A Brief Introduction to the Fundamental Methods in Dynamic Programming

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Foreword

This document aims at providing mathematical details for fundamental methods used to solve dynamic programming problems. Three methods are covered: value function iteration (VFI), envelope condition method (ECM), and policy function iteration (PFI). For implementation in Julia, please refer to Sargent's fantastic website. This document serves as a supplement to the website.

Contents

1. Optimal Growth Model

In this capter, we introduce an optimal growth model. The model is going to be our working example for VFI.

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Consider an agent who seeks to maximize his lifetime expected utility. The agent's problem is to choose his future path of consumption c_t and capital stock k_{t+1}, subject to the constraint:

$$c_t + k_{t+1} \leq y_t, \tag{1.1}$$

where both c_t and k_{t+1} are non-negative. y_t is the agent's income at time t, which follows the law of motion:

$$y_t = z_t f(k_t), \quad z_t \overset{iid}{\sim} \phi, \tag{1.2}$$

where z_t is a random variable that follows a positively supported distribution ϕ. $f(k_t)$ is the production function.

Assumption 1.1.
The production function $f(k_t)$ is continuous and increasing in k_t.

The agent's optimization problem is given by:

$$v(y_t) = \max_{c_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right], \tag{1.3}$$

subject to the constraints Eq. (1.1) and Eq. (1.2), where $\beta \in (0, 1)$ is the discount factor, and $u(c_t)$ is the utility flow in each period. $v(y_t)$ is called the **value function** and y_t is called the **state variable** of v. We further take two assumptions on $u(\cdot)$ and $v(\cdot)$.

Assumption 1.2.
The utility function $u(c_t)$ is continuous and increasing in c_t.

Note that by this assumption, the inequality in Eq. (1.1) is replaced by an equality since if $c_t + k_{t+1}$ is strictly less than y_t, the agent can always increase c_t to improve the utility.

Assumption 1.3.
The value function $v(y_t)$ is bounded.

Note that we may also write value function as follows.

$$
\begin{aligned}
v(y_0) &= \max_{c_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \\
&= \max_{c_t} \mathbb{E}_0 \left[u(c_0) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \right] \\
&= \max_{c_t} u(c_0) + \beta \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{t+1}) \right] \\
&= \max_{c_0} u(c_0) + \beta \mathbb{E}_0 [v(y_1)] \\
&= \max_{c_0} u(c_0) + \beta \int v(z_1 f(y_0 - c_0)) \phi(dz_1).
\end{aligned} \tag{1.4}
$$

The form is called the **Bellman equation**. It is a functional equation regarding v. Note that the true value function would solve this functional equation. The Bellman equation approach has a significant advantage compared to the traditional method of Lagrange multiplier; the Bellman equation approach transforms an infinite horizon problem into a two-period problem, and also deals with the uncertainty. However, there is a clear drawback: How to find v?

2. Value Function Iteration

In this section, we provide mathematical details for value function iteration. We begin by introducing some fundamental concepts in analysis.

Definition.
*A **metric space** is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a function that satisfies the following properties:*

(a) $d(x, y) \geq 0$ *for all* $x, y \in X$; $d(x, y) = 0$ *if and only if* $x = y$.

(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

*d is called a **metric (distance)** on X.*

Definition.
A sequence $\{x_n\}$ in a metric space (X, d) is said to be converge to $x \in X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

Definition.
*A sequence $\{x_n\}$ in a metric space (X, d) is said to be **Cauchy** if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.*

Definition.
*A metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges to a point in X.*

Remark.
$\mathbb{R}^n$ is a complete metric space under the Euclidean metric $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Definition.
A normed space X is a vector space with scalar field $\mathbb{R}$ equipped with a norm $\|\cdot\|$, satisfying that

(a) *$\|x\| \geq 0$ for all $x \in X$; $\|x\| = 0$ if and only if $x = 0$.*

(b) *$\|ax\| = |a| \, \|x\|$ for all $a \in \mathbb{R}$ and $x \in X$.*

(c) *$\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.*

Remark.
The scalar field $\mathbb{R}$ can be replaced by other fields, but for our purpose, we only consider $\mathbb{R}$.

Remark.
The norm induces a metric $d(x, y) = \|x - y\|$. In fact, the Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ induces the Euclidean metric. For this reason, a normed space is automatically a metric space and the metric is defined by its norm.

Definition.
$B(X)$ is the set of all real-valued bounded continuous functions defined on X.

Proposition 2.4.
$B(X)$ is a complete metric space under the supremum norm $\|f\| = \sup\limits_{x \in X} |f(x)|$.

Proof.

Let $\{f_n\}$ be a Cauchy sequence in $B(X)$. For each $x \in X$, define $f(x) = \lim_{n\to\infty} f_n(x)$. The limit exists since $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. We claim that $f \in B(X)$. First, f is bounded since for each $x \in X$, there exists N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$. Letting $m = N$ and $n \to \infty$ yields that $|f(x) - f_N(x)| \leq \epsilon$. Hence, $|f(x)| \leq |f_N(x)| + \epsilon$ for all $x \in X$. Second, f is continuous since for each $x \in X$ and $\epsilon > 0$, we may pick $\delta > 0$ such that $|f_N(x) - f_N(y)| < \epsilon$ for all $y \in X$ with $d(x, y) < \delta$. Hence, $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < 3\epsilon$ for all $y \in X$ with $d(x, y) < \delta$. Since ϵ is arbitrary, f is indeed continuous and hence $f \in B(X)$. This completes the proof. ■

Definition.
An operator $T : X \to X$ is called a ***contraction*** *if there exists $\alpha \in (0,1)$ such that $d(T(x), T(y)) \leq \alpha d(x,y)$ for all $x, y \in X$.*

Theorem 2.5 (Contraction Mapping Theorem).
Let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping with contraction factor $\alpha \in (0,1)$. Then T has an unique fixed point $x^* \in X$. That is, $Tx^* = x^*$. Furthermore, for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to x^*.

Proof.
For each $x_0 \in X$, we define $x_n = T^n(x_0)$. Then

$$d(x_{n+1}, x_n) = d(T^{n+1}(x_0), T^n(x_0)) \leq \alpha^n d(x_1, x_0) \to 0 \quad \text{as } n \to \infty. \tag{2.1}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some $x^* \in X$. Next, suppose both x^* and y^* are fixed points of T. Then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq \alpha d(x^*, y^*) < d(x^*, y^*), \tag{2.2}$$

posing a contradiction. Therefore, x^* is unique. ∎

Theorem 2.6 (Blackwell's Theorem).
Suppose $T : B(X) \to B(X)$ satisfies the following properties:

(a) T is monotone, i.e., $f \leq g$ implies $Tf \leq Tg$.

(b) There exists $\alpha \in (0,1)$ such that for any $c \in \mathbb{R}_+$, $T(f + a) \leq Tf + \alpha c$.

Then T is a contraction.

Proof.
Suppose $f, g \in B(X)$ and $c \in \mathbb{R}_+$ satisfy the conditions (a) and (b). Then notice that

$$g \leq f + \|f - g\|. \tag{2.3}$$

Thus we have

$$Tg \leq T(f + \|f - g\|) \leq Tf + \alpha \|f - g\|. \tag{2.4}$$

Rearranging the terms and taking the norm yiels the desired result. ∎

We now turn back to the Bellman equation.

Definition.
*The **Bellman operator** $T : v \mapsto Tv$ is defined by*

$$Tv(y) = \max_c u(y) + \beta \int v(zf(y-c))\phi(dz). \tag{2.5}$$

Remark.
The solution to the Bellman equation is the fixed point of the Bellman operator T.

Corollary 2.8.
The Bellman operator T is a contraction.

Proof.
Left as an exercise. ∎

Since $B(X)$ is a complete metric space and T is a contraction operator on it, by the contraction mapping theorem, T has an unique fixed point. This fixed point is the solution to the Bellman equation. Also, the proof of the contraction mapping theorem reveals a numerical algorithm to find the fixed point:

(a) Start with a guess $v_0 \in B(X)$.
(b) Apply the Bellman operator T to v_0 to get $v_1 = Tv_0$.
(c) Compare v_1 with v_0. If they are close enough, stop; otherwise, set $v_0 = v_1$ and repeat step 2.

The algorithm is called the **value function iteration**.

The value function iteration is one of the most popular methods to solve dynamic programming problems. One may observe that in the finite-horizon case, the value function iteration is equivalent to the backward induction.

3. Envelope Condition Method

In this section, we introduce the envelope condition method. We begin by adding a few assumptions to the optimal growth model.

4. Policy Function Iteration