Cartesian coordinates of n-D regular simplex

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The rule of cartesian coordinates for n-D regular simplex

The cartesian coordinates of a n-D regular simplex in $\mathbb{R}^n$ can be obtained from these two properties [**?**]

1. For a regular simplex, the distances of its vertices to its center are equal.
2. The angle subtended by any two vertices of ann-dimensional simplex through its center is $\arccos(-1/n)$.

For example in 2D, the vectors (v_0, v_1, v_2) are the vertics of a 2-simplex or regular triangle. Write these as

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Without loss of generality, one can always choose the first vector $v_0 = (1, 0)^T$. Then the vectors become

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

By the second property the dot product of v_0 with all other vectors is $-1/2$, so each of their x components must equal this, and the vectors become

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1/2 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} -1/2 \\ y_2 \end{pmatrix}.$$

Next by the first property, the length of all vectors v_j must be equal, therefore the possible square roots give the two results

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \quad \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}.$$

The process can be carried out in n dimension, using $n + 1$ vectors, applying the first and second properties alternately to determine cartesian coordinates of n-D regular simplex in $\mathbb{R}^n$. The following algorithm is to achieve the above process.

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Algorithm 1 n-D regular simplex coordinates (MATLAB style)

Given $x(1:n, 1:n+1) = 0$
for $i = 1:1:n$ **do**
  $x(i,i) = \sqrt{1 - \sum_{k=1}^{i-1}[x(k,i)]^2}$
  for $j = i+1:1:n+1$ **do**
    $x(i,j) = \frac{1}{x(i,i)}\Big(-\frac{1}{n} - x(1:i-1,i)^T \cdot x(1:i-1,j)\Big)$
  end for
end for

Algorithm Implementation

Ackley function

$$f(\mathbf{x}) = f(x_1, \cdots, x_n) = -a\exp\left(-b\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}\right) - \exp\left(\frac{1}{n}\sum_{i=1}^{n}\cos(cx_i)\right) + a + \exp(1)$$

n is the dimension, where $a = 20$, $b = 0.2$ and $c = 2\pi$. Ackley function has many local minima and a unique global minimum of $(0,0)$ with $f(0,0) = 0$.

Setup:

Search region: $(-10, 10)^n$; initial position: random; $N_{max} = 2^m(n+1)$, $m = 5$. The maximum number of ration is $2^m = 32$. The convergent criterion is $\rho < 1.0^{-5}$.

Case I : d = 100

Table 1: Iterative information of SHC algorithm in optimizing 100 dimensional Ackley function. $N_{max} = 3232$. The ℓ^2-distant between convergent point and global minimizer is 1.907319675315442e-05.

ρ	# rotation	Func. evaluates	Total Func. evaluates	function values
2.500000e+00	1	101	102	1.323139574861084e+01
2.500000e+00	31	3131	3233	1.323139574861084e+01
1.250000e+00	1	101	3334	1.323139574861084e+01
1.250000e+00	31	3131	6465	1.323139574861084e+01
6.250000e-01	1	101	6566	1.323139574861084e+01
6.250000e-01	1	101	6667	9.940649053411246e+00
6.250000e-01	1	101	6768	7.253616867726927e+00
6.250000e-01	1	101	6869	5.371504406271417e+00
6.250000e-01	1	101	6970	4.131601370439101e+00
6.250000e-01	1	101	7071	3.519485425938275e+00
6.250000e-01	1	101	7172	2.509135107511345e+00
6.250000e-01	1	101	7273	1.528942111121421e+00
6.250000e-01	1	101	7374	8.963281283140647e-01
6.250000e-01	1	101	7475	5.953319005442568e-01
6.250000e-01	1	101	7576	4.801822650485095e-01
6.250000e-01	1	101	7677	4.431048117272214e-01
6.250000e-01	1	101	7778	4.353533050822351e-01

6.250000e-01	1	101	7879	4.352008513973376e-01
6.250000e-01	1	101	8081	4.216468351025968e-01
6.250000e-01	1	101	8182	3.964634080189393e-01
6.250000e-01	31	3131	11313	3.964634080189393e-01
3.125000e-01	1	101	11414	3.964634080189393e-01
3.125000e-01	1	101	11515	2.021414646047406e-01
3.125000e-01	1	101	11616	1.742429055941637e-01
3.125000e-01	1	101	11717	1.727263853740242e-01
3.125000e-01	1	101	11919	1.573599740024436e-01
3.125000e-01	31	3131	15050	1.573599740024436e-01
1.562500e-01	1	101	15151	1.573599740024436e-01
1.562500e-01	1	101	15252	7.309458748586994e-02
1.562500e-01	1	101	15454	7.201070066689486e-02
1.562500e-01	31	3131	18585	7.201070066689486e-02
7.812500e-02	1	101	18686	7.201070066689486e-02
7.812500e-02	1	101	18787	3.357460259351885e-02
7.812500e-02	31	3131	21918	3.357460259351885e-02
3.906250e-02	1	101	22019	3.357460259351885e-02
3.906250e-02	1	101	22120	1.642897517429720e-02
3.906250e-02	1	101	22322	1.624038169371023e-02
3.906250e-02	1	101	22423	1.622559763041398e-02
3.906250e-02	1	101	23029	1.620428681899444e-02
3.906250e-02	1	101	23130	1.620281288330805e-02
3.906250e-02	31	3131	26261	1.620281288330805e-02
1.953125e-02	1	101	26362	1.620281288330805e-02
1.953125e-02	1	101	26463	8.016635984284370e-03
1.953125e-02	1	101	26564	8.015283381504368e-03
1.953125e-02	1	101	26766	7.960647521604081e-03
1.953125e-02	1	101	26766	7.960647521604081e-03
1.953125e-02	31	3131	29897	7.960647521604081e-03
9.765625e-03	1	101	29998	7.960647521604081e-03
9.765625e-03	1	101	30099	3.930679837556461e-03
9.765625e-03	31	3131	33230	3.930679837556461e-03
4.882812e-03	1	101	33331	3.930679837556461e-03
4.882812e-03	1	101	33432	1.965649933606706e-03
4.882812e-03	1	101	33634	1.962393110219729e-03
4.882812e-03	1	101	33735	1.962372616152841e-03
4.882812e-03	1	101	34341	1.962044805655871e-03
4.882812e-03	31	3131	37472	1.962044805655871e-03
2.441406e-03	1	101	37573	1.962044805655871e-03
2.441406e-03	1	101	37674	9.778611832702566e-04
2.441406e-03	31	3131	40805	9.778611832702566e-04
1.220703e-03	1	101	40906	9.778611832702566e-04
1.220703e-03	1	101	41007	4.890635637950780e-04
1.220703e-03	1	101	41209	4.888592803462544e-04
1.220703e-03	1	101	41310	4.888589711962155e-04
1.220703e-03	1	101	41916	4.888385156762709e-04
1.220703e-03	31	3131	45047	4.888385156762709e-04

6.103516e-04	1	101	45148	4.888385156762709e-04
6.103516e-04	1	101	45249	2.442210666915301e-04
6.103516e-04	31	3131	48380	2.442210666915301e-04
3.051758e-04	1	101	48481	2.442210666915301e-04
3.051758e-04	1	101	48582	1.221191779139374e-04
3.051758e-04	1	101	48784	1.221008263572188e-04
3.051758e-04	1	101	52016	1.220983608165582e-04
3.051758e-04	1	101	52117	1.220885084309842e-04
3.051758e-04	31	3131	55248	1.220885084309842e-04
1.525879e-04	1	101	55349	1.220885084309842e-04
1.525879e-04	1	101	55450	6.104737234924329e-05
1.525879e-04	1	101	55652	6.104415597096846e-05
1.525879e-04	31	3131	58783	6.104415597096846e-05
7.629395e-05	1	101	58884	6.104415597096846e-05
7.629395e-05	1	101	58985	3.051897912076385e-05
7.629395e-05	31	3131	62116	3.051897912076385e-05
3.814697e-05	1	101	62217	3.051897912076385e-05
3.814697e-05	1	101	62318	1.525955241721277e-05
3.814697e-05	1	101	62520	1.525935175994420e-05
3.814697e-05	1	101	62621	1.525935175283877e-05
3.814697e-05	1	101	63227	1.525933236568022e-05
3.814697e-05	31	3131	66358	1.525933236568022e-05
1.907349e-05	1	101	66459	1.525933236568022e-05
1.907349e-05	1	101	66560	7.629472443149155e-06
1.907349e-05	31	3131	69691	7.629472443149155e-06

Case II : d = 1000

Table 2: Iterative information of SHC algorithm in optimizing 1000 dimensional Ackley function. $N_{max} = 32032$. The ℓ^2-distant between convergent point and global minimizer is 1.907320588287960e-05.

ρ	# rotation	Func. evaluates	Total Func. evaluates	function values
2.500000e+00	1	1001	1002	1.352971471312092e+01
2.500000e+00	31	31031	32033	1.352971471312092e+01
1.250000e+00	1	1001	33034	1.352971471312092e+01
1.250000e+00	31	31031	64065	1.352971471312092e+01
6.250000e-01	1	1001	65066	1.352971471312092e+01
6.250000e-01	1	1001	66067	1.030620236761562e+01
6.250000e-01	1	1001	67068	7.636667217633572e+00
6.250000e-01	1	1001	68069	5.717132892145167e+00
6.250000e-01	1	1001	69070	4.112245570965805e+00
6.250000e-01	1	1001	70071	3.549285769175520e+00
6.250000e-01	1	1001	71072	2.658308818254981e+00
6.250000e-01	1	1001	72073	1.650730630734011e+00
6.250000e-01	1	1001	73074	9.025366799134682e-01
6.250000e-01	1	1001	74075	4.822433298605882e-01
6.250000e-01	1	1001	75076	2.750879702955666e-01
6.250000e-01	1	1001	76077	1.795466282665901e-01

6.250000e-01	1	1001	77078	1.384913050776087e-01
6.250000e-01	1	1001	78079	1.223438728513964e-01
6.250000e-01	1	1001	79080	1.165506788885233e-01
6.250000e-01	1	1001	80081	1.147241998256954e-01
6.250000e-01	1	1001	81082	1.143066572404523e-01
6.250000e-01	1	1001	83084	1.036591755864302e-01
6.250000e-01	1	1001	84085	1.014893953104989e-01
6.250000e-01	31	31031	115116	1.014893953104989e-01
3.125000e-01	1	1001	116117	1.014893953104989e-01
3.125000e-01	1	1001	117118	5.291296750026309e-02
3.125000e-01	1	1001	118119	4.633828080662017e-02
3.125000e-01	1	1001	119120	4.585773393637727e-02
3.125000e-01	1	1001	120121	4.584861398056095e-02
3.125000e-01	1	1001	122123	4.341974243482216e-02
3.125000e-01	1	1001	123124	4.196842200756112e-02
3.125000e-01	31	31031	154155	4.196842200756112e-02
1.562500e-01	1	1001	155156	4.196842200756112e-02
1.562500e-01	1	1001	156157	2.176858636838919e-02
1.562500e-01	1	1001	157158	2.123392104245969e-02
1.562500e-01	1	1001	159160	2.044666717497368e-02
1.562500e-01	31	31031	190191	2.044666717497368e-02
7.812500e-02	1	1001	191192	2.044666717497368e-02
7.812500e-02	1	1001	192193	9.994601321214880e-03
7.812500e-02	31	31031	223224	9.994601321214880e-03
3.906250e-02	1	1001	224225	9.994601321214880e-03
3.906250e-02	1	1001	225226	5.025376691084382e-03
3.906250e-02	1	1001	226227	5.025375563215029e-03
3.906250e-02	1	1001	228229	4.960511766481268e-03
3.906250e-02	31	31031	259260	4.960511766481268e-03
1.953125e-02	1	1001	260261	4.960511766481268e-03
1.953125e-02	1	1001	261262	2.461558036327638e-03
1.953125e-02	31	31031	292293	2.461558036327638e-03
9.765625e-03	1	1001	293294	2.461558036327638e-03
9.765625e-03	1	1001	294295	1.240380565868726e-03
9.765625e-03	1	1001	296297	1.236249745539375e-03
9.765625e-03	1	1001	297298	1.236198660856225e-03
9.765625e-03	1	1001	303304	1.235819268559091e-03
9.765625e-03	31	31031	334335	1.235819268559091e-03
4.882812e-03	1	1001	335336	1.235819268559091e-03
4.882812e-03	1	1001	336337	6.166515313354992e-04
4.882812e-03	31	31031	367368	6.166515313354992e-04
2.441406e-03	1	1001	368369	6.166515313354992e-04
2.441406e-03	1	1001	369370	3.091339892735689e-04
2.441406e-03	1	1001	370371	3.091337680354300e-04
2.441406e-03	1	1001	372373	3.088677265568052e-04
2.441406e-03	31	31031	403404	3.088677265568052e-04
1.220703e-03	1	1001	404405	3.088677265568052e-04
1.220703e-03	1	1001	405406	1.543549107632103e-04

1.220703e-03	31	31031	436437	1.543549107632103e-04
6.103516e-04	1	1001	437438	1.543549107632103e-04
6.103516e-04	1	1001	438439	7.722380626606906e-05
6.103516e-04	1	1001	440441	7.720721338433378e-05
6.103516e-04	1	1001	441442	7.720720143655768e-05
6.103516e-04	1	1001	447448	7.720574954195314e-05
6.103516e-04	31	31031	478479	7.720574954195314e-05
3.051758e-04	1	1001	479480	7.720574954195314e-05
3.051758e-04	1	1001	480481	3.859791772686805e-05
3.051758e-04	31	31031	511512	3.859791772686805e-05
1.525879e-04	1	1001	512513	3.859791772686805e-05
1.525879e-04	1	1001	513514	1.930224566581984e-05
1.525879e-04	1	1001	515516	1.930120707616112e-05
1.525879e-04	1	1001	516517	1.930120688786729e-05
1.525879e-04	1	1001	522523	1.930111624393049e-05
1.525879e-04	31	31031	553554	1.930111624393049e-05
7.629395e-05	1	1001	554555	1.930111624393049e-05
7.629395e-05	1	1001	555556	9.650248174786213e-06
7.629395e-05	31	31031	586587	9.650248174786213e-06
3.814697e-05	1	1001	587588	9.650248174786213e-06
3.814697e-05	1	1001	588589	4.825329914304177e-06
3.814697e-05	1	1001	590591	4.825264992014411e-06
3.814697e-05	1	1001	591592	4.825264988461697e-06
3.814697e-05	1	1001	597598	4.825259325436093e-06
3.814697e-05	31	31031	628629	4.825259325436093e-06
1.907349e-05	1	1001	629630	4.825259325436093e-06
1.907349e-05	1	1001	630631	2.412610290214445e-06
1.907349e-05	31	31031	661662	2.412610290214445e-06