A Brief Introduction to the Fundamental Methods in Dynamic Programming

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1. Optimal Growth Model

In this capter, we introduce an optimal growth model. The model is going to be our working example for the methods.

Consider an agent who seeks to maximize his lifetime expected utility. The agent's problem is to choose his future path of consumption c_t and capital stock k_{t+1}, subject to the constraint:

$$c_t + k_{t+1} \leq y_t, \tag{1.1}$$

where both c_t and k_{t+1} are non-negative. y_t is the agent's income at time t, which follows the law of motion:

$$y_t = z_t f(k_t), \quad z_t \overset{iid}{\sim} \phi, \tag{1.2}$$

where z_t is a random variable that follows a positively supported distribution ϕ. $f(k_t)$ is the production function.

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Assumption 1.1.
The production function $f(k_t)$ is continuous and increasing in k_t.

The agent's optimization problem is given by:

$$v(y_t) = \max_{c_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right], \tag{1.3}$$

subject to the constraints Eq. (1.1) and Eq. (1.2), where $\beta \in (0,1)$ is the discount factor, and $u(c_t)$ is the utility flow in each period. $v(y_t)$ is called the **value function** and y_t is called the **state variable** of v. We further take two assumptions on $u(\cdot)$ and $v(\cdot)$.

Assumption 1.2.
The utility function $u(c_t)$ is continuous and increasing in c_t.

Note that by this assumption, the inequality in Eq. (1.1) is replaced by an equality since if $c_t + k_{t+1}$ is strictly less than y_t, the agent can always increase c_t to improve the utility.

Assumption 1.3.
The value function $v(y_t)$ is bounded.

Note that we may also write value function as follows.

$$\begin{aligned}
v(y_0) &= \max_{c_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \\
&= \max_{c_t} \mathbb{E}_0 \left[u(c_0) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \right] \\
&= \max_{c_t} u(c_0) + \beta \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{t+1}) \right] \\
&= \max_{c_0} u(c_0) + \beta \mathbb{E}_0 [v(y_1)] \\
&= \max_{c_0} u(c_0) + \beta \int v(z_1 f(y_0 - c_0)) \phi(dz_1).
\end{aligned} \tag{1.4}$$

The form is called the **Bellman equation**. It is a functional equation regarding v. Note that the true value function would solve this functional equation. The Bellman equation approach has a significant advantage compared to the traditional method of Lagrange multiplier; the Bellman equation approach transforms an infinite horizon problem into a two-period problem, and also deals with the uncertainty. However, there is a clear drawback: How to find v?

2. Value Function Iteration

A popular method is value function iteration. To see why does the method work, we begin by introducing some fundamental concepts in analysis.

Definition.
*A **metric space** is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a function that satisfies the following properties:*

(a) $d(x, y) \geq 0$ *for all* $x, y \in X$; $d(x, y) = 0$ *if and only if* $x = y$.

(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

*d is called a **metric (distance)** on X.*

Definition.
A sequence $\{x_n\}$ in a metric space (X, d) is said to be converge to $x \in X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

Definition.
*A sequence $\{x_n\}$ in a metric space (X, d) is said to be **Cauchy** if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.*

Definition.
*A metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges to a point in X.*

Remark.
$\mathbb{R}^n$ is a complete metric space under the Euclidean metric $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Definition.
A normed space X is a vector space with scalar field $\mathbb{R}$ equipped with a norm $\|\cdot\|$, satisfying that

(a) $\|x\| \geq 0$ *for all* $x \in X$; $\|x\| = 0$ *if and only if* $x = 0$.

(b) $\|ax\| = |a| \, \|x\|$ *for all* $a \in \mathbb{R}$ *and* $x \in X$.

(c) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

Remark.
The scalar field $\mathbb{R}$ can be replaced by other fields, but for our purpose, we only consider $\mathbb{R}$.

Remark.
The norm induces a metric $d(x,y) = \|x-y\|$. In fact, the Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ induces the Euclidean metric. For this reason, a normed space is automatically a metric space and the metric is defined by its norm.

Definition.
$B(X)$ is the set of all real-valued bounded continuous functions defined on X.

Proposition 2.4.
$B(X)$ is a complete metric space under the supremum norm $\|f\| = \sup\limits_{x \in X} |f(x)|$.

Proof.

Let $\{f_n\}$ be a Cauchy sequence in $B(X)$. For each $x \in X$, define $f(x) = \lim_{n\to\infty} f_n(x)$. The limit exists since $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. We claim that $f \in B(X)$. First, f is bounded since for each $x \in X$, there exists N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$. Letting $m = N$ and $n \to \infty$ yields that $|f(x) - f_N(x)| \leq \epsilon$. Hence, $|f(x)| \leq |f_N(x)| + \epsilon$ for all $x \in X$. Second, f is continuous since for each $x \in X$ and $\epsilon > 0$, we may pick $\delta > 0$ such that $|f_N(x) - f_N(y)| < \epsilon$ for all $y \in X$ with $d(x,y) < \delta$. Hence, $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < 3\epsilon$ for all $y \in X$ with $d(x,y) < \delta$. Since ϵ is arbitrary, f is indeed continuous and hence $f \in B(X)$. This completes the proof. ∎

Definition.
An operator $T : X \to X$ is called a ***contraction*** *if there exists $\alpha \in (0,1)$ such that $d(T(x), T(y)) \leq \alpha d(x,y)$ for all $x, y \in X$.*

Theorem 2.5 (Contraction Mapping Theorem).
Let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping with contraction factor $\alpha \in (0,1)$. Then T has an unique fixed point $x^* \in X$. That is, $Tx^* = x^*$. Furthermore, for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to x^*.

Proof.

For each $x_0 \in X$, we define $x_n = T^n(x_0)$. Then

$$d(x_{n+1}, x_n) = d(T^{n+1}(x_0), T^n(x_0)) \leq \alpha^n d(x_1, x_0) \to 0 \quad \text{as } n \to \infty. \tag{2.1}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some $x^* \in X$. Next, suppose both x^* and y^* are fixed points of T. Then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq \alpha d(x^*, y^*) < d(x^*, y^*), \tag{2.2}$$

posing a contradiction. Therefore, x^* is unique. ∎

Theorem 2.6 (Blackwell's Theorem).
Suppose $T : B(X) \to B(X)$ satisfies the following properties:

(a) T is monotone, i.e., $f \leq g$ implies $Tf \leq Tg$.

(b) There exists $\alpha \in (0,1)$ such that for any $c \in \mathbb{R}_+$, $T(f+c) \leq Tf + \alpha c$.

Then T is a contraction.

Proof.
Suppose $f, g \in B(X)$ and $c \in \mathbb{R}_+$ satisfy the conditions (a) and (b). Then notice that

$$g \leq f + \|f - g\|. \tag{2.3}$$

Thus we have

$$Tg \leq T(f + \|f-g\|) \leq Tf + \alpha \|f - g\|. \tag{2.4}$$

Rearranging the terms and taking the norm yiels the desired result. ∎

We now turn back to the Bellman equation.

Definition.
*The **Bellman operator** $T : v \mapsto Tv$ is defined by*

$$Tv(y) = \max_c u(y) + \beta \int v(zf(y-c))\phi(dz). \tag{2.5}$$

Remark.
The solution to the Bellman equation is the fixed point of the Bellman operator T.

Corollary 2.8.
The Bellman operator T is a contraction.

Proof.
Left as an exercise. ∎

Since $B(X)$ is a complete metric space and T is a contraction operator on it, by the contraction mapping theorem, T has an unique fixed point. This fixed point is the solution to the Bellman equation. Also, the proof of the contraction mapping theorem reveals a numerical algorithm to find the fixed point:

(a) Start with a guess $v_0 \in B(X)$.

(b) Apply the Bellman operator T to v_0 to get $v_1 = Tv_0$.

(c) Compare v_1 with v_0. If they are close enough, stop; otherwise, set $v_0 = v_1$ and repeat step 2.

The algorithm is called the **value function iteration**.

The value function iteration is one of the most popular methods to solve dynamic programming problems. One may observe that in the finite-horizon case, the value function iteration is equivalent to the backward induction.

3. Time Iteration

In this section, we introduce the time iteration method. We begin by adding a few assumptions to the optimal growth model.

Assumption 3.1.
$u(\cdot), f(\cdot) \in C^\infty$ are both strictly concave.

Assumption 3.2.
$u(0) = f(0) = 0$.

Assumption 3.3.
$u(\cdot), f(\cdot)$ satisfies the Inada conditions:

$$\begin{aligned} \lim_{c\to 0} u'(c) = \infty, \quad \lim_{c\to\infty} u'(c) = 0, \\ \lim_{k\to 0} f'(k) = \infty, \quad \lim_{k\to\infty} f'(k) = 0. \end{aligned} \tag{3.1}$$

Definition.
*A function $c^* : \mathbb{R}_+ \to \mathbb{R}_+$ is called the **optimal policy** if*

$$c^*(y) = \arg\max_c u(c) + \beta \int v^*(zf(y-c))\phi(dz), \tag{3.2}$$

where v^* is the value function.

Following this definition, several properties are derived.

Proposition 3.1.
c^* satisfies the following:

(a) c^* *is unique.*

(b) c^* *is continuous and strictly increasing.*

(c) $c^*(y) \in (0, y)$ *for any* $y > 0$.

(d) $(v^*)'(y) = (u' \circ c^*)(y)$.

Proof.

(a), (b) and (c) are omitted. For (d), one may write

$$v^*(y) = \max_k u(y-k) + \beta \int v^*(zf(k))\phi(dz). \tag{3.3}$$

Differentiating with respect to y and evaluating at the maximum yields

$$(v^*)'(y) = u'(c^*(y)). \tag{3.4}$$

■

Remark.
The last property is called the ***envelope condition****.*

Now, by the first order condition, we have

$$\begin{aligned} u'(c^*(y)) &= \beta \int (v^*)'(zf(y - c^*(y)))zf'(y - c^*(y))\phi(dz) \\ &= \beta \int (u' \circ c^*)(zf(y - c^*(y)))zf'(y - c^*(y))\phi(dz). \end{aligned} \tag{3.5}$$

Our goal is to find c^* solving the above functional equation. We first define the set where c^* lies.

Definition.

$$\Sigma := \{\sigma \mid \sigma : y \mapsto \sigma(y) \in (0, y) \textit{ is continuous, strictly increasing.}\} \tag{3.6}$$

And an operator on it.

Definition.
$K : \Sigma \to \Sigma$ *with* $K\sigma$ *defined by the solution* c *of the following functional equation:*

$$u'(c) = \beta \int (u' \circ \sigma)(zf(y - c))zf'(y - c)\phi(dz). \tag{3.7}$$

A careful reader may question whether K is well-defined. The following proposition addresses this concern.

Proposition 3.3.
K is well-defined.

Proof.

To show that K is well-defined, we need to show that the functional equation has an unique solution lying in Σ given any $\sigma \in \Sigma$.

First, observe that the left hand side of the equation is strictly decreasing in c with the value approaching ∞ as $c \to 0$ and approaching 0 as $c \to \infty$. The right hand side is strictly increasing in c with the value approaching 0 as $c \to 0$ and approaching ∞ as $c \to y$. Hence, the equation has a solution by the intermediate value theorem. The strict monotonicity further guarantees the uniqueness of the solution.

Next, we have to show that the solution lies in Σ. By previous discussion, the solution is interior. Also, it is strictly increasing since given any c, the right hand side is strictly decreasing in y, and hence the solution must be strictly increasing. The last piece is the continuity. This is guaranteed by the continuity of u', f, f' and σ. ∎

Having shown that K is well-defined, we are now in a position to examine the convergence of the operator. The operator K has, in fact, a tight connection with the Bellman operator T. We are going to see the connection by introducing the following mapping.

Definition.
Let $\mathcal{F} = \{v : \mathbb{R}_+ \to \mathbb{R}_+ \mid v(0) = 0, v'(y) > u'(y),$ v is strictly concave and differentiable.$\}$. Define $\varphi : \mathcal{F} \to \Sigma$ with $v \mapsto \varphi v = (u')^{-1} \circ v'$.

Proposition 3.4.
φ is a bijection.

Proof.

We first check that φ is well-defined. By assumption, u' is strictly decreasing and continuous; u' is thus a bijection and hence so is $(u')^{-1}$. Note that u' maps $(0, \infty)$ to $(0, \infty)$, and so does v'. Also, for every $v \in \mathcal{F}$, v' is strictly decreasing and continuous, which implies that φv is strictly increasing and continuous with range $(0, \infty)$. Furthermore, since $v' > u'$, $\varphi v(y) = ((u')^{-1} \circ v')(y) < ((u')^{-1} \circ u')(y) = y$. It follows that $\sigma := \phi v \in \Sigma$.

Next, we show that φ is a bijection. Fix $\sigma \in \Sigma$, let

$$v(y) = \int_0^y u'(\sigma(x))dx \in \mathcal{F}. \tag{3.8}$$

Then $\varphi v = (u')^{-1}(u'(\sigma(y))) = \sigma(y)$. Thus φ is surjective. Besides, if $\varphi v = \varphi w$, then $(u')^{-1} \circ v' = (u')^{-1} \circ w'$ and thus $v' = w'$. Since $v(0) = w(0) = 0$, $v = w$. Hence φ is injective. This completes the proof. ∎

Theorem 3.5.
The diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "T"] \arrow[d, "\varphi"] & \mathcal{F} \arrow[d, "\varphi"] \\ \Sigma \arrow[r, "K"] & \Sigma \end{tikzcd}$$

commutes. That is, for any $v \in \mathcal{F}$, $\varphi T v = K \varphi v$.

Proof.
For any $v \in \mathcal{F}$, by the envelope theorem, $(Tv)'(y) = u'(\sigma(y))$, where σ solves

$$u'(\sigma(y)) = \beta \int (u' \circ \sigma)(zf(y - \sigma(y)))zf'(y - \sigma(y))\phi(dz). \tag{3.9}$$

This implies that

$$\varphi T v = ((u')^{-1} \circ u')(\sigma(y)) = \sigma. \tag{3.10}$$

On the other hand, $K\varphi v(y)$ is the σ that solves

$$\begin{aligned} u'(\sigma(y)) &= \beta \int (u' \circ (\varphi v))(zf(y - \sigma(y)))zf'(y - \sigma(y))\phi(dz) \\ &= \beta \int (u' \circ ((u')^{-1} \circ v'))(zf(y - \sigma(y)))zf'(y - \sigma(y))\phi(dz) \\ &= \beta \int v'(zf(y - \sigma(y)))zf'(y - \sigma(y))\phi(dz). \end{aligned} \tag{3.11}$$

The two σ coincide, and hence the diagram commutes. ■

Corollary 3.6.
The sequence of policies $\{\sigma, K\sigma, K^2\sigma, \dots\}$ converges to the optimal policy c^*.

Proof.
With the above theorem, we may write $K^n = \varphi T^n \varphi^{-1}$. Since T is a contraction, $T^n \varphi^{-1} \sigma$ converges to the fixed point of T, which is v^*. Thus $K^n \sigma$ converges to c^*. ■

The above result not only shows the convergence of the operator K but also tells us that the convergent rate is the same as the Bellman operator T.

However, in practice, the time iteration tends to be more efficient than the value function iteration. One of the reason is that the curvature of the policy function tends to be smaller than the value function. While using linear interpolation to approximate the value off the grid points, the error is smaller for the policy function.

Another important reason is that the envelope condition method is often combined with the endogenous grid method. The computationally expensive part of the envelope condition method is to solve the functional equation. This is because of the appearance of c on both sides. The endogenous grid method solves this issue by putting grids on k instead of y. By doing so, in every step, one only needs to evaluate the integral on the right hand side and then apply the inverse of u' to get the updated policy function, which is much faster.

4. Policy Function Iteration

The last method we are going to introduce is the policy function iteration, also known as the Howard's policy improvement algorithm.

Definition.
*Given v, a policy σ is called v-**greedy** if*

$$\sigma(y) = \arg\max_{c \in (0,y)} u(c) + \beta \int v(zf(y-c))\phi(dz). \tag{4.1}$$

Our goal is to find the v^*-greedy policy. The algorithm is thus as follows:

1. Given an initial policy σ, solve the functional equation

$$v_\sigma(y) = u(\sigma(y)) + \beta \int v_\sigma(zf(y-\sigma(y)))\phi(dz) \tag{4.2}$$

   to obtain v_σ.

2. Update the policy to σ' by

$$\sigma'(y) = \arg\max_{c} u(c) + \beta \int v_\sigma(zf(y-c))\phi(dz). \tag{4.3}$$

3. Use σ' as the new initial policy and repeat step 1 and 2. Continue the process until the convergence of σ and σ' is attained.

Remark.
Let $T_\sigma : v \mapsto T_\sigma v$ with

$$T_\sigma v(y) = u(\sigma(y)) + \beta \int v(zf(y-\sigma(y)))\phi(dz). \tag{4.4}$$

Then T_σ is again clearly a contraction operator. One may solve Eq. (4.2) by applying T_σ iteratively.

The convergence of the policy function iteration is guaranteed by the following theorem.

Theorem 4.2.
Let σ be a policy, σ' be the policy updated by a single step of the policy function iteration, and T be the Bellman operator. Then $v_\sigma \leq Tv_\sigma \leq v_{\sigma'}$.

Proof.
By definition, $Tv_\sigma = T_{\sigma'}v_\sigma$. Also, $v_\sigma = T_\sigma v_\sigma \leq Tv_\sigma$. Thus we have $v_\sigma \leq Tv_\sigma = T_{\sigma'}v_\sigma$. Next, we claim that for $n \geq 1$, $v_\sigma \leq Tv_\sigma \leq T^n_{\sigma'}v_\sigma$. The case $n = 1$ has been proven. Now suppose the claim holds for n. Then by applying $T_{\sigma'}$ to both sides of the inequality, we have $Tv_\sigma \leq T^{n+1}_{\sigma'}v_\sigma$. By the monotonicity of T, we have $v_\sigma \leq Tv_\sigma \leq T^{n+1}_{\sigma'}v_\sigma$. By induction, the claim holds for all $n \in \mathbb{N}$. The theorem follows by taking $n \to \infty$. ∎

Corollary 4.3.
The policy function iteration converges to the optimal policy.

Proof.
Let $\{\sigma_k\}$ be the sequence of policies generated by the policy function iteration. One may see the convergence by noticing that

$$T^k v_{\sigma_0} \leq v_{\sigma_k} \leq v^*. \tag{4.5}$$

Thus

$$\|v_{\sigma_k} - v^*\| \leq \left\| T^k v_{\sigma_0} - v^* \right\|. \tag{4.6}$$

By our previous discussion, the right hand side converges to 0 as $k \to \infty$. v_{σ_k} thus converges to v^*. The optimal policy is then the limit of σ_k. ∎