Stein & Shakarchi
Real Analysis Solutions

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Preface

This document contains my solutions to the exercises in *Real Analysis: Measure Theory, Integration, and Hilbert Spaces (Princeton Lectures in Analysis)* by Elias M. Stein and Rami Shakarchi. Some of the theorems that are used in the solutions while not explicitly proved are taken from the book. Also, some of the theorem are used but not explicitly stated in the book. In such cases, the proofs are provided in Appendix, as it would be prolix to present the proofs in the exercises. Finally, the solutions may contain errors and hence should be taken with caution.

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Contents

1. Measure Theory

Exercise 1.1.
Prove that the Cantor set C is totally disconnected and perfect.

Proof.
$\forall\, x, y \in C, \exists\, k$ such that $3^{-k} < |x - y|$. Thus x, y must belong to two different segments in C_k and hence there exist some $z \notin C$ lies between x, y. This proves that C is totally disconnected. To see that C is perfect, for any fix $x \in C$, note that for any $\epsilon > 0$, pick k such that $3^{-k} < \epsilon$. Then there exists a segments of C_k with length 3^{-k} containing x. Choose y from the segment and then $|x - y| \leq 3^{-k} < \epsilon$. This completes the proof. ■

Exercise 1.2.
The Cantor set C can also be described in terms of ternary expressions.

(a) Every number in $[0, 1]$ has a ternary expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}, \quad \text{where } a_k = 0, 1, 2.$$

Prove that $x \in C$ if and only if x has a representation as above where $a_k = 0, 2$.

*(b) The **Cantor-Lebesgue function** is defined on C by*

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad \text{if } x = \sum_k a_k 3^{-k}, \text{ where } b_k = \frac{1}{2} a_k.$$

We choose the representation that $a_k = 0, 2$ in the definition. Show that F is well-defined and continuous on C; moreover, $F(0) = 0$ and $F(1) = 1$.

(c) Prove that $F : C \to [0, 1]$ is surjective.

(d) F can be extended to a function on $[0, 1]$ as follows. For x in the complement of C, x must lie in some open interval (a, b) with $F(a) = F(b)$. Let $F(x) = F(a)$ for such x. Show that F is continuous on $[0, 1]$.

Proof.
For (a), we know that $C = \bigcap_k C_k$, where C_k is the union of closed segments. Thus in every step, we remove the middle and hence all the $a_k = 1$. Note that those numbers with non-unique representations are the terminated points and the segments removed are open sets. Furthermore, such representation is unique if we restrict our choice of a_k

to be 0 or 2 for $x \in C$. To see this, let a_k be the first digit that differs from a'_k with $a_k > a'_k$. Then we have $a_k - a'_k = 2 = \sum_{n=k+1}^{\infty} 3^{-n}(a'_n - a_n) \leq 3^{-k}$, which is absurd. Hence the representation must be unique.

For (b), by the uniqueness of such representation, F is well-defined. Let $x_n \to x \in C$. For any $\epsilon > 0$, $\exists\, N$ such that $2^{-N+1} < \epsilon$. Pick x_n such that the first digit in its representation that differs from x occurs after N. Then $|F(x_n) - F(x)| \leq \sum_k 2^{-k-1} |a_k^n - a_k| \leq \sum_{k \geq N} 2^{-k} = 2^{-N+1} < \epsilon$. Hence F is continuous on C. Furthermore, $F(1) = 1$ since $1 = \sum_k 2^{-k} \times 1$ and $F(0) = 0$ since $0 = \sum_k 2^{-k} \times 0$.

To see (c), note that every number has a binary representation and thus F is surjective.

Finally, for (d), we have shown that F is continuous on C. Observe that F is a piecewise function continuous at each terminated points and constant functions are continuous. We conclude that the extended F is continuous on $[0, 1]$. ∎

Exercise 1.3.

Consider a unit interval $[0, 1]$, and let ξ be a fixed real number with $\xi \in (0, 1)$. C_ξ is defined iteratively as follows. $C_0 = [0, 1]$. At the n-th step, C_n is obtained by removing the open middle interval of length ξ proportion from each of the closed intervals in C_{n-1}.

- *(a) Prove that C_ξ^c in $[0, 1]$ is the union of open intervals of total length 1.*
- *(b) Show directly that $m^*(C_\xi) = 0$.*

Proof.

For (a), let A_n be the complement of C_n in $[0, 1]$. Then $A_n = A_{n-1} \dot{\bigcup} B_n$, where B_n is the union of newly removed open intervals, which are of length $(1-\xi)^{n-1}\xi$. $A_0 = \varnothing$. Thus $m(A_n) = (1-\xi)^{n-1}\xi + m(A_{n-1}) = \sum_{k \leq n} (1-\xi)^{k-1}\xi \to 1$.

For (b), observe that $C_n \subset C_{n-1}$ and C_n are essentially the union of 2^n disjoint intervals of length $(2(1-\xi))^{-n}$. Thus $m^*(C_n) = (1-\xi)^{-n}$ and then $m^*(C_\xi) \leq m^*(C_n) = (1-\xi)^{-n} \to 0$ as $n \to \infty$. Hence $m^*(C_\xi) = 0$. ∎

Exercise 1.4.

Construct a closed set $\hat{C}$ so that at the k-th step, one removes 2^{k-1} central open intervals each of length l_k, with $l_1 + 2l_2 + \cdots + 2^{k-1}l_k < 1$ for any k.

- *(a) If l_j are chosen small enough, then $\sum_k 2^{k-1}l_k < 1$. In this case, show that $m(\hat{C}) = 1 - \sum_k 2^{k-1}l_k$.*
- *(b) Show that if $x \in \hat{C}$, then there exists a sequence of points x_n such that $x_n \notin \hat{C}$, yet $x_n \to x$ and $x_n \in I_n$, where I_n is a subinterval in the complement of $\hat{C}$ with $|I_n| \to 0$.*
- *(c) Prove that $\hat{C}$ is perfect and contains no open intervals.*

(d) *Show that $\hat{C}$ is uncountable.*

Proof.

For (a), we again note that $\hat{C} = \bigcap_n C_n$, where C_n is the union of closed intervals with total length $m(C_n) = m(C_{n-1}) - 2^{n-1}l_n$ and $m(C_0) = [0,1]$. Also, $C_n \searrow \hat{C}$ and hence $m(\hat{C}) = \lim_n m(C_n) = 1 - \sum_k 2^{k-1}l_k$.

To prove (b), let $\epsilon > 0$ be given. A subinterval in C_n is of length $2^{-n}(1 - \sum_{k=1}^{n} 2^{k-1}l_k)$. Then x_n can be chosen from the interval removed in $n+1$-th step in the subinterval containing x. By our construction, $x_n \notin \hat{C}$. Since x_n and x lie in the same subinterval of C_n, $|x_n - x| < \epsilon$ for any $n \geq N$, where N is chosen such that $2^{-N}(1 - \sum_{k=1}^{N} 2^{k-1}l_k) < \epsilon$. Furthermore $|I_n| = l_n \to 0$ as $n \to \infty$ since $\sum_{k=1}^{\infty} 2^{k-1}l_k < \infty$, which implies that $l_k \to 0$ as $k \to \infty$.

To prove (c), note that for any $x \in \hat{C}$, we may pick $x_n \in C_n$, where x_n is the terminated point of the subinterval containing x. Then $|x_n - x| \leq 2^{-n}(1 - \sum_{k=1}^{n} 2^{k-1}l_k) \to 0$ as $n \to \infty$. Hence $\hat{C}$ is perfect. Also, $\hat{C}$ contains no open intervals since the subintervals in C_n have length $2^{-n}(1 - \sum_{k=1}^{n} 2^{k-1}l_k)$, which can be arbitrarily small for large n.

Finally, for (d), if $\sum_{k=1}^{\infty} 2^{k-1}l_k < 1$, since $m(\hat{C}) \neq 0$, $\hat{C}$ must be uncountable. In general, for the case where $\sum_{k=1}^{\infty} 2^{k-1}l_k = 1$, we may construct the bijection to a fat Cantor set, which has each removed intervals with same center but with length $rl_k < l_k$. Such function is essentially a scalar multiplication and hence a bijection. But any fat Cantor set is uncountable. Hence so is any thin Cantor set. ∎

Exercise 1.5.

Suppose E is a given set and $O_n = \{x \mid d(x, E) < 1/n\}$.

(a) *Show that if E is compact, then $m(E) = \lim_{n\to\infty} m(O_n)$.*

(b) *Construct the counterexamples for which E is closed but unbounded and E is bounded but open.*

Proof.

If E is compact, then E is bounded, closed and hence measurable. Notice that O_n is open and bounded. We claim that $O_n \searrow E$. Indeed, $O_n \supset O_{n+1} \supset E$. Also, for any $x \in O_n - E$, let $d = d(x, E)$. Then pick N such that $1/N < d$. Then $x \notin O_N$ and hence $x \notin O_n$. We conclude that $O_n \searrow E$. Thus $m(O_n) \to m(E)$ as $n \to \infty$.

To construct the counterexamples, let $E = \{s_n \mid s_n = 2\sum_{k=1}^{n} 1/k, n \in \mathbb{N}\}$. Then E is closed since every point in E is a isolated point. For any n, $O_n \supset (s_n - 1/n, \infty]$, which has an infinite measure. Thus $m(O_n) = \infty$ for all n. Hence $m(E) \neq \lim_{n\to\infty} m(O_n)$. For the second counterexample, let r_k denote all rational numbers in $(0,1)$. E is the union of $(r_k - 2^{-k}\epsilon, r_k + 2^{-k}\epsilon)$, where $\epsilon \in (0,1)$. Then E is bounded but open with $m(E) \leq \sum_{k=1}^{\infty} 2^{-k}\epsilon = \epsilon$. However, since r_k is dense in $(0,1)$, $O_n \supset (0,1)$ and hence $m(O_n) \geq 1$ for all n. Thus $m(E) \neq \lim_{n\to\infty} m(O_n)$. ■

Exercise 1.6.

Let B be a ball in $\mathbb{R}^d$ of radius r. Prove that $m(B) = v_d r^d$, where $v_d = m(B_1)$ and B_1 is the unit ball centered at the origin with radius 1.

Proof.

By translation and dilation, we know that $B = rB_1 + b$ for some b. Then $m(B) = r^d m(B_1) = v_d r^d$. ■

Exercise 1.7.

Let $\delta \in \mathbb{R}^d_+$ and $E \subset \mathbb{R}^d$. δE is defined as $\{(\delta_1 x_1, \ldots, \delta_d x_d) \mid x \in E\}$. Prove that δE is measurable if E is measurable, and $m(\delta E) = \delta_1 \cdots \delta_d m(E)$.

Proof.

For any measurable $E \subset \mathbb{R}^d$, $\exists\, V$ open such that $E \subset V$ and $m(V - E) < \epsilon$. Let $V - E$ be covered by a sequence of rectangles R_n with $\sum_n m(R_n) < \epsilon$. Furthermore, δV is also open since it is a linear transformation and hence continuous. Also, $\delta V - \delta E$ is covered by $\bigcup_n \delta R_n$ and $m(\delta V - \delta E) = m(\delta(V - E)) \leq m(\delta \bigcup_n R_n) = m(\bigcup_n \delta R_n) = \sum_n m(\delta R_n) = \sum_n \delta_1 \cdots \delta_d m(R_n) < \delta_1 \cdots \delta_d \epsilon$. Since ϵ is arbitrarily small, δE is measurable. Moreover, if Q_n is a sequence of rectangles covering E with $m(E) \leq \sum_n Q_n < \epsilon$, then δQ_n is a sequence of rectangles covering δE with $\sum_n m(\delta Q_n) = \delta_1 \cdots \delta_d \sum_n m(Q_n) < \delta_1 \cdots \delta_d (m(E) + \epsilon)$. Since ϵ is arbitrarily small, $m(\delta E) = \delta_1 \cdots \delta_d m(E)$. ■

Exercise 1.8.

Suppose T is a linear transformation in $\mathbb{R}^d$ and E is a measurable set in $\mathbb{R}^d$. Prove that $T(E)$ is measurable.

Proof.

If E is compact, then so is $T(E)$ since T is linear and hence continuous. Thus if E is an F_σ set, then $T(E)$ is also an F_σ set. This follows from the fact that an F_σ set is the countable union of compact sets. Next, since T is linear, $\exists\, M$ such that $|T(x') - T(x)| \leq M\,|x' - x|$ for any $x, x' \in \mathbb{R}^d$. Then T maps any cube of side length l into a cube of side length $2Ml$. Now suppose that E is a set of measure zero. Let Q_n be a sequence of cubes covering E with $\sum_n m(Q_n) < \epsilon$. Then $m^*(T(E)) \leq \sum_n m^*(T(Q_n)) \leq 2^d M^d \epsilon$ for arbitrarily small

ϵ. Hence $T(E)$ is measurable with measure zero. Finally, for any measurable set E, $\exists\, F$ closed such that $m(E \Delta F) = 0$. Then $m(T(E)\Delta T(F)) = m(T(E\Delta F)) = 0$ and hence $T(E)$ is measurable. ■

Exercise 1.9.
Find an open set V such that $\partial \overline{V}$ has positive Lebesgue measure.

Proof.
Consider the complement of a fat Cantor set in $[0,1]$ obtained by letting $l_k = 4^{-k}$, denoted as V. Being an union of open intervals, V is open. Note that V by Exercise 1.4, V' contains the fat Cantor set. Then $\partial \overline{V} = V' - V$ also contains the fat Cantor set and hence has positive measure. ■

Exercise 1.10.
Construct a decreasing sequence of continuous functions $f_n \to f$ defined on $[0,1]$ while f is not Riemann integrable.

Proof.
Consider the removed intervals V_n in each step in the construction of a fat Cantor set $\hat{C}$ with $l_n = 4^{-n}$. Let I_n^k be the sub-intervals of V_n. Define

$$F_n(x) = \begin{cases} 1 & x \notin V_n, \\ 2 \times 4^n \left|x - p_n^k\right| & x \in I_n^k, \end{cases}$$

where p_n^k is the center of I_n^k. Then F_n is continuous. Put $f_n = \prod_{i=1}^n F_i$. Then f_n is devreasing and continuous. Let $f_n \to f$ pointwisely. f exists since for any x, $f_n(x)$ is a decreasing sequence bounded below. Now we claim that f is discontinuous on $\hat{C}$. Indeed, let $\epsilon = 1/2$ be given. For any $x \in \hat{C}$, any neighborhood of x contains some I_n^k and hence $\exists\, x' \in I_n^k$ such that $f(x') = 0$ and thus $|f(x) - f(x')| = 1 > \epsilon$. The claim thus follows. However, by Exercise 1.4, $\hat{C}$ has positive measure and hence f has positive measure of discontinuity. Thus f is not Riemann integrable. ■

Exercise 1.11.
Let A contains all numbers in $[0,1]$ whose decimal expansion contains no digit 4. Find $m(A)$.

Proof.
Let B_n consist of all numbers in $[0,1]$ in which the first 4 appears at the n-th decimal place. Then we have $A^c = \dot{\bigcup}_n B_n$ and every B_n is simply the union of 9^{n-1} disjoint open intervals of length 10^{-n}. Thus $m(A) = 1 - \sum_{n=1}^{\infty} 9^{n-1} 10^{-n} = 1 - \frac{1}{10}\sum_{n=0}^{\infty}\left(\frac{9}{10}\right)^n = 8/9$. ■

Exercise 1.12.

(a) *Let D be an open disc in $\mathbb{R}^2$ and E. Prove that D is not the disjoint union of open rectangles.*

(b) *Prove that an open connected set Ω is the disjoint union of open rectangles if and only if Ω is an open rectangle.*

Proof.

(a) is proved by (b); hence we dive straight into (b). In (b), the necessity is trivial. For the sufficiency, if not, then Ω must be composed of at least two disjoint open sets. However, since Ω is connected, it cannot be separated into two disjoint open sets. This poses a contradiction and hence Ω must be an open rectangle. ∎

Exercise 1.13.

(a) *Show that a closed set is a G_δ set and an open set is an F_σ set.*

(b) *Give an example of an F_σ set which is not a G_δ set.*

(c) *Give an example of a Borel set which is not a G_δ set nor an F_σ set.*

Proof.

To see (a), let F be a closed set. Consider $V_n = \{x \mid d(x, F) < 1/n\}$. Then $F = \bigcap_n V_n$ since for any $x \in V_n - F$, $d(x, F) \geq \epsilon > 0$ for some ϵ. But $\exists\, N$ such that $1/N < \epsilon$ and $x \notin V_N$. Thus F is a G_δ set. If G is open, G^c is closed and hence a G_δ set. By picking the complement of V_n, G is an F_σ set.

For (b), consider $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is an F_σ set. We now claim that it is not a G_δ set. If $\mathbb{Q}$ is a G_δ set, then $\mathbb{Q} = \bigcap_n V_n$ for open sets V_n. Consider $U_n = V_n - q_n$, where q_n denumerates all rational numbers. Then U_n is dense in $\mathbb{R}$ for each n. By Theorem A.1, $\bigcap_n U_n$ is dense in $\mathbb{R}$. However, $(\bigcap_n V_n) - \mathbb{Q} = \bigcap_n U_n = \varnothing$, which is absurd. Hence $\mathbb{Q}$ is not a G_δ set.

For (c), pick $E = (\mathbb{Q} \bigcap \mathbb{R}_+) \bigcup (\mathbb{Q}^c \bigcap \mathbb{R}_-)$. Then E is a Borel set since $\mathbb{Q}$ is an F_σ set and hence a Borel set. Note that $\mathbb{Q}^c$ is not an F_σ set since if it is, then $\mathbb{Q}$ is a G_δ set. But we have just shown that $\mathbb{Q}$ is not a G_δ set. Thus $\mathbb{Q}^c$ is not an F_σ set. Hence E is neither a G_δ set nor an F_σ set while still a Borel set. ∎

Exercise 1.14.

The ***outer Jordan content*** *of a set $E \subset \mathbb{R}$ is defined by*

$$J^*(E) = \inf \sum_{n=1}^{N} |I_n|,$$

where the infimum is taken over all finite covering rectangles $\{I_n\}$.

(a) *Prove that* $J^*(E) = J^*(\overline{E})$.

(b) *Find a set* E *such that* $J^*(E) = 1$ *while* $m^*(E) = 0$.

Proof.

For (a), it is clear that if $\{I_n\}$ is a cover consisting of rectangles of E, then $E \subset \overline{E} \subset \bigcup_{n=1}^N I_n$ as long as I_n are chosen to be closed. Thus by definition we can pick $\{I_n\}$ such that $J^*(E) \le J^*(\overline{E}) \le \sum_{n=1}^N |I_n| < J^*(E) + \epsilon$ for arbitrarily small ϵ. The first inequality comes from the fact that any cover of a set is also a cover of any of its subset. Then it follows that $J^*(E) = J^*(\overline{E})$.

For (b), consider $E = \mathbb{Q} \cap [0,1]$. Then $J^*(E) = 1$ since E is dense in $[0,1]$ and hence any finite cover of E must cover $[0,1]$. Thus $J^*(E) = 1$. On the other hand, $m^*(E) = 0$ since E is a countable set. ∎

Exercise 1.15.

Define $m_R^*(E)$ *by considering the infimum of volumes among all countable rectangles covering* E. *Prove that* $m_R^*(E) = m^*(E)$ *for any* $E \subset \mathbb{R}^d$.

Proof.

Let I_n be a sequence of rectangles covering E with $\sum_n |I_n| < m_R^*(E) + \epsilon$. Then by extending their borders and subdividing, we obtain a sequence of almost disjoint cubes Q_k with same total volume. Since some cubes may be discarded, we have $\sum_k |Q_k| \le \sum_n |I_n| < m_R^*(E) + \epsilon$. Since ϵ is arbitrary, $m^*(E) \le m_R^*(E)$. The reverse inequality is trivial since every cube is a rectangle. ∎

Exercise 1.16 (Borel-Cantelli Lemma).

Let $\{E_k\}$ *be a sequence of measurable subsets of* $\mathbb{R}^d$ *with* $\sum_k m(E_k) < \infty$. *Let*

$$E = \left\{ x \in \mathbb{R}^d \;\middle|\; x \in E_k \text{ for infinitely many } k \right\} = \limsup_k E_k.$$

(a) *Show that* E *is measurable.*

(b) *Prove that* $m(E) = 0$.

Proof.

We may write E as $\bigcap_n \bigcup_{k \ge n} E_k$. For (a), note that E_k is measurable and hence so is their countable union and countable intersection.

Notice that we have $E \subset \bigcup_{k \ge n} E_k$ for all n. Then $m(E) \le m(\bigcup_{k \ge n} E_k) \le \sum_{k \ge n} m(E_k)$. Letting $n \to \infty$, we must have $\sum_{k \ge n} m(E_k) \to 0$ since the series converges. It follows that $m(E) = 0$. ∎

Exercise 1.17.

Let f_n be a sequence of measurable functions on $[0, 1]$ such that $|f_n(x)| < \infty$ almost everywhere. Show that $\exists$ a sequence of positive numbers c_n such that $\frac{f_n(x)}{c_n} \to 0$ almost everywhere.

Proof.

For any fixed n, pick c_n such that $E_n = \{x \mid |f_n(x)| / c_n > 1/n\}$ has measure less than 2^{-n}. This is valid since $|f_n(x)| < \infty$ almost everywhere. Observe that the set of x such that $|f_n(x)| / c_n \not\to 0$ is a subset of $E = \limsup_n E_n$. Also, $\sum_n m(E_n) < \sum_n 2^{-n} = 1$. By Exercise 1.16, $m(E) = 0$. Thus $|f_n(x)| / c_n \to 0$ almost everywhere. ■

Exercise 1.18.

Prove that every measurable function is the limit of a sequence of continuous functions almost everywhere.

Proof.

Let the measurable function be f defined on $\mathbb{R}^d$. We first consider the case where B_n is the closed ball with radius n centered at the origin and hence has finite measure. Then by Lusin's theorem, for each n, $\exists\, F_{2^{-n}}$ closed such that $m(B_n - F_{2^{-n}}) < 2^{-n}$ and f is continuous on $F_{2^{-n}}$. By Theorem A.4, f can be extended to a continuous function f_n on $\mathbb{R}^d$. Now let E_n denote the set of x such that $f_n(x) \neq f(x)$ and $E = \{x \mid f_n(x) \not\to f\}$. If $x \in E$, then $x \in E_n$ for infinitely many n or $f_n(x)$ would eventually be equal with $f(x)$ after some large n. Observe that $E_n \subset B_n - F_{2^{-n}}$. Thus $\sum_n m(E_n) < \sum_n m(B_n - F_{2^{-n}}) = 1$. By Exercise 1.16, $m(E) = m(\limsup_n E_n) = 0$. Thus $f_n \to f$ almost everywhere. ■

Exercise 1.19.

(a) *Show that if either A and B is open, then $A + B$ is open.*

(b) *Show that if A and B are closed, then $A + B$ is measurable.*

(c) *Show that $A + B$ might not be closed even if A and B are closed.*

Proof.

To prove (a), without loss of generality we may assume that A is open. Then for $x + y \in A + B$ where $x \in A$ and $y \in B$, $\exists$ open set V conatining x such that $V \subset A$. Then $V + y \subset A + B$ and forms a neighborhood of $x + y$. Thus $A + B$ is open.

For (b), note that any closed set is a countable union of compact sets. Next, we show that the sum of two compact is also compact. Let K_1 and K_2 be compact. For any $x_n + y_n \in K_1 + K_2$ where $x_n \in K_1$ and $y_n \in K_2$, $\exists$ convergent subsequences $x_{n_k} \to x \in K_1$ and thus $y_{n_{k_l}} \to y \in K_2$. Thus $x + y \in K_1 + K_2$ is a limit of a subsequence of $x_n + y_n$. Hence $K_1 + K_2$

is compact. Thus $A+B$ is measurable since it is a countable union of compact sets and hence an F_σ set.

For (c), let

$$A = \left\{ \sum_{\text{odd } k<n} \frac{1}{k} \;\middle|\; n \in \mathbb{N} \right\}, \quad B = \left\{ \sum_{\text{even } k<n} \frac{-1}{k} \;\middle|\; n \in \mathbb{N} \right\}.$$

Then A and B are closed since they are subsequences of harmonic series or negative harmonic series. However, the alternating harmonic series does converge to $\log 2$, which is irrational and hence not in $A+B$. Thus $A+B$ is not closed. ∎

Exercise 1.20.
Show that there exist closed sets A and B such that $m(A) = m(B) = 0$ while $m(A+B) > 0$ in both $\mathbb{R}$ and $\mathbb{R}^2$.

Proof.
In $\mathbb{R}$, let A be the Cantor set and $B = \frac{1}{2}A$. Then $m(A) = m(B) = 0$ by Exercise 1.3 and Exercise 1.7. However, for any $x \in [0,1]$ whose ternary expansion is $(x_1, \ldots)$, $x_k \in \{0,1,2\}$. Now simply pick $a \in A$ and $b \in B$ such that their ternary expansions satisfy that if $x_k = 0$, $a_k = b_k = 0$; if $x_k = 1$, $a_k = 0$ and $b_k = 1$; if $x_k = 2$, $a_k = 2$ and $b_k = 0$. Thus $A+B$ contains $[0,1]$ and hence $m(A+B) > 0$.

In $\mathbb{R}^2$, let $A = [0,1] \times \{0\}$ and $B = \{0\} \times [0,1]$. Then $m(A) = m(B) = 0$ but $A+B = [0,1]^2$ and hence $m(A+B) > 0$. ∎

Exercise 1.21.
Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set.

Proof.
Consider the Vitali set V in $[0,1]$ and the Cantor set C. By Exercise 1.2 we know that $\exists$ a surjective continuous function $F : C \to [0,1]$. Then $F^{-1}(V)$ is a subset of C and hence measurable while $F(F^{-1}(V)) = V$ is not measurable. ∎

Exercise 1.22.
Show that there is no function defined on $\mathbb{R}$ that is continuous everywhere in $\mathbb{R}$ and also satisfies that $f(x) = \chi_{[0,1]}(x)$ almost everywhere.

Proof.
Suppose such function f exists. Since $f(x) = \chi_{[0,1]}(x)$ almost everywhere, for any open interval I_δ containing point 0 with length δ, f cannot differ from $\chi_{[0,1]}(x)$ on I_δ since

any nonempty open intervals has non-zero measure. Thus f must take value 1 and 0 on I_δ for every δ. This contradicts to the continuity of f at 0. ∎

Exercise 1.23.
Let $f(x,y)$ be a function on $\mathbb{R}^2$ such that for any fixed x, $f(x,y)$ is continuous in y and vice versa. Prove that f is measurable on $\mathbb{R}^2$.

Proof.

For any given y, let $f_n(x,y) = f(2^{-n}x_n, y)$ where x_n is the largest integer such that $2^{-n}x_n \le x$. We now claim that f_n is measurable. Indeed,

$$\begin{aligned}\{(x,y) \mid f_n(x,y) > a\} &= \bigcup_{k\in\mathbb{Z}} \{(x,y) \mid 2^{-n}k \le x < 2^{-n}(k+1), f(2^{-n}k, y) > a\} \\ &= \bigcup_{k\in\mathbb{Z}} [2^{-n}k, 2^{-n}(k+1)) \times \{y \mid f(2^{-n}k, y) > a\}.\end{aligned}$$

Since $f(2^{-n}k, y)$ is continuous in y, $\{y \mid f(2^{-n}k,y) > a\}$ is open and hence the product is measurable. Thus f_n is measurable. Next, we claim that $f_n \to f$ pointwisely. For any (x,y) and $\epsilon > 0$, $\exists\, N$ such that $|2^{-n}x - x| < \delta$ for all $n \ge N$, where δ is small enough such that $|f(x',y) - f(x,y)| < \epsilon$ when $|x' - x| < \delta$. Thus $|f_n(x,y) - f(x,y)| < \epsilon$ for any $n \ge N$. As f_n is measurable and converges pointwisely to f, f is also measurable. ∎

Exercise 1.24.
Find an enumeration $\{r_n\}$ of $\mathbb{Q}$ such that the complement of $\bigcup_n \left(r_n - \frac{1}{n}, r_n + \frac{1}{n}\right)$ in $\mathbb{R}$ is nonempty.

Proof.

We split the index set $\mathbb{N}$ into $S = \{2^n \mid n \in \mathbb{N}\}$ and $\mathbb{N} - S$. Next, we enumerate those rationals lying outside $[0,1]$ using $\{a_n \mid n \in S\}$ and those inside $[0,1]$ using $\{b_n \mid n \in \mathbb{N} - S\}$. Now let r_n combine the two enumerations. Then observe that

$$m(\bigcup_{n\in S}\left(a_n - \frac{1}{n}, a_n + \frac{1}{n}\right)) \le \sum_{n\in S} \frac{2}{n} < \infty$$

and

$$m(\bigcup_{n\in\mathbb{N}-S}\left(b_n - \frac{1}{n}, b_n + \frac{1}{n}\right)) \le m([-1,2]) < \infty.$$

It follows that the complement of $\bigcup_n (r_n - \frac{1}{n}, r_n + \frac{1}{n})$ must have nonzero measure and hence nonempty. ∎

Exercise 1.25.
Prove that E is measurable if and only if for $\epsilon > 0$, $\exists$ closed set F such that $m^*(E - F) < \epsilon$.

Proof.

Assume that E is measurable, then so does E^c. Thus for any $\epsilon > 0$, $\exists$ open set G such that $m^*(G - E^c) < \epsilon$. Pick $F = G^c \subset E$, then F is closed and $m^*(E - F) = m^*(G - E^c) < \epsilon$.

Conversely, suppose that for any $\epsilon > 0$, $\exists$ closed set $F \subset E$ such that $m^*(E - F) < \epsilon$. Then pick $G = F^c \supset E^c$, which is open. Then $m^*(G - E^c) = m^*(E - F) < \epsilon$. Hence E^c is measurable and so does E. ■

Exercise 1.26.
Suppose $A \subset E \subset B$ where A and B are measurable and $m(A) = m(B)$. Prove that E is measurable.

Proof.

By measurability of B, $\exists$ an open set G such that and $m^*(G - B) < \epsilon$. Then we have $m^*(G - A) = m(G) - m(A) < m(B) + \epsilon - m(A) = m(A) + \epsilon - m(A) = \epsilon$, provided that A is also measurable. Thus $m^*(G - E) \leq m^*(G - A) < \epsilon$, which shows that E is measurable. ■

Exercise 1.27.
Suppose $E_1 \subset E_2$ are compact sets in $\mathbb{R}^d$. Prove that for any c with $m(E_1) < c < m(E_2)$, $\exists$ compact set E such that $m(E) = c$ and $E_1 \subset E \subset E_2$.

Proof.

Let $f(t) = m(E_2 \bigcap D(t))$ where $D(t) = \{x \mid d(x, E) \leq t\}$. Then $f(0) = m(E_1)$ and $f(x) = m(E_2)$ for some $x > 0$. Furthermore, f is continuous since for every $t \in \mathbb{R}_+ \bigcup \{0\}$, $D(t - \frac{1}{n}) \nearrow D(t)$ and hence $\exists\, N$ such that $m(D(t)) - m(D(t - \frac{1}{N})) < \epsilon$. Thus $f(t) - f(t - \frac{1}{N}) = m(E_2 \bigcap D(t)) - m(E_2 \bigcap D(t - \frac{1}{N})) \leq m(D(t)) - m(D(t - \frac{1}{N})) < \epsilon$. Since f is continuous and $m(E_1) < c < m(E_2)$, $\exists\ t \in [0, x]$ such that $f(t) = c$ by intermediate value theorem. Hence $E = E_2 \bigcap D(t)$ is the desired compact set. ■

Exercise 1.28.
Let E be a set with $m^*(E) > 0$. Prove that for each $\alpha \in (0, 1)$, $\exists$ an open interval I such that $m^*(E \bigcap I) \geq \alpha m^*(I)$.

Proof.

Choose an open set V conatining E such that $m^*(E) \geq \alpha m^*(V)$. Consider I_n disjoint open intervals so that their union is V. If all open intervals fail to satisfy the condition, then $m^*(E) = m^*(E \bigcap V) = m^*(\bigcup_n E \bigcap I_n) \leq \sum_n m^*(E \bigcap I_n) < \alpha \sum_n m^*(I_n) = \alpha m^*(V)$, leading to a contradiction. Thus $\exists$ an open interval I such that $m^*(E \bigcap I) \geq \alpha m^*(I)$. ■

Exercise 1.29 (Steinhaus Theorem).
*Suppose $E \subset \mathbb{R}$ is a measurable set with $m(E) > 0$. Prove that the **difference set** of E, $\{x - y \mid x, y \in E\}$ contains an open interval centered at origin.*

Proof.
By Exercise 1.28, $\exists$ an open interval I such that $m^*(E \bigcap I) \geq (9/10)m^*(I)$. Let $S = E \bigcap I$ and suppose that S does not contain an open interval centered at origin. Then for a small enough $\epsilon > 0$, S and $S + \epsilon$ are disjoint. Thus $m^*(S \bigcup (S + \epsilon)) \leq m^*(I \bigcup (I + \epsilon)) = m^*(I) + \epsilon$, but $m^*(S \bigcup (S + \epsilon)) = 2m^*(S) \geq 2(9/10)m^*(I)$. This leads to a contradiction and hence S must contain an open interval centered at origin. ∎

Exercise 1.30.
Let E, F be measurable sets and $m(E), m(F) > 0$. Prove that $E + F$ contains an interval.

Proof.
By Exercise 1.28, given $\alpha, \beta \in (0, 1)$, we can find open intervals I_1, I_2 such that $m(E \bigcap I_1) \geq \alpha m(I_1)$ and $m(-F \bigcap I_2) \geq \beta m(I_2)$. Without loss of generality, we may assume that $m(I_1) \leq m(I_2)$. Then $\exists\, t$ such that $-t + I_1 \subset I_2$. Next, for any $-s \in (0, (1 - \alpha)m(I_1))$, $(-t - s + I_1) \bigcap I_2$ has length at least $m(I_1) + s > \alpha m(I_1)$. Now we claim that $(E - t - s) \bigcap (-F)$ is nonempty. If not, then $(E - t - s) \bigcap (I_1 - s - t)$ and $(-F) \bigcap I_2$ are disjoint. But this implies that $m(((E - t - s) \bigcap (I_1 - s - t)) \bigcup ((-F) \bigcap I_2)) = m(E \bigcap I_1) + m((-F) \bigcap I_2) \geq \alpha m(I_1) + \beta m(I_2)$ and also $m(((E - t - s) \bigcap (I_1 - s - t)) \bigcup ((-F) \bigcap I_2)) \leq m((I_1 - s - t) \bigcup I_2) < m(I_2) + s$, leading to a contradiction by picking $s < \alpha m(I_1) - (1 - \beta)m(I_2)$ and $\alpha > (1 - \beta)m(I_2)/m(I_1)$. Since the intersection is nonempty, so does $I = (E - t - s) \bigcap (I_1 - t - s) \bigcap (-F) \bigcap I_2$. For any $x \in I$, $x = e - t - s = -f$ for some $e \in E$ and $f \in F$. Thus $e + f = t + s$ and $E + F$ contains an interval centered at t with radius $\alpha m(I_1) - (1 - \beta)m(I_2)$. ∎

Exercise 1.31.
Prove that the Vitali set V is not measurable.

Proof.
Suppose V is measurable. Consider translations of V by all rational numbers $\{r_n\}$ in $[0, 1]$. We obtain that $\bigcup_n V + r_n$ is a countable disjoint union of measurable sets covering $[0, 1]$, and hence each V_n must possesses a positive measure or the measure of $\bigcup_n V + r_n$ must be zero. By Exercise 1.29, the difference set of V, denoted as S, contains an open interval centered at origin. Note that S does not contain any rational numbers except 0 by construction and hence cannot have an interval having dense rational numbers in the open interval, posing a contradiction. It follows that V is not measurable. ∎

Exercise 1.32.
Let V be the Vitali set.

(a) *Prove that if $E \subset V$ is measurable, then $m(E) = 0$.*

(b) *Let $G \subset \mathbb{R}$ with $m^*(G) > 0$. Prove that there exists a subset of G that is not measurable.*

Proof.

For (a), the proof is by contradiction. If E has positive measure, then the difference set of E, S, contains an open interval centered at origin by Exercise 1.29. But S does not contain any rational numbers except 0 since $E \subset V$. Then S cannot contain an interval having dense rational numbers in the open interval, leading to a contradiction. Thus $m(E) = 0$.

To see (b), we first note that $m^*(G) > 0$ implies that G is uncountable since any countable set has zero outer measure. This implies that G we can follow the construction of the Vitali set to obtain a subset of G by considering all equivalence classes of $\sim$ where $x \sim y$ if $x - y \in \mathbb{Q}$. This subset is again not measurable by similar arguments as in Exercise 1.31. ■

Exercise 1.33.
Let V be the Vitali set. Show that $V^c = [0,1] - V$ satisfies $m^*(V^c) = 1$ and $m^*(V) + m^*(V^c) \neq m^*(V \bigcup V^c) = m^*([0,1])$ despite that V and V^c are disjoint.

Proof.

Since $V^c \subset [0,1]$, $m^*(V^c) \leq m^*([0,1]) = 1$. Suppose $m^*(V^c) < 1$, then $\exists$ sequence of open intervals U_n whose union U covers V^c such that $m^*(U) < 1$. Since U is open, U is a countable union of disjoint open intervals U'_n and also measurable. Then $1 > 1 - \epsilon > m(U) = \sum_n m(U'_n)$ for some $\epsilon > 0$. Then $\exists$ an open interval lying in U^c and hence in V, contradicting to (a) in Exercise 1.32 since an open interval must have positive measure. We conclude that $m^*(V^c) = 1$.

Next, suppose that $m^*(V) + m^*(V^c) = m^*(V \bigcup V^c) = 1$. Then $m^*(V) = 0$ and hence V is measurable, which is a contradiction to Exercise 1.31. Thus $m^*(V) + m^*(V^c) \neq m^*(V \bigcup V^c)$. ■

Exercise 1.34.
Let C_1, C_2 be the Cantor-like sets as with constant dissection ratios ξ_1, ξ_2 respectively. Prove that $\exists\, F : [0,1] \to [0,1]$ such that F is continuous, bijective, increasing, and $F(C_1) = C_2$.

Proof.

Note that we may write C_1 into $\bigcap_n C_1^n$ where C_1^n is the nth stage of the Cantor set. Then we define f sending any $x \in C_1$ to a sequence a_k such that $a_k = 0$ if x lies in the left interval in the kth stage and $a_k = 1$ otherwise. Then f is a bijection from C_1 to $\{0,1\}^{\mathbb{N}}$. Follow a similar construction, we obtain g from $\{0,1\}^{\mathbb{N}}$ to C_2 being another bijection. f, g are also continuous (by Exercise 1.2) and increasing under the order in $\{0,1\}^{\mathbb{N}}$ set to be the usual order of their summation $\sum_n 2^{-n} a_n$. For the rest disjoint open intervals, just map them through the piecewise linear function h connecting the endpoints. Finally,

$$F = \begin{cases} g \circ f^{-1} & \text{on } C_1 \\ h & \text{on } [0,1] \bigcap (C_1)^c \end{cases}$$

is the desired function. ∎

Exercise 1.35.
Find a measurable function f and a continuous function Φ such that $f \circ \Phi$ is not measurable. Furthermore, show that there exists a Lebesgue measurable set that is not a Borel set.

Proof.

Let $\Phi : C_1 \to C_2$ as in Exercise 1.34 with $m(C_1) > 0$ and $m(C_2) = 0$, $N \subset C_1$ be a non-measurable set by Exercise 1.32. Take $f = \chi_{\Phi(N)}$. Then

$$(f \circ \Phi)^{-1}((0,1]) = \Phi^{-1}(\Phi(N)) = N$$

is not a measurable set. Thus $f \circ \Phi$ is not measurable.Also, $\Phi(N)$ is a Lebesgue measurable set since it is a subset of C_2 and hence has measure zero. However, it is not a Borel set since $\Phi^{-1}(\Phi(N)) = N$ is not measurable. ∎

Exercise 1.36.

(a) Construct a measurable set $E \subset [0,1]$ such that for any nonempty open subinterval $I \subset [0,1]$, both $m(E \bigcap I)$ and $m(E^c \bigcap I)$ have positive measure.

(b) Let $f = \chi_E$. Show that if $g(x) = f(x)$ almost everywhere, then g is discontinuous everywhere.

Proof.

First, we claim that any nonempty open interval I contains a pair of disjoint closed sets S, T such that $m(S), m(T) > 0$ and $S^\circ, T^\circ = \varnothing$. Indeed, For $I = (a,b)$, we can split I into $(a, (a+b)/2)$ and $((a+b)/2, b)$. For each of them, we can build a fat Cantor set lying in the interval. Thus we have found S, T as required.

Now considering all open intervals in $[0,1]$ with rational center and radius. Since such intervals are countable, we can enumerate them as $\{I_n\}$. Pick a pair of disjoint closed sets with empty interiors S_1, T_1 in I_1. The choices of closed disjoint intervals are chosen inductively. Given $S_1, \ldots, S_n$ and $T_1, \ldots, T_n$, $I_{n+1} - S_1 - \cdots - S_n - T_1 - \cdots - T_n$, since the S_i and T_i are closed sets with empty interiors, so does their finite union. Thus we know that there is an open interval in $I_{n+1} - S_1 - \cdots - S_n - T_1 - \cdots - T_n$. Choose disjoint closed sets S_{n+1}, T_{n+1} with empty interiors from the interval. Set $E = \bigcup_n S_n$. Now for any open interval I, I must contain some I_n. Then $\exists\, S_{n+1} \subset I_n \subset I$ is a set with positive measure and hence $m(E \bigcap I) > 0$. Similarly, $T_{n+1} \subset E^c \bigcap I_n$ is also a set with positive measure and hence $m(E^c \bigcap I) > 0$.

For (b), suppose that g is continuous at x. Then for every open $V \subset \mathbb{R}$ containing $g(x)$, $g^{-1}(V)$ is open in $[0,1]$. Choosing an open interval $I \subset g^{-1}(V)$, $m(E \bigcap I) > 0$ and $m(E^c \bigcap I) > 0$. Then $f(I) = \{0,1\}$ since $f = g$ almost everywhere. This means that if we pick x such that $g(x) = f(x)$, then V contains only points $\{0,1\}$, which is absurd. Thus g must be discontinuous everywhere. ∎

Exercise 1.37.
Let Γ be the graph of a continuous function $f : \mathbb{R} \to \mathbb{R}$. Show that Γ has measure zero in $\mathbb{R}^2$.

Proof.

We start by considering the graph of f on $[k, k+1]$, $k \in \mathbb{Z}$. Fixing $k \in \mathbb{Z}$, since $f \in C([k, k+1])$, f is L-Lipschitz for some $L > 0$. Consider sequences of rectangle covers R_n^i as follows. For each n, let

$$R_n^i = [k + \frac{i-1}{n}, k + \frac{i}{n}] \times [f(k + \frac{i-1}{n}) - \frac{L}{n}, f(k + \frac{i-1}{n}) + \frac{L}{n}], \quad i = 1, \ldots, n.$$

Then $m(R_n^i) = \frac{2L}{n^2}$. For given n, R_n^i are almost disjoint and hence $m(\bigcup_i R_n^i) = \frac{2L}{n}$. Letting $n \to \infty$, we obtain that the graph of $f|_{[k,k+1]}$ has measure zero. Since Γ is the countable union of the graphs of $f|_{[k,k+1]}$, $k \in \mathbb{Z}$, Γ has measure zero. ∎

Exercise 1.38.
Let $a, b \geq 0$. Prove that $(a+b)^\gamma \geq a^\gamma + b^\gamma$ if $\gamma \geq 1$ and $(a+b)^\gamma \leq a^\gamma + b^\gamma$ if $0 \leq \gamma \leq 1$. ($a, b > 0$ as $\gamma = 0$)

Proof.

Let $\gamma \geq 0$. Notice that $(a+t)^{\gamma-1} - t^{\gamma-1} \geq 0$. Then

$$0 \leq \int_0^b (a+t)^{\gamma-1} - t^{\gamma-1}\, \mathrm{d}t = \frac{1}{\gamma}((a+t)^\gamma - t^\gamma)\,\Big|_0^b = \frac{1}{\gamma}((a+b)^\gamma - b^\gamma - a^\gamma).$$

Rearranging the terms yields the desired result. The case for reverse inequality is similar. ∎

Exercise 1.39 (A-G inequality).
Prove that $\frac{1}{n}\sum_{i=1}^n x_i \geq \left(\prod_{i=1}^n x_i\right)^{1/n}$ *for all* $x_i \geq 0, i = 1, \ldots, n$.

Proof.

Since log is a concave function, by Jensen's inequality,

$$\log\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \geq \frac{1}{n}\sum_{i=1}^n \log x_i = \frac{1}{n}\log\left(\prod_{i=1}^n x_i\right).$$

Taking exponential on both sides establishes the result. Note that the above proof is valid for all $x_i > 0$; however, the case where some of $x_i = 0$ is trivial since the right hand side of the inequality is 0. ∎

Problem 1.1.
Given an irrational x*, show that there exists infinitely many fractions* $\frac{p}{q}$ *where* p, q *are coprime and* $\left|x - \frac{p}{q}\right| \leq \frac{1}{q^2}$*. In spite of this, prove that the set of those* $x \in \mathbb{R}$ *such that ther exist infinitely many fractions* $\frac{p}{q}$*, where* p, q *are coprime and* $\left|x - \frac{p}{q}\right| \leq \frac{1}{q^{2+\epsilon}}$ *for some small* $\epsilon > 0$*, has measure zero.*

Proof.

Let x be irrational. For $m = 1, \ldots, N+1$, choose n_m to be the greatest integer such that $n_m \leq mx$. Then $mx - n_m \in (0,1)$ since x is irrational. Next, devide $(0,1)$ into N intervals, each of length $1/N$. Since our choice of m produces $N+1$ $mx - n_m \in (0,1)$, by pigeon-hole principle, there is an interval containing two numbers of the form $mx - n_m$. Then

$$\left|(m - m')x - (n_m - n_{m'})\right| = \left|mx - n_m - (m'x - n_{m'})\right| \leq \frac{1}{N}.$$

Without loss of generality, we may assume that $m > m'$. Set $q = m - m'$ and $p = n_m - n_{m'}$. Then $1 \leq q \leq N$ and $\left|x - \frac{p}{q}\right| \leq \frac{1}{qN} \leq \frac{1}{q^2}$. Our choice of N is arbitrary and hence there are infinitely many such fractions.

To see the second part, for each $q \in \mathbb{N}$, let

$$E_q = \bigcup_{p \in \mathbb{Z}, (p,q)=1} \left\{ x \;\middle|\; \left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}} \right\}.$$

Then each E_q is a union of intervals of length $\frac{2}{q^{2+\epsilon}}$. Note that for every given q, there are at most $2q$ integers p such that $(p,q) = 1$. Thus $m(E_q) \leq \frac{4q}{q^{2+\epsilon}} = \frac{4}{q^{1+\epsilon}}$. Then $\sum_q m(E_q) < \infty$.

The set of x such that there exist infinitely many fractions $\frac{p}{q}$, where p, q are coprime is $\limsup_q E_q$ and hence has measure zero by Exercise 1.16. ∎

Problem 1.2.
Prove that every open set $\Omega \subset \mathbb{R}^d$ can be written as an union of almost disjoint closed cubes Q_i with the property: $\exists\, c_1, c_2 > 0$ such that $c_1 l(Q_i) \leq d(Q_i, \Omega^c) \leq c_2 l(Q_i)$ for all i, where $l(Q_i)$ is the side length of Q_i.

Proof.
We find the cubes through the algorithm as follows. First, consider the closed cubes generated by lattice $\mathbb{Z}^d$. Collect those cubes lying in Ω and denote them as Q_1^k. For the rest of cubes which intersect Ω, partition them into cubes of half the original side length. Collect those cubes lying in Ω and denote them as Q_2^k. Repeat the procedure. We obtain a sequence of collections of cubes Q_n^k. This sequence of cubes is clearly almost disjoint. This sequence of cubes also covers Ω since for any $x \in \Omega$, $\exists\, N$ such that $B_{2^{-N}}(x) \subset \Omega$. Then x is contained in some cube in $\left\{Q_n^k\right\}_{n=1}^{N+1}$. $\left\{Q_n^k\right\}$ forms a cover of Ω.

Next, we claim that $d(Q_n^k, \Omega^c)/l(Q_n^k)$ is bounded above. If $d(Q_n^k, \Omega^c) > \sqrt{d}\, l(Q_n^k)$, then Q_n^k is contained in a larger cube within Ω. This contradicts the construction of Q_n^k. Also, $d(Q_n^k, \Omega^c)/l(Q_n^k)$ is bounded below. This is because $d(Q_n^k, \Omega^c) > \epsilon > 0$ for some ϵ since Q_n^k is a compact set contained in Ω and Ω^c is an open set. Thus we have $d(Q_n^k, \Omega^c) > \epsilon 2^{-i+1} = \epsilon l(Q_n^k) > 0$ for some $\epsilon > 0$. This furnishes the proof. ∎

Problem 1.3.
Find an example of a measurable set $C \subset [0,1]$ such that $m(C) = 0$, yet the difference set of C contains a non-trivial interval centered at origin.

Proof.
Let C be the standard Cantor set. Then $m(C) = 0$. We shall prove that the difference set of C is exactly $[-1,1]$. First, write $C = \left\{\sum_k 3^{-k} a_k \;\middle|\; a_k \in \{0,2\}\right\}$ by Exercise 1.2. Then the difference set of C is $S = \left\{\sum_k 3^{-k} c_k \;\middle|\; c_k \in \{-2,0,2\}\right\}$. Given $x \in [-1,1]$, define $s_0 = 0$ and

$$s_{n+1} = \begin{cases} s_n + \frac{2}{3^n} & \text{if } s_n + \frac{2}{3^n} \text{ is closer to } x \text{ comparing to the other two,} \\ s_n & \text{if } s_n \text{ is closer to } x \text{ comparing to the other two,} \\ s_n - \frac{2}{3^n} & \text{if } s_n - \frac{2}{3^n} \text{ is closer to } x \text{ comparing to the other two.} \end{cases}$$

It is clear that $|s_n - x| \leq 3^{-n}$ and hence $s_n \to x$. Thus $x \in S$. This shows that $[-1,1] \subset S$. The converse is trivial. Thus $S = [-1,1]$. As a immediate consequence, S contains a non-trivial interval centered at origin. ∎

Problem 1.4.

Let $E \subset \mathbb{R}^d$ be a rectangle and $f : E \to \mathbb{R}$ be a function bounded by M. Prove that f is Riemann integrable if and only if the set of discontinuities of f has measure zero. [1]

Proof.

Define the oscillation of f at x as

$$\operatorname{osc}(f, x) = \inf_{U : x \in U} \operatorname{diam}(f(U)),$$

where U is open.

We first claim that f is continuous at x if and only if $\operatorname{osc}(f, x) = 0$. Indeed, if f is continuous at x, then $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $y \in B_\delta(x)$. Then $\operatorname{diam}(f(B_\delta(x))) \leq 2\epsilon$. Since ϵ is arbitrary, $\operatorname{osc}(f, x) = 0$. Conversely, if $\operatorname{osc}(f, x) = 0$, then $\forall\, \epsilon > 0$, $\exists$ open U containing x such that $\operatorname{diam}(f(U)) < \epsilon$. This implies that $|f(x) - f(y)| < \epsilon$ for all $y \in U$ and hence f is continuous at x.

Next, let D_ϵ collect all points x such that $\operatorname{osc}(f, x) \geq \epsilon > 0$. We claim that D_ϵ is closed. For any convergent sequence $x_k \in D_\epsilon$, let $x_k \to x$. For any open U containing x, $\exists\, N$ such that $x_k \in U$ for all $k \geq N$. Then $\exists$ an open neighborhood of x_N, U', such that $U' \subset U$ and $\operatorname{diam}(f(U')) \geq \epsilon$. Hence $\operatorname{osc}(f, x) \geq \epsilon$ and $x \in D_\epsilon$, showing that D_ϵ is closed. Observe that $D = \bigcup_{n=1}^\infty D_{1/n}$.

Now suppose that f is Riemann integrable. Then for any $\epsilon > 0$, $\exists\, \mathcal{P}$ such that $\mathrm{U}(f, \mathcal{P}) - \mathrm{L}(f, \mathcal{P}) < \frac{1}{n}$ and $\|\mathcal{P}\| < \frac{1}{n}$. Then

$$\begin{aligned} &\sum_{\substack{Q \in \mathcal{P}, \\ Q \bigcap D_{\frac{1}{n}} \neq \varnothing}} (\sup_Q f - \inf_Q f)\, |Q| + \sum_{\substack{Q \in \mathcal{P}, \\ Q \bigcap D_{\frac{1}{n}} = \varnothing}} (\sup_Q f - \inf_Q f)\, |Q| \\ = &\sum_{Q \in \mathcal{P}} (\sup_Q f - \inf_Q f)\, |Q| = \mathrm{U}(f, \mathcal{P}) - \mathrm{L}(f, \mathcal{P}) < \epsilon. \end{aligned}$$

Note that $\sup_Q f - \inf_Q f = \operatorname{diam}(f(Q))$. This gives that $2Mm^*(D_{\frac{1}{n}}) < \epsilon$ for every n. Since ϵ is arbitrary, we conclude that $m^*(D_{\frac{1}{n}}) = 0$ for each n. Thus D is an union of sets of measure zero and hence also has measure zero.

For the converse, suppose that $m(D) = 0$. Then D_ϵ also has measure zero. Let $\mathcal{P}$ be a

[1] In this problem, I prove a generalized version.

partition on E with $\|\mathcal{P}\| < \delta$ for some $\delta > 0$, which will be determined later. Then

$$\begin{aligned}
\mathrm{U}(f,\mathcal{P}) - \mathrm{L}(f,\mathcal{P}) &= \sum_{Q\in\mathcal{P}} (\sup_Q f - \inf_Q f)\,|Q| \\
&= \sum_{\substack{Q\in\mathcal{P},\\ Q\bigcap D_\epsilon=\varnothing}} (\sup_Q f - \inf_Q f)\,|Q| + \sum_{\substack{Q\in\mathcal{P},\\ Q\bigcap D_\epsilon\neq\varnothing}} (\sup_Q f - \inf_Q f)\,|Q|
\end{aligned}$$

For the first term, $\sup_Q f - \inf_Q f < \epsilon$ for $\|\mathcal{P}\| < \delta_1$ for some $\delta_1 > 0$. And thus the first term is bounded by $\epsilon m(E)$. For the second term, $\sup_Q f - \inf_Q f < 2M$ and since D_ϵ has measure zero, $\exists\, Q_k$ cubic cover of D_ϵ such that $\sum_k |Q_k| < \epsilon$. Now if $\operatorname{diam}(Q) < \delta_2$ for some $\delta_2 > 0$, then those Q intersecting D_ϵ nonempty are subset of $\bigcup_k Q_k$. Thus the second term is bounded by $2M\epsilon$. Choosing $\delta = \min\{\delta_1, \delta_2\}$ yields that

$$\mathrm{U}(f,\mathcal{P}) - \mathrm{L}(f,\mathcal{P}) < \epsilon m(E) + 2M\epsilon$$

whenever $\|\mathcal{P}\| < \delta$. Since ϵ is arbitrary, f is Riemann integrable. ■

Problem 1.5.

Suppose E is measurable with $m(E) < \infty$, $E = E_1 \bigcup E_2$ and $E_1 \bigcap E_2 = \varnothing$. Prove that if $m^*(E_1) + m^*(E_2) = m(E)$, then E_1 and E_2 are measurable.

Proof.

Suppose not. Then $\exists\,\epsilon > 0$ such that $m^*(V_1 - E_1) > \epsilon$ and $m^*(V_2 - E_2) > \epsilon$ for any open sets V_1, V_2 covering E_1 and E_2 respectively. Then $m(V_1 \bigcup V_2 - E) = m(V_1 \bigcup V_2) - m(E) = m(V_1) - m^*(E_1) + m(V_2) - m^*(E_2) = m^*(V_1 - E_1) + m^*(V_2 - E_2) > 2\epsilon$ by assumption. It remains to show that for every open V covering E, $V \supset V_1 \bigcup V_2$ for some open V_1, V_2 covering E_1, E_2 respectively. Indeed, one may simply consider $V_1 = \{x \mid d(x, E_1) < 1/n\} \bigcap V$ and $V_2 = \{x \mid d(x, E_2) < 1/n\} \bigcap V$ for some $n \in \mathbb{N}$. Thus we obtain a contradiction since E is measurable. We conclude that E_1 and E_2 are measurable. ■

Definition.

*A set E is said to be **well-ordered** with respect to a binary relation $\leq$ if for every nonempty subset A of E, $\exists\, a \in A$ such that $a \leq b$ for all $b \in A$.*

Definition.

*Given a nonempty partially ordered set $(E, \leq)$, $A \subset E$ is a **maximal linearly ordered subset** of E if for any B such that $A \subset B \subset E$, B is not linearly ordered.*

Definition.

*If $\mathcal{F}$ is a family of sets and $\mathcal{C} \subset \mathcal{F}$, we call $\mathcal{C}$ a **subchain** of $\mathcal{F}$ if for any $A, B \in \mathcal{C}$, $A \subset B$ or $B \subset A$.*

Problem 1.6.
Prove that the following are equivalent.

(a) ***Axiom of choice****: $\exists$ a choice function $f : \mathcal{P}(X) \setminus \{\varnothing\} \to A$, with $A \mapsto f(A) \in A$ for any $A \in \mathcal{P}(X) \setminus \varnothing$.*

(b) ***Well-ordering principle****: Every set can be well-ordered.*

(c) ***Hausdorff maximal principle****: Every nonempty partially ordered set has a maximal linearly ordered subset.*

(d) ***Zorn's lemma****: Let P be partially ordered. If every linearly ordered subset of P has an upper bound, then P has a maximal element.*

Proof.

We start by proving that (a)$\Rightarrow$(c)[2]. We begin with the following claim.

Claim: Suppose $\mathcal{F} \subset \mathcal{P}(X)$ is a nonempty collection of sets such that the union of every subchain of $\mathcal{F}$ belongs to $\mathcal{F}$. Suppose g is a function which associates to each $A \in \mathcal{F}$ a set $g(A) \in \mathcal{F}$ such that $A \subset g(A)$ and $g(A) - A$ consists of at most one element. Then $\exists\, A \in \mathcal{F}$ such that $g(A) = A$.

Fix $A_0 \in \mathcal{F}$. Call a subcollection $\mathcal{F}'$ of $\mathcal{F}$ a **tower** if (i) $A_0 \in \mathcal{F}'$, (ii) the union of every subchain of $\mathcal{F}'$ belongs to $\mathcal{F}'$, and (iii) $g(A) \in \mathcal{F}'$ for any $A \in \mathcal{F}'$. Then it is clear that the family of all towers is nonempty for if $\mathcal{F}_1$ is the collection of all $A \in \mathcal{F}$ such that $A_0 \subset A$, then $\mathcal{F}_1$ is a tower. Let $\mathcal{F}_0$ be the intersection of all towers. Then $\mathcal{F}_0$ is a minimal tower in the sense that any proper subcollection of $\mathcal{F}_0$ is not a tower. Also, $A_0 \subset A$ for any $A \in \mathcal{F}_0$. We now show that $\mathcal{F}_0$ is a subchain of $\mathcal{F}$.

Let Γ collect all $C \in \mathcal{F}_0$ such that for every $A \in \mathcal{F}_0$, $A \subset C$ or $C \subset A$. For each $C \in \Gamma$, let $\Phi(C)$ be the collection of all $A \in \mathcal{F}_0$ such that either $A \subset C$ or $g(C) \subset A$. Fix $C \in \Gamma$ and $A \in \Phi(C)$. It is clear that (i) and (ii) hold for Γ and $\Phi(C)$. It remains to show that $g(A) \in \Phi(C)$. If $A \in \Phi(C)$, then there are three cases: $A \subsetneq C$, $A = C$, or $g(C) \subset A$. In the case $A \subsetneq C$, C cannot be a proper subset of $g(A)$, otherwise $g(A) - A$ would contain at least two elements; since $C \in \Gamma$, $g(A) \subset C$. In the case $A = C$, $g(A) = g(C) \in \Phi(C)$. In the case $g(C) \subset A$, $g(C) \subset A \subset g(A)$ and hence $g(A) \in \Phi(C)$. Thus $\Phi(C)$ is a tower. Then the minimality of $\mathcal{F}_0$ implies that $\Phi(C) = \mathcal{F}_0$ for every $C \in \Gamma$. Now if $A \in \mathcal{F}_0$ and $C \in \Gamma$, then either $A \subset C$ or $g(C) \subset A$. This implies that $g(C) \in \Gamma$. Thus Γ is a tower and the minimality of $\mathcal{F}_0$ again implies that $\Gamma = \mathcal{F}_0$. We now see that $\mathcal{F}_0$ is linearly ordered by the definition of Γ and thus a subchain of $\mathcal{F}$.

[2] The proof is adapted from the appendix in Rudin's *Real and Complex Analysis*.

Lastly, to show that $\exists\, A \in \mathcal{F}$ such that $g(A) = A$, let A be the union of $\mathcal{F}_0$. By (ii) and (iii), $A \in \mathcal{F}_0$ and $g(A) \in \mathcal{F}_0$. Since A is the largest member in $\mathcal{F}_0$ and $A \subset g(A)$, we have $A = g(A)$. The claim is furnished.

Now we proceed to prove that (a)⇒(c). Let $\mathcal{F}$ be the collection of all linearly ordered subsets of the partially ordered set P. Since every subset of P consisting of exactly one element is linearly ordered, $\mathcal{F}$ is nonempty. Let g be a choice function on $\mathcal{F}$. Note that the union of any chain of linearly ordered sets is linearly ordered. Now let f be a choice function for P. If $A \in \mathcal{F}$, let A^* be the set of all x in the complement of A such that $A \bigcup \{x\} \in \mathcal{F}$. If $A^* \neq \varnothing$, put $g(A) = A \bigcup \{f(A^*)\}$. If $A^* = \varnothing$, put $g(A) = A$. By the claim, $A^* = \varnothing$ for at least one $A \in \mathcal{F}$, and such A is the desired maximal linearly ordered subset.

Next, (c)⇒(d) since an upper bound of a linearly ordered subset is a maximal element of P.

For (d)⇒(b)[3], let E be a set and $\mathcal{A}$ collect all well-orderings of subsets of E. We define a partial order $\prec$ on $\mathcal{A}$: If $\leq_1$ and $\leq_2$ are well-orderings on the subsets E_1 and E_2, then $\leq_1 \prec \leq_2$ if (i) $E_1 \subset E_2$ and $\leq_1, \leq_2$ agree on E_1, and (ii) if $x \in E_2 - E_1$, then $y \leq_2 x$ for all $y \in E_1$. Now every linearly ordered subcollect of $\mathcal{A}$ has an upper bound by taking the union of the subcollection. Applying Zorn's lemma, we obtain a maximal element $\leq$ in $\mathcal{A}$. This maximal element must be a well-ordering on E since if $\leq$ is only a well-ordering on $C \subsetneq E$, and $x \in E - C$, then $\leq$ can be extended to a well-ordering on $C \bigcup \{x\}$, contradicting the maximality of $\leq$.

Finally, to prove that (b)⇒(a), let X be a set, $A \subset X$, and define $f(A)$ be the unique minimal element of A if $A \neq \varnothing$. f is the desired choice function. ■

Problem 1.7.
Let $f \in C^2([0,1])$ and $\Gamma \subset \mathbb{R}^2$ be the graph of f on $[0,1]$. Prove that the following are equivalent.

(a) $m(\Gamma + \Gamma) > 0$.

(b) $\Gamma + \Gamma$ *contains a nonempty open set.*

(c) f *is not linear.*

Proof.

(b)⇒(a) is trivial since every nonempty open set has positive measure.

[3] The proof is from Folland's *Real Analysis: Modern Techniques and Their Applications*.

(a)⇒(c): Suppose f is linear. Then $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. Then

$$\begin{aligned}\Gamma + \Gamma &= \{(x+y, f(x)+f(y)) \mid x, y \in [0,1]\} \\ &= \{(x+y, a(x+y)+2b) \mid x, y \in [0,1]\} \\ &= \{(z, az+2b) \mid z \in [0,2]\}\end{aligned}$$

is measure zero since it is a line segment in $\mathbb{R}^2$.

(c)⇒(b): Suppose f is not linear. Then we may assume that $f'' \neq 0$ has same sign on $(a, b) \subset [0,1]$. Define $\phi(x, y) = (x+y, f(x)+f(y))$. Then

$$D\phi = \begin{pmatrix} 1 & 1 \\ f'(x) & f'(y) \end{pmatrix}$$

is invertible if $x \neq y$. By the inverse function theorem, $\exists$ open sets U, V such that $\phi(U) = V$ is invertible, $U \subset (a,b)^2 - \{(x,y) \mid x = y\}$ and V is open in $\mathbb{R}^2$. Thus $\Gamma + \Gamma$ contains a nonempty open set V. ∎

Definition.
Two sets $A, B \subset \mathbb{R}^d$ are said to be ***similar****, denoted by $A \sim B$, if $\exists\, \delta > 0$ and $a \in \mathbb{R}^d$ such that $A = a + \delta B$.*

Problem 1.8.
Suppose A, B are open sets of finite positive measure in $\mathbb{R}^d$. Then the equality in Brunn-Minkowski inequality holds if and only if A, B are convex and $A \sim B$.

Proof.

Suppose A, B are convex and $A \sim B$. Since A, B are open, $A + B$ is open and also measurable. By the assumption, $\exists\, \delta > 0$ and $a \in \mathbb{R}^d$ such that $A = a + \delta B$. Then $m(A + B) = m((a + \delta B) + B) = m((1+\delta)B) = (1+\delta)^d m(B)$ by the convexity. Thus $m(A + B)^{1/d} = (1+\delta) m(B)^{1/d} = m(B)^{1/d} + m(a + \delta B)^{1/d} = m(B)^{1/d} + m(A)^{1/d}$.

The converse is actually false. One may simply take $A = B = (-1, 0) \bigcup (0, 1)$. Then $m(A + B) = m((-1,1) + (-1,1)) = m((-2,2)) = 4 = 2 + 2 = m(A) + m(B)$. However, A, B are not convex. ∎

2. Integration Theory

Exercise 2.1.
Given a collection of sets $F_1, F_2, \ldots, F_n$, construct $F_1^*, F_2^*, \ldots, F_N^*$ with $N = 2^n - 1$ such that $\bigcup_{i=1}^N F_i^* = \bigcup_{i=k}^n F_k$; F_i^* are disjoint; and $F_k = \bigcup_{F_i^* \subset F_k} F_i^*$ for every k.

Proof.

Let I be a combinatorics of $\{1, \ldots, n\}$. For example, $I = \{1, 3, 4\}$ is a combinatorics. Note that although the length of I is undetermined, the length of I cannot be 0. Now let $\{F_i^*\}$ be the collection of intersections of F_i for $i \in I$ and F_j^c for $j \notin I$, for every given I. Then it is clear that F_i^* are disjoint, and $F_k = \bigcup_{F_i^* \subset F_k} F_i^*$ since for every x in F_k, x must be in some F_i for $i \in I$, where I is over every possible combinatorics containing the one induces F_k. ∎

Exercise 2.2.
Prove that if f is integrable on $\mathbb{R}^d$ and $\delta > 0$, then $f(\delta x) \to f(x)$ in L^1 as $\delta \to 1$.

Proof.

Given $\epsilon > 0$, by theorem 2.4, we know that there exists continuous functions $g \in L^1$ with compact support such that $\|f - g\| < \epsilon/3$. Then $\|f(\delta x) - g(\delta x)\| < \epsilon/3$ as well. Now since g is continuous on compact support, we may assume that g is bounded by M and $\operatorname{supp}(g) \subset I$ where I has finite positive measure S. Then there exists δ such that $|g(x) - g(\delta x)| < \epsilon/3MS$ for some δ sufficiently close to 1. Hence

$$\|g(x) - g(\delta x)\| = \int_{\mathbb{R}^d} |g(x) - g(\delta x)| < \epsilon/3.$$

Then $\|f(x) - f(\delta x)\| \leq \|f(x) - g(x)\| + \|g(x) - g(\delta x)\| + \|f(\delta x) - g(\delta x)\| < \epsilon$. This completes the proof. ∎

Exercise 2.3.
Suppose f is integrable on $(\pi, \pi]$ and extended to $\mathbb{R}$ periodically. Show that

$$\int_{-\pi}^{\pi} f = \int_I f$$

for any interval I of length 2π.

Proof.

We may consider the case where I is contained in two consecutive intervals I_1, I_2 where $I_1 = (k\pi, (k+2)\pi]$ and $I_2 = ((k+2)\pi, (k+4)\pi]$ for some $k \in \mathbb{Z}$. Then

$$\int_I f = \int_{I\cap I_1} f + \int_{I\cap I_2} f = \int_{-(k+1)\pi + I\cap I_1} f + \int_{-(k+1)\pi + I\cap I_2} f = \int_{-\pi}^{\pi} f.$$

The second equality follows from the fact that the measure is invariance under translation and the fact that f is periodic. The third equality follows from the fact that I_1, I_2 are disjoint and thus the length of the two intervals must sum up to 2π. ∎

Exercise 2.4.

Suppose f is integrable on $[0, b]$ and

$$g(x) = \int_x^b \frac{f(t)}{t} dt$$

for $0 < x \leq b$. Show that g is integrable on $[0, b]$ and

$$\int_0^b g(x)dx = \int_0^b f(t)dt.$$

Proof.

A. Appendix

The following are some auxiliary results used in the proofs of the exercises. Most of them are standard results in analysis while have not been encountered in my previous studies. In order to enhance the completeness of the solutions, I have included them in this appendix.

Theorem A.1 (Baire Category Theorem).
Let X be a complete metric space. Then the intersection of countably many dense open sets in X is dense in X.

Proof.

Let G_n be dense open sets in X. We first claim that $\nexists$ perfect F such that G_n^c is dense in F. Suppose not, then we can pick $x \in G_n \bigcap F$ such that $\forall\, B_\epsilon(x)$, $B_\epsilon(x) \bigcap G_n^c \neq \varnothing$ since G_n^c is dense in F. But $B_\epsilon(x) \subset G_n$ for some ϵ, posing a contradiction.

Now pick some perfect E. Since G_n^c is not dense in E, $\exists\, x_1 \in G_1 \bigcap E$. Then $\exists\, \epsilon_1 > 0$ such that $\overline{B_{\epsilon_1}(x_1)} \subset G_1$ and $\overline{B_{\epsilon_1}(x_1)}$ is perfect. Since G_2^c is not dense in E, $\exists\, x_2 \in G_2 \bigcap \overline{B_{\epsilon_1}(x_1)}$. Then $\exists\, \epsilon_2 = \min\{\epsilon_1/2, \delta_2\}$, where δ_2 is a small number such that $\overline{B_{\delta_2}} \subset G_2$. The procedure continues and we obtain a sequence that $\overline{B_{\epsilon_1}(x_1)} \supset \overline{B_{\epsilon_2}(x_2)} \supset \cdots$, which are closed, bounded and nonempty with $\lim_n \operatorname{diam}\left(\overline{B_{\epsilon_n}(x_n)}\right) \le \lim_n 2\epsilon_1 2^{-n+1} = 0$.

We now make the second claim: For any sequence of closed and bounded sets in X with $E_n \supset E_{n+1}$ and $\lim_n \operatorname{diam}(E_n) = 0$, $\bigcap_n E_n$ consists of exactly one point. To see this, suppose $x \neq y$ are distinct points in $\bigcap_n E_n = \lim_n E_n$. Since x and y are distinct, $d(x,y) > 0$ and $\operatorname{diam}(E_n) = \sup_{x',y' \in E_n} d(x',y') \ge d(x,y)$. Let $n \to \infty$, then $\lim_n \operatorname{diam}(E_n) \ge d(x,y) > 0$, which is a contradiction.

By the second claim, $\bigcap_n G_n \supset \bigcap_n \overline{B_{\epsilon_n}(x_n)} \neq \varnothing$. Futhermore, the choice of x_1 is dense in X. This furnishes the proof. ∎

Definition.
A topological space $(X, \mathcal{T})$ is said to be ***normal*** *if for every pair of disjoint closed sets C and D in X, there exist disjoint open sets U and V such that $C \subset U$ and $D \subset V$.*

Lemma A.2 (Urysohn's Lemma).
A topological space $(X, \mathcal{T})$ is normal if and only if for every pair of nonempty disjoint closed sets C and D in X, there is a continuous function $f : X \to [0,1]$ such that $f(C) = \{0\}$ and $f(D) = \{1\}$.

Proof.

We shall first prove the necessity. Let C, D be disjoint closed sets in X and f be the function as in the statement of the lemma. Then $C \subset f^{-1}([0, 1/2))$ and $D \subset f^{-1}((1/2, 1])$. Since f is continuous, the preimages are open. This proves the necessity.

The sufficiency is more complicated. We shall first construct a collection of open sets $\{U_q \mid q \in Q\}$ satisfying that for any $p < q$, $\overline{U_p} \subset U_q$, where $Q = \mathbb{Q} \bigcap [0, 1]$.

To begin, let $U_1 = X - D$ and by the normality of X, pick an open U_0 such that $C \subset U_0 \subset \overline{U_0} \subset U_1$. Note that the desired property holds. Now suppose that U_p has been constructed for $p \in \{p_1, \ldots, p_n\} = P_n$, we are now to construct U_{p_n+1} for p_{n+1}. Since P_n is finite, we may find $q, r \in Q$ such that p_{n+1} is the only numbers in P_n that $q < p_{n+1} < r$. By the inductive hypothesis, we have $\overline{U_q} \subset U_r$. Again, by the normality, we may choose an open $U_{p_{n+1}}$ such that $\overline{U_q} \subset U_{p_{n+1}} \subset \overline{U_{p_{n+1}}} \subset U_r$. With this, the collection of open sets still preserves the desired property that $p < q$ implies $\overline{U_p} \subset U_q$. As the process proceeds, we obtain a collection of open sets $\{U_q \mid q \in Q\}$ preserving the desired property.

Next, we extend the index set of our collection to $\mathbb{Q}$ by defining $U_q = \varnothing$ if $q < 0$ in $\mathbb{Q}$ and $U_q = X$ if $q > 1$. With the extension, we still have the desired property that $p < q$ implies $\overline{U_p} \subset U_q$ for $\{U_q \mid q \in \mathbb{Q}\}$.

For each $x \in X$, we define $\mathbb{Q}(x) = \{q \in \mathbb{Q} \mid x \in U_q\}$. Observe that $\mathbb{Q}(x)$ is nonempty since $x \in U_q = X$ for any $q > 1$ and also $\mathbb{Q}(x)$ is bounded below by 0 since if $q < 0$, then $x \notin U_q = \varnothing$. Remark that for each $x \in X$, $\mathbb{Q}(x)$ contains every rational numbers greater than 1 and some lies between 0 and 1.

By the axiom of completeness, we can define the function $f : X \to [0, 1]$ such that $f(x) = \inf \mathbb{Q}(x)$. Such function is well-defined since $\mathbb{Q}(x)$ is bounded below by 0 and above by 1. It remains to show that f is continuous and seperates C and D.

The first claim is that $f(C) = \{0\}$ and $f(D) = \{1\}$. Indeed, if $x \in C$, then $x \in U_0 \subset U_p$ for all $p > 0$. Thus, $\inf \mathbb{Q}(x) \leq 0$. This gives $f(x) = 0$. If $x \in D$, then $x \notin U_1$ and hence $\inf \mathbb{Q}(x) \geq 1$. Then $f(x) = 1$. Hence we conclude that $f(C) = \{0\}$ and $f(D) = \{1\}$. The claim further implies that f separates C and D.

Before claiming the continuity of f, we shall observe that for any $x \in \overline{U_q}$, $f(x) \leq q$ by the construction that $\overline{U_q} \subset U_r$ for any $r > 0$. Also, for any $x \notin U_q$, $f(x) \geq q$ since $x \notin U_p$ for any $p < q$. With these observations, we are now ready to show the continuity of f.

For any open interval $U = (a, b)$ that lies in $[0, 1]$, we need to show that $f^{-1}(U)$ is open in X. Fixing any $x \in f^{-1}(U)$, we can find rational numbers p, q such that $a < p < f(x) < q < b$. $p < f(x)$ implies that $x \notin \overline{U_p}$; otherwise, $f(x) \geq p$, forming a contradiction. By similar argument, $x \in U_q$. As a result, $x \in U_q - \overline{U_p}$. Let the open set be $V = U_q - \overline{U_p}$.

To show that $f(V) \subset U$, let $y \in V$. Then $y \in U_q \subset \overline{U_q}$ and $f(y) \leq q < b$. Also, $y \notin \overline{U_p} \supset U_p$ and hence $f(y) \geq p > a$. Thus $f(y) \in [p,q] \subset U$. Therefore f is indeed continuous. The proof is now complete. ■

Definition.
A topological space X is said to be a ***Hausdorff space*** *if for any pair of distinct points x and y in X, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.*

Lemma A.3.
Let A and Y be closed subspaces of a normal Hausdorff space X and let U be an open neighbourhood of Y in X. Assume that $C \subset A$ is a closed neighbourhood in A of $Y - A$, contained in $U - A$. Then there exists a closed neighbourhood Z of Y, contained in U, such that $Z - A$ equals C.[4]

Proof.
The case $A = \varnothing$ is elementary and well known; applying it to the situation at hand we obtain first a closed neighbourhood Z''' of Y contained in U. It will suffice to construct (from Z''') a closed neighbourhood Z' of Y contained in U such that $Z' \bigcap A$ is contained in C, since then $Z := Z' \bigcup C$ will have all the properties stated.

Let D be the closure of $A - C$. Since C is assumed to be a neighbourhood of $Y \bigcap A$ in A, the closed sets Y and D are disjoint. Hence Y has a closed neighbourhood Z'' disjoint from D. This implies $Z'' \bigcap A \subset C$, and $Z' := Z'' \bigcap Z'''$ is a closed neighbourhood of Y as required. ■

Theorem A.4 (Tietze Extension Theorem).
Let X be a normal Hausdorff space and A be a closed subset of X. If $f : A \to \mathbb{R}$ is a continuous function, then f can be extended to a continuous function $F : X \to \mathbb{R}$. That is, $F|_A = f$.

Proof.
For $r \in [0,1]$, let $A(r) = \{x \in A \mid f(x) \leq r\}$. Put $B = \{r \in [0,1] \mid 2^n r \text{ is integral for some } n\}$. For $r \in B$ we construct, by induction on the exponent of 2 in the denominator of r, closed subsets $X(r) \subset X$ such that the following two conditions hold:

(a) $X(r) \cap A = A(r)$,

(b) $X(s)$ is a neighbourhood of $X(r)$ if $r < s$.

We may take $X(0) = A(0)$ and $X(1) = X$. Assume that $X(r)$ with the properties above have been constructed for all $r \in (0,1)$ such that $2^n r$ is an integral. Since $X(2^{-n}(i +$

[4]The proof of this lemma and the Tietze Extension Theorem are taken from https://link.springer.com/content/pdf/10.1007/s000130050272.pdf.

1)) is a neighborhood of $X(2^{-n}i)$, there is, by the above lemma, a closed neighborhood $X(2^{-n-1}(2i+1))$ of $X(2^{-n}i)$ within the interior of $X(2^{-n}(i+1))$, having the prescribed intersection $X(2^{-n-1}(2i+1)) \cap A = A(2^{-n-1}2i+1)$.

The rest of the proof follows the classical Urysohn argument that we presented in Lemma A.2:

$$g(x) := \inf\{r \in B \mid x \in X(r)\}$$

defines a continuous function on X, extending f. Continuity follows simply from the fact that, for $r, s \in B$ with $r < s$, this function has values greater than r and less than s on the open set $X^\circ(s) - X(r)$.

Finally, we may decompose f into $g : X \to (0,1)$ and a homeomorphism $h : (0,1) \to \mathbb{R}$. Our previous result shows that $\exists$ continuous extension $G : X \to [0,1]$. Since $G^{-1}(\{0,1\})$ is closed and disjoint from A, by Lemma A.2, the proof is finished. ■