Online Appendix for
"Identification in Auction Models with Interdependent Costs"

Paulo Somaini*

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A Examples of Information Structures

Log-additive models Hong and Shum (2002) use the Wilson (1998) log-additive model where each bidder's log costs is $\log C_i = A_i + V$, where A_i is a private component of i's costs and V is an unknown cost component that is common across bidders. They assume that $A_i \sim N\left(\bar{a}, \sigma_a^2\right)$ and $V \sim N\left(m, \sigma_v^2\right)$. Each bidder has a noisy signal of its cost $S_i = \log C_i + E_i$, where $E_i \sim N\left(0, \sigma_e^2\right)$. This implies that

$$\log C_i | s \sim N\left(\frac{\sigma_v^2}{\sigma_a^2 + \sigma_e^2 + n\sigma_v^2}\left(\bar{a} + m + \frac{\sigma_e^2}{\sigma_a^2 + \sigma_e^2}\Sigma_{j \neq i} s_i\right) + \alpha s_i, \sigma_e^2 \alpha\right)$$

where

$$\alpha = \frac{1}{n}\left(1 + \frac{(n-1)\,\sigma_a^2}{\sigma_a^2 + \sigma_e^2} - \frac{\sigma_v^2}{\sigma_a^2 + \sigma_e^2 + n\sigma_v^2}\right).$$

The model can be renormalized so that each signal has a marginal uniform distribution.

Gaussian Information Structure The log-additive model can be generalized so that bidder i's cost is $C_i = \psi_{i,x_i}^{-1}\left(\beta_i' V + A_i\right)$, where ψ_{i,x_i} is a strictly monotone function that may vary with x_i, V is a vector of common costs distributed $N(0, I)$, β_i is a bidder-specific vector of weights and A_i is a private cost component distributed independently $N\left(0, \sigma_{ia}^2\right)$. Each bidder's signal is $S_i = \psi_{i,x_i}(C_i) + E_i$ where $E_i = N\left(0, \sigma_{ie}^2\right)$. Notice the special cases of independent private costs when $\beta_i = 0$ for all i; the affiliated private costs when $\sigma_{ie}^2 = 0$ for all i; and the pure common values model when $\beta_i = \beta$ and $\sigma_{ia}^2 = 0$ for all i.

In the most general case, the vector of Gaussian signals is $S \sim N(0, \Sigma)$, $\Sigma = B'B + \Sigma_a + \Sigma_e$, where B's i-th column is β_i, Σ_a and Σ_e are diagonal matrices with σ_{ia}^2 and σ_{ie}^2 in the i-th diagonal entry, respectively.

$$\psi_x(C)\,|S \sim N\left(\left(I - \Sigma_e \Sigma^{-1}\right) S, \Sigma_e - \Sigma_e \Sigma^{-1} \Sigma_e\right)$$

*Graduate School of Business, Stanford University, 655 Knight Way E372D, Stanford, CA 94305, soma@stanford.edu

Therefore, the full-information cost of bidder i is $c_i(s_{-i}, s_i, x_i) = E\left(\psi_{i,x_i}^{-1}\left(W + \mu_i s\right)\right)$ where W is a normally distributed random variable with mean zero and variance equal to the i-th diagonal entry of $\Sigma_e - \Sigma_e \Sigma^{-1} \Sigma_e$, and μ_i is the i-th row of $\left(I - \Sigma_e \Sigma^{-1}\right)$. Signals can be renormalized to be marginally uniform. If for a parameter vector α_{i0} and scalar parameters $(\alpha_{ic}, \alpha_{i1})$, $\psi_{i,x_i}(t) = (t - \alpha_{ic} - \alpha_{0i}' x_0 - \alpha_{1i} x_i)\, \psi_i$, then the full-information cost is given by equation (11) in the main text. It can be verified that the coefficients α_{i2} and ν_i in that expression equal to $\sigma_{ie}^2 \psi_i^{-1}$ and $(1 + \sigma_{ie}^2 \tilde{\sigma}_{ii})\psi_i^{-1}$, respectively.

B Proofs

B.1 Proof of Proposition 1

By Theorem 2.1 in Reny and Zamir (2004), Assumptions A.2-A.5 guarantee the existence of an equilibrium in monotone pure strategies. These strategies are the limit of the equilibrium strategies of a sequence of auction games in which bidders never bid above their expected cost conditional on winning. This property also extends to the limit strategies.

Part (i): Take any $b \in [b_*, b^*)$ and suppose that it belongs to the support of only bidder i. The type of bidder i that bids in the neighborhood of b can deviate to some $b' > b$ and win in exactly the same states of the world but earn higher profits. Therefore, b has to belong to the support of at least two bidders. By closedness of supports, this property also extends to b^*.

Part (ii): Suppose that $H_W(b_*) > 0$. There is at least one bidder who submits bid b_* with positive probability. Part (i) and A.5 imply that there is a $\varepsilon > 0$ and another bidder who bids $b_* + \varepsilon$. This bidder can profitably deviate by bidding $b_* - \varepsilon$.

Part (iii): Suppose that there are two bidders that bid b^* with positive probability. Formally, $\exists i, j$ such that $\beta(s_i) = \beta(s_j) = b^*$ for all $s_i \in [\underline{s}_i, \overline{s}_i]$ and $s_j \in \left[\underline{s}_j, \overline{s}_j\right]$, where $\underline{s}_i < \overline{s}_i$ and $\underline{s}_j < \overline{s}_j$. For all $s_i \in [\underline{s}_i, \overline{s}_i]$ bidder i could discontinuously increase its probability of winning by reducing its bid by ε. His expected costs conditional on winning will be weakly lower because the set of competitors signals is now slightly better. The fact that this bidder chooses not to reduce his bid implies that $b^* \le E\left(C_i | s_i, S_{-i} \ge \overline{s}_{-i}\right)$, for all $s_i \in [\underline{s}_i, \overline{s}_i]$. However, because strategies are the limit of a sequence of strategies where each bid has a positive probability of winning, $b^* = E\left(C_i | s_i, S_{-i} \ge \overline{s}_{-i}\right)$ for all $s_i \in [\underline{s}_i, \overline{s}_i]$, which contradicts strict monotonicity of s_i (Assumption A.5). Therefore, there is at most one bidder that bids b^* with positive probability.

Suppose $H_W(b^*) < 1$. If i bids b^* with positive probability, he can deviate to a slightly higher bid and win in exactly the same (positive probability) set of realizations of competitors' signals and receive a higher payment. Instead, if there is no bidder that bids b^* with positive probability, then for every $b' > b^*$ there is an ε and a bidder i such that when i bids $b - \varepsilon$, he has a profitable deviation to b' that results in an arbitrarily small change in the (positive probability) set of realizations of competitors' signals where he wins and a discrete increase in the payment he receives in that set. Thus, $H_W(b^*) < 1$ leads to a contradiction.

Maskin and Riley (2000) show that H_W is continuous and strictly increasing in $[b_*, b^*)$. I have to show only that $\lim_{b\uparrow b^*} H_W(b) = 1$, i.e., that there is no atom at b^*. Suppose that there is an atom at b^*. By the same arguments used in the previous paragraph, there is at most one bidder who bids b^* with positive probability. There has to be a bidder j who bids between b^* and $b^* + \varepsilon$; otherwise, the bidder who bids b^* has a profitable deviation to a slightly higher bid. Bidder j makes non-negative profits conditional on winning when she bids $b^* + \varepsilon$. Bidding $b^* - \varepsilon$ instead would increase the probability of winning and improve discretely the states of the world where she wins. There exists an ε for which this deviation is profitable. This contradicts the existence of an atom at b^*.

Part (iv): Let $b' = \beta_i(s_i')$ and $s_j' = \inf\left\{s_j \in [0,1] : b' \leq \beta_j(s_j)\right\}$. Because $b' \leq b^*$and $s_i > 0$, b' is in the support of bids of some competitor $j \neq i$. By optimality of b',

$$\left[b' - E\left(C_j|s_j', S_{-j} \geq s_{-j}'\right)\right] \Pr\left(S_{-j} \geq s_{-j}'|s_j'\right) \geq \left[b - E\left(C_j|s_j', S_{-j} \geq s_{-j}\right)\right] \Pr\left(S_{-j} \geq s_{-j}|s_j'\right).$$

Solving for $(b' - b)$ and rearranging:

$$\begin{aligned}
\left(b' - b\right) &\geq \left(b' - E\left(C_j|s_j', s_{-j} \leq S_{-j} \ngeq s_{-j}'\right)\right)\left(1 - \Pr\left(s_{-j}' \leq S_{-i}|s_j', S_{-j} \geq s_{-j}\right)\right) \\
&\geq \left(b' - E\left(C_j|s_j', S_{-ij} \geq s_{-ij}, s_i \leq S_i \leq s_i'\right)\right) \Pr\left(S_i \leq s_i'|s_j', S_{-j} \geq s_{-j}\right)
\end{aligned}$$

The second inequality uses the fact that the set $\left\{s_{-j} \leq S_{-j} \ngeq s_{-j}'\right\}$ can be partitioned in two: $\{S_{-ij} \geq s_{-ij}, s_i \leq S_i \leq s_i'\}$ and $\left\{s_{-ij} \leq S_{-ij} \ngeq s_{-ij}', S_i \geq s_i'\right\}$. Thus, it is possible to write the right-hand side of the first line as the sum of two terms. One term conditions on the set $\{S_{-ij} \geq s_{-ij}, s_i \leq S_i \leq s_i'\}$ and the other on the set $\left\{s_{-ij} \leq S_{-ij} \ngeq s_{-ij}', S_i \geq s_i'\right\}$. By optimality of b' and Assumption A.5, both terms are positive and dropping one preserves the inequality.

Let $g(\cdot)$ denote the density of $S_i|s_j', S_{-j} \geq s_{-j}$ and $G(s) = \Pr\left(S_i \leq s|s_j', S_{-j} \geq s_{-j}\right)$. Let $\psi(\sigma) := E\left(C_j|s_j', S_{-ij} \geq s_{-ij}, S_i = \sigma\right)$. Because $b' \geq E\left(C_j|s_j', S_{-j} \geq s_{-j}\right)$,

$$\left(b' - b\right) \geq \left(\frac{\int_{s_i'}^1 \psi(\sigma) g(\sigma) d\sigma}{1 - G(s_i')} - \frac{\int_{s_i}^{s_i'} \psi(\sigma) g(\sigma) d\sigma}{G(s_i')}\right) G(s_i').$$

By Assumption A.5, there exist a positive constant κ such that $|\psi(\sigma') - \psi(\sigma)| \geq \tilde{\kappa}|\sigma' - \sigma|$ for every σ and σ'. The constant $\tilde{\kappa}$ may depend on cost shifter x_j, the joint distribution F and the identity of bidders i and j. This dependence is omited for simplicity. Adding and subtracting $\psi(s_i')$ and using the bounds on $|\psi(\sigma') - \psi(\sigma)|$ yields

$$\left(b' - b\right) \geq \tilde{\kappa}\left(\frac{G(s_i')}{1 - G(s_i')}\int_{s_i'}^1 \left(\sigma - s_i'\right) g(\sigma) d\sigma + \int_{s_i}^{s_i'} \left(s_i' - \sigma\right) g(\sigma) d\sigma\right)$$

Let $\underline{g}$ is the lower bound of the density g in the interval (s_i, s_i') and $\bar{g}$ is the upper bound of g in the interval $(s_i, 1)$. When $\underline{g}2(1 - s_i) \leq 1$, the density g that minimizes the right hand side subject to $\bar{g} \leq g(\sigma) \leq \bar{g}$ for all σ, puts as little probability mass as possible in the interval $[s_i, s_i']$. The minimized value is bounded below by:

$$\left(s_i' - s_i\right) \frac{\tilde{\kappa}}{2} \frac{\underline{g}}{\bar{g}}$$

When $\underline{g}2\left(1 - s_i\right) > 1,$ it is useful to consider the relaxed problem with $\frac{\bar{g}}{2} \leq g\left(\sigma\right) \leq \bar{g}$. The resulting bound is half the previous one. Moreover, the ratio $\frac{\underline{g}}{\bar{g}}$ is bounded below by the ratio $\frac{\min_{\sigma \in [0,1]^n} f(\sigma)}{\max_{\sigma \in [0,1]^n} f(\sigma)}$. Therefore,

$$\left(b' - b\right) \geq \left(s_i' - s_i\right) \kappa_i\left(F, x\right),$$

where $\kappa_i\left(F, x\right) = \frac{\min_j \tilde{\kappa}_{ij}(F,x)}{4} \frac{\min_{\sigma \in [0,1]^n} f(\sigma)}{\max_{\sigma \in [0,1]^n} f(\sigma)}$. □

Proof of the Corollary to Proposition 1 Part (iv) of Proposition 1 implies that the inverse bid function is Lipschitz continuous. Parts (ii) and (iii) yield that $\beta_i^{-1}\left(b_*\right) = 0$ for every bidder, and $\beta_j^{-1}\left(b^*\right) = 1$ for at least one of them. The result for the two-bidder case follows from part (i). □

Corollary summarizing equilibrium bid functions β.

Corollary. *Fix any $x \in \mathbb{R}^n$ and equilibrium monotone pure strategies β such that bidders never bid above their expected cost conditional on winning. For every i, $\beta_i(\cdot)$ is strictly monotone on $[0, \phi_i]$. $\beta_i(0) = b_*$ for at least two bidders. $\beta_i(\phi_i) = b^*$ for at least two bidders. If $n = 2$, $\beta_i(\cdot)$ is also continuous on $[0, \phi_i]$.*

Proof. Immediate from parts (i) and (iv) of Proposition 1. □

Proposition 2.5 in the Supplemental Material shows that inverse bid functions are strictly increasing when $n > 2$ for a large family of primitives. In that case, $\beta_i(\cdot)$ is continuous on $[0, \phi_i]$, $\beta_i(0) = b_*$, and $\beta_i(\phi_i) = b^*$ for every single bidder.

B.2 Proof of Proposition 2

The proof of Proposition 2 has two main parts. The first part shows that there exist a set $X_{-i}^{\varepsilon}\left(x_i, s\right)$ that contains $X_{-i}\left(x_i, s\right)$. The second part relies on Proposition 1.10 (stated and proved in the Supplemental Material document) to show existence a vector of cost shifters $x_{-i} \in X_{-i}^{\varepsilon}\left(x_i, s\right)$ and equilibrium bid functions β for the game with cost shifters $\left(x_i, x_{-i}\right)$ such that for all j, $\beta_j(\sigma) < b < \beta_j(\sigma')$ whenever $\sigma < s_j < \sigma'$. I will also show that in fact $x_{-i} \in X_{-i}\left(x_i, s\right)$.

Part 1. Let

$$\begin{aligned}
X_{-i}^{\varepsilon}\left(x_i, s\right) &= \prod_{j \neq i} \left[\underline{x}_j^{\varepsilon}, \overline{x}_j^{\varepsilon}\right] \\
\underline{x}_j^{\varepsilon} &: \bar{c}_j(\underline{x}_j) = \underline{c}_i\left(x_i\right) - \left(1 - \Pr\left(S_{-j} \geq s_{-j} | s_j\right)\right)^{-1} \max_{k \neq j} h_k - \varepsilon \\
\overline{x}_j^{\varepsilon} &: \underline{c}_j(\overline{x}_j) = \bar{c}_i\left(x_i\right) + \left(1 - \Pr\left(S_{-i} \geq s_{-i} | s_i\right)\right)^{-1} \max_{k \neq i} h_k + \varepsilon
\end{aligned}$$

Notice that if $\varepsilon' > \varepsilon$, $X^{\varepsilon}_{-i}(x_i, s) \subset X^{\varepsilon'}_{-i}(x_i, s)$ and that $X_{-i}(x_i, s) = X^{0}_{-i}(x_i, s) \subset X^{\varepsilon}_{-i}(x_i, s)$ for all $\varepsilon > 0$. Moreover, $X^{\varepsilon}_{-i}(x_i, s)$ exists for all $\varepsilon > 0$ because $\Pr(S_{-i} \geq s_{-i}|s_i) < 1$ and because Assumption A.6 implies that both $\bar{c}_j(\cdot)$ and $\underline{c}_j(\cdot)$ are bijections of $\mathbb{R}$ to $\mathbb{R}$ for all j.

Part 2. This part uses Proposition 1.10 in the supplementary material. To keep notation simple suppose that $i = 1$ for the rest of the proof, i.e., the bidder whose cost shifter is supposed to stay fixed is the first one. Pick any $\varepsilon > 0$ and define $Z = \left[-\bar{b}, -\underline{b}\right] \times X^{\varepsilon}_{-1}(x_1, s)$ where $\underline{b} = \underline{c}_1(x_1) - \varepsilon$ and $\bar{b} = \bar{c}_1(x_1) + (1 - \Pr(S_{-1} \geq s_{-1}|s_1))^{-1} \max_{k \neq 1} h_k + \varepsilon$.

For every $i \in \{1, 2, ..., n\}$, let

$$c_i^{\dagger}(s_{-i}, s_i, z) = c_i^{\dagger}(s_{-i}, s_i, [-b, x_{-1}]) = c_i(s_{-i}, s_i, x_i) - b$$

It follows that for $z_1 = -b$ and $z_{-1} = x_{-1}$

$$\underline{c}_1^{\dagger}(z_1, z_{-1}) = \underline{c}_1(x_1) - b \qquad \bar{c}_1^{\dagger}(z_1, z_{-1}) = \bar{c}_1(x_1) - b;$$

and for all $j > 1$,

$$\underline{c}_j^{\dagger}(z_j, z_{-j}) = \underline{c}_j(x_j) - b \qquad \bar{c}_j^{\dagger}(z_j, z_{-j}) = \bar{c}_j(x_j) - b.$$

I will apply Proposition 1.10 (from the Supplemental Material) to the model given by functions $c_i^{\dagger}$, the set of cost shifters Z, and $a = 0$. The first result is that there are bid functions β and vector of cost shifters $z \in Z$ that are the limit of a sequence of fixed points. Moreover,

$$\begin{aligned} \bar{c}_1^{\dagger}(\underline{z}_1, z_{-1}) &= \bar{c}_1(x_1) - \bar{b} = -(1 - \Pr(S_{-1} \geq s_{-1}|s_1))^{-1} \max_{k \neq 1} h_k - \varepsilon < 0 \\ \underline{c}_1^{\dagger}(\bar{z}_1, z_{-1}) &= \underline{c}_1(x_1) - \underline{b} = \varepsilon > 0. \end{aligned}$$

Similary,

$$\begin{aligned} \bar{c}_j^{\dagger}\left(\underline{z}_j, z_{-j}\right) &= \bar{c}_j\left(\underline{x}_j\right) - b = \underline{b} - b - (1 - \Pr(S_{-j} \geq s_{-j}|s_j))^{-1} \max_{k \neq j} h_k < 0 \\ \underline{c}_j^{\dagger}(\bar{z}_j, z_{-j}) &= \underline{c}_j(\bar{x}_j) - b = \bar{b} - b \geq 0. \end{aligned}$$

Thus, by Proposition 1.10, $\underline{c}_i^{\dagger}(z_i, z_{-i}) = \underline{c}_i(x_i) - b \leq 0$ for all $i = 1, ..., n$, and $\bar{c}_i^{\dagger}(z_i, z_{-i}) = \bar{c}_i(x_i) - b \geq 0$ for all but one bidder. There are three possibilities: (i) $\bar{c}_1(x_1) - b < 0$, (ii) $\bar{c}_k(x_k) - b < 0$ for $k \neq 1$, and (iii) $\bar{c}_k(x_k) - b \geq 0$ for all bidders.

Consider first the case: $\overline{c}_1(x_1) - b < 0$.

$$\begin{aligned}
\overline{c}_1^{\dagger}(\underline{z}_1, z_{-1}) &= -\left(1 - \Pr\left(S_{-1} \geq s_{-1}|s_1\right)\right)^{-1} \max_{k \neq 1} h_k - \varepsilon \\
&< \min_{j \neq 1} \overline{c}_j^{\dagger}\left(z_j, z_{-j}\right) - \left(1 - \Pr\left(S_{-1} \geq s_{-1}|s_1\right)\right)^{-1} \max_{k \neq 1} h_k \\
&\leq \min_{j \neq 1} \overline{c}_j^{\dagger}\left(z_j, z_{-j}\right) - \frac{\min_{j \neq 1} \overline{c}_j^{\dagger}(z_j, z_{-j}) - \min_{j \neq 1} \underline{c}_j^{\dagger}(z_j, z_{-j})}{\left(1 - \Pr\left(S_{-1} \geq s_{-1}|s_1\right)\right)}
\end{aligned}$$

where the first inequality follows because $\min_{j \neq 1} \overline{c}_j\left(z_j, z_{-j}\right) \geq 0$ and $\varepsilon > 0$. The second follows because the maximum difference between $\overline{c}_j^{\dagger}$ and $\underline{c}_j^{\dagger}$ is weakly greater than the difference between the minimum $\overline{c}_j^{\dagger}$ and the minimum $\underline{c}_j^{\dagger}$. By the last part of Proposition 1.10 in the Supplemental Material,

$$\overline{c}_1(x_1) - b = \overline{c}_1^{\dagger}(z_1, z_{-1}) \geq -\left(1 - \Pr\left(S_{-1} \geq s_{-1}|s_1\right)\right)^{-1} \max_{k \neq 1} h_k.$$

For all $j > 1$

$$\begin{aligned}
\underline{c}_j\left(x_j\right) &\leq b \leq \overline{c}_1(x_1) + \left(1 - \Pr\left(S_{-1} \geq s_{-1}|s_1\right)\right)^{-1} \max_{k \neq 1} h_k \\
\overline{c}_j\left(x_j\right) &\geq b \geq \underline{c}_1(x_1).
\end{aligned}$$

It follows that $x_{-i} \in X_{-i}(x_i, s)$.

Consider the second case: $\overline{c}_k(x_k) - b < 0$ for $k \neq 1$.

$$\begin{aligned}
\overline{c}_k^{\dagger}(\underline{z}_k, z_{-k}) &= \underline{c}_1^{\dagger}(z_1, z_{-1}) - \left(1 - \Pr\left(S_{-k} \geq s_{-k}|s_k\right)\right)^{-1} \max_{j \neq k} h_j - \varepsilon \\
&< \min_{j \neq k} \overline{c}_j^{\dagger}\left(z_j, z_{-j}\right) - \left(1 - \Pr\left(S_{-k} \geq s_{-k}|s_k\right)\right)^{-1} \max_{j \neq k} h_j \\
&\leq \min_{j \neq k} \overline{c}_j^{\dagger}\left(z_j, z_{-j}\right) - \frac{\min_{j \neq k} \overline{c}_j^{\dagger}\left(z_j, z_{-j}\right) - \min_{j \neq k} \underline{c}_j^{\dagger}\left(z_j, z_{-j}\right)}{\left[1 - \Pr\left(S_{-i} \geq s_{-i}|s_i\right)\right]}.
\end{aligned}$$

Therefore, by the last part of Proposition 1.10,

$$\overline{c}_k^{\dagger}(\underline{z}_k, z_{-k}) \geq \underline{c}_1^{\dagger}(z_1, z_{-1}) - \left(1 - \Pr\left(S_{-k} \geq s_{-k}|s_k\right)\right)^{-1} \max_{j \neq k} h_j.$$

After substituting $\overline{c}_k^{\dagger}(\underline{z}_k, z_{-k})$ and $\underline{c}_1^{\dagger}(z_1, z_{-1})$:

$$\overline{c}_k\left(\underline{x}_k\right) \geq \overline{c}_1(x_1) - \left(1 - \Pr\left(S_{-k} \geq s_{-k}|s_k\right)\right)^{-1} \max_{j \neq k} h_j$$

while for $j \neq k$

$$\overline{c}_j\left(x_j\right) \geq b \geq \underline{c}_1(x_1).$$

For all j,

$$\underline{c}_j\left(x_j\right) \leq b \leq \overline{c}_1(x_1);$$

therefore, $x_{-i} \in X_{-i}(x_i, s)$.

Consider the third case. For all j, $\bar{c}_j(x_j) \geq b \geq \underline{c}_1(x_1)$ and $\underline{c}_j(x_j) \leq b \leq \bar{c}_1(x_1)$; therefore, $x_{-i} \in X_{-i}(x_i, s)$.

In the three cases, $x_{-i} \in X_{-i}(x_i, s)$ and the last part of Proposition 1.10 also implies that the strategy profile β is an equilibrium of the continuous bid auction game satisfying $\beta_j(\sigma) < b < \beta_j(\sigma')$ whenever $\sigma < s_j < \sigma'$. □

The Corollary follows immediately. By optimality of b, for any $b' > b$

$$(b - E(C_i|s_i, S_{-i} \geq s_{-i}(b), x_i)) \Pr(S_{-i} \geq s_{-i}(b)|s_i) \geq (b' - E(C_i|s_i, S_{-i} \geq s_{-i}(b'), x_i)) \Pr(S_{-i} \geq s_{-i}(b')|s_i).$$

Rearranging,

$$(b - E(C_i|s_i, s_{-i}(b) \leq S_{-i} \ngeq s_{-i}(b'), x_i)) \geq (b' - b) \frac{\Pr(S_{-i} \geq s_{-i}(b')|s_i)}{\Pr(s_{-i}(b) \leq S_{-i} \ngeq s_{-i}(b')|s_i)}.$$

The corollary follows after replacing b by the upper bound $\bar{b}$, and $E(C_i|s_i, s_{-i}(b) \leq S_{-i} \ngeq s_{-i}(b'), x_i)$ by the lesser magnitude $\min_{k \neq i} E(C_i|s_i, s_k, S_{-ik} \geq s_{-ik}(b), x_i)$. □

B.3 Proof of Proposition 3

Pick any $\tilde{x}_i$, set $s = [1, 1, ..., 1]$. By Proposition 2 there are cost shifters $\tilde{x}_{-i}$, bid b and equilibrium strategies β such that: $\tilde{x}_{-i} \in X_{-i}(\tilde{x}_i, [1, 1, ..., 1])$, where

$$X_{-i}(\tilde{x}_i, [1, 1, ..., 1]) = \left\{ \begin{array}{c} x_{-i} \in \mathbb{R}^{n-1} : \forall j \neq i, \\ \underline{c}_j(x_j) \leq \bar{c}_i(\tilde{x}_i) + \max_{k \neq i} h_k \\ \bar{c}_j(x_j) \geq \underline{c}_i(\tilde{x}_i) - \max_{k \neq j} h_k \end{array} \right\}$$

and $\beta_i^{-1}(b) = 1$ for all i. By Assumption A.6, the set $X_{-i}(\tilde{x}_i, [1, 1, ..., 1])$ exists. It follows that $\tilde{x} = [\tilde{x}_i, \tilde{x}_{i0}]$ also exists. For all i, β_i^{-1} is continuous and $\beta_i^{-1}(b) = 1$. Therefore, $H_{B_i|\tilde{x}}(\cdot)$ is continuous and $H_{B_i|\tilde{x}}(b^*) = 1$ for every bidder i. □

B.4 Proof of Theorem 4

The proof consists in approximating

$$\bar{C}_i(s_{-i}|s_i, x_i) := E(C_i \times 1(S_{-i} \geq s_{-i})|s_i, x_i) = \int_{\{\tau:\tau \geq s_{-i}\}} c_i(\tau, s_i, x_i) f_{S_{-i}|s_i}(\tau) d\tau. \quad (1)$$

using a finite number of pivotal or L-shaped sets as those in Figure 2 and showing that the approximation error converges to zero as the number of pivotal sets goes to infinity.

B.4.1 Construction of Pivotal Sets

The first step of the proof is to construct a set of pivotal sets that partition the set $\{S_{-i} \geq s_{-i}\}$. This can be performed recursively. For every $\delta > 0$, construct a sequence of pairs $(\tau_t, x_t)_{t\in\mathbb{N}}$ so that $\tau_1 = s_{-i}$ and for every $t \geq 1$, $x_t \in \{x_i\} \times X^o_{-i}$ is such that $\left[H_{B_j|x_t}\left(Q_{B_i|x_t}(s_i)\right)\right]_{j\neq i} = \tau_t$, which exists because H exhibits sufficient variation, and $\tau_{t+1} = \left[H_{B_j|x_t}\left(Q_{B_i|x_t}(s_i+\delta)\right)\right]_{j\neq i}$ if $Q_{B_i|x_t}(s_i+\delta) < \infty$ and $\tau_{t+1} = [1, ..., 1]$ otherwise. Non-vanishing competition implies that there will be a $T \leq -\frac{\log\delta}{\log(1+\kappa\delta)}$ such that $\Pr(S_{-i} \geq \tau_T | s_i) \leq \delta$. Let $L_t = \{\tau : \tau_t \leq \tau \not\geq \tau_{t+1}\}$, then $\{L_t\}_{t=1}^T$ is the desired partition.

B.4.2 Riemann Integration over Pivotal Sets

The second step of the proof is to compute the Riemann sum across all pivotal sets. Define the expected revenues, costs and profits for bid b_i, signal s_i, and market conditions x as:[1]

$$\begin{aligned} r_i(b_i, s_i, x) &:= b_i \Pr(b_i < M_i | s_i, x) && (2)\\ q_i(b_i, s_i, x) &:= E(C_i \times 1(b_i < M_i) | s_i, x) && (3)\\ U_i(b_i, s_i, x) &:= r_i(b_i, s_i, x) - q_i(b_i, s_i, x). && (4) \end{aligned}$$

Assumptions A.2 and A.3 imply that $r_i(b_i, \cdot, x)$ and $q_i(b_i, \cdot, x)$ are absolutely continuous. By the Envelope Theorem 2 in Milgrom and Segal (2002): if β_i is a best-response to competitors' monotone strategies β_{-i}, then

$$U_i(\beta(s_i+\delta), s_i+\delta, x) - U_i(\beta(s_i), s_i, x) = \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} r_i(\beta_i(\sigma), \sigma, x) - \frac{\partial}{\partial s_i} q_i(\beta_i(\sigma), \sigma, x)\, d\sigma. \quad (5)$$

Take any $t = 1, .., T$ and let $b_t = Q_{B_i|x_t}(s_i)$ and $b'_t = Q_{B_i|x_t}(s_i+\delta)$. The sets $\{s_{-i} : s_{-i} \geq \tau_t\}$ and $\left\{s_{-i} : \forall j \neq i, b_t < \beta_j(s_j, x_t)\right\}$ are equivalent (up to a set of measure zero); thus, by Assumptions A.2 and A.3, $\bar{C}_i(\tau_t | s_i, x_i) = q_i(b_t, s_i, x_t)$. By continuity of f and full-information costs with respect to s_i and left-continuity of the quantile function, $U_i(\beta(s_i), s_i, x_t) = U_i(b_t, s_i, x_t)$.

Using (4) to substitute the terms in the left hand side of (5) and adding over t yields:

$$\bar{C}_i(s_{-i} | s_i, x_i) = \sum_{t=1}^{T}\left(r_i(b_t, s_i, x_t) - r_i\left(b'_t, s_i+\delta, x_t\right) + \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} r_i(\beta_i(\sigma, x_t), \sigma, x_t)\, d\sigma \right) + R. \quad (6)$$

This part concludes by showing that R converges to zero as $\delta \to 0$.

To show that the term R in equation (6) converges to zero as $\delta \to 0$ consider equations (4), (5)

[1] This formulation assumes that bidder i always loses in case of a tie. If this is not the case then there should be an additional term in equation (4) equal to $E((b_i - C_i) \times (1(b_i \prec B_{-i}) - 1(b_i < B_{-i})) | s_i, x)$. Berge's Maximum Theorem implies that $U_i(\beta(s_i), s_i, x_t) = r_i(b_t, s_i, x_t) - q_i(b_t, s_i, x_t)$ where $b_t = \lim_{\tau\uparrow s_i} \beta(\tau)$ and the rest of the proof proceeds unaltered.

and solve for $\bar{C}_i(\tau_t|s_i, x_i)$:

$$\bar{C}_i(\tau_t|s_i, x_i) = r_i(b_t, s_i, x_t) - r_i(b_t', s_i + \delta, x_t) + \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} r_i(\beta_i(\sigma), \sigma, x_t)\, d\sigma + \bar{C}_i(\tau_{t+1}|s_i, x_i) + \nu_t \tag{7}$$

where

$$\nu_t = \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} q_i(b_t', \sigma, x_t) - \frac{\partial}{\partial s_i} q_i(\beta_i(\sigma), \sigma, x_t)\, d\sigma$$

Equation (6) can be obtained replacing recursively in equation (7). The remainder $R = \nu + \eta$, where $\nu = \sum_{t=1}^{T} \nu_t$ and $\eta = \int_{\{\tau:\tau \geq \tau_T\}} c_i(\tau, s_i, x_i)\, f_{S_{-i}|s_i}(\tau)\, d\tau$. By boundedness of $c_i(\cdot, s_i, x_i)$, $\eta \to 0$ as $\delta \to 0$. Moreover,

$$\begin{aligned}
|\nu| &= \left| \sum_{t=1}^{T} \int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} \left(\int_{\{\tau:\beta_i(\sigma) \prec \beta_{-i}(\tau) \not\prec b_t'\}} f(\tau|\sigma)\, c_i(\tau, \sigma, x_i)\, d\tau \right) d\sigma \right| \\
&\leq \sum_{t=1}^{T} \int_{s_i}^{s_i+\delta} \int_{L_t} \left| \frac{\partial}{\partial s_i} \left[f(\tau|\sigma)\, c_i(\tau, \sigma, x_i) \right] \right| d\tau d\sigma \\
&= \int_{s_i}^{s_i+\delta} \int_{\{\tau:\tau \geq s_{-i}\}} \left| \frac{\partial}{\partial s_i} \left[f(\tau|\sigma)\, c_i(\tau, \sigma, x_i) \right] \right| d\tau d\sigma \\
&\leq \delta\kappa.
\end{aligned}$$

The first line obtains after writing q_i as an integral over competitors' signals. The second line line follows from differentiating under the integral sign, passing the absolute value through the summation and integration operators, and integrating over a weakly larger set of τ for each value of σ. Let I be an interval $[s_i, s_i']$. By countinuous differentiability of the density function and the full-information cost over a compact support $I \times [0,1]^n$, the step of differentiating under the integral sign is legitimate, and for some finite and positive κ,

$$\sup_{s_i \in I} \int_{[0,1]^{n-1}} \left| \frac{\partial}{\partial s_i} \left[f(\tau|s_i)\, c_i(\tau, s_i, x_i) \right] \right| d\tau < \kappa. \tag{8}$$

The third line follows from the fact that $\{L_t\}_{t=1}^{T} \subset \{\tau : \tau \geq s_{-i}\}$. The last line follows from the bound in (8). As $\delta \to 0$ while I remains fixed, the approximation (6) becomes arbitrarily precise.

B.4.3 Identification of Expected Costs Conditional on Winning

The third step of the proof is to show that the terms in parenthesis in (6) are identified. The integrand $\frac{\partial}{\partial s_i} r_i(\beta_i(s_i, x_t), s_i, x_t)$ is bounded because f is bounded, and has countably many discontinuities because bid functions are monotone. Thus, it is Riemann integrable and the integral can be approximated up to a desired precision by a Riemann sum of K_t terms:

$$\begin{aligned}\int_{s_i}^{s_i+\delta} \frac{\partial}{\partial s_i} r_i\left(\beta_i\left(\sigma, x_t\right), \sigma, x_t\right) d\sigma &\approx \frac{\delta}{K_t} \sum_{k=1}^{K_t} \frac{\partial}{\partial s_i} r_i\left(\beta_i\left(s_{k-1}, x_t\right), s_{k-1}, x_t\right) \\ &\approx \sum_{k=1}^{K_t} r_i\left(\beta_i\left(s_{k-1}, x_t\right), s_k, x_t\right) - r_i\left(\beta_i\left(s_{k-1}, x_t\right), s_{k-1}, x_t\right)\end{aligned}$$

where $s_k = s_i + \frac{k\delta}{K_t}$. The second line follows from the definition of the derivative. Adding $r_i\left(b_t, s_i, x_t\right) - r_i\left(b_t', s_i + \delta, x_t\right)$ and substituting in (6) yields:

$$\begin{aligned}\bar{C}_i\left(s_{-i}|s_i, x_i\right) &\approx \sum_{t=1}^{T}\sum_{k=1}^{K_t} r_i\left(\beta_i\left(s_{k-1}, x_t\right), s_k, x_t\right) - r_i\left(\beta_i\left(s_k, x_t\right), s_k, x_t\right) & (9)\\ &= \sum_{t=1}^{T}\sum_{k=1}^{K_t} b_{t(k-1)}\left[1 - H_{M|B_i=b_{tk}, x_t}\left(b_{t(k-1)}\right)\right] - b_{tk}\left[1 - H_{M|B_i=b_{tk}, x_t}\left(b_{tk}\right)\right] & (10)\end{aligned}$$

where $b_{tk} = Q_{B_i|x_t}\left(s_k\right) = \beta_i\left(s_k, x_t\right)$. Therefore, $\bar{C}_i\left(s_{-i}|s_i, x_i\right)$ is identified for all $s_{-i} \geq s_{-i}^*$.

B.4.4 Identification of Full Information Costs

The fourth and final step consists in differentiating $\bar{C}_i\left(s_{-i}|s_i, x_i\right)$ to obtain the full-information cost.

$$\frac{d^{n-1}\bar{C}_i\left(s_{-i}|s_i, x_i\right)}{\prod_{j\neq i} ds_j} = (-1)^{n-1} c_i\left(s_i, s_{-i}, x_i\right) f_{S_{-i}|s_i}\left(s_{-i}\right). \tag{11}$$

The full-information cost can be recovered dividing by the conditional density of signals that is also identified. □

B.5 Proof of Proposition 5

Pick any $\delta > 0$ and any triplet (b, b', x_{-i}) such that $x_{-i} \in X_{-i}^o$, $b = Q_{B_i|x}\left(s_i\right)$ for some $s_i \geq s_i^*$, $H_{B_j|x}(b) \geq s_j^*$ for all $j \neq i$, $b' = Q_{B_i|x}\left(H_{B_i|x}\left(b\right) + \delta\right)$. Bids b and b' are optimal for signals s_i and $H_{B_i|x}\left(b\right) + \delta$, respectively. Part (iv) or Proposition 1 implies $b' - b \geq \delta\kappa_i\left(F, x\right)$, and the Corollary of Proposition 2 implies

$$\frac{\Pr\left(s_{-i}\left(b, x\right) \leq S_{-i} \ngeq s_{-i}\left(b', x\right)|s_i\right)}{\Pr\left(S_{-i} \geq s_{-i}\left(b', x\right)|s_i\right)} \geq \delta \begin{bmatrix} \bar{c}_i\left(x_i\right) + \left(1 - \Pr\left(S_{-i} \geq s_{-i}|s_i\right)\right)^{-1} \max_{k\neq i} h_k \\ - \min_{k\neq i} E\left(C_i|s_i, s_k, S_{-ik} \geq s_{-ik}\left(b\right), x_i\right)\end{bmatrix}^{-1} \kappa_i\left(F, x\right)$$

Let

$$\kappa = \begin{bmatrix} \bar{c}_i\left(x_i\right) + \left(1 - \Pr\left(S_{-i} \geq s_{-i}^*|S_i = 1\right)\right)^{-1} \max_{k\neq i} h_k \\ - \min_{k\neq i} E\left(C_i|s_i^*, s_k^*, S_{-ik} \geq s_{-ik}^*, x_i\right)\end{bmatrix}^{-1} \min_{x_{-i}\in X_{-i}^o} \kappa_i\left(F, x\right).$$

The desired result follows because

$$\frac{H_{M_i|B_i=b,x}(b') - H_{M_i|B_i=b,x}(b)}{1 - H_{M_i|B_i=b,x}(b')} = \frac{\Pr\left(s_{-i}(b) \le S_{-i} \ngeq s_{-i}(b') \,|s_i\right)}{\Pr\left(S_{-i} \ge s_{-i}(b') \,|s_i\right)}$$

and

$$\kappa \le \left[\begin{array}{c} \bar{c}_i(x_i) + \left(1 - \Pr\left(S_{-i} \ge s_{-i}|s_i\right)\right)^{-1} \max_{k \neq i} h_k \\ - \min_{k \neq i} E\left(C_i|s_i, s_k, S_{-ik} \ge s_{-ik}(b), x_i\right) \end{array} \right]^{-1} \kappa_i(F, x).$$

This proves a stronger version of the result because κ does not depend on δ. □

B.6 Proof of Proposition 6

Take any bidder i, signal vector $s^* > 0$ and cost shifter x_i. Assumption A.6 implies that the set $\tilde{X}_{-i}(x_i, s^*)$ is bounded. Take any $s \ge s^*$. By Proposition 2, there exist a $x_{-i} \in \mathbb{R}^{n-1}$ a finite bid b and equilibrium bid functions $\left\{\beta_j\right\}_{j=1}^n$ such that $\beta_j^{-1}(b) = s_j$ for all j. Therefore, $H_{B_j|x_i,x_{-i}}(b) = \beta_j^{-1}(b) = s_j$ for all j. Moreover, $x_{-i} \in X_{-i}(x_i, s) \subset \tilde{X}_{-i}(x_i, s) \subset \tilde{X}_{-i}(x_i, s^*)$. The first set inclusion follows because affiliation of signals implies that $\Pr\left(S_{-j} \ge s_{-j}|S_k = s_k\right) \le \Pr\left(S_{-j} \ge s_{-j}|S_k = 1\right)$. The second set inclusion follows because $s \ge s^*$ implies $\Pr\left(S_{-j} \ge s_{-j}|S_k = 1\right) \le \Pr\left(S_{-j} \ge s^*_{-j}|S_k = 1\right)$. □

C Generalization of Theorem 4

C.1 Identification without the assumption of independence

The independence Assumption A.1 can be relaxed significantly. One can assume only local independence at the cost of requiring richer variation in cost shifters. Suppose that there is a finite partition of X^o denoted by $\{X_1^o, X_2^o, ..., X_P^o\}$ and a set of distribution functions $\left\{F_{S|1}, F_{S|2}, ..., F_{S|P}\right\}$ such that $x_p \in X_p^o$ implies that $F_{S|p} = F_{S|x_p}$. Assumption A.1 holds when $P = 1$. Allowing for $P > 1$ relaxes the assumption of independence.

Theorem C.1. *If the independence Assumption A.1 holds within each partition X_p^o with $p \in \{1, ...P\}$, Assumptions A.2 and A.4 hold for each $F_{S|p}$ and Assumptions A.3, A.5 and A.7 hold for all X^o then $c_i(s_{-i}, s_i, x_i)$ is identified from H if $\{X_1^o, X_2^o, ..., X_P^o\}$ is such that for all $\hat{s}_{-i} \ge s_{-i}$, there exist $p \in \{1, ...P\}$ and $\varepsilon > 0$ such that for all s'_{-i} in an ε-neighborhood of $\hat{s}_{-i}$*

$$\left[H_{B_j|x}\left(Q_{B_i|x}(s_i)\right)\right]_{j \neq i} = s'_{-i}$$

for some $x \in \{x_i\} \times X_{-i}^o \cap X_p^o$.

Proof: Suppose that there are two different full-information cost functions consistent with observable H. Let c_i^1 and c_i^2 be these two functions and

$$\bar{C}_i^m(s_{-i}|s_i, x_i, p) := \int_{\{\tau:\tau \ge s_{-i}\}} c_i^m(\tau, s_i, x_i) f(\tau|s_i, x_p)\, d\tau$$

for $m=1,2$. Let Ψ be the support of $\psi(\cdot|s_i,x_i,p) := \left(c_i^1(\tau,s_i,x_i) - c_i^2(\tau,s_i,x_i)\right) f(\tau|s_i,x_p)$. By Assumption A.2, the support is the same for every p. Define $\overline{s}_{-i} = \arg\max_{s_{-i}\in\Psi} \sum_{j\neq i} s_j$. There are $\varepsilon>0$ and $p\in\{1,...P\}$ such that for all $s'_{-i}\in\Delta := \{\tau : \max_{j\neq i}|\tau_j - \overline{s}_j| < \varepsilon\}$

$$\left[H_{B_j|x}\left(Q_{B_i|x}(s_i)\right)\right]_{j\neq i} = s'_{-i}$$

for some $x\in\{x_i\}\times X^o_{-i}\cap X_p$. Restricting the space of competitors' signals to Δ and the set of cost shifters to X^o_p, the Assumptions and Conditions of Theorem 4 hold. Therefore, for all $\tilde{s}_{-i}\in\tilde{\Delta} := \left\{\tau : \max_{j\neq i}|\tau_j - \overline{s}_j| < \varepsilon(n-2)^{-1}\right\}$, it is possible to approximate to arbitrary precision:

$$\bar{C}_i^m(\tilde{s}_{-i}|s_i,x_i,p) \approx \sum_{t=1}^{T}\sum_{k=1}^{K}\left(\overline{r}_i\left(b_{t(k-1)},s_k,x_t\right) - \overline{r}_i\left(b_{tk},s_k,x_t\right)\right) + \bar{C}_i^m\left(s'_{-i}|s_i,x_i,p\right),$$

with $\max_{j\neq i} s'_j \geq \overline{s}_j + \varepsilon$ and $\sum_{j\neq i} s'_j > \sum_{j\neq i}\overline{s}_j$. It follows that $\{\tau : \tau \geq s'_{-i}\}\cap\Psi = \emptyset$, $\bar{C}_i^1\left(s'_{-i}|s_i,x_i,p\right) - \bar{C}_i^2\left(s'_{-i}|s_i,x_i,p\right) = 0$ and

$$\psi(\tilde{s}_{-i}|s_i,x_i,p) = \bar{C}_i^1(\tilde{s}_{-i}|s_i,x_i,p) - \bar{C}_i^2(\tilde{s}_{-i}|s_i,x_i,p) = 0.$$

This is true for all $\tilde{s}_{-i}\in\tilde{\Delta}$ contradicting $\overline{s}_{-i}\in\Psi$. The support of Ψ is empty. Therefore, for any $\tau\geq s_{-i}$ such that $(\tau,s_i)\in\mathcal{S}$, $c_i^1(\tau,s_i,x_i) = c_i^2(\tau,s_i,x_i)$. □

Proposition 2 ensures that for each distribution $F_{S|x_p}$, there is a bounded set $\bar{X}^o_p$ that generates enough variation in equilibrium bid functions. It is possible to construct a partition $\{X^o_1, X^o_2, ..., X^o_P\}$ such that each set $X^o_p \subset \bar{X}^o_p$ and their union covers the whole set of competitors' signals. This partition satisfies the support condition in Theorem C.1.

C.2 Identification under different auction rules

The identification results can be extended to auctions where bidders submit simultaneous or sealed bids and the project is awarded to the bidder with the lowest bid. Let $p_i(b_i,b_{-i})$ be the function that determines i's payment given own and competitors' bids and r be the maximum price the auctioneer is willing to pay. Let $1(b_i\prec b_{-i},r)$ be an indicator that is one if and only if b_i beats bids b_{-i} given the tie-breaking rule in place and is less than or equal to the reserve price and $\min_{j\neq i}(b_j,r) = \min\left(\min_{j\neq i}(b_j),r\right)$. There are no restrictions on the tie-breaking rule as long as the winner is selected among those bidders who tied with the lowest bid.

The payment function for the first five examples in Lizzeri and Persico (2000), and the second price auction, are:

1. First-price auctions: $p_i(b_i,b_{-i}) = b_i 1(b_i\prec b_{-i},r)$

2. All-pay auction: $p_i(b_i,b_{-i}) = \min(b_i,0) + 1(b_i\prec b_{-i},r)\,r$, where $-b_i$ can be thought as a non-refundable bribe or kickback paid to be awarded a contract that pays a fixed amount r to whoever offered the largest bribe (i.e., lowest b_i).

3. War of Attrition: $p_i(b_i, b_{-i}) = \min_{j\neq i}(b_j, 0) + 1(b_i \prec b_{-i}, r)\, r$

4. Combination of first and second-price auction:

$$p_i(b_i, b_{-i}) = \left(\alpha b_i + (1-\alpha)\min_{j\neq i}(b_j, r)\right) 1(b_i \prec b_{-i}, r)$$

5. Combination of all-pay and war of attrition:

$$p_i(b_i, b_{-i}) = \left(\alpha \min(b_i, 0) + (1-\alpha)\min_{j\neq i}(b_j, 0)\right) + 1(b_i \prec b_{-i}, r)$$

6. Second-price auctions: $p_i(b_i, b_{-i}) = \min_{j\neq i}(b_j, r)\, 1(b_i \prec b_{-i}, r)\, r$

The identification results require strategies to be strictly increasing so that higher bids can be associated with higher signals. Moreover, cost shifters have to generate sufficient variation in equilibrium strategies and strategies have to be continuous in signals. Relying on an extensive previous literature, this paper presents results that ensure that these conditions are satisfied for the first-price auction. There are not similar results for alternative auction formats. Focusing on the two-bidder case, Lizzeri and Persico (2000) studied existence and uniqueness of equilibria in nondecreasing strategies. If equilibrium bid functions can be assumed to be strictly increasing and cost shifters induce sufficient variation and non-vanishing competition defined by Definitions 1 and 2, Theorems 4 and C.1 can be applied with minor modifications.

The identification results in this paper only require that the set of bids b_i where $p_i(\cdot, b_{-i})$ is discontinuous is $\cup_{k=1}^{\infty}\{d_k(b_{-i})\}$, where functions $d_k : \mathbb{R}^{n-1} \to \mathbb{R}$ are Borel measurable. All the auctions above satisfy this condition with $d_1(b_{-i}) = r$, $d_2(b_{-i}) = \min_{j\neq i} b_j$ and $d_k(b_{-i}) = \emptyset$ for $k > 2$. This condition implies that $\frac{\partial}{\partial s_i} r_i(\beta_i(s_i, x_t), s_i, x_t)$ has countably many discontinuities which is necessary for the proof of Theorem 4. This proof only requires a couple of notational changes. The function r_i in equation (2) should be

$$r_i(b_i, s_i, x) := E(p_i(b_i, B_i) | s_i, x)$$

Equation (10) now takes a different form depending on the auction rules. The proof of Theorem C.1 remains unaltered.

D Computation of an Equilibrium in monotone pure strategies

The equilibrium inverse bid functions are calculated using a numerical algorithm that is similar to that in Gayle and Richard (2008). The main difference with their approach is that I allow for interdependent costs and correlated signals. Although the computation time grows fast with the number of bidders, auctions with less than 10 bidders are solved within a couple of hours. The algorithm solves the system of differential equations implied by bidders' first-order conditions.

Following Gayle and Richard (2008), I guess the initial conditions and, after the system is solved forward, I verify if the terminal conditions are consistent with equilibrium bidding behavior.

D.1 Technology and information

Each bidder full-information cost is:

$$c_i\left(s_{-i}, s_i, x_0, x_i\right) = \hat{\alpha}_{i0}' x_0 + \hat{\alpha}_{i1} x_i + \hat{\alpha}_{i2} \sum_{j \neq i} \hat{\mu}_{ij} \Phi^{-1}\left(s_j\right) + \hat{\alpha}_{i3} \Phi^{-1}\left(s_i\right) \tag{12}$$

where $\hat{\alpha}$ denote the estimated parameters. The distribution of signals is truncated to avoid numerical problems: instead of using $\Phi^{-1}\left(S_i\right)$, I use $\Phi^{-1}\left(\left(S_i - 0.5\right) 0.999 + 0.5\right)$. The joint copula of signals is still assumed to be Gaussian with correlation matrix equal to $\hat{L}\hat{L}' + \hat{\Lambda}$, where $\hat{L}$ is the estimated loading matrix and $\hat{\Lambda}$ is a diagonal with i-th element equal to $1 - \hat{L}_{i1}^2 - \hat{L}_{i2}^2$. The (rescaled) signals $\Phi^{-1}\left(S\right)$ are assumed to be jointly multivariate normal with covariance matrix $\hat{\Sigma}$.

D.2 System of differential equations

Bidder i's problem is

$$\max_b b \Pr\left(S \geq s_{-i}\left(b\right) | s_i\right) - \bar{C}\left(s_{-i}\left(b\right) | s_i, x_i\right), \tag{13}$$

where $s_{-i}\left(b\right)$ is equal to $\left[H_{B_j|x}\left(b\right)\right]_{j \neq i}$. The first-order condition is

$$P_i + b \nabla P_i s_{-i}'\left(b\right) - \nabla \bar{C}_i s_{-i}'\left(b\right) = 0, \tag{14}$$

where ∇P_i and $\nabla \bar{C}_i$ are the gradients of $P_i = \Pr\left(S_{-i} \geq s_{-i}(b) | s_i\right)$ and $\bar{C}\left(s_{-i}\left(b\right) | s_i, x_i\right)$ with respect to s_{-i}. s_{-i}' is the vector of derivatives of each element of s_{-i} with respect to b. P_i and b are scalars, ∇P_i and $\nabla \bar{C}_i$ are $1 \times (n-1)$ vectors, and $s_{-i}'\left(b\right)$ is an $(n-1) \times 1$ vector.

If all bidders' first-order conditions are considered together:

$$P = M s', \tag{15}$$

where P is an $n \times 1$ vector with typical element: P_i, M is an $n \times n$ matrix with zeros in the main diagonal and typical row: $\nabla \bar{C}_i - b \nabla P_i$, and s' is an $n \times 1$ vector.

D.3 Algorithm

The algorithm makes an initial guess on the lowest bid that firms are willing to submit when they receive signal $s = 0$. So the following variables are initialized: $b^{(0)}$, and $s^{(0)} = [0, 0, ..., 0]'$. The state variables are the scalar $b^{(t)}$ and the vector $s^{(t)}$.

At step t, each bidder's perceived probability of winning $P_i^{(t)} = \Pr\left(S_{-i} \geq s_{-i}^{(t)} | s_i^{(t)}\right)$, expected cost $\bar{C}\left(s_{-i}^{(t)} | s_i^{(t)}, x_i\right)$ and the gradients ∇P_i and $\nabla \bar{C}_i$ are calculated by numerical integration . A bidder is “active” if it finds it profitable to win at the current state— if $b^{(t)}P_i^{(t)} - \bar{C}\left(s_i^{(t)}, s_{-i}^{(t)}\right) \geq 0$. Bidders that do not satisfy this condition are “inactive” and their signal state in the next step is: $s_j^{(t+1)} = s_j^{(t)}$. Construct vector $P^{(t)}$ and matrix $M^{(t)}$ with all “active” bidders and compute $s' = \left(M^{(t)}\right)^{-1} P^{(t)}$. If all s' are positive, set $s_i^{(t+1)} = s_i^{(t)} + s_i'\Delta b$ and $b^{(t+1)} = b^{(t)} + \Delta b$, where Δb is a predetermined bid step. Some elements of the resulting s' may be negative. It means that some firms would prefer to bid higher even if their profits are positive at the current bid. I discuss how to obtain a subset of “active and willing” bidders from the set of “active” bidders below. For the moment assumethat $s' > 0$.

The simulation stops when all but one bidders have nonpositive expected profits, when a bidder reaches $s_i = 1$, or when the system diverges. For low initial $b^{(0)}$, all but one bidders reach their zero profit conditions at low signals, the resulting bidding strategies are consistent with a reserve price equal to $b^{(T)}$, where T is the terminal step. As the initial $b^{(0)}$ increases, the terminal $b^{(T)}$ and signals $s^{(t)}$ also increase. Eventually, the terminal $s^{(t)}$ is such that $s_i = 1$ for some bidder. The resulting bidding strategies are consistent with a nonbinding reserve price. If $b^{(0)}$ is slightly higher, the system diverges and the resulting strategies are not consistent with any equilibrium bidding behavior. The system diverges when all bidders expected profits increase with t. To obtain the results in Section 5.3, I find the initial $b^{(0)}$ consistent with an equilibrium with the highest non-binding reserve price, i.e., the highest reserve price for which the project is procured with probability one.

D.4 “Active and willing” bidders

In a general interdependent asymmetric model the equilibrium bidding strategies may be such that the support of bids is different across bidders. For example, suppose that there are three bidders A, B and C such that A’s and B’s costs are distributed on $[\underline{c}, \bar{c}]$ while C’s costs are distributed on $[\underline{c}^*, \bar{c}]$, for $\underline{c} < \underline{c}^*$. The equilibrium bids can be such that A and B bid $\underline{b} < \underline{c}^*$ when their costs are $\underline{c}$, but C never bids $\underline{b}$. The optimal bid of C when his costs are $\underline{c}^*$ are above $\underline{c}^*$; as a result, there is a range of bids for which bidder C has positive profits, but finds it unprofitable to submit such a bid—he is “active but unwilling”. The first-order condition of “active and unwilling” bidders have to be positive, i.e. they would increase their expected profit by bidding higher. It follows that if k_i is a slack variable,

$$P_i - k_i + b\nabla P s'_{-i}(b) - \nabla\bar{C} s'_{-i}(b) = 0, \tag{16}$$

where $k_i = 0$ if i is willing to submit bid b, and $k_i > 0$ otherwise. The system of equations for all bidders can be written as:

$$P - k = Ms'. \tag{17}$$

where $k_i = 0$ and $s_i' > 0$ for willing bidders, and $k_i > 0$ and $s_i' = 0$ for unwilling bidders. Therefore, the problem of finding all "willing" bidders is the problem of choosing a set of indices J such that:

$$\begin{bmatrix} M_s & 0 \\ M_{sn} & I \end{bmatrix}^{-1} P = \begin{bmatrix} s_{j\in J} \\ k_{j\notin J} \end{bmatrix} \geq 0, \tag{18}$$

where M_s is a square matrix $\#J \times \#J$ that contains element m_{ij} only if $i, j \in J$, while M_{sn} is a $(n - \#J) \times \#J$ matrix that contains element m_{ij} only if $i \notin J$ but $j \in J$. This is a combinatorial problem that can be solved by a brute force approach if there are only a few bidders. Instead, I consider the following algorithm: denote $D_p = diag(P)$ and let $\tilde{k} = D_p k$ and $\tilde{M} = D_p^{-1}$. Equation (17) becomes:

$$1 - \tilde{k} = \tilde{M}s. \tag{19}$$

I find the Perron–Frobenius eigenvector of $\tilde{M}$ and try J equal to the indices of its largest elements. I try first with the first two, then the first three largest elements and so on. This algorithm guides the brute force approach and finds the right set of "willing" bidders faster.

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