Solutions to Stochastic Differential Equations by Øksendal

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Contents

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2. Some Mathematical Preliminaries

Exercise 2.1

Suppose that $X : \Omega \to \mathbb{R}$ *is a function which takes only countably many values* $a_1, a_2, \ldots \in \mathbb{R}$.

(a) Show that X *is a random variable if and only if*

$$X^{-1}(a_i) \in \mathcal{F} \text{ for all } i \in \mathbb{N}.$$

(b) Suppose that X *is a random variable. Show that*

$$E\left[|X|\right] = \sum_{i=1}^{\infty} |a_i| \, P(X = a_i).$$

(c) If X *is a random variable and* $E\left[|X|\right] < \infty$*, show that*

$$E\left[X\right] = \sum_{i=1}^{\infty} a_i P(X = a_i).$$

(d) If X *is a random variable and* $f : \mathbb{R} \to \mathbb{R}$ *is measurable and bounded, show that*

$$E\left[f(X)\right] = \sum_{i=1}^{\infty} f(a_i) P(X = a_i).$$

Solution.

For (a), suppose first that X is a random variable. Since $\{a_i\}$ are Borel sets, $X^{-1}(a_i) \in \mathcal{F}$ for all $i \in \mathbb{N}$. Conversely, assume that $X^{-1}(a_i) \in \mathcal{F}$ for all a_i. Since the range of X is $\{a_i\}_{i \in \mathbb{N}}$, for any Borel set $B \subset \mathbb{R}$, $X^{-1}(B) = \bigcup_{a_i \in B} X^{-1}(a_i) \in \mathcal{F}$, by the definition of σ-algebra. Thus, X is a random variable.

For (b), since X takes only countably many values, so does $|X|$ with $\{|a_i|\}_{i \in \mathbb{N}}$. By the definition of expectation, we have

$$E\left[|X|\right] = \sum_{i=1}^{\infty} |a_i| \, P(X = a_i)$$

in the extended sense.

For (c), since $E\left[|X|\right] < \infty$ and X is a random variable, the series converges absolutely and is well-defined. Hence

$$E\left[X\right] = \sum_{i=1}^{\infty} a_i P(X = a_i).$$

For (d), since f is measurable, $f^{-1}(B)$ is Borel and $X^{-1}f^{-1}(B)$ is measurable. $f(X)$ takes only countably many values, $f(a_1), f(a_2), \ldots$. The definition of expectation gives us

$$E[f(X)] = \sum_{i=1}^{\infty} f(a_i)P(f(X) = f(a_i)) = \sum_{i=1}^{\infty} f(a_i)P(X = a_i).$$

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Exercise 2.2

$X : \Omega \to \mathbb{R}$ is a random variable. The distribution function F of X is defined as

$$F(x) = P(X \leq x).$$

(a) Prove that F has the following properties:

(i) $0 \leq F \leq 1$, $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

(ii) F is non-decreasing.

(iii) F is right-continuous.

(b) $g : \mathbb{R} \to \mathbb{R}$ is measurable such that $E[|g(X)|] < \infty$. Show that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)dF(x).$$

(c) Let $p(x) \geq 0$ be measurable on $\mathbb{R}$ be the density of X, i.e.,

$$F(x) = \int_{-\infty}^{x} p(t)dt.$$

Find density of B_t^2.

Solution.

For (a), since P is a probability measure, $0 \leq P(S) \leq 1$ for any $S \in \mathcal{F}$. In particular, $0 \leq P(X \leq x) \leq 1$ for all $x \in \mathbb{R}$. Also, we can take $x_n \searrow -\infty$ and $|X \leq x_n| \searrow \varnothing$ as $n \to \infty$. Hence

$$\lim_{x\to-\infty} F(x) = \lim_{n\to\infty} P(X \leq x_n) = P(\varnothing) = 0.$$

Similarly, we can take $x_n \nearrow \infty$ and $|X \leq x_n| \nearrow \Omega$ as $n \to \infty$. Hence

$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} P(X \leq x_n) = P(\Omega) = 1.$$

(i) is proved. For (ii), F is non-decreasing because if $x_1 < x_2$, then

$$F(x_1) = P(X \leq x_1) \leq P(X \leq x_2) = F(x_2).$$

For (iii), let $h > 0$.

$$F(x+h) - F(x) = P(X \leq x+h) - P(X \leq x) = P(x < X \leq x+h).$$

For any $y > x$, there exists $h > 0$ such that $y > x + h$. Thus $(x, x+h] \searrow \emptyset$ as $h \to 0$. Hence

$$F(x+h) - F(x) = P(x < X \leq x+h) \to P(\emptyset) = 0$$

as $h \to 0$. Therefore, F is right-continuous.

For (b), by definition of expectation, the left-hand side is

$$E\left[g(X)\right] = \int_{\mathbb{R}} g(x) d\mu_X(x),$$

where $\mu_X(B) = P(X^{-1}(B))$ for any Borel set $B \subset \mathbb{R}$. ∎