Solutions to Stochastic Differential Equations by Øksendal

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Contents

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2. Some Mathematical Preliminaries

Exercise 2.1

Suppose that $X : \Omega \to \mathbb{R}$ is a function which takes only countably many values $a_1, a_2, \ldots \in \mathbb{R}$.

(a) Show that X is a random variable if and only if

$$X^{-1}(a_i) \in \mathcal{F} \text{ for all } i \in \mathbb{N}.$$

(b) Suppose that X is a random variable. Show that

$$E\left[|X|\right] = \sum_{i=1}^{\infty} |a_i|\, P(X = a_i).$$

(c) If X is a random variable and $E\left[|X|\right] < \infty$, show that

$$E\left[X\right] = \sum_{i=1}^{\infty} a_i P(X = a_i).$$

(d) If X is a random variable and $f : \mathbb{R} \to \mathbb{R}$ is measurable and bounded, show that

$$E\left[f(X)\right] = \sum_{i=1}^{\infty} f(a_i) P(X = a_i).$$

Solution.

For (a), suppose first that X is a random variable. Since $\{a_i\}$ are Borel sets, $X^{-1}(a_i) \in \mathcal{F}$ for all $i \in \mathbb{N}$. Conversely, assume that $X^{-1}(a_i) \in \mathcal{F}$ for all a_i. Since the range of X is $\{a_i\}_{i\in\mathbb{N}}$, for any Borel set $B \subset \mathbb{R}$, $X^{-1}(B) = \bigcup_{a_i \in B} X^{-1}(a_i) \in \mathcal{F}$, by the definition of σ-algebra. Thus, X is a random variable.

For (b), since X takes only countably many values, so does $|X|$ with $\{|a_i|\}_{i\in\mathbb{N}}$. By the definition of expectation, we have

$$E\left[|X|\right] = \sum_{i=1}^{\infty} |a_i|\, P(X = a_i)$$

in the extended sense.

For (c), since $E\left[|X|\right] < \infty$ and X is a random variable, the series converges absolutely and is well-defined. Hence

$$E\left[X\right] = \sum_{i=1}^{\infty} a_i P(X = a_i).$$

For (d), since f is measurable, $f^{-1}(B)$ is Borel and $X^{-1} f^{-1}(B)$ is measurable. $f(X)$ takes only countably many values, $f(a_1), f(a_2), \ldots$. The definition of expectation gives us

$$E[f(X)] = \sum_{i=1}^{\infty} f(a_i) P(f(X) = f(a_i)) = \sum_{i=1}^{\infty} f(a_i) P(X = a_i).$$

■

Exercise 2.2

$X : \Omega \to \mathbb{R}$ is a random variable. The distribution function F of X is defined as

$$F(x) = P(X \leq x).$$

(a) Prove that F has the following properties:

(i) $0 \leq F \leq 1$, $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

(ii) F is non-decreasing.

(iii) F is right-continuous.

(b) $g : \mathbb{R} \to \mathbb{R}$ is measurable such that $E[|g(X)|] < \infty$. Show that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) dF(x).$$

(c) Let $p(x) \geq 0$ be measurable on $\mathbb{R}$ be the density of X, i.e.,

$$F(x) = \int_{-\infty}^{x} p(t) dt.$$

Find density of B_t^2.

Solution.

For (a), since P is a probability measure, $0 \leq P(S) \leq 1$ for any $S \in \mathcal{F}$. In particular, $0 \leq P(X \leq x) \leq 1$ for all $x \in \mathbb{R}$. Also, we can take $x_n \searrow -\infty$ and $|X \leq x_n| \searrow \emptyset$ as $n \to \infty$. Hence

$$\lim_{x\to-\infty} F(x) = \lim_{n\to\infty} P(X \leq x_n) = P(\emptyset) = 0.$$

Similarly, we can take $x_n \nearrow \infty$ and $|X \leq x_n| \nearrow \Omega$ as $n \to \infty$. Hence

$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} P(X \leq x_n) = P(\Omega) = 1.$$

(i) is proved. For (ii), F is non-decreasing because if $x_1 < x_2$, then

$$F(x_1) = P(X \leq x_1) \leq P(X \leq x_2) = F(x_2).$$

For (iii), let $h > 0$.

$$F(x+h) - F(x) = P(X \leq x+h) - P(X \leq x) = P(x < X \leq x+h).$$

For any $y > x$, there exists $h > 0$ such that $y > x + h$. Thus $(x, x+h] \searrow \varnothing$ as $h \to 0$. Hence

$$F(x+h) - F(x) = P(x < X \leq x+h) \to P(\varnothing) = 0$$

as $h \to 0$. Therefore, F is right-continuous.

For (b), by definition of expectation, the left-hand side is

$$E\left[g(X)\right] = \int_{\mathbb{R}} g(x) d\mu_X(x),$$

where $\mu_X(B) = P(X^{-1}(B))$ for any Borel set $B \subset \mathbb{R}$.

For (c),

$$F(x) = P(B_t^2 \leq x) = P(B_t \leq \sqrt{x}) = \int_{-\infty}^{\sqrt{x}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{u^2}{2t}\right) du.$$

Hence,

$$p(u) = \frac{d}{dx} F(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x}{2t}\right) \frac{1}{2\sqrt{x}}.$$

■

Exercise 2.3

Let $\{\mathcal{F}_i\}_{i \in \mathcal{I}}$ be a collection of σ-algebras on Ω. Prove that

$$\mathcal{F} = \bigcap_{i \in \mathcal{I}} \mathcal{F}_i$$

is again a σ-algebra.

Solution.

First, since $\mathcal{F}_i$ are σ-algebras, they contain $\varnothing$ and hence $\varnothing \in \mathcal{F}$. For any $A \in \mathcal{F}$, $A \in \mathcal{F}_i$ for all $i \in \mathcal{I}$ and hence $A^c \in \mathcal{F}_i$ for all $i \in \mathcal{I}$. Thus $A^c \in \mathcal{F}$. Finally, for any countable collection $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$, we have $A_n \in \mathcal{F}_i$ for all $i \in \mathcal{I}$ and all $n \in \mathbb{N}$. Then $\bigcup_n A_n \in \mathcal{F}_i$ for all $i \in \mathcal{I}$. Hence $\bigcup_n A_n \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a σ-algebra. ■

Exercise 2.4

(a) Let $X : \Omega \to \mathbb{R}$ be a random variable such that $E\left[|X|^p\right] < \infty$ for some $p \in (0, \infty)$. Prove the Chebyshev's inequality:

$$P(|X| \geq \lambda) \leq \frac{1}{\lambda^p} E\left[|X|^p\right]$$

for any $\lambda > 0$.

(b) Suppose there exists $k > 0$ such that $M = E\left[\exp(k\,|X|)\right] < \infty$. Prove that $P(|X| \geq \lambda) \leq Me^{-k\lambda}$ for any $\lambda > 0$.

Solution.

For (a), directly estimate that

$$P(|X| \geq \lambda) = \int_\Omega \chi_{\{|X|^p \geq \lambda^p\}} dP \leq \int_\Omega \frac{|X|^p}{\lambda^p} dP = \frac{1}{\lambda^p} E\left[|X|^p\right].$$

(b) is similar:

$$P(|X| \geq \lambda) = \int_\Omega \chi_{\{\exp(k|X|) \geq \exp(k\lambda)\}} dP \leq \int_\Omega \exp(k\,|X|)\exp(-k\lambda) dP = M\exp(-k\lambda).$$

■

Exercise 2.5
Let $X, Y : \Omega \to \mathbb{R}$ be two independent random variables and assume for simplicity that X, Y are bounded. Prove that

$$E[XY] = E[X]\,E[Y].$$

Solution.

For any $\epsilon > 0$, by definition of the expectation, we can find simple functions s and t on Ω such that

$$\int |s - X|\,dP < \epsilon, \quad \int |t - Y|\,dP < \epsilon, \quad \Rightarrow \quad \left|E[X] - \int s dP\right| < \epsilon, \quad \left|E[Y] - \int t dP\right| < \epsilon,$$

where s and t can be written as

$$s = \sum_{i=1}^n s_i \chi_{X^{-1}[s_i, s_{i+1})} \quad \text{and} \quad t = \sum_{j=1}^m t_j \chi_{Y^{-1}[t_j, t_{j+1})},$$

with s_i and t_j being arranged in ascending order. Thus,

$$\begin{aligned}
\int st dP &= \sum_{i=1}^n \sum_{j=1}^m s_i t_j P(\{X \in [s_i, s_{i+1})\} \cap \{Y \in [t_j, t_{j+1})\}) \\
&= \sum_{i=1}^n \sum_{j=1}^m s_i t_j P(X \in [s_i, s_{i+1}))P(Y \in [t_j, t_{j+1})) \\
&= \left(\sum_{i=1}^n s_i P(X \in [s_i, s_{i+1}))\right)\left(\sum_{j=1}^m t_j P(Y \in [t_j, t_{j+1}))\right) = \left(\int s dP\right)\left(\int t dP\right).
\end{aligned}$$

Also,

$$\left|E[XY] - \int st dP\right| \leq \left|\int |X - s|\,|t|\,dP\right| + \left|\int |Y - t|\,|X|\,dP\right|.$$

X and Y are bounded, say by M and N respectively. Then t is also bounded by N from our construction. Thus

$$\left|E[XY] - \int st dP\right| \leq M\epsilon + N\epsilon.$$

Combine the results above, we arrive at

$$\begin{aligned}
|E[XY] - E[X]E[Y]| &\le \left|E[XY] - \int stdP\right| + \left|E[X]E[Y] - \int sdP \int tdP\right| \\
&\le (M+N)\epsilon + \left|E[X] - \int sdP\right|\left|\int tdP\right| + \left|E[Y] - \int tdP\right| |E[X]| \\
&\le (M+N)\epsilon + \epsilon N + \epsilon M.
\end{aligned}$$

Since ϵ is arbitrary, we conclude that $E[XY] = E[X]E[Y]$. ∎

Exercise 2.6

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_1, \ldots \in \mathcal{F}$ be sets such that

$$\sum_{i=1}^{\infty} P(A_i) < \infty.$$

Prove the Borel-Cantelli lemma:

$$P\left(\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i\right) = 0.$$

Solution.

Set $B_m = \bigcup_{i=m}^{\infty} A_i$ be measurable. Then

$$P(B_m) \le \sum_{i=m}^{\infty} P(A_i) \to 0$$

as $m \to \infty$ by the assumption. Thus

$$P\left(\bigcap_{m=1}^{\infty} B_m\right) \le \lim_{n\to\infty} P\left(\bigcap_{m=1}^{n} B_m\right) \le \lim_{n\to\infty} P(B_n) = 0.$$

∎

Exercise 2.7

(a) Suppose $G_1, \ldots, G_n$ are disjoint sets in $\mathcal{F}$ such that $\bigcup_{i=1}^{n} G_i = \Omega$. Prove that the family

$$\mathcal{G} = \{G \mid G \textit{ is a union of some } G_i\} \cup \{\varnothing\}$$

is a σ-algebra.

(b) Prove that every finite σ-algebra is of type $\mathcal{G}$ as in (a).

(c) Let $\mathcal{F}$ be a finite σ-algebra on Ω and $X : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Prove that X is simple.

Solution.

For (a), first, $\varnothing \in \mathcal{G}$ by definition. Let $G \in \mathcal{G}$. Then $G = \cup_{i \in \mathcal{I}} G_i$ for some $\mathcal{I} \subset \{1, \ldots, n\}$, with the convention that $\cup_{i \in \varnothing} G_i = \varnothing$. Then $G^c = \bigcup_{i \notin \mathcal{I}} G_i \in \mathcal{G}$. Lastly, for countably many $G_i \in \mathcal{G}$, since $\mathcal{G}$ is finite, there are in fact finitely many distinct G_i and the union must lie in $\mathcal{G}$ by the definition. Hence $\mathcal{G}$ is a σ-algebra.

For (b), let $\mathcal{F}$ be a finite σ-algebra. Consider the collection

$$\mathcal{S} = \{S \in \mathcal{F} \mid S \cap F = \varnothing \text{ or } S \text{ for all } F \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is finite, $\mathcal{S}$ is also finite. We first check that every distinct sets in $\mathcal{S}$ are disjoint. Suppose not. There are $S_1, S_2 \in \mathcal{S}$ such that $S_1 \cap S_2$ is non-empty. Then $S_1 \cap S_2 = S_1 = S_2$, contradicting the assumption that S_1 and S_2 are distinct. Thus every distinct sets in $\mathcal{S}$ are disjoint. Next, we check that $\bigcup_{S \in \mathcal{S}} S = \Omega$. If not, let $A = \Omega \setminus \bigcup_{S \in \mathcal{S}} S$ be non-empty and $A \cap F$ is a non-empty proper subset of A for some $F \in \mathcal{F}$. But then $A \cap F$ or $A \cap F^c$ must satisfy the condition that there is some $F' \in \mathcal{F}$ such that $A \cap F \cap F'$ or $A \cap F^c \cap F'$ is non-empty, proper subset of $A \cap F$ or $A \cap F^c$ respectively. Note that $F' \neq F$ and the process continues. In the end, we can find a infinite sequence of distinct sets lying in $\mathcal{F}$, contradicting the finiteness of $\mathcal{F}$. Thus $\bigcup_{S \in \mathcal{S}} S = \Omega$. Finally, by (a),

$$\mathcal{G} = \{G \mid G \text{ is a union of some } S \in \mathcal{S}\} \cup \{\varnothing\}$$

is a σ-algebra. It remains to show that $\mathcal{G} = \mathcal{F}$. Clearly, $\mathcal{G} \subset \mathcal{F}$ since $\mathcal{S} \subset \mathcal{F}$. For any $F \in \mathcal{F}$, we can write $F = \bigcup_{i=1}^{n} S_i$ for some $S_i \in \mathcal{S}$. Thus $F \in \mathcal{G}$. We end up with $\mathcal{G} = \mathcal{F}$.

For (c), suppose that X can take infinitely many values $\{a_i\}_{i \in \mathcal{I}}$. Since X is $\mathcal{F}$-measurable, $X^{-1}(\{a_i\}) \in \mathcal{F}$ for all $i \in \mathcal{I}$. In particular, $X^{-1}(\{a_i\})$ and $X^{-1}(\{a_j\})$ are disjoint for all $i \neq j$. This implies that $\mathcal{F}$ contains infinitely many disjoint sets, contradicting the finiteness of $\mathcal{F}$. Thus X can only take finitely many values and is simple. ■

Exercise 2.8

Let B_t be Brownian motion on $\mathbb{R}$, $B_0 = 0$. Put $E = E^0$.

(a) *Prove that*

$$E\left[e^{iuB_t}\right] = e^{-\frac{u^2 t}{2}} \text{ for all } u \in \mathbb{R}.$$

(b) *Use the power series expansion of the exponential function to show that*

$$E\left[B_t^{2k}\right] = \frac{(2k)!}{2^k k!} t^k \text{ for all } k \in \mathbb{N}.$$

(c) *Prove that*

$$E\left[f(B_t)\right] = \int_{\mathbb{R}} f(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx$$

for all measurable functions f on $\mathbb{R}$ such that the integral is finite. Deduce (b) by setting $f(x) = x^{2k}$.

(d) *Now suppose that B_t is a n-dimensional Brownian motion. Prove that*

$$E^x\left[|B_t - B_s|^4\right] = n(n+2)\,|t-s|^2$$

for all $n \in \mathbb{N}$ and $0 \le s, t \le T$.

Solution.

For (a), directly compute the expectation:

$$E\left[e^{iuB_t}\right] = \int_{-\infty}^{\infty} e^{iux}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,dx = e^{-\frac{u^2t}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-iut)^2}{2t}}\,dx = e^{-\frac{u^2t}{2}}.$$

For (b), note that the power series expansion of the exponential function gives

$$e^{iuB_t} = \sum_{k=0}^{\infty}\frac{(iuB_t)^k}{k!} \quad\Rightarrow\quad E\left[e^{iuB_t}\right] = \sum_{k=0}^{\infty}\frac{(iu)^k}{k!}E\left[B_t^k\right].$$

The right-hand side is

$$e^{-\frac{u^2t}{2}} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{u^2t}{2}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k u^{2k}}{2^k k!}t^k.$$

For those two expressions to be equal, as a function of u, $E\left[B_t^k\right] = 0$ for odd k and we may rewrite the first expression as

$$E\left[e^{iuB_t}\right] = \sum_{k=0}^{\infty}\frac{(-1)^k u^{2k}}{(2k)!}E\left[B_t^{2k}\right].$$

By comparing the coefficients,

$$E\left[B_t^{2k}\right] = \frac{(-1)^k t^k}{2^k k!}\cdot\frac{(2k)!}{(-1)^k} = \frac{(2k)!}{2^k k!}t^k.$$

For (c), it is clear that B_t has the density $p(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$. It follows that

$$E\left[f(B_t)\right] = \int f(x)\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,dx.$$

By setting $f : x \mapsto x^{2k}$, we have

$$\begin{aligned}
E\left[B_t^{2k}\right] &= \int x^{2k}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,dx \\
&= x^{2k+1}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\Big|_{-\infty}^{\infty} - \int 2kx^{2k}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}} - x^{2k}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\frac{x^2}{t}\,dx \\
&= \frac{1}{t}E\left[B_t^{2(k+1)}\right] - 2kE\left[B_t^{2k}\right] \quad\Rightarrow\quad E\left[B_t^{2(k+1)}\right] = (2k+1)tE\left[B_t^{2k}\right].
\end{aligned}$$

And also when $k = 1$, $E\left[B_t^2\right] = t$. Suppose $E\left[B_t^{2k}\right] = \frac{(2k)!}{2^k k!}t^k$. Then

$$E\left[B_t^{2(k+1)}\right] = (2k+1)t\frac{(2k)!}{2^k k!}t^k = \frac{(2(k+1))!}{2^{k+1}(k+1)!}t^{k+1}.$$

The conclusion follows by induction.

For (d), if $n = 1$, $B_t - B_s \sim N(0, |t-s|)$ and $E^x\left[|B_t - B_s|^4\right] = 3|t-s|^2$. Now suppose that for n-dimensional B_t, $E^x\left[|B_t - B_s|^4\right] = n(n+2)|t-s|^2$. Then for $n+1$-dimensional B_t,

$$\begin{aligned}
E^x\left[|B_t - B_s|^4\right] &= \int_{\mathbb{R}^n}\int_{\mathbb{R}}(|y|^2+z^2)^2\frac{1}{\sqrt{(2\pi)^{n+1}|t-s|^{n+1}}}\exp\left(-\frac{|y|^2+z^2}{2|t-s|}\right)dydz \\
&= \int_{\mathbb{R}^n}|y|^4\frac{1}{\sqrt{(2\pi)^n|t-s|^n}}\exp\left(-\frac{|y|^2}{2|t-s|}\right)dy \\
&\quad + 2|t-s|\int_{\mathbb{R}^n}|y|^2\frac{1}{\sqrt{(2\pi)^n|t-s|^n}}\exp\left(-\frac{|y|^2}{2|t-s|}\right)dy + 3|t-s|^2 \\
&= n(n+2)|t-s|^2 + 2n|t-s|^2 + 3|t-s|^2 = (n+1)(n+3)|t-s|^2.
\end{aligned}$$

Thus by induction, the conclusion holds for all $n \in \mathbb{N}$. It follows from the Kolmogorov's continuity theorem that we can always set B_t to be a continuous process. ■

Exercise 2.9

Let $(\Omega, \mathcal{F}, P) = ([0,\infty), \mathcal{B}, \mu)$ be a probability space where μ is a probability measure such that there is no mass at single points. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega, \\ 0 & \text{if } t \neq \omega. \end{cases} \quad \text{and} \quad Y_t(\omega) = 0 \text{ for all } (t,\omega) \in [0,\infty)\times[0,\infty).$$

Prove that $\{X_t\}$ and $\{Y_t\}$ have the same distributions and X_t is a version of Y_t. And yet $t \mapsto Y_t(\omega)$ is continuous for all ω, while $t \mapsto X_t(\omega)$ is discontinuous for all ω.

Solution.

First, given any t, X_t and Y_t are both random variables. For $t_1, \ldots, t_k \in [0,\infty)$, consider the sets $F_1, \ldots, F_k \in \mathcal{B}$. Since Y_t is constant,

$$P(Y_{t_1} \in F_1, \ldots, Y_{t_k} \in F_k) = \mathbb{1}\left\{0 \in \cap_{i=1}^k F_i\right\}.$$

Also, since μ has no mass at single points,

$$P(X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k) = \mathbb{1}\left\{0 \in \cap_{i=1}^k F_i\right\}.$$

Thus, X_t and Y_t have the same distributions. Furthermore, for any $t \in [0,\infty)$, $\{X_t = Y_t\} = \Omega \setminus \{t\}$, which has zero measure and hence X_t is a version of Y_t. Now since Y_t is constant, $t \mapsto Y_t(\omega)$ is continuous for all ω. On the other hand, for any $\omega \in [0,\infty)$, $X(t,\omega)$ is discontinuous at $t = \omega$, proving that $t \mapsto X_t(\omega)$ is discontinuous for all ω. ■

Exercise 2.10

Prove that the Brownian motion B_t has stationary increments, i.e., given $h > 0$, the process $\{B_{t+h} - B_t\}$ has the same distribtuions for all t.

Solution.

For any $h > 0$, $B_{t+h} - B_t \sim N(0, h)$. The stationarity follows immediately. ■

Exercise 2.11

If $B_t = (B_t^{(1)}, \dots, B_t^{(n)})$ is an n-dimensional Brownian motion, then the component processes $B_t^{(i)}$, $1 \le i \le n$, are independent Brownian motions.

Solution.

Since the B_t is continuous almost surely, the component processes $B_t^{(i)}$ are continuous almost surely as well. Now we may regard the component processes as projections of B_t and hence they are normally distributed. $E\left[B_t^{(i)}\right] = 0$ since $E\left[B_t\right] = 0$. Also, $\mathrm{Cov}(B_t^{(i)}, B_t^{(j)}) = t\delta_{ij}$ as $\mathrm{Var}(B_t) = tI$. Since for $i \neq j$ the covariance is zero, the component processes are independent. ■

Exercise 2.12

Let B_t be a Brownian motion and fix $t_0 \ge 0$. Prove that the process $\tilde{B}_t = B_{t+t_0} - B_{t_0}$ is a Brownian motion.

Solution.

Since B_t is almost surely continuous, $B_{t+t_0} - B_{t_0}$ is also almost surely continuous. Also, $B_{t+t_0} - B_{t_0} \sim N(0, t)$. For $s < t < u$,

$$\mathrm{Cov}(\tilde{B}_u - \tilde{B}_t, \tilde{B}_t - \tilde{B}_s) = \mathrm{Cov}(B_{u+t_0} - B_{t+t_0}, B_{t+t_0} - B_{s+t_0}) = 0.$$

Thus $\tilde{B}_t$ has independent increments. This shows that $\tilde{B}_t$ is a Brownian motion. ■

Exercise 2.13

Let B_t be 2-dimensional Brownian motion and put

$$D_\rho = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| < \rho\right\}$$

for $\rho > 0$. Compute $P^0(B_t \in D_\rho)$.

Solution.

Since $B_t \sim N(0, tI)$,

$$P^0(B_t \in D_\rho) = \int_{D_\rho} \frac{1}{2\pi t} \exp\left(-\frac{|x|^2}{2t}\right) dx = \int_0^{2\pi} \int_0^{\rho} \frac{1}{2\pi t} \exp\left(-\frac{r^2}{2t}\right) r dr d\theta = 1 - \exp\left(-\frac{\rho^2}{2t}\right).$$

■

Exercise 2.14

Let B_t be n-dimensional Brownian motion and $K \subset \mathbb{R}^n$ be a measure zero set under the Lebesgue measure. Prove that the expected total length of time that B_t spends in K is zero.