Solutions to Stochastic Differential Equations by Øksendal

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Contents

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2. Some Mathematical Preliminaries

Exercise 2.1

Suppose that $X : \Omega \to \mathbb{R}$ is a function which takes only countably many values $a_1, a_2, \ldots \in \mathbb{R}$.

(a) Show that X is a random variable if and only if

$$X^{-1}(a_i) \in \mathcal{F} \text{ for all } i \in \mathbb{N}.$$

(b) Suppose that X is a random variable. Show that

$$E\left[|X|\right] = \sum_{i=1}^{\infty} |a_i|\, P(X = a_i).$$

(c) If X is a random variable and $E\left[|X|\right] < \infty$, show that

$$E\left[X\right] = \sum_{i=1}^{\infty} a_i P(X = a_i).$$

(d) If X is a random variable and $f : \mathbb{R} \to \mathbb{R}$ is measurable and bounded, show that

$$E\left[f(X)\right] = \sum_{i=1}^{\infty} f(a_i) P(X = a_i).$$

Solution.

For (a), suppose first that X is a random variable. Since $\{a_i\}$ are Borel sets, $X^{-1}(a_i) \in \mathcal{F}$ for all $i \in \mathbb{N}$. Conversely, assume that $X^{-1}(a_i) \in \mathcal{F}$ for all a_i. Since the range of X is $\{a_i\}_{i\in\mathbb{N}}$, for any Borel set $B \subset \mathbb{R}$, $X^{-1}(B) = \bigcup_{a_i \in B} X^{-1}(a_i) \in \mathcal{F}$, by the definition of σ-algebra. Thus, X is a random variable.

For (b), since X takes only countably many values, so does $|X|$ with $\{|a_i|\}_{i\in\mathbb{N}}$. By the definition of expectation, we have

$$E\left[|X|\right] = \sum_{i=1}^{\infty} |a_i|\, P(X = a_i)$$

in the extended sense.

For (c), since $E\left[|X|\right] < \infty$ and X is a random variable, the series converges absolutely and is well-defined. Hence

$$E\left[X\right] = \sum_{i=1}^{\infty} a_i P(X = a_i).$$

For (d), since f is measurable, $f^{-1}(B)$ is Borel and $X^{-1}f^{-1}(B)$ is measurable. $f(X)$ takes only countably many values, $f(a_1), f(a_2), \ldots$. The definition of expectation gives us

$$E[f(X)] = \sum_{i=1}^{\infty} f(a_i) P(f(X) = f(a_i)) = \sum_{i=1}^{\infty} f(a_i) P(X = a_i).$$

∎

Exercise 2.2

$X : \Omega \to \mathbb{R}$ is a random variable. The distribution function F of X is defined as

$$F(x) = P(X \leq x).$$

(a) Prove that F has the following properties:

(i) $0 \leq F \leq 1$, $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

(ii) F is non-decreasing.

(iii) F is right-continuous.

(b) $g : \mathbb{R} \to \mathbb{R}$ is measurable such that $E[|g(X)|] < \infty$. Show that

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) dF(x).$$

(c) Let $p(x) \geq 0$ be measurable on $\mathbb{R}$ be the density of X, i.e.,

$$F(x) = \int_{-\infty}^{x} p(t) dt.$$

Find density of B_t^2.

Solution.

For (a), since P is a probability measure, $0 \leq P(S) \leq 1$ for any $S \in \mathcal{F}$. In particular, $0 \leq P(X \leq x) \leq 1$ for all $x \in \mathbb{R}$. Also, we can take $x_n \searrow -\infty$ and $|X \leq x_n| \searrow \emptyset$ as $n \to \infty$. Hence

$$\lim_{x\to-\infty} F(x) = \lim_{n\to\infty} P(X \leq x_n) = P(\emptyset) = 0.$$

Similarly, we can take $x_n \nearrow \infty$ and $|X \leq x_n| \nearrow \Omega$ as $n \to \infty$. Hence

$$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} P(X \leq x_n) = P(\Omega) = 1.$$

(i) is proved. For (ii), F is non-decreasing because if $x_1 < x_2$, then

$$F(x_1) = P(X \leq x_1) \leq P(X \leq x_2) = F(x_2).$$

For (iii), let $h > 0$.

$$F(x+h) - F(x) = P(X \leq x+h) - P(X \leq x) = P(x < X \leq x+h).$$

For any $y > x$, there exists $h > 0$ such that $y > x + h$. Thus $(x, x+h] \searrow \emptyset$ as $h \to 0$. Hence

$$F(x+h) - F(x) = P(x < X \leq x+h) \to P(\emptyset) = 0$$

as $h \to 0$. Therefore, F is right-continuous.

For (b), by definition of expectation, the left-hand side is

$$E\left[g(X)\right] = \int_{\mathbb{R}} g(x) d\mu_X(x),$$

where $\mu_X(B) = P(X^{-1}(B))$ for any Borel set $B \subset \mathbb{R}$.

For (c),

$$F(x) = P(B_t^2 \leq x) = P(B_t \leq \sqrt{x}) = \int_{-\infty}^{\sqrt{x}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{u^2}{2t}\right) du.$$

Hence,

$$p(u) = \frac{d}{dx} F(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x}{2t}\right) \frac{1}{2\sqrt{x}}.$$

■

Exercise 2.3

Let $\{\mathcal{F}_i\}_{i \in \mathcal{I}}$ be a collection of σ-algebras on Ω. Prove that

$$\mathcal{F} = \bigcap_{i \in \mathcal{I}} \mathcal{F}_i$$

is again a σ-algebra.

Solution.

First, since $\mathcal{F}_i$ are σ-algebras, they contain $\emptyset$ and hence $\emptyset \in \mathcal{F}$. For any $A \in \mathcal{F}$, $A \in \mathcal{F}_i$ for all $i \in \mathcal{I}$ and hence $A^c \in \mathcal{F}_i$ for all $i \in \mathcal{I}$. Thus $A^c \in \mathcal{F}$. Finally, for any countable collection $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$, we have $A_n \in \mathcal{F}_i$ for all $i \in \mathcal{I}$ and all $n \in \mathbb{N}$. Then $\bigcup_n A_n \in \mathcal{F}_i$ for all $i \in \mathcal{I}$. Hence $\bigcup_n A_n \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a σ-algebra. ■

Exercise 2.4

(a) Let $X : \Omega \to \mathbb{R}$ be a random variable such that $E\left[|X|^p\right] < \infty$ for some $p \in (0, \infty)$. Prove the Chebyshev's inequality:

$$P(|X| \geq \lambda) \leq \frac{1}{\lambda^p} E\left[|X|^p\right]$$

for any $\lambda > 0$.

(b) Suppose there exists $k > 0$ such that $M = E\left[\exp(k|X|)\right] < \infty$. Prove that $P(|X| \geq \lambda) \leq Me^{-k\lambda}$ for any $\lambda > 0$.

Solution.

For (a), directly estimate that

$$P(|X| \geq \lambda) = \int_\Omega \chi_{\{|X|^p \geq \lambda^p\}} dP \leq \int_\Omega \frac{|X|^p}{\lambda^p} dP = \frac{1}{\lambda^p} E\left[|X|^p\right].$$

(b) is similar:

$$P(|X| \geq \lambda) = \int_\Omega \chi_{\{\exp(k|X|) \geq \exp(k\lambda)\}} dP \leq \int_\Omega \exp(k\,|X|)\exp(-k\lambda) dP = M\exp(-k\lambda).$$

■

Exercise 2.5

Let $X, Y : \Omega \to \mathbb{R}$ be two independent random variables and assume for simplicity that X, Y are bounded. Prove that

$$E[XY] = E[X]\,E[Y].$$

Solution.

For any $\epsilon > 0$, by definition of the expectation, we can find simple functions s and t on Ω such that

$$\int |s - X|\, dP < \epsilon, \quad \int |t - Y|\, dP < \epsilon, \quad \Rightarrow \quad \left|E[X] - \int s dP\right| < \epsilon, \quad \left|E[Y] - \int t dP\right| < \epsilon,$$

where s and t can be written as

$$s = \sum_{i=1}^n s_i \chi_{X^{-1}[s_i, s_{i+1})} \quad \text{and} \quad t = \sum_{j=1}^m t_j \chi_{Y^{-1}[t_j, t_{j+1})},$$

with s_i and t_j being arranged in ascending order. Thus,

$$\begin{aligned}
\int st dP &= \sum_{i=1}^n \sum_{j=1}^m s_i t_j P(\{X \in [s_i, s_{i+1})\} \cap \{Y \in [t_j, t_{j+1})\}) \\
&= \sum_{i=1}^n \sum_{j=1}^m s_i t_j P(X \in [s_i, s_{i+1})) P(Y \in [t_j, t_{j+1})) \\
&= \left(\sum_{i=1}^n s_i P(X \in [s_i, s_{i+1}))\right)\left(\sum_{j=1}^m t_j P(Y \in [t_j, t_{j+1}))\right) = \left(\int s dP\right)\left(\int t dP\right).
\end{aligned}$$

Also,

$$\left|E[XY] - \int st dP\right| \leq \left|\int |X - s|\,|t|\, dP\right| + \left|\int |Y - t|\,|X|\, dP\right|.$$

X and Y are bounded, say by M and N respectively. Then t is also bounded by N from our construction. Thus

$$\left|E[XY] - \int st dP\right| \leq M\epsilon + N\epsilon.$$

Combine the results above, we arrive at

$$
\begin{aligned}
|E[XY] - E[X]E[Y]| &\leq \left|E[XY] - \int stdP\right| + \left|E[X]E[Y] - \int sdP \int tdP\right| \\
&\leq (M+N)\epsilon + \left|E[X] - \int sdP\right|\left|\int tdP\right| + \left|E[Y] - \int tdP\right||E[X]| \\
&\leq (M+N)\epsilon + \epsilon N + \epsilon M.
\end{aligned}
$$

Since ϵ is arbitrary, we conclude that $E[XY] = E[X]E[Y]$. ■

Exercise 2.6

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_1, \ldots \in \mathcal{F}$ be sets such that

$$\sum_{i=1}^{\infty} P(A_i) < \infty.$$

Prove the Borel-Cantelli lemma:

$$P\left(\bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i\right) = 0.$$

Solution.

Set $B_m = \bigcup_{i=m}^{\infty} A_i$ be measurable. Then

$$P(B_m) \leq \sum_{i=m}^{\infty} P(A_i) \to 0$$

as $m \to \infty$ by the assumption. Thus

$$P\left(\bigcap_{m=1}^{\infty} B_m\right) \leq \lim_{n\to\infty} P\left(\bigcap_{m=1}^{n} B_m\right) \leq \lim_{n\to\infty} P(B_n) = 0.$$

■

Exercise 2.7

(a) *Suppose $G_1, \ldots, G_n$ are disjoint sets in $\mathcal{F}$ such that $\bigcup_{i=1}^{n} G_i = \Omega$. Prove that the family*

$$\mathcal{G} = \{G \mid G \text{ is a union of some } G_i\} \cup \{\emptyset\}$$

is a σ-algebra.

(b) *Prove that every finite σ-algebra is of type $\mathcal{G}$ as in (a).*

(c) *Let $\mathcal{F}$ be a finite σ-algebra on Ω and $X : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Prove that X is simple.*

Solution.

For (a), first, $\varnothing \in \mathcal{G}$ by definition. Let $G \in \mathcal{G}$. Then $G = \cup_{i\in\mathcal{I}} G_i$ for some $\mathcal{I} \subset \{1, \dots, n\}$, with the convention that $\cup_{i\in\varnothing} G_i = \varnothing$. Then $G^c = \bigcup_{i\notin\mathcal{I}} G_i \in \mathcal{G}$. Lastly, for countably many $G_i \in \mathcal{G}$, since $\mathcal{G}$ is finite, there are in fact finitely many distinct G_i and the union must lie in $\mathcal{G}$ by the definition. Hence $\mathcal{G}$ is a σ-algebra.

For (b), let $\mathcal{F}$ be a finite σ-algebra. Consider the collection

$$\mathcal{S} = \{S \in \mathcal{F} \mid S \cap F = \varnothing \text{ or } S \text{ for all } F \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is finite, $\mathcal{S}$ is also finite. We first check that every distinct sets in $\mathcal{S}$ are disjoint. Suppose not. There are $S_1, S_2 \in \mathcal{S}$ such that $S_1 \cap S_2$ is non-empty. Then $S_1 \cap S_2 = S_1 = S_2$, contradicting the assumption that S_1 and S_2 are distinct. Thus every distinct sets in $\mathcal{S}$ are disjoint. Next, we check that $\bigcup_{S\in\mathcal{S}} S = \Omega$. If not, let $A = \Omega \setminus \bigcup_{S\in\mathcal{S}} S$ be non-empty and $A \cap F$ is a non-empty proper subset of A for some $F \in \mathcal{F}$. But then $A \cap F$ or $A \cap F^c$ must satisfy the condition that there is some $F' \in \mathcal{F}$ such that $A \cap F \cap F'$ or $A \cap F^c \cap F'$ is non-empty, proper subset of $A \cap F$ or $A \cap F^c$ respectively. Note that $F' \neq F$ and the process continues. In the end, we can find a infinite sequence of distinct sets lying in $\mathcal{F}$, contradicting the finiteness of $\mathcal{F}$. Thus $\bigcup_{S\in\mathcal{S}} S = \Omega$. Finally, by (a),

$$\mathcal{G} = \{G \mid G \text{ is a union of some } S \in \mathcal{S}\} \cup \{\varnothing\}$$

is a σ-algebra. It remains to show that $\mathcal{G} = \mathcal{F}$. Clearly, $\mathcal{G} \subset \mathcal{F}$ since $\mathcal{S} \subset \mathcal{F}$. For any $F \in \mathcal{F}$, we can write $F = \bigcup_{i=1}^n S_i$ for some $S_i \in \mathcal{S}$. Thus $F \in \mathcal{G}$. We end up with $\mathcal{G} = \mathcal{F}$.

For (c), suppose that X can take infinitely many values $\{a_i\}_{i\in\mathcal{I}}$. Since X is $\mathcal{F}$-measurable, $X^{-1}(\{a_i\}) \in \mathcal{F}$ for all $i \in \mathcal{I}$. In particular, $X^{-1}(\{a_i\})$ and $X^{-1}(\{a_j\})$ are disjoint for all $i \neq j$. This implies that $\mathcal{F}$ contains infinitely many disjoint sets, contradicting the finiteness of $\mathcal{F}$. Thus X can only take finitely many values and is simple. ■

Exercise 2.8

Let B_t be Brownian motion on $\mathbb{R}$, $B_0 = 0$. Put $E = E^0$.

(a) Prove that

$$E\left[e^{iuB_t}\right] = e^{-\frac{u^2 t}{2}} \text{ for all } u \in \mathbb{R}.$$

(b) Use the power series expansion of the exponential function to show that

$$E\left[B_t^{2k}\right] = \frac{(2k)!}{2^k k!} t^k \text{ for all } k \in \mathbb{N}.$$

(c) Prove that

$$E\left[f(B_t)\right] = \int_{\mathbb{R}} f(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx$$

for all measurable functions f on $\mathbb{R}$ such that the integral is finite. Deduce (b) by setting $f(x) = x^{2k}$.

(d) *Now suppose that B_t is a n-dimensional Brownian motion. Prove that*

$$E^x\left[|B_t - B_s|^4\right] = n(n+2)\,|t-s|^2$$

for all $n \in \mathbb{N}$ and $0 \le s, t \le T$.

Solution.

For (a), directly compute the expectation:

$$E\left[e^{iuB_t}\right] = \int_{-\infty}^{\infty} e^{iux}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,dx = e^{-\frac{u^2 t}{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-iut)^2}{2t}}\,dx = e^{-\frac{u^2 t}{2}}.$$

For (b), note that the power series expansion of the exponential function gives

$$e^{iuB_t} = \sum_{k=0}^{\infty}\frac{(iuB_t)^k}{k!} \quad\Rightarrow\quad E\left[e^{iuB_t}\right] = \sum_{k=0}^{\infty}\frac{(iu)^k}{k!}E\left[B_t^k\right].$$

The right-hand side is

$$e^{-\frac{u^2 t}{2}} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{u^2 t}{2}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k u^{2k}}{2^k k!}t^k.$$

For those two expressions to be equal, as a function of u, $E\left[B_t^k\right] = 0$ for odd k and we may rewrite the first expression as

$$E\left[e^{iuB_t}\right] = \sum_{k=0}^{\infty}\frac{(-1)^k u^{2k}}{(2k)!}E\left[B_t^{2k}\right].$$

By comparing the coefficients,

$$E\left[B_t^{2k}\right] = \frac{(-1)^k t^k}{2^k k!}\cdot\frac{(2k)!}{(-1)^k} = \frac{(2k)!}{2^k k!}t^k.$$

For (c), it is clear that B_t has the density $p(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$. It follows that

$$E\left[f(B_t)\right] = \int f(x)\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,dx.$$

By setting $f : x \mapsto x^{2k}$, we have

$$\begin{aligned}
E\left[B_t^{2k}\right] &= \int x^{2k}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\,dx \\
&= x^{2k+1}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\Big|_{-\infty}^{\infty} - \int 2kx^{2k}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}} - x^{2k}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\frac{x^2}{t}\,dx \\
&= \frac{1}{t}E\left[B_t^{2(k+1)}\right] - 2kE\left[B_t^{2k}\right] \quad\Rightarrow\quad E\left[B_t^{2(k+1)}\right] = (2k+1)tE\left[B_t^{2k}\right].
\end{aligned}$$

And also when $k = 1$, $E\left[B_t^2\right] = t$. Suppose $E\left[B_t^{2k}\right] = \frac{(2k)!}{2^k k!} t^k$. Then

$$E\left[B_t^{2(k+1)}\right] = (2k+1)t\frac{(2k)!}{2^k k!}t^k = \frac{(2(k+1))!}{2^{k+1}(k+1)!}t^{k+1}.$$

The conclusion follows by induction.

For (d), if $n = 1$, $B_t - B_s \sim N(0, |t-s|)$ and $E^x\left[|B_t - B_s|^4\right] = 3\,|t-s|^2$. Now suppose that for n-dimensional B_t, $E^x\left[|B_t - B_s|^4\right] = n(n+2)\,|t-s|^2$. Then for $n+1$-dimensional B_t,

$$\begin{aligned}
E^x\left[|B_t - B_s|^4\right] &= \int_{\mathbb{R}^n}\int_{\mathbb{R}} (|y|^2 + z^2)^2 \frac{1}{\sqrt{(2\pi)^{n+1}\,|t-s|^{n+1}}} \exp\left(-\frac{|y|^2 + z^2}{2\,|t-s|}\right) dydz \\
&= \int_{\mathbb{R}^n} |y|^4 \, \frac{1}{\sqrt{(2\pi)^n\,|t-s|^n}} \exp\left(-\frac{|y|^2}{2\,|t-s|}\right) dy \\
&\quad + 2\,|t-s| \int_{\mathbb{R}^n} |y|^2 \, \frac{1}{\sqrt{(2\pi)^n\,|t-s|^n}} \exp\left(-\frac{|y|^2}{2\,|t-s|}\right) dy + 3\,|t-s|^2 \\
&= n(n+2)\,|t-s|^2 + 2n\,|t-s|^2 + 3\,|t-s|^2 = (n+1)(n+3)\,|t-s|^2\,.
\end{aligned}$$

Thus by induction, the conclusion holds for all $n \in \mathbb{N}$. It follows from the Kolmogorov's continuity theorem that we can always set B_t to be a continuous process. ■

Exercise 2.9

Let $(\Omega, \mathcal{F}, P) = ([0,\infty), \mathcal{B}, \mu)$ be a probability space where μ is a probability measure such that there is no mass at single points. Define

$$X_t(\omega) = \begin{cases} 1 & \text{if } t = \omega, \\ 0 & \text{if } t \neq \omega. \end{cases} \quad \text{and} \quad Y_t(\omega) = 0 \text{ for all } (t,\omega) \in [0,\infty) \times [0,\infty).$$

Prove that $\{X_t\}$ and $\{Y_t\}$ have the same distributions and X_t is a version of Y_t. And yet $t \mapsto Y_t(\omega)$ is continuous for all ω, while $t \mapsto X_t(\omega)$ is discontinuous for all ω.

Solution.

First, given any t, X_t and Y_t are both random variables. For $t_1, \ldots, t_k \in [0,\infty)$, consider the sets $F_1, \ldots, F_k \in \mathcal{B}$. Since Y_t is constant,

$$P(Y_{t_1} \in F_1, \ldots, Y_{t_k} \in F_k) = \mathbb{1}\left\{0 \in \cap_{i=1}^k F_i\right\}.$$

Also, since μ has no mass at single points,

$$P(X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k) = \mathbb{1}\left\{0 \in \cap_{i=1}^k F_i\right\}.$$

Thus, X_t and Y_t have the same distributions. Furthermore, for any $t \in [0,\infty)$, $\{X_t = Y_t\} = \Omega \setminus \{t\}$, which has zero measure and hence X_t is a version of Y_t. Now since Y_t is constant, $t \mapsto Y_t(\omega)$ is continuous for all ω. On the other hand, for any $\omega \in [0,\infty)$, $X(t,\omega)$ is discontinuous at $t = \omega$, proving that $t \mapsto X_t(\omega)$ is discontinuous for all ω. ■

Exercise 2.10
Prove that the Brownian motion B_t has stationary increments, i.e., given $h > 0$, the process $\{B_{t+h} - B_t\}$ has the same distribtuions for all t.

Solution.
For any $h > 0$, $B_{t+h} - B_t \sim N(0, h)$. The stationarity follows immediately. ■

Exercise 2.11
If $B_t = (B_t^{(1)}, \dots, B_t^{(n)})$ is an n-dimensional Brownian motion, then the component processes $B_t^{(i)}$, $1 \le i \le n$, are independent Brownian motions.

Solution.
Since the B_t is continuous almost surely, the component processes $B_t^{(i)}$ are continuous almost surely as well. Now we may regard the component processes as projections of B_t and hence they are normally distributed. $E\left[B_t^{(i)}\right] = 0$ since $E\left[B_t\right] = 0$. Also, $\mathrm{Cov}(B_t^{(i)}, B_t^{(j)}) = t\delta_{ij}$ as $\mathrm{Var}(B_t) = tI$. Since for $i \neq j$ the covariance is zero, the component processes are independent. ■

Exercise 2.12
Let B_t be a Brownian motion and fix $t_0 \ge 0$. Prove that the process $\tilde{B}_t = B_{t+t_0} - B_{t_0}$ is a Brownian motion.

Solution.
First, it is clear that $\tilde{B}_0 = B_{t_0} - B_{t_0} = 0$. Since B_t is almost surely continuous, $B_{t+t_0} - B_{t_0}$ is also almost surely continuous. Also, $B_{t+t_0} - B_{t_0} \sim N(0, t)$. For $s < t < u$,

$$\mathrm{Cov}(\tilde{B}_u - \tilde{B}_t, \tilde{B}_t - \tilde{B}_s) = \mathrm{Cov}(B_{u+t_0} - B_{t+t_0}, B_{t+t_0} - B_{s+t_0}) = 0.$$

Thus $\tilde{B}_t$ has independent increments. This shows that $\tilde{B}_t$ is a Brownian motion. ■

Exercise 2.13
Let B_t be 2-dimensional Brownian motion and put

$$D_\rho = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| < \rho\right\}$$

for $\rho > 0$. Compute $P^0(B_t \in D_\rho)$.

Solution.
Since $B_t \sim N(0, tI)$,

$$P^0(B_t \in D_\rho) = \int_{D_\rho} \frac{1}{2\pi t} \exp\left(-\frac{|x|^2}{2t}\right) dx = \int_0^{2\pi} \int_0^{\rho} \frac{1}{2\pi t} \exp\left(-\frac{r^2}{2t}\right) r\,dr\,d\theta = 1 - \exp\left(-\frac{\rho^2}{2t}\right).$$

■

Exercise 2.14

Let B_t be n-dimensional Brownian motion and $K \subset \mathbb{R}^n$ be a measure zero set under the Lebesgue measure. Prove that the expected total length of time that B_t spends in K is zero.

Solution.

Given $\omega \in \Omega$, the process $B_t(\omega)$ spends

$$\int_0^\infty \mathbb{1}\left\{B_t(\omega) \in K\right\} dt$$

amount of time in K. The expected total length of time that B_t spends in K is

$$\int_\Omega \int_0^\infty \mathbb{1}\left\{B_t(\omega) \in K\right\} dt dP(\omega) = \int_0^\infty \int_\Omega \mathbb{1}\left\{B_t(\omega) \in K\right\} dP(\omega) dt = 0$$

by the Fubini-Tonelli theorem, since K has measure zero under the Lebesgue measure and $B_t^{-1}(K)$ is measure zero under the probability measure P for all $t \geq 0$. ■

Exercise 2.15

Let B_t be an n-dimensional Brownian motion starting at 0 *and let $U \in \mathbb{R}^{n \times n}$ be a orthogonal matrix, i.e., $U^T U = I$. Prove that $\tilde{B}_t = U B_t$ is also a Brownian motion.*

Solution.

Since B_t is a Brownian motion and U is a linear transformation on a finite-dimensional space, $\tilde{B}_t$ must be contiuous almost surely and $\tilde{B}_0 = UB_0 = 0$. Also, since $B_t \sim N(0, tI)$, we have $\tilde{B}_t \sim N(0, tU^T U) = N(0, tI)$. For $s < t < u$,

$$\operatorname{Cov}(\tilde{B}_u - \tilde{B}_t, \tilde{B}_t - \tilde{B}_s) = \operatorname{Cov}(UB_u - UB_t, UB_t - UB_s) = U \operatorname{Cov}(B_u - B_t, B_t - B_s) U^T = 0.$$

Thus $\tilde{B}_t$ has independent increments. Therefore, $\tilde{B}_t$ is a Brownian motion. ■

Exercise 2.16

Let B_t be a Brownian motion on $\mathbb{R}$ and $c > 0$. Prove that the process $X_t = \frac{1}{c} B_{c^2 t}$ is a Brownian motion.

Solution.

First, $X_0 = \frac{1}{c} B_0 = 0$. Suppose $\omega \in \Omega$ is such that $B_t(\omega)$ is continuous. For $\epsilon > 0$, there exists $\delta > 0$ such that $|B_t(\omega) - B_s(\omega)| < c\epsilon$ as long as $|t - s| < \delta$. Then we may set $\delta' = \delta / c^2$ and see that

$$|X_t(\omega) - X_s(\omega)| = \frac{1}{c} |B_{c^2 t}(\omega) - B_{c^2 s}(\omega)| < \epsilon$$

whenever $|t - s| < \delta' \Leftrightarrow \left|c^2 t - c^2 s\right| < \delta$. Thus X_t is continuous almost surely. Also, since $B_{c^2 t} \sim N(0, c^2 tI)$, $X_t \sim N(0, tI)$. For $s < t < u$,

$$\operatorname{Cov}(X_u - X_t, X_t - X_s) = \operatorname{Cov}\left(\frac{1}{c} B_{c^2 u} - \frac{1}{c} B_{c^2 t}, \frac{1}{c} B_{c^2 t} - \frac{1}{c} B_{c^2 s}\right) = \frac{1}{c^2} \operatorname{Cov}(B_{c^2 u} - B_{c^2 t}, B_{c^2 t} - B_{c^2 s}) = 0.$$

We conclude that X_t is a Brownian motion. ■

Exercise 2.17

Let B_t be a Brownian motion on $\mathbb{R}$. Show that the quadratic variation process is $\langle B,\ B\rangle_t^2(\omega) = t$ almost surely by the following steps:

(a) Define $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and put $Y(t,\omega) = \sum_{t_k \le t} (\Delta B_k)^2$ for a partition $\{t_k\}_{k=1}^n$ of $[0,t]$. Show that

$$E\left[(Y_t - t)^2\right] = 2\sum_{t_k \le t} (\Delta t_k)^2$$

and deduce that $Y(t,\cdot) \to t$ in L^2 as $\Delta t_k \to 0$.

(b) Use (a) to show that almost every paths of B_t does not have a bounded total variation on $[0,t]$.

Solution.

For (a), note that ΔB_k are independent and $\Delta B_k \sim N(0,\Delta t_k)$.

$$\begin{aligned}
E\left[(Y_t - t)^2\right] &= E\left[\left(\sum_{t_k\le t}(\Delta B_k)^2 - \Delta t_k\right)^2\right] = E\left[\sum_{t_k\le t}((\Delta B_k)^2 - \Delta t_k)^2\right] \\
&= \sum_{t_k\le t} E\left[(\Delta B_k)^4\right] - 2E\left[(\Delta B_k)^2\right]\Delta t_k + (\Delta t_k)^2 \\
&= \sum_{t_k\le t} 3(\Delta t_k)^2 - 2(\Delta t_k)^2 + (\Delta t_k)^2 \\
&= 2\sum_{t_k\le t}(\Delta t_k)^2 \le 2\max_k \Delta t_k \sum_{t_k\le t}\Delta t_k = 2t\max_k \Delta t_k \to 0
\end{aligned}$$

as $\max_k \Delta t_k \to 0$, the right-hand side converges to 0 and hence $Y(t,\cdot) \to t$ in $L^2(P)$. Hence $\langle B,\ B\rangle_t^2(\omega) = t$ almost surely.

For (b), let $\mathcal{P}$ be a partition of $[0,t]$. With respect to $\mathcal{P}$, define the total variation of B_t as

$$Z_t^{\mathcal{P}}(\omega) = \sum_{t_k\le t}\left|B_{t_{k+1}}(\omega) - B_{t_k}(\omega)\right| \le N(\mathcal{P})\sum_{t_k\le t}\left|B_{t_{k+1}}(\omega) - B_{t_k}(\omega)\right|^2 \to \infty$$

as $\|\mathcal{P}\| \to 0$ by the Cauchy inequality and (a) that $Y(t,\cdot) \to t$ in $L^2(P)$. ■

Exercise 2.18

Let $\Omega = \{1,2,3,4,5\}$ and $\mathcal{U} = \{\{1,2,3\},\{3,4,5\}\}$ be a collection of subsets of Ω.

(a) Find $\sigma(\mathcal{U})$.

(b) Define $X : \Omega \to \mathbb{R}$ by

$$X(1) = X(2) = 0, \quad X(3) = 10, \quad X(4) = X(5) = 1.$$

Is X $\sigma(\mathcal{U})$-measurable?

(c) *Define $Y : \Omega \to \mathbb{R}$ by*

$$Y(1) = 0, \quad Y(2) = Y(3) = Y(4) = Y(5) = 1.$$

Find $\sigma(Y)$.

Solution.

For (a),

$$\sigma(\mathcal{U}) = \{\varnothing, \{1,2\}, \{3\}, \{4,5\}, \{1,2,3\}, \{3,4,5\}, \{1,2,4,5\}, \Omega\}.$$

For (b), it is not hard to verify that X is $\sigma(\mathcal{U})$-measurable.

For (c),

$$\sigma(Y) = \{\varnothing, \{1\}, \{2,3,4,5\}, \Omega\}.$$

■

Exercise 2.19

Prove that every convergent sequence is a Cauchy sequence.

Solution.

Let $\{x_n\}$ be a convergent sequence with limit x. For any $\epsilon > 0$, we may find $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \epsilon/2$. Then for all $m, n \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence. ■

Exercise 2.20

Let B_t be a 1-dimensional Brownian motion, $\sigma \in \mathbb{R}$ and $0 \leq s < t$. Prove that

$$E\left[\exp(\sigma(B_s - B_t))\right] = \exp\left(\frac{\sigma^2(s-t)}{2}\right).$$

Solution.

Since $B_s - B_t \sim N(0, t-s)$,

$$\begin{aligned} E\left[\exp(\sigma(B_s - B_t))\right] &= \int_{-\infty}^{\infty} e^{\sigma x} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}}\, dx \\ &= e^{\frac{\sigma^2(t-s)}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-\sigma(t-s))^2}{2(t-s)}}\, dx = e^{\frac{\sigma^2(t-s)}{2}}. \end{aligned}$$

■

3. Itô Integrals

Exercise 3.1

Prove that

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds.$$

Solution.

Let $\mathcal{P}_k = \{0 = t_0 < t_1 < \ldots < t_k = t\}$ be a partition of the interval $[0, t]$. Denote B_{t_i} as B_i. We have that

$$\sum_i t_i \Delta B_i = t_{k-1} B_k - \sum_i \Delta t_i B_{i+1}.$$

Now take $\|\mathcal{P}_k\| \to 0$. Then

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds.$$

■

Exercise 3.2

Prove that

$$\int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds.$$

Solution.

Let $\mathcal{P}_k = \{0 = t_0 < \ldots < t_k = t\}$ be a partition. Denote B_{t_i} as B_i and $\Delta B_i = B_{t_{i+1}} - B_{t_i}$. Now

$$\begin{aligned} B_t^3 &= \sum_i B_{i+1}^3 - B_i^3 = \sum_i (B_i + \Delta B_i)^3 - B_i^3 \\ &= \sum_i B_i^3 + 3B_i^2 \Delta B_i + 3B_i (\Delta B_i)^2 + (\Delta B_i)^3 - B_i^3 \\ &= \sum_i 3B_i^2 \Delta B_i + 3B_i (\Delta B_i)^2 + (\Delta B_i)^3. \end{aligned}$$

Now take $\|\mathcal{P}_k\| \to 0$. Then

$$\sum_i 3B_i^2 (\Delta B_i) \to 3 \int_0^t B_s^2 dB_s$$

by the definition. For the second term, let $X_t = 3B_t$. Notice that by (d) of exercise 2.8,

$$\begin{aligned} E\left[\left(\sum_i X_i (\Delta B_i)^2 - \sum_i X_i \Delta t_i\right)^2\right] &= E\left[\sum_i X_i^2 \Big((\Delta B_i)^2 - \Delta t_i\Big)^2 + 2\sum_{i>j} X_i X_j \Big((\Delta B_i)^2 - \Delta t_i\Big)\Big((\Delta B_j)^2 - \Delta t_j\Big)\right] \\ &= \sum_i E\left[X_i^2 \Big((\Delta B_i)^2 - \Delta t_i\Big)^2\right] = \sum_i E\left[X_i^2\right] (3(\Delta t_i)^2 - 2(\Delta t_i)^2 + (\Delta t_i)^2) \\ &= \sum_i E\left[X_i^2\right] (\Delta t_i)^2 \to 0 \end{aligned}$$

as $\|\mathcal{P}_k\| \to 0$ since $E\left[X_i^2\right]$ is bounded. But $\sum_i X_i \Delta t_i \to \int_0^t X_s ds$ by definition, so

$$\sum_i 3B_i(\Delta B_i)^2 \to 3\int_0^t B_s ds.$$

Finally, we note that

$$\begin{aligned} E\left[\left(\sum_i (\Delta B_i)^3\right)^2\right] &= E\left[\sum_i (\Delta B_i)^6 + 2\sum_{i>j} (\Delta B_i)^3 (\Delta B_j)^3\right] \\ &= \sum_i E\left[(\Delta B_i)^6\right] = 15\sum_i (\Delta t_i)^3 \to 0 \end{aligned}$$

as $\|\mathcal{P}_k\| \to 0$. Hence we see that

$$B_t^3 = 3\int_0^t B_s^2 dB_s + 3\int_0^t B_s ds \quad \Rightarrow \quad \int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

■

Exercise 3.3

Let $X_t : \Omega \to \mathbb{R}^n$ is a stochastic process and $\mathcal{H}_t$ is the filtration of X_t.

(a) *Show that if X_t is a martingale with respect to some filtration $\mathcal{N}_t$, then it is also a martingale with respect to $\mathcal{H}_t$.*

(b) *Show that if X_t is a martingale with respect to $\mathcal{H}_t$, then*

$$E\left[X_t\right] = E\left[X_0\right]$$

for all $t \geq 0$.

(c) *Give an example of stochastic process X_t such that $E\left[X_t\right] = E\left[X_0\right]$ for all $t \geq 0$ but X_t is not a martingale with respect to $\mathcal{H}_t$.*

Solution.

For (a), for X_t being a martingale with respect to $\mathcal{N}_t$, we have $\mathcal{H}_t \subset \mathcal{N}_t$ for all $t \geq 0$. Also, X_t is integrable. Finally,

$$E\left[X_s \mid \mathcal{H}_t\right] = E\left[E\left[X_s \mid \mathcal{N}_t\right] \mid \mathcal{H}_t\right] = E\left[X_t \mid \mathcal{H}_t\right] = X_t$$

for all $s \geq t$ by the tower property. Hence X_t is a martingale with respect to $\mathcal{H}_t$ as well.

For (b), note that $E\left[X_t \mid \mathcal{H}_0\right] = X_0$. Hence $E\left[X_t\right] = E\left[E\left[X_t \mid \mathcal{H}_0\right]\right] = E\left[X_0\right]$.

For (c), consider the process defined by

$$X_0 = \begin{cases} -1 & \text{with prob.} = 0.5 \\ 1 & \text{with prob.} = 0.5 \end{cases}, \quad X_t = (t+1)\operatorname{sgn}(X_0) \text{ for } t > 0.$$

Then $E[X_t] = t \cdot 0.5 + (-t) \cdot 0.5 = 0$ for all $t \geq 0$, but $E[X_t \mid X_0] = E[(t+1)\operatorname{sgn}(X_0) \mid X_0] = (t+1)\operatorname{sgn}(X_0) \neq X_0$ for $t > 0$. Hence X_t is not a martingale with respect to $\mathcal{H}_t$. ■

Exercise 3.4

Check whether the following processes are martingale with respect to $\{\mathcal{F}_t\}$.

(a) $X_t = B_t + 4t$.

(b) $X_t = B_t^2$.

(c) $X_t = t^2 B_t - 2\int_0^t sB_s ds$.

(d) $X_t = B_1(t)B_2(t)$*, where* $(B_1(t), B_2(t))$ *is a 2-dimensional Brownian motion.*

Solution.

For (a),

$$E[X_s \mid \mathcal{F}_t] = E[B_s \mid \mathcal{F}_t] + 4s = B_t + 4s \neq X_t.$$

Hence X_t is not a martingale.

For (b),

$$\begin{aligned} E[X_s \mid \mathcal{F}_t] &= E\left[B_s^2 \mid \mathcal{F}_t\right] = E\left[(B_t + (B_s - B_t))^2 \mid \mathcal{F}_t\right] \\ &= E\left[B_t^2 + 2B_t(B_s - B_t) + (B_s - B_t)^2 \mid \mathcal{F}_t\right] \\ &= B_t^2 + (s-t) \neq X_t. \end{aligned}$$

Hence X_t is not a martingale.

For (c),

$$\begin{aligned} E[X_s \mid \mathcal{F}_t] = E\left[s^2 B_s - 2\int_0^s uB_u du \mid \mathcal{F}_t\right] &= s^2 B_t - 2\int_0^t uB_u du - 2\int_t^s uE[B_u \mid \mathcal{F}_t]\, du \\ = s^2 B_t - 2\int_0^t uB_u du - 2B_t \int_t^s u du &= t^2 B_t - 2\int_0^t uB_u du = X_t. \end{aligned}$$

Hence X_t is a martingale.

For (d),

$$E[B_1(s)B_2(s) \mid \mathcal{F}_t] = E[B_1(s) \mid \mathcal{F}_t]\, E[B_2(s) \mid \mathcal{F}_t] = B_1(t)B_2(t).$$

Hence X_t is a martingale. ■

Exercise 3.5

Prove that $M_t = B_t^2 - t$ *is an* $\mathcal{F}_t$*-martingale.*

Solution.

Compute that

$$E[M_s \mid \mathcal{F}_t] = E\left[B_s^2 \mid \mathcal{F}_t\right] - s = E\left[(B_t + (B_s - B_t))^2 \mid \mathcal{F}_t\right] - s = B_t^2 + (s-t) - s = B_t^2 - t = M_t.$$

Hence M_t is a martingale. ■

Exercise 3.6

Prove that $N_t = B_t^3 - 3tB_t$ is a martingale.

Solution.

Compute that

$$\begin{aligned}
E\left[N_s \mid \mathcal{F}_t\right] &= B_t^3 - 3tB_t + E\left[B_s^3 - B_t^3 \mid \mathcal{F}_t\right] - 3E\left[sB_s - tB_t \mid \mathcal{F}_t\right] \\
&= N_t + E\left[(B_s - B_t)^3 + 3B_s^2 B_t - 3B_s B_t^2 \mid \mathcal{F}_t\right] - 3B_t(s-t) \\
&= N_t + 3E\left[B_s^2 B_t - B_s B_t^2 \mid \mathcal{F}_t\right] - 3B_t(s-t) \\
&= N_t + 3B_t E\left[(B_s - B_t)^2 - 2B_t(B_s - B_t) + B_t^2 \mid \mathcal{F}_t\right] - 3B_t^3 - 3B_t(s-t) \\
&= N_t + 3B_t(s-t) + 3B_t^3 - 3B_t^3 - 3B_t(s-t) = N_t.
\end{aligned}$$

Hence N_t is a martingale. ■

Exercise 3.7

A famous result from Itô gives the following formula

$$n! \int_{0 \le u_1 \le u_2 \le \cdots \le u_n \le t} dB_{u_1} dB_{u_2} \cdots dB_{u_n} = t^{n/2} h_n\left(\frac{B_t}{\sqrt{t}}\right),$$

where h_n is the Hermite polynomial of degree n, defined by

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left[e^{-x^2/2}\right].$$

(a) *Verify that for each $n \in \mathbb{N}$, the integrand satisfies the requirement of definition 3.1.4; in other words, the Itô integral is well-defined.*

(b) *Verify the formula for $n = 1, 2, 3$.*

(c) *Use (b) to give another proof of exercise 3.6.*

Solution.

For (a), we show this by induction. If $n = 1$, the integral becomes $\int_0^t dB_{u_1}$ and the integrand is constant. Since $(t, \omega) \to 1$ is clearly $\mathcal{B} \times \mathcal{F}$-measurable, $\omega \to 1$ is also $\mathcal{F}_t$-measurable for each $t \ge 0$ and thus $\mathcal{F}_t$-adapted. Finally, $E\left[\int_0^t 1^2 dt\right] = t < \infty$, so the integrand is in L^2 and the Itô integral is well-defined. Suppose the claim holds for n, for $n+1$, the integrand is

$$f(u_1, \omega) = \int_{u_1}^t \int_{u_2}^t \cdots \int_{u_n}^t dB_{u_{n+1}} \cdots dB_{u_3} dB_{u_2}.$$

■