa$:

$$baaba \xRightarrow{ba(ab \Rightarrow \varepsilon)a} baa \xRightarrow{b(\varepsilon \Rightarrow ab)aa} babaa$$

The algebra of rewriting. To ensure that the term equivalence relation ultimately satisfies the necessary axioms, we introduce some algebraic operations on rewrite steps and paths, called inversion, composition, and whiskering.

Definition 17.18. The *inverse* of a rewrite step $s := x(u \Rightarrow v)y$, denoted s^{-1}, is defined as $x(v \Rightarrow u)y$, so we swap the u and the v. The inverse of a positive rewrite step is negative, and vice versa, which means that the edges in the rewrite graph come in complementary pairs:



We can also invert a rewrite *path*, to get a new path which undoes the transformation described by the original path. The empty rewrite path is its own inverse. For a non-empty rewrite path, we invert each rewrite step and perform them in reverse order:

$$p^{-1} := \begin{cases} 1_t & \text{if } p = 1_t \\ s_n^{-1} \circ \cdots \circ s_1^{-1} & \text{if } p = s_1 \circ \cdots \circ s_n \end{cases}$$

When shown in the rewrite graph, the inverse path p^{-1} is p but backwards:

$$\mathrm{src}(p) \xleftarrow{s_1} \cdots \xleftarrow{\cdots} \cdots \xleftarrow{s_n} \mathrm{dst}(p)$$

We can think of a rewrite step as a rewrite path with length 1. Then, if p is *either* a rewrite step or a rewrite path, we see that taking the inverse of p interchanges its source and destination:

$$\mathrm{src}(p^{-1}) = \mathrm{dst}(p)$$
$$\mathrm{dst}(p^{-1}) = \mathrm{src}(p)$$

Example 17.19. We exhibited this rewrite path in Example 17.17, with source *baaba* and destination *babaa*:

$$baaba \xRightarrow{ba(ab \Rightarrow \varepsilon)a} baa \xRightarrow{b(\varepsilon \Rightarrow ab)aa} babaa$$

We can construct the inverse rewrite path from *babaa* to *baaba* by inverting both steps and swapping the order in which we apply them:

$$babaa \xRightarrow{b(ab \Rightarrow \varepsilon)aa} baa \xRightarrow{ba(\varepsilon \Rightarrow ab)a} baaba$$

Definition 17.20. If p_1 and p_2 are two rewrite paths with $\mathrm{dst}(p_1) = \mathrm{src}(p_2)$, we define their *composition* $p_1 \circ p_2$ as the rewrite path consisting of the initial term of p_1, followed by the rewrite steps of p_1, and finally the steps of p_2. This gives us a new rewrite path with the same source as p_1 and the same destination as p_2:

$$\mathrm{src}(p_1 \circ p_2) = \mathrm{src}(p_1)$$
$$\mathrm{dst}(p_1 \circ p_2) = \mathrm{dst}(p_2)$$

Shown as a diagram, composition joins two paths (the "=" in the middle is not an edge in the graph, but denotes we have two identical terms):

$$\mathrm{src}(p_1) \Longrightarrow \cdots \Longrightarrow (\mathrm{dst}(p_1) = \mathrm{src}(p_2)) \Longrightarrow \cdots \Longrightarrow \mathrm{dst}(p_2)$$

Empty rewrite paths act as the identity with respect to composition:

$$1_{\mathrm{src}(p)} \circ p = p$$
$$p \circ 1_{\mathrm{dst}(p)} = p$$

The inverse of the composition is the composition of the inverses, but flipped:

$$(p_1 \circ p_2)^{-1} = p_2^{-1} \circ p_1^{-1}$$

Finally, rewrite path composition is associative. If p_3 is another rewrite path such that $\mathrm{dst}(p_2) = \mathrm{src}(p_3)$:

$$p_1 \circ (p_2 \circ p_3) = (p_1 \circ p_2) \circ p_3$$

Example 17.21. We continue with Example 17.19. Consider these two rewrite paths in the bicyclic monoid, which we will call p_1 and p_2:

$$baabb \xRightarrow{ba(ab \Rightarrow \varepsilon)b} bab \xRightarrow{b(ab \Rightarrow \varepsilon)} b$$

$$aabbb \xRightarrow{a(ab \Rightarrow \varepsilon)bb} abb \xRightarrow{(ab \Rightarrow \varepsilon)b} b$$

The two paths have the same destination term b, but different source terms. If we invert the second path and compose it with the first, we get the rewrite path $p_1 \circ p_2^{-1}$, which transforms $baabb$ into $aabbb$:

$$baabb \xRightarrow{ba(ab \Rightarrow \varepsilon)b} bab \xRightarrow{b(ab \Rightarrow \varepsilon)} b \xRightarrow{(\varepsilon \Rightarrow ab)b} abb \xRightarrow{a(\varepsilon \Rightarrow ab)bb} aabbb$$

Definition 17.22. If $s := x(u \Rightarrow v)y$ and $z \in A^*$, we define the left and right *whiskering* action of z on s. Whiskering extends a rewrite step by lengthening its whiskers:

$$z \triangleleft s := zx(u \Rightarrow v)y$$
$$s \triangleright z := x(u \Rightarrow v)yz$$

The whiskering operation generalizes to rewrite paths as follows. First, we apply the monoid operation to the initial term, and then we whisker each rewrite step. If the path is empty, whiskering produces another empty rewrite path on a new term:

$$z \triangleleft 1_t = 1_{z \cdot t}$$
$$1_t \triangleright z = 1_{t \cdot z}$$

If the path is non-empty, the distributive law relates the whiskering of paths and steps:

$$z \triangleleft (s_1 \circ \cdots \circ s_n) = (z \triangleleft s_1) \circ \cdots \circ (z \triangleleft s_n)$$
$$(s_1 \circ \cdots \circ s_n) \triangleright z = (s_1 \triangleright z) \circ \cdots \circ (s_n \triangleright z)$$

Here is a diagram for $z \triangleleft p$ and $p \triangleright z$:

$$z \cdot \mathrm{src}(p) \xRightarrow{z \triangleleft s_1} z \cdots \Longrightarrow z \cdots \xRightarrow{z \triangleleft s_n} z \cdot \mathrm{dst}(p)$$

$$\mathrm{src}(p) \cdot z \xRightarrow{s_1 \triangleright z} \cdots z \Longrightarrow \cdots z \xRightarrow{s_n \triangleright z} \mathrm{dst}(p) \cdot z$$

Once again, if we let p denote either a rewrite path or a single rewrite step, we see that whiskering is related to the monoid operation of A^* in two ways:

$$\mathrm{src}(z \triangleleft p) = z \cdot \mathrm{src}(p)$$
$$\mathrm{dst}(z \triangleleft p) = z \cdot \mathrm{dst}(p)$$

$$\mathrm{src}(p \triangleright z) = \mathrm{src}(p) \cdot z$$
$$\mathrm{dst}(p \triangleright z) = \mathrm{dst}(p) \cdot z$$

Also, if $z' \in A^*$ is some other term:

$$z \triangleleft (z' \triangleleft p) = (z \cdot z') \triangleleft p$$
$$(p \triangleright z) \triangleright z' = p \triangleright (z \cdot z')$$

The left and right whiskering actions are compatible with each other:

$$(z \triangleleft p) \triangleright z' = z \triangleleft (p \triangleright z')$$

Finally, the distributive law generalizes to rewrite path composition:

$$z \triangleleft (p_1 \circ p_2) = (z \triangleleft p_1) \circ (z \triangleleft p_2)$$
$$(p_1 \circ p_2) \triangleright z = (p_1 \triangleright z) \circ (p_2 \triangleright z)$$

Proposition 17.23. Let $\langle A \mid R \rangle$ be a monoid presentation. Let P be the set of paths in the rewrite graph of $\langle A \mid R \rangle$. The following axioms characterize P:

1. (Base case) For each $(u, v) \in R$, there is a rewrite path $(u \Rightarrow v) \in P$.
2. (Closure under inversion) If $p \in P$, then $p^{-1} \in P$.
3. (Closure under composition) If $p_1, p_2 \in P$ with $\mathrm{dst}(p_1) = \mathrm{src}(p_2)$, then $p_1 \circ p_2 \in P$.
4. (Closure under whiskering) If $p \in P$ and $z \in A^*$, then $z \triangleleft p$ and $p \triangleright z \in P$.

Proof. By Definition 17.14 there is a positive rewrite step $(u \Rightarrow v)$ for each $(u, v) \in R$. Since every rewrite step is also a rewrite path of length 1, this establishes (1). Also, rewrite steps are closed under inverses and whiskering, which establishes (2) and (4) for rewrite paths of length 1. The general case of (2), (3) and (4) can be shown by induction on the length of the rewrite path. □

Recall that $x \sim_R y$ means our rewrite graph has a path from x to y.

Proposition 17.24. Let $\langle A \mid R \rangle$ be a monoid presentation. Then $\sim_R$ is an equivalence relation on A^*.

Proof. We check each axiom in turn.

1. (Reflexivity) For each $x \in A^*$, there is an empty rewrite path $1_x \in P$. Since $\mathrm{src}(1_x) = \mathrm{dst}(1_x) = x$, it follows that $x \sim x$.
2. (Symmetry) Assume $x \sim y$. Then there is a $p \in P$ such that $x = \mathrm{src}(p)$ and $y = \mathrm{dst}(p)$. Since P is closed under taking inverses, $p^{-1} \in P$. But $y = \mathrm{src}(p^{-1})$ and $x = \mathrm{dst}(p^{-1})$, so we have $y \sim x$.
3. (Transitivity) Assume $x \sim y$ and $y \sim z$. Then there is a $p_1 \in P$ with $x = \mathrm{src}(p_1)$ and $y = \mathrm{dst}(p_1)$, and a $p_2 \in P$ with $y = \mathrm{src}(p_2)$ and $z = \mathrm{dst}(p_2)$. We have $\mathrm{dst}(p_1) = \mathrm{src}(p_2)$, and P is closed under composition so $p_1 \circ p_2 \in P$. We also know that:

   $$\mathrm{src}(p_1 \circ p_2) = \mathrm{src}(p_1) = x$$
   $$\mathrm{dst}(p_1 \circ p_2) = \mathrm{dst}(p_2) = z$$

   Thus, $x \sim z$.

Notice how we used inversion and composition above, but not whiskering. □

We will use whiskering to show that $\sim$ is *compatible* with term concatenation:

Definition 17.25. Let M be a monoid. A relation $R \subseteq M \times M$ is *translation-invariant* if for all $(x, y) \in R$ and $z \in M$, we have both (zx, zy) and $(xz, yz) \in R$.

A translation-invariant equivalence relation is known as a *monoid congruence*.

Example 17.26. Consider the monoid $(\mathbb{N}, +, 0)$. The standard linear order $<$ on $\mathbb{N}$ is translation-invariant (but not an equivalence relation). For instance $5 < 7$ also implies that $5 + 2 < 7 + 2$. (Hence we're using "translation" in the geometric sense.)

Theorem 17.27. Let $\langle A \mid R \rangle$ be a monoid presentation. Then the term equivalence relation $\sim_R$ is a monoid congruence.

Proof. We've seen that $\sim$ is an equivalence relation, so it remains to establish translation invariance. Suppose that we're given x, y, $z \in A^*$ where $x \sim y$. We must show that $zx \sim zy$ and $xz \sim yz$. To show that $zx \sim zy$, take $p \in P$ such that $x = \mathrm{src}(p)$ and $y = \mathrm{dst}(p)$. Since P is closed under whiskering, we have $z \triangleleft p \in P$, so $zx \sim zy$:

$$\mathrm{src}(z \triangleleft p) = z \cdot \mathrm{src}(p) = zx$$
$$\mathrm{dst}(z \triangleleft p) = z \cdot \mathrm{dst}(p) = zy$$

To show that $xz \sim yz$, consider $p \triangleright z$ instead. □

Thus, *the set of equivalence classes* of $\sim_R$ has the structure of a monoid:

Theorem 17.28. Let $\langle A \mid R \rangle$ be a monoid presentation. Term concatenation is well-defined on the equivalence classes of $\sim_R$; that is, if we pick two representative terms and concatenate them together, the equivalence class of the result does not depend on the choice of representatives.

Proof. Suppose we're given x, x', y, $y' \in A^*$ such that $x \sim x'$ and $y \sim y'$. We must show that $xy \sim x'y'$. By assumption, there exist a pair of rewrite paths p_x, p_y such that:

$$\mathrm{src}(p_x) = x$$
$$\mathrm{dst}(p_x) = x'$$

$$\mathrm{src}(p_y) = y$$
$$\mathrm{dst}(p_y) = y'$$

Now, we form the rewrite path $p := (p_x \triangleright y) \circ (x' \triangleleft p_y)$ by whiskering and composing p_x and p_y. The composition is valid:

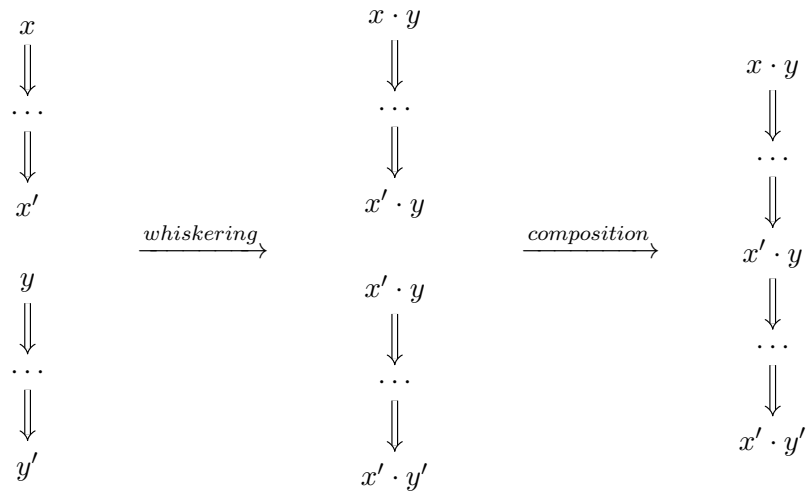
$$\mathrm{dst}(p_x \triangleright y) = \mathrm{dst}(p_x) \cdot y = x' \cdot y$$
$$\mathrm{src}(x' \triangleleft p_y) = x' \cdot \mathrm{src}(p_y) = x' \cdot y$$

We also have:

$$\mathrm{src}(p) = \mathrm{src}(p_x \triangleright y) = \mathrm{src}(p_x) \cdot y = x \cdot y$$
$$\mathrm{dst}(p) = \mathrm{dst}(x' \triangleleft p_y) = x' \cdot \mathrm{dst}(p_y) = x' \cdot y'$$

The existence of this rewrite path establishes that $xy \sim x'y'$. A pictorial proof is shown in Figure 17.1. □

Figure 17.1.: Term concatenation is well-defined on the equivalence classes of $\sim$



Example 17.29. We can apply this theorem to the bicyclic monoid $\langle a, b \mid ab \sim \varepsilon \rangle$ from Example 17.12. Consider the four terms $x := baba$, $x' := ba$, $y := a$, $y' := aba$. We have $x \sim x'$ and $y \sim y'$ via these two rewrite paths:

$$p_x := b(ab \Rightarrow \varepsilon)a$$
$$p_y := (\varepsilon \Rightarrow ab)a$$

Now, $(p_x \triangleright a) \circ (ba \triangleleft p_y)$ is a rewrite path from $babaa$ to $baaba$, showing that $xy \sim x'y'$:

$$b(ab \Rightarrow \varepsilon)aa \circ ba(\varepsilon \Rightarrow ab)a$$

Our original example showed that the bicyclic monoid can encode the problem of matching parentheses, by checking if a term is equivalent to ε. This was somewhat trivial, because we only looked within a single equivalence class, without making use of the monoid operation. A more impressive party trick is to use the bicyclic monoid to model the *stack effects* of programs in a *stack language*. This encoding relies on the monoid operation.

A stack language program is a sequence of tokens, where each token is either a *literal* or a *word*. To run the program, we evaluate each token sequentially from left to right:

1. A literal is immediately pushed on the stack.
2. A word, which is analogous to a function in other languages, pops some number of input values from the stack, processes those values, and then pushes zero or more results back on the stack.

To each token, we assign a *stack effect*, which is a pair of natural numbers (m, n) that encodes the number of inputs m and the number of outputs n that result from evaluating this token on the stack. It turns out if we represent a stack effect as an element $b^m a^n$ in the bicyclic monoid, then composition of stack effects is the same as the monoid operation.

Consider a stack language whose literals are integers and whose words are as below.

Token	**Description**	**Stack effect**
A literal	Any integer value.	a
`neg`	Pop the top of the stack and push its negation.	ba
`+`	Pop two values from the stack and push their sum.	b^2a
`*`	Pop two values from the stack and push their product.	b^2a
`dup`	Duplicate the top of the stack.	ba^2
`.`	Pop the top of the stack and print it out.	b

We can use this language to evaluate simple postfix arithmetic expressions:

```
5 dup * 3 neg + .
```

The above program has no overall effect on the stack; it will execute correctly when run on an empty stack, but if there were values on the stack, they remain unchanged. On the other hand, we can also have a program like "`dup * .`" which expects at least one input value to be present, otherwise the evaluation of "`dup`" will attempt to pop the top of an empty stack. We can also write a program like "`5 7 + 1`" which expects no inputs, but leaves values behind.

We can formalize this as follows. Given a stack language program, we map each token to its stack effect, which is known, and then compose them together. This gives us the stack effect of the overall program, as an element of the bicyclic monoid. The following closed form expression for the monoid operation can be used:

$$b^i a^j \cdot b^k a^l = \begin{cases} b^{i-j+k} a^l & \text{if } j < k \\ b^i a^{j-k+l} & \text{if } j \geq k \end{cases}$$

We can calculate that the stack effect of "`5 dup * 3 neg + .`" is ε, confirming that this program has no effect on the stack:

$$a \cdot ba^2 \cdot b^2 a \cdot a \cdot ba \cdot b^2 a \cdot b = \varepsilon$$

The stack effect of the *concatenation* of two programs is the composition of their stack effects. For this reason, stack-based languages are also known as *concatenative languages*. Examples of stack-based languages include Forth [107], Joy [108], and Factor [109]. The Factor compiler implements a static stack effect checker based on this idea.

17.3. A Swift Connection

In this section, we will show that finitely-presented monoids map to Swift protocols in a very natural way. We make use of the derived requirements formalism we defined in Section 5.2, and some further results from Section 11.3. By now, it should be apparent that there are some similarities between the two theories, even if the concepts do not completely overlap:

Swift generics	**F-P monoids**
Type parameters	Terms
Explicit requirements	Rewrite rules
Derived requirements	Rewrite paths
Reduced type equality	Term equivalence
Type parameter graph	Cayley graph

Free monoids. The below protocol—or rather the protocol generic signature $G_{\texttt{M}}$—models the free monoid over the two generators $\{a, b\}^*$:

```
protocol M {
  associatedtype A: M
  associatedtype B: M
}
```

The requirement signature of `M` declares two associated conformance requirements:

$$[\texttt{Self.A: M}]_{\texttt{M}}$$
$$[\texttt{Self.B: M}]_{\texttt{M}}$$

We define a function φ that sends each element of $\{a, b\}^*$ to a type parameter of $G_{\texttt{M}}$ by recursively constructing a dependent member type for each "a" and "b", until we finally get to the empty term, which becomes the protocol `Self` type:

$$\varphi(\varepsilon) := \texttt{Self}$$
$$\varphi(ua) := \varphi(u)\texttt{.A}$$
$$\varphi(ub) := \varphi(u)\texttt{.B}$$

For example, the four distinct terms of length 2 map to the following type parameters:

$$\varphi(aa) = \texttt{Self.A.A}$$
$$\varphi(ab) = \texttt{Self.A.B}$$
$$\varphi(ba) = \texttt{Self.B.A}$$
$$\varphi(bb) = \texttt{Self.B.B}$$

Every type parameter output by φ is a valid type parameter of $G_{\texttt{M}}$, and every such type parameter also conforms to `M`:

Proposition 17.30. If $t \in \{a, b\}^*$ and $\texttt{T} := \varphi(t)$, then $G_{\texttt{M}} \vdash \texttt{T}$ and $G_{\texttt{M}} \vdash [\texttt{T: M}]$.

Proof. We proceed by induction on the length of t.

Base case. If $|t| = 0$, then $t = \varepsilon$, and $\varphi(\varepsilon) = \texttt{Self}$. We can derive `Self` via a Generic elementary statement:

1. `Self` (Generic)

Likewise we get the conformance via a Conf elementary statement:

2. `[Self: M]` (Conf)

Inductive step. We know that $t = ua$ or $t = ub$ for some $u \in A^*$, so let $\texttt{U} := \varphi(u)$. By induction, $G_{\texttt{M}} \vdash \texttt{U}$ and $G_{\texttt{M}} \vdash [\texttt{U: M}]$. (We actually only need the latter.) First, we derive `U.A` or `U.B` by applying the AssocName inference rule to `[U: M]`:

1. `[U: M]` (...)
2. `U.A` (AssocName 1)
3. `U.B` (AssocName 1)

To derive `[U.A: M]` or `[U.B: M]` from `[U: M]`, we apply the AssocConf inference rule for the appropriate associated requirement:

1. `[U: M]` (...)
2. `[U.A: M]` (AssocConf 1)
3. `[U.B: M]` (AssocConf 1)

This completes the induction. □

We wish to equip the type parameters of $G_{\texttt{M}}$ with the structure of a monoid. The identity element we take to be `Self`, since this is $\varphi(\varepsilon)$.

The binary operation " $\cdot$ " is defined by *formal substitution*, replacing `Self` in the type parameter on the right-hand side with the entire type parameter on the left. Consider the two type parameters `Self.A.A` and `Self.B.B`:

$$(\texttt{Self.A.A}) \cdot (\texttt{Self.B.B}) = \texttt{Self.A.A.B.B}$$

To ensure this gives us a valid type parameter, we turn to Lemma 11.7. Suppose we are given two valid type parameters $G_{\texttt{M}} \vdash \texttt{U}$ and $G_{\texttt{M}} \vdash \texttt{V}$. By Proposition 17.30, we know that $G_{\texttt{M}} \vdash [\texttt{U: M}]$. The conditions of Lemma 11.7 are satisfied, thus $G_{\texttt{M}} \vdash \texttt{U} \cdot \texttt{V}$.

We now have a set of elements, a binary operation, and an identity element. The final question is, in what sense is this monoid "equivalent" to the free monoid $\{a, b\}^*$? The answer lies in the fact that φ is an *isomorphism* of monoids.

Definition 17.31. Let M and N be monoids.

1. A *monoid homomorphism* is a function $f\colon M \to N$ that sends the identity to the identity, and is compatible with the binary operation. That is, $f(\varepsilon_M) = \varepsilon_N$, and for all $x, y \in M$, $f(x \cdot_M y) = f(x) \cdot_N f(y)$.
2. A *monoid isomorphism* is a monoid homomorphism f which has an inverse f^{-1}, so $f^{-1}(f(x)) = x$ for all $x \in M$ and $f(f^{-1}(f(x)) = x$ for all $x \in N$. In this case we also say that M and N are *isomorphic*.

We already saw that φ maps the identity element ε of $\{a, b\}^*$ to the identity element `Self` of $G_{\texttt{M}}$. It also respects the monoid operation. For example,

$$\begin{aligned}&\varphi(aa) \cdot \varphi(bb)\\ &\quad = (\texttt{Self.A.A}) \cdot (\texttt{Self.B.B})\\ &\quad = \texttt{Self.A.A.B.B}\\ &\quad = \varphi(aabb)\end{aligned}$$

Finally, there is an inverse φ^{-1} mapping the type parameters of $G_{\texttt{M}}$ to elements of $\{a, b\}^*$:

$$\begin{aligned}&\varphi^{-1}(\texttt{Self}) := \varepsilon\\ &\varphi^{-1}(\texttt{U.A}) := \varphi^{-1}(\texttt{U}) \cdot a\\ &\varphi^{-1}(\texttt{U.B}) := \varphi^{-1}(\texttt{U}) \cdot b\end{aligned}$$

We've shown that the type parameters of $G_{\texttt{M}}$ have a monoid structure isomorphic to $\{a, b\}^*$. Furthermore, nothing in our construction relied on there being exactly two generators; we can add or remove associated type declarations in `M` as needed. For example, if we apply this construction to the free monoid with one generator, we get protocol `N` from Example 8.8. This monoid is isomorphic to $(\mathbb{N}, +, 0)$, which justifies the name `N`.

Finitely-presented monoids. We extend our encoding from a free monoid A^* to a finitely-presented monoid $\langle A \,|\, R\rangle$. We start with a protocol `M` that encodes A^*, then for each $(u, v) \in R$, we apply φ to u and v to obtain an associated same-type requirement $[\varphi(u)$ `==` $\varphi(v)]_{\texttt{M}}$. Finally, we attach a trailing `where` clause to `M` where we state these requirements. Let's try the bicyclic monoid $\langle a, b \,|\, ab \sim \varepsilon\rangle$ from Example 17.12:

```
protocol M {
  associatedtype A: M
  associatedtype B: M where Self.A.B == Self
}
```

The same-type requirements of `M` give the protocol generic signature $G_{\texttt{M}}$ a non-trivial equivalence class structure under the reduced type equality relation.

Example 17.32. Recalling Example 17.17, we saw that $baaba \sim babaa$ in the bicyclic monoid, with $ba(ab \Rightarrow \varepsilon)a \circ b(\varepsilon \Rightarrow ab)aa$ being one possible rewrite path from $baaba$ to $babaa$. Let `M` be the protocol that encodes this monoid. We can construct a derivation of $G_{\texttt{M}} \vdash$ [`Self.B.A.A.B.A == Self.B.A.B.A.A`] by translating each rewrite step:

$ba(ab \Rightarrow \varepsilon)a$	$G_{\texttt{M}} \vdash$ [`Self.B.A.A.B.A == Self.B.A.A`]
$b(\varepsilon \Rightarrow ab)aa$	$G_{\texttt{M}} \vdash$ [`Self.B.A.A == Self.B.A.B.A.A`]

The first rewrite step, $ba(ab \Rightarrow \varepsilon)a$, proves an equivalence between $baaba$ and baa. We will build a derivation piecemeal. We start with the left whisker ba:

1. [`Self: M`] (Conf)
2. [`Self.B: M`] (AssocConf 1)
3. [`Self.B.A: M`] (AssocConf 2)

We then apply the AssocSame inference rule to get $ba(ab \Rightarrow \varepsilon)$:

4. [`Self.B.A.A.B == Self.B.A`] (AssocSame 3)

Finally, we apply the SameName inference rule to get $ba(ab \Rightarrow \varepsilon)a$:

5. [`Self.B.A.A.B.A == Self.B.A.A`] (SameName 3 4)

The second rewrite step, $b(\varepsilon \Rightarrow ab)aa$, proves an equivalence between baa and $babaa$. We can reuse some of the above derivation steps to get $b(\varepsilon \Rightarrow ab)$:

6. [`Self.B.A.B == Self.B`] (AssocSame 2)
7. [`Self.B == Self.B.A.B`] (Sym 6)

We then build $b(\varepsilon \Rightarrow ab)a$:

8. [`Self.B.A.B: M`] (AssocConf 3)
9. [`Self.B.A == Self.B.A.B.A`] (SameName 8 7)

One more a, and we get $b(\varepsilon \Rightarrow ab)aa$:

10. [`Self.B.A.B.A: M`] (AssocConf 8)
11. [`Self.B.A.A == Self.B.A.B.A.A`] (SameName 10 9)

Finally, we compose the two rewriting steps with Trans, and we're done:

12. [`Self.B.A.A.B.A == Self.B.A.B.A.A`] (Trans 9 11)

We now prove that for any monoid presentation $\langle A \mid R \rangle$, the corresponding generic signature $G_{\mathtt{M}}$ has the same equivalence class structure as $\sim$:

1. If $x \sim y$, then $\varphi(x)$ and $\varphi(y)$ are equivalent type parameters in $G_{\mathtt{M}}$.
2. If $\varphi(x)$ and $\varphi(y)$ are equivalent type parameters in $G_{\mathtt{M}}$, then $x \sim y$.

Theorem 17.33. Let $\langle A \mid R \rangle$ be a monoid presentation, and let `M` be the protocol that encodes $\langle A \mid R \rangle$. Then $x \sim_R y$ implies $G_{\mathtt{M}} \vdash [\varphi(x) \texttt{ == } \varphi(y)]$ for all $x, y \in A^*$.

Proof. We are given a rewrite path p with $\mathrm{src}(p) = x$ and $\mathrm{dst}(p) = y$, and we must construct a derivation of $G_{\mathtt{M}} \vdash [\varphi(x) \texttt{ == } \varphi(y)]$. We proceed by induction on the length of p.

Base case. We have an empty rewrite path, so $p = 1_t$ for some $t \in A^*$. We construct a derivation of $G_{\mathtt{M}} \vdash \varphi(t)$ using Proposition 17.30, and then apply the Reflex inference rule to derive $[\varphi(t) \texttt{ == } \varphi(t)]$:

1. $\varphi(t)$ $(\ldots)$
2. $[\varphi(t) \texttt{ == } \varphi(t)]$ (Reflex 1)

Inductive step. Suppose that p has length at least 1, so that $p = p' \circ s$ for some shorter rewrite path p' and rewrite step s. Let's say that $s = t(u \Rightarrow v)w$ where $t, w \in A^*$, and (u, v) or $(v, u) \in R$. Let's look at the source and destination of p' and s:

$$\mathrm{src}(p') = x \qquad \mathrm{src}(s) = tuw$$
$$\mathrm{dst}(p') = tuw \qquad \mathrm{dst}(s) = tvw = y$$

By induction, we know that $G_{\mathtt{M}} \vdash [\varphi(x) \texttt{ == } \varphi(tuw)]$. It remains to construct a derivation of $G_{\mathtt{M}} \vdash [\varphi(tuw) \texttt{ == } \varphi(tvw)]$. We do this by first constructing $G_{\mathtt{M}} \vdash [\varphi(tu) \texttt{ == } \varphi(tv)]$.

By Proposition 17.30, we know $G_{\mathtt{M}} \vdash [\varphi(t)\texttt{: M}]$. Now, consider the direction of s:

1. If s is positive, then `M` states an associated same-type requirement $[\varphi(u) \texttt{ == } \varphi(v)]_{\mathtt{M}}$. We apply AssocSame to $[\varphi(t)\texttt{: M}]$, so we get $G_{\mathtt{M}} \vdash [\varphi(tu) \texttt{ == } \varphi(tv)]$.
2. If s is negative, AssocSame gives us $G_{\mathtt{M}} \vdash [\varphi(tv) \texttt{ == } \varphi(tu)]$ instead, so we use the Sym inference rule to get $G_{\mathtt{M}} \vdash [\varphi(tu) \texttt{ == } \varphi(tv)]$.

Next, Proposition 17.30 gives us $G_{\mathtt{M}} \vdash \varphi(w)$. Since we have $G_{\mathtt{M}} \vdash [\varphi(tu) \texttt{ == } \varphi(tv)]$ and $G_{\mathtt{M}} \vdash \varphi(w)$, Lemma 11.7 implies that $G_{\mathtt{M}} \vdash [\varphi(tuw) \texttt{ == } \varphi(tvw)]$.

Finally, we use Trans to derive $G_{\mathtt{M}} \vdash [\varphi(x) \texttt{ == } \varphi(y)]$ (recall that $y = tvw$):

1. $[\varphi(x) \texttt{ == } \varphi(tuw)]$ $(\ldots)$
2. $[\varphi(tuw) \texttt{ == } \varphi(y)]$ $(\ldots)$
3. $[\varphi(x) \texttt{ == } \varphi(y)]$ (Trans 1 2)

Thus, every rewrite path p maps to a derived same-type requirement in $G_{\mathtt{M}}$. □

Theorem 17.34. Let $\langle A \mid R \rangle$ be a monoid presentation, and let `M` be the protocol that encodes $\langle A \mid R \rangle$. Then $G_{\texttt{M}} \vdash [\varphi(x) \texttt{ == } \varphi(y)]$ implies that $x \sim y$ for all x, $y \in A^*$.

Proof. We are given a derivation of $[\varphi(x) \texttt{ == } \varphi(y)]$, and we must show that $x \sim y$ by constructing a rewrite path p with source x and destination y. We proceed by structural induction on derived requirements.

The following derivation steps produce same-type requirements (however, Same does not come up in our case, because $G_{\texttt{M}}$ does not have explicit same-type requirements; all same-type requirements are derived from the requirement signature of `M`):

Same	AssocSame	SameName
Reflex	Sym	Trans

In each case, we must construct a new rewrite path p whose source and destination match the same-type requirement in the conclusion of the step, from the rewrite paths corresponding to each assumption used by the step, if any.

Base case. The base case is when our derivation step does not have any same-type requirement assumptions, but the conclusion is some same-type requirement. There are two such kinds of steps, and in both we construct the rewrite path p directly.

An application of the AssocSame inference rule, when written in the form below, implies that we have a rewrite rule $(u, v) \in R$. We set $p := t(u \Rightarrow v)$:

$$[\varphi(tu) \texttt{ == } \varphi(tv)] \qquad (\text{AssocSame } [\varphi(t)\texttt{: M}])$$

The other base case is a Reflex step, where we set $p := 1_t$:

$$[\varphi(t) \texttt{ == } \varphi(t)] \qquad (\text{Reflex } \varphi(t))$$

Inductive step. We handle Sym by inverting our rewrite path. Induction gives us p' with $\mathrm{src}(p') = t$ and $\mathrm{dst}(p') = u$. We set $p := (p')^{-1}$, because $\mathrm{src}(p) = u$ and $\mathrm{dst}(p) = t$:

$$[\varphi(u) \texttt{ == } \varphi(t)] \qquad (\text{Sym } [\varphi(t) \texttt{ == } \varphi(u)])$$

We handle Trans by composition. Induction gives us p_1, p_2 with $\mathrm{src}(p_1) = t$, $\mathrm{dst}(p_1) = \mathrm{src}(p_2) = u$, and $\mathrm{dst}(p_2) = v$. We set $p := p_1 \circ p_2$, because $\mathrm{src}(p) = t$ and $\mathrm{dst}(p) = v$:

$$[\varphi(t) \texttt{ == } \varphi(v)] \qquad (\text{Trans } [\varphi(t) \texttt{ == } \varphi(u)]\ [\varphi(u) \texttt{ == } \varphi(v)])$$

Finally, we handle SameName by whiskering. Here, $g \in A$ is a generator, corresponding to an associated type declaration in `M`. Induction gives us p' with $\mathrm{src}(p') = t$ and $\mathrm{dst}(p') = u$. We set $p := p' \triangleright g$, because $\mathrm{src}(p) = tg$ and $\mathrm{dst}(p) = ug$:

$$[\varphi(tg) \texttt{ == } \varphi(ug)] \qquad (\text{SameName } [\varphi(u)\texttt{: M}]\ [\varphi(t) \texttt{ == } \varphi(u)])$$

Thus, every derived same-type requirement of $G_{\texttt{M}}$ maps to a rewrite path p. □

In general, the Cayley graph is not preserved by a monoid isomorphism, because it depends on the choice of generating set. However, in our Swift encoding, the Cayley graph manifests directly. We now know that `N` and `Z4` encode the monoids $\{a\}^*$ and $\langle a \,|\, a^4 \sim \varepsilon \rangle$ respectively. We also previously remarked that:

- The Cayley graph of $\{a\}^*$ (Example 17.7) looks like the type parameter graph of $G_{\texttt{N}}$ (Example 8.8).
- The Cayley graph of $\langle a \,|\, a^4 \sim \varepsilon \rangle$ (Example 17.10) looks like the type parameter graph of $G_{\texttt{Z4}}$ (Example 8.9).

This always works, and furthermore the conformance path graph of Section 12.2 coincides with the other two graphs in this case as well:

Theorem 17.35. Let $\langle A \,|\, R \rangle$ be a monoid presentation, and let `M` be the protocol that encodes $\langle A \,|\, R \rangle$. The following three graphs are isomorphic:

1. The Cayley graph of $\langle A \,|\, R \rangle$.
2. The type parameter graph of $G_{\texttt{M}}$.
3. The conformance path graph of $G_{\texttt{M}}$.

Proof. First, consider the vertex set of each graph. We've already established that terms and type parameters have the same equivalence class structure. From this it follows the only difference between the vertex sets is essentially our choice of labels:

1. A vertex in (1) is an equivalence class of terms: $[\![t]\!]$ for some $t \in A^*$.
2. A vertex in (2) is an equivalence class of type parameters: $[\![\varphi(t)]\!]$ for some $t \in A^*$.
3. A vertex in (3) is an equivalence class of abstract conformances. Since every type parameter of $G_{\texttt{M}}$ conforms to exactly one protocol—`M`—every abstract conformance has the form $[\varphi(t)\texttt{: M}]$ for some $t \in A^*$.

The same can be said about the edge sets as well:

1. An edge in (1) joins each $[\![t]\!]$ with $[\![tg]\!]$ for all $a \in A$.
2. An edge in (2) joins each $[\![\varphi(t)]\!]$ with $[\![\varphi(t)\texttt{.A}]\!]$ for all associated types `A` of `M`.
3. An edge in (3) joins each $[[\![\varphi(t)]\!]\texttt{: M}]$ with $[[\![\varphi(t)\texttt{.A}]\!]\texttt{: M}]$ for all associated conformance requirements $[\texttt{Self.A: M}]_{\texttt{M}}$ of `M`.

One final remark. Recall that in the Cayley graph of a monoid presentation $\langle A \,|\, R \rangle$, every vertex has the same number of successors, and we just showed that the same is true of the type parameter graph of $G_{\texttt{M}}$. On the other hand, in an arbitrary generic signature, the number of successors of a vertex will vary, because equivalence classes will conform to various sets of protocols with distinct associated types. The type parameter graph of a "typical" generic signature is almost never the Cayley graph of a monoid. □

17.4. The Word Problem

The previous section's Theorem 17.33 and Theorem 17.34 tell us that we can write a program that type checks if and only if two terms in a finitely-presented monoid are equivalent. The compiler must be able to verify the equivalence, or reject the program if it does not hold.

For example, we can start with our protocol encoding the bicyclic monoid, and also declare a method in a protocol extension:

```
protocol M {
  associatedtype A: M
  associatedtype B: M where Self.A.B == Self
}

extension M {
  static func testBicyclicMonoid() {
    sameType(Self.B.A.A.B.A.self, Self.B.A.B.A.A.self)  // ok
    sameType(Self.A.A.A.self, Self.B.B.B.self)          // error!
  }
}

func sameType<T>(_: T.Type, _: T.Type) {}
```

A call to `sameType()` requires that both arguments are metatypes with the same instance type. For this program to type check, the compiler must verify two statements, corresponding to each call of `sameType()`:

$$G_{\texttt{M}} \vdash [\texttt{Self.B.A.A.B.A == Self.B.A.B.A.A}]$$
$$G_{\texttt{M}} \vdash [\texttt{Self.B.A.A == Self.B.B.B}]$$

The first statement is true, because $baaba \sim babaa$, as we've seen. However, the second statement is actually false, and in fact $baa \not\sim bbb$. The compiler accepts the first call, and diagnoses an error on the second call to `sameType()`, as we would expect.

We can write similar programs to solve word problems in every other finitely-presented monoid we previously introduced, and the Swift compiler will accept them all. We now ask, does this always work, or are there some protocol declarations constructed this way that we cannot accept?

This is the well-known *word problem*:

- **Input:** A monoid presentation $\langle A \,|\, R \rangle$, and two terms x, $y \in A^*$.

  Result: True or false: $x \sim_R y$?

The word problem was perhaps first posed in the early 20th century by Axel Thue, who studied various formalisms defined by the replacement of substrings within larger strings via a fixed set of rules. Thue described what later became known as *Thue systems* and *semi-Thue systems* in a 1914 paper [110]. A Thue system is a finitely-presented monoid, while a semi-Thue system is what we get if we replace the symmetric term equivalence relation with a *reduction relation*, which we will study in the next section.

It seems that Thue's goal was in many ways similar to ours: to analyze a formal system by encoding its statements as terms in a finitely-presented monoid, such that the proof of the statement becomes a sequence of rewrite steps. Questions about the formal system then become instances of the word problem. The final task was to figure out this missing piece, and—voilá—we've solved all of mathematics.

Thue was able to solve the word problem in certain restricted instances. For example, if the rewrite rules preserve term length, then every equivalence class must be finite, and $x \sim y$ can be decided by exhaustive enumeration of $[\![y]\!]$. What eluded Thue though, was a general approach that worked in all cases.

The next twist in this saga came after the development of computability theory. We already sketched out Turing machines and the halting problem in Section 12.4, where we saw that the question of *termination checking* is undecidable in our type substitution algebra. Now, we will see another such undecidable problem. In a 1947 paper [111], Emil Post established that no computable algorithm can solve the word problem. Post did this by defining an encoding of a Turing machine as a finitely-presented monoid. At a very high level, this encoding works as follows:

1. At any point in time, the complete state of the Turing machine, consisting of the current state symbol and the contents of the tape, can be described by a term in the finitely-presented monoid.

2. The single-step transitions, where the machine replaces the symbol at the current head position and moves left or right, define the monoid's rewrite rules.

3. A sequential execution defines a rewrite path. If the machine halts, we know there exists a rewrite path from the initial state to the halting state. Thus, the Turing machine halts if and only if the initial and halting states are equivalent terms.

If we could solve the word problem in an *arbitrary* finitely-presented monoid, we could also solve the halting problem for an arbitrary Turing machine—a contradiction, so no such algorithm exists. Even more is true. Not only is there no "universal" algorithm that takes the monoid presentation as part of its input, but we can construct a *specific* finitely-presented monoid with an undecidable word problem.

One approach is to start with Turing's construction of a *universal machine*. This is a Turing machine that simulates the execution of another Turing machine one step at a time, receiving the standard description of the other Turing machine as input on its tape. These days, we might call this an "emulator" or "interpreter."

By applying Post's construction to a universal Turing machine, we can get a specific monoid presentation with an undecidable word problem, because we can now encode an arbitrary Turing machine as a term in this specific monoid, and then determine if the Turing machine halts by solving the word problem. The "universal Turing machine" monoid has a large number of generators and rewrite rules, though.

A much simpler monoid with an undecidable word problem appeared in a 1958 paper by G. S. Tseitin [112]. For an English translation with commentary, see [113].

Theorem 17.36. This finitely-presented monoid has an undecidable word problem:

$$\begin{aligned}\mathfrak{C}_1 := \langle a, b, c, d, e \,|\, & ac \sim ca,\ ad \sim da,\ bc \sim cb,\ bd \sim db, \\ & eca \sim ce,\ edb \sim de, \\ & cca \sim ccae \rangle\end{aligned}$$

Corollary 17.37. No computable algorithm can decide if two arbitrary type parameters are equivalent in the protocol generic signature $G_{\texttt{C1}}$:

```
protocol C1 {
  associatedtype A: C1
  associatedtype B: C1
  associatedtype C: C1
  associatedtype D: C1
  associatedtype E: C1
    where A.C == C.A, A.D == D.A, B.C == C.B, B.D == D.B,
          E.C.A == C.E, E.D.B == D.E,
          C.C.A == C.C.A.E
}
```

The Swift compiler *rejects* this protocol and diagnoses an error:

```
error: cannot build rewrite system for protocol;
rule length limit exceeded
```

The "rule length" in the error refers not to the rules we wrote down, but the rules generated while constructing a *convergent* presentation for $\mathfrak{C}_1$, something we get to in Section 17.5. If our monoid has an undecidable word problem, this cannot be done, and the procedure for generating such a presentation uses various heuristics to know when to give up. On the author's machine, the Swift compiler rejects protocol `C1` after about 30 milliseconds of processing.

To understand $\mathfrak{C}_1$, we need to introduce a new concept. A *special monoid presentation* is one where the right-hand side of each rewrite rule is ε, and a *special monoid* is one that has such a presentation:

$$\langle a_1, \ldots, a_m \,|\, u_1 \sim \varepsilon,\ \ldots,\ u_n \sim \varepsilon \rangle$$

The bicyclic monoid from Example 17.12 is special. Every group is special, because of the inverse operation; we can always replace a rule $u \sim v$ with $uv^{-1} \sim \varepsilon$.

Tseitin's monoid $\mathfrak{C}_1$ is a *universal machine for solving word problems in special monoids*. If we take a special monoid presentation $\langle A \mid R \rangle$ and a pair of terms $x, y \in A^*$, we can encode $\langle A \mid R \rangle$, x, and y as three terms W, X, and Y over the generators of $\mathfrak{C}_1$. Then, Tseitin's key result is the following:

$$x \sim y \quad \text{in } \langle A \mid R \rangle \qquad \text{if and only if} \qquad WX \sim WY \quad \text{in } \mathfrak{C}_1$$

We will sketch this out by encoding a word problem in the *dihedral group of order 12*, which describes the symmetries of a regular hexagon, as a word problem in $\mathfrak{C}_1$. Think of s as rotation by 60°, and t as reflection by an axis of symmetry:

$$D_{12} := \langle s, t \mid s^6 \sim \varepsilon,\ t^2 \sim \varepsilon,\ (st)^2 \sim \varepsilon \rangle$$

(First step.) We encode the presentation of D_{12} first. We translate each rewrite rule of D_{12} into an element of $\mathfrak{C}_1$, by replacing s with cd and t with cd^2 (if D_{12} had a third generator, it would map to cd^3, and so on):

$$\begin{aligned} s^6 &\mapsto (cd)^6 \\ t^2 &\mapsto (cd^2)^2 \\ (st)^2 &\mapsto (cdcd^2)^2 \end{aligned}$$

(Second step.) We append c at the end of each term above, concatenate them together, and add one more c at the end. We've encoded our *presentation* of D_{12} as an *element* of $\mathfrak{C}_1$. This is our term W:

$$W := \underbrace{(cd)^6}_{s^6} \cdot c \cdot \underbrace{(cd^2)^2}_{t^2} \cdot c \cdot \underbrace{(cdcd^2)^2}_{(st)^2} \cdot c \cdot c$$

(Third step.) Now we pick a pair of terms from D_{12}. Let's take $x := ts$ and $y := s^5t$. A rotation followed by a reflection is the same as a reflection followed by a rotation in the opposite direction, so $ts \sim s^5t$. We again translate these terms into $\mathfrak{C}_1$, except using a and b instead of c and d. This gives us our X and Y:

$$\begin{aligned} X &:= ab^2ab \\ Y &:= (ab)^5ab^2 \end{aligned}$$

Tseitin's rewrite rules define a (rather long) rewrite path from WX to WY.

Our example does not exhibit undecidability, because the word problem in D_{12} is very easy to solve. There are only 12 elements, so we can just write down the Cayley table. However, as with monoids, the word problem for groups is undecidable in the general case [114] (for a more accessible introduction, see [115], or Chapter 12 of [116]). Because $\mathfrak{C}_1$ can encode the word problem in *all* groups, we certainly cannot hope to write down an algorithm to solve the word problem in $\mathfrak{C}_1$.

The big picture. We've shown that every finitely-presented monoid can be encoded as a specific kind of Swift protocol, almost as if Swift's protocols are a generalization of the finitely-presented monoids! This doesn't immediately help us implement Swift generics, though. However, Chapter 18 shows that by taking a suitable alphabet and set of rewrite rules, we can go in the other direction and encode any Swift generic signature as a finitely-presented monoid. While the undecidability of the word problem prevents us from accepting *all* generic signatures, the next section will describe a large class of finitely-presented monoids where the word problem is easy to solve. All reasonable generic signatures fit in with our model.

Closing remarks. In Section 12.4, we showed that the type substitution algebra can encode arbitrary computation, using only recursive conformance requirements [78]. On the other hand, the encoding of a finitely-presented monoid as a protocol needs both recursive conformance requirements and protocol `where` clauses [66].

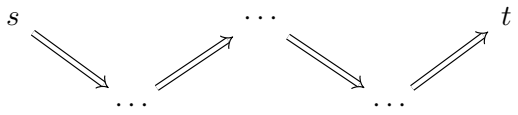
Tseitin's monoid is not a special monoid, so it cannot encode its own word problem. D. J. Collins [117] later discovered a "more" universal word problem interpreter having only a few more rules; this can encode word problems in $\mathfrak{C}_1$ or even itself:

$$\begin{aligned}\mathfrak{U} := \langle a, b, c, d, e_1, e_2, f, g \,|\, & ac \sim ca,\ ad \sim da,\ af \sim fa,\ ag \sim ga,\\ & bc \sim cb,\ bd \sim db,\ bf \sim fb,\ bg \sim gb,\\ & ce_1 \sim ec_1a,\ de_1 \sim e_1db,\\ & e_2c \sim cae_2,\ e_2d \sim dbe_2,\\ & e_1g \sim ge_2,\\ & f \sim fe_1,\ e_2f \sim f\rangle\end{aligned}$$

17.5. The Normal Form Algorithm

There is a "naive" solution to the word problem: we start from our first term s, and attempt different combinations of rewrite steps, until we end up at t. In fact, we can arrange the search so that if such a path does exist, the search always terminates with the appropriate rewrite path.

However, if there is no such path, we will never know when to stop. Recall that a rewrite rule $(u, v) \in R$ is a *symmetric* equivalence. A rewrite path from s to t, if it exists, may contain multiple occurrences of the same rewrite rule, applied in either or both directions over the course of the path. The rewrite graph is infinite, and the intermediate terms visited along our path might be arbitrarily long.



We now change our point of view, and think of a rewrite rule (u, v) as a directed *reduction* from u to v. That is, we're allowed to replace u with v anywhere it appears in a term, but not vice versa. Recall that a rewrite step $x(u \Rightarrow v)y$ is positive if $(u, v) \in R$, and negative if $(v, u) \in R$.

Definition 17.38. A *positive rewrite path* is one where each step taken is positive.

An empty rewrite path is positive, because it contains no steps. Positive rewrite paths are also closed under composition and whiskering, meaning that a) if p_1 and p_2 are positive, then so is $p_1 \circ p_2$; b) if p is positive and $z \in A^*$, so are $z \triangleleft p$ and $p \triangleright z$. (On the other hand, if p is positive, then p^{-1} is positive if and only if p is empty.)

Definition 17.39. Let $\langle A \mid R \rangle$ be a monoid presentation.

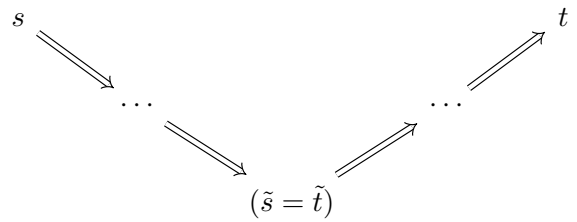
1. The *reduction relation* on A^* generated by R, denoted $s \to_R t$ or $s \to t$ when R is clear from context, relates each pair of terms s and t such that there exists a positive rewrite path with source s and destination t.
2. A term s is *irreducible* or *reduced* if $s \to t$ implies $s = t$ for all $t \in A^*$.
3. A term s *reduces* to t if $s \to t$, and t is irreducible. We also say t is a *normal form* of s, denoted by $\tilde{s} = t$.

We can describe an algorithm that attempts to find normal forms.

Algorithm 17A (Normal form algorithm)**.** As input, takes a term $t \in A^*$ and a list of rewrite rules $R := \{(u_1, v_1), \ldots (u_n, v_n)\}$. As output, returns some normal form of t, which we denote $\tilde{t}$; and a positive rewrite path p, such that $\mathrm{src}(p) = t$ and $\mathrm{dst}(p) = \tilde{t}$.

1. Initialize the return value p with the empty rewrite path 1_t.
2. If $t = xuy$ for some x, $y \in A^*$ and $(u, v) \in R$, append the rewrite step $x(u \Rightarrow v)y$ at the end of p, replace t with xvy, and go back to Step 1.
3. Otherwise, return t and p.

Suppose we are given a pair of terms s, $t \in A^*$ and we wish to decide if $s \sim t$. If our algorithm tells us that both have the same normal form, it means that $s \to \tilde{s}$ via p_s, $t \to \tilde{t}$ via p_t, and also $\tilde{s} = \tilde{t}$. We can then conclude that $s \sim t$, because $p_s \circ p_t^{-1}$ is a rewrite path (no longer positive, unless p_t is empty) with source s and destination t:



Solving the word problem by comparing normal forms is very appealing, but it cannot work in the general case, because we already know the word problem is undecidable. Broadly speaking, two things can go wrong:

1. **Existence:** Our description of the normal form algorithm apparently returns a value in Step 3, so what does it mean for the normal form of some term t to not exist? The trouble is that the algorithm might not *terminate*. In this case, we say that t does not have a normal form.

2. **Uniqueness:** We demonstrated that if $\tilde{s} = \tilde{t}$ for some s, $t \in A^*$, then $s \sim t$. The converse does not always hold, however: a single equivalence class of $\sim$ may contain more than one irreducible term of $\to$. In other words, there may exist s and t such that $s \sim t$, and yet $\tilde{s} \neq \tilde{t}$.

To understand when and how we can overcome these difficulties, we turn to the theory of *string rewriting*. The text by Book and Otto [118] is a classic. Alternatively, instead of introducing string rewriting by way of finitely-presented monoids as we do here, one could instead start with an *abstract* reduction relation operating on "tree-like" expressions, of which strings are a special case. This is the theory of *term rewriting*, and we invite the reader to also read [119] and [120] for a thorough study of this approach.

Existence. Consider the monoid with two elements $\{0, 1\}$ and the binary operation $x \cdot y := \max(x, y)$. (The identity is 0.) Here is one way to present this monoid:

$$\langle a \,|\, a \sim aa \rangle$$

Every non-empty term is *equivalent* to a, but *no* non-empty term has a normal form:

$$a \to aa \to aaa \to aaaa \to \dots$$

The normal form algorithm is quite useless here. However, if we replace our rewrite rule (a, aa) with (aa, a), the normal form algorithm now *always* terminates, with no change to the term equivalence relation. For example:

$$aaaa \to aaa \to aa \to a$$

Now, every non-empty term has the normal form a. In fact, if $\langle A \,|\, R \rangle$ has the property that for each $(u, v) \in R$, either $|u| < |v|$ or $|v| < |u|$, we can always *orient* our rewrite rules so that $|v| < |u|$. We say such rewrite rules are *length-reducing*. The ensures that the intermediate term t becomes shorter on each iteration of Step 2 in Algorithm 17A, after we replace u with v. This can only go on for finitely many steps, so we must always get a normal form.

Limiting ourselves to length-reducing rewrite rules is too restrictive; for example, recall that the free commutative monoid has the *length-preserving* rewrite rule (ba, ab). Thus, we must generalize the idea of comparing term length to establish termination.

Definition 17.40. Let A be a finite alphabet of symbols. A *reduction order* $<$ is a partial order on A^* that is also well-founded (Definition 5.23) and translation-invariant (Definition 17.25).

Definition 17.41. Let $\langle A \mid R \rangle$ be a monoid presentation. A rewrite rule $(u, v) \in R$ is *oriented* with respect to some reduction order $<$, if the right-hand side is smaller with respect to this order, so $u > v$. (Of course, if $<$ is not a linear order, it may happen that we have a *non-orientable* rewrite rule $(u, v) \in R$ where $u \not< v$ and $v \not< u$.)

If we can orient the rewrite rules of our monoid presentation using a reduction order, then the normal form algorithm will always find *some* normal form in a finite number of steps. This solves the first of our two problems:

Theorem 17.42. Let $\langle A \mid R \rangle$ be a monoid presentation. If the rewrite rules of R are oriented with respect to a reduction order $<$, then every equivalence class of $\sim$ contains *at least* one reduced term. We also say that the reduction relation $\to$ is *terminating* or *Noetherian* in this case.

Proof. Assume we have a non-terminating execution of Algorithm 17A, starting with some term t. This means we have an infinite sequence of terms, each one reducing to the next by a positive rewrite step:

$$t \to t_1 \to t_2 \to \cdots \to t_n \to \cdots$$

Now, let $s = x(u \Rightarrow v)y$ be a positive rewrite step. Since $(u, v) \in R$ is oriented, we have $u > v$. Also, $<$ is translation-invariant, so $xuy > xvy$. Thus, $\mathrm{src}(s) > \mathrm{dst}(s)$. This means our infinite reduction sequence gives us an infinite descending chain of terms under our reduction order:

$$t > t_1 > t_2 > \cdots > t_n > \cdots$$

However, this contradicts the assumption that $<$ is well-founded. Thus, no such infinite sequence of terms exists, and the normal form algorithm must terminate. □

If we equip our alphabet of symbols A with a well-founded partial order, we can always define a reduction order on A^*, called the *shortlex order*:

1. Terms of shorter length precede terms of longer length.
2. Otherwise, if two terms have the same length, we compare the symbols pairwise using the order on A.

For example, if $A := \{a, b, c\}$, we might say that $a < b < c$ to define a linear order on A; then we can order various terms of A^* using the shortlex order:

$$a < b < ab < ac < aab$$

Algorithm 17B (Shortlex order)**.** Takes two terms x, $y \in A^*$ as input, and returns one of "$<$", "$>$", "$=$" or "$\perp$" as output.

1. (Shorter) If $|x| < |y|$, return "$<$".
2. (Longer) If $|x| > |y|$, return "$>$".
3. (Initialize) If $|x| = |y|$, we compare elements. Let $i := 0$.
4. (Equal) If $i = |x|$, we didn't find any differences, so $x = y$. Return "$=$".
5. (Subscript) Let x_i and $y_i \in A$ be the ith symbol of x and y, respectively.
6. (Compare) Compare x_i with y_i using the order on A. Return the result if it is "$<$", "$>$", or "$\perp$". Otherwise, we must have $x_i = y_i$, so keep going.
7. (Next) Increment i and go back to Step 4.

The shortlex order will return "$\perp$" if and only if two equal-length terms x and y have an incomparable pair of symbols at the same position, under the partial order on A. In particular, if we have a linear order on A, the shortlex order is always a linear order on A^*. We omit two simple proofs here. It can also be shown that the shortlex order is well-founded using the same argument as in Proposition 5.24. Finally, one can show that the shortlex order is translation-invariant by induction on term length.

Example 17.43. The shortlex order is *not* the same as the lexicographic "dictionary" order on strings. For example, $b < ab$ under the shortlex order, but $ab < b$ under the lexicographic order. The lexicographic order is *not* well-founded, thus it is unsuitable for use as a reduction order:

$$b > ab > aab > aaab > aaaab > \cdots$$

To summarize, we can always guarantee termination by introducing a reduction order. At this point, we pause to observe that we can now apply the normal form algorithm to solve the word problem in our prior examples:

1. The monoid $\langle a \,|\, a^4 \sim \varepsilon \rangle$ from Example 17.10 has a single length-reducing rewrite rule. The normal form algorithm will output one of $\{\varepsilon, a, a^2, a^3\}$.
2. The bicyclic monoid $\langle a, b \,|\, ab \sim \varepsilon \rangle$ from Example 17.12 has a single length-reducing rewrite rule. The normal form algorithm will find the unique term of the form $b^m a^n$ in each equivalence class.
3. The free commutative monoid $\langle a, b \,|\, ba \sim ab \rangle$ from Example 17.11 can be equipped with the shortlex order where $a < b$ to establish termination. The normal form algorithm will find in each equivalence class a unique term of the form $a^m b^n$.

Uniqueness. We now describe monoid presentations with unique normal forms:

Theorem 17.44 (Church-Rosser Theorem)**.** Let $\langle A \mid R \rangle$ be a monoid presentation. The following two conditions are equivalent:

1. (Confluence) For all $t \in A^*$, if $t \to u$ and $t \to v$ for some u, $v \in A^*$, there exists a term $z \in A^*$ such that $u \to z$ and $v \to z$.
2. (Church-Rosser property) For all x, $y \in A^*$ such that $x \sim y$, there exists $z \in A^*$ such that $x \to z$ and $y \to z$.

The Church-Rosser Theorem was discovered by Alonzo Church and John Rosser [121]. The original result was about the lambda calculus, which is a rewriting system whose terms are combinations of variables, function applications, and anonymous functions. Lambda calculus is Turing-complete, so while reduction may not terminate with a given lambda term, it is always confluent, so the Church-Rosser property holds. Haskell Curry (whom the Haskell language was named after) subsequently generalized the theorem away from the lambda calculus in [122]. We use the form of the statement where the terms are elements in a free monoid; a proof can be found in [118].

Example 17.45. Consider the following monoid presentation:

$$\langle a, b, c \mid ab \sim a,\ bc \sim b \rangle$$

The reduction relation is Noetherian, because the rules are length-reducing. However, we will see the reduction relation is *not* confluent, so the Church-Rosser property fails: we can find two irreducible terms in the same equivalence class.

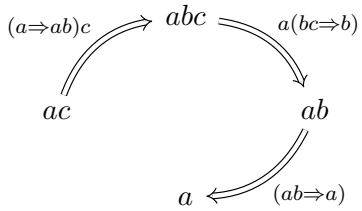
First, observe that both ac and a are irreducible: neither term has any subterms equal to the left hand side of either rule. Now, let's play with the term abc. We can begin by applying either of the two rewrite rules:

1. Applying the first rule to abc gives us ac, which is irreducible.
2. Applying the second rule to abc gives us ab, which reduces to a via the first rule, which is irreducible.

It appears that Algorithm 17A is not deterministic; in Step 2, we have a choice of rewrite rules to apply, and so we output one of two rewrite paths that both end in distinct irreducible terms:

$$p_1 := (ab \Rightarrow a)c$$
$$p_2 := a(bc \Rightarrow b) \circ (ab \Rightarrow a)$$

We've found a *confluence violation*. Note that $\mathrm{src}(p_1) = \mathrm{src}(p_2) = abc$, so $p_1^{-1} \circ p_2$ is a rewrite path from ac to a. Thus $ac \sim a$, and yet, they're distinct and irreducible. If we draw the rewrite path $p_1^{-1} \circ p_2$ as a diagram, we see that no positive rewrite path can take us from ac to a; we must go "over the hump":



Now, let's *add* the rewrite rule (ac, a) to our monoid presentation:

$$\langle a, b, c \,|\, ab \sim a,\ bc \sim b,\ ac \sim a \rangle$$

We *already had* a rewrite path from ac to a, so adding (ac, a) does not change the equivalence relation $\sim$ (we'll see this is a so-called Tietze transformation in Section 19.4). It *does* change $\rightarrow$; in particular, the reduction relation becomes confluent.

In our new presentation, the non-determinism disappears; the normal form algorithm always reduces abc to a, regardless of which rewrite rule we choose to apply first. In fact, every equivalence class now has a unique normal form $c^i b^j a^k$ for i, j, $k \in \mathbb{N}$.

We give a name to this kind of monoid presentation where the normal form algorithm always terminates and outputs a unique reduced term from each equivalence class.

Definition 17.46. A *convergent rewriting system* is a monoid presentation $\langle A \,|\, R \rangle$ where the reduction relation $\rightarrow_R$ is terminating and confluent.

A monoid presented by a convergent rewriting system has a decidable word problem:

Corollary 17.47. Let $\langle A \,|\, R \rangle$ be a convergent rewriting system. Then for all x, $y \in A^*$, we have $x \sim y$ if and only if $\tilde{x} = \tilde{y}$.

We add two final entries to the Rosetta stone we started in Section 17.3:

Swift generics	**F-P monoids**
Type parameter order	Shortlex order
Reduced type parameter	Reduced term

In fact, the type parameter order of Algorithm 5D is basically a specific instance of the shortlex order, except we described it with a recursive algorithm, whereas Algorithm 17B is iterative.

Completion. The process of repairing confluence violations by adding rules is called *completion*. If completion succeeds, we get a convergent rewriting system. Chapter 19 will explain the Knuth-Bendix algorithm that is used for this purpose. This algorithm allows the Swift compiler to accept a protocol `M` that encodes the original presentation of Example 17.45, and then prove the equivalence $G_{\texttt{M}} \vdash$ [`Self.A.C == Self.A`].

Completion is only a *semi-decision procedure* that may fail to terminate; in practice, we impose an iteration limit. This cannot be improved upon, because ultimately, it is undecidable if a monoid can be presented by a convergent rewriting system [123]. A monoid with an undecidable word problem cannot have such a presentation, so completion must fail in that case. Is the converse true? That is, if a finitely-presented monoid is known to have a decidable word problem, will completion always terminate in a finite number of steps and output a convergent rewriting system? The answer is also no:

1. With a fixed presentation $\langle A \,|\, R \rangle$, completion may succeed or fail depending on the choice of reduction order $<$.
2. Two presentations $\langle A \,|\, R \rangle$ and $\langle A' \,|\, R' \rangle$ over different generating sets (alphabets) may present the same monoid up to isomorphism, but completion may succeed with one and fail with the other.
3. Some finitely-presented monoids are known to have a decidable word problem, but no convergent presentation over *any* generating set.

We will see in Section 19.5 that the avoidance of the first two issues motivates several design decisions in the lowering of generic signatures to monoid presentations. The second issue was originally explored by [124], where we learn that the monoid presentation $\langle a, b \,|\, aba \sim bab \rangle$ cannot be extended to a convergent presentation on the same alphabet, but by adding a new symbol c together with the rewrite rule $ab \sim c$, completion will quickly arrive at a convergent presentation.

The third case is more difficult. The below example is from a paper by Craig C. Squier [106]; also, see [125] and [126].

Theorem 17.48. The following finitely-presented monoid has a word problem decidable via a "bespoke" algorithm, but no convergent presentation over any generating set:

$$S_1 := \langle a, b, t, x, y \,|\, ab \sim \varepsilon,\ xa \sim atx,\ xt \sim tx,\ xb \sim bx,\ xy \sim \varepsilon \rangle$$

To prove this theorem, Squier introduced a new combinatorial property of the rewrite graph (the term *derivation graph* is used in the paper), called *finite derivation type*. It is shown that the rewrite graph of a convergent rewriting system has finite derivation type, and that finite derivation type does not depend on the choice of presentation, so it is an invariant of the monoid itself, and not just a particular presentation.

For example, $\langle a, b \,|\, aba \sim bab \rangle$ from above has finite derivation type, because a *different* presentation of the same monoid happens to be convergent.

On the other hand, if it can be shown than some monoid does not have finite derivation type, we can conclude that no convergent presentation exists for this monoid, even if its word problem is decidable by other means. Squier was able to show that while S_1 does not have finite derivation type, it does have a decidable word problem, and an explicit decision procedure is given.

So, each of the below classes is a proper subset of the next, and we shall be content to play in the shallow end of the pool from here on out:

$$\begin{gathered}\text{convergent rewriting system} \\ \subsetneq \\ \text{decidable word problem} \\ \subsetneq \\ \text{all finitely-presented monoids}\end{gathered}$$

More conditions satisfied by monoids presented by convergent rewriting systems were explored in [127], [128], and [129]. For a survey of results about convergent rewriting systems, see [130]. Many open questions remain. Notice how many of the example monoids we saw only had a single rewrite rule. Even with such "one-relation" monoids, the situation is far from trivial. For example, it is not known if every one-relation monoid can be presented by a convergent rewriting system, or more generally, if the word problem for such monoids is decidable or not. However, one-relation monoids are known to have finite derivation type [131]. For a survey of results about one-relation monoids and a good overview of the word problem in general, see [132]. Apart from string rewriting, another approach to solving the word problem is to use finite state automata to describe the structure of a monoid or group [133]. Other interesting undecidable problems in abstract algebra appear in [134].

Type substitution. In fact, we could have formalized the type substitution algebra as a term rewriting system. This would be a rewriting system over tree-structured terms, because the types, substitution maps and conformances in this algebra all recursively nest within one another.

The "evaluation rules" for the $\otimes$ operator would instead define a reduction relation, and the associativity of $\otimes$ would be equivalent to proving that the reduction relation is confluent. (The intrepid reader can quickly review Appendix C to see why this might be the case.) This reduction relation is not terminating, though, because as Section 12.4 demonstrates, it can encode arbitrary computation.

When viewed as a term rewriting system, type substitution is a lot like the lambda calculus—confluent but non-terminating. While this is an interesting observation, we will not pursue this direction further.

* * *

This chapter does not have a **Source Code Reference** section, because we only talked about math.

18. Symbols, Terms, and Rules

OUR MOTIVATION for encoding a finitely-presented monoid as a generic signature was theoretical—it gave us interesting test cases, and showed that there are limits to what we can accept. In comparison, the encoding of a generic signature as a finitely-presented monoid is an eminently practical matter—it solves generic signature queries. As with the derived requirements formalism in Section 5.2, we gradually reveal the full encoding. We begin with the core Swift generics model: unbound type parameters, conformance requirements, and same-type requirements between type parameters. After we prove that the core model is correct, we will extend it to cover bound type parameters, and also layout, superclass, and concrete type requirements.

Core model. Let G be a generic signature. We recall from Section 16.1 that if $\texttt{P}$ is some protocol, then $G \prec \texttt{P}$ means that G depends on $\texttt{P}$. We now introduce a certain monoid presentation $\langle A \,|\, R \rangle$, called the *requirement machine* for G.

We take our "chewy" recursive inputs—type parameters, conformance requirements, and same-type requirements—and cook them until the "connective tissue" breaks down. This yields a finite alphabet A where each symbol is one of the three kinds below:

$\texttt{τ\_d\_i}$	Generic parameter symbol	for some d, $i \in \mathbb{N}$
$\texttt{A}$	Name symbol	for some identifier $\texttt{A}$
$[\texttt{P}]$	Protocol symbol	for some $\texttt{P}$ such that $G \prec \texttt{P}$

To get R, we define a "term" function to translate a type parameter of G into a term, and a "rule" function to translate each explicit requirement of G into a rewrite rule:

$$\begin{aligned} \mathsf{term}(\texttt{τ\_d\_i}) &:= \texttt{τ\_d\_i} \\ \mathsf{term}(\texttt{U.A}) &:= \mathsf{term}(\texttt{U}) \cdot \texttt{A} \end{aligned} \qquad \begin{aligned} \mathsf{rule}\,[\texttt{T: P}] &:= \mathsf{term}(\texttt{T}) \cdot [\texttt{P}] \sim \mathsf{term}(\texttt{T}) \\ \mathsf{rule}\,[\texttt{T == U}] &:= \mathsf{term}(\texttt{T}) \sim \mathsf{term}(\texttt{U}) \end{aligned}$$

In addition, for each protocol $\texttt{P}$ such that $G \prec \texttt{P}$, we define a "$\mathsf{term}_{\texttt{P}}$" function to translate a type parameter of $G_{\texttt{P}}$ into a term starting with $[\texttt{P}]$, and a "$\mathsf{rule}_{\texttt{P}}$" function to translate each associated requirement of $\texttt{P}$ into a rewrite rule, in a manner analogous to "rule":

$$\begin{aligned} \mathsf{term}_{\texttt{P}}(\texttt{Self}) &:= [\texttt{P}] \\ \mathsf{term}_{\texttt{P}}(\texttt{V.A}) &:= \mathsf{term}_{\texttt{P}}(\texttt{V}) \cdot \texttt{A} \end{aligned} \qquad \begin{aligned} \mathsf{rule}_{\texttt{P}}\,[\texttt{U: Q}]_{\texttt{P}} &:= \mathsf{term}_{\texttt{P}}(\texttt{U}) \cdot [\texttt{Q}] \sim \mathsf{term}_{\texttt{P}}(\texttt{U}) \\ \mathsf{rule}_{\texttt{P}}\,[\texttt{U == V}]_{\texttt{P}} &:= \mathsf{term}_{\texttt{P}}(\texttt{U}) \sim \mathsf{term}_{\texttt{P}}(\texttt{V}) \end{aligned}$$

This completely describes a very remarkable object, where the *derived requirements* of G are *rewrite paths* in the requirement machine for G.

Example 18.1. Let G be the generic signature below:

```
<τ_0_0, τ_0_1 where τ_0_0: Sequence, τ_0_1: Sequence,
                    τ_0_0.Element: Equatable,
                    τ_0_0.Element == τ_0_1.Element>
```

We met this signature in Example 5.1 and then studied it in Section 5.2 and Section 5.3. We will now construct its requirement machine.

Our generic signature depends on `Sequence` and `IteratorProtocol`. Both declare an associated type named "`Element`", and `Sequence` also declares an associated type named "`Iterator`". Our alphabet has six symbols—two of each kind:

$$A := \{\underbrace{\texttt{τ\_0\_0}, \texttt{τ\_0\_1}}_{\text{generic param}}, \underbrace{\texttt{Element}, \texttt{Iterator}}_{\text{name}}, \underbrace{[\texttt{Sequence}], [\texttt{IteratorProtocol}]}_{\text{protocol}}\}$$

Here are the requirements of G and `Sequence` that will define our rewrite rules:

$[\texttt{τ\_0\_0: Sequence}]$

$[\texttt{τ\_0\_1: Sequence}]$

$[\texttt{τ\_0\_0.Element: Equatable}]$

$[\texttt{τ\_0\_0.Element == τ\_0\_1.Element}]$

$[\texttt{Self.Iterator: IteratorProtocol}]_{\texttt{Sequence}}$

$[\texttt{Self.Element == Self.Iterator.Element}]_{\texttt{Sequence}}$

We build our set of rewrite rules R by applying "rule" or "rule_P" to each requirement:

$$\texttt{τ\_0\_0} \cdot [\texttt{Sequence}] \sim \texttt{τ\_0\_0} \tag{1}$$

$$\texttt{τ\_0\_1} \cdot [\texttt{Sequence}] \sim \texttt{τ\_0\_1} \tag{2}$$

$$\texttt{τ\_0\_0} \cdot \texttt{Element} \cdot [\texttt{Equatable}] \sim \texttt{τ\_0\_0} \cdot \texttt{Element} \tag{3}$$

$$\texttt{τ\_0\_0} \cdot \texttt{Element} \sim \texttt{τ\_0\_1} \cdot \texttt{Element} \tag{4}$$

$$[\texttt{Sequence}] \cdot \texttt{Iterator} \cdot [\texttt{IteratorProtocol}] \sim [\texttt{Sequence}] \cdot \texttt{Iterator} \tag{5}$$

$$[\texttt{Sequence}] \cdot \texttt{Element} \sim [\texttt{Sequence}] \cdot \texttt{Iterator} \cdot \texttt{Element} \tag{6}$$

Note that "term" is just a flattening of the linked list structure of a type parameter into an array; the resulting term is a generic parameter symbol followed by zero or more name symbols. To distinguish the two representations, we separate the components of a type parameter with "`.`" as before, while terms join their symbols with the requirement machine monoid operation " $\cdot$ ". The protocol variants "term_P" and "rule_P" are needed to encode the original protocol of each associated requirement. This is why "term_P" replaces `Self` with the protocol symbol [P] instead of the generic parameter symbol `τ_0_0`.

One more remark. Name symbols and protocol symbols exist in different namespaces, so even if we renamed `IteratorProtocol` to `Iterator`, the name symbol `Iterator` would remain distinct from the protocol symbol [`Iterator`]. On the other hand, two associated types with the same name in different protocols always define the *same* name symbol.

Now, let's continue with our example and think about the equivalence relation these rewrite rules generate. Notice how there is a certain symmetry behind the appearance of protocol symbols. The conformance requirements (1), (2), (3) and (5) rewrite a term that *ends* in a protocol symbol. The associated requirements (5) and (6) rewrite a term that *begins* with a protocol symbol. (The associated conformance requirement (5) is both; its left-hand side begins and ends with a protocol symbol.)

We know that $G \vdash$ [`τ_0_0.Iterator.Element == τ_0_1.Iterator.Element`] from Example 5.9, even though this requirement is not explicitly stated. Applied to this requirement, "rule" outputs this ordered pair of terms:

$$(\texttt{τ\_0\_0} \cdot \texttt{Iterator} \cdot \texttt{Element}, \texttt{τ\_0\_1} \cdot \texttt{Iterator} \cdot \texttt{Element})$$

Again, this is not an explicit requirement, so this is not one of the rules in R. However, there is a rewrite path from the first term to the second; they're equivalent via $\sim_R$:

$$\begin{aligned}
&(\texttt{τ_0_0} \Rightarrow \texttt{τ_0_0} \cdot [\texttt{Sequence}]) \triangleright \texttt{Iterator} \cdot \texttt{Element} \\
&\circ \texttt{τ_0_0} \triangleleft ([\texttt{Sequence}] \cdot \texttt{Iterator} \cdot \texttt{Element} \Rightarrow [\texttt{Sequence}] \cdot \texttt{Element}) \\
&\circ (\texttt{τ_0_0} \cdot [\texttt{Sequence}] \Rightarrow \texttt{τ_0_0}) \triangleright \texttt{Element} \\
&\circ (\texttt{τ_0_0} \cdot \texttt{Element} \Rightarrow \texttt{τ_0_1} \cdot \texttt{Element}) \\
&\circ (\texttt{τ_0_1} \Rightarrow \texttt{τ_0_1} \cdot [\texttt{Sequence}]) \triangleright \texttt{Element} \\
&\circ \texttt{τ_0_1} \triangleleft ([\texttt{Sequence}] \cdot \texttt{Element} \Rightarrow [\texttt{Sequence}] \cdot \texttt{Iterator} \cdot \texttt{Element}) \\
&\circ (\texttt{τ_0_1} \cdot [\texttt{Sequence}] \Rightarrow \texttt{τ_0_1}) \triangleright \texttt{Iterator} \cdot \texttt{Element}
\end{aligned}$$

We can also understand this by writing down the intermediate terms between every consecutive pair of rewrite steps. We use conformance requirements to insert and delete protocol symbols before and after invoking our associated same-type requirement:

$$\begin{aligned}
&\texttt{τ_0_0} \cdot \texttt{Iterator} \cdot \texttt{Element} \\
&\sim \texttt{τ_0_0} \cdot [\texttt{Sequence}] \cdot \texttt{Iterator} \cdot \texttt{Element} \\
&\sim \texttt{τ_0_0} \cdot [\texttt{Sequence}] \cdot \texttt{Element} \\
&\sim \texttt{τ_0_0} \cdot \texttt{Element} \\
&\sim \texttt{τ_0_1} \cdot \texttt{Element} \\
&\sim \texttt{τ_0_1} \cdot [\texttt{Sequence}] \cdot \texttt{Element} \\
&\sim \texttt{τ_0_1} \cdot [\texttt{Sequence}] \cdot \texttt{Iterator} \cdot \texttt{Element} \\
&\sim \texttt{τ_0_1} \cdot \texttt{Iterator} \cdot \texttt{Element}
\end{aligned}$$

Now let's try with $G \vdash [\texttt{τ\_0\_0.Iterator: IteratorProtocol}]$. We apply "rule" to get an ordered pair of terms:

$$(\texttt{τ\_0\_0} \cdot \texttt{Iterator} \cdot [\texttt{IteratorProtocol}], \texttt{τ\_0\_0} \cdot \texttt{Iterator})$$

To prove this equivalence, we make use of the conformance requirement (1), and then the associated conformance requirement (5):

$$\begin{aligned}&(\texttt{τ\_0\_0} \Rightarrow \texttt{τ\_0\_0} \cdot [\texttt{Sequence}]) \triangleright \texttt{Iterator} \cdot [\texttt{IteratorProtocol}] \\ &\circ \texttt{τ\_0\_0} \triangleleft ([\texttt{Sequence}] \cdot \texttt{Iterator} \cdot [\texttt{IteratorProtocol}] \Rightarrow [\texttt{Sequence}] \cdot \texttt{Iterator}) \\ &\circ (\texttt{τ\_0\_0} \cdot [\texttt{Sequence}] \Rightarrow \texttt{τ\_0\_0}) \triangleright \texttt{Iterator}\end{aligned}$$

Once again, we can show the intermediate terms in our path:

$$\begin{aligned}&\texttt{τ\_0\_0} \cdot \texttt{Iterator} \cdot [\texttt{IteratorProtocol}] \\ &\sim \texttt{τ\_0\_0} \cdot [\texttt{Sequence}] \cdot \texttt{Iterator} \cdot [\texttt{IteratorProtocol}] \\ &\sim \texttt{τ\_0\_0} \cdot [\texttt{Sequence}] \cdot \texttt{Iterator} \\ &\sim \texttt{τ\_0\_0} \cdot \texttt{Iterator}\end{aligned}$$

18.1. Correctness

We saw some examples of encoding derived requirements as word problems. We now make this precise. We prove two theorems, to establish that we can translate derivations into rewrite paths, and vice versa. We use the algebra of rewriting from Section 17.2.

We begin with a preliminary result. In Section 5.2, we defined the AssocConf and AssocSame inference rules via formal substitution: if `T` is some type parameter of G known to conform to some protocol `P`, and `Self.U` is some type parameter of $G_{\texttt{P}}$, then `T.U` denotes the replacement of `Self` with `T` in `Self.U`. We can relate this to the requirement machine monoid operation.

Lemma 18.2. Let G be a generic signature, and let $\langle A \mid R \rangle$ be the requirement machine for G. Further, suppose that:

1. `T` is some type parameter of G,
2. `P` is some protocol, and $\mathsf{term}(\texttt{T}) \cdot [\texttt{P}] \sim \mathsf{term}(\texttt{T})$,
3. `Self.U` is some type parameter of $G_{\texttt{P}}$.

Then $\mathsf{term}(\texttt{T}) \cdot \mathsf{term}_{\texttt{P}}(\texttt{Self.U}) \sim \mathsf{term}(\texttt{T.U})$.

Proof. Note that $\mathsf{term}_{\texttt{P}}(\texttt{Self.U}) = [\texttt{P}] \cdot u$ for some $u \in A^*$, by definition of "$\mathsf{term}_{\texttt{P}}$". We also let $t := \mathsf{term}(\texttt{T})$, so that $\mathsf{term}(\texttt{T.U}) = t \cdot u$. Now, if p is a rewrite path from $t \cdot [\texttt{P}]$ to t, then $p \triangleright u$ is a rewrite path from $t \cdot [\texttt{P}] \cdot u$ to $t \cdot u$. □

Example 18.3. In Example 18.1, we can see the second rewrite path in a new way:

$$\begin{aligned}
&\mathsf{term}(\texttt{τ_0_0.Iterator}) \cdot [\texttt{IteratorProtocol}] \\
&\sim \mathsf{term}(\texttt{τ_0_0}) \cdot \mathsf{term}_{\texttt{Sequence}}(\texttt{Self.Iterator}) \cdot [\texttt{IteratorProtocol}] \\
&\sim \mathsf{term}(\texttt{τ_0_0}) \cdot \mathsf{term}_{\texttt{Sequence}}(\texttt{Self.Iterator}) \\
&\sim \mathsf{term}(\texttt{τ_0_0.Iterator})
\end{aligned}$$

This idea of "cleaving" a term in two will be key in the inductive step of the below proof, when we consider the AssocConf and AssocSame inference rules.

Theorem 18.4. Let G be a generic signature, and let $\langle A \mid R \rangle$ be the requirement machine for G.

1. If $G \vdash [\texttt{T: P}]$ for some type parameter $\texttt{T}$ and protocol $\texttt{P}$, then $\mathsf{term}(\texttt{T}) \cdot [\texttt{P}] \sim \mathsf{term}(\texttt{T})$.
2. If $G \vdash [\texttt{T == U}]$ for some pair of type parameters $\texttt{T}$ and $\texttt{U}$, then $\mathsf{term}(\texttt{T}) \sim \mathsf{term}(\texttt{U})$.

Proof. We use structural induction on derived requirements (Section 11.3) to define a "path" mapping that assigns to each derived requirement $G \vdash D$ a rewrite path p such that $\mathsf{rule}(D) = (\mathrm{src}(p), \mathrm{dst}(p))$. The conclusion then follows.

Base case. To handle the elementary statements, and those inference rules that do not have any requirements among their assumptions, we construct a rewrite path directly.

A Conf step invokes an explicit conformance requirement of G:

$$[\texttt{T: P}] \qquad \text{(Conf)}$$

Note that $\mathsf{rule}\,[\texttt{T: P}] = (\mathsf{term}(\texttt{T}) \cdot [\texttt{P}], \mathsf{term}(\texttt{T})) \in R$. The path is a single rewrite step:

$$\mathsf{path}\,[\texttt{T: P}] := (\mathsf{term}(\texttt{T}) \cdot [\texttt{P}] \Rightarrow \mathsf{term}(\texttt{T}))$$

A Same step invokes an explicit same-type requirement of G:

$$[\texttt{T == U}] \qquad \text{(Same)}$$

Note that $\mathsf{rule}\,[\texttt{T == U}] = (\mathsf{term}(\texttt{T}), \mathsf{term}(\texttt{U})) \in R$. The path is a single rewrite step:

$$\mathsf{path}\,[\texttt{T == U}] := (\mathsf{term}(\texttt{T}) \Rightarrow \mathsf{term}(\texttt{U}))$$

A Reflex step derives a trivial same-type requirement from a valid type parameter:

$$[\texttt{T == T}] \qquad \text{(Reflex T)}$$

We don't need the fact that $G \vdash \texttt{T}$ at all, because in a finitely-presented monoid, every term is already equivalent to itself via the empty rewrite path, so we let $t := \mathsf{term}(\texttt{T})$, and we set:

$$\mathsf{path}\,[\texttt{T == T}] := 1_t$$

Inductive step. In each case, the "path" of the conclusion is defined from the "path" of the step's assumptions. For an AssocConf step, the induction hypothesis gives us $p_1 := \mathsf{path}\,[\texttt{T: P}]$, so $\mathrm{src}(p_1) = \mathsf{term}(\texttt{T}) \cdot [\texttt{P}]$ and $\mathrm{dst}(p_1) = \mathsf{term}(\texttt{T})$:

$$[\texttt{T.U: Q}] \qquad (\textsc{AssocConf}\ [\texttt{Self.U: Q}]_{\texttt{P}}\ [\texttt{T: P}])$$

Note that $\mathsf{rule}_{\texttt{P}}\,[\texttt{Self.U: Q}]_{\texttt{P}} \in R$ is of the form $([\texttt{P}] \cdot u \cdot [\texttt{Q}], [\texttt{P}] \cdot u)$ for some $u \in A^*$. Using Lemma 18.2, we insert a $[\texttt{P}]$ in the right spot, apply our associated requirement, and then delete the $[\texttt{P}]$. We let $t := \mathsf{term}(\texttt{T})$, let $s := ([\texttt{P}] \cdot u \cdot [\texttt{Q}] \Rightarrow [\texttt{P}] \cdot u)$, and we set:

$$\mathsf{path}\,[\texttt{T.U: Q}] := (p_1^{-1} \triangleright u \triangleright [\texttt{Q}]) \circ (t \triangleleft s) \circ (p_1 \triangleright u)$$

For an AssocSame step, the induction hypothesis gives us $p_1 := \mathsf{path}\,[\texttt{T: P}]$:

$$[\texttt{T.U == T.V}] \qquad (\textsc{AssocSame}\ [\texttt{Self.U == Self.V}]_{\texttt{P}}\ [\texttt{T: P}])$$

Note that $\mathsf{rule}_{\texttt{P}}\,[\texttt{Self.U == Self.V}]_{\texttt{P}} \in R$ is of the form $([\texttt{P}] \cdot u, [\texttt{P}] \cdot v)$ for some $u, v \in A^*$. Again, we use Lemma 18.2. We let $t := \mathsf{term}(\texttt{T})$, let $s := ([\texttt{P}] \cdot u \Rightarrow [\texttt{P}] \cdot v)$, and we set:

$$\mathsf{path}\,[\texttt{T.U == T.V}] := (p_1^{-1} \triangleright u) \circ (t \triangleleft s) \circ (p_1 \triangleright v)$$

For a Sym or Trans step, we use the rewrite path inverse and composition operations:

$$[\texttt{U == T}] \qquad (\textsc{Sym}\ [\texttt{T == U}])$$
$$[\texttt{T == V}] \qquad (\textsc{Trans}\ [\texttt{T == U}]\ [\texttt{U == V}])$$

$$\mathsf{path}\,[\texttt{U == T}] := \mathsf{path}\,[\texttt{T == U}]^{-1}$$
$$\mathsf{path}\,[\texttt{T == V}] := \mathsf{path}\,[\texttt{T == U}] \circ \mathsf{path}\,[\texttt{U == V}]$$

For a SameConf step, the induction hypothesis gives us a pair of rewrite paths $p_1 := \mathsf{path}\,[\texttt{U: P}]$, and $p_2 := \mathsf{path}\,[\texttt{T == U}]$:

$$[\texttt{T: P}] \qquad (\textsc{SameConf}\ [\texttt{U: P}]\ [\texttt{T == U}])$$

We rewrite $\mathsf{term}(\texttt{T}) \cdot [\texttt{P}]$ into $\mathsf{term}(\texttt{U}) \cdot [\texttt{P}]$ using p_2, delete $[\texttt{P}]$ from the end of the term using p_1, and rewrite $\mathsf{term}(\texttt{U})$ back to $\mathsf{term}(\texttt{T})$ using p_2 again:

$$\mathsf{path}\,[\texttt{T: P}] := (p_2 \triangleright [\texttt{P}]) \circ p_1 \circ p_2^{-1}$$

For a SameName step, the induction hypothesis gives us $p_1 := \mathsf{path}\,[\texttt{U: P}]$, and $p_2 := [\texttt{T == U}]$:

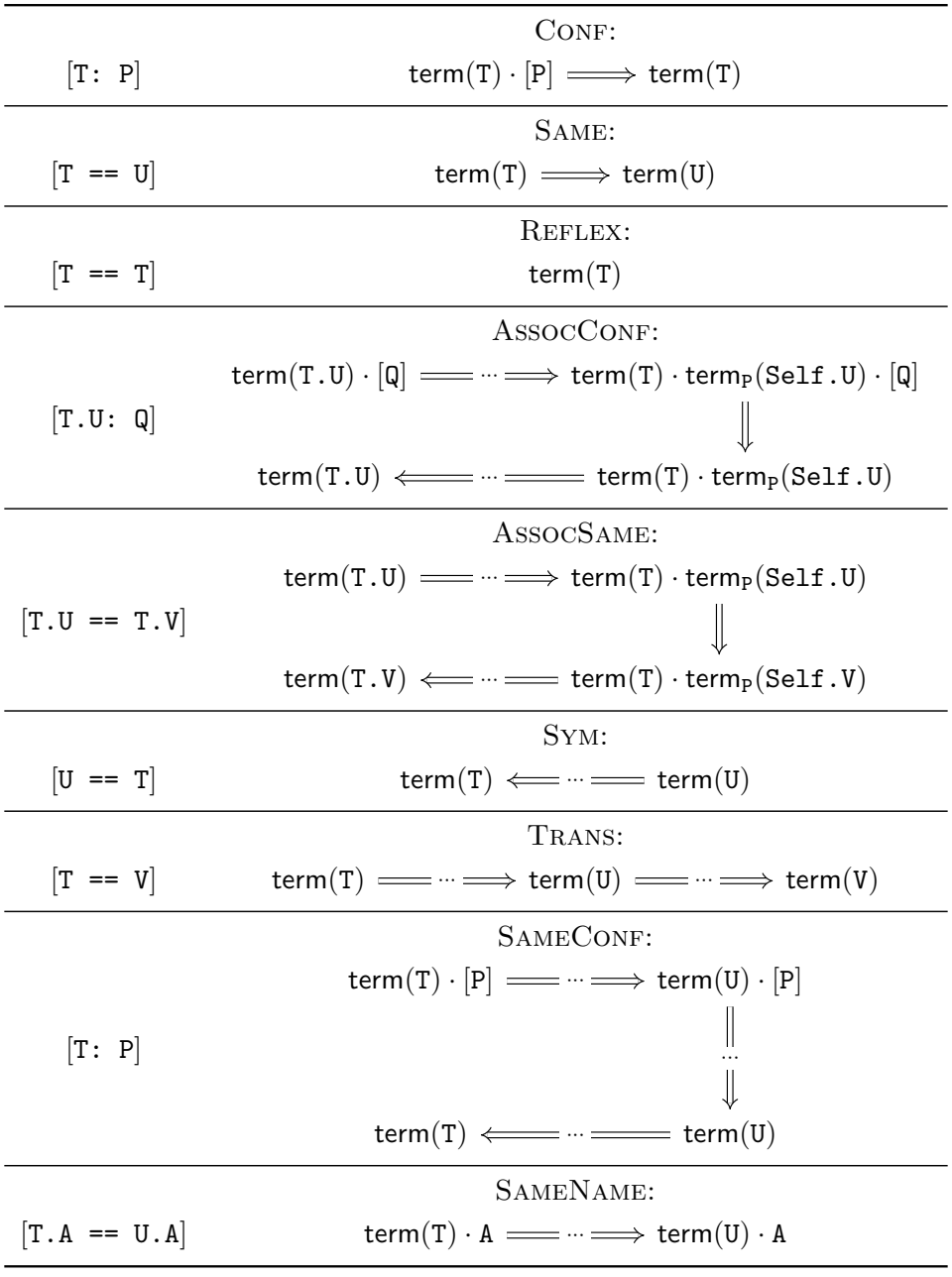
$$[\texttt{T.A == U.A}] \qquad (\textsc{SameName}\ [\texttt{U: P}]\ [\texttt{T == U}])$$

We only need p_2 to construct our new rewrite path, by whiskering:

$$\mathsf{path}\,[\texttt{T.A == U.A}] := p_2 \triangleright \texttt{A}$$

By induction, this completely defines "path", so we apply it to our original requirement and get the desired result. Figure 18.1 shows each rewrite path we constructed. □

Figure 18.1.: Rewrite paths defined by Theorem 18.4, as paths in the rewrite graph



We now go in the other direction, and describe how rewrite paths in the requirement machine define derived requirements in our generic signature. Consider how we might reverse our "path" mapping. The main difficulty to overcome is the fact that we did not actually use every assumption in each step:

1. The "path" of a Reflex step [T == T] disregards the proof of the validity of the type parameter T, because in the monoid, we can immediately construct an empty rewrite path for *any* term.

   However, this means that if we start from the empty rewrite path at a "nonsensical" term, such as $\mathtt{A} \cdot \mathtt{τ\_0\_1} \cdot [\mathtt{Sequence}] \cdot \mathtt{τ\_0\_0}$, we cannot "go backwards" and hope to prove the validity of any type parameter in G.

2. The "path" of a SameName step [T.A == U.A] is defined from path [T == U] by whiskering, ignoring path [T: P], because we can whisker *any* rewrite path.

   That is, if we have an equivalent pair of terms $\mathsf{term}(\mathtt{T}) \sim \mathsf{term}(\mathtt{U})$, and A is any name symbol, it is *always* the case that $\mathsf{term}(\mathtt{T}) \cdot \mathtt{A} \sim \mathsf{term}(\mathtt{U}) \cdot \mathtt{A}$ in the monoid.

   However, we may not be able to derive $G \vdash [\mathtt{T.A == U.A}]$, because T might not conform any protocol with an associated type named "A". By the same reasoning, we might not even be able to derive $G \vdash [\mathtt{T == U}]$.

Thus, not every equivalence class of $\sim_R$ corresponds to a valid type parameter of our generic signature G. Before we can see how to translate a specific rewrite path into a derivation, we need to understand which rewrite paths are meaningful to us. We begin by describing the equivalence class of $\mathsf{term}(\mathtt{T})$ for a type parameter T.

Definition 18.5. Let $t \in A^*$ be a term.

- If the first symbol of t is a generic parameter symbol and all subsequent symbols are name symbols, then we say that t is a *standard* term.
- If the first symbol of t is a generic parameter symbol and all subsequent symbols are either name or protocol symbols, we say that t is an *admissible* term.

Lemma 18.6. Some consequences of the above:

1. Every standard term is $\mathsf{term}(\mathtt{T})$ for some unbound type parameter T.
2. Every standard term is an admissible term.
3. If (u, v) is a rewrite rule output by "rule", both u and v are admissible terms.
4. If $t \in A^*$ is an admissible term and $u \in A^*$ is a combination of name and protocol symbols (possibly empty), then tu is an admissible term.

The next lemma states that admissibility is preserved by rewriting. In particular, this means that rewriting a standard term always outputs an admissible term.

Lemma 18.7. Let G be a generic signature, and let $\langle A \mid R \rangle$ be the requirement machine for G. Let $t \in A^*$ be an admissible term. If $t \sim z$ for some other term $z \in A^*$, then z is an admissible term.

Proof. Let p be a rewrite path from t to z. We argue by induction on the length of p.

Base case. If we have an empty rewrite path, then $t = z$, so z is an admissible term.

Inductive step. Otherwise, $p = p' \circ s$ for some rewrite path p' and rewrite step s. We write $s := x(u \Rightarrow v)y$, where x, $y \in A^*$, and one of (u, v) or $(v, u) \in R$.

By the induction hypothesis, $\mathrm{dst}(p') = \mathrm{src}(s) = xuy$ is an admissible term. Because the prefix xu is non-empty, the right whisker y contains only name and protocol symbols.

If the rule applied by s is the "rule" for an explicit requirement, then the left whisker x is empty, because u starts with a generic parameter symbol. Furthermore, v is an admissible term. We see that $\mathrm{dst}(s) = vy$ is an admissible term.

If this rule applied by s is the "$\mathsf{rule_P}$" for an associated requirement, then the left whisker x is non-empty, and furthermore an admissible term, and u and v must contain only name and protocol symbols. We see that $\mathrm{dst}(s) = xvy$ is an admissible term. □

Just like a standard term describes an unbound type parameter, an admissible term more generally describes a type parameter and *a list of conformance requirements*.

Definition 18.8. The "type" function maps each admissible term to an unbound type parameter, by ignoring protocol symbols, and translating generic parameter symbols and name symbols into generic parameter types and dependent member types:

$$\begin{aligned}
&\mathsf{type}(\texttt{τ_d_i}) &&:= \texttt{τ_d_i}\\
&\mathsf{type}(\texttt{U} \cdot \texttt{A}) &&:= \mathsf{type}(\texttt{U}).\texttt{A}\\
&\mathsf{type}(\texttt{U} \cdot [\texttt{P}]) &&:= \mathsf{type}(\texttt{U})
\end{aligned}$$

Definition 18.9. Let $z \in A^*$ be an admissible term. The *conformance chart* of z is a *multiset* (that is, we allow duplicates) of conformance requirements:

1. Each requirement corresponds to an occurrence of a protocol symbol in z.
2. The subject type of each requirement is the prefix before the protocol symbol.
3. Duplicate elements represent identical consecutive protocol symbols.

Definition 18.10. The "chart" function maps each admissible term to its conformance chart, by ignoring generic parameter and name symbols, and converting protocol symbols into requirements:

$$\begin{aligned}
&\mathsf{chart}(\texttt{τ_d_i}) &&:= \{\}\\
&\mathsf{chart}(\texttt{U} \cdot \texttt{A}) &&:= \mathsf{chart}(\texttt{U})\\
&\mathsf{chart}(\texttt{U} \cdot [\texttt{P}]) &&:= \mathsf{chart}(\texttt{U}) \cup \{[\mathsf{type}(\texttt{U})\texttt{: P}]\}
\end{aligned}$$

Example 18.11. If $z = \mathsf{term}(\texttt{T})$ is a standard term:

$$\mathsf{type}(z) = \texttt{T}$$
$$\mathsf{chart}(z) = \{\}$$

If $z = \mathsf{term}(\texttt{T}) \cdot [\texttt{P}]$ for some unbound type parameter `T` and protocol `P`:

$$\mathsf{type}(z) = \texttt{T}$$
$$\mathsf{chart}(z) = \{[\texttt{T: P}]\}$$

If $z = \texttt{τ\_0\_1} \cdot [\texttt{Sequence}] \cdot [\texttt{Sequence}] \cdot \texttt{Iterator} \cdot [\texttt{IteratorProtocol}]$,

$$\mathsf{type}(z) = \texttt{τ\_0\_1.Iterator},$$
$$\mathsf{chart}(z) = \{[\texttt{τ\_0\_1: Sequence}],$$
$$[\texttt{τ\_0\_1: Sequence}],$$
$$[\texttt{τ\_0\_1.Iterator: IteratorProtocol}]\}.$$

We can say a lot more if we add the assumptions that `T` is a *valid* type parameter, and our generic signature G is well-formed. (We lose nothing by doing this, because otherwise we diagnose an error and reject the program; see Section 11.3). Having done so, we see that rewriting a standard term into an admissible term is tantamount to simultaneously deriving a same-type requirement *together with every conformance requirement in the destination term's chart*.

Theorem 18.12. Let G be a well-formed generic signature, let $\langle A \mid R \rangle$ be the requirement machine for G, and let t be a standard term such that $\mathsf{type}(t)$ is a valid type parameter of G. Suppose we are given a term $z \in A^*$ such that $t \sim z$. Then z is an admissible term, and furthermore:

1. $G \vdash [\mathsf{type}(t) \texttt{ == } \mathsf{type}(z)]$.
2. $G \vdash [\mathsf{type}(u) \texttt{: P}]$ for each $[\mathsf{type}(u) \texttt{: P}] \in \mathsf{chart}(z)$.

Proof. Lemma 18.7 already shows that z is an admissible term. To establish the principal conclusion, we reason as follows:

1. Because we started with a standard term, we know that every protocol symbol appearing in z must have been introduced by some rewrite step in our path.

   When a rewrite step introduces a protocol symbol, we must be able to derive this conformance requirement, and record it alongside our chart.

2. The other rewrite steps apply same-type requirements, and we must be able to build a derived same-type requirement corresponding to this transformation.

   We must also understand the effect that same-type requirement rules have on the conformance chart, so that we can update our collected set of derived conformance requirements when needed.

Let p be a rewrite path from t to z. We argue by induction on the length of p.

Base case. If p is empty, then $t = z$, so $\mathsf{type}(t) = \mathsf{type}(z)$, and $\mathsf{chart}(z) = \{\}$. We use the assumption that $\mathsf{type}(t)$ is a valid type parameter to derive the same-type requirement $[\mathsf{type}(t)\ \texttt{==}\ \mathsf{type}(t)]$, by applying the Reflex inference rule to a derivation of $G \vdash \mathsf{type}(t)$. We do not need to derive any conformance requirements.

Inductive step. Otherwise, $p = p' \circ s$ for some rewrite path p' and rewrite step s. Let $z' := \mathrm{dst}(p') = \mathrm{src}(s)$, and note that $z = \mathrm{dst}(s)$. By the induction hypothesis:

1. $G \vdash [\mathsf{type}(t)\ \texttt{==}\ \mathsf{type}(z')]$.
2. $G \vdash [\mathsf{type}(u')\texttt{:}\ \texttt{P}]$ for each $[\mathsf{type}(u')\texttt{:}\ \texttt{P}] \in \mathsf{chart}(z')$.

The rewrite step s applies a conformance or same-type requirement; this requirement is explicit ("rule") or associated ("$\mathsf{rule}_\texttt{P}$"); and the rewrite step may be positive or negative. We consider each case to derive the desired conclusion about z:

	Kind:	**Domain:**	**Sign:**	**General Form:**
1.	conformance	explicit	$+$	$(u \cdot [\texttt{P}] \Rightarrow u) \triangleright y$
2.		explicit	$-$	$(u \Rightarrow u \cdot [\texttt{P}]) \triangleright y$
3.		associated	$+$	$x \triangleleft ([\texttt{P}] \cdot u \cdot [\texttt{Q}] \Rightarrow [\texttt{P}] \cdot u) \triangleright y$
4.		associated	$-$	$x \triangleleft ([\texttt{P}] \cdot u \Rightarrow [\texttt{P}] \cdot u \cdot [\texttt{Q}]) \triangleright y$
5.	same-type	explicit	$+$	$(u \Rightarrow v) \triangleright y$
6.		explicit	$-$	$(v \Rightarrow u) \triangleright y$
7.		associated	$+$	$x \triangleleft ([\texttt{P}] \cdot u \Rightarrow [\texttt{P}] \cdot v) \triangleright y$
8.		associated	$-$	$x \triangleleft ([\texttt{P}] \cdot v \Rightarrow [\texttt{P}] \cdot u) \triangleright y$

(★ Case 1.) A positive rewrite step for an explicit conformance requirement has the following general form, where the left whisker is empty, $(u \cdot [\texttt{P}], u) \in R$ for some standard term u and protocol symbol $[\texttt{P}]$, and the right whisker y is a combination of name and protocol symbols:

$$(u \cdot [\texttt{P}] \Rightarrow u) \triangleright y$$

The rewrite step deletes an occurrence of the protocol symbol $[\texttt{P}]$:

$$z' = \mathrm{src}(s) = u \cdot [\texttt{P}] \cdot y$$
$$z = \mathrm{dst}(s) = u \cdot y$$

$$\mathsf{type}(z) = \mathsf{type}(z')$$
$$\mathsf{chart}(z) = \mathsf{chart}(z') \setminus \{[\mathsf{type}(u)\texttt{:}\ \texttt{P}]\}$$

Therefore our conclusion already holds; we just "discard" our derivation of $[\mathsf{type}(u)\texttt{:}\ \texttt{P}]$.

(★ Case 2.) A negative rewrite step for an explicit conformance requirement has the following general form, where again the left whisker is empty, $(u \cdot [\mathtt{P}], u) \in R$ for some standard term u and protocol symbol $[\mathtt{P}]$, and the right whisker y is as before:

$$(u \Rightarrow u \cdot [\mathtt{P}]) \triangleright y$$

The rewrite step inserts an occurrence of the protocol symbol $[\mathtt{P}]$:

$$z' = \mathrm{src}(s) = u \cdot y$$
$$z = \mathrm{dst}(s) = u \cdot [\mathtt{P}] \cdot y$$

$$\mathsf{type}(z) = \mathsf{type}(z')$$
$$\mathsf{chart}(z) = \mathsf{chart}(z') \cup \{[\mathsf{type}(u)\colon\ \mathtt{P}]\}$$

To establish the conclusion, we must derive $[\mathsf{type}(u)\colon\ \mathtt{P}]$. We do this with a Conf elementary statement, because $\mathsf{rule}\,[\mathsf{type}(u)\colon\ \mathtt{P}] = (u \cdot [\mathtt{P}], u) \in R$:

1. $[\mathsf{type}(u)\colon\ \mathtt{P}]$ (Conf)

(★ Case 3.) A positive rewrite step for an associated conformance requirement has the following general form, where x is an admissible term, $([\mathtt{P}] \cdot u \cdot [\mathtt{Q}], [\mathtt{P}] \cdot u) \in R$ for some u that contains only name symbols, and the right whisker y is as before:

$$x \triangleleft ([\mathtt{P}] \cdot u \cdot [\mathtt{Q}] \Rightarrow [\mathtt{P}] \cdot u) \triangleright y$$

The rewrite step deletes a protocol symbol $[\mathtt{Q}]$. Note that $\mathsf{type}(x \cdot [\mathtt{P}] \cdot u) = \mathsf{type}(x \cdot u)$:

$$z' = \mathrm{src}(s) = x \cdot [\mathtt{P}] \cdot u \cdot [\mathtt{Q}] \cdot y$$
$$z = \mathrm{dst}(s) = x \cdot [\mathtt{P}] \cdot u \cdot y$$

$$\mathsf{type}(z) = \mathsf{type}(z')$$
$$\mathsf{chart}(z) = \mathsf{chart}(z') \setminus \{[\mathsf{type}(x \cdot u)\colon\ \mathtt{Q}]\}$$

As in Case 1, we "discard" our prior derivation of $[\mathsf{type}(x \cdot u)\colon\ \mathtt{Q}]$.

(★ Case 4.) A negative rewrite step for an associated conformance requirement has the following general form, where x is an admissible term, $([\mathtt{P}] \cdot u, [\mathtt{P}] \cdot u \cdot [\mathtt{Q}]) \in R$ for some u that contains only name symbols, and the right whisker y is as before:

$$x \triangleleft ([\mathtt{P}] \cdot u \Rightarrow [\mathtt{P}] \cdot u \cdot [\mathtt{Q}]) \triangleright y$$

The rewrite step inserts a protocol symbol $[\mathtt{Q}]$. Again, $\mathsf{type}(x \cdot [\mathtt{P}] \cdot u) = \mathsf{type}(x \cdot u)$:

$$z' = \mathrm{src}(s) = x \cdot [\mathtt{P}] \cdot u \cdot y$$
$$z = \mathrm{dst}(s) = x \cdot [\mathtt{P}] \cdot u \cdot [\mathtt{Q}] \cdot y$$

$$\mathsf{type}(z) = \mathsf{type}(z')$$
$$\mathsf{chart}(z) = \mathsf{chart}(z') \cup \{[\mathsf{type}(x \cdot u)\colon\ \mathtt{Q}]\}$$

We must derive $[\mathsf{type}(x \cdot u)\colon\ \mathtt{Q}]$. Since $[\mathsf{type}(x)\colon\ \mathtt{P}] \in \mathsf{chart}(z')$, the induction hypothesis tells us that $G \vdash [\mathsf{type}(x)\colon\ \mathtt{P}]$, and the AssocConf inference rule then gives us:

1. $[\mathsf{type}(x)\colon\ \mathtt{P}]$ (...)
2. $[\mathsf{type}(x \cdot u)\colon\ \mathtt{Q}]$ (AssocConf 1)

(★ Case 5.) A positive rewrite step for an explicit same-type requirement has the following general form, where the left whisker is empty, $(u, v) \in R$ for standard terms u and v, and the right whisker y is a combination of name and protocol symbols:

$$(u \Rightarrow v) \triangleright y$$

The rewrite step replaces the prefix u with v:

$$z' = \mathrm{src}(s) = u \cdot y$$
$$z = \mathrm{dst}(s) = v \cdot y$$

In this case, $\mathsf{type}(z)$ and $\mathsf{type}(z')$ are not identical, so we must first derive the same-type requirement $G \vdash [\mathsf{type}(t)\ \texttt{==}\ \mathsf{type}(z)]$. We now write $u \cdot y$ instead of z', so the induction hypothesis gave us this same-type requirement:

1. $[\mathsf{type}(u \cdot y)\ \texttt{==}\ \mathsf{type}(v \cdot y)]$ (...)

We now use the assumption that G is well-formed. Because $\mathsf{type}(u \cdot y)$ appears in the above requirement, the well-formedness of G allows us to derive $G \vdash \mathsf{type}(u \cdot y)$:

2. $\mathsf{type}(u \cdot y)$ (...)

Next, we derive $[\mathsf{type}(u)\ \texttt{==}\ \mathsf{type}(v)]$ with a Same elementary statement:

3. $[\mathsf{type}(u)\ \texttt{==}\ \mathsf{type}(v)]$ (Same)

Furthermore, $\mathsf{type}(u)$ is a prefix of $\mathsf{type}(u \cdot y)$. We apply Lemma 12.1 to $G \vdash \mathsf{type}(u \cdot y)$ and $G \vdash [\mathsf{type}(u)\ \texttt{==}\ \mathsf{type}(v)]$ to derive $G \vdash [\mathsf{type}(u \cdot y)\ \texttt{==}\ \mathsf{type}(v \cdot y)]$, or in other words, $[\mathsf{type}(z')\ \texttt{==}\ \mathsf{type}(z)]$:

4. $[\mathsf{type}(z')\ \texttt{==}\ \mathsf{type}(z)]$ (...)

Finally, we apply the Trans inference rule to compose the above requirement with the same-type requirement given to us by the induction hypothesis:

5. $[\mathsf{type}(t)\ \texttt{==}\ \mathsf{type}(z)]$ (Trans 1 4)

We now have our same-type requirement, but we must also derive each conformance requirement in $\mathsf{chart}(z)$ from the conformance requirements in $\mathsf{chart}(z')$.

Notice how z' and z have the same number and relative ordering of protocol symbols, and they all occur in the right whisker y, so there is a one-to-one correspondence between the elements of $\mathsf{chart}(z')$ and $\mathsf{chart}(z)$.

At each occurrence of a protocol symbol in y, we split up y into a pair of terms, by writing $y = y_1 \cdot [\mathtt{P}] \cdot y_2$ for some y_1, $y_2 \in A^*$ and some protocol $\mathtt{P}$. Then:

$$[\mathsf{type}(u \cdot y_1)\colon\ \mathtt{P}] \in \mathsf{chart}(z')$$
$$[\mathsf{type}(v \cdot y_1)\colon\ \mathtt{P}] \in \mathsf{chart}(z)$$

For each $[\mathsf{type}(u \cdot y_1)\colon\ \mathtt{P}] \in \mathsf{chart}(z')$, we know that $\mathsf{type}(u \cdot y_1)$ is a prefix of $\mathsf{type}(z')$. Because G is well-formed, Proposition 11.4 says that $G \vdash \mathsf{type}(u \cdot y_1)$. Then, $\mathsf{type}(u)$ is also a prefix of $\mathsf{type}(u \cdot y_1)$, so Lemma 12.1 says that $G \vdash [\mathsf{type}(u \cdot y_1)\ \mathtt{==}\ \mathsf{type}(v \cdot y_1)]$. Finally, we flip this requirement with Sym, and apply SameConf to derive the desired conformance requirement in $\mathsf{chart}(z)$:

1. $[\mathsf{type}(u \cdot y_1)\ \mathtt{==}\ \mathsf{type}(v \cdot y_1)]$ (...)
2. $[\mathsf{type}(u \cdot y_1)\colon\ \mathtt{P}]$ (...)
3. $[\mathsf{type}(v \cdot y_1)\ \mathtt{==}\ \mathsf{type}(u \cdot y_1)]$ (Sym 1)
4. $[\mathsf{type}(u \cdot y_1)\colon\ \mathtt{P}]$ (SameConf 3 1)

(★ Case 6.) A negative rewrite step for an explicit same-type requirement has the following general form, where $(u, v) \in R$ for standard terms u and v, the left whisker is empty, and the right whisker y is is a combination of name and protocol symbols:

$$(v \Rightarrow u) \triangleright y$$

We proceed as in Case 5, except that when the Same elementary statement gives us $[\mathsf{type}(u)\ \mathtt{==}\ \mathsf{type}(v)]$, we must apply the Sym inference rule to flip it around to get $[\mathsf{type}(v)\ \mathtt{==}\ \mathsf{type}(u)]$ before we proceed:

1. $[\mathsf{type}(u)\ \mathtt{==}\ \mathsf{type}(v)]$ (Same)
2. $[\mathsf{type}(v)\ \mathtt{==}\ \mathsf{type}(u)]$ (Sym 1)

We can then derive our final same-type requirement, and all conformance requirements listed in the chart, as before.

(★ Case 7.) A positive rewrite step for an associated same-type requirement has the following general form, where the left whisker x is an admissible term, $([\mathtt{P}] \cdot u, [\mathtt{P}] \cdot v) \in R$, u and v only contain name symbols, and the right whisker y is a combination of name and protocol symbols:

$$x \triangleleft ([\mathtt{P}] \cdot u \Rightarrow [\mathtt{P}] \cdot v) \triangleright y$$

The rewrite step replaces $[\texttt{P}] \cdot u$ with $[\texttt{P}] \cdot v$ in the middle of the term:

$$z' = \mathrm{src}(s) = x \cdot [\texttt{P}] \cdot u \cdot y$$
$$z = \mathrm{dst}(s) = x \cdot [\texttt{P}] \cdot v \cdot y$$

Note that $[\mathsf{type}(x)\texttt{: P}] \in \mathsf{chart}(z')$, which we can derive by the induction hypothesis. Also, observe that $\mathsf{type}(x \cdot [\texttt{P}] \cdot u) = \mathsf{type}(x \cdot u)$. We derive $[\mathsf{type}(x \cdot u)\ \texttt{==}\ \mathsf{type}(x \cdot v)]$ from $[\mathsf{type}(x)\texttt{: P}]$ via the AssocSame inference rule:

1. $[\mathsf{type}(x)\texttt{: P}]$ $(\ldots)$
2. $[\mathsf{type}(x \cdot u)\ \texttt{==}\ \mathsf{type}(x \cdot v)]$ (AssocSame 1)

We then apply Lemma 12.1 to $G \vdash \mathsf{type}(x \cdot u \cdot y)$ and $G \vdash [\mathsf{type}(x \cdot u)\ \texttt{==}\ \mathsf{type}(x \cdot v)]$ to derive $G \vdash [\mathsf{type}(x \cdot u \cdot y)\ \texttt{==}\ \mathsf{type}(x \cdot v \cdot y)]$, or in other words, $[\mathsf{type}(z')\ \texttt{==}\ \mathsf{type}(z)]$:

4. $[\mathsf{type}(z')\ \texttt{==}\ \mathsf{type}(z)]$ $(\ldots)$

Finally, we apply the Trans inference rule to compose the above requirement with the same-type requirement given to us by the induction hypothesis:

5. $[\mathsf{type}(t)\ \texttt{==}\ \mathsf{type}(z)]$ (Trans 1 4)

We must also update the conformance chart. Both x and y may contain occurrences of protocol symbols, but since $\mathsf{chart}(x) \subseteq \mathsf{chart}(z') \cap \mathsf{chart}(z)$, the requirements in $\mathsf{chart}(x)$ that are common to both remain unchanged. We derive new conformance requirements corresponding to occurrences of protocol symbols in y as in Case 5.

(★ Case 8.) A negative rewrite step for an associated same-type requirement has the following general form, where the left whisker x is an admissible term, $([\texttt{P}] \cdot u, [\texttt{P}] \cdot v) \in R$, u and v contain only name symbols, and the right whisker $y \in A^*$ is a combination of name and protocol symbols:

$$x \triangleleft ([\texttt{P}] \cdot v \Rightarrow [\texttt{P}] \cdot u) \triangleright y$$

We proceed as in Case 7, except we must flip the same-type requirement around via Sym as in Case 6. □

The specific statement we are looking for falls out from the preceding theorem:

Corollary 18.13. Let G be a generic signature, and let $\langle A \,|\, R \rangle$ be the requirement machine for G. Assume G is well-formed.

1. If $\mathsf{term}(\texttt{T}) \cdot [\texttt{P}] \sim \mathsf{term}(\texttt{T})$ for some valid type parameter $\texttt{T}$, then $G \vdash [\texttt{T: P}]$.
2. If $\mathsf{term}(\texttt{T}) \sim \mathsf{term}(\texttt{U})$ for valid type parameters $\texttt{T}$ and $\texttt{U}$, then $G \vdash [\texttt{T == U}]$.

Proof. In both cases, we have an equivalence between a standard term and an admissible term, so we can apply the previous theorem.

(★ Case 1.) Let $t := \mathsf{term}(\texttt{T})$, and $z := \mathsf{term}(\texttt{T}) \cdot [\texttt{P}]$. We have:

$$\mathsf{type}(z) = \texttt{T}$$
$$\mathsf{chart}(z) = \{[\texttt{T: P}]\}$$

Translating our rewrite path gives us $G \vdash [\texttt{T == T}]$ which we discard, together with $G \vdash [\texttt{T: P}]$ which we keep.

(★ Case 2.) Let $t := \mathsf{term}(\texttt{T})$, and $z := \mathsf{term}(\texttt{U})$. We have:

$$\mathsf{type}(z) = \texttt{U}$$
$$\mathsf{chart}(z) = \{\}$$

Translating our rewrite path gives us $G \vdash [\texttt{T == U}]$ which we keep, together with an empty list of derived conformance requirements which we discard. □

The decision procedure. Recall the four basic generic signature queries that describe the equivalence class structure of a generic signature, from Section 5.5. We saw the first two are fundamental from a theoretical standpoint.

Taken together, Corollary 17.47, Theorem 18.4, and Corollary 18.13 say that if we have a well-formed generic signature G presented by a convergent rewriting system $\langle A \,|\, R \rangle$, and the given type parameters are valid, then:

1. $G \vdash [\texttt{T: P}]$ **if and only if** $\mathsf{term}(\texttt{T}) \cdot [\texttt{P}]$ has the same normal form as $\mathsf{term}(\texttt{T})$.
2. $G \vdash [\texttt{T == U}]$ **if and only if** $\mathsf{term}(\texttt{T})$ has the same normal form as $\mathsf{term}(\texttt{U})$.

Also, crucially, the preconditions here are all things we can *check*, and we can compute the normal form of a term using Algorithm 17A. As we now quite clearly can see:

When implemented as below, the two basic generic signature queries always output the correct answer in a finite number of steps.

- **Query:** $\texttt{requiresProtocol}(\texttt{G}, \texttt{T}, \texttt{P})$

  Apply the normal form algorithm to $\mathsf{term}(\texttt{T}) \cdot [\texttt{P}]$ and $\mathsf{term}(\texttt{T})$ and check if we get a pair of identical terms.

- **Query:** $\texttt{areReducedTypeParametersEqual}(\texttt{G}, \texttt{T}, \texttt{U})$

  Apply the normal form algorithm to $\mathsf{term}(\texttt{T})$ and $\mathsf{term}(\texttt{U})$ and check if we get a pair of identical terms.

18.2. Symbols

Given a generic signature, our "core model" describes a monoid presentation whose alphabet consists of generic parameter, name, and protocol symbols. This gave us an elegant formalism for a subset of Swift generics. The remainder of this chapter describes the full implementation, with less mathematical rigor than before, and instead a focus on practical concerns.

We begin by expanding our alphabet with a few more symbol kinds on top of the three found in the core model. We will also define a reduction order on this alphabet. While our correctness result only needed the symmetric term equivalence relation, we need a reduction order to make use of the normal form algorithm in the implementation.

Symbols are formed by the rewrite context, the global singleton from Section 16.1 which manages the lifecycle of requirement machine instances. There are eight symbol kinds, with each kind having its own set of structural components. This resembles how we modeled types in Chapter 3.

For each symbol kind, a constructor function takes the structural components and returns a pointer to the unique symbol formed from those structural components. The identity of a symbol is determined by its structural components, so symbols can be compared for pointer equality.

Definition 18.14. A *symbol* in the Requirement Machine alphabet is an instance of one of the following:

- A **generic parameter symbol** `τ_d_i`, for some depth d and index $i \in \mathbb{N}$.
- A **name symbol** `A`, where `A` is an identifier appearing in the program.
- A **protocol symbol** [`P`], where `P` is a reference to a protocol declaration.
- An **associated type symbol** [`P`|`A`], where `P` is some protocol, and `A` is the name of an associated type declaration.
- A **layout symbol** [layout: `L`], where `L` is a layout constraint, either `AnyObject` or `_NativeClass`.
- A **superclass symbol** [superclass: `C`; $t_0, \ldots, t_n$] where `C` is an interface type, called the *pattern type* of the symbol, and the t_i are terms, called *substitution terms*. In a superclass symbol, the pattern type is a class or generic class type.
- A **concrete type symbol** [concrete: `X`; $t_0, \ldots, t_n$] where the pattern type `X` is now any interface type that is not a type parameter, and the t_i are substitution terms.
- A **concrete conformance symbol** [concrete: `X`: `P`; $t_0, \ldots, t_n$] where `X` is any interface type that is not a type parameter, the t_i are substitution terms, and `P` is a reference to a protocol declaration.

Name symbols. We already introduced name symbols at the beginning of this chapter and there isn't much else to say, but we add one remark. In the requirement machine for a well-formed generic signature G, every name symbol "`A`" has to be the name of some associated type or type alias of some `P` such that $G \prec \texttt{P}$—for otherwise, "`A`" cannot appear in any valid type parameter. However, we allow name symbols for arbitrary identifiers to be formed, so that we can model invalid programs as well. In any case, only a finite set of name symbols are constructed.

Protocol symbols. The reduction order on protocol symbols has the property that if a protocol `Q` inherits from a protocol `P`, then $[\texttt{P}] < [\texttt{Q}]$.

Consider the directed graph where the vertices are protocol declarations, and for each pair of protocols `Q` and `P` such that `Q` inherits from `P`, there is an edge with source `Q` and destination `P`. Define $\mathsf{depth}(\texttt{P}) \in \mathbb{N}$ as the number of protocols we can reach from `P`, excluding `P` itself. To compute the "depth" of a protocol, we evaluate the **all inherited protocols request** with `P`, which returns the set of all protocols reachable from `P`, and then take the number of elements in this set.

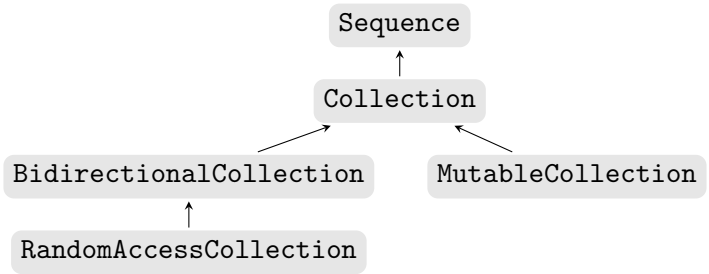
Algorithm 18A (Protocol reduction order)**.** Takes two protocol symbols $[\texttt{P}]$ and $[\texttt{Q}]$ as input, and returns one of "<", ">" or "=" as output.

1. If $\mathsf{depth}(\texttt{P}) > \mathsf{depth}(\texttt{Q})$, return "<". (Deeper protocols are smaller.)
2. If $\mathsf{depth}(\texttt{P}) < \mathsf{depth}(\texttt{Q})$, return ">".
3. If $\mathsf{depth}(\texttt{P}) = \mathsf{depth}(\texttt{Q})$, compare the protocols with Algorithm 5B.

Example 18.15. Various standard library protocols inherit from `Sequence`:

```
public protocol Sequence {...}
public protocol Collection: Sequence {...}
public protocol BidirectionalCollection: Collection {...}
public protocol MutableCollection: Collection {...}
public protocol RandomAccessCollection: BidirectionalCollection {...}
```

The protocol inheritance relationships above define this graph:



The corresponding protocol symbols are ordered as follows—notice how deeper protocols precede shallower protocols:

$$[\texttt{RandomAccessCollection}] < [\texttt{BidirectionalCollection}] < [\texttt{MutableCollection}] < [\texttt{Collection}] < [\texttt{Sequence}]$$

Associated type symbols. We encode the relationship between a protocol symbol and the name symbols of its associated type declarations by extending our alphabet with *associated type symbols*, and our set of rewrite rules with *associated type rules*.

If $[\texttt{P}|\texttt{A}]$ is an associated type symbol, it must be the case that `A` is the name of an associated type declared either by `P` itself, or some base protocol of `P`. The notation $[\texttt{P}|\texttt{A}]$ fuses a protocol symbol $[\texttt{P}]$ with a name symbol `A`, and in fact an associated type rule has the form $[\texttt{P}] \cdot \texttt{A} \sim [\texttt{P}|\texttt{A}]$.

Why do we need associated type symbols? There are two reasons:

1. In Section 5.5, we saw that we can implement `isValidTypeParameter`() using the `areReducedTypeParametersEqual`() and `requiresProtocol`() generic signature queries. Once we add associated type symbols, we can instead directly decide if a type parameter is valid using the normal form algorithm.

2. More importantly, it also turns out that we need associated type symbols to obtain a convergent rewriting system when our generic signature has an infinite set of equivalence classes.

We will study both of these points in great detail in Chapter 19.

The reduction order on associated type symbols is defined using the reduction order on protocol symbols:

Algorithm 18B (Associated type reduction order)**.** Takes two associated type symbols $[\texttt{P}|\texttt{A}]$ and $[\texttt{Q}|\texttt{B}]$ as input, and returns one of "<", ">" or "=" as output.

1. Compare the identifiers `A` and `B` using the lexicographic order. Return the result if it is "<" or ">". Otherwise, both associated types have the same name.

2. Compare the protocol symbols $[\texttt{P}]$ and $[\texttt{Q}]$ using Algorithm 18A and return the result.

An associated type symbol does not directly refer to an associated type *declaration*, because the associated type might be *inherited* by the protocol. That is, every associated type declaration of a base protocol also gives rise to a distinct symbol when "viewed" from the derived protocol.

Example 18.16. We continue Example 18.15. The `Sequence` protocol declares an `Element` associated type, so its derived protocols inherit this associated type. In the requirement machine for the protocol generic signature $G_{\texttt{RandomAccessCollection}}$, the alphabet contains various associated type symbols, ordered as follows:

$$\begin{aligned}&[\texttt{RandomAccessCollection}|\texttt{Element}]\\ &< [\texttt{BidirectionalCollection}|\texttt{Element}]\\ &< [\texttt{Collection}|\texttt{Element}]\\ &< [\texttt{MutableCollection}|\texttt{Element}]\\ &< [\texttt{Sequence}|\texttt{Element}]\end{aligned}$$

We will see *why* the order must respect protocol inheritance in Section 19.5.

The next three symbol kinds appear when we build rewrite rules for layout, superclass and concrete type requirements. Just like a conformance requirement [`T: P`] defines a rewrite rule $\mathsf{term}(\texttt{T}) \cdot [\texttt{P}] \sim \mathsf{term}(\texttt{T})$, these other requirement kinds define rewrite rules of the form $\mathsf{term}(\texttt{T}) \cdot s \sim \mathsf{term}(\texttt{T})$, where `T` is the requirement's subject type, and s is a layout, superclass or concrete type symbol.

Layout symbols. The [layout: `AnyObject`] layout symbol represents the `AnyObject` layout constraint we introduced in Definition 4.1. It appears when we translate a layout requirement [`T: AnyObject`] into a rule $\mathsf{term}(\texttt{T}) \cdot [\mathsf{layout}\texttt{: AnyObject}] \sim \mathsf{term}(\texttt{T})$.

We also mentioned the `_NativeClass` layout constraint in Section 5.5. This constraint cannot be stated directly by the user, but it is implied by a superclass requirement to a Swift-native class. These two layout symbols are ordered as follows:

$$[\mathsf{layout}\texttt{: AnyObject}] < [\mathsf{layout}\texttt{: \_NativeClass}]$$

Superclass and concrete type symbols. Superclass and concrete type symbols encode a concrete type that may contain type parameters. To form the symbol, we first decompose the concrete type into a *pattern type* and a list of *substitution terms*:

$$\text{concrete type} = \text{pattern type} + \text{substitution terms}$$

We're forming the symbol because we're translating an explicit or associated requirement into a rewrite rule, so we use the "term" or "$\mathsf{term}_{\texttt{P}}$" mapping as appropriate to translate each type parameter appearing in our concrete type. (The next section will introduce Algorithm 18G and Algorithm 18H used for this purpose, but for now, the definitions from the start of this chapter remain valid.) We then replace each type parameter with a "phantom" generic parameter $\texttt{τ\_0\_}i$, where the depth is always zero, and the index $i \in \mathbb{N}$ is the index of the corresponding substitution term in the list.

Algorithm 18C (Build concrete type symbol)**.** Receives an interface type `X` as input, and optionally, a protocol `P`. As output, returns a pattern type together with a list of substitution terms. Note that the type `X` must not itself be a type parameter.

1. Initialize S with an empty list of terms, and let $i := 0$.
2. Perform a pre-order walk over the tree structure of `T`, transforming each type parameter `T` contained in `X` to form the pattern type `Y`:
   a) If we're lowering an explicit requirement, let $t :=$ $\mathsf{term}(\texttt{T})$. Otherwise, let $t :=$ $\mathsf{term}_{\texttt{P}}(\texttt{T})$. Set $S \leftarrow S + \{t\}$.
   b) Replace `T` with the generic parameter type `τ_0_i` in `X`.
   c) Set $i \leftarrow i + 1$.
3. Return the type `Y`, and the array of substitution terms S.

Example 18.17. Given the explicit requirement [`τ_0_0 == Array<τ_0_1.Element>`], we decompose the right-hand side as follows:

$$\texttt{Array<τ\_0\_1.Element>} = \texttt{Array<τ\_0\_0>} + \{\texttt{τ\_0\_1} \cdot \texttt{Element}\}$$

We can then form the concrete type symbol [$\mathsf{concrete}$: `Array<τ_0_0>`; `τ_0_1` · `Element`] from the pattern type `Array<τ_0_0>` and list of substitution terms {`τ_0_1` · `Element`}.

Example 18.18. If we have a fully concrete type that does not contain type parameters, then our list of substitution terms will be empty. For example, the right-hand side of the superclass requirement [`Self: NSObject`] defines the symbol [$\mathsf{superclass}$: `NSObject`].

Example 18.19. The pattern type output by Algorithm 18C satisfies certain conditions:

1. The pattern type itself cannot be a type parameter.
2. The pattern type does not contain any dependent member types.
3. Every generic parameter type appearing in the pattern type has depth zero.
4. Every generic parameter type has a unique index.
5. Indices are consecutive and start from zero.

None of the following are valid pattern types:

```
τ_0_0
Array<τ_0_0.Element>
Array<τ_1_0>
Dictionary<τ_0_0, τ_0_0>
(τ_0_1) -> τ_0_0
```

To compare a pair of superclass or concrete type symbols, we first compare their pattern types for canonical type equality, and then compare their substitution terms using the reduction order on terms, which we define in the next section. Note that this is a partial order; symbols with distinct pattern types are incomparable.

Algorithm 18D (Concrete type reduction order)**.** Takes two superclass or concrete type symbols s_1 and s_2 as input. Returns one of "<", ">", "=" or "⊥" as output.

1. (Invariant) We assume the two symbols already have the same kind; the case of comparing different kinds is handled by the general symbol order we define next.
2. (Incomparable) Compare the pattern type of s_1 and s_2 with canonical type equality. If the pattern types are distinct, return "⊥".
3. (Initialize) Both symbols have the same pattern type, so they must have the same number of substitution terms, say n. Also, let $i := 0$.
4. (Equal) If $i = n$, all substitution terms are identical. Return "=".
5. (Compare) Compare the ith substitution terms of s_1 and s_2 using Algorithm 18I. Return the result if it is "<" or ">".
6. (Next) Otherwise, set $i \leftarrow i + 1$ and go back to Step 4.

Note that superclass and concrete type symbols contain terms, but those terms cannot recursively contain superclass and concrete type symbols, because terms corresponding to type parameters only contain generic parameter, name, protocol and associated type symbols.

Concrete conformance symbols. Such a symbol consists of a pattern type, a list of substitution terms, and a protocol declaration. We will see in Chapter 21 that when a type parameter `T` is subject to the combination of a conformance requirement `[T: P]` and a concrete same-type requirement `[T == X]`, we introduce a rewrite rule containing a concrete conformance symbol:

$$\mathsf{term}(\texttt{T}) \cdot [\mathsf{concrete}\texttt{: X: P; ...}] \sim \mathsf{term}(\texttt{T})$$

To compare two concrete conformance symbols, we first compare their protocol, then we compare their pattern type and substitution terms.

Algorithm 18E (Concrete conformance reduction order)**.** Takes two concrete conformance symbols s_1 and s_2 as input. Returns one of "<", ">", "=" or "⊥" as output.

1. Compare the protocol of s_1 and s_2 using Algorithm 18A. Return the result if it is "<" or ">".
2. Otherwise, compare the pattern type and substitution terms as in Algorithm 18D.

Symbol order. To compare an arbitrary pair of symbols that may have different kinds, we must define an order among the symbol kinds.

Algorithm 18F (Symbol reduction order)**.** Takes two symbols as input, and returns one of "<", ">", "=" or "⊥" as output. If the two symbols have different kinds, map their kinds to natural numbers and compare the numbers with the usual linear order on $\mathbb{N}$:

Concrete conformance:	0
Protocol:	1
Associated type:	2
Generic parameter:	3
Name:	4
Layout:	5
Superclass:	6
Concrete type:	7

Otherwise, both symbols have the same kind, which we already know how to compare:

- For generic parameter symbols, use Algorithm 5A.
- For name symbols, use the lexicographic order on their stored identifiers.
- For protocol symbols, use Algorithm 18A.
- For associated type symbols, use Algorithm 18B.
- For layout symbols, use the order previously described.
- For superclass symbols, use Algorithm 18D.
- For concrete type symbols, use Algorithm 18D.
- For concrete conformance symbols, use Algorithm 18E.

18.3. Terms

In the theory, a term is an element of a free monoid. In the implementation, a term is a sequence of symbols. There are two flavors. A *mutable term* is a value type which store its own heap-allocated buffer. Mutable terms are cheap to modify, but expensive to copy and store; the normal form algorithm and Knuth-Bendix completion both operate on mutable terms. An *immutable term* immutable term is expensive to allocate, because it is uniqued by the rewrite context, but cheap to compare for equality, because immutable terms with the same length and symbols are equal as pointers. Immutable terms are used in the representation of rewrite rules.

Because of how requirements map to rules, the rewrite rules in a requirement machine always have non-empty terms on both sides. For this reason, there is no immutable term representation of the empty term ε. Mutable terms may be empty though, and that is the initial state of a mutable term upon creation, before any symbols are added.

We now describe a pair of algorithms for translating type parameters to mutable terms, implementing the "term" and "$\mathsf{term}_{\mathtt{P}}$" mappings from the start of the present chapter. We start with the "term" mapping, used for lowering type parameters to terms in query machines and minimization machines. We extend this mapping to handle bound dependent member types, which translate to associated type symbols. They appear when we're building a query machine from the minimal requirements of an existing generic signature. Here is the general idea:

Type parameter:	**Term:**
`τ_0_0.A.B`	`τ_0_0` · `A` · `B`
`τ_0_0.[P]A.[Q]B`	`τ_0_0` · `[P|A]` · `[Q|B]`

A type parameter is a single-linked list where each interior node is a dependent member type, and the tail of the list is a generic parameter type. The head of the list is the *outermost* member type, so "`B`" in `τ_0_0.A.B`, and the base type of each dependent member type is the "next" pointer.

The below is an instance of the classical algorithm for converting a linked list into an array. We traverse the list from head to tail, at each step adding a symbol at the end of the term; this leaves us with a term where the *last* element is the generic parameter symbol. We then reverse the order of symbols to get the final result.

Algorithm 18G (Build term for explicit requirement)**.** Takes a type parameter `T` as input, and outputs a (non-empty) mutable term.

1. (Initialize) Let t be a new empty mutable term.
2. (Base case) If `T` is a generic parameter type `τ_d_i`:
   a) First, add the generic parameter symbol `τ_d_i` to t.
   b) Then, reverse the symbols in t, and return t.
3. (Recursive case) Otherwise, `T` is a dependent member type:
   a) If `T` is an unbound dependent member type `U.A` for some type parameter `U`, add the name symbol `A` to t.
   b) Otherwise, `T` is a bound dependent member type `U.[P]A`, again for some type parameter `U`. Add the associated type symbol `[P|A]` to t.

   Set `T` $\leftarrow$ `U`, and go back to Step 2.

Theorem 11.8 showed that every valid type parameter has a bound and unbound type parameter form. The above mapping is defined such that the terms obtained from the bound and unbound form of the same type parameter become equivalent under the term equivalence relation that will be generated by our rewrite rules.

An alternate valid implementation of Algorithm 18G would instead map both bound and unbound dependent member types to name symbols, ignoring the protocol of an associated type declaration referenced by a bound dependent member type. Instead, our implementation constructs associated type symbols directly, which has the effect of short-circuiting some work that would otherwise be performed later, in the normal form algorithm or in completion. We will say more about this in Section 19.2.

Next, consider the "$\mathsf{term}_{\mathtt{P}}$" mapping, used for building terms in a protocol machine or protocol minimization machine. Once again, we define the mapping on both unbound and bound type parameters. We've already seen that an unbound type parameter of $G_{\mathtt{P}}$ maps to a term that starts with a protocol symbol, followed by a series of name symbols. For bound type parameters, the protocol symbol disappears, and we're left with only associated type symbols:

Type parameter:	**Term:**
`Self`	[P]
`Self.A.B`	[P] · A · B
`Self.[P]A.[Q]B`	[P|A] · [Q|B]

Algorithm 18H (Build term for associated requirement)**.** Takes a type parameter `T` and a protocol declaration `P` as input, and outputs a (non-empty) mutable term.

1. (Initialize) Let t be a new empty mutable term.
2. (Base case) If `T` is the protocol `Self` type (`τ_0_0`), consider the term t constructed so far:
   a) If t ends with an associated type symbol [Q|A], then either `Q` = `P`, or `P` inherits from `Q`. In both cases, *replace* the last symbol of t with [P|A].
   b) Otherwise, either t is empty, or it ends with a name symbol. Add the protocol symbol [P] to t.

   Finally, reverse the symbols in t, and return t.
3. (Recursive case) Otherwise, `T` is a dependent member type.
   a) If `T` is an unbound dependent member type `U.A`, add the name symbol `A` to t.
   b) Otherwise, `T` is a bound dependent member type `U.[Q]A`. Add the associated type symbol [Q|A] to t.

   Set `T` ← `U`, and go back to Step 2.

Example 18.20. Note the behavior of Algorithm 18H with inherited associated types:

```
protocol Base {
  associatedtype Element
}

protocol Derived: Base {
  // associatedtype Element
}
```

Let's apply Algorithm 18H to the bound and unbound form of the type parameter `Self.Element` of $G_{\texttt{Derived}}$:

$$\text{term}_{\texttt{Derived}}(\texttt{Self.[Base]Element}) = [\texttt{Derived}|\texttt{Element}]$$
$$\text{term}_{\texttt{Derived}}(\texttt{Self.Element}) = [\texttt{Derived}] \cdot \texttt{Element}$$

Notice the appearance of an inherited associated type symbol. Now, suppose we uncomment the declaration of `Element` in `Derived`. We have a new bound type parameter `Self.[Derived]Element`, whose term is actually identical to the other one:

$$\text{term}_{\texttt{Derived}}(\texttt{Self.[Derived]Element}) = [\texttt{Derived}|\texttt{Element}]$$

In `Derived`, both of these pairs of terms are equivalent:

$$[\texttt{Derived}] \cdot \texttt{Element} \sim [\texttt{Derived}|\texttt{Element}]$$
$$[\texttt{Derived}] \cdot [\texttt{Base}|\texttt{Element}] \sim [\texttt{Derived}|\texttt{Element}]$$

We will see why in the next section, when we define the rewrite rules for a protocol machine.

Reduction order. In the previous section, we defined a partial order on symbols. We now extend this to a partial order on terms. Recall the discussion of reduction orders from Section 17.5.

We start with the standard shortlex order from Algorithm 17B, but add an additional check to ensure that terms containing name symbols are always ordered after terms without name symbols, even if the term with name symbols is shorter. This is an instance of a *weighted shortlex order*, where our chosen weight function counts the number of name symbols appearing in a term. In the next section, we will see that the weighted shortlex order is necessary for correctness once we introduce the rewrite rules for protocol type aliases.

Definition 18.21. Let A be any set. We say that $w\colon A^* \to \mathbb{N}$ is a *weight function* if it satisfies $w(xy) = w(x) + w(y)$ for all x, $y \in A^*$. That is, w is a monoid homomorphism from A^* to $\mathbb{N}$, where the latter is viewed as a monoid under addition.

Algorithm 18I (Weighted shortlex order)**.** Takes two terms t and u as input, and returns one of "$<$", "$>$", "$=$" or "$\perp$" as output.

1. (Weight) Compute $w(t)$ and $w(u)$.
2. (Less) If $w(t) < w(u)$, return "$<$".
3. (More) If $w(t) > w(u)$, return "$>$".
4. (Shortlex) Otherwise $w(t) = w(u)$, so we compare the terms using Algorithm 17B and return the result.

The weighted shortlex order is a suitable reduction order under Definition 17.40.

Proposition 18.22. Let $w\colon A^* \to \mathbb{N}$ be a weight function. Then the weighted shortlex order induced by w is translation-invariant and well-founded.

Proof. Let $<$ be the standard shortlex order on A^*, and let $<_w$ be the weighted shortlex order induced by w.

We prove translation invariance first. We're given x, y, $z \in A^*$ with $x <_w y$. We will show that $zx <_w zy$; the proof that $xz <_w yz$ is similar. Since $x <_w y$, by definition we either have $w(x) < w(y)$, or $w(x) = w(y)$ and $x < y$. We consider each case:

1. If $w(x) < w(y)$, then $w(z) + w(x) < w(z) + w(y)$ because the linear order on $\mathbb{N}$ is translation-invariant, and since $w(zx) = w(z) + w(x)$ and $w(zy) = w(z) + w(y)$, we see that $w(zx) < w(zy)$, therefore $zx <_w zy$.
2. If $w(x) = w(y)$ and $x < y$, then $w(zx) = w(zy)$, and the translation invariance of $<$ on A^* implies that $zx < zy$. Once again, we conclude that $zx <_w zy$.

Now, we demonstrate that $<_w$ is well-founded, by contradiction. Assume we have an infinite descending chain:

$$t_1 >_w t_2 >_w t_3 >_w \cdots$$

If we apply the weight function w to each term t_i in this chain, we see that:

$$w(t_1) \geq w(t_2) \geq w(t_3) \geq \cdots$$

Since $w(t_i) \in \mathbb{N}$, we can only "step down" finitely many times. So there exists some index i such that for all $j > i$, $w(t_j) = w(t_i)$. Together with $t_j <_w t_i$, we conclude that actually, $t_j < t_i$. Thus, we get an infinite descending chain for the *standard* shortlex order $<$:

$$t_i > t_{i+1} > t_{i+2} > \cdots$$

However, this is a contradiction, because the standard shortlex order $<$ is well-founded. Thus, $<_w$ must also be well-founded. □

We now dispose of the notation $<_w$. We will denote the reduction order on terms by $<$, with the understanding that it is the weighted shortlex order of Algorithm 18I.

18.4. Rules

We now completely describe the "rule" and "rule$_P$" mappings for translating requirements to rewrite rules. Recall the requirement machine kinds from Chapter 16.

We start with the "rule" mapping, which defines the local rules of **generic signature** and **generic signature minimization** machines. We can assume the input requirement is desugared already (Section 11.2). This mapping uses Algorithm 18G, denoted "term", to translate type parameters to terms.

The handling of the interesting requirement kinds—layout, superclass and concrete type requirements—resembles conformance requirements. Let's say that a *property rule* is a rewrite rule of the form $(\mathsf{term}(\mathtt{T}) \cdot s, \mathsf{term}(\mathtt{T}))$, where s is a symbol. Every requirement kind, with the exception of same-type requirements between type parameters, maps to a property rule. The following table summarizes the algorithm:

Property rules: $\mathsf{term}(\mathtt{T}) \cdot s \sim \mathsf{term}(\mathtt{T})$	
rule [T: P]	s = [P]
rule [T: AnyObject]	s = [layout: AnyObject]
rule [T: C]	s = [superclass: ...]
rule [T == X]	s = [concrete: ...]
Same-type rules:	
[T == U]	$\mathsf{term}(\mathtt{T}) \sim \mathsf{term}(\mathtt{U})$

Algorithm 18J (Build rule from explicit requirement)**.** Takes a desugared requirement as input, and outputs a pair of terms.

1. For a **conformance requirement** [T: P], build a property rule from term(T) and the protocol symbol [P].

2. For a **layout requirement** [T: AnyObject], build a property rule from term(T) and the layout symbol [layout: AnyObject].

3. For a **superclass requirement** [T: C], build a superclass symbol s using Algorithm 18C, and then build a property rule from from term(T) and s.

4. For a **same-type requirement** [T == X], where X is a **concrete type**, build a concrete type symbol s using Algorithm 18C, and then build a property rule from from term(T) and s.

5. For a **same-type requirement** [T == U] where U is a **type parameter**, build a rule from term(T) and term(U).

For protocol machines and protocol minimization machines, the translation is the same, except for the use of Algorithm 18H (denoted "$\mathsf{term}_{\mathtt{P}}$") instead of Algorithm 18G (denoted "term") to translate type parameters to terms.

Property rules: $\mathsf{term}_{\mathtt{P}}(\mathtt{Self.U}) \cdot s \sim \mathsf{term}_{\mathtt{P}}$	
$\mathsf{rule}_{\mathtt{P}}\,[\mathtt{Self.U:\ Q}]_{\mathtt{P}}$	$s = [\mathtt{Q}]$
$\mathsf{rule}_{\mathtt{P}}\,[\mathtt{Self.U:\ AnyObject}]_{\mathtt{P}}$	$s = [\mathsf{layout}\mathtt{:\ AnyObject}]$
$\mathsf{rule}_{\mathtt{P}}\,[\mathtt{Self.U:\ C}]_{\mathtt{P}}$	$s = [\mathsf{superclass}\mathtt{:\ ...}]$
$\mathsf{rule}_{\mathtt{P}}\,[\mathtt{Self.U\ ==\ X}]_{\mathtt{P}}$	$s = [\mathsf{concrete}\mathtt{:\ ...}]$
Same-type rules:	
$\mathsf{rule}_{\mathtt{P}}\,[\mathtt{Self.U\ ==\ Self.V}]_{\mathtt{P}}$	$\mathsf{term}_{\mathtt{P}}(\mathtt{Self.U}) \sim \mathsf{term}_{\mathtt{P}}(\mathtt{Self.V})$

Algorithm 18K (Build rule from associated requirement)**.** Takes a desugared requirement and a protocol declaration `P` as input, and outputs a pair of terms.

1. For an **associated conformance requirement** $[\mathtt{Self.U:\ Q}]_{\mathtt{P}}$, build a property rule from $\mathsf{term}_{\mathtt{P}}(\mathtt{Self.U})$ and the protocol symbol $[\mathtt{P}]$.
2. For a **associated layout requirement** $[\mathtt{Self.U:\ AnyObject}]_{\mathtt{P}}$, build a property rule from $\mathsf{term}_{\mathtt{P}}(\mathtt{Self.U})$ and the layout symbol $[\mathsf{layout}\mathtt{:\ AnyObject}]$.
3. For a **associated superclass requirement** $[\mathtt{Self.U:\ C}]_{\mathtt{P}}$, build a superclass symbol s using Algorithm 18C, and then build a property rule from from $\mathsf{term}_{\mathtt{P}}(\mathtt{Self.U})$ and s.
4. For a **associated same-type requirement** $[\mathtt{Self.U\ ==\ X}]_{\mathtt{P}}$, where `X` is a **concrete type**, build a concrete type symbol s using Algorithm 18C, and then build a property rule from from $\mathsf{term}_{\mathtt{P}}(\mathtt{Self.U})$ and s.
5. For a **associated same-type requirement** $[\mathtt{Self.U\ ==\ Self.V}]_{\mathtt{P}}$ where `Self.V` is a **type parameter**, build a rule from $\mathsf{term}_{\mathtt{P}}(\mathtt{Self.U})$) and $\mathsf{term}_{\mathtt{P}}(\mathtt{Self.V})$.

In the core model, we lower each associated requirement of each protocol dependency to a rewrite rule. In the implementation, we add a few more rules for each protocol, to describe the structure of each protocol itself:

Identity conformance rule:	
`protocol P`	$[\mathtt{P}] \cdot [\mathtt{P}] \sim [\mathtt{P}]$
Associated type rules:	
`associatedtype A`	$[\mathtt{P}] \cdot \mathtt{A} \sim [\mathtt{P}\vert\mathtt{A}]$
Inherited ssociated type rules:	
`associatedtype A`	$[\mathtt{P}] \cdot \mathtt{A} \sim [\mathtt{P}\vert\mathtt{A}]$

The *identity conformance rule* conceptually states that the `Self` type of an associated requirement conforms to the protocol itself. It plays a minor role. We can mostly ignore it, but we will justify it in Example 19.10. We also add a series of *associated type rules* for the protocol's own associated type declarations, and finally, *inherited associated type rules* for those it inherits. The last two kinds convert between name symbols and associated type symbols.

Algorithm 18L (Build protocol rules)**.** Takes a protocol declaration `P` as input, and outputs a list of rules.

1. Initialize R with an empty list of rules, represented as pairs of terms.
2. Add the identity conformance rule ([P] · [P], [P]) to R.
3. For each associated type `A` of `P`, build the associated type rule ([P] · A, [P|A]) and add it to R.
4. (Inherited associated types) For each associated type `A` of each protocol `Q` inherited by `P`, add the inherited associated type rule ([P] · A, [P|A]) to R.

In the previous section, we claimed that our extension of "term" and "$\mathsf{term}_{\mathtt{P}}$" to bound type parameters has the property that if we build a pair of terms from the bound and unbound form of the same type parameter, we get an equivalent pair of terms. Associated type rules ensure that this equivalence holds.

We will say more about these symbols in Section 19.3. Tietze transformations. This also does not change either the alphabet or rewrite rules of the requirement machine:

```
protocol Base {
  associatedtype Element
}

protocol Derived: Base {
  // associatedtype Element
}
```

Suppose we're building a requirement machine for $G_{\texttt{Derived}}$. If we then uncomment the redundant associated type declaration, we generate the same list of rewrite rules either way, except now ([Derived] · Element, [Derived|Element]) is an associated type rule and not an *inherited* associated type rule.

Protocol type aliases. Derived requirements don't capture protocol type aliases

The second algorithm concerns protocol type aliases.

Protocol type aliases are part of the rewrite system because they can appear in generic requirements. Here is an example, They must be in the protocol itself and not inside an extension. Link to here from other sections and vice versa.

Written protocols:

- Underlying type is resolved with structural resolution
- Protocol type aliases are in fact analogous to same-type requirements where the subject type is an unresolved type `Self.A`.
- For protocol minimization machines, we evaluate the **type alias requirements** request to collect all type alias declarations together.

Minimized protocols:

- For protocol machines, protocol type aliases are encoded in the requirement signature.

Like same-type requirements, we must separately consider the case where the underlying type is a concrete type, and where it is a type parameter. Mention the reduction order thing.

Algorithm 18M (Build rule for protocol type alias)**.** Takes a protocol `P`, and a type alias member `A` of `P` as input. Returns a pair of terms (u, v) as output.

1. Get the underlying type of `A`.
2. If the underlying type of `A` is concrete, use Algorithm 18C to build a concrete type symbol, say s, return the pair:

   $$([\texttt{P}] \cdot \texttt{A} \cdot s,\ [\texttt{P}] \cdot \texttt{A})$$

3. If the underlying type of `A` is some type parameter `Self.U`, return the pair:

   $$([\texttt{P}] \cdot \texttt{A},\ \mathsf{term}_{\texttt{P}}(\texttt{Self.U}))$$

Representation of rules. A rewrite rule is represented by a pair of immutable terms—the *left-hand side* and *right-hand side*—together with some flags. The right-hand side term must precede the left-hand side term in the reduction order, as required by the theory.

The flags associated with each rule are important for completion and minimization:

- **Permanent** rules are created from associated type declarations and the implicit conformance of the protocol `Self` type to the protocol itself. Permanent rules are not subject to minimization. This bit is set by the rule builder (Section 18.4).

- **Left-simplified**, **right-simplified** and **substitution-simplified** rules were replaced by new rules (Section 19.1).
- **Conflicting** rules and **Recursive** rules are discovered during property map construction, and indicate a problem with user-written requirements that needs to be diagnosed (Chapter 20).
- **Explicit** rules are directly created from user-written requirements, or related to other explicit rules in a very specific way. This bit is set by the rule builder and then propagated to other rules during minimization. Explicit rules have a special behavior with the minimal conformances algorithm (Section 22.4).
- **Redundant** rules are discovered during minimization, after which the requirement builder constructs the final list of minimal requirements from all remaining rules not marked as redundant (Chapter 22.5).
- **Frozen** rules cannot have any of their flags changed. After construction but before a requirement machine is installed in the rewrite context (Chapter 16), all rules are marked frozen to establish the invariant that no further updates can be made to any rule flags.

Once a flag has been set, it cannot be cleared, and the frozen flag prevents any more flags from being set.

A requirement machine stores its rewrite rules in an array. Imported rules appear first, followed by local rules. Often we need to iterate over just the local rules, so we track the index of the first local rule in the array. During rewrite system construction and completion, new rules are appended at the end of the array, but we require that indices into the array remain stable, so rules are never inserted in the middle. Various auxiliary data structures represent references to rules as indices into the rules array:

- Term reduction repeatedly looks for rules whose left-hand side matches a subterm of a given term; these lookups are made efficient with a prefix trie where each entry references a rule (Section 18.5).
- Property map entries reference the rules they were constructed from.
- Completion and property map construction both need to derive rewrite paths describing equivalences between terms, and these rewrite paths are comprised of rewrite steps which reference rules.

18.5. The Normal Form Algorithm

rewrite steps only store the length of each whisker, index of rule, and inverse flag.

rule trie. Section 6.3 of [135].

Algorithm 18N (Insert rule in rule trie)**.** Takes the index of a rewrite rule as input. Has side effects.

1. (Initialize) Let N be the root node of the trie. Let $i := 0$.
2. (End) If $i = |t|$, store the index of our rewrite rule in N and return.
3. (Traverse) Let s_i be the ith symbol of t. Look up s_i in N. Create a new node if no such node exists.
4. (Child) Call this child node M. Set $N \leftarrow M$. Set $i \leftarrow i + 1$. Go back to Step 2.

Algorithm 18O (Lookup in rule trie)**.** Takes a term t and an offset i such that $0 \leq i < |t|$. Return the index of a rewrite rule (u, v) such that u is a prefix of $t[i :]$, or a null value if no such rule exists.

1. (Initialize) Let N be the root node of the trie.
2. (End) If $i = |t|$, we've reached the end of the term. Return null.
3. (Traverse) Let s_i be the ith symbol of t. Look up s_i in N. Return null if no such node exists.
4. (Child) Otherwise, call this child node M. If M stores the index of a rewrite rule, return this index.
5. (Advance) Set $N \leftarrow M$. Set $i \leftarrow i + 1$. Go back to Step 2.

Algorithm 18P (Normal form algorithm using rule trie)**.** Takes a mutable term t as input. Mutates t, and outputs a positive rewrite path p where $\mathrm{src}(p)$ is the original term t, and $\mathrm{dst}(p)$ is the normal form of t.

1. (Initialize) Initialize the return value p with the empty rewrite path 1_p. Let $i := 0$.
2. (End) If $i = |t|$, return p.
3. (Lookup) Invoke Algorithm 18O with t and i.
4. (Rewrite) If we found a rewrite rule (u, v), then let $x := t[i :]$, $y := t[: i + |u|]$. Replace the subterm $t[i : i + |u|]$ with v, set $i \leftarrow i + 1$, set $p \leftarrow p \circ x(u \Rightarrow v)y$, and go back to Step 2.
5. (Done) Otherwise, t is reduced. Return p.

Algorithm 18Q (Record rewrite rule)**.** Takes a pair of terms (u, v) as input. Has side effects.

1. (Reduce) Apply Algorithm 18P to u and v to obtain $\tilde{u}$ and $\tilde{v}$.
2. (Trivial) If $\tilde{u} = \tilde{v}$, return.
3. (Compare) Use Algorithm 18I to compare $\tilde{u}$ with $\tilde{v}$.
4. (Error) If $\tilde{u}$ and $\tilde{v}$ are incomparable, signal an error.
5. (Orient) If $\tilde{v} > \tilde{u}$, swap $\tilde{u}$ and $\tilde{v}$.
6. (Record) Add $(\tilde{u}, \tilde{v})$ to the list of local rules.
7. (Trie) Invoke Algorithm 18N to insert the new local rule in the trie.

Our implementation explicitly stores the source and destination terms in a critical pair, as well as the rewrite path itself. As we will learn in Section 19.6, we use a compressed representation for rewrite paths in memory, where a rewrite step $x(u \Rightarrow v)y$ only stores the *length* of each whisker x and y and not the actual terms x and y. Thus, we must store the source and destination terms separately since they cannot be recovered from the rewrite path alone.

Aho-Corasick algorithm [136].

18.6. Source Code Reference

Symbols

Key source files:

- `lib/AST/RequirementMachine/Symbol.h`
- `lib/AST/RequirementMachine/Symbol.cpp`
- `lib/AST/RequirementMachine/Term.h`
- `lib/AST/RequirementMachine/Term.cpp`
- `lib/AST/RequirementMachine/Trie.h`

`rewriting::Symbol::Kind` *enum class*

The symbol kind. The order of the cases is significant; it is from the reduction order on symbols (Algorithm 18F).

- `ConcreteConformance`
- `Protocol`

- `AssociatedType`
- `GenericParam`
- `Layout`
- `Superclass`
- `ConcreteType`

`rewriting::Symbol` *class*

Represents an immutable, uniqued symbol. Meant to be passed as a value, it wraps a single pointer to internal storage. Symbols are logically variant types, but C++ does not directly support that concept so there is a bit of boilerplate in their definition.

Symbols are constructed with a set of static factory methods which take the structural components and the `RewriteContext`:

- `forName()` takes an `Identifier`.
- `forProtocol()` takes a `ProtocolDecl *`.
- `forAssociatedType()` takes a `ProtocolDecl *` and `Identifier`.
- `forGenericParam()` takes a canonical `GenericTypeParamType *`.
- `forLayout()` takes a `LayoutConstraint`.
- `forSuperclass()` takes a pattern type and substitution terms.
- `forConcreteType()` takes a pattern type and substitution terms.
- `forConcreteConformance()` also takes a `ProtocolDecl *`.

The last three methods take the pattern type as a `CanType` and the substitution terms as an `ArrayRef<CanType>`. The `RewriteContext::getSubstitutionSchemaFromType()` method implements Algorithm 18C to build the pattern type and substitution terms from an arbitrary `Type`. Note that the pattern type is always a canonical type, so type sugar is not preserved when round-tripped through the Requirement Machine, for example when building a new generic signature.

Taking symbols apart:

- `getKind()` returns the `Symbol::Kind`.
- `getName()` returns the `Identifier` stored in a name or associated type symbol.

- `getProtocol()` returns the `ProtocolDecl *` stored in a protocol, associated type or concrete conformance symbol.
- `getGenericParam()` returns the `GenericTypeParamDecl *` stored in a generic parameter symbol.
- `getLayoutConstraint()` returns the `LayoutConstraint` stored in a layout symbol.
- `getConcreteType()` returns the pattern type stored in a superclass, concrete type or concrete conformance symbol.
- `getSubstitutions()` returns the substitution terms stored in a superclass, concrete type or concrete conformance symbol.
- `getRootProtocol()` returns `nullptr` if this is a generic parameter symbol, or a protocol declaration if this is a protocol or associated type symbol. Otherwise, asserts. This is the "domain" of the symbol.

Comparing symbols:

- `operator==` tests for equality.
- `compare()` is the symbol reduction order (Algorithm 18F). The return type of `Optional<int>` encodes the result as follows:
  - `None`: $\perp$
  - `Some(0)`: $=$
  - `Some(-1)`: $<$
  - `Some(1)`: $>$

Debugging:

- `dump()` prints a human-readable representation of a symbol.

`rewriting::Term` *class*

An immutable term. Meant to be passed as a value, it wraps a single pointer to internal storage. Immutable terms are created by a static factory method:

- `get()` creates a `Term` from a `MutableTerm`, which must be non-empty.

Looking at terms:

- `containsUnresolvedSymbols()` returns true if the term contains name symbols.

- `getRootProtocol()` returns `nullptr` if the first symbol is a generic parameter symbol, otherwise returns the protocol of the first protocol or associated type symbol. Asserts if the first symbol has any other kind.
- `size()`, `begin()`, `end()`, `rbegin()`, `rend()`, `operator[]` are the standard C++ operations for iteration and indexing into the term, as an array of `Symbol`.
- `back()` returns the last `Symbol` in the term.

Comparing terms:

- `operator==` tests for equality.
- `compare()` is the term reduction order (Algorithm 18I). The return type of `Optional<int>` encodes the result in the same manner as `Symbol::compare()`.

Debugging:

- `dump()` prints a human-readable representation of a term.

`rewriting::MutableTerm` *class*

A mutable term. Meant to be passed as a value or const reference. Owns a heap-allocated buffer for storing its own elements. The default constructor creates an empty `MutableTerm`. The other constructors allow specifying an initial list of symbols as an iterator pair, `ArrayRef`, or the symbols of an immutable `Term`.

Supports the same operations as `Term`, together with various mutating methods:

- `add()` pushes a single `Symbol` on the end of the term.

`rewriting::Rule` *class*

`rewriting::RewriteSystem` *class*

See also Section 19.6.

- `simplify()` reduces a term using Algorithm 18P.

`rewriting::RewriteContext` *class*

`rewriting::RuleBuilder` *class*

---

collectRulesFromReferencedProtocols() is actually the algorithm from the previous section.

`rewriting::Trie` *template class*

See also Section 19.6.

- `insert()` inserts an entry using Algorithm 18N.
- `find()` finds an entry using Algorithm 18O.

19. Completion

Knuth-Bendix completion is the central algorithm in the Requirement Machine. Completion attempts to construct a convergent rewriting system from a list of rewrite rules, and a convergent rewriting system allows us to decide if two terms have the same normal form in a finite number of steps, solving the word problem. As we saw in the previous chapter, our initial rewrite rules are defined by the explicit requirements of a generic signature and its protocol dependencies. A desirable property of this mapping was given by Theorem 17.34: a *derived* requirement defines a rewrite path over these rewrite rules representing explicit requirements. All of this means that completion gives us a *decision procedure* for the derived requirements formalism: the question of whether any given derived requirement is satisfied—that is, if there exists a valid derivation built from explicit requirements—is easily solved by term reduction in a convergent rewriting system. This is the foundation on which we build both generic signature queries and minimization.

The algorithm. We'll give a self-contained description first, with much of the rest of the chapter devoted to examples. Our description can be supplemented with any text on rewrite systems, such as [118] or [119]. The algorithm is somewhat clever; to really "get it" might require several attempts. Donald E. Knuth and Peter Bendix described the algorithm for term rewrite systems in a 1970 paper [137]; a correctness proof was later given by Gérard Huet in [138]. In our application, the terms are elements of a free monoid, so we have a string rewrite system; this special case was studied in [139]. A survey of related techniques appears in [140].

The entry point into the Knuth-Bendix completion procedure is Algorithm 19E, but we break off four smaller pieces before we get there, so that only the top-level loop remains:

- Algorithm 19C finds all rules that overlap with a fixed rule at a fixed position.
- Algorithm 19D finds all pairs of rules that overlap at any position.
- Algorithm 19A builds a critical pair from a pair of overlapping rules.
- Algorithm 19B resolves a critical pair.

We begin with Algorithms 19A and 19B, proceeding from the inside out. The twin concepts of overlapping rule and critical pair are fundamental to the algorithm, and they provide the theoretical justification for the rest.

Local confluence. We would like our reduction relation $\to$ to satisfy the Church-Rosser property: if $x \sim y$ are two equivalent terms, then $x \to z$ and $y \to z$ for some term z. By Theorem 17.44, this is equivalent to $\to$ being confluent, meaning any two positive rewrite paths diverging from a common source can be extended to meet each other. This is difficult to verify directly, but a 1941 paper by Max Newman [141] shows there is a simpler equivalent condition when the reduction relation is terminating.

Definition 19.1. A reduction relation $\to$ is *locally confluent*, if whenever s_1 and s_2 are two positive rewrite steps with $\mathrm{src}(s_1) = \mathrm{src}(s_2)$, there exists a term z such that $\mathrm{dst}(s_1) \to z$ and $\mathrm{dst}(s_2) \to z$.

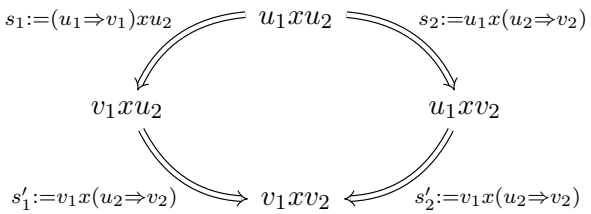
To test for local confluence, we "diverge" from a term by only one step in two different directions, and then check if both sides reduce to some common term. We will see this can be decided algorithmically, and also that we can "repair" any local confluence violations we do find. Thus, Newman's result is fundamental:

Theorem 19.2 (Newman's Lemma)**.** If a reduction relation $\to$ is terminating and locally confluent, then $\to$ is confluent.

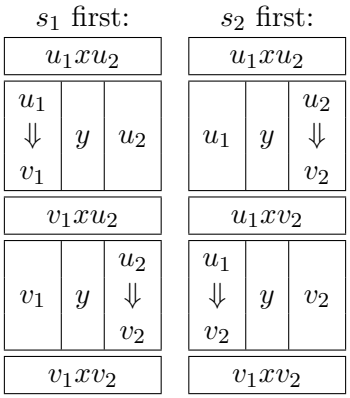
Overlapping rules. A pair of positive rewrite steps with a common source define a *critical pair*. A critical pair shows that some term can be reduced in "two different ways." We can answer if our rewrite rules define a locally confluent reduction relation by inspecting each critical pair. With any non-trivial list of rewrite rules, there are infinitely many such critical pairs, however, all but a finite subset can be disregarded. Suppose we have a rewrite system over an alphabet A with two rules:

$$u_1 \Rightarrow v_1$$
$$u_2 \Rightarrow v_2$$

For any term $x \in A^*$, we can form the "sandwich" term $t := u_1 x u_2$. Every such choice of x defines a new critical pair; the occurrences of u_1 and u_2 within t can be rewritten in two ways, by $s_1 := (u_1 \Rightarrow v_1)xu_2$ and $s_2 := u_1 x(u_2 \Rightarrow v_2)$. However, since s_1 and s_2 rewrite disjoint subterms of t, we say this critical pair is *orthogonal*. Orthogonal critical pairs are not interesting because they cannot witness a local confluence violation. To see why, notice that regardless of whether we apply s_1 or s_2 first, there exists a complementary rewrite step s_1' or s_2' to rewrite $\mathrm{dst}(s_1)$ or $\mathrm{dst}(s_2)$ into the "reduced sandwich" $v_1 x v_2$. In fact, we get a commutative diagram like this for any orthogonal critical pair:



We can also visualize an orthogonal critical pair using the "pictorial" notation for rewrite steps we devised in Section 17.2:



Clearly, in our quest to uncover local confluence violations, we only need to inspect critical pairs that are *not* orthogonal; that is, they must rewrite *overlapping* subterms of their common source term. There are only finitely many such critical pairs, and they are all generated by inspecting the left-hand sides of rewrite rules in our rewrite system. We can completely characterize them with the below definition.

Definition 19.3. Two rules (u_1, v_1) and (u_2, v_2) *overlap* if one of the following holds:

1. The left-hand side of the second rule is contained entirely within the left-hand side of the first. That is, $u_1 = xyz$ and $u_2 = y$ for some x, y, $z \in A^*$. If we write down the terms u_1 and u_2 and shift u_2 over until they line up, we get this:

$$\begin{array}{l} xyz \\ \ \ y \end{array}$$

2. The left-hand side of the second rule has a prefix equal to a suffix of the left-hand side of the first. That is, $u_1 = xy$ and $u_2 = yz$ for some x, y, $z \in A^*$, with $|x| > 0$ and $|z| > 0$. If we write down the terms u_1 and u_2 and shift u_2 over until they line up, we get this:

$$\begin{array}{l} xy \\ \ \ yz \end{array}$$

The above are the two ways in which a non-orthogonal critical pair can rewrite the same term. When the case distinction is important, we can talk about an overlap of the *first kind*, or *second kind*, respectively. In both cases, after a suitable assignment of x, y and z, we define the *overlap term* to be xyz, and the *overlap position* to be $|x|$.

Example 19.4. Consider these three rules:

$$\texttt{τ\_0\_0} \cdot [\texttt{Collection}|\texttt{SubSequence}] \cdot [\texttt{Equatable}] \quad (1)$$

$$\texttt{τ\_0\_0} \cdot [\texttt{Collection}|\texttt{SubSequence}] \Rightarrow \texttt{τ\_0\_1} \quad (2)$$

$$[\texttt{Collection}|\texttt{SubSequence}] \cdot [\texttt{Collection}|\texttt{Element}] \Rightarrow [\texttt{Collection}|\texttt{Element}] \quad (3)$$

There is an overlap of the first kind between (1) and (2) at position 0; the left-hand side of (2) is contained entirely in the left-hand side of (1). Here, the overlap term is $\texttt{τ\_0\_0} \cdot [\texttt{Collection}|\texttt{SubSequence}] \cdot [\texttt{Equatable}]$:

$$\texttt{τ\_0\_0} \cdot [\texttt{Collection}|\texttt{SubSequence}] \cdot [\texttt{Equatable}]$$
$$\texttt{τ\_0\_0} \cdot [\texttt{Collection}|\texttt{SubSequence}]$$

There is an overlap of the second kind between (1) and (3) at position 1; the left-hand side of (3) begins with the prefix $[\texttt{Collection}|\texttt{SubSequence}]$, which is also a suffix of the left-hand side of (1). The overlap term is $\texttt{τ\_0\_0} \cdot [\texttt{Collection}|\texttt{SubSequence}] \cdot [\texttt{Equatable}]$:

$$\texttt{τ\_0\_0} \cdot [\texttt{Collection}|\texttt{SubSequence}]$$
$$[\texttt{Collection}|\texttt{SubSequence}] \cdot [\texttt{Collection}|\texttt{Element}]$$

The definition of an overlap of the second kind requires both x and z to be non-empty. If we relaxed this condition, then we would *also* have an overlap of the second kind between rule (2) and (1) at position 0:

$$\texttt{τ\_0\_0} \cdot [\texttt{Collection}|\texttt{SubSequence}]$$
$$\texttt{τ\_0\_0} \cdot [\texttt{Collection}|\texttt{SubSequence}] \cdot [\texttt{Equatable}]$$

The overlap term and position is the same as the first case we saw already, so attempting to resolve this critical pair would not reveal anything new. We adjust our definition to avoid duplicated work in this situation.

It can happen that two rules overlap more than once at *different* positions; in this case we must consider every possible overlap. For example, these two rules generate four overlaps in total:

$$[\texttt{P}|\texttt{A}] \cdot [\texttt{P}|\texttt{B}] \cdot [\texttt{P}|\texttt{A}] \cdot [\texttt{P}|\texttt{B}] \Rightarrow [\texttt{P}|\texttt{C}] \quad (4)$$

$$[\texttt{P}|\texttt{B}] \cdot [\texttt{P}|\texttt{A}] \cdot [\texttt{P}|\texttt{B}] \cdot [\texttt{P}|\texttt{A}] \Rightarrow [\texttt{P}|\texttt{D}] \quad (5)$$

Rule (4) overlaps with (5) at position 1:

$$[\texttt{P}|\texttt{A}] \cdot [\texttt{P}|\texttt{B}] \cdot [\texttt{P}|\texttt{A}] \cdot [\texttt{P}|\texttt{B}]$$
$$[\texttt{P}|\texttt{B}] \cdot [\texttt{P}|\texttt{A}] \cdot [\texttt{P}|\texttt{B}] \cdot [\texttt{P}|\texttt{A}]$$

Rrule (4) overlaps with (5) at position 3:

$$[\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}]$$
$$[\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}]$$

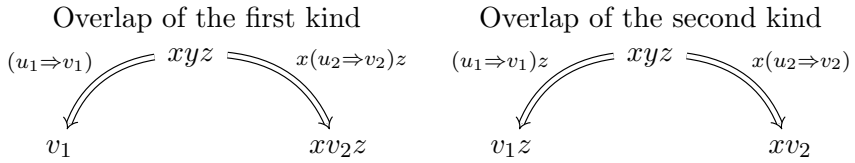
Rule (5) overlaps with (4) at position 1:

$$[\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}]$$

Last but not least, rule (5) overlaps with (4) at position 3:

$$[\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}]$$

Resolving critical pairs. A critical pair exhibits some term t being rewritten in two distinct ways. If we take the destination term of each of the two rewrite steps, we get a pair of terms that are known to be equivalent to t, and each other. For an overlap of the first kind, the two terms are $(v_1,\ xv_2z)$; for the second kind, $(v_1z,\ xv_2)$:



We *resolve* a critical pair (t_1, t_2) by reducing both sides with the reduction relation as constructed so far and comparing the reduced terms. There are four possible outcomes:

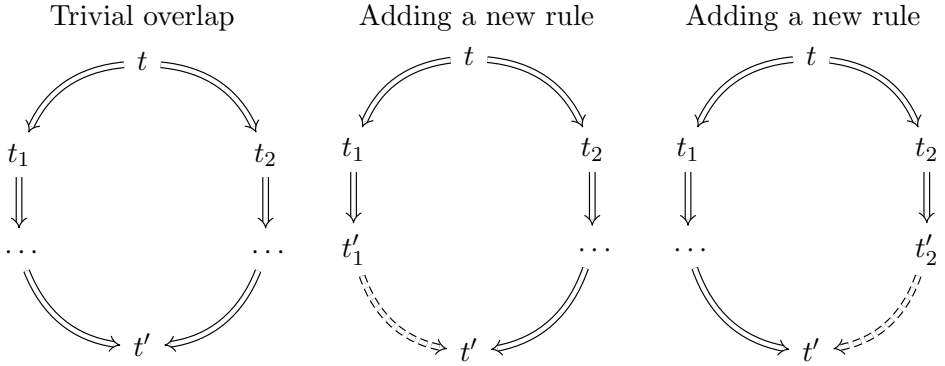
1. If t_1 and t_2 reduce to the same term t', we say the critical pair is *trivial*. (Note that this terminology offers an alternate definition of local confluence: a reduction relation is locally confluent if all critical pairs are trivial.)

2. If $t_1 \to t_1'$ and $t_2 \to t_2'$ with $t_1' \neq t_2'$, we found two distinct reduced terms in the same equivalence class: a *confluence violation*. If $t_2' < t_1'$, we repair the confluence violation by adding a new rewrite rule (t_1', t_2'). Having done so, if we then define $t' := t_2'$, we once again see that both t_1 and t_2 reduce to the same term t'.

3. If instead $t_1' < t_2'$, we have the same situation except we resolve it by adding a new rewrite rule (t_2', t_1'), and we let $t' := t_1'$.

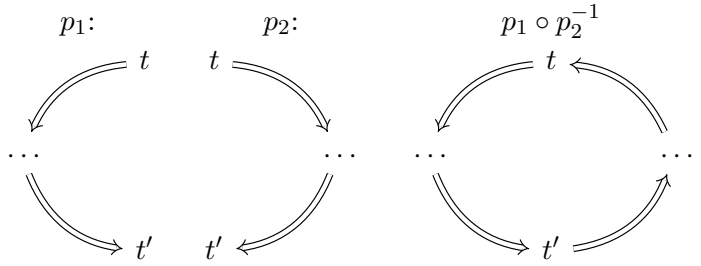
4. If t_1' and t_2' are distinct but incomparable, we have a non-orientable relation, and we must report an error. This cannot happen under the reduction order used by the Requirement Machine.

We can draw a diagram for each case, showing a subgraph of the rewrite graph; dashed arrows indicate new rules being added:



After adding a new rewrite rule if necessary, we have a pair of rewrite paths p_1 and p_2, and a term t'. Note that $\mathrm{src}(p_1) = \mathrm{src}(p_2) = t$ and $\mathrm{dst}(p_1) = \mathrm{dst}(p_2) = t'$, so both paths have the same source and destination. We say p_1 and p_2 are *parallel* rewrite paths.

Rewrite loops. Now we slightly change our point of view and introduce a new concept. If we take two parallel rewrite paths p_1 and p_2, we can form the rewrite path $\ell := p_1 \circ p_2^{-1}$ by composing the first path with the inverse of the second:



This new rewrite path ℓ has the property that it begins and ends at the *same* term t:

$$\mathrm{src}(\ell) = \mathrm{src}(p_1) = t \qquad \mathrm{dst}(\ell) = \mathrm{dst}(p_2^{-1}) = \mathrm{src}(p_2) = t$$

We say that ℓ is a *rewrite loop* with basepoint t. A rewrite loop applies some sequence of rewrite rules to the basepoint term t, then rewrites it "back" to t, via a possibly *different* mix of rewrite rules. In the rewrite graph, a rewrite loop is a cycle (sometimes called a "closed path" by graph theorists). Note that for each $t \in A^*$, the empty rewrite path 1_t can also be seen as a trivial rewrite loop with basepoint t.

Now, we give two algorithms that together form the inner loop of the completion procedure. It is also convenient to view a critical pair as a rewrite path of length 2, rather than a pair of rewrite steps with the same source term—we compose the first step with the inverse of the second. In this new formulation, critical pair resolution "completes" the critical pair to form a rewrite loop.

Algorithm 19A (Construct critical pair)**.** Takes two rules $u_1 \Rightarrow v_1$ and $u_2 \Rightarrow v_2$, together with an overlap position i where $0 \leq i < |u_1|$. Returns a triple (t_1, t_2, p) where t_1 and t_2 are terms, and p is a rewrite path with $\mathrm{src}(p) = t_1$ and $\mathrm{dst}(p) = t_2$.

1. If $i + |u_2| \leq |u_1|$, we have an overlap of the first kind; $u_1 = xu_2z$ for some x and z.
   a) Let $x := u_1[: i]$ (the prefix of u_1 of length i), and $z := u_1[i + |u_2| :]$ (the suffix of u_1 of length $|u_1| - |u_2| - i$).
   b) Set $t_1 := v_1$, the result of rewriting u_1 with the first rule.
   c) Set $t_2 := xv_2z$, the result of rewriting u_1 with the second rule.
   d) Set $p := (v_1 \Rightarrow u_1) \circ x(u_2 \Rightarrow v_2)z$. This is a rewrite path from t_1 to t_2.
2. Otherwise, we have an overlap of the second kind; $u_1 = xy$ and $u_2 = yz$ for some x, y and z.
   a) Let $x := u_1[: i]$ (the prefix of u_1 of length i), $z := u_2[|u_1| - i :]$ (the suffix of u_2 of length $|u_2| - |u_1| + i$). (We don't actually need $y := u_1[i :] = u_2[: |u_1| - i]$.)
   b) Set $t_1 := v_1z$, the result of rewriting xyz with the first rule.
   c) Set $t_2 := xv_2$, the result of rewriting xyz with the second rule.
   d) Set $p := (v_1 \Rightarrow u_1)z \circ x(u_2 \Rightarrow v_2)$.
3. Return the triple (t_1, t_2, p).

Algorithm 19B (Resolve critical pair)**.** As input, takes terms u and v, and a rewrite path p with $\mathrm{src}(p) = u$ and $\mathrm{dst}(p) = v$. Records a rewrite loop, and possibly adds a new rule, returning true if a rule was added.

1. (Fast path) If $u = v$, p is already a loop; record it and return false.
2. (Left) Apply Algorithm 18P to u, obtaining $\tilde{u}$ and p_1.
3. (Right) Apply Algorithm 18P to v, obtaining $\tilde{v}$ and p_2.
4. (Compare) Use Algorithm 18I to compare $\tilde{u}$ with $\tilde{v}$.
5. (Trivial) If $\tilde{u} = \tilde{v}$, record a loop $p_1^{-1} \circ p \circ p_2$ with basepoint $\tilde{u} = \tilde{v}$, and return false.
6. (Smaller) If $\tilde{v} < \tilde{u}$, add the rule $(\tilde{u}, \tilde{v})$, record a loop $p_2^{-1} \circ p^{-1} \circ p_1 \circ (\tilde{u} \Rightarrow \tilde{v})$ with basepoint $\tilde{u}$, and return true.
7. (Larger) If $\tilde{u} < \tilde{v}$, add the rule $(\tilde{v}, \tilde{u})$, record a loop $p_1^{-1} \circ p \circ p_2 \circ (\tilde{v} \Rightarrow \tilde{u})$ with basepoint $\tilde{v}$, and return true.
8. (Error) Otherwise, $\tilde{u}$ and $\tilde{v}$ are incomparable; signal an error.

Rewrite loops are not just a theoretical tool; our implementation of the Knuth-Bendix algorithm follows [142] and [143] in encoding and recording the rewrite loops that describe resolved critical pairs. This enables the computation of minimal requirements in Section 22.1. Only local rules are subject to minimization, so we only record rewrite loops involving local rules. If a requirement machine instance is only to be used for generic signature queries and not minimization, rewrite loops are not recorded at all.

An optimization. Now that we know how to process a single overlap and resolve a critical pair, the next chunk of code concerns enumerating all candidate overlaps. In an arbitrary string rewrite system, we must consider all possible combinations: for every rewrite rule $u_1 \Rightarrow v_1$, for every rewrite rule $u_2 \Rightarrow v_2$, and for every position $i < |u_1|$, we would need to check if the corresponding subterms of u_1 and u_2 are identical. In our application, the bottom-up construction of a rewrite system from protocol components, and the partition of rewrite rules into imported rules and local rules, enables an optimization where overlaps between certain pairs of rules need not be considered at all:

- We don't need to look for overlaps between imported rules. While an imported rule can overlap with another imported rule, all such critical pairs are trivial and do not need to be resolved again.
- We don't need to look for overlaps between an imported rule and a local rule. An imported rule cannot overlap with a local rule.

Only two interesting pairings remain:

- A local rule can overlap with another local rule.
- A local rule can overlap with an imported rule.

We now proceed to prove that this is indeed the case. First, suppose two imported rules overlap. We will consider three possibilities in turn:

- There is an overlap of the first kind.
- There is an overlap of the second kind, and the second rule's left-hand side starts with an associated type symbol.
- There is an overlap of the second kind, and the second rule's left-hand side starts with a protocol symbol.

In all three cases, we will conclude that both rules either originate from the same protocol component, or two distinct components where one imports the other. This implies that the critical pair is now trivial, having been resolved by completion of that protocol component which contains the other.

In the first case, it is immediate that both rules were imported from the same protocol component for `P`:

$$[\texttt{P}|\texttt{A}] \cdot [\texttt{Q}|\texttt{B}]$$
$$[\texttt{P}|\texttt{A}]$$

In the second case, `P` and `Q` are either in the same protocol component, or `Q` is a protocol dependency of `P`, because we have $G_{\texttt{P}} \vdash [\texttt{Self.A: Q}]$:

$$[\texttt{P}|\texttt{A}] \cdot [\texttt{Q}|\texttt{B}]$$
$$[\texttt{Q}|\texttt{B}] \cdot [\texttt{R}|\texttt{C}]$$

The final case similarly implies that `Q` is a protocol dependency of `P`:

$$[\texttt{P}|\texttt{A}] \cdot [\texttt{Q}]$$
$$[\texttt{Q}] \cdot [\texttt{R}]$$

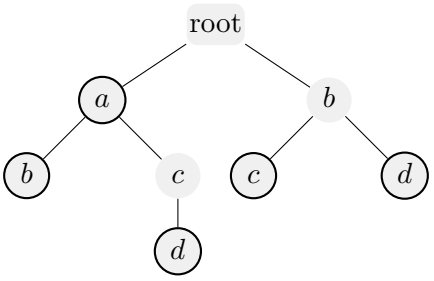
Next, we claim that imported rules cannot overlap with local rules. In a generic signature rewrite system, the local rules are precisely those whose left-hand side starts with a generic parameter symbol. The left-hand side of an imported rule cannot start with, or contain, a generic parameter symbol, proving the claim. In a protocol component rewrite system, we can similarly rule out overlap of the first kind between an imported rule and a local rule, because the two rules must start with the same protocol or associated type symbol, and thus originate from the same protocol component. Now, suppose a protocol component rewrite system has an overlap of the second kind. We show that if the second rule, the one with left-hand side $[\texttt{Q}|\texttt{B}] \cdot [\texttt{R}|\texttt{C}]$, is a local rule, then the first rule is as well:

$$[\texttt{P}|\texttt{A}] \cdot [\texttt{Q}|\texttt{B}]$$
$$[\texttt{Q}|\texttt{B}] \cdot [\texttt{R}|\texttt{C}]$$

As before, `Q` must be a protocol dependency of `P`. But this time, we have the further assumption that `Q` belongs to the *current* protocol component, so the appearance of `P` means that `P` must *also* be a protocol dependency of `Q`. Thus, `P` and `Q` depend on each other, and are actually part of the same component; therefore both rules are local.

Another optimization. If we fix a rule and position, we can find all overlaps involving this rule and position by performing a lookup into the rule trie, which we previously used to speed up term reduction in Section 18.5. This further cuts down the work in enumerating overlaps. While the underlying data structure is the same, the lookup algorithm here differs from that used by term reduction; we must enumerate all matches, instead of stopping after the first one.

Consider a set of rewrite rules having the terms a, ab, bc, bd and acd as their left-hand sides. The rule with left-hand side ab has overlaps with all other rules except for acd. Here is the rule trie, with thick borders denoting nodes associated with rewrite rules:



When checking the left-hand side ab for overlaps, we perform two lookups:

- At position 0, we look up ab. Starting from the root, we encounter the rule for a, followed by ab (which is the left-hand side of our rule itself, so we skip it). The latter node is a leaf so we end the search.
- At position 1, we look up b. The node for b doesn't store a rewrite rule, however, it has child nodes. We've reached the end of our input sequence, so we recursively visit all children, and find the two final overlap candidates, bc and bd.

This new kind of trie lookup can be thought of as a coroutine or an iterator, yielding zero or more results as the search proceeds. We implement it as a higher-order function taking a callback.

Algorithm 19C (Overlap lookup in rule trie)**.** Takes a term t, an offset i such that $0 \leq i < |t|$, and a callback. For each rule (u, v) where $t[i:]$ is a prefix of u or u is a prefix of $t[i:]$, invokes the callback with the rule (u, v).

1. (Initialize) Let N be the root node of the trie.
2. (End) If $i = |t|$, we've reached the end of the term. Perform a pre-order traversal of all child nodes of N, and for those children that have an associated rewrite rule, invoke the callback with that rule (this is the case where $t[i:]$ is a prefix of each u).
3. (Traverse) Let s_i be the ith symbol of t. Look up s_i in N. If there is no such child node, return.
4. (Child) Otherwise, call this child node M. If M has an associated rule $u \Rightarrow v$, invoke the callback with this rule (in this case, u is a prefix of $t[i:]$).
5. (Advance) Set $N \leftarrow M$. Set $i \leftarrow i + 1$. Go back to Step 2.

The next algorithm feeds the results of Algorithm 19C into Algorithm 19A to build a list of all critical pairs in the rewrite system. After resolving these critical pairs, we will have to check for overlaps again, in case there are any involving the newly-added rules. To avoid repeated work, the below algorithm maintains a set of visited overlaps, so that we can avoid building and resolving a critical pair we already know has been resolved.

Algorithm 19D (Find overlapping rules)**.** Takes a list of rewrite rules as input, and outputs a list of critical pairs. Also queries and updates the set of visited overlaps.

1. (Initialize) Set $i := 0$. Let n be the total number of local rules. Initialize an empty output list.
2. (Outer check) If $i = n$, we're done. Return the output list.
3. (Get rule) Denote the ith local rule by (u_1, v_1) (the subscript says 1 not i). If this rule is marked as **left-simplified**, **right-simplified** or **substitution-simplified**, skip it entirely and go to Step 7. Otherwise, let $j := 0$.
4. (Inner check) If $j = |u_1|$, we're done. Go to Step 9.
5. (Find overlaps) Use Algorithm 19C to find all rules whose left-hand side is a prefix of the symbol range $u_1[j :]$ or vice versa.
6. (Visit) For each matching rule (u_2, v_2) found above, check if this rule is marked as **left-simplified**, **right-simplified** or **substitution-simplified**. Also, check if $((u_1, v_1), (u_2, v_2), j)$ is already in the visited set. In either case, move on to the next matching rule.
7. (Record) Otherwise, insert $((u_1, v_1), (u_2, v_2), j)$ to the visited set, and build a critical pair for this overlap using Algorithm 19A. This produces a triple (t_1, t_2, p) describing the critical pair; add this critical pair to the output list.
8. (Inner loop) Set $j \leftarrow j + 1$, and go back to Step 3.
9. (Outer loop) Set $i \leftarrow i + 1$, and go back to Step 2.

We can now describe the main loop of the Knuth-Bendix completion procedure, which repeatedly finds and resolves critical pairs until no more non-trivial critical pairs remain. This process might not terminate, and we might find ourselves discovering new critical pairs and adding new rules to resolve them, forever. To prevent an infinite loop in the case of failure, we implement a termination check; if we think we've done too much work already, we give up on constructing a covergent rewrite system. We already mentioned the **left-simplified**, **right-simplified**, and **substitution-simplified** flags a few times; they are set by the rule simplification passes, with the first two described in Section 19.1 and the third one in Section 20.1. These passes are invoked at appropriate times in the main loop below.

Algorithm 19E (Knuth-Bendix completion procedure)**.** Takes a rewrite system as input, and outputs success or failure. On success, the rewrite system is now convergent, with all resolved critical pairs described by recorded rewrite loops. Failure occurs if we encounter a non-orientable rewrite rule, or if we trigger the termination check.

1. (Initialize) Clear the flag.
2. (Overlaps) Use Algorithm 19D to build a list of critical pairs.
3. (Left simpify) Invoke Algorithm 19F.
4. (Resolve) For each critical pair, invoke Algorithm 19B and set the flag if any new rewrite rules were added.
5. (Right simplify) Invoke Algorithm 19G.
6. (Substitution simplify) Invoke Algorithm 20A.
7. (Termination) If we exceeded the rule count, term length or concrete nesting limits, return failure.
8. (Repeat) If the flag was set, go back to Step 1. Otherwise, return success.

Termination. The termination check is controlled by a handful of frontend flags:

- `-requirement-machine-max-rule-count=<value>` sets the maximum number of local rules to `value`. Imported rules do not count toward this total, so this is hard to hit with realistic code. The default is a maximum of 4000 local rules.
- `-requirement-machine-max-rule-length=<value>` sets the maximum length of the terms in a rule to `value`. To compute the actual quantity, we add the length of the longest *user-written* rule; so the restriction in on the relative "growth" and not on the length of a type parameter written by the user. The default value is 12.
- `-requirement-machine-max-concrete-nesting=<value>` sets the upper bound on nested concrete types to `value`, to prevent substitution simplification from constructing an infinite type like `G<G<G<...>>>`. As with the limit on rule length, we add the maximum nesting depth of user-written rules to get the actual limit. The default value is 30.

While the first of the three is sufficient to detect non-termination, it takes a second or two for completion to record that many rules. The other two limits improve user experience in this case by rejecting clearly invalid programs sooner. The rule length limit being relative instead of just a total ban on terms of length 12 allows various pathological cases to succeed which would otherwise be needlessly rejected.

The following protocol, for example, presents the monoid $\mathbb{Z}_{14}$ and defines a rule of length 14, so the absolute rule length limit is really $14 + 12 = 26$. Completion does not add any longer rules so we accept it without issues:

```
protocol Z14 {
  associatedtype A: Z14
    where Self == Self.A.A.A.A.A.A.A.A.A.A.A.A.A.A
}
```

If completion fails when building a rewrite system for minimization, we have a source location associated with some protocol or generic declaration. An error is diagnosed at this source location, and we proceed with minimization producing an empty list of requirements. If completion fails on a rewrite system built from an existing generic signature or protocol component, there is no source location we can use for diagnostics; the compiler dumps the entire rewrite system and aborts with a fatal error. The latter scenario is unusual; if we successfully constructed a generic signature from user-written requirements, we should be able to build a rewrite system for it again.

Debugging flags. A pair of debugging options can help us understand the operation of the completion procedure; both can be set together[1], but be warned that they produce a large volume of output:

- `-debug-requirement-machine=completion` will dump all overlapping rules and critical pairs.
- `-debug-requirement-machine=add` will dump all rewrite rules and rewrite loops obtained while resolving critical pairs.

19.1. Rule Simplification

We impose two further conditions on our convergent rewriting system:

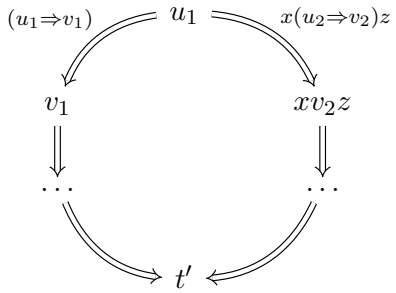
1. No rule has a left-hand side that can be reduced by any other rule.
2. No rule has a right-hand side that can be reduced by any other rule.

Such rewrite systems are called *left-reduced* or *right-reduced*, respectively, or simply *reduced* if both conditions are met. Any convergent rewriting system can be transformed into a reduced rewriting system with a pair of simplification passes that possibly delete and add rules.

[1] With a single `-debug-requirement-machine=` flag, separating the subflags with commas.

In our implementation, we don't *actually* delete rules, because we use the index of each rule as a stable reference elsewhere; instead, we set a pair of rule flags, **left-simplified** and **right-simplified**, and delete the rule from the rule trie. We've seen these flags mentioned already, so now we reveal their purpose. This will motivate the subsequent theory, setting the stage for the remaining two sections of this chapter.

Left simplification. If the left-hand side of a rewrite rule $u_1 \Rightarrow v_1$ can be reduced by another rewrite rule $u_2 \Rightarrow v_2$, then $u_1 = xu_2z$ for some x, $z \in A^*$, so we have an overlap of the first kind in the sense of Definition 19.3. Once we resolve all critical pairs, we don't need the first rule at all; we know that in a convergent rewriting system, both ways of reducing the overlap term $u_1 := xu_2z$ produce the same result:

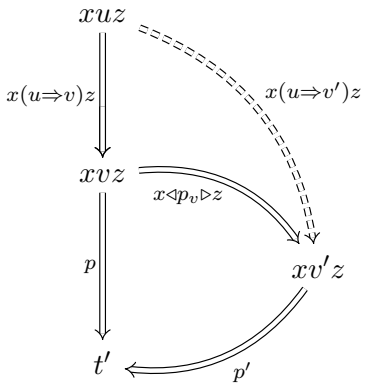


The left simplification algorithm considers the left-hand side of each rule, and marks the rule if it finds a subterm matching some other rule.

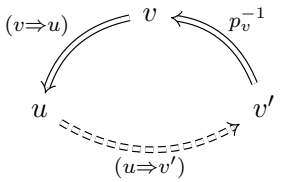
Algorithm 19F (Left-simplify rewrite rules)**.** Takes the list of local rules as input. Has side effects.

1. (Initialize) Let n be the total number of local rules, and set $i := 0$.
2. (Outer check) If $i = n$, return. Otherwise, say (u, v) is the ith local rule, and set $j := 0$.
3. (Inner check) If $j = |u|$, go to Step 8.
4. (Search) Look up $u[: j]$ in the rule trie, where $u[: j]$ is the suffix of u of length $|u| - j$.
5. (Decide) If the trie lookup returns no results, or it returns (u, v) itself (which can only happen if $j = 0$, so $u[: j] = u$), go to Step 7.
6. (Mark) Otherwise, u has a subterm equal to the left-hand side of some other rule. Mark $u \Rightarrow v$ as **left-simplified** and go to Step 7.
7. (Inner loop) Set $j \leftarrow j + 1$ and go back to Step 3.
8. (Outer loop) Set $i \leftarrow i + 1$ and go back to Step 2.

Right simplification. The other case is when we have a rule (u, v) whose right-hand side v is not reduced, so $v \to v'$ via some positive rewrite path p_v. Now suppose we have a positive rewrite path $x(u \Rightarrow v)z \circ p$, where $\mathrm{dst}(p)$ is some reduced term t'. If we reduce xuz to xvz via $x(u \Rightarrow v)z$, we have two choices on how to proceed: we can follow p, or reduce xvz to $xv'z$ via $x \triangleleft p_v \triangleright z$. By confluence, the second choice must take us to t' via a positive rewrite path p' with $\mathrm{src}(p') = xv'z$ and $\mathrm{dst}(p') = t'$. Thus, we record a new rule $u \Rightarrow v'$ that obsoletes $u \Rightarrow v$:



The right simplification algorithm outputs a right-reduced rewrite system by attempting to reduce the right-hand side of each rewrite rule. While left simplification does not need to record new rewrite rules because completion has already resolved all overlaps of the first kind, right simplification actually records new rules as well as marking existing rules as having been simplified. The new rule is related with the existing rule by a rewrite loop:



Algorithm 19G (Right-simplify rewrite rules)**.** Takes the list of local rules as input. Has side effects.

1. (Initialize) Let n be the total number of local rules, and set $i := 0$.
2. (Check) If $i = n$, return. Otherwise, let (u, v) be the ith local rule.
3. (Reduce) Apply Algorithm 18P to v to get a term $\tilde{v}$ and a rewrite path p_v. If $v = \tilde{v}$, the right-hand side v is already reduced, so go to Step 7.
4. (Record) Invoke Algorithm 18Q to add a new rewrite rule $(u, \tilde{v})$.

5. (Relate) Add the rewrite loop $(u \Rightarrow v) \circ p \circ (v' \Rightarrow u)$ with basepoint u, relating the old rule $u \Rightarrow v$ with the new rule $u \Rightarrow v'$.
6. (Mark) Mark the old rule as **right-simplified**.
7. (Loop) Increment i and go back to Step 2.

Our justification for the validity of these passes worked from the assumption that we had a convergent rewriting system; that is, that completion had already been performed. In practice, Algorithm 19E repeatedly runs both passes during completion, once per round of critical pair resolution. This is advantageous, because we can subsequently avoid considering overlaps that involve simplified rules. This strategy remains sound as long as we perform left simplification after computing critical pairs, but *before* resolving them, which might add new rules. This narrows the candidates for left simplification to those rules whose overlaps have already been considered. As for the right simplification pass, it is actually fine to run it at any point; we choose to run it after resolving critical pairs.

Related concepts. We previously saw in Section 11.4 that the same-type requirements in a generic signature are subject to similar conditions of being left-reduced and right-reduced. There is a connection here, because as we will see in Section 22.5, the minimal requirements of a generic signature are ultimately constructed from the rules of a reduced rewrite system. However, there are a few notational differences:

- The roles of "left" and "right" are reversed because requirements use a different convention; in a reduced same-type requirement [`U == V`], we have `U` $<$ `V`, whereas in a rewrite rule (u, v) we have $v < u$.
- Reduced same-type requirements have the "shortest" distance between the left-hand and right-hand side, so if `T`, `U` and `V` are all equivalent and `T` $<$ `U` $<$ `V`, the corresponding requirements are [`T == U`], [`U == V`]. On the other hand, if we have three terms t, u and v with $t < u < v$, then the two corresponding rewrite rules would be (u, t) and (v, t).

These differences are explained by the original `GenericSignatureBuilder` minimization algorithm, described as finding a minimum spanning tree for a graph of connected components. The notion of reduced requirement output by this algorithm became part of the Swift stable ABI. In Section 22.5 we show that a list of rewrite rules in a reduced rewrite system defines a list of reduced requirements via a certain transformation.

What we call reduced rewrite systems are sometimes "normalized", "canonical", or "inter-reduced" in the literature. Our rewrite system also implements a third *substitution simplification* pass for rewrite system simplification. Substitution simplification reduces substitution terms appearing in superclass, concrete type and concrete conformance symbols. We will discuss it in Chapter 20.

19.2. Associated Types

A conformance rule $t \cdot [\texttt{P}] \Rightarrow t$ always overlaps with an associated type rule $[\texttt{P}] \cdot \texttt{A} \Rightarrow [\texttt{P}|\texttt{A}]$, and resolving this critical pair defines a rule $t \cdot \texttt{A} \Rightarrow t \cdot [\texttt{P}|\texttt{A}]$ (unless the right-hand side reduces further). These rewrite rules reduce those terms representing unbound type parameters to bound type parameters. Thus, we will see how the bound and unbound type parameters from Section 5.3 manifest in our rewriting system.

Example 19.5. We're going to look at this pair of protocol declarations, and the protocol generic signature $G_{\texttt{P}}$:

```
protocol Q {
  associatedtype B
}

protocol P {
  associatedtype A: Q
}
```

Protocol `P` has an associated conformance requirement $[\texttt{Self.[P]A: Q}]_{\texttt{P}}$, and $G_{\texttt{P}}$ has a conformance requirement $[\texttt{τ\_0\_0: P}]$, so the protocol dependency graph has the two edges $G_{\texttt{P}} \prec \texttt{P}$ and $\texttt{P} \prec \texttt{Q}$. We first build a rewrite system for `Q`. The rewrite system for $G_{\texttt{P}}$ imports rules from `P` and `Q`:

$[\texttt{Q}] \cdot \texttt{B} \Rightarrow [\texttt{Q}|\texttt{B}]$	(1)
$[\texttt{P}] \cdot \texttt{A} \Rightarrow [\texttt{P}|\texttt{A}]$	(2)
$[\texttt{P}|\texttt{A}] \cdot [\texttt{Q}] \Rightarrow [\texttt{P}|\texttt{A}]$	(3)
$[\texttt{P}|\texttt{A}] \cdot \texttt{B} \Rightarrow [\texttt{P}|\texttt{A}] \cdot [\texttt{Q}|\texttt{B}]$	(*4)
$\texttt{τ\_0\_0} \cdot [\texttt{P}] \Rightarrow \texttt{τ\_0\_0}$	(5)
$\texttt{τ\_0\_0} \cdot \texttt{A} \Rightarrow \texttt{τ\_0\_0} \cdot [\texttt{P}|\texttt{A}]$	(*6)

We can categorize these rewrite rules as follows:

- Associated type rules: (1) and (2).
- Conformance requirements: (3) and (5).
- Rules added by completion are indicated with an asterisk: (*4) and (*6).

We omit the identity conformance rules $[\texttt{Q}] \cdot [\texttt{Q}] \Rightarrow [\texttt{Q}]$ and $[\texttt{P}] \cdot [\texttt{P}] \Rightarrow [\texttt{P}]$; in this example they would only clutter the presentation. They will serve a useful purpose later, in Example 19.10.

We now go through the construction step by step. Protocol `Q` does not depend on any other protocols. The rewrite system for `Q` is just rule (1):

$$[\mathtt{Q}] \cdot \mathtt{B} \Rightarrow [\mathtt{Q}|\mathtt{B}] \tag{1}$$

Protocol `P` imports the single rule of `Q`, and adds rules (2) and (3):

$$[\mathtt{P}] \cdot \mathtt{A} \Rightarrow [\mathtt{P}|\mathtt{A}] \tag{2}$$
$$[\mathtt{P}|\mathtt{A}] \cdot [\mathtt{Q}] \Rightarrow [\mathtt{P}|\mathtt{A}] \tag{3}$$

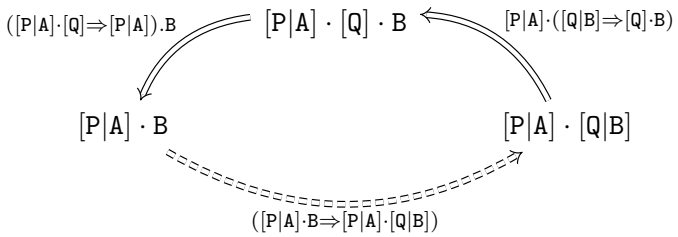
We need to check the left-hand side of rules (2) and (3) for overlaps with other rules. Rule (2) does not overlap with any other rules. Rule (3) overlaps with rule (1) on the term $[\mathtt{P}|\mathtt{A}] \cdot [\mathtt{Q}] \cdot \mathtt{B}$:

$$\begin{array}{l} [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{Q}] \\ \qquad [\mathtt{Q}] \cdot \mathtt{B} \end{array}$$

Rule (3) is local to `P`, while (1) was imported from `Q`. The two sides of the critical pair reduce to $[\mathtt{P}|\mathtt{A}] \cdot \mathtt{B}$ and $[\mathtt{P}|\mathtt{A}] \cdot [\mathtt{Q}|\mathtt{B}]$. Resolving this critical pair introduces rule (*4):

$$[\mathtt{P}|\mathtt{A}] \cdot \mathtt{B} \Rightarrow [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{Q}|\mathtt{B}] \tag{*4}$$

We also record a rewrite loop that defines rule (*4) via rules (3) and (1):



Once again, we check for overlaps in the left-hand sides of our local rules—now (2), (3) and (4). We see that no more critical pairs remain, and we have a convergent rewriting system for `P`.

Finally, we build the rewrite system for $G_{\mathtt{P}}$. We import all rules from `Q` and `P`, and add one new local rule corresponding to the conformance requirement $[\tau\_0\_0\colon \mathtt{P}]$:

$$\tau\_0\_0 \cdot [\mathtt{P}] \Rightarrow \tau\_0\_0 \tag{5}$$

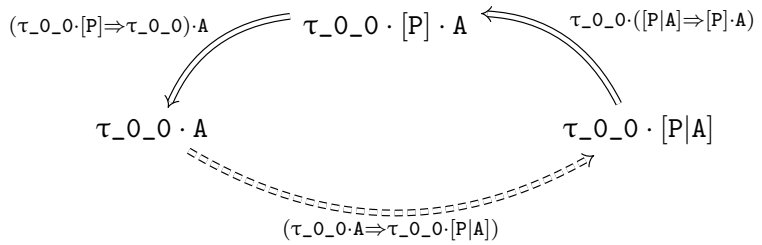
We need to check the left-hand side of rule (5) for overlaps with other rules. Indeed, rule (5) overlaps with rule (2) on the term $\tau\_0\_0 \cdot [\mathtt{P}] \cdot \mathtt{A}$:

$$\begin{array}{l} \tau\_0\_0 \cdot [\mathtt{P}] \\ \qquad [\mathtt{P}] \cdot \mathtt{A} \end{array}$$

Resolving this critical pair introduces rule (*6):

$$\texttt{τ\_0\_0} \cdot \texttt{A} \Rightarrow \texttt{τ\_0\_0} \cdot [\texttt{P}|\texttt{A}] \tag{*6}$$

We also record a rewrite loop that defines rule (*6) via rules (5) and (2):



At this point, there might be overlaps involving one of rule (5) or (*6); a quick check shows none remain, so we have a convergent rewriting system for $G_{\texttt{P}}$.

Now, we get to the interesting part. Consider these two type parameters of $G_{\texttt{P}}$, and their corresponding terms:

1. The unbound type parameter $\texttt{τ\_0\_0.A.B}$, and term $\texttt{τ\_0\_0} \cdot \texttt{A} \cdot \texttt{B}$.
2. The bound type parameter $\texttt{τ\_0\_0.[P]A.[Q]B}$, and term $\texttt{τ\_0\_0} \cdot [\texttt{P}|\texttt{A}] \cdot [\texttt{Q}|\texttt{B}]$.

The second term is reduced, so the first term must reduce to the second term. Term reduction outputs a positive rewrite path p with $\mathrm{src}(p) = \texttt{τ\_0\_0} \cdot \texttt{A} \cdot \texttt{B}$ and $\mathrm{dst}(p) = \texttt{τ\_0\_0} \cdot [\texttt{P}|\texttt{A}] \cdot [\texttt{Q}|\texttt{B}]$. This path involves rules (4) and (6) added by completion:

$$p := (\texttt{τ\_0\_0} \cdot \texttt{A} \Rightarrow \texttt{τ\_0\_0} \cdot [\texttt{P}|\texttt{A}]) \cdot \texttt{B} \circ \texttt{τ\_0\_0} \cdot ([\texttt{P}|\texttt{A}] \cdot \texttt{B} \Rightarrow [\texttt{P}|\texttt{A}] \cdot [\texttt{B}|\texttt{Q}])$$

Here is a diagram for p:

$$\texttt{τ\_0\_0} \cdot \texttt{A} \cdot \texttt{B} \Longrightarrow \texttt{τ\_0\_0} \cdot [\texttt{P}|\texttt{A}] \cdot \texttt{B} \Longrightarrow \texttt{τ\_0\_0} \cdot [\texttt{P}|\texttt{A}] \cdot [\texttt{Q}|\texttt{B}]$$

There is a "telescoping" effect, as term reduction processes the name symbols from left to right, replacing them with associated type symbols. As an aside, if we have an *invalid* type parameter, such as $\texttt{τ\_0\_0.A.A}$, the term reduces to $\texttt{τ\_0\_0} \cdot [\texttt{P}|\texttt{A}] \cdot \texttt{A}$ but no further; the final name symbol `A` cannot be "resolved" because the type parameter $\texttt{τ\_0\_0.A}$ does not *have* a member type named `A`.

The standard term $\texttt{τ\_0\_0} \cdot \texttt{A} \cdot \texttt{B}$ reduces to $\texttt{τ\_0\_0} \cdot [\texttt{P}|\texttt{A}] \cdot [\texttt{Q}|\texttt{B}]$ because the unbound type parameter $\texttt{τ\_0\_0.A.B}$ is equivalent to the bound type parameter $\texttt{τ\_0\_0.[P]A.[Q]B}$, and the latter is the reduced type of the former. The equivalence follows because we can derive a same-type requirement, and the left-hand side of this same-type requirement is the reduced type because no smaller type parameter can be shown to be equivalent:

$$G_{\texttt{P}} \vdash [\texttt{τ\_0\_0.[P]A.[Q]B == τ\_0\_0.A.B}]$$

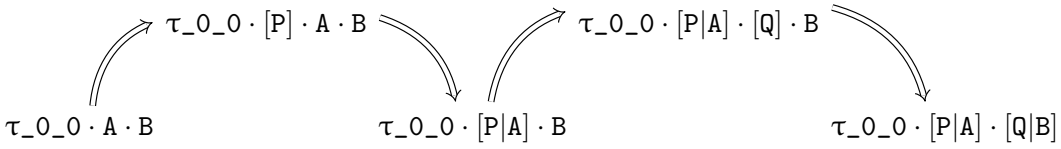
Here is one possible derivation for this same-type requirement:

1. [τ_0_0: P] (Conf)
2. [τ_0_0.[P]A == τ_0_0.A] (AssocBind 1)
3. [τ_0_0.A == τ_0_0.[P]A] (Sym 2)
4. [τ_0_0.[P]A: Q] (AssocConf 1)
5. [τ_0_0.A.B == τ_0_0.[P]A.B] (SameName 3 4)
6. [τ_0_0.[P]A.B == τ_0_0.A.B] (Sym 5)
7. [τ_0_0.[P]A.[Q]B == τ_0_0.[P]A.B] (AssocBind 4)
8. [τ_0_0.[P]A.[Q]B == τ_0_0.A.B] (Trans 7 6)

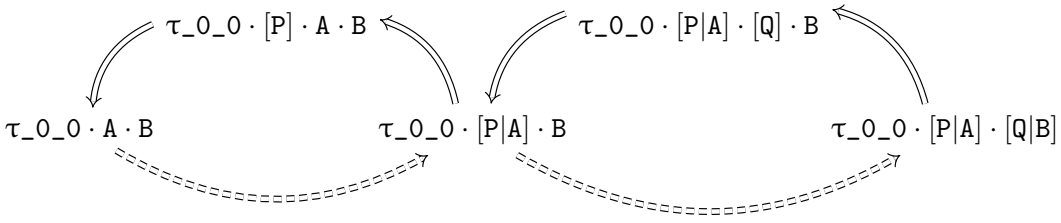
By Theorem 18.4, we can transform this derivation into a rewrite path from $\tau\_0\_0 \cdot A \cdot B$ to $\tau\_0\_0 \cdot [P|A] \cdot [Q|B]$, involving the initial rewrite rules only. We call this path p':

$$p' := (\tau\_0\_0 \Rightarrow \tau\_0\_0 \cdot [P]) \cdot A \cdot B \circ \tau\_0\_0 \cdot ([P] \cdot A \Rightarrow [P|A]) \cdot B \\ \circ \tau\_0\_0 \cdot ([P|A] \Rightarrow [P|A] \cdot [Q]) \cdot B \circ \tau\_0\_0 \cdot [P|A] \cdot ([Q] \cdot B \Rightarrow [Q|B])$$

Unlike p, p' is *not* a positive rewrite path, because the first and third steps are negative; each one applies a conformance rule backwards. We can visualize p' as a path in the rewrite graph, with the negative rewrite steps going up:



We now have two parallel rewrite paths: p (output by the normal form algorithm), and p' (constructed from our derivation). Their composition $p' \circ p^{-1}$ is a rewrite loop; we can represent it as a diagram, with rules (*4) and (*6) indicated by dashed arrows:



Recall the two rewrite loops we recorded in completion. We take the first rewrite loop, and whisker it on the right by B; then, we whisker the second rewrite loop on the left by τ_0_0. Now, the two rewrite loops visit the common term $\tau\_0\_0 \cdot [P|A] \cdot B$. Informally, we can "glue" them together at this point, and then we get $p' \circ p^{-1}$.

In fact, given any two parallel rewrite paths in a convergent rewriting system, we can always "tile" the two-dimensional space between them if we take the finite set of rewrite loops generated by resolving critical pairs, and glue them together along common edges or vertices after putting the loops "in context" via whiskering. We will develop this idea further in Chapter 22, but here is a thought to ruminate on in the meantime:

> Terms, generated by a finite set of symbols, are the zero-dimensional objects in the rewrite graph. Rewrite paths, generated by a finite set of rewrite rules, define an equivalence relation on terms; they are the one-dimensional objects. Rewrite loops, generated by a finite set of critical pairs, define an equivalence relation on paths; they are the two-dimensional objects.

An equivalence relation on paths is called a *homotopy relation*.

Example 19.6. Let's build on our previous example and look at how unbound type parameters reduce when they appear in requirements. We add a new protocol `R`, and declare a constrained protocol extension of `P` where `Self.A.B` conforms to `R`:

```
protocol R {}

protocol Q {
  associatedtype B
}

protocol P {
  associatedtype A: Q
}

extension P where Self.A.B: R {}
```

When type checking these declarations, we need to build a generic signature for this protocol extension. We start with requirement [τ_0_0: P] from the extended type, and add the user-written requirement [τ_0_0.A.B: R]. The rewrite system looks like the previous example but with two new rules:

$$\tau\_0\_0 \cdot A \cdot B \cdot [R] \Rightarrow \tau\_0\_0 \cdot A \cdot B \quad (7)$$

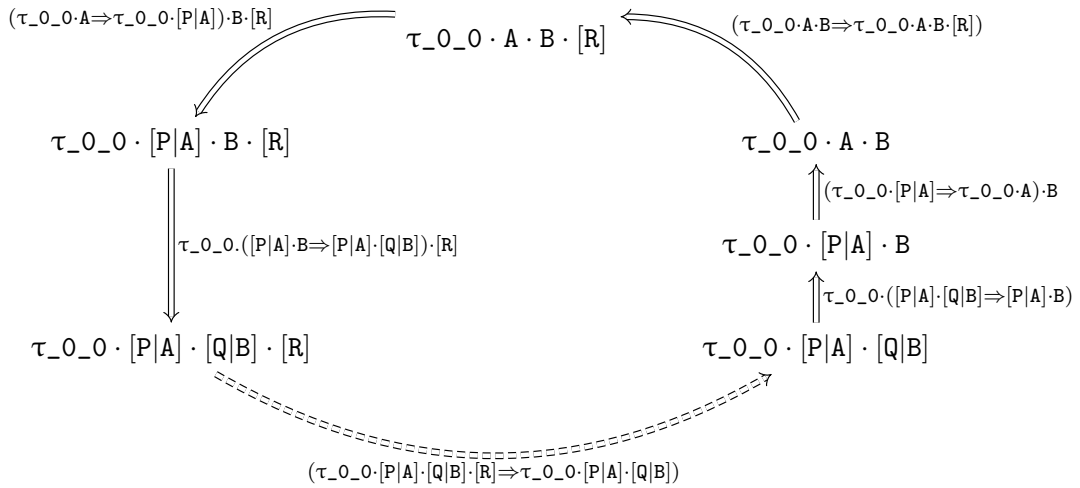
$$\tau\_0\_0 \cdot [P|A] \cdot [Q|B] \cdot [R] \Rightarrow \tau\_0\_0 \cdot [P|A] \cdot [Q|B] \quad (8^*)$$

Rule (7) corresponds to the user-written requirement. Rule (*8) is added by completion, resolving the overlap between rule (7) and (6) on the term $\tau\_0\_0 \cdot A \cdot B \cdot [R]$:

$\tau\_0\_0 \cdot A \cdot B \cdot [R]$

$\tau\_0\_0 \cdot A$

We record a rewrite loop defining rule (*8) via rule (4), (6) and (7):



Note that this was an overlap of the first kind, so rule (7) is now marked **left-simplified**. Thus, rule (*8) completely supersedes rule (7). Rule (*8) survives minimization and maps to the requirement [τ_0_0.[P]A.[Q]B: R], which appears in the generic signature that we output for this protocol extension:

```
<τ_0_0 where τ_0_0: P, τ_0_0.[P]A.[Q]B: R>
```

If our compiler invocation is generating a serialized module, we serialize the protocol extension's generic signature and do not compute it again when this module is imported in the future. We might still need a rewrite system for generic signature queries though, in which scenario we build a "second-generation" rewrite system:

1. We originally built a convergent rewriting system from the user-written requirements of the protocol extension.
2. After finding a minimal set of rules, we convert minimal rules to requirements, and serialize the resulting generic signature.
3. In a subsequent compiler invocation, we deserialize this generic signature.
4. Then, we build a convergent rewriting system from the requirements of this generic signature; rule (8) is now one of our initial rules.

The rewrite system after step (1) is equivalent to that after step (4). That is, if we ignore **left-simplified** and **right-simplified** rules, both will have the same rules, up to permutation. (We can't completely prove this fact, because minimization is tricky. But this is the intended invariant here.)

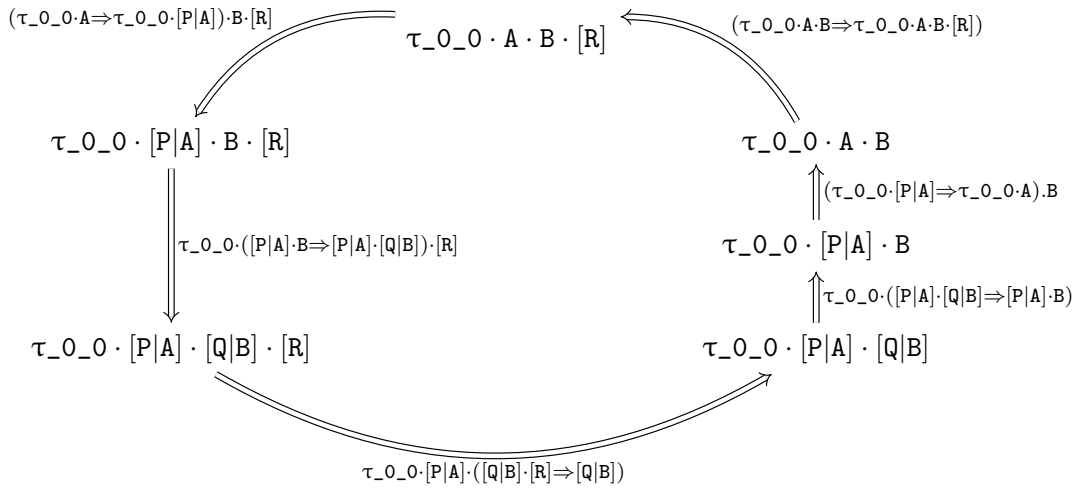
Example 19.7. Now we're going to change `Q` to add an associated requirement stating that `B` conform to `R`. The conformance requirement [τ_0_0.A.B: R] in `where` clause of our protocol extension becomes redundant as a result, because every `B` (of a type conforming to `Q`) now conforms to `R`:

```
protocol R {}

protocol Q {
  associatedtype B: R
}

protocol P {
  associatedtype A: Q
}

extension P where Self.A.B: R {}
```

The rewrite system for building the protocol extension's generic signature starts out as in the previous example, but now one more imported rule comes from protocol `Q`:

$$[\mathtt{Q}|\mathtt{B}] \cdot [\mathtt{R}] \Rightarrow [\mathtt{Q}|\mathtt{B}]$$

The conformance rule $\tau\_0\_0 \cdot \mathtt{A} \cdot \mathtt{B} \cdot [\mathtt{R}] \Rightarrow \tau\_0\_0 \cdot \mathtt{A} \cdot \mathtt{B}$ overlaps with $\tau\_0\_0 \cdot \mathtt{A} \Rightarrow \tau\_0\_0 \cdot [\mathtt{P}|\mathtt{A}]$ on the term $\tau\_0\_0 \cdot \mathtt{A} \cdot \mathtt{B} \cdot [\mathtt{R}]$, as before. This is an overlap of the first kind, so the original rule $\tau\_0\_0 \cdot \mathtt{A} \cdot \mathtt{B} \cdot [\mathtt{R}] \Rightarrow \tau\_0\_0 \cdot \mathtt{A} \cdot \mathtt{B}$ is marked **left-simplified**. This time though, the critical pair is trivial; both sides already reduce to $\tau\_0\_0 \cdot [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{Q}|\mathtt{B}]$:



Example 19.8. Our next goal is to understand what happens when a type parameter conforms to two unrelated protocols that both declare an associated type with the same name:

```
protocol P1 {
  associatedtype A
}

protocol P2 {
  associatedtype A
}
```

We're going to look at this generic signature:

```
<τ_0_0 where τ_0_0: P1, τ_0_0: P2>
```

Every equivalence class of type parameters can be uniquely identified by some unbound type parameter. In the above, the type parameters `τ_0_0.[P1]A` and `τ_0_0.[P2]A` must therefore be equivalent to `τ_0_0.A`. We're going to see how this works out in the rewrite system. Here is the convergent rewriting system for the above generic signature:

$[\texttt{P1}] \cdot \texttt{A} \Rightarrow [\texttt{P1}|\texttt{A}]$	(1)
$[\texttt{P2}] \cdot \texttt{A} \Rightarrow [\texttt{P2}|\texttt{A}]$	(2)
$\texttt{τ\_0\_0} \cdot [\texttt{P1}] \Rightarrow \texttt{τ\_0\_0}$	(3)
$\texttt{τ\_0\_0} \cdot [\texttt{P2}] \Rightarrow \texttt{τ\_0\_0}$	(4)
$\texttt{τ\_0\_0} \cdot \texttt{A} \Rightarrow \texttt{τ\_0\_0} \cdot [\texttt{P1}|\texttt{A}]$	(*5)
$\texttt{τ\_0\_0} \cdot [\texttt{P2}|\texttt{A}] \Rightarrow \texttt{τ\_0\_0} \cdot [\texttt{P1}|\texttt{A}]$	(*6)

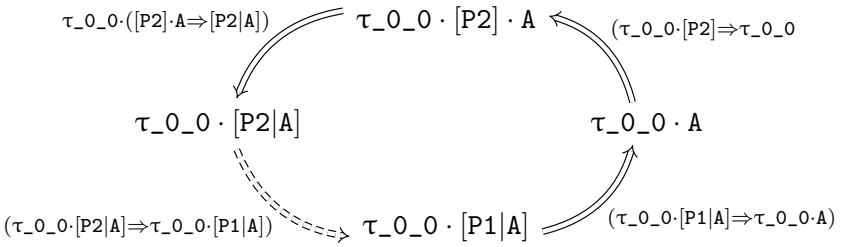
Rules (1) and (2) are imported from `P1` and `P2`, and rules (3) and (4) correspond to the conformance requirements [`τ_0_0: P1`] and [`τ_0_0: P2`]. Completion also adds two additional rules. Rule (*5) is added when resolving the overlap between rule (3) and rule (1), exactly as in Example 19.5.

Rule (4) overlaps with rule (2) on the term $\texttt{τ\_0\_0} \cdot [\texttt{P2}] \cdot \texttt{A}$:

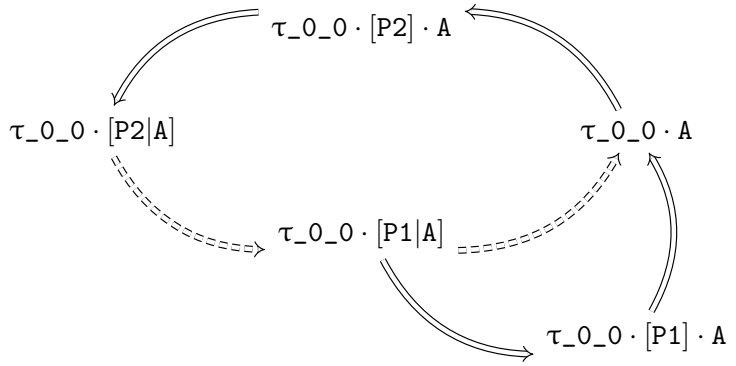
$$\begin{array}{l} \texttt{τ\_0\_0}.[\texttt{P2}] \\ \qquad [\texttt{P2}] \cdot \texttt{A} \end{array}$$

This gives us rule (*6), which has a new form we haven't seen before. One side of the critical pair reduces to $\texttt{τ\_0\_0} \cdot [\texttt{P2}|\texttt{A}]$, and the other reduces to $\texttt{τ\_0\_0} \cdot [\texttt{P1}|\texttt{A}]$ via rule (5).

We record a rewrite loop defining rule (*6) via (2), (4) and (5):



We can "glue" the rewrite loops defining rule (*5) and (*6) together into a single diagram:



The two dashed arrows are rules (*5) and (*6). The solid arrows indicate a rewrite path joining `τ_0_0·[P1|A]` with `τ_0_0·[P2|A]` involving the initial rules only. This rewrite path corresponds to the following derivation:

1. [`τ_0_0: P1`] (Conf)
2. [`τ_0_0.[P1]A == τ_0_0.A`] (AssocBind 1)
3. [`τ_0_0: P2`] (Conf)
4. [`τ_0_0.[P2]A == τ_0_0.A`] (AssocBind 3)
5. [`τ_0_0.A == τ_0_0.[P2]A`] (Sym 4)
6. [`τ_0_0.[P1]A == τ_0_0.[P2]A`] (Trans 2 5)

Indeed, we see that the equivalence class with reduced type `τ_0_0.[P1]A` contains three type parameters. Here they are with corresponding terms, in type parameter order. In this case, the reduced type and the reduced term coincide:

Type	**Term**
`τ_0_0.[P1]A`	`τ_0_0 · [P1|A]`
`τ_0_0.[P2]A`	`τ_0_0 · [P2|A]`
`τ_0_0.A`	`τ_0_0 · A`

19.3. More Critical Pairs

Example 19.9. Consider this protocol inheritance hierarchy, where `Bot` inherits from `Mid`, which inherits from `Top`, and `Mid` declares an associated type:

```
protocol Top {}

protocol Mid: Top {
  associatedtype A
}

protocol Bot: Mid {}
```

We're going to examine the rewrite system for the generic signature $G_{\texttt{Bot}}$, made up rules imported from `Mid` and `Bot`, and the local rules (as before, we omit the identity conformance rules):

$[\texttt{Mid}] \cdot \texttt{A} \Rightarrow [\texttt{Mid}|\texttt{A}]$	(1)
$[\texttt{Mid}] \cdot [\texttt{Top}] \Rightarrow [\texttt{Mid}]$	(2)
$[\texttt{Bot}] \cdot \texttt{A} \Rightarrow [\texttt{Bot}|\texttt{A}]$	(3)
$[\texttt{Bot}] \cdot [\texttt{Mid}] \Rightarrow [\texttt{Bot}]$	(4)
$[\texttt{Bot}] \cdot [\texttt{Mid}|\texttt{A}] \Rightarrow [\texttt{Bot}|\texttt{A}]$	(*5)
$[\texttt{Bot}] \cdot [\texttt{Top}] \Rightarrow [\texttt{Bot}]$	(*6)
$\tau\_0\_0 \cdot [\texttt{Bot}] \Rightarrow \tau\_0\_0$	(7)
$\tau\_0\_0 \cdot [\texttt{Mid}] \Rightarrow \tau\_0\_0$	(*8)
$\tau\_0\_0 \cdot [\texttt{Top}] \Rightarrow \tau\_0\_0$	(*9)
$\tau\_0\_0 \cdot \texttt{A} \Rightarrow \tau\_0\_0 \cdot [\texttt{Bot}|\texttt{A}]$	(*10)
$\tau\_0\_0 \cdot [\texttt{Mid}|\texttt{A}] \Rightarrow \tau\_0\_0 \cdot [\texttt{Bot}|\texttt{A}]$	(*11)

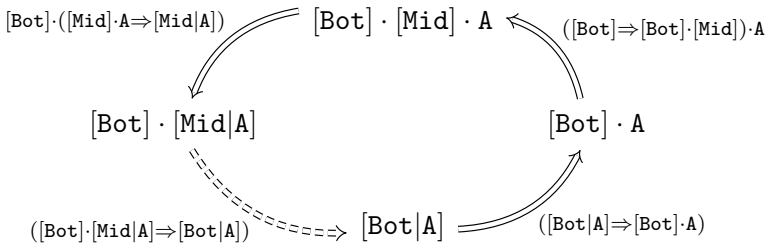
We're going to focus on a handful of interesting rules:

- While `Bot` does not declare any associated types, it inherits `A` from `Mid`; rule (3) is the inherited associated type rule for `A`.
- Rules (*5) and (*11) are added by completion as are a consequence of the inherited associated type rule.
- Rules (*6), (*8) and (*9) are also from completion. They express the "transitive" conformance requirements $[\texttt{Self: Top}]_{\texttt{Bottom}}$, $[\tau\_0\_0\texttt{: Top}]$ and $[\tau\_0\_0\texttt{: Mid}]$.

Rule (4) overlaps with rule (1) on the term $[\texttt{Bot}] \cdot [\texttt{Mid}] \cdot \texttt{A}$:

$$\begin{array}{l} [\texttt{Bot}] \cdot [\texttt{Mid}] \\ \qquad [\texttt{Mid}] \cdot \texttt{A} \end{array}$$

Resolving this critical pair records a rewrite loop defining rule (*5) via (1), (3) and (4):

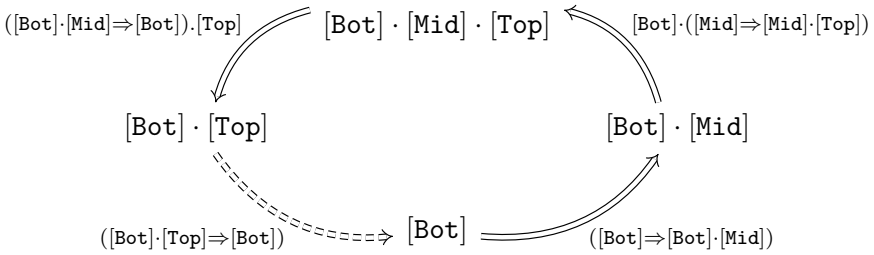


Rule (*5) says, if a term conforming to `Bot` is followed by $[\texttt{Mid}|\texttt{A}]$, we can reduce it to $[\texttt{Bot}|\texttt{A}]$.

Rule (4) overlaps with rule (3) on the term $[\texttt{Bot}] \cdot [\texttt{Mid}] \cdot [\texttt{Top}]$:

$$\begin{array}{l} [\texttt{Bot}] \cdot [\texttt{Mid}] \\ \qquad [\texttt{Mid}] \cdot [\texttt{Top}] \end{array}$$

Resolving this critical pair records a rewrite loop defining rule (*6) via rule (2) and (4):

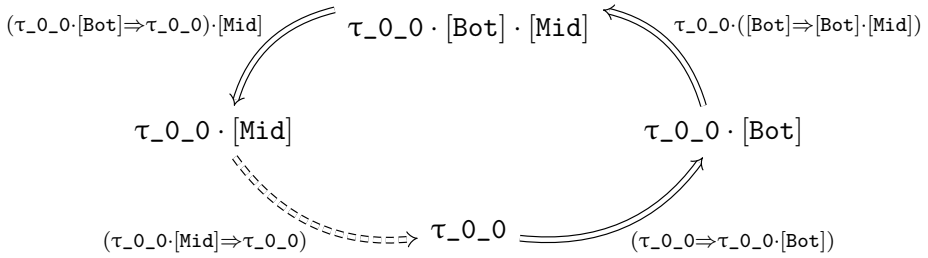


Rule (*6) says, if a term conforms to `Bot`, it also conforms to `Top`. In general, after completion, rewrite rules of the form $[\texttt{P}] \cdot [\texttt{Q}] \Rightarrow [\texttt{P}]$ will encode the transitive closure of the protocol inheritance relation.

This completes the rewrite system for `Bot`. Moving on to $G_{\texttt{Bot}}$, we see that rule (7) overlaps with rule (4) on the term $\texttt{τ\_0\_0} \cdot [\texttt{Bot}] \cdot [\texttt{Mid}]$:

$$\begin{array}{l} \texttt{τ\_0\_0} \cdot [\texttt{Bot}] \\ \qquad [\texttt{Bot}] \cdot [\texttt{Mid}] \end{array}$$

Resolving this critical pair defines rule (*8) via rules (4), (7) and (*8):



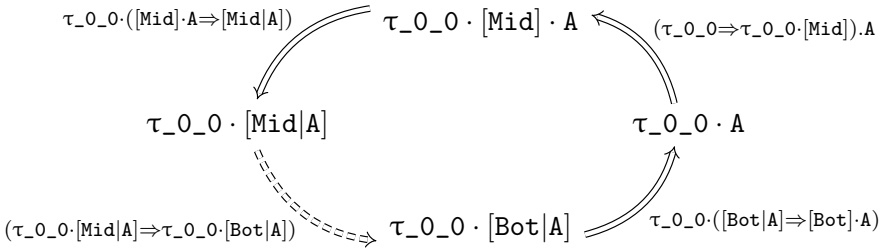
Rule (*8) is the derived requirement [τ_0_0: Mid]. Rule (7) overlaps with rule (5) in the same way, defining rule (*9), which is the derived requirement [τ_0_0: Top]. Once again, we see the transitive closure of the protocol inheritance relation being computed.

Rule (7) overlaps with rule (3) on the term τ_0_0 · [Bot] · A, defining rule (*10), by the same general principle as in Example 19.5.

Finally, rule (*8) overlaps with rule (1) on the term τ_0_0 · [Mid] · A:

τ_0_0 · [Mid]
[Mid] · A

Resolving this critical pair defines rule (*11) via rule (1), (3) and (*8):



In this example, the relationship between type parameters and terms is more subtle than we've previously seen, because the two distinct associated type symbols [Mid|A] and [Bot|A] both correspond to the same associated type *declaration*. On one hand, the equivalence class of τ_0_0.[Mid]A contains one other type parameter, τ_0_0.A. However, we have *three* equivalent terms: τ_0_0 · [Mid|A], τ_0_0 · [Bot|A], and τ_0_0 · A. Not only that, but applying Algorithm 18G to the reduced type parameter τ_0_0.[Mid]A outputs τ_0_0 · [Mid|A], which is *not* reduced; the reduced term here is τ_0_0 · [Bot|A], because [Bot|A] < [Mid|A] in the reduction order (Algorithm 18A):

Type	Term
τ_0_0.[Mid]A	τ_0_0 · [Bot|A]
	τ_0_0 · [Mid|A]
τ_0_0.A	τ_0_0 · A

Now, suppose we change the declaration of `Bot` to *re-state* the associated type:

```
protocol Bot: Mid {
  associatedtype A
}
```

The rewrite system is identical; we now call rule (3) an associated type rule instead of an *inherited* associated type rule, but the rule remains the same. Our equivalence class has three type parameters and three terms, because now every associated type symbol corresponds to a declaration:

Type	Term
$\tau\_0\_0.[\texttt{Bot}]\texttt{A}$	$\tau\_0\_0 \cdot [\texttt{Bot}\vert\texttt{A}]$
$\tau\_0\_0.[\texttt{Mid}]\texttt{A}$	$\tau\_0\_0 \cdot [\texttt{Mid}\vert\texttt{A}]$
$\tau\_0\_0.\texttt{A}$	$\tau\_0\_0 \cdot \texttt{A}$

We defined Algorithm 5C of the type parameter order so that a root associated type declaration always precedes such a re-stated associated type declaration; therefore the reduced type parameter of our equivalence class remains $\tau\_0\_0.[\texttt{Mid}]\texttt{A}$. The reduced term is also unchanged, $\tau\_0\_0 \cdot [\texttt{Bot}|\texttt{A}]$. This means that re-stating an associated type (or removing a re-stated associated type) does not change either the rewrite system or reduced type relation, and in particular, has no effect on calling convention, witness table layout, or any other aspect of the ABI.

Note that because the correspondence is not immediate, the mapping of terms to type parameters must be defined in such a way to output a reduced type parameter given a reduced term, by a careful choice of associated type declaration at each step. This will be explained in Section 20.4. This minor complication could be avoided if it weren't for inherited associated type symbols, and the fact that the reduction order on associated type symbols differs from the type parameter order on associated type declarations. However, both of those behaviors have important redeeming qualities, as we will see in Section 19.5.

Example 19.10. We now ask if `f()` type checks; is `T.C.B` equivalent to `T.A`?

```
protocol S {
  associatedtype A
  associatedtype B
  associatedtype C: S where Self == Self.C.C, Self.B == Self.C.A
}

func f<T: S>(val: T.C.B) {
  let val2: T.A = val
}
```

Our protocol states three associated requirements:

$$[\texttt{Self.C: S}]_{\texttt{S}}$$
$$[\texttt{Self == Self.C.C}]_{\texttt{S}}$$
$$[\texttt{Self.B == Self.C.A}]_{\texttt{S}}$$

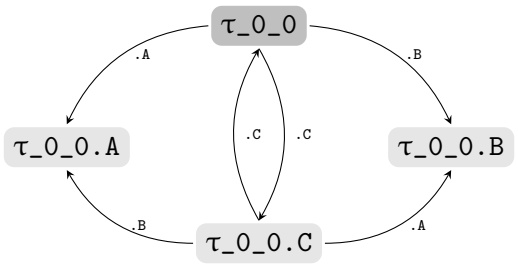
A consequence of the two same-type requirements is that `τ_0_0.A` is equivalent to `τ_0_0.C.C.A`, which is equivalent to `τ_0_0.C.B`. Here is the full derivation:

1. [`τ_0_0: S`] (Conf)
2. [`τ_0_0.C: S`] (AssocConf 1)
3. [`τ_0_0.C.B == τ_0_0.C.C.A`] (AssocSame 2)
4. [`τ_0_0 == τ_0_0.C.C`] (AssocSame 1)
5. [`τ_0_0.C.C == τ_0_0`] (Sym 4)
6. [`τ_0_0.C.C.A == τ_0_0.A`] (SameName 1 5)
7. [`τ_0_0.C.B == τ_0_0.A`] (Trans 3 6)

Our generic signature defines four infinite equivalence classes:

`τ_0_0`	`τ_0_0.A`	`τ_0_0.C`	`τ_0_0.B`
`τ_0_0.C.C`	`τ_0_0.C.B`	`τ_0_0.C.C.C`	`τ_0_0.C.A`
`τ_0_0.C.C.C.C`	`τ_0_0.C.C.A`	`τ_0_0.C.C.C.C.C`	`τ_0_0.C.C.B`
...	...	...	...

For a visual perspective, we turn to the type parameter graph of $G_{\texttt{S}}$:



Here are two concrete conforming types:

```
struct X: S {
  typealias A = Int
  typealias B = String
  typealias C = Y
}
```

```
struct Y: S {
  typealias A = String
  typealias B = Int
  typealias C = X
}
```

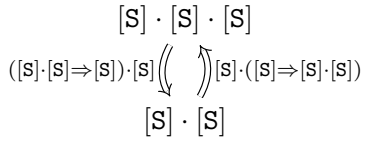
Now, let's look at our fundamental source of truth, the rewrite system. We start with an identity conformance rule, three associated type rules, and finally, three rules corresponding to user-written requirements:

$[\mathtt{S}] \cdot [\mathtt{S}] \Rightarrow [\mathtt{S}]$	(1)
$[\mathtt{S}] \cdot \mathtt{A} \Rightarrow [\mathtt{S}\vert\mathtt{A}]$	(2)
$[\mathtt{S}] \cdot \mathtt{B} \Rightarrow [\mathtt{S}\vert\mathtt{B}]$	(3)
$[\mathtt{S}] \cdot \mathtt{C} \Rightarrow [\mathtt{S}\vert\mathtt{C}]$	(4)
$[\mathtt{S}] \cdot \mathtt{C} \cdot [\mathtt{S}] \Rightarrow [\mathtt{S}] \cdot \mathtt{C}$	(5)
$[\mathtt{S}] \cdot \mathtt{C} \cdot \mathtt{A} \Rightarrow [\mathtt{S}] \cdot \mathtt{B}$	(6)
$[\mathtt{S}] \cdot \mathtt{C} \cdot \mathtt{C} \Rightarrow [\mathtt{S}]$	(7)

Completion also adds *ten* new rules; we will reveal a few at a time. Let's consider the first rule, $[\mathtt{S}] \cdot [\mathtt{S}] \Rightarrow [\mathtt{S}]$. Every protocol has an identity conformance rule, but in the previous examples it didn't play an important role so we ignored it. The identity conformance rule always overlaps with itself:

$$[\mathtt{S}] \cdot [\mathtt{S}]$$
$$[\mathtt{S}] \cdot [\mathtt{S}]$$

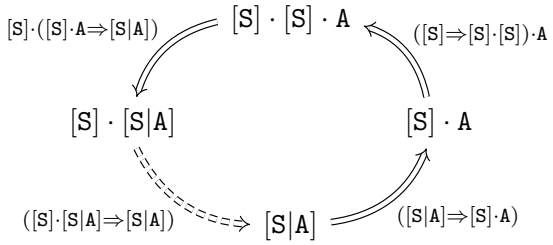
The critical pair is always trivial:



Identity conformance rules also overlap with associated type rules. For example, rule (1) overlaps with (2) on the term $[\mathtt{S}] \cdot [\mathtt{S}] \cdot \mathtt{A}$:

$$[\mathtt{S}] \cdot [\mathtt{S}]$$
$$[\mathtt{S}] \cdot \mathtt{A}$$

One side of this critical pair reduces to $[\mathtt{S}] \cdot [\mathtt{S}\vert\mathtt{A}]$, while the other reduces to $[\mathtt{S}\vert\mathtt{A}]$. We define a new rule $[\mathtt{S}] \cdot [\mathtt{S}\vert\mathtt{A}] \Rightarrow [\mathtt{S}\vert\mathtt{A}]$ and record a rewrite loop:



In the same way we get such a rule for every associated type:

$[\mathtt{S}] \cdot [\mathtt{S}|\mathtt{A}] \Rightarrow [\mathtt{S}|\mathtt{A}]$	(*8)
$[\mathtt{S}] \cdot [\mathtt{S}|\mathtt{B}] \Rightarrow [\mathtt{S}|\mathtt{B}]$	(*9)
$[\mathtt{S}] \cdot [\mathtt{S}|\mathtt{C}] \Rightarrow [\mathtt{S}|\mathtt{C}]$	(*10)

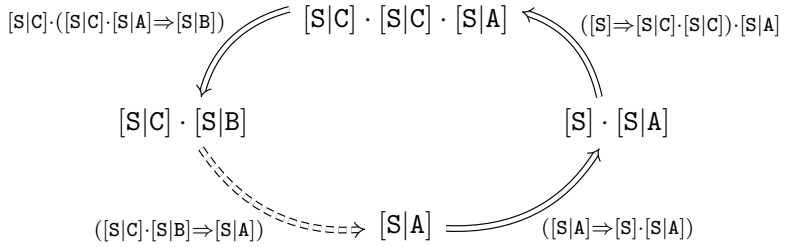
While we could have also written down similar rules in previous examples, they would not have illustrated anything of consequence. They will shortly; but first, we have a few overlaps between conformance rules and associated type rules. As in previous examples, we get the usual rules (*11), (*12), and (*13) for reducing unbound type parameters; we also left-simplify (5), (6) and (7), replacing them with rules (*14), (*15), and (*16):

$[\mathtt{S}|\mathtt{C}] \cdot \mathtt{A} \Rightarrow [\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{A}]$	(*11)
$[\mathtt{S}|\mathtt{C}] \cdot \mathtt{B} \Rightarrow [\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{B}]$	(*12)
$[\mathtt{S}|\mathtt{C}] \cdot \mathtt{C} \Rightarrow [\mathtt{S}]$	(*13)
$[\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}] \Rightarrow [\mathtt{S}|\mathtt{C}]$	(*14)
$[\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{C}] \Rightarrow [\mathtt{S}]$	(*15)
$[\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{A}] \Rightarrow [\mathtt{S}|\mathtt{B}]$	(*16)

Here is the main event: rule (*15) overlaps with (*16) on the term $[\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{A}]$:

$$\begin{array}{l} [\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{C}] \\ \qquad [\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{A}] \end{array}$$

The right-hand side of this critical pair reduces the overlap term $[\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{A}]$ with rule (*15), which gives us $[\mathtt{S}] \cdot [\mathtt{S}|\mathtt{A}]$. Previously, terms for bound type parameters would begin with *either* a protocol symbol or an associated type symbol, but not both; the associated type symbol already encodes the protocol. Indeed, this funny-looking term reduces to just $[\mathtt{S}|\mathtt{A}]$ via rule (*8), defined as a consequence of that mysterious identity conformance rule (1).



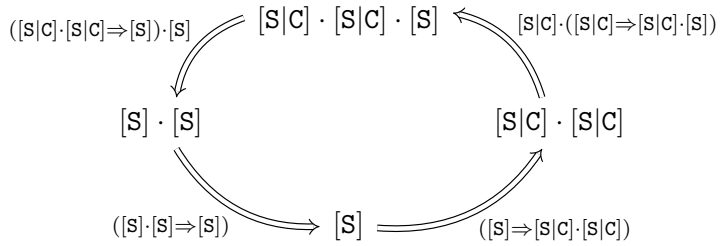
At last, and as predicated, we get the rule that proves $G_{\mathtt{S}} \vdash [\tau\_0\_0.\mathtt{A} == \tau\_0\_0.\mathtt{C.B}]$:

$$[\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{B}] \Rightarrow [\mathtt{S}|\mathtt{A}] \qquad (*17)$$

Now the *encore*: rule (*15) overlaps with rule (*14) on the term $[\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}]$:

$$\begin{array}{l} [\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}|\mathtt{C}] \\ \qquad [\mathtt{S}|\mathtt{C}] \cdot [\mathtt{S}] \end{array}$$

This critical pair resolves trivially, by way of the identity conformance rule:



If the identity conformance rule had not been part of that initial set, this last critical pair would *define* the identity conformance rule, and we would obtain the same rewrite system in the end. We don't need to explicitly add the identity conformance rule, after all. There is a practical consideration though. By making this rule part of the initial set, and cruicially, marking it **permanent**, we remove it from consideration in the rewrite system minimization algorithm. This cuts out a lot of unnecessary work.

Protocol `S` is an interesting test case demonstrating the rewrite system's ability to discover non-trivial identities. It originates from a developer's bug report in 2020 [144]; at the time, it broke the `GenericSignatureBuilder`'s minimization algorithm. Amusingly, the Rust compiler's generics implementation is unable to prove the derived requirement:

```
trait S {
  type A;
  type B;
  type C: S<A = Self::B, C = Self>;
}

fn f<T: S>(val: <T::C as S>::B) {
  let val2: T::A = val;
  // note: expected associated type `<T as S>::A`
  // found associated type `<<T as S>::C as S>::B`
}
```

19.4. Tietze Transformations

Both the Knuth-Bendix completion and the rule simplification algorithms preserve the equivalence relation on terms. When completion adds a new rewrite rule, it is because the corresponding pair of terms are already joined by a rewrite path. There is an analogous guarantee when rule simplification deletes a rewrite rule: the two terms are known to be joined by at least one other rewrite path that does not involve this rule. These transformations knead the reduction relation into a better form, while the equivalence relation on terms remains completely determined by the rewrite rules of our initial monoid presentation.

Mathematically speaking, both algorithms are defined as a transformation on monoid presentations. This transformation decomposes into a composition of sequential steps, where at each step the monoid presentation is changed in such a way that monoid isomorphism is preserved. So far, we've seen two of the four isomorphism-preserving transformations below, so named after Heinrich Tietze, who introduced them in 1908:

Definition 19.11. Let $\langle A \mid R \rangle$ be a finitely-presented monoid. An *elementary Tietze transformation*, or just Tietze transformation, is one of the following:

1. (Adding a rewrite rule) If a pair of terms u, $v \in A^*$ are already joined by a rewrite path from u to v—that is, if $u \sim v$ as elements of $\langle A \mid R \rangle$—we can add (u, v):

   $$\langle A \mid R \cup \{(u, v)\} \rangle$$

2. (Removing a rewrite rule) If $(u, v) \in R$ and we have a rewrite path from u to v that does not contain the rewrite step $x(u \Rightarrow v)y$ or $x(v \Rightarrow u)y$ for any x, $y \in A^*$, we can remove (u, v):

   $$\langle A \mid R \setminus \{(u, v)\} \rangle$$

3. (Adding a generator) If a is some symbol distinct from all other symbols of A, and $t \in A^*$ is any term, we can simultaneously add a and make it equivalent to t:

   $$\langle A \cup \{a\} \mid R \cup \{(t, a)\} \rangle$$

4. (Removing a generator) If $a \in A$, $(t, a) \in R$ for some term t not involving a, and no other $(u, v) \in R$ has a term u or v involving a, we can simultaneously remove a and (t, a):

   $$\langle A \setminus \{a\} \mid R \setminus \{(t, a)\} \rangle$$

The following is the key result here: two finitely-presented monoids are isomorphic if and only if they are *Tietze-equivalent*, meaning we can obtain one from the other by a finite sequence of elementary Tietze transformations.

Some finer points:

- For every Tietze transformation, there is a complementary transformation which undoes the change; in this way (1) and (2) are inverses, and similarly (3) and (4).
- We already know that changing the order of the terms in a rewrit erule—replacing $(u, v) \in R$ with (v, u)—does not change the set of rewrite steps generated, and thus presents the same monoid. Now we see this operation is actually a composition of two elementary Tietze transformations; we first add (v, u), and remove (u, v).
- Some definitions of (4) drop the condition that the removed symbol a not appear in any other rewrite rule $(u, v) \in R$. Our restriction does not cost us any generality, because if a occurs in any other rewrite rule, we can always first perform a series of Tietze transformations to eliminate such occurrences: for each such (u, v), we replace occurrences of a with t in u and v, add the new rewrite rule, and finally remove the old rule (u, v). However, we must still require that a not occur in t; otherwise, replacing a with t does not eliminate all occurrences of a.

Associated type symbols. Tietze transformations give us a new way to understand associated type symbols. The ultimate equivalence between rewrite systems built from user-written requirements and minimal requirements, shown in Section 19.2, means we could have defined Algorithms 18G and 18H to only ever build terms from protocol and name symbols. After completion, we end up with the same rewrite system, it just takes a little bit more work to get there. In this setup, among the initial rewrite rules, the only occurrences of associated type symbols are on the right-hand sides of associated type rules:

$$[\texttt{P}] \cdot \texttt{A} \Rightarrow [\texttt{P}|\texttt{A}]$$

Thus, we can understand the associated type rules as having been added by a sequence of Tietze transformations of the third kind, applied to some monoid presentation. This "primordial" monoid presentation involves only protocol and name symbols, and it encodes the same monoid that ours does, up to isomorphism. So why do we bother with associated type symbols at all? In the next section, we will see the answer has to do with the interaction between recursive conformance requirements and completion.

Further discussion. The elementary Tietze transformations are "higher dimensional" rewrite steps, in that they define an edge relation on a graph whose vertices are *monoid presentations*. A path in this graph witnesses the fact that the source and destination define an isomorphic pair of monoids. The problem of deciding if two presentations are joined by a path is the *monoid isomorphism problem*. Like the word problem, this is of course undecidable in general.

Tietze transformations are fundamental to the study of *combinatorial group theory*, and are described in any book on the subject, such as [145]. Recall that in a group presentation, a rewrite rule (u, v) can always be written as (uv^{-1}, ε), with the identity element on the right hand side. The term uv^{-1} is called a *relator*; a set of relators takes the place of rewrite rules in a group presentation. Tietze transformations of monoid presentations are described in [118] and [146].

19.5. Recursive Conformances

From previous examples, it may look like completion essentially enumerates *all* derived requirements, but this cannot be true in general. We know from Section 12.3 that recursive conformance requirements allow us to define generic signatures with an infinite set of derived requirements. If the compiler is to allow such generic signatures, this infinite set of derived requirements must be encoded by a finite number of rewrite rules.

We will see that with recursive conformance requirements, successful termination of the Knuth-Bendix algorithm depends both on the existence of associated type symbols (and rules), and our choice of reduction order. This furnishes the concrete example for a scenario mentioned in Section 17.4.

Example 19.12. Our first example is the protocol `N` that we encountered several times already, most recently in Section 17.3:

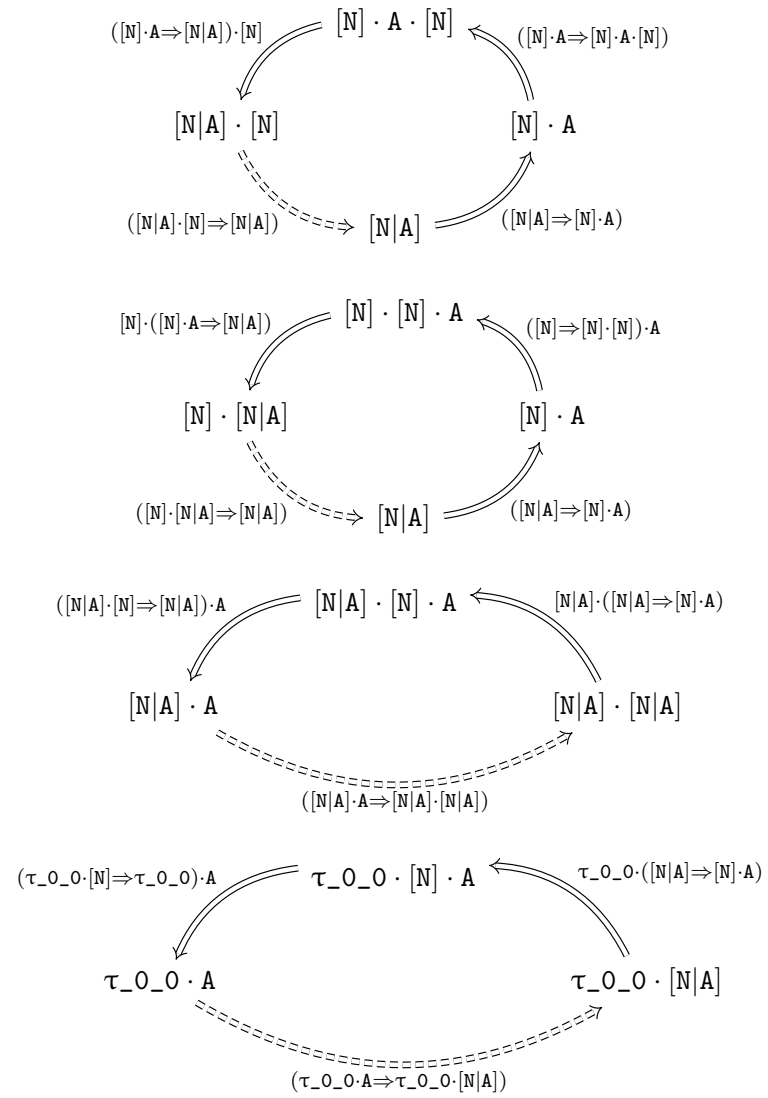
```
protocol N {
  associatedtype A: N
}
```

Let's look at the convergent rewriting system for $G_{\texttt{N}}$:

$[\texttt{N}] \cdot [\texttt{N}] \Rightarrow [\texttt{N}]$	(1)
$[\texttt{N}] \cdot \texttt{A} \Rightarrow [\texttt{N}|\texttt{A}]$	(2)
$[\texttt{N}] \cdot \texttt{A} \cdot [\texttt{N}] \Rightarrow [\texttt{N}] \cdot \texttt{A}$	(3)
$[\texttt{N}|\texttt{A}] \cdot [\texttt{N}] \Rightarrow [\texttt{N}|\texttt{A}]$	(*4)
$[\texttt{N}] \cdot [\texttt{N}|\texttt{A}] \Rightarrow [\texttt{N}|\texttt{A}]$	(*5)
$[\texttt{N}|\texttt{A}] \cdot \texttt{A} \Rightarrow [\texttt{N}|\texttt{A}] \cdot [\texttt{N}|\texttt{A}]$	(*6)
$\tau\_0\_0 \cdot [\texttt{N}] \Rightarrow \tau\_0\_0$	(7)
$\tau\_0\_0 \cdot \texttt{A} \Rightarrow \tau\_0\_0 \cdot [\texttt{N}|\texttt{A}]$	(*8)

Rules (*4), (*5), (*6) and (*8) are defined by the rewrite loops shown in Figure 19.1. Rule (3) is marked **left-simplified** because its left-hand side $[\texttt{N}] \cdot \texttt{A} \cdot [\texttt{N}]$ can always be reduced by the rewrite path $([\texttt{N}] \cdot \texttt{A} \Rightarrow [\texttt{N}|\texttt{A}]).[\texttt{N}] \circ ([\texttt{N}|\texttt{A}] \cdot [\texttt{N}] \Rightarrow [\texttt{N}|\texttt{A}])$.

Figure 19.1.: Critical pairs in $G_{\mathtt{N}}$



In our rewrite system, the type parameter terms have $\texttt{τ\_0\_0}$ as the first symbol, and either $\texttt{A}$ or $[\texttt{N}|\texttt{A}]$ as each subsequent symbol. Any such term reduces to a term of the same length:

$$\texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}] \cdot \texttt{A} \cdot \texttt{A} \to \texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}] \cdot [\texttt{N}|\texttt{A}] \cdot [\texttt{N}|\texttt{A}]$$

The reduced terms are those that do not contain $\texttt{A}$; that is, a reduced term has the form $\texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}]^n$. Also, every type parameter of $G_{\texttt{N}}$ conforms to $\texttt{N}$. We have two infinite families of derived conformance requirements, where the subject types are the bound and unbound type parameters from each equivalence class:

[τ_0_0.[N]A: N]
[τ_0_0.[N]A.[N]A: N]
[τ_0_0.[N]A.[N]A.[N]A: N]
...
[τ_0_0.A: N]
[τ_0_0.A.A: N]
[τ_0_0.A.A.A: N]
...

Each derived requirement corresponds to an equivalence of terms $t.[\texttt{N}] \sim t$, where t is a type parameter term. We can explicitly construct rewrite paths that witness these equivalences. Let's temporarily denote the positive rewrite step for rules (*4), (*6) and (*8) by α, β and γ respectively:

$$\alpha := ([\texttt{N}|\texttt{A}] \cdot [\texttt{N}] \Rightarrow [\texttt{N}|\texttt{A}])$$
$$\beta := ([\texttt{N}|\texttt{A}] \cdot \texttt{A} \Rightarrow [\texttt{N}|\texttt{A}] \cdot [\texttt{N}|\texttt{A}])$$
$$\gamma := (\texttt{τ\_0\_0} \cdot \texttt{A} \Rightarrow \texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}])$$

For a derived requirement with bound subject type, $t = \texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}]^n$. We can rewrite $t.[\texttt{N}]$ to t with a single rewrite step, using rule α to eliminate the suffix $[\texttt{N}]$ while leaving the rest of the term unchanged:

$$\texttt{τ\_0\_0} \triangleleft \alpha$$
$$\texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}] \triangleleft \alpha$$
$$\texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}] \cdot [\texttt{N}|\texttt{A}] \triangleleft \alpha$$
$$\ldots$$

For those derived requirements with unbound subject types, $t = \texttt{τ\_0\_0} \cdot \texttt{A}$. To rewrite $t.[\texttt{N}]$ to t, we first reduce $t \to t'$, eliminate $[\texttt{N}]$ with rule α, and reverse the reduction to obtain t.

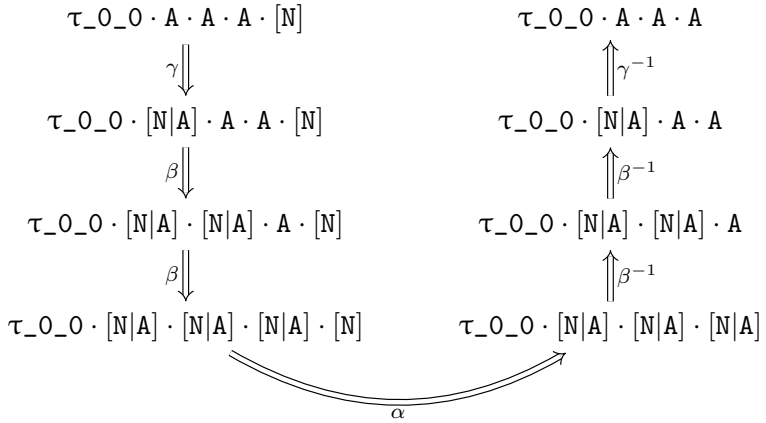
The reduction is given by a path p_n, where each p_n rewrites $\texttt{τ\_0\_0} \cdot \texttt{A}^n \to \texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}]^n$:

$$
\begin{aligned}
&p_1 := \gamma \\
&p_2 := (\gamma \rhd \texttt{A} \cdot [\texttt{N}]) \circ (\texttt{τ\_0\_0} \lhd \beta \rhd [\texttt{N}]) \\
&p_3 := (\gamma \rhd \texttt{A} \cdot \texttt{A} \cdot [\texttt{N}]) \circ (\texttt{τ\_0\_0} \lhd \beta \rhd \texttt{A} \cdot [\texttt{N}]) \circ (\texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}] \lhd \beta \rhd [\texttt{N}]) \\
&\ldots
\end{aligned}
$$

Thus, the derived requirements with unbound subject types correspond to the following rewrite paths:

$$
\begin{aligned}
&(p_1 \rhd [\texttt{N}]) \circ (\texttt{τ\_0\_0} \lhd \alpha) \circ p_1^{-1} \\
&(p_2 \rhd [\texttt{N}]) \circ (\texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}] \lhd \alpha) \circ p_2^{-1} \\
&(p_3 \rhd [\texttt{N}]) \circ (\texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}] \cdot [\texttt{N}|\texttt{A}] \lhd \alpha) \circ p_3^{-1} \\
&\ldots
\end{aligned}
$$

For example, $\texttt{τ\_0\_0} \cdot \texttt{A} \cdot \texttt{A} \cdot \texttt{A} \cdot [\texttt{N}]$ and $\texttt{τ\_0\_0} \cdot \texttt{A} \cdot \texttt{A} \cdot \texttt{A}$ both reduce to $\texttt{τ\_0\_0} \cdot [\texttt{N}|\texttt{A}] \cdot [\texttt{N}|\texttt{A}] \cdot [\texttt{N}|\texttt{A}]$; we can join them with this rewrite path:



Thus, our eight rules encode two infinite families of derived requirements.

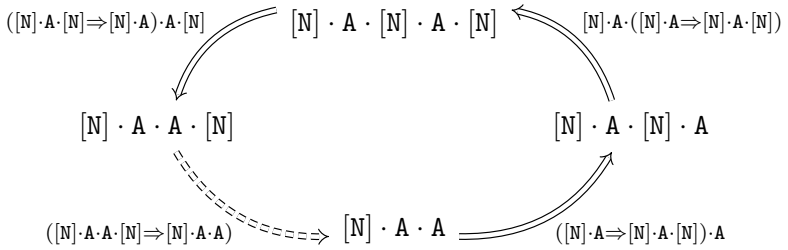
Now, we will show that the associated type rule $[\texttt{N}] \cdot \texttt{A} \Rightarrow [\texttt{N}|\texttt{A}]$ was essential. Let's start over with our the initial rules, but delete the symbol $[\texttt{N}|\texttt{A}]$ and rule (2) before attempting completion. Only two rules remain:

$[\texttt{N}] \cdot [\texttt{N}] \Rightarrow [\texttt{N}]$	(1)
$[\texttt{N}] \cdot \texttt{A} \cdot [\texttt{N}] \Rightarrow [\texttt{N}] \cdot \texttt{A}$	(3)

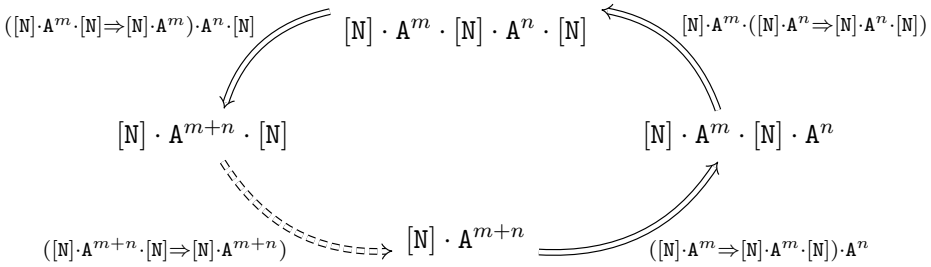
We see that rule (3) rule overlaps with itself on the term $[\mathtt{N}] \cdot \mathtt{A} \cdot [\mathtt{N}] \cdot \mathtt{A} \cdot [\mathtt{N}]$:

$$\begin{array}{l} [\mathtt{N}] \cdot \mathtt{A} \cdot [\mathtt{N}] \\ \qquad [\mathtt{N}] \cdot \mathtt{A} \cdot [\mathtt{N}] \end{array}$$

Resolving this critical pair introduces a new rule $[\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot [\mathtt{N}] \Rightarrow [\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A}$:



After adding this rule, we check for overlaps again. The new rule overlaps with rule (3) on $[\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot [\mathtt{N}] \cdot \mathtt{A} \cdot [\mathtt{N}]$, and also with itself on $[\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot [\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot [\mathtt{N}]$. Rule (3) also overlaps with the new rule on $[\mathtt{N}] \cdot \mathtt{A} \cdot [\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot [\mathtt{N}]$. It is apparent that resolving those critical pairs will introduce new rules, and this process will never end. We get an infinite family of critical pairs, indexed by m, $n \in \mathbb{N}$:



These critical pairs define an infinite sequence of rewrite rules:

$$\begin{array}{l} [\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot [\mathtt{N}] \Rightarrow [\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \\ [\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot \mathtt{A} \cdot [\mathtt{N}] \Rightarrow [\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot \mathtt{A} \\ [\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot \mathtt{A} \cdot \mathtt{A} \cdot [\mathtt{N}] \Rightarrow [\mathtt{N}] \cdot \mathtt{A} \cdot \mathtt{A} \cdot \mathtt{A} \cdot \mathtt{A} \\ \ldots \end{array}$$

Adding the symbol $[\mathtt{N}|\mathtt{A}]$ together with the rule $[\mathtt{N}] \cdot \mathtt{A} \Rightarrow [\mathtt{N}|\mathtt{A}]$ is a Tietze transformation, so we know it does not change the equivalence relation on terms. However, it ensures convergence. Convergence also depends on associated type symbols preceding name symbols, so $[\mathtt{N}|\mathtt{A}] < \mathtt{A}$. Otherwise, we once again end up with an infinite sequence of

rewrite rules:

$$[\texttt{N}|\texttt{A}] \cdot \texttt{A} \cdot [\texttt{N}] \Rightarrow [\texttt{N}|\texttt{A}] \cdot \texttt{A}$$
$$[\texttt{N}|\texttt{A}] \cdot \texttt{A} \cdot \texttt{A} \cdot [\texttt{N}] \Rightarrow [\texttt{N}|\texttt{A}] \cdot \texttt{A} \cdot \texttt{A}$$
$$[\texttt{N}|\texttt{A}] \cdot \texttt{A} \cdot \texttt{A} \cdot \texttt{A} \cdot [\texttt{N}] \Rightarrow [\texttt{N}|\texttt{A}] \cdot \texttt{A} \cdot \texttt{A} \cdot \texttt{A}$$
$$\ldots$$

Let's summarize this example using monoid presentations and Tietze transformations. We write n, a, b instead of $[\texttt{N}]$, $\texttt{A}$, $[\texttt{N}|\texttt{A}]$. Together with the identity conformance rule, the user-written requirement $[\texttt{Self.A: N}]_{\texttt{N}}$, with an unbound type parameter as the subject type, defines this presentation over the alphabet a, n:

$$M_1 := \langle a, n \,|\, nn \sim n,\ nan \sim na \rangle$$

We saw that completion does not terminate on M_1; abstractly, we can understand it as producing an *infinite* convergent presentation M_1', containing a family of rewrite rules parameterized by $i \in \mathbb{N}$:

$$M_1' := \langle a, n \,|\, na^i n \sim na^i \rangle$$

To ensure convergence, we added the symbol b and its defining rewrite rule $na \sim b$ to M_1; a Tietze transformation which gave us the presentation M_2 over a, b, n:

$$M_2 := \langle a, b, n \,|\, nn \sim n,\ nan \sim na,\ na \sim b \rangle$$

Performing completion on M_2 with the reduction order $n < b < a$ succeeds after adding the two rewrite rules $bn \sim b$ and $ba \sim bb$, giving us the finite convergent presentation M_2':

$$M_2' := \langle a, b, n \,|\, nn \sim n,\ nan \sim na,\ na \sim b,\ bn \sim b,\ ba \sim bb \rangle$$

Rule simplification removes the rewrite rule $nan \sim na$, giving us a reduced presentation:

$$M_2'' := \langle a, b, n \,|\, nn \sim n,\ na \sim b,\ bn \sim b,\ ba \sim bb \rangle$$

We haven't seen rewrite system minimization yet, but to minimize M_2'', we eliminate all rewrite rules added by completion except for $bn \sim b$, which we need because the original rewrite rule $nan \sim na$ was deleted. This leaves us with M_3:

$$M_3 := \langle a, b, n \,|\, nn \sim n,\ na \sim b,\ bn \sim b \rangle$$

The rule $bn \sim b$ is the minimal conformance requirement $[\texttt{Self.[N]A: N}]_{\texttt{N}}$ with a bound type parameter as the subject type. All of the above presentations define the *same* monoid, up to isomorphism. Furthermore, if we perform completion on M_3, we get M_2''.

Example 19.13. Let's say we inherit from `N`, and impose our own recursive conformance requirement on `A`:

```
protocol Q: N where A: Q {}
```

Even though protocol `Q` does not declare an associated type named `A`, we define an associated type symbol $[\texttt{Q}|\texttt{A}]$ and corresponding rule $[\texttt{Q}] \cdot \texttt{A} \Rightarrow [\texttt{Q}|\texttt{A}]$. We will see that convergence is contingent on this rule.

The generic signature $G_{\texttt{Q}}$ has all the derived requirements of $G_{\texttt{N}}$, and adds two more infinite families of conformances to `Q`:

$$\begin{array}{l}
\hline
[\tau_0_0.[\texttt{N}]\texttt{A}: \texttt{Q}] \\
[\tau_0_0.[\texttt{N}]\texttt{A}.[\texttt{N}]\texttt{A}: \texttt{Q}] \\
[\tau_0_0.[\texttt{N}]\texttt{A}.[\texttt{N}]\texttt{A}.[\texttt{N}]\texttt{A}: \texttt{Q}] \\
\ldots \\
\hline
[\tau_0_0.\texttt{A}: \texttt{Q}] \\
[\tau_0_0.\texttt{A}.\texttt{A}: \texttt{Q}] \\
[\tau_0_0.\texttt{A}.\texttt{A}.\texttt{A}: \texttt{Q}] \\
\ldots \\
\hline
\end{array}$$

Listing 19.1 shows the convegent rewrite system for `Q`, which imports rules from protocol `N`. Completion discovers the following critical pairs; we will be content to just summarize because they are similar to previous examples:

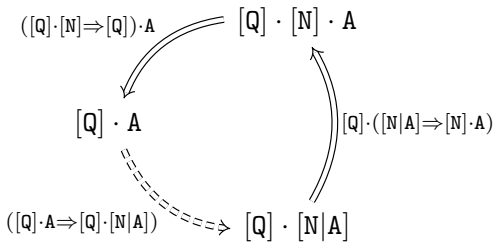
- Rule (6) overlaps with (7) on $[\texttt{Q}] \cdot [\texttt{Q}] \cdot \texttt{A}$. We define rule (*10) (this general principle was established in Example 19.10).
- Rule (8) overlaps with (2) on $[\texttt{Q}] \cdot [\texttt{N}] \cdot \texttt{A}$. We define rule (*11) (Example 19.9).
- Rule (9) overlaps with (7) on $[\texttt{Q}] \cdot \texttt{A} \cdot [\texttt{Q}]$. We define rule (*12), and mark rule (9) as **left-simplified** (Example 19.6).
- Rule (*12) overlaps with (8) on $[\texttt{Q}|\texttt{A}] \cdot [\texttt{Q}] \cdot [\texttt{N}]$. We define rule (*13).
- Rule (*12) overlaps with (7) on $[\texttt{Q}|\texttt{A}] \cdot [\texttt{Q}] \cdot \texttt{A}$. We define rule (*14) (Example 19.5).
- Rule (*13) overlaps with (2) on $[\texttt{Q}|\texttt{A}] \cdot [\texttt{N}] \cdot \texttt{A}$. We define rule (*15).

Here's a question. What if we only had associated type symbols that directly correspond to associated type declarations? That is, what if we got rid of the symbol $[\texttt{Q}|\texttt{A}]$, together with rule (7), the inherited associated type rule $[\texttt{Q}] \cdot \texttt{A} \Rightarrow [\texttt{Q}|\texttt{A}]$? The resulting monoid would again be equivalent, because this is a Tietze transformation. However, we will see that completion fails.

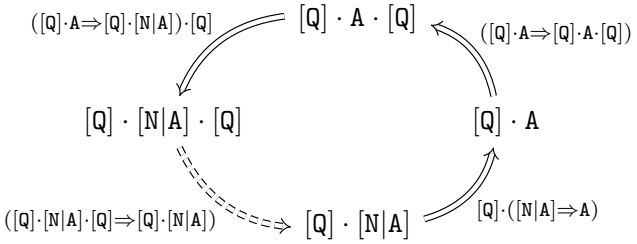
Listing 19.1.: Rewrite system for protocol `Q`

$[\mathtt{N}] \cdot [\mathtt{N}] \Rightarrow [\mathtt{N}]$	(1)
$[\mathtt{N}] \cdot \mathtt{A} \Rightarrow [\mathtt{N}\vert\mathtt{A}]$	(2)
$[\mathtt{N}\vert\mathtt{A}] \cdot [\mathtt{N}] \Rightarrow [\mathtt{N}\vert\mathtt{A}]$	(3)
$[\mathtt{N}] \cdot [\mathtt{N}\vert\mathtt{A}] \Rightarrow [\mathtt{N}\vert\mathtt{A}]$	(*4)
$[\mathtt{N}\vert\mathtt{A}] \cdot \mathtt{A} \Rightarrow [\mathtt{N}\vert\mathtt{A}] \cdot [\mathtt{N}\vert\mathtt{A}]$	(*5)
$[\mathtt{Q}] \cdot [\mathtt{Q}] \Rightarrow [\mathtt{Q}]$	(6)
$[\mathtt{Q}] \cdot \mathtt{A} \Rightarrow [\mathtt{Q}\vert\mathtt{A}]$	(7)
$[\mathtt{Q}] \cdot [\mathtt{N}] \Rightarrow [\mathtt{Q}]$	(8)
$[\mathtt{Q}] \cdot \mathtt{A} \cdot [\mathtt{Q}] \Rightarrow [\mathtt{Q}] \cdot \mathtt{A}$	(9)
$[\mathtt{Q}] \cdot [\mathtt{Q}\vert\mathtt{A}] \Rightarrow [\mathtt{Q}\vert\mathtt{A}]$	(*10)
$[\mathtt{Q}] \cdot [\mathtt{N}\vert\mathtt{A}] \Rightarrow [\mathtt{Q}\vert\mathtt{A}]$	(*11)
$[\mathtt{Q}\vert\mathtt{A}] \cdot [\mathtt{Q}] \Rightarrow [\mathtt{Q}\vert\mathtt{A}]$	(*12)
$[\mathtt{Q}\vert\mathtt{A}] \cdot [\mathtt{N}] \Rightarrow [\mathtt{Q}\vert\mathtt{A}]$	(*13)
$[\mathtt{Q}\vert\mathtt{A}] \cdot \mathtt{A} \Rightarrow [\mathtt{Q}\vert\mathtt{A}] \cdot [\mathtt{Q}\vert\mathtt{A}]$	(*14)
$[\mathtt{Q}\vert\mathtt{A}] \cdot [\mathtt{N}\vert\mathtt{A}] \Rightarrow [\mathtt{Q}\vert\mathtt{A}] \cdot [\mathtt{Q}\vert\mathtt{A}]$	(*15)

Rule (8) still overlaps with rule (2), but instead of rule (*11), we now get a rule $[\mathtt{Q}] \cdot \mathtt{A} \Rightarrow [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}]$ when we resolve this critical pair:



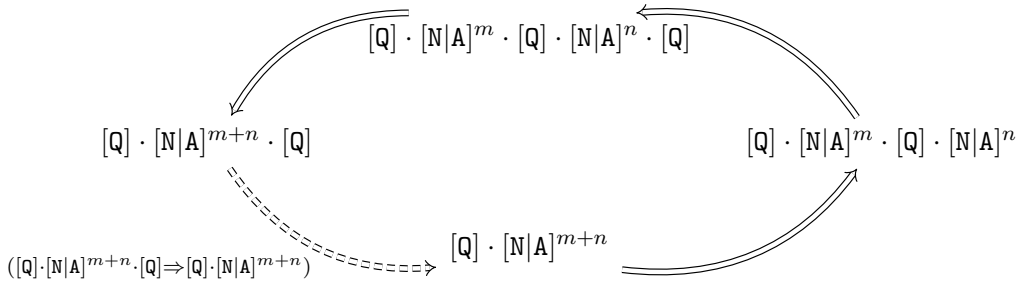
Rule (9) now overlaps with $[\mathtt{Q}] \cdot \mathtt{A} \Rightarrow [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}]$. We get a new rule $[\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{Q}] \Rightarrow [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}]$:



Now, the rule $[\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{Q}] \Rightarrow [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}]$ overlaps with itself on the term $[\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{Q}]$:

$$\begin{array}{l} [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{Q}] \\ \qquad\quad [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{Q}] \end{array}$$

The overlaps of this rule with itself generate an infinite family of critical pairs indexed by m, $n \in \mathbb{N}$:



These critical pairs define an infinite family of rewrite rules:

$$\begin{array}{l} [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{Q}] \Rightarrow [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \\ [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{Q}] \Rightarrow [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{N}|\mathtt{A}] \\ [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{Q}] \Rightarrow [\mathtt{Q}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{N}|\mathtt{A}] \cdot [\mathtt{N}|\mathtt{A}] \\ \ldots \end{array}$$

Recall that the reduction order on protocols was defined to compare the number of elements in the protocol inheritance closure before comparing names. So if neither `N` nor `Q` inherited any other protocols, we would have $[\texttt{N}] < [\texttt{Q}]$. However, since `Q` inherits from `N`, we have $[\texttt{Q}] < [\texttt{N}]$ and thus $[\texttt{Q|A}] < [\texttt{N|A}]$. We won't go into details, but the convergence of the above rewrite system *also* depends on the reduction order being so. If instead $[\texttt{N|A}] < [\texttt{Q|A}]$, our rewrite system would not have a (finite) convergent presentation (we will see below that we can still describe an *infinite* convergent presentation; but we have no way to compute with one).

To summarize the theory, let's write a, b, c, p, q instead of `A`, `[N|A]`, `[Q|A]`, `[P]`, `[Q]`. The identity conformance rules and user-written requirements of `N` and `Q` define the following presentation:

$$M_1 := \langle a, n, q \,|\, nn \sim n,\ nq \sim n,\ qq \sim q,\ nan \sim na,\ qaq \sim qa \rangle$$

Completion doesn't terminate with the above; abstractly, we get an infinite convergent presentation over the same generators:

$$M_1' := \langle a, n, q \,|\, na^i n \sim na^i,\ qa^i n \sim qa^i,\ qa^i q \sim qa^i \rangle$$

The first Tietze transformation adds b with defining rewrite rule $na \sim b$, as in the previous example:

$$M_2 := \langle a, b, n, q \,|\, nn \sim n,\ qq \sim q,\ qn \sim q,\ nan \sim na,\ qaq \sim qa,\ na \sim b \rangle$$

We saw that completion still fails; we get the following infinite convergent presentation:

$$M_2' := \langle a, b, n, q \,|\, nn \sim n,\ na \sim b,\ bn \sim b,\ ba \sim bb,\ qb^i q \sim qb^i,\ qb^i a \sim qb^{i+1} \rangle$$

However, we then add c with defining rewrite rule $qa \sim c$:

$$M_3 := \langle a, b, c, n, q \,|\, nn \sim n,\ qq \sim q,\ qn \sim q,\ nan \sim na,\ qaq \sim qa,\ na \sim b,\ qa \sim c \rangle$$

Completion succeeds on M_3 using a reduction order where $c < b < a$ and $p < n$, and we get a finite convergent rewriting system. We've already listed the numerous rules once, so we won't do it again. Let's call this finite convergent presentation M_2':

$$M_3' := \langle a, b, c, n, q \,|\, \ldots 15 \text{ rewrite rules } \ldots \rangle$$

The minimized rewrite system corresponds to this presentation:

$$M_4 := \langle a, b, c, n, q \,|\, nn \sim n,\ na \sim b,\ bn \sim b,\ qq \sim q,\ qa \sim c,\ qn \sim q,\ cq \sim c \rangle$$

If we start with M_3 and perform completion, we again get M_2'.

Example 19.14. Our final example lays bare a limitation of the Requirement Machine in the form a generic signature whose rewrite system is not convergent. It is probably the simplest possible such instance, so we will study it in detail. We will attempt to combine the recursion of protocol `N`, with an equivalence between two associated types that have the same name in unrelated protocols, as in Example 19.8. The setup is the same as in that example, but now both associated types are subject to recursive conformance requirements:

```
protocol P1 {
  associatedtype A: P1
}

protocol P2 {
  associatedtype A: P2
}
```

We define a type parameter conforming to both `P1` and `P2`:

```
<τ_0_0 where τ_0_0: P1, τ_0_0: P2>
```

We have the following initial rewrite rules:

$[\texttt{P1}] \cdot [\texttt{P1}] \Rightarrow [\texttt{P1}]$	(1)
$[\texttt{P1}] \cdot \texttt{A} \Rightarrow [\texttt{P1}\vert\texttt{A}]$	(2)
$[\texttt{P1}\vert\texttt{A}] \cdot [\texttt{P1}] \Rightarrow [\texttt{P1}\vert\texttt{A}]$	(3)
$[\texttt{P1}\vert\texttt{A}] \cdot \texttt{A} \Rightarrow [\texttt{P1}\vert\texttt{A}] \cdot [\texttt{P1}\vert\texttt{A}]$	(4)
$[\texttt{P2}] \cdot [\texttt{P2}] \Rightarrow [\texttt{P2}]$	(5)
$[\texttt{P2}] \cdot \texttt{A} \Rightarrow [\texttt{P2}\vert\texttt{A}]$	(6)
$[\texttt{P2}\vert\texttt{A}] \cdot [\texttt{P2}] \Rightarrow [\texttt{P2}\vert\texttt{A}]$	(7)
$[\texttt{P2}\vert\texttt{A}] \cdot \texttt{A} \Rightarrow [\texttt{P2}\vert\texttt{A}] \cdot [\texttt{P2}\vert\texttt{A}]$	(8)
$\tau\_0\_0 \cdot [\texttt{P1}] \Rightarrow \tau\_0\_0$	(9)
$\tau\_0\_0 \cdot [\texttt{P2}] \Rightarrow \tau\_0\_0$	(10)

In our generic signature, there are infinitely many type parameters, all of the form `τ_0_0` followed by a combination of `A`, `[P1]A` and `[P2]A`:

```
τ_0_0.A
τ_0_0.[P2]A.[P1]A
τ_0_0.[P1]A.A.[P2]A
...
```

Membership in an equivalence class is determined by the number of `A`'s appearing in the type parameter, so $\texttt{τ\_0\_0.[P1]A.[P1]A}$, $\texttt{τ\_0\_0.A.A}$ and $\texttt{τ\_0\_0.[P2]A.[P2]A}$ are all equivalent, so we can derive same-type requirements for these equivalences.

Rules (9) and (10) overlap with (2) and (6), respectively. Resolving these overlaps proceeds as in Example 19.8:

$$\texttt{τ\_0\_0} \cdot \texttt{A} \Rightarrow \texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \tag{11}$$
$$\texttt{τ\_0\_0} \cdot \texttt{[P2|A]} \Rightarrow \texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \tag{12}$$

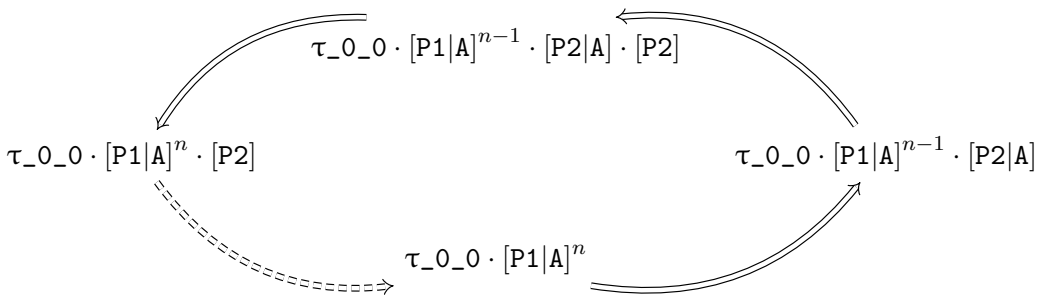
We added new rules, so we must check for overlapping rules again. Now, (12) overlaps with (7) on the term $\texttt{τ\_0\_0} \cdot \texttt{[P2|A]} \cdot \texttt{[P2]}$, and rule (12) also overlaps with (8) on the term $\texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P2|A]} \cdot \texttt{A}$. Resolving both critical pairs adds two new rules:

$$\texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P2]} \Rightarrow \texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \tag{13}$$
$$\texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P2|A]} \Rightarrow \texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P1|A]} \tag{14}$$

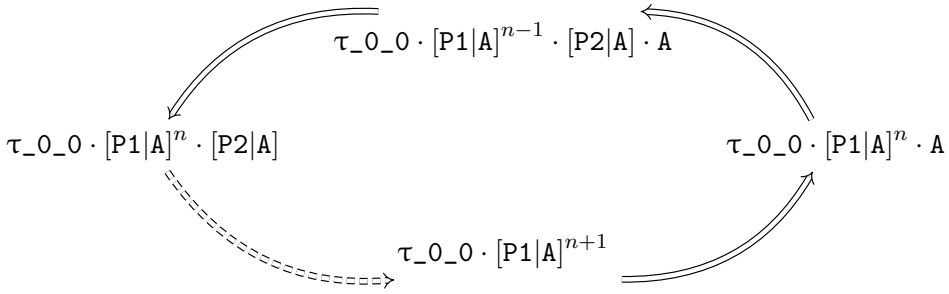
We now begin the third round, with another two critical pairs: rule (14) overlaps with (7) on the term $\texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P2|A]} \cdot \texttt{[P2]}$, and with (8) on the term $\texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P2|A]} \cdot \texttt{A}$. A pattern begins to emerge. We add two new rules, but we have two new critical pairs:

$$\texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P1|A]} \cdot \texttt{[P2]} \Rightarrow \texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P1|A]} \tag{15}$$
$$\texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P1|A]} \cdot \texttt{[P2|A]} \Rightarrow \texttt{τ\_0\_0} \cdot \texttt{[P1|A]} \cdot \texttt{[P1|A]} \cdot \texttt{[P1|A]} \tag{16}$$

Completion eventually hits a limit and fails. Our rewrite system actually has two infinite families of critical pairs; the first family defines a rule $\texttt{τ\_0\_0} \cdot \texttt{[P1|A]}^n \cdot \texttt{[P2]} \Rightarrow \texttt{τ\_0\_0} \cdot \texttt{[P1|A]}^n$ for each $n \in \mathbb{N}$:



The second defines $\texttt{τ\_0\_0} \cdot [\texttt{P1}|\texttt{A}]^n \cdot [\texttt{P2}|\texttt{A}] \Rightarrow \texttt{τ\_0\_0} \cdot [\texttt{P1}|\texttt{A}]^{n+1}$:



In protocol `N`, associated type rules allowed us an infinite set of derived requirements with a finite set of rewrite rules. In the next example of protocol `Q` inheriting from `N`, inherited associated type rules saved the day. This time though, we don't have any tricks left.

Let's write a, b, c, p, q, t instead of `A`, [P|A], [Q|A], [P], [Q], τ_0_0. Our generic signature defines the following monoid presentation over the alphabet a, p, q, t:

$$M_1 := \langle a, p, q, t \,|\, pp \sim p,\ pap \sim pa,\ qq \sim q,\ qaq \sim qa,\ tp \sim t,\ tq \sim t \rangle$$

The Requirement Machine added the symbols b and c with defining rewrite rules $pa \sim b$ and $qa \sim c$, respectively:

$$M_2 := \langle a, b, c, p, q, t \,|\, pp \sim p,\ pap \sim pa,\ qq \sim q,\ qaq \sim qa,\ tp \sim t,\ tq \sim t,\ pa \sim b,\ qa \sim c \rangle$$

We saw that completion fails; we got an infinite convergent presentation over the same alphabet:

$$M_2' := \langle a, b, c, p, q, t \,|\, pp \sim p,\ pa \sim b,\ bp \sim b,\ ba \sim bb,\ qq \sim q,\ qa \sim c,\ cq \sim c,\ ca \sim cc,\ ta \sim tb,\ tb^i q \sim tb^i,\ tb^i c \sim tb^{i+1} \rangle$$

In fact, the associated type rules don't simplify our understanding of this monoid at all. We can also write down an infinite convergent presentation for the completion of M_1:

$$M_1' := \langle a, p, q, t \,|\, pa^i p \sim pa^i,\ qa^i q \sim qa^i,\ ta^i p \sim ta^i,\ ta^i q \sim ta^i \rangle$$

This infinite presentation is simple enough that one can implement a one-off algorithm for solving the word problem in M_1'. We can simplify the original monoid presentation further by removing $pp \sim p$ and $qq \sim q$; the result is *not* Tietze-equivalent, but it runs into the same problems with completion.

Future directions. We will now state an open question about this strikingly simple monoid presentation of only four symbols and rewrite rules:

> Can the monoid $\langle a, p, q, t \,|\, pap \sim pa,\ qaq \sim qa,\ tp \sim t,\ tq \sim t \rangle$ be presented by a finite convergent rewriting system?

Either of the possible outcomes would be of interest:

- If we can answer in the affirmative, then the Requirement Machine might be able to support our generic signature by tweaking how we construct the symbols and rules of our rewrite system.
- If instead we prove that no such convergent rewriting system exists, we have another example of a finitely-presented monoid with decidable word problem that cannot be presented by a finite convergent rewriting system, much like Theorem 17.48.

One approach to the study of infinite convergent rewriting systems is via formal language theory; this is explored in [147]. We recall the standard definitions, which can be found in [148]. A *language* over an alphabet A is a subset of the free monoid A^*; a language is *recursive* if there is a total computable function to determine membership in this set, and a *regular language* is a set of strings recognized by a finite state automaton (or equivalently, matching a regular expression—the classical kind, without back references). The left-hand sides of the rules in a (finite or infinite) convergent rewriting system define a language in this sense, and a finite convergent rewriting system is then the special case where this language is finite; this direction is explored in.

Every finitely-presented monoid can be presented by an infinite convergent rewriting system; but if the word problem in this monoid is undecidable, the rewrite system's language is not recursive, so of course this viewpoint does not fundamentally change the difficulty of the problem. However, if we consider infinite convergent presentations with a sufficiently restricted class of language (for instance, regular languages), we can find an effective procedure for term reduction with a rewrite system of this type. Completion then becomes a highly non-trivial problem. A completion procedure for a certain kind of infinite convergent rewriting system was proposed in [149]; there, the left-hand sides involve exponents, much like the notation $ta^ip \sim ta^i$ and $ta^iq \sim ta^i$ used in our examples here. In principle, such an extension could one day be considered for the Requirement Machine.

Example 19.15. We end this chapter with one final curiosity. We proved the derived requirements formalism to be undecidable in Section 17.3 by showing that an arbitrary finitely-presented monoid $\langle A \,|\, R \rangle$ can be encoded in the form of a protocol declaration. In Chapter 18 we defined a lowering of a generic signature and its protocol dependencies into a finitely-presented monoid. If we chain both transformations, we see that we can

map a finitely-presented monoid to a protocol declaration and then back to a finitely-presented monoid. In what sense does the latter monoid encode the original monoid?

Let $A^* := \{a, b, c\}$, $R := \{(ab, c), (bc, \varepsilon)\}$, and consider the monoid $M := \langle A \,|\, R \rangle$. Written down as a Swift protocol, $M := \langle a, b, c \,|\, ab \sim c,\ bc \sim \varepsilon \rangle$ looks like this:

```
protocol P {
  associatedtype A: P
  associatedtype B: P
  associatedtype C: P
    where A.B == C, B.C == Self
}
```

Now we pretend to comment out the `where` clause, so now our protocol `P` just presents the free monoid A^*. We will list the convergent rewriting system for `P`, but partition the rules into two sets in a new way. The first set of rules involves name symbols; we call this set $\mathcal{N}$:

$$[\mathtt{P}] \cdot \mathtt{A} \Rightarrow [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}] \cdot \mathtt{B} \Rightarrow [\mathtt{P}|\mathtt{B}]$$
$$[\mathtt{P}] \cdot \mathtt{C} \Rightarrow [\mathtt{P}|\mathtt{C}]$$
$$[\mathtt{P}|\mathtt{A}] \cdot \mathtt{A} \Rightarrow [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}|\mathtt{A}] \cdot \mathtt{B} \Rightarrow [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{B}]$$
$$[\mathtt{P}|\mathtt{A}] \cdot \mathtt{C} \Rightarrow [\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}|\mathtt{C}]$$
$$[\mathtt{P}|\mathtt{B}] \cdot \mathtt{A} \Rightarrow [\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}|\mathtt{B}] \cdot \mathtt{B} \Rightarrow [\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{B}]$$
$$[\mathtt{P}|\mathtt{B}] \cdot \mathtt{C} \Rightarrow [\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{C}]$$
$$[\mathtt{P}|\mathtt{C}] \cdot \mathtt{A} \Rightarrow [\mathtt{P}|\mathtt{C}] \cdot [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}|\mathtt{C}] \cdot \mathtt{B} \Rightarrow [\mathtt{P}|\mathtt{C}] \cdot [\mathtt{P}|\mathtt{B}]$$
$$[\mathtt{P}|\mathtt{C}] \cdot \mathtt{C} \Rightarrow [\mathtt{P}|\mathtt{C}] \cdot [\mathtt{P}|\mathtt{C}]$$

The second set involves protocol and associated type symbols only. We call this set $\mathcal{S}$:

$$[\mathtt{P}] \cdot [\mathtt{P}] \Rightarrow [\mathtt{P}]$$
$$[\mathtt{P}] \cdot [\mathtt{P}|\mathtt{A}] \Rightarrow [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}] \cdot [\mathtt{P}|\mathtt{B}] \Rightarrow [\mathtt{P}|\mathtt{B}]$$
$$[\mathtt{P}] \cdot [\mathtt{P}|\mathtt{C}] \Rightarrow [\mathtt{P}|\mathtt{C}]$$
$$[\mathtt{P}|\mathtt{A}] \cdot [\mathtt{P}] \Rightarrow [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}] \Rightarrow [\mathtt{P}|\mathtt{B}]$$
$$[\mathtt{P}|\mathtt{C}] \cdot [\mathtt{P}] \Rightarrow [\mathtt{P}|\mathtt{C}]$$

The union $\mathcal{N} \cup \mathcal{S}$ is a rewrite system for a protocol encoding the free monoid $\{a, b, c\}^*$. Here, $\mathcal{N}$ contains 12 elements, and $\mathcal{S}$ contains 7 elements. More generally, if $|A| = n$, we have $|\mathcal{N}| = n(n+1)$ and $|\mathcal{S}| = 2n + 1$. Now, if we identify a, b, c with [P|A], [P|B], [P|C], and define a new symbol $e :=$ [P], we see that $\langle A \cup \{e\} \,|\, \mathcal{S} \rangle$ is a convergent presentation. But to understand the object it presents, we need a few related definitions.

A *semigroup* is a set together with an associative binary operation, but without a distinguished identity element. The *free semigroup* over a set A, denoted A^+, is the set of all *non-empty* strings from the symbols of A. We can define a *finitely-presented semigroup* analogously to a finitely-presented monoid; here the rewrite rules have the form (u, v) where u, $v \in A^+$.

Every monoid trivially satisfies the semigroup axioms, and thus a free monoid A^* is also a semigroup. However a free monoid A^* is not a *free* semigroup. For example, taking $A := \{a, b, c\}$, we see among the elements of A^* *viewed as a semigroup*, we have non-trivial identities involving ε, such as $a = \varepsilon a$, $b\varepsilon = b$, and so on. Thus, this cannot be a free semigroup, where $xy \neq x$ and $xy \neq y$ for all x, $y \in A^+$. The free monoid A^* *is* a finitely-presented semigroup though; we must add the symbol e to the alphabet, and a pair of rewrite rules $ex \sim e$, $xe \sim e$ for each $x \in A$. For example, as a semigroup, $\{a, b, c\}^*$ has four generators and seven rewrite rules:

$$\langle a, b, c, e \,|\, e \sim e,\ ea \sim a,\ eb \sim b,\ ec \sim c,\ ae \sim a,\ be \sim b,\ ce \sim c \rangle$$

Indeed, we see this set of rewrite rules is just $\mathcal{S}$, so our embedding of the free monoid as a Swift protocol ends up encoding the free monoid as the finitely-presented semigroup $\langle A \cup \{e\} \,|\, \mathcal{S} \rangle$, but embedded in an even larger semigroup that also contains name symbols and the set of rules $\mathcal{N}$.

Now, we imagine that we uncomment the `where` clause of our protocol `P`, with the two requirements [`Self.A.B == C`] and [`Self.B.C == Self`]. These requirements contribute two new rewrite rules to our convergent rewriting system; we will call this set $\mathcal{R}$:

$$[\texttt{P}|\texttt{A}] \cdot [\texttt{P}|\texttt{B}] \Rightarrow [\texttt{P}|\texttt{C}]$$
$$[\texttt{P}|\texttt{B}] \cdot [\texttt{P}|\texttt{C}] \Rightarrow [\texttt{P}]$$

We can construct $\mathcal{R}$ from R by replacing ε with e anywhere that it appears in a rewrite rule. This gives us a set of semigroup rewrite rules, and we see that $\langle A \cup \{e\} \,|\, \mathcal{S} \cup \mathcal{R} \rangle$ is a semigroup presentation of $\langle A \,|\, R \rangle$:

$$\langle a, b, c, e \,|\, ee \sim e,\ ea \sim a,\ eb \sim b,\ ec \sim c,\ ae \sim a,\ be \sim b,\ ce \sim c,\ ab \sim c,\ bc \sim e \rangle$$

Neither the monoid presentation M nor the semigroup presentation $\langle A \cup \{e\} \,|\, \mathcal{S} \cup \mathcal{R} \rangle$ is convergent. Now consider the following set of rewrite rules $\hat{R}$:

$$\hat{R} := \{(cc, a),\ (ba, c),\ (ca, a),\ (cb, \varepsilon)\}$$

A trivial but technical calculation shows that $\langle A \,|\, R \cup \hat{R} \rangle$ is a convergent monoid presentation of $\langle A \,|\, R \rangle$. Next, we transform $\hat{R}$ into a set of semigroup rewrite rules, replacing ε with e. We call this set $\hat{\mathcal{R}}$:

$$[\mathtt{P}|\mathtt{C}] \cdot [\mathtt{P}|\mathtt{C}] \Rightarrow [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}|\mathtt{B}] \cdot [\mathtt{P}|\mathtt{A}] \Rightarrow [\mathtt{P}|\mathtt{C}]$$
$$[\mathtt{P}|\mathtt{C}] \cdot [\mathtt{P}|\mathtt{A}] \Rightarrow [\mathtt{P}|\mathtt{A}]$$
$$[\mathtt{P}|\mathtt{C}] \cdot [\mathtt{P}|\mathtt{B}] \Rightarrow [\mathtt{P}]$$

We now claim that $\langle A \cup \{e\} \,|\, \mathcal{S} \cup \mathcal{R} \cup \hat{\mathcal{R}} \rangle$ is a convergent semigroup presentation, because we can mechanically transform any positive rewrite path over $\langle A \,|\, R \cup \hat{R} \rangle$ into a positive rewrite path over this semigroup presentation. The only rewrite step that does not map trivially is $x(cb \Rightarrow \varepsilon)y$, for some x, $y \in A^*$. The corresponding rewrite step over the semigroup presentation, $x(cb \Rightarrow e)y$, has destination term xey instead of xy. If either one of x or y is non-empty, we can eliminate the occurrence of e by adding a single positive rewrite step from $\mathcal{S}$. Otherwise, if $xey = e$, there is nothing to do. In any given positive rewrite path, this transformation can only add finitely many new rewrite steps involving rewrite rules from $\mathcal{S}$. Thus, confluence and termination are preserved.

It is of course not true in general that the union of two convergent rewriting systems is convergent, however in our construction it happens to work. There are no non-trivial overlaps between the elements of $\mathcal{N}$, $\mathcal{S}$ and $\mathcal{R}$. We've proven the following:

Theorem 19.16. Suppose $\langle A \,|\, R \rangle$ is a finitely-presented monoid, $\hat{R}$ is a finite set of rewrite rules such that $\langle A \,|\, R \cup \hat{R} \rangle$ is a convergent monoid presentation, and both R and $\hat{R}$ are oriented with respect to the shortlex order on A^*. We define a protocol $\mathtt{P}$ encoding $\langle A \,|\, R \rangle$, naming the associated types in correspondence with the fixed linear order on the generating set A (for example, if $a < b < c < \cdots$, we have $[\mathtt{P}|\mathtt{A}] < [\mathtt{P}|\mathtt{B}] < [\mathtt{P}|\mathtt{C}]$). Then our protocol $\mathtt{P}$ has the following *convergent* presentation:

$$\langle A \cup \{e\} \cup B \,|\, \mathcal{N} \cup \mathcal{S} \cup \mathcal{R} \cup \hat{\mathcal{R}} \rangle$$

Here, $e := [\mathtt{P}]$, B is a set of name symbols, $\mathcal{N}$ and $\mathcal{S}$ only depend on A, and $\mathcal{R}$ and $\hat{\mathcal{R}}$ are constructed from R and $\hat{R}$ as above. The semigroup presented above also contains a sub-semigroup with the following convergent presentation:

$$\langle A \cup \{e\} \,|\, \mathcal{S} \cup \mathcal{R} \cup \hat{\mathcal{R}} \rangle$$

This sub-semigroup is isomorphic to the monoid $\langle A \,|\, R \cup \hat{R} \rangle$ and thus $\langle A \,|\, R \rangle$.

Hence, we see that if we encode a finitely-presented monoid $\langle A \,|\, R \rangle$ as a protocol $\mathtt{P}$, the Requirement Machine will accept $\mathtt{P}$ and solve the word problem in $\langle A \,|\, R \rangle$ if and only if $\langle A \,|\, R \rangle$ has a convergent presentation compatible with the shortlex order on A^*. That's a satisfying conclusion to "Swift type checking is undecidable."

Our double encoding of "a monoid as a protocol as a rewrite system" introduces many new symbols and rewrite rules not found in the original presentation. It is a remarkable fact of our construction then, that a convergent monoid presentation always maps to a convergent rewriting system where the added detritus can always be "factored out."

19.6. Source Code Reference

Key source files:

- `lib/AST/RequirementMachine/KnuthBendix.cpp`
- `lib/AST/RequirementMachine/RewriteSystem.cpp`
- `lib/AST/RequirementMachine/Trie.h`

`rewriting::RewriteSystem` *class*

See also Section 18.6.

- `addRule()` implements Algorithm 19B.
- `recordRewriteLoop()` records a rewrite loop if this rewrite system is used for minimization.
- `computeCriticalPair()` implements Algorithm 19A.
- `computeConfluentCompletion()` implements Algorithm 19E.
- `simplifyLeftHandSides()` implements Algorithm 19F.
- `simplifyRightHandSides()` implements Algorithm 19G.

`rewriting::Trie` *template class*

- `findAll()` finds all overlapping rules using Algorithm 19D.

20. The Property Map

20.1. Substitution Simplification

Algorithm 20A (Simplify substitution terms)**.**

20.2. Property Unification

20.3. Concrete Type Unification

20.4. Generic Signature Queries

20.5. Reduced Types

20.6. Source Code Reference

21. Concrete Conformances

22. Rewrite System Minimization

A. Mathematical Conventions

The more formal sections in this book are written in "Euclidean style", where the text is further subdivided into elements of the below kinds. All elements are numbered consecutively within a single chapter, except for algorithms which use a letter:

- A **Definition** introduces terminology or notation.
- An **Example** shows how this terminology and notation arises in practice.
- A **Proposition** states a logical consequence of one or more definitions.
- A **Lemma** is an intermediate proposition in service of proving another theorem.
- A **Theorem** is a "deeper" proposition which is more profound in some sense.
- A **Corollary** is an immediate consequence of an earlier theorem.
- An **Algorithm** is a step-by-step description of a computable function, defined in terms of inputs and outputs.

A handful of Greek letters are used for variable names and other notation: lowercase α (alpha), β (beta), γ (gamma), ε (epsilon), η (eta), π (pi), σ (sigma), τ (tau), φ (phi); and uppercase Σ (sigma). The "=" operator means two existing things are identical or equivalent; "$\neq$" means they are not. The ":=" operator *defines* the new thing on the left as a combination of existing things on the right. The "$\leftarrow$" operator denotes an imperative assignment statement in an algorithm.

Sets

A *set* is a collection of elements without regard to order or duplicates. Sets can be finite or infinite. A finite set can be specified by listing its elements in any order, for example $\{a, b, c\}$. The empty set $\varnothing$ is the unique set with no elements. The set of *natural numbers* $\mathbb{N} := \{0, 1, 2, \ldots\}$ is the infinite set containing zero and all of its successors. The set of *integers* $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ also contains the negative whole numbers.

The notation $x \in S$ means "x is an element of a set S," and $x \notin S$ is its negation. Properties of sets can be defined using *existential* quantification ("there exists (at least one) $x \in S$ such that x has this property…") or *universal* quantification ("for all $x \in S$, the following property is true of x…").

A set X is a *subset* of another set Y, written as $X \subseteq Y$, if for all $x \in X$, it is also true that $x \in Y$. If $X \subseteq Y$ and $Y \subseteq X$, then the two sets have the same elements, so they're equal; $X = Y$.

If $X \subseteq Y$ and $X \neq Y$, then X is a *proper* subset of Y, written as $X \subsetneq Y$. The *union* $X \cup Y$ is the set of all elements belonging to either X or Y. The *intersection* $X \cap Y$ is the set of all elements belonging to both X and Y. The *difference* $X \setminus Y$ is the set of all elements of X not also in Y.

The *Cartesian product* of two sets X and Y, denoted $X \times Y$, is the set of all *ordered pairs* (x, y) where $x \in X$, $y \in Y$. Note that the ordered pair (x, y) is not the same as the set $\{x, y\}$, because $(x, y) \neq (y, x)$. The Cartesian product construction generalizes to any finite number of sets, to give *ordered tuples* or *sequences*.

Functions

A *function* (or *mapping*) $f \colon X \to Y$ assigns to each $x \in X$ a unique element $f(x) \in Y$. If the sets X and Y are equipped with some kind of additional structure (which will be explicitly defined), then f is a *homomorphism* if it preserves this structure.

A function $f \colon X \times Y \to Z$ defined on the Cartesian product can be thought of as taking a pair of values $x \in X$, $y \in Y$ to an element $f(x, y) \in Z$. A *binary operation* is a function named by a symbol like "$\otimes$", "$\triangleleft$", or "$\cdot$" defined on the Cartesian product of two sets. The application of a binary operation is denoted by writing the symbol in between the two elements, like $x \otimes y$.

The *cardinality* of a finite set S, denoted $|S|$, is the number of elements in S. The notation $|x|$ sometimes denotes certain other functions taking values in $\mathbb{N}$, such as the length of a sequence; this will always be explicitly defined when needed.

More

- Formal systems (Section 5.2).
- Equivalence relations (Section 5.3, Section 17.2).
- Partial and linear orders (Section 5.4, Section 17.5, Section 18.3).
- Category theory (Section 6.2).
- Directed graphs (Section 8.3, Section 12.2, Section 12.3, Section 16.1).
- Proof by induction (Section 11.3).
- Computability theory (Section 12.4, Section 17.4).
- Finitely-presented monoids and string rewriting (Chapter 17, Chapter 19).

B. Derived Requirements

Let G be a generic signature. We generate the theory of G by repeated application of inference rules, starting from a finite set of elementary statements. A derivation proves that some element belongs to this set by giving a list of derivation steps where the assumptions in each step are conclusions of previous steps. Nomenclature:

Symbol	Description
`T`, `U`, and `V`	type parameters
`Self.U` and `Self.V`	type parameters rooted in the protocol `Self` type
`X`	a concrete type
`C`	a concrete class type
`X`′ and `C`′	obtained from `X` and `C` by replacing `Self` with `T`
`P` and `Q`	protocols
`A`	the name of an associated type of `P`
`[P]A`	an associated type declaration of `P`
`T.[P]A` and `T.A`	bound and unbound dependent member type
[`T: P`]	a conformance requirement
[`T == U`]	a same-type requirement between type parameters
[`T == X`]	a concrete same-type requirement
[`T: C`]	a superclass requirement
[`T: AnyObject`]	a layout requirement
$[\texttt{Self.U: Q}]_{\texttt{P}}$	an associated requirement of protocol `P`

See Section 5.2 and Section 5.3 for details.

Elementary statements. For each generic parameter `τ_d_i` of G:

`τ_d_i` (Generic)

For each explicit requirement of G by kind:

[`T: P`] (Conf)

[`T == U`] (Same)

[`T == X`] (Concrete)

[`T: C`] (Super)

[`T: AnyObject`] (Layout)

Requirement signatures. Assume $G \vdash$ [T: P]. For each associated type A of P:

T.A	(AssocName [T: P])
T.[P]A	(AssocDecl [T: P])
[T.[P]A == T.A]	(AssocBind [T: P])

For each associated requirement of P by kind:

[T.U: Q]	(AssocConf [Self.U: Q]$_{\texttt{P}}$ [T: P])
[T.U == T.V]	(AssocSame [Self.U == Self.V]$_{\texttt{P}}$ [T: P])
[T.U == X′]	(AssocConcrete [Self.U == X]$_{\texttt{P}}$ [T: P])
[T.U: C′]	(AssocSuper [Self.U: C]$_{\texttt{P}}$ [T: P])
[T.U: AnyObject]	(AssocLayout [Self.U: AnyObject]$_{\texttt{P}}$ [T: P])

Equivalence. Same-type requirements generate an equivalence relation:

[T == T]	(Reflex T)
[U == T]	(Sym [T == U])
[T == V]	(Trans [T == U] [U == V])

Compatibility. A derived conformance, superclass or layout requirement applies to all type parameters in an equivalence class:

[T: P]	(SameConf [U: P] [T == U])
[T == X]	(SameConcrete [U == X] [T == U])
[T: C]	(SameSuper [U: C] [T == U])
[T: AnyObject]	(SameLayout [U: AnyObject] [T == U])

If two type parameters are equivalent, so are all corresponding member types:

[T.A == U.A]	(SameName [U: P] [T == U])
[T.[P]A == U.[P]A]	(SameDecl [U: P] [T == U])

C. Type Substitution

We've been using the type substitution algebra for describing identities among types, conformances and substitution maps. The formalism was introduced over the course of Chapters 6, 7, and 12.

The entities being operated upon:

- $\textsc{Proto}$ is the set of all protocols.
- $\textsc{Type}(G)$ is the set of all interface types valid in the generic signature G (see Section 5.2).
- $\textsc{Sub}(G, H)$ is the set of all substitution maps with input generic signature G and output generic signature H.
- $\textsc{Conf}(G)$ is the set of all conformances with output generic signature G.
- $\textsc{Conf}_{\texttt{P}}(G)$ is the set of all conformances to protocol `P` with output generic signature G.
- $\textsc{AssocType}_{\texttt{P}}$ is the set of all associated type declarations in protocol `P`.
- $\textsc{AssocConf}_{\texttt{P}}$ is the set of all associated conformance requirements in protocol `P`.
- $\textsc{Req}(G)$ is the set of all requirements containing interface types of G.

A composition operation is defined on various kinds of entities, written as $A \otimes B$. If a juxtaposition of three entities can be evaluated in two different ways, the result is always the same, so the composition operator is *associative*: $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.

Substitution maps act on the right of types, conformances, other substitution maps, and requirements. This gives us type substitution (Section 6), conformance substitution (Chapter 7), substitution map composition (Section 6.2), and requirement substitution (Section 9.3). This action produces a new entity of the same kind as the original:

$$\textsc{Type}(G) \otimes \textsc{Sub}(G, H) \longrightarrow \textsc{Type}(H)$$
$$\textsc{Conf}(G) \otimes \textsc{Sub}(G, H) \longrightarrow \textsc{Conf}(H)$$
$$\textsc{Sub}(F, G) \otimes \textsc{Sub}(G, H) \longrightarrow \textsc{Sub}(F, H)$$
$$\textsc{Req}(G) \otimes \textsc{Sub}(G, H) \to \textsc{Req}(H)$$

Protocols act on the left of types (global conformance lookup, Section 7.1):

$$\textsc{Proto} \otimes \textsc{Type}(G) \longrightarrow \textsc{Conf}(G)$$

Associated type declarations act on the left of conformances (type witness projection, Section 7.3):

$$\textsc{AssocType}_{\mathtt{P}} \otimes \textsc{Conf}_{\mathtt{P}}(G) \longrightarrow \textsc{Type}(G)$$

Associated conformance requirements act on the left of conformances (associated conformance projection, Section 7.5):

$$\textsc{AssocConf}_{\mathtt{P}} \otimes \textsc{Conf}_{\mathtt{P}}(G) \longrightarrow \textsc{Conf}_{\mathtt{P}}(G)$$

Substitution maps. A substitution map is completely determined by its action on each generic parameter type and root abstract conformance. Type substitution with a generic parameter type projects the replacement type from the substitution map:

$$\mathtt{τ\_d\_i} \otimes \{\ldots, \mathtt{τ\_d\_i} \mapsto \mathtt{Int}, \ldots; \ldots\} := \mathtt{Int}$$

Conformance substitution with a root abstract conformance projects the corresponding conformance from the substitution map:

$$[\mathtt{T:\ Q}] \otimes \{\ldots; \ldots, [\mathtt{T:\ Q}] \mapsto [\mathtt{G<Int>:\ Q}], \ldots\} := [\mathtt{G<Int>:\ Q}]$$

If $\mathtt{T} \in \textsc{Type}(G_1)$, $[\mathtt{T:\ P}] \in \textsc{Conf}_{\mathtt{P}}(G_1)$, $\Sigma_1 \in \textsc{Sub}(G_1, G_2)$, $\Sigma_2 \in \textsc{Sub}(G_2, G_3)$, $\Sigma_3 \in \textsc{Sub}(G_3, G_4)$, then

$$(\mathtt{T} \otimes \Sigma_1) \otimes \Sigma_2 = \mathtt{T} \otimes (\Sigma_1 \otimes \Sigma_2)$$
$$([\mathtt{T:\ P}] \otimes \Sigma_1) \otimes \Sigma_2 = [\mathtt{T:\ P}] \otimes (\Sigma_1 \otimes \Sigma_2)$$
$$(\Sigma_1 \otimes \Sigma_2) \otimes \Sigma_3 = \Sigma_1 \otimes (\Sigma_2 \otimes \Sigma_3)$$

Conformances. Type witness and associated conformance projection is compatible with conformance substitution. Suppose $[\mathtt{T:\ P}] \in \textsc{Conf}_{\mathtt{P}}(G)$, $\pi(\mathtt{[P]A}) \in \textsc{AssocType}_{\mathtt{P}}$, $\pi(\mathtt{Self.U:\ Q}) \in \textsc{AssocConf}_{\mathtt{P}}$, and $\Sigma \in \textsc{Sub}(G, H)$, then

$$\big(\pi(\mathtt{[P]A}) \otimes [\mathtt{T:\ P}]\big) \otimes \Sigma = \pi(\mathtt{[P]A}) \otimes ([\mathtt{T:\ P}] \otimes \Sigma)$$
$$\big(\pi(\mathtt{Self.U:\ Q}) \otimes [\mathtt{T:\ P}]\big) \otimes \Sigma = \pi(\mathtt{Self.U:\ Q}) \otimes \big([\mathtt{T:\ P}] \otimes \Sigma\big)$$

If d is a nominal type declaration conforming to $\mathtt{P}$, and G is the generic signature of the conformance context, then there exists a normal conformance $[\mathtt{T}_d\mathtt{:\ P}] \in \textsc{Conf}_{\mathtt{P}}(G)$, and

$$[\mathtt{P}] \otimes \mathtt{T}_d = [\mathtt{T}_d\mathtt{:\ P}]$$

If H is some other generic signature, $\Sigma \in \textsc{Sub}(G, H)$, and $\texttt{T} = \texttt{T}_d \otimes \Sigma$, then there exists a specialized conformance $[\texttt{T: P}] \in \textsc{Conf}_{\texttt{P}}(H)$, and

$$[\texttt{P}] \otimes \texttt{T} = [\texttt{T}_d\texttt{: P}] \otimes \Sigma = [\texttt{T: P}]$$

If $\texttt{T}$ is a type parameter of G and the conformance requirement $[\texttt{T: P}]$ can be derived from G, then there exists an abstract conformance $[\texttt{T: P}] \in \textsc{Conf}_{\texttt{G}}(P)$, and

$$[\texttt{P}] \otimes \texttt{T} = [\texttt{T: P}]$$

Global conformance lookup is compatible with type substitution:

$$([\texttt{P}] \otimes \texttt{T}) \otimes \Sigma = \texttt{P} \otimes (\texttt{T} \otimes \Sigma)$$

Abstract conformances. The type witnesses of an abstract conformance are dependent member types, and the associated conformances are other abstract conformances:

$$\pi(\texttt{[P]A}) \otimes [\texttt{T: P}] = \texttt{T.[P]A}$$
$$\pi(\texttt{Self.U: Q}) \otimes [\texttt{T: P}] = [\texttt{T.U: Q}]$$

Applying a substitution map to an abstract conformance is called local conformance lookup (Section 7.4). The first identity above defines substitution of dependent member types in terms of local conformance lookup:

$$\texttt{T.[P]A} \otimes \Sigma = \pi(\texttt{[P]A}) \otimes ([\texttt{T: P}] \otimes \Sigma)$$

Every abstract conformance has a unique factorization as a reduced conformance path. A conformance path is a sequence of zero or more associated conformance projections applied to an root abstract conformance:

$$\pi(\texttt{Self.[P}_{n+1}\texttt{]A}_n\texttt{: P}_n) \otimes \cdots \otimes \pi(\texttt{Self.[P}_2\texttt{]A}_1\texttt{: P}_1) \otimes [\texttt{T}_0\texttt{: P}_0]$$

When applied to a non-root abstract conformance, local conformance lookup first factors the abstract conformance with a conformance path. Thus, applying a substitution map to a dependent member type breaks down into this juxtaposition of entities, from *right* to *left*:

- A substitution map.
- Then, a conformance path, consisting of a root abstract conformance, followed by zero or more associated conformance projections.
- Finally, a type witness projection.

The substituted type of a dependent member type is computed by taking a root conformance from the substitution map, following a chain of zero or more associated conformance projections, and finally projecting the type witness.

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