Probability Theory I – Homework 2

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Exercise 2.1

Let X be a random variable that takes values in $[0,1]$. Find $\lim_{n\to\infty} \mathrm{E}[X^n]$.

Solution. Since $|X^n| \le 1$, which is integrable on $[0,1]$, the LDCT gives

$$\lim_{n\to\infty} \mathrm{E}[X^n] = \mathrm{E}\left[\lim_{n\to\infty} X^n\right] = \mathrm{P}(X=1).$$

The last equality follows from the fact that $X^n \to 0$ for every $0 \le X < 1$. ■

Exercise 2.2

Let $A_1, \dots A_n$ be events and let $A = \cup_{i=1}^n A_i$. First, prove that $\mathbf{1}_A = 1 - \prod_{i=1}^n (1 - \mathbf{1}_{A_i})$. Then, expand the RHS and take expectations to conclude that

$$\mathrm{P}(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mathrm{P}(A_i) - \sum_{i<j} \mathrm{P}(A_i \cap A_j) + \cdots + (-1)^{n-1} \mathrm{P}(\cap_{i=1}^n A_i).$$

Solution. $A = \cup_i A_i = (\cap_i A_i^c)^c = \Omega - \cap_i(\Omega - A_i)$. Thus $\mathbf{1}_A = 1 - \prod_{i=1}^n (1 - \mathbf{1}_{A_i})$. The RHS is

$$1-(1-\sum_{i=1}^n \mathbf{1}_{A_i} + \sum_{i<j} \mathbf{1}_{A_i}\mathbf{1}_{A_j} - \cdots + (-1)^n \mathbf{1}_{A_1}\cdots\mathbf{1}_{A_n}) = \sum_{i=1}^n \mathbf{1}_{A_i} - \sum_{i<j} \mathbf{1}_{A_i}\mathbf{1}_{A_j} + \cdots + (-1)^{n+1}\mathbf{1}_{A_1}\cdots\mathbf{1}_{A_n}.$$

Note that $\mathbf{1}_{A_1}\cdots\mathbf{1}_{A_n} = \mathbf{1}_{\cap_{i=1}^n A_n}$. We have that the RHS is

$$\sum_{i=1}^n \mathbf{1}_{A_i} - \sum_{i<j} \mathbf{1}_{A_i\cap A_j} + \cdots + (-1)^{n-1}\mathbf{1}_{A_1\cap\cdots\cap A_n}.$$

Taking expectation gives

$$\begin{aligned}
\mathrm{P}(A) = \mathrm{E}[\mathbf{1}_A] &= \mathrm{E}\left[\sum_{i=1}^n \mathbf{1}_{A_i} - \sum_{i<j} \mathbf{1}_{A_i\cap A_j} + \cdots + (-1)^{n-1}\mathbf{1}_{A_1\cap\cdots\cap A_n}\right] \\
&= \sum_{i=1}^n \mathrm{P}(A_i) - \sum_{i<j} \mathrm{P}(A_i \cap A_j) + \cdots + (-1)^{n-1} \mathrm{P}(\cap_{i=1}^n A_i).
\end{aligned}$$

■

Exercise 2.3

Let π_n be a uniformly chosen random permutation of $\{1,\dots,n\}$. Let X_n be the number of fixed points of π_n. Find $\mathrm{P}(X_n = 0)$ and evaluate its limit as $n \to \infty$.

Solution. Take $A_i = \{\pi_n(i) = i\}$ for $i = 1,\dots,n$. $\{X_n = 0\} = (\cup_i A_i)^c$. Hence

$$\mathrm{P}(X_n = 0) = 1 - \mathrm{P}(\cup_i A_i).$$

By the previous exercise,

$$\begin{aligned}
\mathrm{P}(\cup_i A_i) &= \sum_{i=1}^{n} \mathrm{P}(A_i) - \sum_{i<j} \mathrm{P}(A_i \cap A_j) + \cdots + (-1)^{n-1}\,\mathrm{P}(\cap_{i=1}^n A_i) \\
&= \sum_{i=1}^{n} \frac{(n-1)!}{n!} - \sum_{i<j} \frac{(n-2)!}{n!} + \cdots + (-1)^{n-1}\frac{1}{n!}.
\end{aligned}$$

Note that for $k \le n$,

$$\sum_{1\le i_1<\cdots<i_k\le n} 1 = \binom{n}{k}.$$

Thus

$$\mathrm{P}(\cup_i A_i) = \binom{n}{1}\frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \cdots + (-1)^{n-1}\frac{1}{n!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1}\frac{1}{n!}.$$

So

$$\mathrm{P}(X_n = 0) = 1 - \frac{1}{1!} + \cdots + (-1)^n \frac{1}{n!}.$$

When $n \to \infty$,

$$\mathrm{P}(X_n = 0) = 1 - \frac{1}{1!} + \cdots + (-1)^n \frac{1}{n!} \to e^{-1}$$

by the Taylor expansion of e^x. ■

Exercise 2.4

Suppose that $\mathrm{E}\left[|X|\right] < \infty$ *and that* A_n *are disjoint sets with* $\cup_n A_n = A$*. Show that*

$$\sum_{n=1}^{\infty} \mathrm{E}\left[X\mathbf{1}_{A_n}\right] = \mathrm{E}\left[X\mathbf{1}_A\right].$$

Solution. Put $E_n = \cup_{i=1}^n A_i$. Then $E_n \nearrow A$ and $\mathbf{1}_{E_n} = \sum_{i=1}^n \mathbf{1}_{A_i} \to \mathbf{1}_A$ pointwisely since A_i are disjoint. Then

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathrm{E}\left[X\mathbf{1}_{A_i}\right] &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{E}\left[X\mathbf{1}_{A_i}\right] = \lim_{n\to\infty} \mathrm{E}\left[X \sum_{i=1}^{n} \mathbf{1}_{A_i}\right] \\
&= \lim_{n\to\infty} \mathrm{E}\left[X\mathbf{1}_{E_n}\right] = \mathrm{E}\left[X \lim_{n\to\infty} \mathbf{1}_{E_n}\right] = \mathrm{E}\left[X\mathbf{1}_A\right],
\end{aligned}$$

where the second last equality follows from LDCT, since $\left|X\mathbf{1}_{E_n}\right| \le |X|$ is integrable. ■

Exercise 2.5

Let our sample space $\Omega = \{1,2,3,4\}$, $\mathcal{F} = 2^{\Omega}$, *and suppose that* $\mathrm{P}(\{k\}) = \frac{1}{4}$ *for each* $k \in \Omega$*. Find two collection of subsets* $\mathcal{A}_1, \mathcal{A}_2$ *such that they are independent while* $\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2)$ *are not.*

Solution. Let $\mathcal{A}_1 = \{\{1,2\},\{2,4\}\}$ and $\mathcal{A}_2 = \{\{2,3\}\}$. Then

$$\sigma(\mathcal{A}_1) = 2^\Omega, \quad \text{and} \quad \sigma(\mathcal{A}_2) = \{\emptyset, \{2,3\}, \{1,4\}, \Omega\}.$$

Let $A_2 = \{2,3\} \in \mathcal{A}_2$. For each $A_1 \in \mathcal{A}_1$,

$$\mathrm{P}(A_1 \cap A_2) = \mathrm{P}(\{2\}) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = \mathrm{P}(A_1)\,\mathrm{P}(A_2).$$

Hence they are independent. However, we can take $\{2,3\} \in \sigma(\mathcal{A}_1) \cap \sigma(\mathcal{A}_2)$ and

$$\mathrm{P}(\{2,3\} \cap \{2,3\}) = \frac{1}{2} \neq \frac{1}{4} = \mathrm{P}(\{2,3\})\,\mathrm{P}(\{2,3\}).$$

Hence $\sigma(\mathcal{A}_1)$ and $\sigma(\mathcal{A}_2)$ are not independent. ■

Exercise 2.6

(a) *Show that if X and Y are independent, integer-valued random variables, then for any integer n,*

$$\mathrm{P}(X+Y=n) = \sum_{m=-\infty}^{\infty} \mathrm{P}(X=m)\,\mathrm{P}(Y=n-m).$$

(b) *Recall that a random variable Y has a Poisson distribution with parameter λ if*

$$\mathrm{P}(Y=k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

for $k = 0,1,\ldots$ and is zero otherwise. Show that if $X = \text{Poisson}(\lambda)$ and $Y = \text{Poisson}(\mu)$, then $X+Y = \text{Poisson}(\lambda+\mu)$.

Solution. For (a), write

$$\{X+Y=n\} = \cup_{m=-\infty}^{\infty}(\{X=m\} \cap \{Y=n-m\}),$$

where the union is disjoint. Thus

$$\mathrm{P}(X+Y=n) = \sum_{m=-\infty}^{\infty} \mathrm{P}(X=m, Y=n-m) = \sum_{m=-\infty}^{\infty} \mathrm{P}(X=m)\,\mathrm{P}(Y=n-m)$$

by the independence.

For (b), apply (a).

$$\begin{aligned}
\mathrm{P}(X+Y=n) &= \sum_{m=0}^{n} \mathrm{P}(X=m)\,\mathrm{P}(Y=n-m) = \sum_{m=0}^{n} e^{-\lambda}\frac{\lambda^m}{m!}e^{-\mu}\frac{\mu^{n-m}}{(n-m)!} \\
&= e^{-(\lambda+\mu)}\frac{1}{n!}\sum_{m=0}^{n} n!\frac{\lambda^m}{m!}\frac{\mu^{n-m}}{(n-m)!} = e^{-(\lambda+\mu)}\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}\lambda^m\mu^{n-m} = e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!}.
\end{aligned}$$

We conclude that $X + Y = \text{Poisson}(\lambda + \mu)$. ∎

Exercise 2.7

Suppose $\mathrm{E}[X_n] = 0$ *and* $\mathrm{E}[X_n X_m] \le r(n-m)$ *for* $m \le n$ *with* $r(k) \to 0$ *as* $k \to \infty$. *Show that*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} 0.$$

Solution. Estimate that

$$\begin{aligned}
0 \le \mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i - 0\right)^2\right] &= \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{E}\left[X_i^2\right] + 2\sum_{i<j}\mathrm{E}\left[X_i X_j\right]\right) \\
&= \frac{1}{n^2}\left[n(r(0)) + 2\left(\sum_{i=1}^{n-1}(n-i)r(i)\right)\right] \\
&\le \frac{1}{n}r(0) + \frac{2}{n}\sum_{i=1}^{n-1} r(i).
\end{aligned}$$

For any $\epsilon > 0$, there is N such that $r(n) < \epsilon$ for every $n > N$ and

$$\frac{2}{n}\sum_{i=1}^{n-1} r(i) = \frac{2}{n}\sum_{i=1}^{N} r(i) + \frac{2}{n}\sum_{i=N+1}^{n-1} r(i) \le \frac{2}{n}\sum_{i=1}^{N} r(i) + \frac{2}{n}n\epsilon \to \epsilon.$$

Since ϵ can be arbitrarily small, we have that $\frac{2}{n}\sum_{i=1}^{n-1} r(i) \to 0$ and

$$\mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i - 0\right)^2\right] \le \frac{1}{n}r(0) + \frac{2}{n}\sum_{i=1}^{n-1} r(i) \to 0$$

as $n \to \infty$. Thus

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 0$$

in L^2 and hence in probability. ∎

Exercise 2.8

(a) *Let* f *be a measurable function on* $[0,1]$ *with*

$$\int_0^1 |f(x)|\,dx < \infty.$$

Let $U_1, \ldots$ *be independent and uniformly distributed on* $[0,1]$, *and*

$$I_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

Show that $I_n \xrightarrow{p} I := \int_0^1 f(x)dx$.

(b) *Suppose that* $\int_0^1 |f(x)|^2 \, dx < \infty$. *Use the Chebyshev inequality to estimate*

$$\mathrm{P}\left(|I_n - I| > \frac{a}{\sqrt{n}}\right).$$

Solution. For (a), notice that $\mathrm{E}\left[f(U_i)\right] = \int_0^1 f(x)dx = I$. By WLLN,

$$I_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i) \xrightarrow{p} \int_0^1 f(x)dx = I.$$

For (b), by the Chebyshev inequality,

$$\begin{aligned}
\mathrm{P}\left(|I_n - I| > \frac{a}{\sqrt{n}}\right) &= \mathrm{P}\left(|I_n - I|^2 > \frac{a^2}{n}\right) \leq \frac{n}{a^2}\,\mathrm{E}\left[(I_n - I)^2\right] \\
&= \frac{1}{na^2}\,\mathrm{E}\left[\left(\sum_{i=1}^{n} f(U_i) - I\right)^2\right] \\
&= \frac{1}{na^2}\left[\sum_{i=1}^{n} \mathrm{E}\left[(f(U_i) - I)^2\right] + \sum_{i<j} \mathrm{E}\left[(f(U_i) - I)\right]\mathrm{E}\left[(f(U_j) - I)\right]\right] \\
&= \frac{1}{a^2}\left(\int_0^1 |f(x)|^2 \, dx - \left(\int_0^1 f(x)dx\right)^2\right)
\end{aligned}$$

by the independence. ■