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Math189R SU17
Homework 5
Monday, June 5, 2017

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

1 (Murphy 12.5 - Deriving the Residual Error for PCA) It may be helpful to reference section 12.2.2 of Murphy.

(a) Prove that

$$\left\| \mathbf{x}_i - \sum_{j=1}^{k} z_{ij} \mathbf{v}_j \right\|^2 = \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j.$$

Hint: first consider the case when $k = 2$. Use the fact that $\mathbf{v}_j^\top \mathbf{v}_j$ is 1 if $i = j$ and 0 otherwise. Recall that $z_{ij} = \mathbf{x}_i^\top \mathbf{v}_j$.

(b) Now show that

$$J_k = \frac{1}{n} \sum_{i=1}^{n} \left(\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j \right) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \lambda_j.$$

Hint: recall that $\mathbf{v}_j^\top \boldsymbol{\Sigma} \mathbf{v}_j = \lambda_j \mathbf{v}_j^\top \mathbf{v}_j = \lambda_j$.

(c) If $k = d$ there is no truncation, so $J_d = 0$. Use this to show that the error from only using $k < d$ terms is given by

$$J_k = \sum_{j=k+1}^{d} \lambda_j.$$

Hint: partition the sum $\sum_{j=1}^{d} \lambda_j$ into $\sum_{j=1}^{k} \lambda_j$ and $\sum_{j=k+1}^{d} \lambda_j$.

a.) $\| x_i - \sum_{j=1}^{k} z_{ij} v_j \|^2 = (x_i - \sum z_{ij} v_j)^T (x_i - \sum z_{ij} v_j) = x_i^T x_i - x_i^T \sum z_{ij} v_j - (\sum z_{ij} v_j)^T x_i + (\sum z_{ij} v_j)^T (\sum z_{ij} v_j)$

$= x_i^T x_i - 2 x_i^T \sum z_{ij} v_j + \sum v_j^T z_{ij}^T z_{ij} v_j = x_i^T x_i - 2 x^T \sum z_{ij} v_j + \sum v_j^T x_i x_i^T v_j$

($v_j^T x_i v^T v x^T v_j$; $x_i^T v_j = z_{ij}$)

■

$= x_i^T x_i - 2 x_i^T \sum z_{ij} v_j + x_i \sum v_j v_j^T$ 1

$= x_i^T x_i - x_i^T \sum z_{ij} v_j = \boxed{x_i^T x_i - \sum_{j=1}^{k} v_j x_i x_i^T v_j}$

b.) $J_K = \frac{1}{n}\sum_{i=1}^{n}\left(x_i^T x_i - \sum_{j=1}^{K} v_j^T x_i x_i^T v_j\right)$

$= \frac{1}{n}\sum_{i}^{n} x_i^T x_i - \frac{1}{n}\sum_{i}^{n}\sum_{j}^{K} v_j^T x_i x_i^T v_j$

$= \frac{1}{n}\sum_{i}^{n} x_i^T x_i - \sum_{j}^{K} v_j^T \underbrace{\left(\frac{1}{n}\sum_{i}^{n} x_i x_i^T\right)}_{\text{This is the covariance matrix } \Sigma} v_j$

$= \frac{1}{n}\sum_{i}^{n} x_i^T x_i - \sum_{j}^{K} \underbrace{v_j^T \Sigma v_j}_{\lambda_j} = \boxed{\frac{1}{n}\sum_{i=1}^{n} x_i^T x_i - \sum_{j=1}^{K}\lambda_j}$

c.) $J_K = \frac{1}{n}\sum_{i}^{n} x_i^T x_i - \sum_{j=1}^{K=d}\lambda_j$

$J_K = \frac{1}{n}\sum_{i}^{n} x_i^T x_i - \sum_{j}^{d}\lambda_j + \sum_{j=K+1}^{d}\lambda_j$

$J_d = 0 = \frac{1}{n}\sum_{i}^{n} x_i^T x_i - \sum_{j=1}^{K=d}\lambda_j$

$\sum_{j=1}^{K=d}\lambda_j = \frac{1}{n}\sum_{i=1}^{n} x_i^T x_i$

$J_K = \cancel{\frac{1}{n}\sum_{i}^{n} x_i^T x_i} - \cancel{\frac{1}{n}\sum_{i}^{n} x_i^T x_i} + \sum_{j=K+1}^{d}\lambda_j \rightarrow \boxed{J_K = \sum_{j=K+1}^{d}\lambda_j}$ ✓

2 (ℓ_1-Regularization) Consider the ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$:

$$\|\mathbf{x}\|_1 = \sum_i |\mathbf{x}_i|.$$

Draw the norm-ball $B_k = \{\mathbf{x} : \|\mathbf{x}\|_1 \leq k\}$ for $k = 1$. On the same graph, draw the Euclidean norm-ball $A_k = \{\mathbf{x} : \|\mathbf{x}\|_2 \leq k\}$ for $k = 1$ behind the first plot. (Do not need to write any code, draw the graph by hand).

Show that the optimization problem

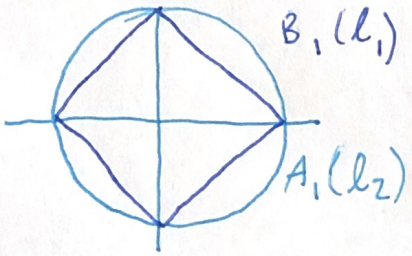
$$\begin{aligned} &\text{minimize: } f(\mathbf{x}) \\ &\text{subj. to: } \|\mathbf{x}\|_p \leq k \end{aligned}$$

is equivalent to

$$\text{minimize: } f(\mathbf{x}) + \lambda\|\mathbf{x}\|_p$$

(hint: create the Lagrangian). With this knowledge, and the plots given above, argue why using ℓ_1 regularization (adding a $\lambda\|\mathbf{x}\|_1$ term to the objective) will give sparser solutions than using ℓ_2 regularization for suitably large λ.

norm Ball :



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Extra Credit (Lasso) Show that placing an equal zero-mean Laplace prior on each element of the weights $\boldsymbol{\theta}$ of a model is equivelent to ℓ_1 regularization in the Maximum-a-Posteriori estimate

$$\text{maximize: } \mathbb{P}(\boldsymbol{\theta}|\mathcal{D}) = \frac{\mathbb{P}(\mathcal{D}|\boldsymbol{\theta})\mathbb{P}(\boldsymbol{\theta})}{\mathbb{P}(\mathcal{D})}.$$

Note the form of the Laplace distribution is

$$\text{Lap}(x|\mu, b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

where μ is the location parameter and $b > 0$ controls the variance. Draw (by hand) and compare the density $\text{Lap}(x|0,1)$ and the standard normal $\mathcal{N}(x|0,1)$ and suggest why this would lead to sparser solutions than a Gaussian prior on each elements of the weights (which correspond to ℓ_2 regularization).

laplacian is sparser, since the laplace distribution has the most probability at 0, with much less probability at the other values.

Gaussian has a wide distribution of probabilities at many different values.

Thus the laplacian makes it easier to get to 0 than the gaussian.

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