Notes on Ergodic Theory

Kai-Jyun Wang*

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Abstract

This note is based on Olav Kallenberg's *Foundations of Modern Probability*, Chapter 25, 26, and 27. The solutions to the exercises are planned to be included.

1. Stationary Process and Ergodic Theorems

Definition 1.1
Let $(S, \mathcal{S}, \mu)$ be a measure space. A measurable mapping $T : S \to S$ ***preserves*** *μ or μ-**preserving** if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{S}$.*

Remark
Let ξ be a random variable on S. Then T preserves μ if and only if $\xi \circ T \stackrel{d}{=} \xi$.

Definition 1.2
*Consider the left shift mapping θ defined on S^∞ by $\theta(x_1, x_2, \ldots) = \theta(x_2, x_3, \ldots)$. A random variable ξ on S^∞ is called **stationary** if $\theta\xi \stackrel{d}{=} \xi$.*

Lemma 1.3 (Stationary and Invariance)
For any random variable $\xi : S \to E$ and a measurable mapping $T : S \to S$,

(a) $\xi T \stackrel{d}{=} \xi$ if and only if the sequence $\{\xi T^n\}$ is stationary.

(b) If $\xi T \stackrel{d}{=} \xi$, then for every measurable function f defined on E, the sequence defined by $\{f(\xi T^n)\}$ is stationary.

(c) Any stationary random sequence $\eta \in S^\infty$ can be represented as $\eta_n = f(\eta T^n)$.

Proof. Suppose first that $\xi T \stackrel{d}{=} \xi$. Then

$$\theta(\{f(\xi T^n)\}) = \left\{f\xi T^{n+1}\right\} = \{f\xi T^n \circ T\} \stackrel{d}{=} \{f(\xi T^n)\},$$

*National Taiwan University, Department of Economics.

where θ is the left shift. This proves (b) and the suffiency of (a) by taking f to be the identity function. Conversely, if $\eta = \{\eta_n\}$ is a stationary sequence, then $\{\eta_n\} \overset{d}{=} \theta(\{\eta_n\}) = \{\eta_{n+1}\}$. Let $\eta_n = \pi_0(\theta^n \eta)$, where π_0 is the projection onto the first coordinate. This proves (c). And, taking $\eta = \{\xi T^n\}$, we have

$$\xi = \pi_0(\{\xi T^n\}) \overset{d}{=} \pi_0(\theta(\{\xi T^n\})) = \pi_0(\{\xi T^{n+1}\}) = \xi T.$$

This proves the necessity of (a). ■

Definition 1.4
We extend the definition of stationarity to a sequence indexed by $\mathbb{Z}$ by requiring that $\theta\xi \overset{d}{=} \xi$, where θ is the left shift on $S^{\mathbb{Z}}$.

Remark
The advantage of such definition is that the left shift on $S^{\mathbb{Z}}$ is now invertible and hence forms a group instead of a semigroup. Observe that lemma 1.3 does not depend on the choice of our index set and hence still applies.

Definition 1.5
*Let $(S, \mathcal{S})$ be a measurable space and μ be a measure on $\mathcal{S}$. μ is said to be **complete** if $A \subset B$ and $\mu(B) = 0$ implies $A \in \mathcal{S}$.*

Remark
The Lebesgue measure on $\mathbb{R}$ is complete, but the Borel measure is not. In fact, the completion of the Borel measure is the Lebesgue measure.

Definition 1.6
*Let $(S, \mathcal{S}, \mu)$ be a measure space and $T : S \to S$ be a measurable mapping. A set $I \subset S$ is said to be **invariant** under T if $T^{-1}(I) = I$ and **almost invariant** if $\mu(T^{-1}(I) \triangle I) = 0$.*

Definition 1.7
*Let $(S, \mathcal{S}, \mu)$ be a measure space, $T : S \to S$ be a measurable mapping, and $\mathcal{S}^\mu$ be the completion of $\mathcal{S}$ with respect to μ. The **invariant σ-algebra** of T is the collection $\mathcal{I} \subset \mathcal{S}$ of all sets that are invariant under T. Similarly, the **almost invariant σ-algebra** of T is the collection $\mathcal{I}' \subset \mathcal{S}^\mu$ of all sets that are almost invariant under T.*

Definition 1.8
*A measurable function $f : S \to E$ is said to be **invariant** if $f \circ T = f$ and **almost invariant** if $f \circ T = f$ μ-a.e.*