Probability Theory I – Homework 1

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Exercise 1.1

Suppose f is a measurable mapping from one measurable space S to another measurable space U. If A is a measurable subset of S, does it follow that the image $f(A)$ is a measurable subset of U?

Solution. No. Consider the case where $S = U = \{0,1\}$. Define the σ-algebra on S to be $\mathcal{S} = 2^S$ and the one on U to be $\mathcal{U} = \{\varnothing, U\}$. f is the identity mapping, which is measurable. However, $f(\{0\}) = \{0\} \notin \mathcal{U}$. ■

Exercise 1.2

Let Ω be a sample space and let I be an arbitrary index set, which could be uncountable.

(a) For each $i \in I$, let $\mathcal{F}_i$ be a σ-algebra on Ω. Show that $\bigcap_{i\in I} \mathcal{F}_i$ is a σ-algebra on Ω.

(b) Let $\mathcal{A}$ be a collection of subsets of Ω. Show that there is a smallest σ-algebra containing $\mathcal{A}$.

Solution. For (a), first note that $\varnothing \in \mathcal{F}_i$ for all $i \in I$ and thus $\varnothing \in \cap_{i\in I}\mathcal{F}_i$. Next, if $A \in \cap_{i\in I}\mathcal{F}_i$, then $A \in \mathcal{F}_i$ for all $i \in I$. Since $\mathcal{F}_i$ are σ-algebras, $A^c \in \mathcal{F}_i$ for all i; hence $A^c \in \cap_{i\in I}\mathcal{F}_i$. Finally, if $\{A_n\} \subset \cap_{i\in I}\mathcal{F}_i$ is a countable subcollection, then $\{A_n\} \subset \mathcal{F}_i$ for all i. Since $\mathcal{F}_i$ are σ-algebras, $\cup_n A_n \in \mathcal{F}_i$ for all i; hence $\cup_n A_n \in \cap_{i\in I}\mathcal{F}_i$. Therefore, $\cap_{i\in I}\mathcal{F}_i$ is a σ-algebra.

For (b), set $\mathcal{F}$ to be the intersection of all σ-algebras containing $\mathcal{A}$. Since the power set of Ω is a σ-algebra containing $\mathcal{A}$, $\mathcal{F}$ is non-empty. By (a), we know that $\mathcal{F}$ is a σ-algebra. By definition, if $\mathcal{G}$ is a σ-algebra containing $\mathcal{A}$, then $\mathcal{F} \subset \mathcal{G}$ since $\mathcal{F}$ is the intersection of all such σ-algebras. Hence $\mathcal{F}$ is the smallest σ-algebra containing $\mathcal{A}$. ■

Exercise 1.3

Let $\Omega = \mathbb{R}$ and $\mathcal{F}$ be all subsets of $\mathbb{R}$ such that either A or A^c is countable. Define $P(A) = 0$ if A is countable and $P(A) = 1$ if uncountable. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Solution. I first claim that $\mathcal{F}$ is a σ-algebra. Since $\varnothing$ is countable, $\varnothing \in \mathcal{F}$. If $A \in \mathcal{F}$, then either A or A^c is countable; either A^c or $A = (A^c)^c$ is countable, respectively. Thus $A^c \in \mathcal{F}$. Suppose that $\{A_n\} \subset \mathcal{F}$ is a countable collection of sets. If all A_n are countable, then $\cup_n A_n$ is countable and thus in $\mathcal{F}$. If there is some A_n, say A_1, is uncountable, then A_1^c is countable. Now $(\cup_n A_n)^c = \cap_n A_n^c \subset A_1^c$ is countable, so $\cup_n A_n \in \mathcal{F}$. Therefore, $\mathcal{F}$ forms a σ-algebra.

Next, I show that P is a probability measure. Clearly every set in $\mathcal{F}$ is either countable or uncountable, so P is well-defined and $P(A) \geq 0 = P(\varnothing)$ for all $A \in \mathcal{F}$ since $\varnothing$ is countable. Also, $P(\Omega) = 1$ since $\mathbb{R}$ is uncountable. Finally, suppose that $\{A_n\} \subset \mathcal{F}$ is a countable collection of disjoint sets. If all A_n are countable, then $\cup_n A_n$ is countable and

$$P(\cup_n A_n) = 0 = \sum_n 0 = \sum_n P(A_n).$$

If some A_n, say A_1, is uncountable, then A_n are countable for all $n \neq 1$; otherwise, $\cup_{n\neq 1} A_n$ would be uncountable and $A_1^c \supset \cup_{n\neq 1} A_n$ is uncountable, contradiction. Now $\cup_n A_n$ is uncountable and

$$P(\cup_n A_n) = 1 = 1 + \sum_{n\neq 1} 0 = P(A_1) + \sum_{n\neq 1} P(A_n) = \sum_n P(A_n).$$

Therefore, P is a probability measure and $(\Omega, \mathcal{F}, P)$ is a probability space. ■

Exercise 1.4

Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then Z is a random variable.

Solution. For any Borel set $B \subset \mathbb{R}$,

$$Z^{-1}(B) = \Big(A \cap X^{-1}(B)\Big) \cup \Big(A^c \cap Y^{-1}(B)\Big).$$

Since X and Y are random variables, $X^{-1}(B), Y^{-1}(B) \in \mathcal{F}$. The σ-algebra $\mathcal{F}$ is closed under finite intersections and unions, so $Z^{-1}(B) \in \mathcal{F}$. Z is thus a random variable. ■

Exercise 1.5

Show that if $\mathcal{S} = \sigma(\mathcal{A})$, then $X^{-1}(\mathcal{A}) = \{X^{-1}(A) \mid A \in \mathcal{A}\}$ generates $\sigma(X) = \{X^{-1}(B) \mid B \in \mathcal{S}\}$.

Solution. Set

$$\mathcal{F} = \left\{E \in \mathcal{S} \mid X^{-1}(E) \in \sigma(X^{-1}(\mathcal{A}))\right\}.$$

Clearly $\varnothing \in \mathcal{S}$ and $X^{-1}(\varnothing) = \varnothing \in \sigma(X^{-1}(\mathcal{A}))$, so $\varnothing \in \mathcal{F}$. If $E \in \mathcal{F}$, then $E^c \in \mathcal{S}$ and $X^{-1}(E^c) = (X^{-1}(E))^c \in \sigma(X^{-1}(\mathcal{A}))$ since $\sigma(X^{-1}(\mathcal{A}))$ is a σ-algebra. Thus $E^c \in \mathcal{F}$. Suppose that $\{E_n\} \subset \mathcal{F}$ is a countable collection of sets. Then $\{E_n\} \subset \mathcal{S}$ and $\cup_n E_n \in \mathcal{S}$. Also,

$$X^{-1}(\cup_n E_n) = \cup_n X^{-1}(E_n) \in \sigma(X^{-1}(\mathcal{A}))$$

since $X^{-1}(E_n) \in \sigma(X^{-1}(\mathcal{A}))$ and $\sigma(X^{-1}(\mathcal{A}))$ is a σ-algebra. Hence $\cup_n E_n \in \mathcal{F}$ and $\mathcal{F}$ is a σ-algebra.

By the construction, $\mathcal{A} \subset \mathcal{F} \subset \mathcal{S}$ and we must have $\mathcal{S} = \sigma(\mathcal{A}) \subset \mathcal{F} \subset \mathcal{S}$; hence $\mathcal{F} = \mathcal{S}$. But by the definition of $\mathcal{F}$, $X^{-1}(\mathcal{F}) = \{X^{-1}(E) \mid E \in \mathcal{F}\}$ is a σ-algebra such that $X^{-1}(\mathcal{A}) \subset X^{-1}(\mathcal{F}) \subset \sigma(X^{-1}(\mathcal{A}))$. Thus $\sigma(X) = X^{-1}(\mathcal{S}) = X^{-1}(\mathcal{F}) = \sigma(X^{-1}(\mathcal{A}))$. ■

Exercise 1.6

Conclude that a random variable Y is measurable with respect to $\sigma(X)$ if and only if $Y = f(X)$, where $f : \mathbb{R} \to \mathbb{R}$ is measurable.

Solution. If $Y = f(X)$ for some measurable function $f : \mathbb{R} \to \mathbb{R}$, then for any Borel set $B \subset \mathbb{R}$, $Y^{-1}(B) = X^{-1}(f^{-1}(B)) \in \sigma(X)$ since $f^{-1}(B)$ is also a Borel set and X is $\sigma(X)$-measurable. Thus Y is $\sigma(X)$-measurable.

Conversely, suppose that Y is $\sigma(X)$-measurable. If Y is simple, $Y = \sum_{i=1}^n y_i \mathbf{1}_{A_i}$ where $A_i \in \sigma(X)$ are disjoint and y_i are distinct. Now $A_i = X^{-1}(B_i)$ for disjoint Borel sets B_i. Set $f = \sum_i y_i \mathbf{1}_{B_i}$, which is measurable. Then

$$f(X) = \sum_i y_i \mathbf{1}_{X^{-1}(B_i)} = \sum_i y_i \mathbf{1}_{A_i} = Y.$$

For general measurable Y, there is a sequence of simple measurable functions $Y_k \longrightarrow Y$ pointwise. For each k, there is a corresponding measurable $f_k : \mathbb{R} \to \mathbb{R}$ such that $Y_k = f_k(X)$. Now define f to be the pointwise limit of f_k; then f is measurable and

$$Y = \lim_{k\to\infty} Y_k = \lim_{k\to\infty} f_k(X) = f(X).$$

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