An Introduction to Value Function Iteration

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July 23, 2024

Foreword

This document aims at providing mathematical details for value function iteration (VFI) in economics. For implementation in Julia, please refer to Sargent's fantastic website. This document serves as a supplement to the website.

Contents

1. Optimal Growth Model

In this capter, we introduce an optimal growth model. The model is going to be our working example for VFI.

Consider an agent who seeks to maximize his lifetime expected utility. The agent's problem is to choose his future path of consumption c_t and capital stock k_{t+1}, subject to the constraint:

$$c_t + k_{t+1} \leq y_t, \tag{1.1}$$

where both c_t and k_{t+1} are non-negative. y_t is the agent's income at time t, which follows the law of motion:

$$y_t = z_t f(k_t), \quad z_t \overset{iid}{\sim} \phi, \tag{1.2}$$

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where z_t is a random variable that follows a positively supported distribution ϕ. $f(k_t)$ is the production function.

Assumption 1.1.
The production function $f(k_t)$ is continuous and increasing in k_t.

The agent's optimization problem is given by:

$$v(y_t) = \max_{c_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right], \tag{1.3}$$

subject to the constraints Eq. (1.1) and Eq. (1.2), where $\beta \in (0,1)$ is the discount factor, and $u(c_t)$ is the utility flow in each period. $v(y_t)$ is called the **value function** and y_t is called the **state variable** of v. We further take two assumptions on $u(\cdot)$ and $v(\cdot)$.

Assumption 1.2.
The utility function $u(c_t)$ is continuous and increasing in c_t.

Note that by this assumption, the inequality in Eq. (1.1) is replaced by an equality since if $c_t + k_{t+1}$ is strictly less than y_t, the agent can always increase c_t to improve the utility.

Assumption 1.3.
The value function $v(y_t)$ is bounded.

Note that we may also write value function as follows.

$$\begin{aligned}
v(y_0) &= \max_{c_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \\
&= \max_{c_t} \mathbb{E}_0 \left[u(c_0) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \right] \\
&= \max_{c_t} u(c_0) + \beta \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{t+1}) \right] \\
&= \max_{c_0} u(c_0) + \beta \mathbb{E}_0 [v(y_1)] \\
&= \max_{c_0} u(c_0) + \beta \int v(z_1 f(y_0 - c_0)) \phi(dz_1).
\end{aligned} \tag{1.4}$$

The form is called the **Bellman equation**. It is a functional equation regarding v. Note that the true value function would solve this functional equation. The Bellman equation approach has a significant advantage compared to the traditional method of Lagrange multiplier; the Bellman equation approach transforms an infinite horizon problem into a two-period problem, and also deals with the uncertainty. However, there is a clear drawback: How to find v?

2. Mathematical Details

In this section, we provide mathematical details for solving the Bellman equation. We begin by introducing some fundamental concepts in analysis.

Definition.
*A **metric space** is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a function that satisfies the following properties:*

(a) $d(x, y) \geq 0$ *for all* $x, y \in X$*;* $d(x, y) = 0$ *if and only if* $x = y$.

(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(c) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

*d is called a **metric (distance)** on X.*

Definition.
A sequence $\{x_n\}$ in a metric space (X, d) is said to be converge to $x \in X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

Definition.
*A sequence $\{x_n\}$ in a metric space (X, d) is said to be **Cauchy** if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.*

Definition.
*A metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges to a point in X.*

Remark.
$\mathbb{R}^n$ is a complete metric space under the Euclidean metric $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

Definition.
A normed space X is a vector space with scalar field $\mathbb{R}$ equipped with a norm $\|\cdot\|$, satisfying that

(a) $\|x\| \geq 0$ *for all* $x \in X$*;* $\|x\| = 0$ *if and only if* $x = 0$.

(b) $\|ax\| = |a| \, \|x\|$ *for all* $a \in \mathbb{R}$ *and* $x \in X$.

(c) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

Remark.
The scalar field $\mathbb{R}$ can be replaced by other fields, but for our purpose, we only consider $\mathbb{R}$.

Remark.
The norm induces a metric $d(x,y) = \|x-y\|$. In fact, the Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ induces the Euclidean metric. For this reason, a normed space is automatically a metric space and the metric is defined by its norm.

Definition.
$B(X)$ is the set of all real-valued bounded continuous functions defined on X.

Proposition 2.4.
$B(X)$ is a complete metric space under the supremum norm $\|f\| = \sup\limits_{x\in X} |f(x)|$.

Proof.

Let $\{f_n\}$ be a Cauchy sequence in $B(X)$. For each $x \in X$, define $f(x) = \lim_{n\to\infty} f_n(x)$. The limit exists since $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. We claim that $f \in B(X)$. First, f is bounded since for each $x \in X$, there exists N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$. Letting $m = N$ and $n \to \infty$ yields that $|f(x) - f_N(x)| \leq \epsilon$. Hence, $|f(x)| \leq |f_N(x)| + \epsilon$ for all $x \in X$. Second, f is continuous since for each $x \in X$ and $\epsilon > 0$, we may pick $\delta > 0$ such that $|f_N(x) - f_N(y)| < \epsilon$ for all $y \in X$ with $d(x,y) < \delta$. Hence, $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < 3\epsilon$ for all $y \in X$ with $d(x,y) < \delta$. Since ϵ is arbitrary, f is indeed continuous and hence $f \in B(X)$. This completes the proof. ∎

Definition.
*An operator $T : X \to X$ is called a **contraction** if there exists $\alpha \in (0,1)$ such that $d(T(x), T(y)) \leq \alpha d(x,y)$ for all $x, y \in X$.*

Theorem 2.5 (Contraction Mapping Theorem).
Let (X,d) be a complete metric space and $T : X \to X$ be a contraction mapping with contraction factor $\alpha \in (0,1)$. Then T has an unique fixed point $x^* \in X$. That is, $Tx^* = x^*$. Furthermore, for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to x^*.

Proof.

For each $x_0 \in X$, we define $x_n = T^n(x_0)$. Then

$$d(x_{n+1}, x_n) = d(T^{n+1}(x_0), T^n(x_0)) \leq \alpha^n d(x_1, x_0) \to 0 \quad \text{as } n \to \infty. \tag{2.1}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some $x^* \in X$. Next, suppose both x^* and y^* are fixed points of T. Then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq \alpha d(x^*, y^*) < d(x^*, y^*), \tag{2.2}$$

posing a contradiction. Therefore, x^* is unique. ■

Theorem 2.6 (Blackwell's Theorem).
Suppose $T : B(X) \to B(X)$ satisfies the following properties:

(a) T is monotone, i.e., $f \leq g$ implies $Tf \leq Tg$.

(b) There exists $\alpha \in (0, 1)$ such that for any $c \in \mathbb{R}_+$, $T(f + a) \leq Tf + \alpha c$.

Then T is a contraction.

Proof.
Suppose $f, g \in B(X)$ and $c \in \mathbb{R}_+$ satisfy the conditions (a) and (b). Then

$$f \leq g + \|f - g\|. \tag{2.3}$$

Thus we have

$$Tf \leq T(g + \|f - g\|) \leq Tg + \alpha \|f - g\|. \tag{2.4}$$

Rearranging the terms yiels the desired result. ■

We now turn back to the Bellman equation.

Definition.
*The **Bellman operator** $T : v \mapsto Tv$ is defined by*

$$Tv(y) = \max_c u(y) + \beta \int v(zf(y - c))\phi(dz). \tag{2.5}$$

Remark.
The solution to the Bellman equation is the fixed point of the Bellman operator T.

Corollary 2.8.
The Bellman operator T is a contraction.

Proof.
We are going to check the conditions in Blackwell's Theorem are satisfied. First, if $v \leq w$, $v, w \in B(X)$, let c_v and c_w be the optimal consumption for v and w, respectively.

Then

$$\begin{aligned}
Tv(y) &= u(c_v) + \beta \int v(zf(y - c_v))\phi(dz) \\
&\leq u(c_v) + \beta \int w(zf(y - c_v))\phi(dz) \\
&\leq u(c_w) + \beta \int w(zf(y - c_w))\phi(dz) = Tw(y).
\end{aligned} \tag{2.6}$$

Thus the monotonicity is satisfied. Second, for any $c \in \mathbb{R}_+$,

$$\begin{aligned}
T(v + c)(y) &= u(c) + \beta \int v(zf(y - c))\phi(dz) \\
&\leq u(c) + \beta \int v(zf(y - c)) + c\phi(dz) = Tv(y) + \alpha c.
\end{aligned} \tag{2.7}$$

This completes the proof. ■

Since $B(X)$ is a complete metric space and T is a contraction operator on it, by the contraction mapping theorem, T has an unique fixed point. This fixed point is the solution to the Bellman equation. Also, the proof of the contraction mapping theorem reveals a numerical algorithm to find the fixed point:

(a) Start with a guess $v_0 \in B(X)$.

(b) Apply the Bellman operator T to v_0 to get $v_1 = Tv_0$.

(c) Compare v_1 with v_0. If they are close enough, stop; otherwise, set $v_0 = v_1$ and repeat step 2.

The algorithm is called the **value function iteration**.