Econometric Methods for Auctions[1]

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Abstract
This chapter offers a comprehensive overview of econometric methods for the structural analysis of auction data. We survey the classical direct approach as well as the indirect

approach which relies on the bidder's first-order conditions thereby avoiding the computation of the equilibrium strategies. Consequently, the latter allows to estimate a large class of auction models. Estimation methods can be parametric, semiparametric or nonparametric. We also address the fundamental issues of identification of the model primitives and the restrictions imposed by the model on observables. Beyond the standard auction mechanisms such as first-price sealed-bid and ascending auctions, we cover the estimation of advanced auction models such as sequential auctions, multi-unit auctions, scoring auctions and auctions of contracts to name a few.

Keywords
Auctions; identification; structural estimation; parametric, semiparametric and nonparametric estimation methods

JEL Classification: C14, C57, D44

Section 1: Introduction

The past twenty five years have witnessed tremendous developments in the econometrics of auctions which has become one of the most successful areas of structural econometric research. The well defined auction rules, an abundant theoretical literature and the availability of data constitute altogether a strong foundation for the structural analysis of auction data. The empirical literature on auctions is motivated by two objectives. On one hand, auctions constitute a unique opportunity for economists to confront game theoretic models of incomplete information to field data. At the 1995 Marshall Lecture, Professor Jean-Jacques Laffont stated that "the methodologial progresses which will be made in the future of the econometrics of auctions, will be a useful indicator of what game theory can hope to bring to empirical industrial organization." See Laffont (1997). On the other hand, economists are interested in a large number of policy issues such as the empirical assessment of reserve prices, changes in auction rules and increased competition to name a few, or questions whether one should use auctions instead of bargaining, contracting or posting prices. This trend toward empirical concerns is also noticeable in auction theory with the books by Klemperer (2004) and Milgrom (2004) linking auction theory and practical issues complementing traditional surveys on auction theory by McAfee and McMillan (1987a), Wilson (1992) and the textbook by Krishna (2010). In this respect, the recent sale of the spectrum and emission permits, the booming online platforms and the markets for electricity following the deregulation of the sector have called for new expertise with novel questions to address.

As in many economic fields, the empirical analysis of auction data has started with a reduced form approach which culminated with the work by Professor Kenneth Hendricks and Professor Robert Porter as surveyed by Porter (1995) showing the relevance of strategic auction models under incomplete information. Given such an empirically grounded motivation, the structural approach has been quickly adopted as it allows richer policy conclusions through counterfactual exercises. The structural approach requires that an explicit model defined by some primitives be chosen to explain the data. Such a model results from the combination of the observed auction rules and a reduced form data analysis. With a model in hand, the next step is to derive an econometric model to estimate the primitives and potentially test the model validity. Econometrics enters at this stage. The complexity and nonlinearity of auction models call for the development of new econometric methods. We can distinguish two general points of view to estimate the model primitives. The classical direct methodology relies on the equilibrium strategies that link the observed bids to the unobserved bidders' private information and model primitives. With equilibrium strategies that are often not in explicit form, direct econometric methods often become highly computational thereby limiting the models that can be estimated. In contrast, the recent indirect methology relies on the first-order conditions characterizing the equilibrium strategies and hence circumvents the computation of the latter. Moreover, the equilibrium bidding strategies in an auction model depend on the whole distribution of bidders' private information. Thus estimating one or more moments of this distribution is not sufficiently informative. Because the analyst has little prior information on the distribution of bidders' private information, whereas most parametric families can lead to an inadequate fit to the observed bid distributions, these call for nonparametric methods. A similar situation arises in labor economics with equilibrium models of wage dispersion, where the distributions of firms' and/or workers' productivities play the role of agents' private information and are thus left unspecified. See Bontemps, Robin, Van den Berg (2000) and Postel-Vinay and Robin (2002).

The econometrics of auctions tackles fundamental questions at the core of the structural approach. Following Koopmans (1945), a first issue is to determine whether the model primitives can be uniquely recovered from the observables which are usually the bids. In view of the previous discussion, identification is studied without parametric assumptions on the model primitives and exploits the monotonicity of the equilibrium strategies. In case of nonidentification, parametric assumptions are introduced with parsimony or additional identifying restrictions such as classical exclusion restrictions are imposed. An alternative approach initiated by Manski (1995) relies on partial identification and bounds the model primitives or other features of interest. See the recent survey by Molinari (2020) in the companion volume of the Handbook of Econometrics. Often, identification is constructive and provides key equations to estimate the model primitives. A second issue relates to testing the model validity on the data. This requires the characterization of the restrictions imposed by the model on observables which are then used to develop specification tests. A related question is whether competing models can be distinguished in view of the observables. A third issue is to develop estimation procedures that are not numerically complex and/or computationally demanding. These challenges call for fruitful collaborations with econometricians. It is then frequent in the econometrics of auctions to observe coauthorships where one of the coauthor's main research field is applied econometrics or microeconometrics.

An interesting feature of auctions is the wide variety of mechanisms used in practice thereby providing an endless source of inspiration for research. To name a few, beyond the traditional first-price sealed-bid and ascending auctions, auctions may also involve a scoring rule, an ex post round of bargaining and can be multi-unit or subject to contingent payments upon ex post realization. Also, when the observed auction mechanism significantly deviates from the theoretical model or becomes too difficult to model, an incomplete approach becomes useful based on minimal assumptions of bidders' rational behavior. Each case constitutes an econometric challenge leading to an original solution. Moreover, auctions can be viewed as a mechanism of price formation and allocation under private information. A number of economic problems can be analyzed through the lens of auctions and thus estimated using methods from the econometrics of auctions. The latter has also paved the way to the structural estimation of more general models under incomplete information such as bargaining and nonlinear pricing. In addition to the development of econometric techniques, the estimation methods cover a large span ranging from sieve estimation to deconvolution problems. In this respect, the econometrics of auctions provides interesting material for teaching microeconometrics. More generally, the field belongs to the expanding literature on the econometrics of games which started with the estimation of discrete games. See the surveys by Bajari, Hong, Nekipelov (20013) on the econometrics of discrete games and Berry and Reiss (2007) and Berry and Tamer (2007) on entry models in empirical industrial organization.

This chapter attempts to be as complete as possible from the first to recent developments of the econometrics of auctions. There is, however, some material that we voluntarily exclude from this chapter. This includes the estimation of auctions under bounded rationality such as level k auctions and auctions under ambiguity. Since the field is rapidly expanding, we have also applied strict criteria for the selection of papers we include in this chapter. We survey papers only when there is a significant new econometric component. As a result, we excluded papers whose main purpose is empirical or policy oriented. See Hortacsu and Perrigne (2021) for a comprehensive survey of the empirical auction literature. Lastly, given the length of the current chapter, we excluded papers on testing auction models as such a topic would deserve a survey by itself though we indicate the restrictions implied by some models which are used in model specification testing.

There are several surveys preceding this one. Early surveys by Hendricks and Paarsch (1995) and Hong and Shum (2000) focus on parametric direct estimation with an emphasis on private values for the former and common values for the latter for both ascending and first-price sealed-bid auctions. The survey by Perrigne and Vuong (1999) highlights the promising indirect approach for estimating nonparametrically first-price sealed-bid auction models with private values. The textbook by Paarsch and Hong (2006) offers a complete survey of the early structural literature that relies mostly on parametric econometric models. Athey and Haile (2006, 2007) offer the most comprehensive surveys on the structural analysis of auction data up to the mid 2000s. Athey and Haile (2007) in the previous volume of the Handbook of Econometrics focus on nonparametric identification, whereas Athey and Haile (2006) are more empirically oriented with some applications. The same year, Hendricks and Porter (2007) in the previous volume of the Handbook of Industrial Economics survey a mix of methods and empirical results and emphasize the reduced form versus structural approaches as well as the positive and normative goals of the empirical literature. More recently, Hickman, Hubbard and Saglam's (2012) survey informally covers a wide range of recent topics, whereas Gentry, Hubbard, Nekipelov and Paarsch's (2018) survey is the most up to date and favors empirical illustrations to well known auction data sets.

This chapter constitutes a solid reference for researchers wishing to expand the literature on the econometrics of auctions. Relative to previous surveys, our chapter is more econometrically oriented with a focus on estimation in addition to identification. It also makes the links with existing econometric methods and statistical properties. Lastly, our chapter is self-contained and clarifies various contributions. Thus it could be a support for teaching graduate field courses in microeconometrics and the structural analysis of auction data. The chapter is organized as follows. Section 2 briefly reviews the basic auction models for first-price sealed-bid and ascending auctions. Section 3 considers the benchmark first-price sealed-bid auction model with independent private values and surveys classical direct and recent indirect estimation methods with a discussion of identification and model restrictions. The following Sections 4, 5 and 6 cover the econometrics of first-price sealed-bid auctions within the paradigm of private values. Section 4 reviews basic extensions with the introduction of a reserve price, bidders' asymmetry, affiliated private values and unobserved heterogeneity, whereas Section 5 reviews advanced extensions with risk aversion, bidders' entry, sequential and simultaneous auctions. Section 6 considers alternative auction mechanisms with share auctions, scoring auctions and auctions of contracts in which additional variables beyond bids are part of the auction mechanism. Section 7 surveys the econometrics of ascending auctions which relies on order statistics as well as basic and advanced extensions. It also covers incomplete models with the estimation of bounds. Section 8 deviates from private values to cover direct and indirect estimation methods for first-price sealed-bid, ascending and share auctions with common value. Lastly, Section 9 contains some concluding remarks.

Section 2: A General Model

As mentioned in the introduction, the auction mechanisms used in practice are very diverse. In view of the existing literature on the econometrics of auction data, we first focus on single-object auctions with two widespread selling mechanisms, namely, the ascending (button) auction and the first-price sealed-bid auction. See Sections 5 and 6 for multiple-object auctions such as sequential and simultaneous auctions as well as alternative allocation rules including share auctions, scoring auctions and auction of contracts. This section introduces a general framework that defines the bidders' valuations for the auctioned object. This framework is called the interdependent value model though in most papers two polar cases are often entertained. It provides a fundamental understanding of the model structure and its economic content. It also allows us to build important extensions in Sections 4 and 5 such as bidders' asymmetry, risk aversion and entry among others. The theoretical material in this section and following ones can be found in Krishna (2010) book on auction theory, which is remarkable in its clarity.

Section 2.1: Interdependent Values

Notations and Assumptions

A single and indivisible object is put for sale by an auctioneer. When the auctioneer is the buyer, we refer to procurement auctions commonly called reverse auctions. The auctioneer decides the auction mechanism, first-price sealed-bid or ascending in our case, and also possible features such as the level of the reserve price as studied in Section 4.1. For now, there is no reserve price or the reserve price is not binding in the sense that it does not prevent any bidder from participating. There are I risk neutral bidders participating in the auction.

We consider the pioneering model by Wilson (1977). This model is interesting as it is simple but rich enough to encompass a large class of auction models. An auction is modeled as a game of incomplete information. Each bidder possesses some private information known to himself only. Specifically, each bidder receives a noisy signal σ_i taking values in $[\underline{\sigma}, \overline{\sigma}] \subset I\!R$. Each bidder also has a valuation or utility U_i for the object. This utility depends on his own signal σ_i as well as a common component C that is unknown to bidders. This common component C affects all bidders' utilities in a similar way and takes values in $[\underline{c}, \overline{c}] \subset I\!R$. Formally, bidders' utility is $U_i = U(\sigma_i, C) < \infty$, where the utility function $U(\cdot, \cdot)$ is nonnegative and increasing in both arguments. Bidders are risk neutral. A classical and early example of signal and common component originates from gas lease auctions. Each biddder has a private estimate σ_i which aggregates geological and engineering surveys on the amount of oil in the tract, his opportunity cost, a value estimate of the oil tract, etc whereas the common component C is the actual amount of oil in the tract. The latter affects all bidders but is unknown to them at the time of bidding.

Bidders' signals and common component $(\sigma_1, \ldots, \sigma_I, C)$ are random variables drawn from a joint distribution $F(\cdot, \ldots, \cdot)$ with a positive density $f(\cdot, \ldots, \cdot)$ on the support $[\underline{\sigma}, \overline{\sigma}]^I \times [\underline{c}, \overline{c}]$. The interdependent value model draws its name from the nature of dependence among the signals and the common component.[2] Indeed, $(\sigma_1, \ldots, \sigma_I, C)$ are affiliated as defined next. Let $\vee$ ($\wedge$) denote the component wise minimum (maximum) between two vectors z and z'.

Affiliation: *Let Z be a random vector with joint density $f_Z(\cdot)$ on a rectangular support $\mathcal{S}_Z$. The variables in Z are affiliated if and only if $f_Z(z \vee z')f_Z(z \wedge z') \geq f_Z(z)f_Z(z')$ for every $(z, z') \in \mathcal{S}_Z^2$.*

In simple terms, affiliation is a concept of positive dependence. It means that if a variable in Z takes a large value, the other variables in Z are more likely to take large values as well. Also, a twice-differentiable joint density $f_Z(\cdot, \ldots, \cdot)$ of $Z = (Z_1, \ldots, Z_n)$ is affiliated if and only if $\partial^2 \log f_Z(z)/\partial z_j \partial z_k \geq 0$ for any pair (j, k) with $j \neq k$. Affiliation is also called log-supermodularity. See Milgrom and Weber (1982) and Krishna (2010) for properties.

Under affiliation, the bidders' signals and the common component are positively dependent. Within an incomplete information setting, each bidder only knows his own

[2] A more general formulation is to let $U_i = \mathrm{E}[V_i|\sigma_1, \ldots, \sigma_I]$ where $(V_1, \ldots, V_I, \sigma_1, \ldots, \sigma_I)$ are jointly distributed as $F(\cdot, \ldots, \cdot)$. See e.g. Krishna (2010, Chapter 6).

signal σ_i but he also knows that his competitors' signals σ_j and the common component C are likely to take large values if he has a strong signal σ_i. Art auctions provide a classical economic example. There is a prestige effect to own a piece of art from a valued artist which explains the affiliation among the bidders' signals and the common value, which could be the resale value. The model structure $[U(\cdot,\cdot), F(\cdot,\ldots,\cdot), I]$ is common knowledge to all bidders and to the auctioneer. A special case arises when bidders are symmetric, i.e., when the joint density $f(\cdot,\ldots,\cdot)$ is exchangeable in its first I arguments. Equilibrium bidding strategies with symmetric bidders are reviewed in Sections 2.2 and 2.3. See Sections 4, 5, 6, 7 and 8 for models with asymmetric bidders.

Wilson's (1977) model has been extended by Milgrom and Weber (1982). The latter consider that the bidders' utilities depend not only on their own signal and the common component but also on the signals of their competitors, i.e., $U_i = U(\sigma_i, \sigma_{-i}, C)$, where $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_I)$. Moreover, the common component C is more general as it might be multidimensional so as to include variables which may influence the value of the object to all bidders while some can be observed by the seller. This more general model will be studied in Section 8. Nonetheless, Wilson's (1977) model contains the key ingredients and encompasses the most classical models of private value and pure common value as explained next.

The Paradigms of Private Value and Common Value

The empirical literature has focused on two paradigms that are polar cases of the general interdependent value model. In the paradigm of private value, the utility of each bidder is equal to his own signal, i.e., $U(\sigma_i, C) = \sigma_i$. In this case, it is customary to replace σ_i by V_i for private value. The interpretation is that each bidder knows his monetary value or willingness-to-pay V_i for the auctioned object. These values can be independent or affiliated among bidders. The former is called the Independent Private Value (IPV) paradigm while the latter is called the Affiliated Private Value (APV) paradigm. In Section 2.3, we review the symmetric IPV model which is the most commonly used in the empirical auction literature. See Section 4.3 for the APV model.

Within the paradigm of Pure Common Value (PCV), the bidder's utility is equal to the (unknown) common component C, i.e., $U(\sigma_i, C) = C$. This implies that all bidders receive the same utility from owning the good. This model abstracts away from any private component such as opportunity cost, value estimate, etc affecting the bidder's utility. It has been initially developed to analyze gas lease auction data and is frequently called the Mineral Rights Model. This model assumes that the signals are conditionally independent upon the common value C. Specifically, each bidder's signal σ_i is drawn from a conditional density $f_{\sigma|C}(\cdot|\cdot)$. Thus the joint density of signals and common value satisfies $f(\sigma_1, \ldots, \sigma_I, C) = f_C(C)f_{\sigma|C}(\sigma_1|C) \ldots f_{\sigma|C}(\sigma_I|C)$, where $f_C(\cdot)$ denotes the marginal density of the common value. It can be easily verified that this joint density is affiliated if $\partial^2 \log f_{\sigma|C}(\sigma|c)/\partial\sigma\partial c \geq 0$. Typically, this model also assumes that each signal σ_i is an unbiased estimator of the common value C, i.e., $\mathrm{E}[\sigma_i|C] = C$.

In the early empirical literature, researchers have focused on the issue of the winner's curse. Each bidder knows only his signal σ_i and based on this information alone he estimates the common value by $\mathrm{E}[C|\sigma_i]$. If the bid is increasing in the signal, winning the object reveals to the winner that he has the highest signal among the I bidders. Thus, winning the object conveys bad news as he is the most 'optimistic' bidder with the highest estimate leading to the winner's curse. As expected, this bad news becomes worse when the number of bidders is large. Fortunately, the winner's curse can be alleviated if bidders are rational and incorporate this effect by shading appropriately their bids below their estimates. Thus, in equilibrium the winner's curse does not arise. See Section 8.

Section 2.2: First-Price Sealed-Bid and Ascending Auctions

The theoretical auction literature mainly considers four auction mechanisms that are the second-price sealed-bid or Vickrey auction, the ascending or English auction, the first-price sealed-bid auction and the descending or Dutch auction. Since the second and third mechanisms are most frequently used in practice, the empirical literature and estimation methods have mostly focused on the ascending auction and the first-price sealed-bid auction. In the first-price sealed-bid auction, the I bidders submit their bids in sealed envelopes. The auctioneer opens the envelopes and ranks the bids to allocate the object to the highest bidder who pays his bid. In the ascending auction, rules can differ but theorists use the so-called button auction that is analytically simple. All bidders press a button at the beginning of the auction. The price rises continuously and bidders release the button to indicate that they are no longer in the competition. Once they leave the auction, they cannot come back. The remaining bidders observe the price at which each bidder leaves the auction. The last bidder who has not released the button wins the object and pays the price at which the bidder before him released the button.

Equilibrium in Symmetric Ascending Auctions

In an ascending (button) auction, the values at which other bidders drop out and the number of bidders who drop out are valuable information. Indeed, each bidder updates the expectation of his valuation in view of this information that evolves along the auction. By doing so, he avoids the winner's curse.

Milgrom and Weber (1982) characterize the Bayesian Nash equilibrium in ascending auctions with symmetric bidders. Without loss of generality, we consider bidder 1 since all bidders are symmetric. Let $\sigma_{-1}^{(1)} \leq \sigma_{-1}^{(2)} \leq \ldots \leq \sigma_{-1}^{(I-1)}$ be the largest to the smallest of the signals $\sigma_2, \sigma_3, \ldots, \sigma_I$ of bidder's 1 competitors. Bidder 1 needs to decide whether to remain active or drop out at every price. This decision depends on his signal σ_1, the number of bidders k who quit and the price levels at which they quit $(p_1, \ldots, p_k)$ with $p_1 < \ldots < p_k$. Thus, the bidder's strategy is history dependent. Let x be the value of bidder's 1 signal, i.e., $\sigma_1 = x$. If no bidder has quit, bidder 1 quits when the price reaches

$$s_0(x) \quad = \quad \mathrm{E}[U_1 | \sigma_1 = x, \sigma_{-1}^{(I-1)} = x, \ldots, \sigma_{-1}^{(1)} = x]. \tag{2.1}$$

When $k = 1, \ldots, I-2$ bidders have quit, bidder 1 quits when the price reaches

$$\begin{aligned} s_k(x|p_1, \ldots, p_k) \quad = \quad & \mathrm{E}[U_1 | \sigma_1 = x, \sigma_{-1}^{(I-1)} = x, \ldots, \sigma_{-1}^{k+1} = x, \\ & s_{k-1}(\sigma_{-1}^{(k)} | p_1, \ldots, p_{k-1}) = p_k, \ldots, s_0(\sigma_{-1}^{(1)}) = p_1]. \end{aligned} \tag{2.2}$$

The strategy $s(\cdot) \equiv [s_0(\cdot), \ldots, s_{I-2}(\cdot)]$ as defined in (2.1)-(2.2) constitutes a symmetric Bayesian Nash equilibrium since $(s_0, \ldots, s_{I-2})$ are best responses given each bidder's private information. A simple way to understand the ascending auction is to view it in two phases. In the first phase, the $I-2$ losing bidders, i.e., those with the lowest estimates, reveal their signals. In the second phase, the last two bidders engage in a second-price sealed-bid auction, where the information revealed in the first phase raises the expected price of the second phase. Therefore, the expected revenue for the seller in an ascending auction is larger than in a second-price sealed-bid auction in which the winner is the highest bidder who pays the second highest bid. In the latter mechanism, the equilibrium strategy is to bid $v(\sigma_1, \sigma_1)$ where $v(x, y) \equiv \mathrm{E}[U_1 | \sigma_1 = x, \sigma_{-1}^{\max} = y]$ is the expectation of bidder's 1 utility $U_1 = U(\sigma_1, C)$ when his signal σ_1 is equal to x and the highest signal among his competitors $\sigma_{-1}^{\max} \equiv \sigma_{-1}^{(I-1)} = \max_{j \neq 1} \sigma_j$ is equal to y. Because of affiliation, the function $v(\cdot, \cdot)$ is increasing in both arguments.

This equilibrium has some key properties. First, the equilibrium strategy $s(\cdot)$ depends on the utility function $U(\cdot, \cdot)$, the conditional distribution $F_{C|\sigma_1, \ldots, \sigma_I}(\cdot|\cdot, \ldots, \cdot)$ but not on the joint signal distribution $F_{\sigma_1, \ldots, \sigma_I}(\cdot, \ldots, \cdot)$. Thus, several signal distributions may lead to the same expectations (2.1)-(2.2) and the same equilibrium strategy $s(\cdot)$. Hence, the strategy $s(\cdot)$ leads to an ex post equilbrium since $s(\cdot)$ is a Nash equilibrium of the complete information game that results if the signals $(\sigma_1, \ldots, \sigma_I)$ were commonly known. In particular, bidders do not suffer from any regret in the incomplete information game. See Krishna (2010). Second, the strategy (2.1)-(2.2) is increasing in one's own signal. This means that the winning bidder is the one with the largest signal. However, allocating the object to the bidder with the highest signal does not necessarily guarantee an efficient allocation. Indeed, efficiency means that the good is allocated to the bidder with the largest valuation. To achieve efficiency, the utility function needs to satisfy a single crossing condition. In Wilson's (1977) model, this condition is satisfied since $\partial U(\sigma_i, C)/\partial\sigma_i \geq \partial U(\sigma_j, C)/\partial\sigma_i = 0$. In Milgrom and Weber's (1982) general model, the single crossing property, namely $\partial U(\sigma_i, \sigma_{-i}, C)/\partial\sigma_i \geq \partial U(\sigma_j, \sigma_{-j}, C)/\partial\sigma_i$ needs to be satisfied to attain efficiency for any pair of bidders $i \neq j$. Third, as noted above, the ascending auction generates a higher expected revenue for the seller than the second-price sealed-bid auction. This property is related to the positive effect of information revealed along the auction. More generally, any valuable information revealed by the seller such as information about the common component has a positive impact on the expected price as it reduces the uncertainty about the common value. Fourth, it is easy to verify that the equilibrium strategy $s(\cdot)$ has the following location-scale invariance property. If signals, common value and utility function were $\tilde{\sigma}_i = \kappa_0 + \kappa_1\sigma_i$ for all i, $\tilde{C} = \kappa_0 + \kappa_1 C$ and $\tilde{U}(\cdot, \cdot) = \kappa_0 + \kappa_1 U[(\cdot - \kappa_0, \cdot - \kappa_0)/\kappa_1]$ for some κ_0 and $\kappa_1 > 0$, then $\tilde{s}(\cdot) = \kappa_0 + \kappa_1 s[(\cdot - \kappa_0)/\kappa_1]$ is a symmetric equilibrium strategy in the tilde setting. This property is used later.

Equilibrium in Symmetric First-Price Sealed-Bid Auctions

Wilson (1977) and Milgrom and Weber (1982) characterize the Bayesian Nash equilibrium in first-price sealed-bid auctions with symmetric bidders. Again, without loss of generality we consider bidder 1 since bidders are symmetric. At such an equilibrium bidder 1 chooses a bid B_1 that maximizes his expected profit from winning the auction. His net profit is $U(\sigma_1, C) - B_1$. In addition, he needs to consider his probability of winning the auction, i.e., that his bid is larger than the bids of his opponents. As usual in theory, we assume that the equilibrium bidding strategy $s(\cdot)$ is increasing in the signal. This monotonicity property is checked after the derivation of the equilibrium. The bidder's ex post profit is $[U(\sigma_1, C) - B_1]\mathbb{1}[B_1 \geq s(\sigma_{-1}^{\max})]$. Thus, conditional on his signal σ_1, bidder's 1 expected profit is

$$\begin{aligned} \mathrm{E}\Big\{[U(\sigma_1, C) - B_1]\mathbb{1}[B_1 \geq s(\sigma_{-1}^{\max})]|\sigma_1\Big\} &= \mathrm{E}\Big\{\mathrm{E}\Big[(U(\sigma_1, C) - B_1)\mathbb{1}[B_1 \geq s(\sigma_{-1}^{\max})]|\sigma_1, \sigma_{-1}^{\max}\Big]\,|\sigma_1\Big\} \\ &= \int_{\underline{\sigma}}^{s^{-1}(B_1)} [v(\sigma_1, y) - B_1] f_{\sigma^{\max}|\sigma}(y|\sigma_1)dy, \qquad (2.3) \end{aligned}$$

with $v(x, y) \equiv \mathrm{E}[U_1|\sigma_1 = x, \sigma_{-1}^{\max} = y]$ and $F_{\sigma^{\max}|\sigma}(\cdot|\cdot)$ is the distribution of $\sigma^{\max}$ given σ with density $f_{\sigma^{\max}|\sigma}(\cdot|\cdot)$, where the subscript 1 is dropped by symmetry. Maximizing (2.3) with respect to B_1 leads to the differential equation

$$s'(\sigma_1) = [v(\sigma_1, \sigma_1) - s(\sigma_1)] \frac{f_{\sigma^{\max}|\sigma}(\sigma_1|\sigma_1)}{F_{\sigma^{\max}|\sigma}(\sigma_1|\sigma_1)} \tag{2.4}$$

using $s^{-1}(B_1) = \sigma_1$. The quantity $\nu(\sigma_1) \equiv v(\sigma_1, \sigma_1)$ is called bidder 1's (expected) pivotal value. The boundary condition is $s(\underline{\sigma}) = \nu(\underline{\sigma})$. Solving (2.4) subject to this boundary condition gives the closed form solution

$$B_1 = s(\sigma_1) = \nu(\sigma_1) - \int_{\underline{\sigma}}^{\sigma_1} L(a|\sigma_1) d\nu(a), \tag{2.5}$$

where $L(a|\sigma_1) = \exp[-\int_a^{\sigma_1} f_{\sigma^{\max}|\sigma}(\sigma|\sigma)/F_{\sigma^{\max}|\sigma}(\sigma|\sigma)d\sigma]$. Since $L(\cdot|\sigma_1) > 0$, we have $\nu(\sigma_1) > s(\sigma_1)$ for $\sigma_1 > \underline{\sigma}$. The equilibrium strategy $s(\cdot)$ is increasing as desired since $\nu(\sigma)$ is an increasing function of σ. The second term in the right-hand side (RHS) of (2.5) is interpreted economically as the shading of the pivotal valuation $\nu(\sigma)$. It is the rent left to the bidder. The strategy (2.5) gives a symmetric Bayesian Nash equilibrium.

In addition to its monotonicity, this equilibrium has four key properties. First, regarding efficiency, the same argument as in ascending auctions applies here. The single crossing condition is satisfied in Wilson's (1977) model since the bidder's utility does not depend on the opponents' signals. In Milgrom and Weber's (1982) model, the single crossing condition needs to be satisfied for first-price sealed-bid auctions to be efficient. Second, a first-price sealed-bid auction is dominated by a second-price sealed-bid auction in terms of expected revenue. Thus, the ascending auction generates more expected revenue than the first-price sealed-bid auction in view of the revenue ranking indicated previously. As in ascending auctions, any valuable information on the common component revealed by the seller increases the expected revenue in a first-price sealed-bid auction. Third, a descending auction in which the price decreases continuously and the winner is the first bidder expressing his willingness to buy at the current price is informationally equivalent to a first-price sealed-bid auction. Indeed, there is no revealed information during the bidding process of a descending auction. Hence, the equilibrium strategy is also given by (2.5). Fourth, it is easy to verify that the equilibrium strategy $s(\cdot)$ satisfies the location-scale invariance property. That is, $\tilde{s}(\cdot) = \kappa_0 + \kappa_1 s[(\cdot - \kappa_0)/\kappa_1]$ if $\tilde{\sigma}_i = \kappa_0 + \kappa_1 \sigma_i$ for all i, $\tilde{C} = \kappa_0 + \kappa_1 C$ and $\tilde{U}(\cdot, \cdot) = \kappa_0 + \kappa_1 U[(\cdot - \kappa_0, \cdot - \kappa_0)/\kappa_1]$ for some κ_0 and $\kappa_1 > 0$.

Section 2.3: The symmetric IPV paradigm

The Independent Private Value paradigm is one of the earliest considered in theory and one of the most used in the empirical literature. In this case, a bidder's utility reduces to his signal, i.e. $U(\sigma_i, C) = \sigma_i$, where the signals σ_i are mutually independent. As noted earlier, it is then customary to use the notation V_i instead of σ_i and to view V_i as the private value or willingness-to-pay. The equilibrium strategies in the ascending and first-price sealed-bid auctions greatly simplify. Though these strategies can be derived from (2.1)-(2.2) and (2.5), it is instructive to derive them directly.

Ascending Auctions

Though this auction can take different forms, following Section 2.2, we consider the button format in which a price increases continuously and bidders release a button to drop out. The auction ends when there is a single bidder left and this bidder pays the price at which the previous bidder drops out. Because the private values are identically and independently distributed (i.i.d.), it is easy to verify from (2.1)-(2.2) that $s_0(x) = \ldots = s_{I-2}(x) = x$. Thus the strategy $s(V_i) = V_i$ gives a symmetric Bayesian Nash Equilibrium. As a matter of fact, the strategy $s(V_i) = V_i$, which is to bid his private value, is a dominant strategy for every bidder. Indeed, a bidder will never release the button when the price is above his valuation V_i, while there is no gain by releasing the button at a value below V_i whatever the strategy used by other bidders. It is worthnoting that this strategy also applies if the private values V_i were affiliated and even asymmetric as reviewed in Section 7. As expected, the same strategy applies to a second-price sealed-bid auction.

First-Price Sealed-Bid Auctions

Our benchmark model is the first-price sealed-bid auction with symmetric independent private values. Despite its simplicity, this model contains the fundamentals upon which we can build several extensions to make the auction model richer. It is sufficiently simple to understand the main issues arising in the structural estimation of auction models and more generally of models with asymmetric information.

As above, an indivisible object is offered for sale through a first-price sealed-bid auction with no minimum price or reserve price. The $I \geq 2$ bidders are risk neutral. Bidder i has a positive value or willingness-to-pay for the object denoted V_i. The values $V_1, \ldots, V_I$ are i.i.d. from a distribution $F(\cdot)$ with density $f(\cdot) > 0$ on $[\underline{v}, \overline{v}]$ with $0 \leq \underline{v} < \overline{v} < \infty$. Each bidder knows his own value but not his competitors' values. The pair $[I, F(\cdot)]$ is common knowledge to all parties. This defines the symmetric IPV first-price sealed-bid auction model. Since this model is a special case of the interdependent value model by setting $U_i(\sigma_i, C) = \sigma_i = V_i$ where $(V_1, \ldots, V_I)$ are i.i.d. as $F(\cdot)$, we can obtain its symmetric equilibrium strategy from (2.5). It is instructive to derive it directly.

Because the number of bidders is finite, each bidder thinks strategically when choosing his bid B_i. Given that bidders have imperfect information on their competitors' values, we adopt a Bayesian Nash equilibrium as developed by Harsanyi (1967). The derivation of the symmetric equilibrium is as follows. See also Riley and Samuelson (1981). Bidder i's profit from winning the object is the monetary gain $V_i - B_i$. His probability of winning is $\Pr(B_i \geq B_{-i}^{\max})$, where $B_{-i}^{\max}$ denotes the maximum of his competitors' bids. Thus, letting his gain of not winning being zero, his expected gain from the auction is $(V_i - B_i)\Pr(B_i \geq B_{-i}^{\max})$. By symmetry, all bidders use the same strategy $s(\cdot)$ which maps the bidder's private value V_i to his bid B_i, i.e., $B_i = s(V_i)$. Assuming that $s(\cdot)$ is increasing, the probability of winning becomes $F[s^{-1}(B_i)]^{I-1}$ since bidders' values are i.i.d. Thus, assuming that $s(\cdot)$ is differentiable, maximizing the expected profit $(V_i - B_i)F[s^{-1}(B_i)]^{I-1}$ with respect to B_i and requiring that $B_i = s(V_i)$ lead to the first-order condition (FOC)

$$1 = [V_i - s(V_i)](I-1)\frac{f(V_i)}{F(V_i)}\frac{1}{s'(V_i)}, \tag{2.6}$$

for every $V_i \in [\underline{v}, \overline{v}]$. The equilibrium strategy $s(\cdot)$ solves this differential equation with boundary condition $s(\underline{v}) = \underline{v}$. Solving (2.6) gives the closed form solution

$$B_i = V_i - \frac{1}{F(V_i)^{I-1}}\int_{\underline{v}}^{V_i} F(x)^{I-1}dx \equiv s(V_i; F, I), \tag{2.7}$$

where the second term in the RHS represents the value shading or equivalently the bidder's information rent. The equilibrium strategy depends on the value distribution $F(\cdot)$ and the number of bidders I. From Maskin and Riley (1984, 2000b, 2003), the Bayesian Nash equilibrium exists, is unique, symmetric, increasing, continuous on $[\underline{v}, \overline{v}]$, and differentiable on $(\underline{v}, \overline{v}]$. This strategy is given by (2.7).

Equilibrium Properties

First, it can be verified that the symmetric equilibrium strategy (2.7) is increasing in the value V_i as desired. This property is crucial for the econometrics of incomplete information models. It also leads to efficiency of the equilibrium as the winner is the bidder with the highest valuation for the auctioned object. The latter property does not always hold under interdependent values. Also, non identically distributed values leading to asymmetry among bidders may result in an inefficient allocation. See Section 4.2. Second, it is straightforward to show that the number of bidders I has a positive impact on bidding. In other words, more competition in the auction reduces the bidders' rents and increases the expected price for the seller. Thus, more competition is desirable for the seller. This contrasts with the effect of competition when values are interdependent where the effect of increasing competition can be positive up to some value of competition and negative afterward. In the presence of common value, an increase in competition exacerbates the winner's curse. As a result, in equilibrium, bidders need to be less aggressive and correct downward their bids to account for the winner's curse. When the number of bidders becomes large, the winner's curse effect may dominate the competitive effect so that bids decrease with competition. See Krishna and Morgan (1997) and Hong and Shum (2002). The latter property also holds when private values are affiliated, suggesting that the decrease of bids with competition is not specific to the winner's curse as first noted by Laffont (1997) and formalized by Pinkse and Tan (2005).

A third property which was first discovered by Vickrey (1961) and further elaborated by Riley and Samuelson (1981) and Myerson (1981) is the Revenue Equivalence Theorem (RET). It states that within the IPV paradigm with risk neutral bidders, any symmetric and increasing equilibrium of any standard auction yields the same expected revenue to the seller provided the expected payment of a bidder with value $\underline{v}$ is zero. An auction is standard if the winner is the bidder with the highest value. Clearly, second-price sealed-bid, ascending, first-price sealed-bid and descending auctions are standard auctions. Thus, when e.g. there is no reserve price, the seller's expected revenue in each auction are all equal to the expectation of the second highest bidders' valuation. The RET has been used in the early literature on the econometrics of auctions. It has, however, limitations due to the strong assumptions necessary for this result to hold.

Procurement Auctions

Up to now, we have studied the case of a seller facing potential buyers. An auction in which a buyer faces several sellers is called a procurement or reverse auction. In practice, a buyer who can be a government agency, a private company or an individual procures a wide range of goods such as projects, equipment, services, etc. In this case, each seller i has a cost C_i for executing the project. Procurement auctions are mostly organized through first-price sealed-bid auctions in which the seller with the lowest bid wins the auction. It is straightforward to derive the Bayesian Nash equilibrium in the benchmark model with costs i.i.d. as $F(\cdot)$ with support $[\underline{c}, \overline{c}] \subset I\!R_+$. Assuming that the symmetric equilibrium strategy $s(\cdot)$ is increasing, seller i's probability of winning with a bid B_i is $\Pr[B_i < B_j; j \neq i] = [1 - F(s^{-1}(B_i))]^{I-1}$. Thus, when $s(\cdot)$ is differentiable, maximizing the expected profit $(B_i - C_i)[1 - F(s^{-1}(B_i))]^{I-1}$ with respect to B_i and requiring that $B_i = s(C_i)$ lead to the FOC

$$1 = [s(C_i) - C_i](I-1)\frac{f(C_i)}{1 - F(C_i)}\frac{1}{s'(C_i)} \tag{2.8}$$

for every $C_i \in [\underline{c}, \overline{c}]$. The equilibrium strategy $s(\cdot)$ solves this differential equation with boundary condition $s(\overline{c}) = \overline{c}$. This gives the closed form solution

$$B_i = C_i + \frac{1}{[1 - F(C_i)]^{I-1}} \int_{C_i}^{\overline{c}} [1 - F(x)]^{I-1} dx \equiv s(C_i; F, I), \tag{2.9}$$

where the second term in the RHS represents seller i's information rent or profit. The strategy (2.9) is similar to (2.7) with V_i and $F(\cdot)$ replaced by C_i and $[1-F(\cdot)]$, respectively, together with a change of sign and boundaries. The properties of the equilibrium strategies are also similar except that bids are now decreasing in competition.

To conclude this section, we give a preview of the different models we study further in the following sections. Sections 4-7 maintain the assumption of private values in first-price sealed-bid auctions with the exception of Section 7 which is devoted to ascending auctions. Section 8 addresses the PVC and interdependent value models. Sections 4 and 5 relax some assumptions of the benchmanrk IPV model. In Section 4, a reserve price limits bidders' participation to those with values above the announced reserve price. When the reserve price is kept secret, bidders need to account for the possibility that their bids might not match the secret reserve price. In the case of asymmetric bidders, the values are not identically distributed as some bidders' exogenous characteristics may affect the distribution of their values. Private values might also be affiliated leading to the first-price sealed-bid APV model. In Section 5, bidders' utility function is no longer the identity function and displays risk aversion. Also some factors occuring ex post the auction may affect bidders' values thereby introducing additional uncertainty at the time of the auction. When entry is endogenous, bidding is the second stage of a game whereas the first stage determines the number of participating bidders. Auctions can also involve multiple objects. In sequential auctions, bidders might be interested in buying several goods and their values for the potentially different goods may be dependent and exhibit synergies. This might also lead to dynamic considerations. In simultaneous auctions, bidders may bid on bundles of goods as in combinatorial auctions.

Section 6 studies more complex allocation rules. In share auctions, the good is divisible and bidders bid for a share of it. The bidding strategy is then a demand function relating quantity and price. In scoring auctions, the auctioneer considers not only the bids but other factors such as the project quality while the allocation rule expresses the trade-off between price and quality. In auctions of contracts with scale auctions and royalty auctions, bids are multidimensional and payments are contingent upon realization of the project. As in scoring auctions, bidders' private information can be multidimensional which renders the model more complicated. Section 7 studies ascending auctions within the private value paradigm and includes extensions to reserve price, asymmetry, affiliation, risk aversion, entry as well as sequential auctions. This section also introduces an alternative incomplete modeling approach to account for deviations of ascending auctions from the button model. Lastly, Section 8 covers models with common values for both first-price and ascending auctions as well as share auctions.

In most of these extensions, equilibrium strategies are complex and quite often cannot be obtained in closed forms in contrast to the benchmark model. The model properties are also different though the monotonicity of the equilibrium often remains. In the next section we explain how the econometrics of auctions has developed by exploiting the monotonicity of the equilibrium strategies even when the latter do not have explicit forms and their differential equations are hard to solve numerically.

Section 3: First-Price Sealed-Bid Auctions

In this section, we present structural estimation methods of the benchmark model of Section 2.3, namely the symmetric IPV first-price sealed-bid auction model. Broadly, they are classified as direct and indirect methods depending on whether they use the equilibrium strategy (2.7) or the first-order condition (2.6), respectively. In Section 3.1, we review direct methods as they are the most natural in structural estimation and historically preceded indirect methods. In view of its numerous advantages, however, most empirical papers have adopted the indirect approach. In particular, the latter allows to address fundamental issues in structural modeling such as the nonparametric identification of the model primitives and the characterization of the restrictions imposed by the model on observables, as presented in Section 3.2. In Sections 3.3 and 3.4, we present two nonparametric estimation methods in the spirit of the nonparametric identification argument. Section 3.3 is devoted to kernel-based estimators, which have been developed first, while Section 3.4 is devoted to quantile-based estimators. Both methods have similar asymptotic properties but their implementations differ. We also present methods that correct for issues arising with nonparametric estimators such as boundary effects and the curse of dimensionality.

A common feature to both approaches is to assume that the observed bids are the outcomes of a Bayesian Nash equilibrium of the model under consideration. This is the foundation of structural econometrics which defined an econometric model closely derived from a theoretical model. Throughout this section, there are L auctions and each auction is indexed by ℓ. Auctioned objects are potentially heterogeneous in terms of size, quality, value and many other aspects. For now, we consider that this hetereogeneity is observed by the analyst and is captured by a vector X_ℓ of dimension d. These relevant characteristics can be discrete or continuous. We review in Section 4.4 the case of unobserved heterogeneity. The value distribution is now conditional on (X_ℓ, I_ℓ) as $F(\cdot|X_\ell, I_\ell)$. In addition to X_ℓ, we note that the conditioning vector contains the number of bidders I_ℓ, which is not part *per se* of the model of Section 2.3. This is more general as it allows the bidders' values to depend on the number of bidders. Indeed, I_ℓ can capture some unobserved heterogeneity as when higher valued objects attract more bidders.

The observables are $\{B_{i\ell}, i = 1, \ldots, I_\ell, X_\ell, I_\ell, \ell = 1, \ldots, L\}$. The structural econometric model relies on the equilibrium strategy (2.7) which relates the observed bids $B_{i\ell}, i = 1, \ldots, I_\ell, \ell = 1, \ldots, L$ to the unobserved private values $V_{i\ell}, i = 1, \ldots, I_\ell, \ell = 1, \ldots, L$. Namely, $B_{i\ell} = s(V_{i\ell}; F_\ell, I_\ell)$, where $V_{i\ell}$ is i.i.d. as $F_\ell(\cdot) \equiv F(\cdot|X_\ell, I_\ell)$ given (X_ℓ, I_ℓ) with support $[\underline{v}(X_\ell, I_\ell), \overline{v}(X_\ell, I_\ell)]$. Since the unobserved values $V_{i\ell}$ are random, the observed bids $B_{i\ell}$ are random as well. A special feature of auction models is that the (unknown) conditional distribution $F(\cdot|X_\ell, I_\ell)$ of the unobservables also appears in the equilibrium strategy $s(\cdot;\cdot,\cdot)$. An alternative and frequent situation arises when the analyst only observes the winning bid $B_\ell^w \equiv \max_{i=1,\ldots,I_\ell} B_{i\ell}$ for each auction. In this case, the same remark applies as $B_\ell^w = s(V_\ell^w; F_\ell, I_\ell)$, where $V_\ell^w \equiv \max_{i=1,\ldots,I_\ell} V_{i\ell}$ is the winner's private value which is distributed as $F^{I_\ell}(\cdot|X_\ell, I_\ell)$ in auction ℓ. The purpose of the structural approach is to estimate the primitive(s) of the economic model, which is the distribution of private values $F(\cdot|\cdot,\cdot)$ in the benchmark model.

Section 3.1: Direct Estimation Methods

The direct approach has a long history in the econometrics of structural models dating back to the work of the Cowles Commission. See Koopmans (1950). It starts from specifying a family of private value distributions, often parametric $F(\cdot|X, I; \theta)$ with parameters θ. Since $B_{i\ell} = s(\cdot; F_\ell, I_\ell)$ where $F_\ell(\cdot) = F(\cdot|X_\ell, I_\ell; \theta)$, this family induces a family of distributions for the observed bids $B_{i\ell}$ given the covariates X_ℓ and the number of bidders I_ℓ. The direct approach was first used by Paarsch (1992) who initiated the structural econometrics of auctions.[3]

Maximum Likelihood (ML) Based Methods

As indicated by Donald and Paarsch (1993), a major difficulty of ML estimation is that the support of the observed bid distribution depends on the parameter θ. This also arises in labor models as first noted by Flinn and Heckman (1982). Indeed, from (2.7) the upper bound of the bid distribution given (X, I) is $\overline{b}(X, I; \theta) = \overline{v}(X, I) - \int_{\underline{v}(X,I)}^{\overline{v}(X,I)} F(u|X, I; \theta)^{I-1} du$ which depends on θ even if the bounds $\underline{v}(X, I)$ and $\overline{v}(X, I)$ of the parametric family for the value distribution are independent of θ. This violates a regularity condition for asymptotic

[3]Paarsch (1992) uses a standard method of moments upon restricting $F(\cdot|\cdot,\cdot)$ to simple parametric families of distributions such as the exponential, Pareto and Weibull distributions to obtain explicit forms for the moments of observed bids.

normality of ML estimation leading to a non-standard ML estimator whose consistency is not immediate to establish. See e.g. Amemiya (1985).

Donald and Paarsch (1993, 1996) modify ML estimation to address this difficulty by proposing a Piecewise Pseudo Maximum Likelihood (PPML) estimation procedure. Let $\overline{b}(X, I; \theta)$ be the upper bound of the support of this density. A key assumption is that there exists a partition of the parameter vector θ in (θ_1, θ_2), where θ_1 is a scalar, such that the upper bound $\overline{b}(X, I; \theta_1, \theta_2) = s[\overline{v}(X, I; \theta_1, \theta_2); X, I, \theta_1, \theta_2]$ is invertible in θ_1, where $s(\cdot; X, I, \theta) \equiv s(\cdot; F(\cdot|X, I; \theta), I)$ is the equilibrium strategy (2.7). Let $\hat{\overline{b}}(X, I)$ be an estimator of the upper bound $\overline{b}(X, I; \theta_1, \theta_2)$ such as the maximum of the winning bids in auctions with I bidders and characteristics X when the latter are all discrete. This estimator is used to eliminate θ_1 in the likelihood function by letting $\theta_1 = \theta_1[\hat{\overline{b}}(X, I), X, I; \theta_2]$, where $\theta_1(\cdot, X, I; \theta_2)$ is the inverse of $\overline{b}(X, I; \cdot, \theta_2)$. Suppose that only the winning bids B_ℓ^w are observed. The density $g^w(\cdot|X, I; \theta)$ of the winning bid B^w given (X, I) induced by the parametric specification $F(\cdot|\cdot, \cdot; \theta)$ is

$$g^w(\cdot|X, I; \theta) = \frac{If[s^{-1}(\cdot; X, I, \theta)|X, I; \theta]F^{I-1}[s^{-1}(\cdot; X, I, \theta)|X, I; \theta]}{s'[s^{-1}(\cdot; X, I, \theta); X, I, \theta]},$$

where $f(\cdot|X, I; \theta)$ and $s'(\cdot; X, I)$ are the density and derivative of $F(\cdot|X, I; \theta)$ and $s(\cdot; X, I)$, respectively, whereas $s^{-1}(\cdot; X, I)$ is the inverse of $s(\cdot; X, I)$. The PPML procedure consists in maximizing the log-likelihood function

$$\frac{1}{L}\sum_{\ell=1}^{L} \log g^w[B_\ell^w|X_\ell, I_\ell; \theta_1(\hat{\overline{b}}(X_\ell, I_\ell), X_\ell, I_\ell; \theta_2), \theta_2], \tag{3.1}$$

with respect to the remaining parameter θ_2. Donald and Paarsch (1993) establish the consistency and $\sqrt{L}$-asymptotic normality of the resulting estimator $\hat{\theta}_2$ and hence of $\hat{\theta}_1 = \theta_1[\hat{\overline{b}}(X, I), X, I; \hat{\theta}_2]$ when all characteristics X are discrete so that the boundary estimator $\hat{\overline{b}}(X, I)$ converges at a rate faster than $\sqrt{L}$.

In a second paper, Donald and Paarsch (1996) address directly the properties of the ML estimator when the boundaries of the support depend on unknown parameters. The ML estimator is obtained by maximizing the likelihood of the winning bids subject to the constraints that all winning bids be consistent with the implied upper bounds $\overline{b}(X, I; \theta)$ of the bid distributions. This gives the following non-linear programming problem

$$\max_\theta \sum_{\ell=1}^{L} \log g^w(B_\ell^w|X_\ell, I_\ell; \theta), \tag{3.2}$$

$$\text{subject to} \quad \begin{cases} B_1^w \le \overline{b}(X_1, I_1; \theta), \\ \vdots \\ B_L^w \le \overline{b}(X_L, I_L; \theta). \end{cases}$$

They show that the ML estimator is consistent. Moreover, under appropriate assumptions including that all characteristics X are discrete, they establish its asymptotic distribution which is typically nonstandard with a L-convergence rate. This is because the ML estimator is then determined by the winning bids associated with the values (X_ℓ, I_ℓ) for which the constraints $B_\ell^w \leq \bar{b}(X_\ell, I_\ell; \theta)$ are binding. See Donald and Paarsch (2002) and Chernozhukov and Hong (2004) for ML estimation under nonstandard conditions. See also Hirano and Porter (2003) for efficient estimation.

The PPML and ML procedures can be easily adapted by modifying (3.1) and (3.2) when all bids are observed. It suffices to sum over the observed bids $\{B_{i\ell}; i = 1, \ldots, I_\ell, \ell = 1, \ldots, L\}$ and to use the induced bid density $g(\cdot|X, I; \theta) = f[s^{-1}(\cdot; X, I, \theta)|X, I; \theta]/s'[s^{-1}(\cdot; X, I, \theta); X, I, \theta]$. As recognized by Hendricks and Paarsch (1995), however, a major disadvantage of the ML-based methods is their heavy computational burden. Indeed, they require the computation of the equilibrium strategy $s(\cdot; X, I, \theta)$ and its inverse $s^{-1}(\cdot; X, I, \theta)$ to determine the upper bound $\bar{b}(X, I; \theta)$ and the bid density $g^w(\cdot|X, I; \theta)$ or $g(\cdot|X, I; \theta)$ for any value (X_ℓ, I_ℓ) in the data and any trial value θ in the maximization of (3.1) or in the resolution of the non-linear programming problem (3.2). In addition, (3.1) requires the inversion of the upper bound $\bar{b}(X, I; \cdot, \theta_2)$ to obtain $\theta_1(\cdot, X, I; \theta_2)$. Hence, as in Paarsch (1992), only a few parametric specifications of $F(\cdot|X, I; \theta)$ can be entertained so as to obtain simple expressions for the bidding strategies.

Simulated Non-Linear Least-Squares Methods

In view of the computational drawbacks associated with ML-based methods, Laffont, Ossard and Vuong (1995) develop a simulation-based estimator in the spirit of McFadden (1989) and Pakes and Pollard (1989). While still parameterizing the private value distribution, the main advantage of their method is to avoid the computation of the equilibrium strategy and hence to allow for arbitrary families of distributions $F(\cdot|\cdot, \cdot; \theta)$. Moreover, because it exploits the RET, standard auctions such as descending, second-price sealed-bid and ascending auctions can be analyzed as well within the IPV paradigm. Specifically, the authors consider a descending auction where only the winning bids $B_\ell^w, \ell = 1, \ldots, L$ are observed. As noted in Section 2.3, the equilibrium strategy is still given by (2.7). Thus, $B_\ell^w = s(V_\ell^w; X_\ell, I, \theta)$. The number of bidders I is assumed to be constant across auctions. Because I is usually unobserved in a descending auction, it is treated as a parameter.

Let $m_\ell^w(\theta) \equiv \mathrm{E}[B_\ell^w|X_\ell, I] = \int b g^w(b|X_\ell, I; \theta) db$ be the expectation of the winning bid B_ℓ^w given (X_ℓ, I) when the value distribution is $F(\cdot|X_\ell, I; \theta)$. The usual Non-Linear Least Squares (NLLS) estimator minimizes the objective function $Q_L(\theta) = (1/L) \sum_{\ell=1}^{L} [B_\ell^w - m_\ell^w(\theta)]^2$ with respect to θ. Because $m_\ell^w(\theta)$ is typically not available in explicit form, they replace it by an unbiased simulator $\bar{M}_\ell^w(\theta) = (1/S)\sum_{s=1}^S M_{s\ell}^w(\theta)$. That is, $\mathrm{E}[\bar{M}_\ell^w(\theta)|X_\ell, I]$ $= m_\ell^w(\theta)$ where S is the number of simulations. They note that minimizing $Q_L(\theta)$ with $m_\ell^w(\theta)$ replaced by $\bar{M}_\ell^w(\theta)$ leads to an inconsistent estimator of θ for any fixed S as L increases to infinity. Thus they propose an adjustment of the objective function leading to the Simulated Non-Linear Least-Squares (SNLLS) objective function

$$\tilde{Q}_{S,L}^w(\theta) = \frac{1}{L}\sum_{\ell}^{L}\left([B_\ell^w - \bar{M}_\ell^w(\theta)]^2 - \frac{1}{S(S-1)}\sum_{s=1}^{S}[M_{s\ell}^w(\theta) - \bar{M}_\ell^w(\theta)]^2\right). \tag{3.3}$$

The second sum corrects for the use of the simulator $\bar{M}_\ell^w(\theta)$ instead of the conditional mean $m_\ell(\theta)$.

To construct the simulator $\bar{M}_\ell^w(\theta)$, they invoke the RET. This gives $\mathrm{E}[B_\ell^w|X_\ell, I] =$ $\mathrm{E}[U_\ell^{(I-1:I)}|X_\ell, I]$, where $U_\ell^{(I-1:I)}$ denotes the second-highest order statistic among I random draws from $F(\cdot|X_\ell, I;\theta)$. Thus

$$m_\ell^w(\theta) = \int_0^{+\infty}\cdots\int_0^{+\infty} u^{(I-1:I)} f(u_1|X_\ell, I;\theta)\cdots f(u_I|X_\ell, I;\theta)du_1\ldots du_I,$$

where $f(\cdot|X_\ell, I;\theta)$ is the density of $F(\cdot|X_\ell, I;\theta)$. To implement the simulator, the simulation noise should not depend on the parameters θ to ensure consistency as noted by Gourieroux and Monfort (1997). Importance sampling satisfies this requirement. The method consists in drawing randomly SLI values $u_{s\ell i}, i = 1,\ldots,I, \ell = 1,\ldots,L, s = 1,\ldots,S$ from a known density $\phi(\cdot)$ on support $[0,+\infty)$. Thus, $m_\ell^w(\theta)$ is estimated unbiasedly by $\bar{M}_\ell^w(\theta) = \frac{1}{S}\sum_{s=1}^S M_{s\ell}^w(\theta)$, where

$$M_{s\ell}^w(\theta) = u_{s\ell}^{(I-1:I)}\frac{f(u_{s\ell 1}|X_\ell, I;\theta)\cdots f(u_{s\ell I}|X_\ell, I;\theta)}{\phi(u_{s\ell 1})\cdots\phi(u_{s\ell I})}. \tag{3.4}$$

See e.g. Hammersley and Handscomb (1964). The SNLLS estimator is obtained by minimizing (3.3) with respect to (θ, I) using the simulations (3.4) treating I as an unknown (discrete) parameter.[4] Laffont, Ossard and Vuong (1995) establish the consistency and $\sqrt{L}$-asymptotic normality of the SNLLS estimator of θ when the number of simulations S is fixed and the number of auctions L tends to infinity. The main advantage of this method is that it requires neither the computation of the equilibrium strategy $s(\cdot; X, I, \theta)$ nor of its inverse. Consequently, the method is computationally convenient and can accommodate

[4]See also Li (2009) who determines the number I of bidders using a Vuong (1989)-type model selection test that compares Simulated Mean Square Errors of Prediction (SMSEP) across values of I. For each I, the SMSEP uses simulated bids from a model estimated by indirect inference as in Li (2010). See Gourieroux, Monfort and Renault (1993) and Gallant and Tauchen (1996) for indirect inference. This requires, however, evaluating the equilibrium strategy (2.7) in model selection and estimation.

any parametric family of distributions for $F(\cdot|\cdot,\cdot)$. In addition, this estimator allows for heterogeneity across auctioned objects with discrete and continuous exogenous variables.

The SNLLS estimator uses only the winning bids and the RET to simulate their expectations. Li and Vuong (1997) extend this method when all bids are observed. With no reserve price, the expectation $m_\ell(\theta)$ of a bid $B_{i\ell}$ given (X_ℓ, I_ℓ) satisfies

$$m_\ell(\theta) = \frac{I_\ell - 1}{I_\ell - 2}\mathrm{E}[V_{i\ell}|X_\ell, I_\ell;\theta] - \frac{\bar{b}(X_\ell, I_\ell;\theta)}{I_\ell - 2},$$

when $I_\ell > 2$. Because the conditional mean $\mathrm{E}[V_i|X, I;\theta]$ of private values given (X, I) is typically specified in explicit form (such as linear in some parameters), the difficulty arises from evaluating the upper bound $\bar{b}(X, I;\theta) = s[\bar{v}(X, I;\theta); X, I, \theta]$. Noting that $\bar{b}(X, I;\theta) = \mathrm{E}[U^{(I-1:I-1)}]$ where $U^{(I-1:I-1)}$ is the highest order statistic in $I-1$ random draws from $F(\cdot|X, I;\theta)$, they propose an unbiased estimator $\bar{M}^{\bar{b}}_\ell(\theta) = \frac{1}{S}\sum_{s=1}^{S} M^{\bar{b}}_{s\ell}(\theta)$ of $\bar{b}(X_\ell, I_\ell;\theta)$, where

$$M^{\bar{b}}_{s\ell}(\theta) = u_{s\ell}^{(I_\ell-1:I_\ell-1)} \frac{f(u_{s\ell 1}|X_\ell, I_\ell;\theta)\cdots f(u_{s\ell(I_\ell-1)}|X_\ell, I_\ell;\theta)}{\phi(u_{s\ell 1})\cdots\phi(u_{s\ell(I_\ell-1)})},$$

similarly to (3.4), where the $u_{s\ell k}$ are $I_\ell - 1$ random draws from the importance sampler $\phi(\cdot)$ for each simulation-auction pair (s,ℓ). The SNLLS objective function becomes

$$\begin{aligned}\tilde{Q}^{\bar{b}}_{S,L}(\theta) \quad &= \quad \frac{1}{L}\sum_{\ell}^{L}\frac{1}{I_\ell}\sum_{i=1}^{I_\ell}\left(B_{i\ell} - \frac{I_\ell-1}{I_\ell-2}\mathrm{E}[V_{i\ell}|X_\ell, I_\ell;\theta] + \frac{\bar{M}^{\bar{b}}_\ell(\theta)}{I_\ell-2}\right)^2 \\ &\quad -\frac{1}{L}\sum_{\ell}^{L}\frac{1}{I_\ell(I_\ell-2)^2}\sum_{i=1}^{I_\ell}\frac{1}{S(S-1)}\sum_{s=1}^{S}[M^{\bar{b}}_{s\ell}(\theta) - \bar{M}^{\bar{b}}_\ell(\theta)]^2,\end{aligned}$$

in the spirit of (3.3). Minimizing this adjusted SNLLS objective function provides an estimator of θ which is $\sqrt{L}$-asymptotic normal for S fixed.

A weakness of the SNLLS methods is that they only use the first moment of the bids or winning bids for identification and estimation. Laffont and Vuong (1993) extend these methods to higher moments when only the winning bids are observed. Specifically, they exploit the property that the equilibrium strategy (2.7) itself can be written as a conditional expectation, namely $s(\cdot;F,I) = \mathrm{E}[U^{(I-1:I)}|U^{(I:I)} = \cdot]$ where $U^{(I:I)}$ and $U^{(I-1:I)}$ are respectively the highest and second highest order statistics in I random draws from $F(\cdot|X,I)$. See Laffont, Ossard and Vuong (1995). Because $B^w = s(V^w; X, I, \theta)$ where V^w is distributed as $F^I(\cdot|X, I;\theta)$, the kth moment $m^w_k(\theta)$ of the winning bid given (X, I) is

$$m^w_k(\theta) = \int_0^{+\infty}\left(\int_0^{+\infty} 1\!\!1[u \le v^w]\frac{dF^{I-1}(u|X, I;\theta)}{F^{I-1}(v^w|X, I;\theta)}\right)^k dF^I(v^w|X, I;\theta),$$

for any integer $k \geq 1$ because the distribution of $U^{(I-1:I)}$ given $U^{(I:I)} = v^w$ and (X, I) is $F^{I-1}(u|X, I, \theta)/F^{I-1}(v^w|X, I, \theta)$ for $u \leq v^w$. Upon viewing the integrand as the product of k (identical) univariate integrals, the preceding equation shows that $m_k(\theta)$ can be approximated by an unbiased simulator. Thus, using a Simulated Method of Moments (SMM) as studied by McFadden (1989) and Pakes and Pollard (1989), Laffont and Vuong (1993) obtain a $\sqrt{L}$-asymptotic normal estimator of θ for any fixed number of simulations. Besides avoiding the exact computation of the moments $m_k^w(\theta)$ for every trial value of θ in an optimization routine, the simulated methods control for the approximation error arising from a fixed number of simulations through the modified asymptotic variances of the estimators. This SMM clearly extends to the case when one observes all the bids since $B_i = s(V_i; X, I, \theta)$ where $s(\cdot; X, I) = \mathrm{E}[U^{(I-1:I)}|U^{(I:I)} = \cdot, X, I]$.

Augmented Least-Squares (LS) Regressions

The preceding direct estimation methods are fully parametric. The next method developed by Rezende (2008) is amenable to semiparametric estimation under potentially restrictive conditions. The observations are (B^w, X, I). As for the SNLLS method, it relies on the RET and does not require the computation of the equilibrium strategies $s(\cdot; X, I)$ for every auction in the sample. Specifically, the author assumes that (i) the number of bidders I is exogenous so that $F(\cdot|X, I) = F(\cdot|X)$, (ii) the private value distribution is location-scale invariant, i.e., $F(\cdot|X) = \tilde{F}[(\cdot - m(X))/\sigma(X)]$ for some distribution $\tilde{F}(\cdot)$ and functions $m(\cdot)$ and $\sigma(\cdot)$, and (iii) $m(\cdot)$ and $\sigma(\cdot)$ are parameterized as $m(X) = m(X; \theta_1)$ and $\sigma(X) = \sigma(X; \theta_2)$ where $m(\cdot; \cdot)$ and $\sigma(\cdot; \cdot)$ are known functions and $\theta = (\theta_1, \theta_2)$ is a vector of unknown parameters.

First, assume that the distribution $\tilde{F}(\cdot)$ is known. Thus the model is fully parametric. Condition (ii) is equivalent to $V_i = m(X) + \sigma(X)\tilde{V}_i$ where $\tilde{V}_i$ is distributed as $\tilde{F}(\cdot)$ independently of (X, I). Since $B_\ell^w = s(V_\ell^w; F(\cdot|X_\ell), I_\ell)$, it follows from the location-scale invariance property of the equilibrium strategy in Section 2.3 that $B_\ell^w = \mu(X_\ell) + \sigma(X_\ell)s(\tilde{V}_\ell^w; \tilde{F}, I_\ell)$. Rezende (2008) considers only the first moment of B_ℓ^w conditional on (X_ℓ, I_ℓ) for identification and estimation. Using (iii) and the RET gives

$$\mathrm{E}[B_\ell^w|X_\ell, I_\ell] \quad = \quad \mu(X_\ell; \theta_1) + \sigma(X_\ell; \theta_2)\mathrm{E}[\tilde{V}^{(I_\ell-1:I_\ell)}], \tag{3.5}$$

where $\tilde{V}^{(I-1:I)}$ is the second-highest order statistic in I random draws from $\tilde{F}(\cdot)$. Since $\tilde{F}(\cdot)$ is known, then $\mathrm{E}[\tilde{V}^{(I-1:I)}] = I(I-1)\int v\tilde{F}^{I-2}(v)[1-\tilde{F}(v)]d\tilde{F}(v)$ is known for every I. Thus, (3.5) shows that θ is consistently estimated at the $\sqrt{L}$-rate from a NLLS regression of the winning bids B^w on the covariates X provided the usual identification conditions of NLLS are satisfied. See e.g., Amemiya (1985). For instance, Rezende (2008) assumes that $m(X;\theta_1) = X'\theta_1$ and $\sigma(X;\theta_2) = X'\theta_2$. Estimation of θ reduces to an ordinary LS regression of B^w on X augmented by the interactions $\tilde{X} = \mathrm{E}[\tilde{V}^{(I-1:I)}] \cdot X$ provided the $2d$ variables $[X, \tilde{X}]$ are linearly independent. The private value distribution $F(\cdot|X)$ is then estimated by $\tilde{F}[(\cdot - m(X;\hat{\theta}_1))/\sigma(X;\hat{\theta}_2)]$.

When $\tilde{F}(\cdot)$ is unknown, the model becomes semiparametric. Let $\overline{I} < \infty$ denote the maximum number of possible bidders. Thus the unknown expectations $\mathrm{E}[\tilde{V}^{(I-1:I)}]$ can be viewed as additional parameters in (3.5) and estimated together with (θ_1, θ_2) by NLLS upon introducing dummies $1\!\!I[I_l = k]$ for $k = 2, \ldots, \overline{I}$. Identification is more demanding than when $\tilde{F}(\cdot)$ is known. This can be seen with the linear specifications of $m(X;\theta_1)$ and $\sigma(X;\theta_2)$ in Rezende (2008) who relies on exclusion restrictions and normalizations to complete the identification and estimation of θ. The author also shows how to identify the distribution $\tilde{F}(\cdot)$ from the LS estimates of $\mathrm{E}[\tilde{V}^{(I-1:I)}]$ so as to recover the value distribution $F(\cdot|X)$. The identification argument is constructive and follows Hoeffding (1953) and Pollak (1973) but requires that all the expectations $\mathrm{E}[\tilde{V}^{(I-1:I)}]$, $I = 2, 3, \ldots$ be identified and hence that $\overline{I} = \infty$.

The method extends when all bids are observed. Using the location-scale invariance property of the equilibrium strategy and taking expectation conditional on (X_ℓ, I_ℓ) give

$$\mathrm{E}[B_{i\ell}|X_\ell, I_\ell] = \mu(X_\ell;\theta_1) + \sigma(X_\ell, \theta_2)\mathrm{E}[s(\tilde{V}_{i\ell}; \tilde{F}, I_\ell)],$$

similarly to (3.5). Thus, as in Rezende (2008) we can treat $\mathrm{E}[s(\tilde{V}_i; \tilde{F}, I)]$ as known or unknown depending on the knowledge of $\tilde{F}(\cdot)$. As for the SNLLS methods, the augmented LS regression methods do not require computation of the equilibrium strategies. They are simpler but not as general as they crucially depend on the location-scale invariance of the private value distribution. Though the augmented LS regression methods do not rely on a parameterization of $F(\cdot|\cdot)$, they achieve identification of the latter only when the number of bidders grows to infinity, which is typically not satisfied in practice.

Nonparametric Methods

Bierens and Song (2012) propose a Semi-NonParametric Integrated Simulated Moment (SNP-ISM) estimator that allows the value distribution $F(\cdot|I)$ to be fully nonparametric. The authors study the case where all the bids are observed and there are no covariates X. Extensions to the case where one observes only the winning bid and there are covariates are possible but computationally demanding. The SNP-ISM estimator is simulation-based but in contrast to the SNLLS methods, which use only a finite number of moments of the bids or winning bids, the method accounts for all the moments through their characteristic functions. Specifically, given I and L_I auctions with I bidders, the SNP-ISM objective function is the L_2-distance between the empirical bid characteristic function $\hat{\phi}(t;I)$ and the empirical characteristic function $\hat{\phi}^S(t;F,I)$ constructed from IL_I bids $B^S_{i\ell}$ simulated from a candidate value distribution $F(\cdot|I)$ in $\mathcal{F}_L$, namely,

$$\hat{Q}^S_L(F) \equiv \frac{1}{T}\int_{-T}^{+T} \left|\hat{\phi}^S(t;F,I) - \hat{\phi}(t;I)\right|^2 dt,$$

where $\hat{\phi}(t;I) \equiv \frac{1}{IL_I}\sum_{\ell:I_\ell=I}^{L_I}\sum_{i=1}^{I}\exp(\iota t B_{i\ell})$ and $\hat{\phi}^S(t;F,I) \equiv \frac{1}{IL_I}\sum_{\ell:I_\ell=I}^{L_I}\sum_{i=1}^{I}\exp(\iota t B^S_{i\ell})$ with $\iota^2 = -1$, $B^S_{i\ell} \equiv s(V^S_{i\ell};F,I)$ and $V^S_{i\ell}$ independently drawn from $F(\cdot|I)$.[5] The method relies on a sieve $\{\mathcal{F}_L\}_{L=1}^{\infty}$ of increasing compact sets of absolutely continuous distributions. Let $F_o(\cdot|I)$ be a pilot absolutely continuous distribution with support $[\underline{v}_o(I), \overline{v}_o(I)]$ and $\{\delta_{oj}\}_{j=1}^{\infty}$ be a sequence of positive numbers satisfying $\sum_{j=0}^{\infty}\delta^2_{oj} < \infty$. The authors then define $\mathcal{F}_L$ as the set of distributions of the form $H_L[F_o(\cdot)]$ where $H_L(\cdot)$ has density

$$h_L(u) = \frac{\left(1 + \sum_{j=1}^{L}\delta_j \mathcal{L}_j(u)\right)^2}{1 + \sum_{j=1}^{L}\delta_j^2},$$

satisfying $\max_{j=1,\dots,L}|\delta_j|/\delta_{oj} \leq 1$. The $\mathcal{L}_j(\cdot)s$ are the orthonormal Legendre polynomials on $[0,1]$. They show that $\{\mathcal{F}_L\}$ is a sequence of compact sets dense in $\mathcal{F}_\infty$ with respect to the sup-norm. Similarly, the corresponding sets of densities constitute a sequence of compact sets that is dense with respect to the L_1-norm. Minimization of $\hat{Q}^S_L(F)$ over the finite-dimensional set $\mathcal{F}_L$ delivers the SNP-ISM estimators $\hat{F}(\cdot|I)$ and $\hat{f}(\cdot|I)$ of the value distribution and density. Assuming that the true value distribution belongs to $\mathcal{F}_\infty$, the authors establish the consistency of $\hat{F}(\cdot|I)$ and $\hat{f}(\cdot|I)$ over $[\underline{v}_o(I), \overline{v}_o(I)]$ with respect to the sup-norm and L_1-norm, respectively. See also Chen (2007) for a survey on sieve estimation. Though the method greatly relaxes the parametric assumptions of the preceding methods, a drawback is that it requires evaluating the equilibrium strategy (2.7) for every trial value of the sieve parameters δ_j during the minimization routine.

Section 3.2: The Indirect Approach

The previous section shows the limitations of the direct approach. The estimators are computationally demanding. In addition, they mostly require an explicit form of the equilibrium strategy or a property such as the RET to avoid its computation. As soon as we deviate from the benchmark model, auction models do not always provide explicit forms for the equilibrium strategies and the RET no longer holds. Moreover, some fundamental issues have been addressed only partially under some parameterization. These are the

[5] Because the bid distribution has bounded support, the constant T can be any positive number. Moreover, to ensure that the simulation noise is independent of the candidate $F(\cdot|I)$, one should generate $V^S_{\imath\ell}$ as $V^S_{i\ell} = F^{-1}(U^S_{i\ell}|I)$ where the $U^S_{i\ell}$s are IL_I random draws from a uniform $\mathcal{U}[0,1]$.

nonparametric identification of the model primitive(s) and the restrictions implied by the model on the observables. This calls for a different approach which we present next.

The Indirect Approach

The econometrician faces three main challenges when adopting a structural approach. A first issue concerns the identifiability of the model primitives under minimal functional forms and/or distributional assumptions. See Hsiao (1983) and Matzkin (2007) for surveys on parametric and nonparametric identification. The question is whether the observations are sufficient to recover uniquely the model primitive(s) while minimizing the use of parameterization. In this chapter we mainly focus on global point identification. We refer to Florens and Sbaï (2010) for local identification results and to Sections 5.2, 7.2 and 7.3 for examples of set identification. Without identification, there is no ground for estimating the model primitives. A second issue is related to the characterization of the model restrictions on observables. In other words, what are the conditions on observables guaranteeing that the data are explained or *rationalized* by the model under consideration. For instance, can any bid distribution be explained by the benchmark model? Equivalently, is the latter testable from observed bids, in which case what are the restrictions imposed by the model on observables? Lastly, the analyst needs to find a computationally feasible estimation procedure given the complexity/nonlinearity of the model. The indirect approach addresses these three fundamental challenges in an integrated framework.

The development of the indirect approach was initiated by Guerre, Perrigne and Vuong (2000). The basic idea is to exploit the differential equations characterizing the equilibrium strategies of the model. For instance, in the benchmark model, the indirect approach does not use the explicit form of the equilibrium strategy (2.7) but instead the first-order condition of the bidder's profit maximization leading to the differential equation (2.6). It then exploits a key property of the equilibrium strategy, i.e., its monotonicity, to rewrite this first-order condition in terms of observables. The main advantage of this approach is that it does not requires an explicit form of the equilibrium strategy. As a matter of fact, it does not even require solving numerically the differential equation characterizing the equilibrium strategy. Throughout this chapter, we review several auction models for which there is no explicit form for the equilibrium strategies and/or the differential equations characterizing the equilibrium strategies are hard to solve numerically.

This subsection reviews the results in Guerre, Perrigne and Vuong (2000) on the identification and restrictions of the benchmark model. The authors consider identification without imposing distributional and/or functional assumptions on the model primitive which is the distribution of private values. This is referred as nonparametric identification. More generally, the econometrics of structural models has revived the interest in identification as auctions provide several interesting examples. See Athey and Haile (2007) for a survey on identification of standard auction models. Identification is fundamental for three reasons. First, lack of identification implies that different model primitives can explain the same observations. Thus identification should be addressed before estimation. Second, identification allows us to understand which data variations are needed to recover the model structure. When the model is not identified, then we may consider additional information under the form of restrictions or data to restore point identification. Third, identification is often constructive thereby providing important insights for developing estimation methods.

Characterization of the model restrictions or *rationalization* should not be confused with identification. The former characterizes the collection of data generating processes (DGP) that are rationalized by the model, while identification implicitly assumes that the DGP belongs to this collection. Thus rationalization should precede identification in principle. In particular, rationalization answers whether the theoretical model imposes some restrictions on observables. Without such restrictions, the model could explain any data. These restrictions can then be used to develop tests of the model validity or to distinguish it from alternative models. Violation of any such restrictions would lead to reject the model in view of the data and should induce the analyst to revisit the model or consider another model. It is interesting to make a parallel between these econometrics concepts and economic theory. Identification is the empirical analog of the uniqueness of the equilibrium, while characterizing the model restrictions is the empirical analog of establishing the existence of equilibria.

IDENTIFICATION

When studying identification, it is important to clearly specify the model structure and the observables whose variations are used to identify the model primitives. In the benchmark model, the only unknown model primitive is the value distribution $F(\cdot|I)$ with support $[\underline{v}(I), \overline{v}(I)]$ since the number of bidders I is observed. To simplify, we omit the covariates X as the analysis can be conducted conditional on X. In contrast, we maintain the conditioning on I to capture some unobserved heterogeneity. The observables are the bids $(B_1, \dots, B_I)$. Though bids are observed while private values are not, the increasing equilibrium strategy $s(\cdot; F, I)$ relates the former to the latter which are distributed as $F(\cdot|I)$. Hence, bids are random with a distribution $G(\cdot|I)$ that is determined by the distribution of private values. Because the equilibrium strategy (2.7) depends on the number of bidders, the bid distribution is conditional on the number of bidders even if the value distribution was independent of I. Let $[\underline{b}(I), \overline{b}(I)]$ be the support of $G(\cdot|I)$ where $\underline{b}(I) = s(\underline{v}(I); F, I) = \underline{v}(I)$ from the boundary condition of the differential equation (2.6) and $\overline{b}(I) = s(\overline{v}(I); F, I)$. As a matter of fact, the boundaries $\underline{b}(I)$and $\overline{b}(I)$ also depend on the latent value distribution $F(\cdot)$. We omit this dependence to simplify the notation.

Because $B_i = s(V_i; F, I)$, the bid distribution $G(\cdot|I)$ is related to the value distribution $F(\cdot|I)$. Specifically, we have $G(b|I) = \Pr[B \le b|I] = \Pr[s^{-1}(B; F, I) \le s^{-1}(b; F, I)|I]$ since the inverse of the equilibrium strategy $s^{-1}(\cdot; F, I)$ is increasing. Because $s^{-1}(B; F, I) = V$, the latter probability is also equal to $\Pr[V \le s^{-1}(b; F, I)|I] = F(s^{-1}(b)|I) = F(v|I)$ upon letting $v = s^{-1}(b; F, I)$. Hence, $G(b|I) = F(v|I)$. This relationship is the key to the identification of the benchmark model. Differentiating with respect to b gives the density $g(b|I) = f(s^{-1}(b)|I)/s'(s^{-1}(b), F, I) = f(v|I)/s'(v; F, I)$ for any arbitrary v and corresponding $b = s(v; F, I)$. In particular, the ratio $g(b|I)/G(b|I)$ equals $[f(v|I)/F(v|I)] \times [1/s'(v; F, I)]$ for all $b = s(v; F, I)$ and $v \in [\underline{v}(I), \overline{v}(I)]$.

Allowing for the dependence on I, the FOC (2.6) of the benchmark model is

$$1 = (V_i - s(V_i, F, I))(I-1)\frac{f(V_i|I)}{F(V_i|I)}\frac{1}{s'(V_i; F, I)}$$

for $i = 1, \dots, I$. Guerre, Perrigne and Vuong (2000) rewrite this differential equation in terms of observables. Specifically, in view of the relationships derived above, the product of the last two terms is equal to the ratio $[g(B_i|I)/G(B_i|I)]$. This gives

$$V_i = B_i + \frac{1}{I-1}\frac{G(B_i|I)}{g(B_i|I)} \equiv \xi(B_i; G, I), \tag{3.6}$$

where $B_i = s(V_i; F, I)$, for $i = 1, \dots, I$. In particular, $\xi(\cdot; G, I)$ is increasing because the equilibrium strategy (2.7) is increasing. In contrast to (2.7), (3.6) expresses each bidder's private value V_i as a function of observables, namely, the bidder's bid B_i, its distribution $G(\cdot|I)$ with density $g(\cdot|I)$ and the number of bidders I since $G(\cdot|I)$ and $g(\cdot|I)$ are identified from the observed bids and the number of bidders. It can be viewed as the inverse of the equilibrium strategy $s^{-1}(\cdot; F, I)$ as it maps B_i to V_i though a difference is that $\xi(\cdot; G, I)$ depends on the bid distribution $G(\cdot|I)$ and not on $F(\cdot|I)$ directly. The second term of the RHS is bidder's information rent or value shading in terms of observables. An alternative but economically important derivation of (3.6) that is more akin to the competitive bidding literature initiated by Friedman (1956) in Operations Research is as follows. See Laffont (1997) for a discussion of the game and decision theoretic approaches to auctions. Indeed, bidder i's expected profit can be written as $(V_i - B_i)G(B_i|I)^{I-1}$. This follows because his winning probability is $\Pr(B_{-i}^{\max} \le B_i|I) = G(B_i|I)^{I-1}$ where $B_{-i}^{\max} = \max_{j \ne i} B_j$ and B_j are i.i.d. distributed as $G(\cdot|I)$. Thus, taking as given competitors' (equilibrium) strategies and maximizing the expected profit with respect to B_i gives (3.6). In particular, this derivation shows that bidding $B_i = s(V_i; F, I)$ is bidder i's best response to his competitors' equilibrium behavior.

If the observed bid B_i is the equilibrium bid, as is maintained in the structural approach, then the corresponding bidder's value V_i must satisfy (3.6). This immediately shows that the value distribution $F(\cdot|I)$ is identified since the bidders' values V_i are recovered from their bids B_i, while $G(\cdot|I)$ and its density $g(\cdot|I)$ are identified from the observed bids and the number of bidders.

Identification: *The value distribution $F(\cdot|\cdot)$ in the benchmark model is nonparametrically identified. Specifically, the value distribution satisfies $F(v|I) = G[\xi^{-1}(v; G, I)|I]$ for all $v \in [\underline{v}(I), \overline{v}(I)]$ and all observed I.*

This is a powerful result as addressing the identification of the benchmark model was not immediate a priori. Indeed, the bid distribution $G(\cdot|I)$ depends on the underlying distribution $F(\cdot|I)$ in two ways, directly through the private value V_i, which is distributed as $F(\cdot|I)$ and indirectly through the equilibrium strategy $s(\cdot; F, I)$, which depends on $F(\cdot|I)$. The rewriting (3.6) of the differential equation defining the equilibrium strategy allows us to address identification straightforwardly. It is worthnoting that if the value distribution does not depend on the number of bidders as assumed generally in theoretical models, then $F(\cdot)$ would be overidentified from $F(\cdot) = G[\xi^{-1}(\cdot; G, I)|I]$ as I varies.

It is useful to extend this result when one only observes the winning bid B^w as in descending auctions while still assuming that the number I of bidders is observed. The winning bid is distributed as $G^w(\cdot|I)$ with a density $g^w(\cdot|I)$. Because $g^w(\cdot|I)/G^w(\cdot|I) = Ig(\cdot|\cdot)/G(\cdot|I)$ and $\xi(\cdot; G, I)$ is increasing, it follows from (3.6) that

$$V^w = B^w + \frac{I}{I-1} \frac{G^w(B^w|I)}{g^w(B^w|I)},$$

where V^w is the winner's private value. Similarly to (3.6), this equation expresses the winner's private value as a function of his bid, the distribution and density of the winning bid which are identified from the observed winning bids. This identifies the distribution of the winner's private value $F^w(\cdot|I)$. By independence of the private values, we have $F^w(\cdot|I) = F^I(\cdot|I)$. Thus, observing the number of bidders identifies the underlying distribution of private values $F(\cdot|I)$.

Model Restrictions

Characterizing all the model restrictions is an important problem to (i) determine the set of DGPs that the model can rationalize, (ii) test model validity, and (iii) discriminate among competing models. Let $G(b_1, \ldots, b_I|I)$ be the joint distribution of observed bids. Let $\mathcal{P}^k$ be the set of distributions with positive densities on hypercube supports in $I\!R_+^k$ for $k \geq 1$. Hereafter, we restrict ourselves to strictly increasing and differentiable Bayesian Nash equilibrium strategies. The following result from Guerre, Perrigne and Vuong (2000) provides necessary and sufficient conditions on the joint bid distribution $G(\cdot, \ldots, \cdot|I)$ to be rationalized by the benchmark model.

Rationalization: *Let $I \geq 2$ and $G(\cdot, \ldots, \cdot|I) \in \mathcal{P}^I$ with support $[\underline{b}(I), \overline{b}(I)]^I$. There exists a value distribution $F(\cdot|I) \in \mathcal{P}^1$ with support $[\underline{v}(I), \overline{v}(I)]$ that rationalizes $G(\cdot, \ldots, \cdot|I)$ in the benchmark model if and only if*

C1: $G(b_1, \ldots, b_I|I) = \Pi_{i=1}^I G(b_i|I)$ for any $(b_1, \ldots, b_I) \in [\underline{b}(I), \overline{b}(I)]^I$,

C2: The function $\xi(\cdot; G, I)$ defined in (3.6) is increasing on $[\underline{b}(I), \overline{b}(I)]$ and its inverse is differentiable on $[\underline{v}(I), \overline{v}(I)] = [\xi(\underline{b}(I); G, I), \xi(\overline{b}(I); G, I)]$.

Because C1 and C2 are necessary and sufficient conditions, this result determines the set of distributions $G(\cdot, \ldots, \cdot|I)$ that are rationalized by the benchmark model. These restrictions are intuitive. Condition C1 says that bids are independent and identically distributed as $G(\cdot|I)$. This is a direct consequence of private values being i.i.d. Hence, C1 is related to the paradigm, namely symmetric IPV in the benchmark model. Condition C2 says that the function $\xi(\cdot; G, I)$ is increasing in b given the number of bidders I. Indeed $\xi(\cdot; G, I)$ is the inverse of the equilibrium strategy since $\xi(b; G, I) = s^{-1}(b; F, I)$ for all $b \in [\underline{b}(I), \overline{b}(I)]$. Condition C2 is a direct consequence of the monotonicity and differentiability of the equilibrium strategy $s(\cdot; F, I)$. Hence, C2 is related to the equilibrium concept, namely monotone Bayesian Nash.

An important consequence of the previous result is that C1 and C2 reduce the set of joint bid distributions that can be rationalized by the symmetric IPV model. Thus, we can confront the model with the data. The monotonicity of $\xi(\cdot; G, I)$ in C2 is equivalent to the ratio $g(\cdot|I)/G^I(\cdot|I)$ being decreasing. In particular, any log-concave bid distribution satisfies it. This includes, e.g., Normal, Laplace, Logistic, Uniform, Exponential and Extreme Value distributions for any value of their parameters. In addition, the Gamma and the Weibull distributions are log-concave when the shape parameter is greater than or equal to one, whereas the Beta distribution requires that both parameters are greater than or equal to one. All these distributions have unimodal densities. In contrast, as indicated in Guerre, Perrigne and Vuong (2000) some highly peaked multimodal densities violate C2. When the restrictions C1 and/or C2 are not empirically satisfied, the analyst should investigate alternate models. For instance, violation of independence may suggest some dependence among private values, whereas violation of identically distributed values may suggest asymmetry among private values. See Sections 4.3 and 4.2.

Procurement Auctions

In the case of procurement auctions, the bid distribution $G(\cdot|I)$ is related to the cost distribution $F(\cdot|I)$ by $G(b|I) = F(c|I)$ where $c = s^{-1}(b; F, I)$ since $s(\cdot; F, I)$ is increasing. Hence, similarly to (3.6) and allowing for the dependence of the cost distribution on I, we can rewrite the differential equation (2.8) as

$$C_i = B_i - \frac{1}{I-1}\frac{1-G(B_i|I)}{g(B_i|I)} \equiv \xi(B_i; F, I),$$

where the second term of the RHS is the bidder's rent/profit. Thus all the results of this section apply to procurement auctions. In this chapter we consider alternatively auctions and procurement auctions. Indeed, because public institutions often use reverse auctions, an important proportion of papers have analyzed procurement data which are readily available through these institutions' websites. Identification and estimation methods need to be slightly adapted as the same ideas carry over.

Section 3.3: Kernel-Based Estimators

Equation (3.6) naturally suggests the following estimation procedure. In a first step, one uses the observed bids to estimate the bid distribution $G(\cdot|I)$ and its density $g(\cdot|I)$. In a second step, one plugs these estimates in (3.6) to obtain values that are used to estimate the value density $f(\cdot|I)$. This has become known as the GPV procedure in the empirical auction literature. In view of the nonparametric nature of the identification argument, we present nonparametric estimators in this subsection and the following one.[6]

Nonparametric methods have pros and cons. Their main advantage is to leave the econometric model free of any functional and/or distributional assumptions. The data then reveal the shapes of these functions and/or distributions. Thus, using a nonparametric estimation method, when data size permits, can constitute a preliminary step that helps the analyst in his/her choice of parametric specifications. The main cons of nonparametric estimators are related to their slow rates of convergence and the curse of dimensionality as well as the difficulty of deriving asympotic properties. We indicate at

[6] One could also use parametric estimators. This would consist in parameterizing first the bid distribution which should satisfy C2 and then in parameterizing the value distribution in line with the relation $F(\cdot|I) = G[\xi^{-1}(\cdot; G, I)|I]$. Alternatively, one can consider a nonparametric first step and a parametric specification of $f(\cdot|I)$ or some of its moments in the second step. See Aryal, Gabrielli and Vuong (2019) who use local polynomial estimation in the first step to avoid boundary effects. See Fan and Gijbels (1996) for a monograph on local polynomial estimation. The resulting two-step estimator converges at the $\sqrt{L}$-rate as for the class of semiparametric estimators considered in Newey and McFadden (1994).

the end of this subsection a simple approach to circumvent the curse of dimensionality by demeaning the bids. If the analyst has a data set of respectable size, which is becoming more frequent, we recommend nonparametric estimators as they are simple to implement and do not require any optimization routine. Regarding asymptotic properties, researchers have mostly focused on establishing uniform consistency of the estimators though in some cases, they have also established their rates of convergence and asymptotic normality. In practice, to obtain confidence intervals to estimated functions/values, we recommend bootstrap methods that are commonly used in empirical industrial organization. See, e.g., the monograph by Efron and Tibshirani (1993) and the survey by Horowitz (2001) on bootstrap methods.

Smoothness Assumptions

Nonparametric estimation requires some smoothness of the functions of interest. In the benchmark model, it is the value distribution $F(V|X, I)$ and in particular its density $f(V|X, I)$. The next assumption clarifies the underlying data generating process as well as the smoothness properties of $F(\cdot|\cdot,\cdot)$. Let $\mathcal{S}_A$ and $\mathcal{S}_{A|B}$ denote the supports of A and A conditional on B where A and B are two generic random variables.

Assumptions on $F(\cdot|\cdot,\cdot)$**:** *The following assumptions hold*

- *(i) The $d+1$-dimensional vectors $(X_\ell, I_\ell), \ell = 1, \ldots, L$ are i.i.d. with conditional density $f_{X|I}(\cdot|\cdot)$,*
- *(ii) For each auction ℓ, the values $V_{i\ell}, i = 1, \ldots, I_\ell$ are i.i.d. conditionally upon (X_ℓ, I_ℓ) as $F(\cdot|\cdot,\cdot)$ with density $f(\cdot|\cdot,\cdot)$,*
- *(iii) The support of $F(\cdot|\cdot, I)$ is $\mathcal{S}_{V,X|I} = \{(v, x) : x \in [\underline{x}, \overline{x}], v \in [\underline{v}(x, I), \overline{v}(x, I)]$ with $0 \le \underline{v}(x, I) < \overline{v}(x, I) < \infty$ and $\underline{x} < \overline{x}$,*
- *(iv) $f(v|x, I) \ge c_f > 0$ for $(v, x) \in \mathcal{S}_{V,X|I}$ and $f_{X|I}(x|I) \ge c_f > 0$ for $x \in \mathcal{S}_{X|I}$,*
- *(v) $F(\cdot|\cdot, I)$ and $f_{X|I}(\cdot|I)$ admit up to $R+1$ continuous partial derivatives on their respective supports with $R \ge 1$,*

for each $I \in \mathcal{S}_I$, where $\mathcal{S}_I$ is a finite subset of $\{2, 3, \ldots\}$.

Part (ii) is the extension of the i.i.d. assumption of private values conditionally on observed characteristics X. Part (iv) requires that the underlying value density $f(\cdot|\cdot,\cdot)$ is bounded away from zero. This assumption agrees with the theoretical assumption that the private value density is positive on its compact support. Part (v) specifies the smoothness of the primitive $F(\cdot|\cdot, I)$ and implies that its density $f(\cdot|\cdot, I)$ is R continuously differentiable on its support.

The identification argument suggests using the bid density $g(\cdot|\cdot, I)$ to estimate the density $f(\cdot|\cdot, I)$. Because these two densities are related through the equilibrium strategy, parts (iii,iv,v) on $f(\cdot|\cdot, I)$ imply some properties on $g(\cdot|\cdot, I)$. The next result from Guerre, Perrigne and Vuong (2000) provides such properties.

Smoothness: *Given parts (iii,iv,v) above, for each $I \in \mathcal{S}_I$ the bid distribution $G(\cdot|\cdot, I)$ satisfies*

(i) Its support is $\mathcal{S}_{B,X|I} = \{(b,x) : x \in [\underline{x}, \overline{x}], b \in [\underline{b}(x,I), \overline{b}(x,I)]\}$ with $\underline{b}(x,I) = \underline{v}(x,I)$. Moreover, the bid boundaries $\underline{b}(\cdot, I)$ and $\overline{b}(\cdot, I)$ are $R+1$ continuously differentiable on $[\underline{x}, \overline{x}]$,

(ii) $g(b|x,I) \geq c_g > 0$ for $(b,x) \in \mathcal{S}_{B,X|I}$,

(iii) $G(\cdot|\cdot, I)$ is $R+1$ continuously differentiable on $\mathcal{S}_{B,X|I}$, while $g(\cdot|\cdot, I)$ is $R+1$ continuously differentiable on any closed inner subset of $\mathcal{S}_{B,X|I}$.

The above properties are in line with parts (iii,iv,v) above with the exception of the smoothness property of the bid density. In particular, the bid density is smoother than the value density. The intuition arises from the key equation (3.6) rewitten as

$$g(b|x,I) = \frac{G(b|x,I)}{(I-1)[\xi(b;x,I) - b]},$$

where $\xi(b;x,I) \equiv \xi(b; G(\cdot|x,I), I)$. Since $\xi(b;x,I)$ is the inverse of the equilibrium strategy, its smoothness follows the smoothness of the equilibrium strategy. By (2.7), the latter is given by $F(\cdot|\cdot,\cdot)$ which is $R+1$ continuously differentiable. Thus, the bid density is also $R+1$ continuously differentiable. This smoothness property of $g(\cdot|\cdot, I)$ arises in most auction models reviewed in this chapter and has important consequences on the convergence rate at which we can estimate the underlying value density. Lastly, part (ii) is convenient since the bid density appears in the denominator of (3.6).

A Two-Step Kernel Estimator

To estimate $f(\cdot|\cdot,\cdot)$, Guerre, Perrigne and Vuong (2000) use kernel estimators in both steps. With covariates X, they rewrite (3.6) as

$$V_{i\ell} = B_{i\ell} + \frac{1}{I_\ell - 1} \frac{G(B_{i\ell}, X_\ell | I_\ell)}{g(B_{i\ell}, X_\ell | I_\ell)} \equiv \xi(B_{i\ell}; X_\ell, I_\ell), \tag{3.7}$$

for $i = 1, \ldots, I_\ell$, $\ell = 1, \ldots, L$, where $G(b,x|I) \equiv G(b|x,I) f_{X|I}(x|I)$ and $g(b,x|I) = g(b|x,I) f_{X|I}(x|I)$. Thus in the first step, one estimates $G(b,x|I)$ and $g(b,x|I)$ as

$$\begin{aligned} \hat{G}(b,x|I) &= \frac{1}{IL_I h_G^d} \sum_{\ell: I_\ell = I}^{L_I} \sum_{i=1}^{I} 1\!\!1(B_{i\ell} \leq b) K_G\left(\frac{x - X_\ell}{h_G}\right), \\ \hat{g}(b,x|I) &= \frac{1}{IL_I h_g^{d+1}} \sum_{\ell: I_\ell = I}^{L_I} \sum_{i=1}^{I} K_g\left(\frac{b - B_{i\ell}}{h_g}\right) K_g\left(\frac{x - X_\ell}{h_g}\right), \end{aligned} \tag{3.8}$$

where L_I is the number of auctions with I bidders, $K_G(\cdot)$ and $K_g(\cdot)$ are kernel functions and (h_G, h_g) are smoothing parameters called bandwidths. See Hardle (1991) and Silverman (1986) for monographs on kernel estimation. We discuss below the choice of the bandwidths (h_G, h_g). Since I is discrete, (3.8) uses the subset of auctions with I bidders.

In the second step, one estimates each private value by plugging (3.8) in (3.7), namely

$$\hat{V}_{i\ell} = B_{i\ell} + \frac{1}{I_\ell - 1}\frac{\hat{G}(B_{i\ell}, X_\ell, I_\ell)}{\hat{g}(B_{i\ell}, X_\ell, I_\ell)}, \quad i = 1, \ldots, I_\ell, \ \ell = 1, \ldots, L. \tag{3.9}$$

Once the values are recovered, one can estimate their conditional density by

$$\hat{f}(v|x, I) = \frac{\hat{f}_{V,X|I}(v, x|I)}{\hat{f}_{X|I}(x|I)} = \frac{\frac{1}{IL_I h_f^{d+1}} \sum_{\ell:I_\ell=I}^{L_I} \sum_{i=1}^{I} K_f\left(\frac{v-\hat{V}_{i\ell}}{h_f}\right) K_X\left(\frac{x-X_\ell}{h_f}\right)}{\frac{1}{L_I h_X^d} \sum_{\ell:I_\ell=I}^{L_I} K_X\left(\frac{x-X_\ell}{h_X}\right)}, \tag{3.10}$$

where $K_f(\cdot)$ and $K_X(\cdot)$ are kernel functions and (h_f, h_X) are bandwidths. Again, the choice of the bandwidths is discussed below. If $f(\cdot|X, I)$ does not depend on I, we can pool (3.10) by averaging over the different values of $I \in \mathcal{S}_I$ to obtain an estimator of $f(v|x)$. Comparing the resulting estimator $\hat{f}(\cdot|\cdot)$ with $\hat{f}(\cdot, \cdot|I)$ can be the basis of a test for the exogeneity of the number of bidders.

The kernels $K_G(\cdot), K_g(\cdot), K_f(\cdot)$ and $K_X(\cdot)$ satisfy standard conditions. Namely, they are symmetric on hypercube supports, admit twice continuous derivatives, and integrate to one on their supports. They are of order $R + 1$ with the exception of $K_f(\cdot)$, which is of order R. These conditions agree with the smoothness of the densities.[7] Because of the idiosyncratic smoothness of the bid density as shown above, the asymptotic properties of this two-step estimator of the value density deserve special attention.

Optimal Convergence Rate

Since the value density $f(\cdot|\cdot, I)$ is R continuously differentiable, the optimal (fastest) uniform convergence rate at wich one can estimate it would be $(L/\log L)^{R/(2R+d+1)}$ if private values were observed. See Stone (1982). This is not the case since we only observe their corresponding bids. Relying on the minimax approach developed by Kashminskii (1979), Guerre, Perrigne and Vuong (2000) derive the optimal uniform convergence rate for estimating the value density $f(\cdot|\cdot, I)$ from observed bids. A rationale for deriving the optimal rate is that it applies to a large class of nonparametric estimators, not only to the kernel-based two-step estimator defined previously. Specifically, the uniform convergence rate of any estimator in this class cannot be faster than this optimal rate.

[7]In practice, we recommend setting $R = 2$ to avoid higher order kernels which may entail negative estimated values for densities. Nonetheless, we provide general formulas in this chapter as it is easy to obtain the corresponding rates by simply replacing R by its chosen value.

In order to derive the optimal convergence rate r_L^*, they consider the minimax value

$$\inf_{\hat{f}(\cdot|\cdot,I)} \sup_{f \in U_\epsilon(f_0)} \Pr_g \left[r_L \times \sup_{(v,x) \in \mathcal{C}_{V,X|I}} |\hat{f}(v|x,I) - f(v|x,I)| > \kappa \right], \tag{3.11}$$

for a given $I \in \mathcal{S}_I$, where κ is a positive constant and $\mathcal{C}_{V,X|I}$ is a compact inner subset of the support of $f_0(v,x|I)$. The conditional density $f(v|x,I)$ corresponds to a joint density $f(v,x|I)$ that belongs to an ϵ-neighborhood $U_\epsilon(f_0)$ of $f_0(\cdot,\cdot|I)$. Since they consider uniform convergence, they use the sup-norm to define the neighborhood $U_\epsilon(f_0)$ as well as to assess the discrepancy between the estimator $\hat{f}(\cdot|\cdot,I)$ and the density $f(\cdot|\cdot,I)$. The estimator $\hat{f}(\cdot|\cdot,I)$ is any estimator that uses data generated by the benchmark model with value density $f(\cdot|\cdot,I)$. That is, $B_{i\ell}$ is related to $V_{i\ell}$ by the equilibrium strategy under $F(\cdot|\cdot,I)$, i.e., $B_{i\ell} = s(V_{i\ell}; F(\cdot|X_\ell, I_\ell), I_\ell)$. The probability $\Pr_g[\cdot]$ is the probability associated with the data $\{(B_{i\ell}, X_\ell, I_\ell); i = 1, \ldots, I_\ell, \ell = 1, \ldots, L\}$ hence generated.

Intuitively, if r_L diverges to infinity very fast then (3.11) approaches one as $L \to \infty$. If r_L diverges to infinity too slowly, then (3.11) approaches zero. The optimal rate r_L^* is the slowest rate r_L for which (3.11) is bounded away from zero as $L \to \infty$ (and $\epsilon \to 0$). The next result from Guerre, Perrigne and Vuong (2000) provides such a rate r_L^*.

Optimal Rate: *The optimal uniform convergence rate r_L^* for estimating the value density $f(\cdot|\cdot,I)$ from observed bids is $(L/\log L)^{R/(2R+d+3)}$ on any compact inner subset of its support and every $I \in \mathcal{S}_I$.*

This result tells us that any nonparametric estimator of $f(\cdot|\cdot,I)$ converges at most at the rate $(L/\log L)^{R/(2R+d+3)}$. This rate is slower than the optimal rate $(L/\log L)^{R/(2R+d+1)}$ if private values were observed. It also says that observing bids is not as good as observing private values to estimate the private value density despite the one-to-one mapping between bids and private values. It is instructive to understand how this optimal rate arises. From the relationship between the bid and private value densities, we have

$$f(v|x,I) = \frac{g(\xi^{-1}(v;x,I)|x,I)}{\xi'[\xi^{-1}(v;x,I);x,I]}.$$

We know that $g(\cdot|\cdot,I)$ is $R+1$ continuously differentiable. Thus, $g(\cdot|\cdot,I)$ and $\xi(\cdot;\cdot,I)$ can be estimated at the optimal rate $(L/\log L)^{(R+1)/(2R+d+3)}$ by Stone (1982). But $\xi'(\cdot;x,I)$ involves the derivative of the bid density $g(\cdot|x,I)$ by (3.7). Thus, the optimal rate for estimating $\xi'(\cdot;x,I)$ is that for $g'(b|x,i)$, i.e., $(L/\log L)^{R/(2R+d+3)}$. This slower rate gives the optimal rate for estimating the value density $f(\cdot|\cdot,\cdot)$.

Bandwidth Rates and Uniform Consistency

The next task is to find an estimator of $f(\cdot|\cdot,\cdot)$ that attains the optimal rate. Guerre, Perrigne and Vuong (2000) show that an appropriate choice of the vanishing rates of the bandwidths (h_G, h_g, h_f, h_X) in the kernel estimators (3.8) and (3.10) achieves the optimal rate r_L^* for $\hat{f}(\cdot|\cdot,\cdot)$ as stated next.

Optimal Bandwidths: *Let the bandwidths in (3.8) and (3.10) be*

$$\begin{aligned} h_G &= \lambda_G(\log L/L)^{1/(2R+d+2)}, \quad h_g = \lambda_g(\log L/L)^{1/(2R+d+3)} \\ h_f &= \lambda_f(\log L/L)^{1/(2R+d+3)}, \quad h_X = \lambda_X(\log L/L)^{1/(2R+d+2)} \end{aligned}$$

where the λs are positive constants. The estimator (3.10) attains the optimal uniform convergence rate $(L/\log L)^{R/(2R+d+3)}$ since

$$\sup_{(v,x)\in\mathcal{C}(V)} |\hat{f}(v|x,I) - f(v|x,I)| = O[(\log L/L)^{R/(2R+d+3)}]$$

for any compact inner subset $\mathcal{C}_{V|X,I}$ of the support of $f(\cdot|\cdot,I)$ for every $I \in \mathcal{S}_I$.

In view of their smoothness, the bid distribution, bid density, and covariate density are estimated at their optimal rates. In contrast, the bandwidth h_f for the value density leads to oversmoothing relative to the usual rate $(\log L/L)^{1/(2R+d+1)}$. Regarding the constants, one can use the standard rule of thumb, i.e., the product of a constant associated with the kernel function and the empirical standard deviation of the variable of interest. See Hardle (1991) for a table providing these constants for standard kernel functions. Also, uniform convergence is restricted to closed subsets of $S_{V|X,I}$. This issue relates to boundary effects associated with kernel estimators which we have omitted for the sake of presentation. We discuss this issue below. Using these optimal bandwidths, Ma, Marmer and Shneyerov (2019) establish the asymptotic normality of the two-step estimator (3.10) and provide its asympotic variance. In particular, asymptotic normality can be obtained at the rate $L^{R/(2R+d+3)}$.

Boundary Estimation and Boundary Effects

It is well known that kernel estimators suffer from asymptotic bias close to the boundaries. Indeed, the standard kernel estimator does not account for the absence of data outside the boundaries of the support and hence underestimates the density within an h-neighborhood of the boundaries. There exist several remedies to this problem.

Guerre, Perrigne and Vuong (2000) propose a simple and intuitive trimming that eliminates observations too close to the boundaries. Specifically, they consider the conditions

$$\begin{aligned} \hat{\underline{b}}(X_\ell, I) + (1+\epsilon)\max(h_G, h_g) \leq \quad & B_{i\ell} \quad \leq \hat{\overline{b}}(X_\ell, I) - (1+\epsilon)\max(h_G, h_g), \\ \underline{x} + (1+\epsilon)\max(h_G, h_g) \leq \quad & X_\ell \quad \leq \overline{x} - (1+\epsilon)\max(h_G, h_g), \end{aligned} \tag{3.12}$$

where ϵ is a small positive number and $\hat{\underline{b}}(X,I)$ and $\hat{\overline{b}}(X,I)$ are estimators of the lower and upper boundaries of the conditional bid distribution $G(\cdot|X,I)$, respectively. Instead of (3.9) they define the estimated private values as

$$\hat{V}_{i\ell} = \begin{cases} B_{i\ell} + \frac{1}{I_\ell - 1}\frac{\hat{G}(B_{i\ell},X_\ell|I_\ell)}{\hat{g}(B_{i\ell},X_\ell|I_\ell)} & \text{if (3.12) is satisfied} \\ +\infty & \text{otherwise.} \end{cases}$$

With a kernel $K(\cdot)$ defined on $[-1,1]$, this eliminates the pseudo values close to the boundaries from the second step. In practice, one applies the trimming on bids only as they are correlated with the characteristics X. Regarding the choice of ϵ, it can be any small value but we recommend a visual check of the estimated private value density. Though simple, the trimming can suppress important information. To mitigate this effect, one can perform the two-step estimator after taking the logarithm (or even double logarithm) of the bids when the bid density is skewed so as to obtain a bell-shaped bid density. This transformation entails a rewriting of the inverse equilibrium strategy as the derivative of $G(\log b|x,I)$ is equal to $g(\log b|x,I)/b$.

The trimming method requires estimating the boundaries of the bid distribution. Guerre, Perrigne and Vuong (2000) use a simple boundary estimator. They partition $I\!R^d$ in hypercubes of sides h_δ, which induces a partition of $[\underline{x},\overline{x}]$. Given a number of bidders I, the upper boundary $\overline{b}(x,I)$ is estimated as the supremum of the bids $B_{i\ell}$ whose corresponding X_ℓ falls in the same hypercube as x. Similarly, the lower boundary $\underline{b}(x,I)$ is estimated as the infimum of those bids. When h_δ is proportional to $(\log L/L)^{1/(d+1)}$, the boundary estimators converge uniformly at the rate $(L/\log L)^{1/(d+1)}$. On one hand, this rate is faster than than the optimal rate for estimating $f(\cdot|\cdot,I)$ and thus does not affect the convergence rate of the two-step estimator. On the other hand, this estimator does not provide a smooth estimate of the boundaries.

Campo, Guerre, Perrigne and Vuong (2011) use instead Korostelev and Tsybakov (1993) estimator for image reconstruction which exploits the smoothness of the boundaries. Given the $R+1$ differentiability of the boundaries, the estimator for the upper boundary $\overline{b}(x,I)$ minimizes the volume of the cylinder whose base is the bin containing x and whose upper surface is defined by a polynomial of degree R in x subject to the constraint that the observations are contained in such a cylinder. The lower boundary estimator is defined similarly by a maximization subject to the constraint that observations are above the surface. The optimal polynomials evaluated at x give the estimators $\hat{\overline{b}}(x,I)$ and $\hat{\underline{b}}(x,I)$. The boundary estimators converge uniformly at the rate $(L/\log L)^{(R+1)/(R+1+d)}$ when h_δ is proportional to $(\log L/L)^{1/(R+1+d)}$. This rate is faster than the rates of the previous boundary estimators. It is also faster than $\sqrt{L}$ when $R \geq d$. These estimators are piecewise smooth but they are not continuous. Using smoothing splines of degree R delivers continuous estimators.

Alternatively to the trimming used by Guerre, Perrigne and Vuong (2000), Hickman and Hubbard (2015) use a boundary correction proposed by Karunamuni and Zhang (2008). This correction is a hybrid of reflection and transformation methods to reduce the bias close to the boundaries. The former artificially 'reflects' data near the boundaries by adding a kernel term, while the latter transforms the data onto an unbounded support. Both methods reduce the bias at the price of increasing the variance near the endpoints. Consider observations $X = (X_1, \ldots, X_L)$ and the estimation of their density $f_X(\cdot)$, which is bounded away from zero on its support $[0, \infty)$. The corrected kernel density estimator for $x \in [0, h)$ is

$$\hat{f}_X(x) = \frac{1}{Lh} \sum_{\ell=1}^{L} \tilde{K}(x; X_\ell, h)$$

with

$$\tilde{K}(x; X_\ell, h) = K\left(\frac{x - X_\ell}{h}\right) + K\left(\frac{x + \hat{\tau}(X_\ell)}{h}\right),$$

where the transformation is $\hat{\tau}(y) = y + \hat{d}y^2 + A\, \hat{d}^2\, y^3$. The estimated parameter $\hat{d}$ is given by $\hat{d} = [\log \phi_L(h_1; X) - \log \Psi_L(h_0; X)]/h_1$, where

$$\phi_L(h_1; X) = \frac{1}{L^2} + \frac{1}{Lh_1} \sum_{\ell=1}^{L} K\left(\frac{h_1 - X_\ell}{h_1}\right), \quad \Psi_L(h_0; X) = \max\left(\frac{1}{L^2}, \frac{1}{Lh_0} \sum_{\ell=1}^{L} K_0\left(\frac{-X_\ell}{h_0}\right)\right).$$

Karunamuni and Zhang (2008) consider a constant $A > 1/3$, a bandwidth $h_1 = o(h)$, a bandwidth $h_0 = \eta h_1$ and a one-sided kernel $K_0(\cdot)$ on support $[-1, 0]$. See these authors' paper for the determination of the constant η and the choice of the one-sided kernel.

Hickman and Hubbard (2015) adapt this boundary correction to the benchmark model where the lower boundary may not vanish and the upper boundary is finite. They use it in each step of the GPV procedure. Their Monte Carlo study shows significant improvements. Because of the data loss in the trimming technique, they consider the region of valid inference and find it to be above 99% of the private value support under the boundary correction while this percent can drop below 50% under trimming. The square root of the Mean Integrated Squared Error (MISE) over the region of valid inference also improves with the boundary correction. They illustrate the boundary correction method to the gas lease auctions data used in Campo, Perrigne and Vuong (2003) where private values are affiliated and asymmetric. Their empirical analysis shows that trimming could entail an important loss of valuable information on the positive dependence among private values with an underestimation of information rents and inefficiency.

Curse of Dimensionality

A well-known issue with nonparametric estimators is the curse of dimensionality arising from the dimension d of the vector of exogenous variables X which exponentially decreases the rates of convergence. As a consequence, considering $d > 1$ would require large data sets to obtain satisfactory results. Haile, Hong and Shum (2006) are the first to propose a remedy by first homogeneizing bids across auctions. The idea is to demean the bids upon assuming that $V_{i\ell} = m(X_\ell, I_\ell; \beta) + \tilde{V}_{i\ell}$, $i = 1, \ldots, I_\ell$, $\ell = 1, \ldots, L$, where $m(\cdot, \cdot; \cdot)$ is a known function, β is a vector of unknown parameters and the residuals $\tilde{V}_{i\ell}$ are independent of X_ℓ conditional on I_ℓ with $\mathrm{E}[\tilde{V}_{i\ell}|I_\ell] = 0$. This is equivalent to assuming that the value distribution belongs to the location family $F(\cdot|X, I) = \tilde{F}(\cdot - m(X, I; \beta)|I)$ where $\tilde{F}(\cdot|I)$ is the distribution of the demeaned private values $\tilde{V}_{i\ell}$ conditional on I. From the location-scale invariance of the equilibrium strategy in Section 2.2, we have $B_{i\ell} = m(X_\ell, I_\ell; \beta) + \tilde{B}_{i\ell}$ where $\tilde{B}_{i\ell} = s(\tilde{V}_{i\ell}; \tilde{F}, I_\ell)$ is the 'demeaned' bid. Let $\gamma_I \equiv \mathrm{E}[\tilde{B}_i|I] = \mathrm{E}[s(\tilde{V}_i|\tilde{F}, I)|I]$ and $\gamma \equiv \{\gamma_I; I \in \mathcal{S}_I\}$. We obtain

$$B_{i\ell} = m(X_\ell, I_\ell; \beta) + \gamma_{I_\ell} + \epsilon_{i\ell},$$

where $\epsilon_{i\ell} \equiv \tilde{B}_{i\ell} - \gamma_{I_\ell}$ has zero mean conditional on (X_ℓ, I_ℓ). Thus LS regression of the observed bids on (X_ℓ, I_ℓ) and the set of dummies $\{\mathbb{1}(I_\ell = I); I \in \mathcal{S}_I\}$ gives estimated parameters $\hat{\beta}$, estimated residuals $\hat{\epsilon}_{i\ell}$ and hence estimated demeaned bids $\hat{\tilde{B}}_{i\ell} = \hat{\epsilon}_{i\ell} + \hat{\gamma}_{I_\ell}$ provided the parameters (β, γ) are identified. Since $\tilde{B}_{i\ell} = s(\tilde{V}_{i\ell}; \tilde{F}, I_\ell)$, the estimated demeaned bids $\hat{\tilde{B}}_{i\ell}$ can be used as bids in the GPV procedure without any covariates X so as to estimate the density of $\tilde{f}(\cdot|I_\ell)$ of the demeaned values $\tilde{V}_{i\ell}$. The density of private values is then estimated by $\hat{\tilde{f}}(\cdot - m(X, I; \hat{\beta})|I)$. This demeaning method became popular in empirical studies as it drops the d dimension in the convergence rates thereby eliminating the curse of dimensionality. It assumes, however, that the covariates affect only the mean of the private value distribution and not its higher moments.

An alternative dimension reduction device borrows from single-index models. The idea is to reduce the d dimension into a single one. Specifically, one assumes that the value distribution is of the form $F(\cdot|X, I) = \tilde{F}(\cdot|X'\beta_I, I)$, where one parameter in β_I is normalized to one for each $I \in \mathcal{S}_I$. Since $\mathrm{E}[B_{i\ell}|X_\ell, I_\ell] = m(X_\ell'\beta_{I_\ell}, I_\ell)$ for some unknown function $m(\cdot, \cdot)$, the coefficients β_I can be estimated by a single-index estimator on the subsample of auctions with I bidders. See e.g., Powell, Stock and Stoker (1989), Ichimura (1993) and Hardle, Hall and Ichimura (1993). This gives a $\sqrt{L}$-consistent estimator $\hat{\beta}_I$ for each $I \in \mathcal{S}_I$. The GPV procedure then applies with the estimated distribution and density $\hat{G}(\cdot|X'\hat{\beta}_I, I)$ and $\hat{g}(\cdot|X'\hat{\beta}_I, I)$ of bids given the estimated index $X'\hat{\beta}_I$ and I. The second step estimates the value density conditional on $(X'\hat{\beta}_I, I)$ giving $\hat{f}(\cdot|X, I) =$

$\hat{\hat{f}}(\cdot|X'\hat{\beta}_I, I)$, which corresponds to an effective dimension equal to one. The vector of exogenous variables can now affect all the moments of the value distribution though all the moments depend on X only through the same linear index $X'\beta_I$ given I.

Section 3.4: Quantile-Based Estimators

An alternative nonparametric estimation method exploits the monotonicity of the equilibrium strategy using quantiles. Let $v(\alpha; x, I)$ and $b(\alpha; x, I)$ be the α-quantiles of the private value and bid distributions $F(\cdot|x, I)$ and $G(\cdot|x, I)$ given (x, I), respectively. Thus, $F[v(\alpha; x, I)|x, I] = G[b(\alpha; x, I)|x, I] = \alpha$ by definition. Because the inverse equilibrium strategy $\xi(\cdot; x, I)$ is increasing, it follows from (3.6) and the rank invariance property of quantiles that

$$v(\alpha; x, I) = b(\alpha; x, I) + \frac{1}{I-1}\frac{\alpha}{g[b(\alpha; x, I)|x, I]} \equiv \xi(\alpha; x, I) \tag{3.13}$$

for $\alpha \in [0, 1]$. See Milgrom (2004) and Haile, Hong and Shum (2006). Several papers rely on (3.13) to propose quantile-based estimators for the benchmark model.

Differentiating the Value Quantile

Marmer and Shneyerov (2012) estimate the value density by differentiating the value quantile (3.13). The authors invoke a well-known property relating a density evaluated at an α-quantile and the derivative of this α-quantile. Namely, differentiating $F[v(\alpha; x, I)|x, I] = \alpha$ with respect to α leads to $f[v(\alpha; x, I)|x, I] = 1/v'(\alpha; x, I)$, where $v'(\cdot; x, I)$ is the derivative of $v(\cdot; x, I)$. A similar property applies to the bid quantile and density. Thus, differentiating (3.13) gives

$$f[v|x, I] = \left[\frac{I}{I-1}\frac{1}{g[b(\alpha; x, I)|x, I]} - \frac{1}{I-1}\frac{\alpha g'[b(\alpha; x, I)|x, I]}{g^3[b(\alpha; x, I)|x, I]}\right]^{-1},$$

where $v = v(\alpha; x, I)$ or equivalently $\alpha = F(v|x, I)$. This suggests estimating $f(v|x, I)$ by a plug-in method upon estimating the bid quantile $b(\alpha; x, I)$, the bid density $g(\cdot|x, I)$ and its derivative $g'(\cdot|x, I)$ evaluated at this quantile.

Marmer and Shneyerov (2012) use the kernel density estimator for the bid density of Section 3.3, namely $\hat{g}(\cdot|x, I) = \hat{g}(\cdot, x|I)/\hat{f}_{X|I}(x|I)$ where the latter are given by (3.8) and (3.10). Differentiating $\hat{g}(\cdot|x, I)$ provides an estimator $\hat{g}'(\cdot|x, I)$ of the derivative. For the bid quantile $b(\cdot|x, I) \equiv \inf_b\{G(b|x, I) \geq \cdot\}$, they use the standard bid quantile estimator $\hat{b}(\cdot; x, I) = \inf_b\{b : \hat{G}(b|x, I) \geq \cdot\}$ where $\hat{G}(\cdot|x, I) = \hat{G}(\cdot, x|I)/\hat{f}_{X|I}(x|I)$ with $\hat{G}(\cdot, x|I)$ given in (3.8). To estimate $\alpha = F(v|x, I)$, they use $F(\cdot|x, I) \equiv \sup_a\{v(a|x, I) \leq \cdot\}$ where $v(a|x, I)$ is given by (3.13). Because plugging-in $\hat{g}(\cdot|x, I)$ and $\hat{b}(\cdot|x, I)$ in the RHS of (3.13) leads to an estimator $\hat{v}(\cdot;x,I)$ that is not necessarily increasing, the authors propose a monotone version $\tilde{v}_{MS}(\cdot;x,I)$ of the value quantile function. Namely, $\tilde{v}_{MS}(a;x,I)$ equals $\sup_{\tilde{a}\in[a_0,a]}\hat{v}(\tilde{a};x,I)$ for $a\in[a_0,1]$ and $\inf_{\tilde{a}\in[a,a_0]}\hat{v}(\tilde{a};x,I)$ for $a\in[0,a_0]$ where $a_0=0.5$. For another monotone estimator of the value quantile function, see below.

Letting the bandwidths h_G, h_g and h_X in $\hat{G}(\cdot,\cdot|I)$, $\hat{g}(\cdot,\cdot|I)$ and $\hat{f}_{X|I}(\cdot|I)$ be proportional to $(\log L/L)^{1/(2R+d+3)}$, Marmer and Shneyerov (2012) show that their quantile-based estimator of the value density $f(\cdot|\cdot,I)$ achieves the optimal uniform rate.[8] They also derive its asymptotic distribution at the rate $L^{R/(2R+d+3)}$. A Monte Carlo study then compares the kernel-based two-step estimator (3.10) and their quantile-based estimator. None of the two estimators dominates the other though the former tends to have a smaller bias because the quantile-based estimator involves the estimate of the derivative of a density. Overall, the former estimator tends to be more efficient when the derivative of the value density is positive and the number of bidders is large while the latter tends to be more efficient in the reverse case. As for the two-step estimator (3.10), the quantile-based estimator suffers from boundary effects and uniform convergence is obtained on compact inner subsets of the support.

Differentiating the Bid Quantile

Using $b'(\alpha;x,I)=1/g[b(\alpha;x,I)|x,I]$, Guerre and Sabbah (2012) note that (3.13) can be rewritten as

$$v(\alpha;x,I)=b(\alpha;x,I)+\frac{\alpha b'(\alpha;x,I)}{I-1}, \tag{3.14}$$

for $\alpha\in[0,1]$. Gimenes and Guerre (2021) then focus on the estimation of the bid quantile and especially its derivative. Quantile regression is the subject of a vast literature since the landmark paper by Koenker and Bassett (1978). However, estimating the derivative $b'(\cdot;x,I)$ has attracted less interest apart from the natural estimator based on $b'(\alpha;x,I)=1/g[b(\alpha;x,I)|x,I]$ which requires estimating the bid density $g(\cdot|x,I)$ nonparametrically.

Gimenes and Guerre (2021) propose a local polynomial quantile regression method that estimates simultaneously the quantile $b(\cdot;x,I)$ and its derivative $b'(\cdot;x,I)$. The authors first note that the equilibrium strategy (2.7) expressed in quantiles combined with a change of variable and an integration by parts gives

$$b(\alpha;x,I)=\frac{I-1}{\alpha^{I-1}}\int_0^\alpha u^{I-2}v(u;x,I)du. \tag{3.15}$$

Thus, if the private value quantile satisfies $v(\alpha;x,I)=\sum_{k=1}^{K}P_{kK}(x)\gamma_{kK}(\alpha,I)$ with $P_{kK}(x)$ a function of x and $\gamma_{kK}(\alpha,I)$ its coefficients which depends on (α,I), then (3.15) shows

[8] Their bandwidths h_G and h_X are slightly smaller than the optimal bandwidths. This has no effect as the rate of the quantile-based estimator comes from estimating the derivative $g'(\cdot|x,I)$.

that the bid quantile satisfies $b(\alpha;x,I)=\sum_{k=1}^{K}P_{kK}(x)\beta_{kK}(\alpha,I)$ with coefficients $\beta_{kK}(\alpha,I)$ related to $\gamma_{kK}(\alpha,I)$ by the equality $\beta_{kK}(\alpha,I)=[(I-1)/\alpha^{I-1}]\int_0^\alpha u^{I-2}\gamma_{kK}(u;I)du$ for $k=1,\dots,K$. This linear invariance property of the equilibrium strategy suggests that approximating the value quantile $v(\alpha;\cdot,I)$ by a linear sieve with basis functions $P_{kK}(\cdot)$ leads to approximating the bid quantile $b(\alpha;\cdot,I)$ by the same linear sieve. See Chen (2007) for a survey on sieve estimation. Moreover, estimating the coefficients $\beta_{kK}(\alpha,I)$ provides estimates of the coefficients $\gamma_{kK}(\alpha,I)$ through the inverse relation (3.14), namely

$$\gamma_{kK}(\alpha,I)=\beta_{kK}(\alpha,I)+\frac{\alpha\beta'_{kK}(\alpha,I)}{I-1}, \tag{3.16}$$

where $\beta'_{kK}(\cdot,I)$ is the derivative of $\beta_{kK}(\cdot,I)$.

To estimate the coefficients $\beta_{kK}(\alpha,I)$ and their derivatives $\beta_{kK}^{(r)}(\alpha,I)$ up to order $(R{+}1)$ when $v(\cdot;\cdot\alpha)$ is $(R+1)$ continuously differentiable, the authors consider an Augmented Quantile Regression (AQR). Indeed, in addition to the usual term $\sum_{k=1}^{K}P_{kK}(x)\beta_{kK}(\alpha,I)$ in the check function $\rho_\alpha(x)=[\alpha-1\!\!I(x<0)]x$ of the quantile regression, one considers the other terms in a Taylor expansion of order $(R+1)$ of $\sum_{k=1}^{K}P_{kK}(x)\beta_{kK}(\alpha,I)$ in an h -neighborhood of α. For any given I, this leads to the objective function

$$\begin{aligned}\frac{1}{IL_I}\sum_{\ell:I_\ell=I}^{L_I}\sum_{i=1}^{I}\int_0^1\rho_u\Bigg[B_{i\ell}-\sum_{k=1}^{K}P_{kK}(X_i)\beta_{0k}-\sum_{k=1}^{K}P_{kK}(X_i)\frac{h(u-\alpha)}{1!}\beta_{1k}-\dots\\ -\sum_{k=1}^{K}P_{kK}(X_i)\frac{[h(u-\alpha)]^{R+1}}{(R+1)!}\beta_{R+1,k}\Bigg]\times\frac{1}{h}K\left(\frac{u-\alpha}{h}\right)du,\end{aligned} \tag{3.17}$$

where $K(\cdot)$ is a kernel and h a vanishing bandwidth. As the authors indicate, integrating over $u\in[0,1]$ allows to estimate $\underline{b}(x,I)$ and $\overline{b}(x,I)$ as well as to smooth out the check function around zero. For the basis functions $P_{kK}(\cdot)$, they use localized multivariate Cardinal B-splines of degree $R+2$ and set their number K to be inversely proportional to h^d. See also Belloni, Chernozhukov, Chetverikov and Fernandez-Val (2019). Minimization of (3.17) with respect to $\{(\beta_{0k},\dots,\beta_{R+1.k});k=1,\dots,K\}$ gives estimated bid quantile and its first derivative as $\hat{b}(\alpha;x,I)=\sum_{k=1}^{K}P_{kK}(x)\hat\beta_{0k}(\alpha,I)$ and $\hat{b}'(\alpha;x,I)=\sum_{k=1}^{K}P_{kK}(x)\hat\beta_{1k}(\alpha,I)$. Estimation of the coefficients $\gamma_{kK}(\alpha,I)$ follows from (3.16) with $[\beta_{kK}(\alpha,I),\beta'_{kK}(\alpha,I)]$ replaced by $[\hat\beta_{0k}(\alpha,I),\hat\beta_{1k}(\alpha,I)]$. This defines the AQR estimator of the value quantile as $\hat{v}(\alpha;x,I)=\sum_{k=1}^{K}P_{kK}(x)\hat\gamma_{kK}(\alpha,I)$ for every α on a grid $[0,1]$ and for every $I\in\mathcal{S}_I$.

With $R\geq d/2$ and $h\propto(\log L/L)^{1/(2R+d+3)}$, the authors show that the bid quantile estimator $\hat{b}(\cdot;x,I)$ and its first derivative estimator $\hat{b}'(\cdot;x,I)$ converge at the rate $(L/\log L)^{(R+1)/(2R+d+3)}$ uniformly on $[0,1]$. It follows from (3.14) that the AQR estimator of the value quantile $v(\cdot;\cdot,I)$ converges uniformly at the rate $(L/\log L)^{(R+1)/(2R+d+3)}$, which is the optimal uniform convergence rate for estimating the value distribution $F(\cdot|\cdot, I)$. An appealing feature of the AQR estimator is that its convergence rate is valid on the interval $[0,1]$, i.e., there is no boundary effect and thus no need for trimming. The authors also establish its asymptotic distribution. To estimate the value density, they exploit the identity

$$F(v|x,I) = \mathrm{E}[\mathbb{1}(v(U;x,I) \leq v)|x,I] = \int_0^1 \mathbb{1}[v(\alpha;x,I) \leq v] d\alpha,$$

where U is distributed uniformly on $[0,1]$. See also Chernozhukov, Fernandez-Val and Galichon (2010). A smoothed version of this identity leads to $F(v|x,I) \approx \int_0^1 \mathbb{1}_h[v - v(\alpha;x,I)]d\alpha$ where $\mathbb{1}_h(t) = \int_{-\infty}^{t/h} K(u)du$ with $K(\cdot)$ a kernel and h a bandwidth. The value density is then estimated by $(1/h)\int_0^1 K[(v - \hat{v}(\alpha;x,I))/h]d\alpha$, which converges to $f(v|x,I)$ when h converges to zero.

Integrating the Value Quantile

Following Liu and Vuong (2020) who use the integrated quantile for testing, Luo and Wan (2018) use it to impose monotonicity of the equilibrium strategy in estimation. Dropping the covariates X to simplify and integrating (3.14) give the integrated quantile

$$D(t;I) \equiv \int_0^t \xi(\alpha;I)\ d\alpha = \frac{I}{I-1} t \cdot b(t;I) + \frac{I-2}{I-1}\int_0^t b(\alpha;I)\ d\alpha \tag{3.18}$$

for $t \in [0,1]$ upon integration by parts. Because $\xi(\cdot;I)$ is increasing, then $D(\cdot;I)$ is convex, and conversely. This leads to considering the Greatest Convex Minorant (GCM) of the estimated integrated quantile $\hat{D}(\cdot;I)$ to test and/or impose monotonicity of $\xi(\cdot;I)$. In contrast to the previous quantile-based methods, an appealing feature of (3.18) is that it involves neither the density in a denominator nor the derivative of the bid quantile. Estimation of $D(\cdot;I)$ achieves the $\sqrt{L}$-convergence rate by plugging-in the standard quantile estimator $\hat{b}(\alpha;I) = \inf_b\{\hat{G}(b|I) \geq \alpha\}$ in (3.18) with $\hat{G}(\cdot|I)$ being the empirical bid distribution for auctions with I bidders. In particular, there is no boundary effect and no need for trimming. There is also no need for a smoothing parameter such as a bandwidth. Moreover, because $\hat{b}(\cdot;I)$ is a step function then $\hat{D}(\cdot;I)$ is a continuous piecewise linear function. Thus the GCM of the estimated integrated quantile denoted $\underline{\mathcal{C}}(\hat{\xi}_I)(\cdot)$ is straightforward to determine as it is continuous and piecewise linear.

Luo and Wan (2018) take the piecewise (left) derivatives of $\underline{\mathcal{C}}(\hat{\xi}_I)(\cdot)$ as an estimator $\hat{v}_{LW}(\cdot;I)$ of the value quantile since $v(\cdot;I) = D'(\cdot;I)$. Because $\underline{\mathcal{C}}(\hat{\xi}_I)(\cdot)$ is convex, the estimator $\hat{v}_{LW}(\cdot;I)$ is weakly increasing.[9] Assuming that the value distribution $F(\cdot|I)$ is continuously differentiable, they show that $\hat{v}_{LW}(\cdot;I)$ and hence the estimator $\hat{F}(\cdot|I) \equiv$

[9]Though the rearrangement method proposed by Chernozhukov, Fernadez-Val and Galichon (2010)

$\sup_\alpha\{\alpha; \hat{v}_{LW}(\alpha; I) \leq \cdot\}$ converge pointwise at the $L^{1/3}$-rate. They derive their asymptotic normal distributions. Because $\hat{v}_{LW}(\cdot; I)$ is a step function, they smooth it by

$$\tilde{v}_{LW}(\cdot; I) = \int_0^1 \frac{1}{h} K\left(\frac{\cdot - a}{h}\right) \hat{v}_{LW}(a; I) da,$$

where $K(\cdot)$ is a kernel with compact support and h is a bandwidth following Parzen (1979). With $F(\cdot|I)$ twice continuously differentiable and $h \propto L^{-1/5}$, the authors show that the kernel smoothed estimator $\tilde{v}_{LW}(\cdot; I)$ converges pointwise at the rate $L^{2/5}$ which corresponds to the optimal uniform convergence rate $(L/\log L)^{1/5}$ for estimating $f(\cdot|I)$ when $R = 1$ and $d = 0$.

Dimension Reduction

In view of the curse of dimensionality, Gimenes and Guerre (2021) propose a dimension reduction technique. Following Horowitz and Lee (2005), they assume that the bid quantile $b(\cdot; X, I)$ is additively separable in the covariates X, i.e., $b(\cdot; X, I) = \sum_{j=1}^d b_j(\cdot; X_j, I)$. This corresponds to assuming that the value quantile $v(\cdot; X, I)$ is additively separable in $X = (X_1, \ldots, X_d)$ by the linear invariance property of the equilibrium strategy noted after (3.15). The estimator follows from (3.17) with $P_{kK}(X_i)\beta_{0k}$ replaced by $\sum_{j=1}^d P_{kK}(X_{ji})\beta_{0jk}$ and similarly for the other terms. The number of coefficients to be estimated is now $(R+1)dK$ instead of $(R+1)K$. Because of the additive separability of $b(\alpha; \cdot, I)$, however, the number K of splines $P_{kK}(\cdot)$ need not be as large so that the number of coefficients to be estimated may be actually smaller. The authors show that the uniform convergence rate applies but now with the effective dimension $d = 1$. This method clearly extends to higher order interactions such as bivariate interactions among the d covariates in which case the effective dimension equals 2.

Section 4: Basic Extensions

In this section, we review some basic extensions of the benchmark model of Section 2.3 and present the corresponding estimation methods. These extensions apply to the first-price sealed-bid auction mechanism within the private value paradigm. Common value and more generally interdependent values are reviewed in Section 8. The first three extensions include reserve prices, asymmetric values and affiliated values, which lead to straight applications of the GPV procedure. We consider the indirect approach. Direct

could be used in principle to achieve monotonicity of $\hat{v}(\cdot; x, I)$, this would require to estimate the density $g(\cdot|x, I)$ in the denominator of (3.13). See also Henderson, List, Millimet, Parmeter and Price (2012) who are the first to impose monotonicity of the inverse equilibrium strategy $\xi(b, x, I)$ by tilting the empirical distribution of the data.

estimation methods of Section 3.1 can be extended but are computionally feasible only in a few cases. We also focus on kernel-based estimators though one could adapt as well the quantile-based estimators reviewed in Section 3.4. These extensions are 'basic' but address recurrent practical issues in the empirical analysis of auctions.

In many auctions there is a minimum bid, i.e., a value below which the seller refuses to sell. This minimum bid or reserve price is often announced to bidders before the auction but it can also be kept secret in some auctions. Section 4.1 covers both cases. The main econometric difficulty is that the announced reserve price acts as a truncation of the bid distribution and thus limits identification. Moreover, the number I of bidders may not be observed. In Section 4.2, we consider asymmetry among bidders, which is also known as *ex ante* asymmetry. This refers to a situation where bidders are heterogeneous in their private value distributions so that they no longer draw their values from the same distribution. Empirically, this arises because of differences in some exogenous bidders' characteristics such as size, distance in construction procurements, etc. The auction model then becomes more complex with no closed form solutions for the equilibrium strategies. Nonetheless, the indirect approach offers a practical and clean solution. In Section 4.3, we address the third extension which departs from the independence of private values and considers affiliated private values as defined in Section 2.1. On empirical grounds, for instance, affiliation arises when there is some prestige effect in owning the object.

A fourth extension introduces unobserved heterogeneity across auctioned objects. Unobserved heterogeneity is not part of the theoretical model but is instead an econometric issue. It may happen that some object characteristics are unobserved because they are impossible to quantify. For instance, construction projects in procurement auctions include numerous technical information that the analyst does not necessarily observe. Another example arises with used goods such as collectible stamps or coins where a visual examination provides valuable information. In Section 3.1, we mention how the number I of bidders as a conditioning variable in the private value distribution captures some unobserved heterogeneity. This may not be sufficient. Section 4.4 addresses unobserved heterogeneity through measurement errors and deconvolution techniques.

To unify the presentation, we rely on the best-response interpretation given after (3.6) of the equilibrium strategy. The same idea carries through the basic extensions as well as to other auction formats in subsequent sections. For instance in first-price auctions, bidder i's expected profit is in general of the form $(V_i - B_i)\,\Pr_{win}(B_i|V_i)$ where $\Pr_{win}(B_i|V_i) \equiv \Pr(B_{-i}^{\max} \leq B_i|V_i)$ denotes bidder i's winning probability conditional on his private information V_i for a bid B_i. Taking as given competitors' behavior and maximizing the expected profit with respect to B_i then gives the first-order condition

$$V_i = B_i + \frac{\Pr_{win}(B_i|V_i)}{\partial \Pr_{win}(B_i|V_i)/\partial B_i}.$$

Such a necessary condition combined with the model restrictions on the bid distribution then delivers a valid monotone Bayesian Nash equilibrium. Hereafter, we omit the conditioning on the characteristics X in the value distribution $F(\cdot|I)$ when introducing an extension and studying its identification but reintroduce them in estimation.

Section 4.1: Reserve Price

Announced Reserve Price

From a theoretical point of view, the presence of an announced reserve price $p_0 > 0$ may induce bidders to bid more aggressively by reducing their information rent. If $p_0 \leq \underline{v}(I)$, the reserve price is nonbinding and the bidding strategy remains as in (2.7). In contrast, when the reserve price is binding, i.e., $p_0 > \underline{v}(I)$, the bidding strategy (2.7) changes with a lower bound p_0 for the integral instead of $\underline{v}$, namely,

$$B_i = V_i - \frac{1}{F(V_i|I)^{I-1}} \int_{p_0}^{V_i} F(x|I)^{I-1} dx \equiv s(V_i; F, I, p_0), \tag{4.1}$$

for $V_i \geq p_0$. This increasing strategy solves the differential equation (2.6) with the boundary condition $p_0 = s(p_0; F, I)$.

From an econometric point of view, the reserve price induces a truncation of the bid distribution as p_0 acts as a screening device on bidders' participation. Specifically, when there is an announced binding reserve price $p_0 \in (\underline{v}(I), \overline{v}(I))$, only bidders with values above p_0 actually tender a bid. Thus the number I^* of actual (or active) bidders is smaller or equal than the number I of potential bidders. Let $\tilde{B}_j = 0$ if $V_j < p_0$ and $\tilde{B}_j = B_j$ if $V_j \geq p_0$, where $B_j = s(V_j; F, I, p_0)$ is the observed bid. The observed bid B_j follows the truncated distribution $G^*(\cdot|I) \equiv \Pr[s(V_j; F, I, p_0) \leq \cdot | I, V_j \geq p_0]$ with support $[p_0, \overline{b}(I)] = [p_0, s(\overline{v}(I); F, I, p_0)]$. Thus the distribution of $\tilde{B}_j$ is $\tilde{G}(\cdot|I) = F(p_0|I) + [1 - F(p_0|I)]G^*(\cdot|I)\mathbb{1}(\cdot \geq p_0)$, where $F(p_0|I)$ is the probability that a potential bidder does not bid. Hence bidder i's probability of winning is $\Pr(B_i > \tilde{B}_{-i}^{\max}|I) = \{F(p_0|I) + [1 - F(p_0|I)]G^*(\cdot|I)\}^{I-1}$ for $B_i \geq p_0$. Maximizing the expected profit $(V_i - B_i)\Pr(B_i > \tilde{B}_{-i}^{\max}|I)$ with respect to B_i gives

$$V_i = B_i + \frac{1}{I-1}\left(\frac{G^*(B_i|I)}{g^*(B_i|I)} + \frac{F(p_0|I)}{1 - F(p_0|I)}\frac{1}{g^*(B_i|I)}\right) \equiv \xi(B_i; G^*, I, F(p_0|I)), \tag{4.2}$$

for $i = 1, \ldots, I^*$, where $g^*(\cdot|I)$ is the truncated bid density. Equation (4.2) reduces to (3.6) when there is no reserve price.

The RHS of (4.2) depends on the number of potential bidders I and the probability of not participating $F(p_0|I) \in (0,1)$, which are both unknown to the analyst. This complicates the identification problem. Because I^* follows a Binomial distribution $\mathcal{B}[I, 1-F(p_0|I)]$, I and $F(p_0|I)$ are identified using two moments of the Binomial. This result exploits variations in the observed number of actual bidders while the number I of potential bidders is assumed constant across auctions. Once I and $F(p_0|I)$ are identified, it is straightforward to identify the truncated value distribution $F^*(\cdot|I) = [F(\cdot|I) - F(p_0|I)]/[1-F(p_0|I)]$ from (4.2). Since $F(p_0|I)$ is identified, identification of $F(\cdot|I)$ is achieved on $[p_0, \overline{v}] = [\xi(p_0; G^*, I, F(p_0|I)), \xi(\overline{b}(I); G^*, I, F(p_0|I)]$. We note that $F(\cdot|I)$ can be identified on its full support if there is sufficient variation in the reserve price across auctions so that p_0 approaches $\underline{v}$.

Regarding the model restrictions, beside that the number of actual bidders must be Binomial distributed, conditions similar to C1 and C2 in Section 3.2 must hold. Specifically, the observed bids are i.i.d. as some distribution $G^*(\cdot|I)$ and the function $\xi(\cdot; G^*, I, F(p_0|I))$ defined in (4.2) is increasing in b on $[p_0, \overline{b}(I)]$ with a differentiable inverse on $[p_0, \overline{v}(I)]$, where $\overline{v}(I) = \xi(\overline{b}(I); G^*, I, F(p_0|I))$. Moreover, examining the differential equation defining the equilibrium strategy shows that $\lim_{v\downarrow p_0} f(v|I)/s'(v; F, I) = +\infty$ since $F(p_0|I) \in (0,1)$. Therefore, the truncated bid density should satisfy $\lim_{b\downarrow p_0} g^*(b|I) = +\infty$ as well.[10] These conditions are necessary and sufficient and thus characterize all the restrictions on observables imposed by the benchmark IPV model with an announced binding reserve price.

Estimation follows the identification argument. The observations are $\{B_{i\ell}, p_{0\ell}, X_\ell, i = 1,\ldots,I_\ell^*,\ \ell = 1,\ldots,L\}$ upon allowing the reserve price to vary across auctions. We first need to estimate the number of potential bidders I and the participating probability $1-F(p_{0\ell}|X_\ell, I)$. We recall that the number of potential bidders is constant across auctions or more generally on some subsets of auctions that are determined by a data-driven analysis. We estimate I by the maximum of the observed I_ℓ^* across auctions or on each corresponding subset of auctions. This gives $\hat{I}$, which converges at a rate faster than $\sqrt{L}$. Regarding the estimation of $1-F(p_{0\ell}|X_\ell, I_\ell)$, we have $\mathrm{E}[I_\ell^*|X_\ell] = I[1-F(p_{0\ell}|X_\ell, I)]$. Thus, using a nonparametric regression estimator of I_ℓ^* on X_ℓ, we obtain an estimator of $F(p_{0\ell}|X_\ell, I)$ as $1-\hat{\mathrm{E}}[I_\ell^*|X_\ell]/\hat{I}$. With these estimates in hand, we apply the GPV procedure to (4.2).[11] The resulting estimator $\hat{f}(\cdot|\cdot, I)$ is uniformly consistent and attains

[10] In contrast, $\lim_{b\downarrow \underline{b}(I)} g(b|I) = f(\underline{v}(I)|I)/s'(\underline{v}(I); F, I) = If(\underline{v}(I)|I)/(I-1) < \infty$ when the reserve price is nonbinding. This can be used to assess whether the reserve price is binding.

[11] As a matter of fact, Guerre, Perrigne and Vuong (2000) note that the truncated density $g^*(b|x, I)$ behaves as $1/\sqrt{b-p_0}$ in the neighborhood of the reserve price. To account for this, they recommend the

the optimal rate of Section 3.3.

Instead of assuming a constant number of potential bidders across auctions, An, Hu and Shum (2010) develop a methodology when the unobserved number of potential bidders may vary across auctions but the reserve price remains constant. The argument draws from Hu (2008) on misclassification errors. One needs a proxy and an instrument with the same support $\mathcal{S}_I = \{2, \ldots, \overline{I}\}$ as the number I of potential bidders. The proxy and instrument could be discretized continuous variables but conditionally on I must be mutually independent and not affect the truncated bid distribution $G^*(\cdot|I)$. Because the latter does not depend on I^*, a natural proxy for I is the observed number I^* of actual bidders upon restricting auctions to have at least two bidders. As an instrument Z, the authors propose either a discretized second bid or a discretized variable that affects the number I of potential bidders but not private values conditional on I. Thus, for any $b \geq p_0$ and omitting auction covariates X, we have

$$\begin{aligned} g^*(b, I^*, Z) &= \sum_{I \in \mathcal{S}_I} g^*(b|I^*, I, Z) \Pr(I^*|I, Z) \Pr(I, Z) \equiv \sum_{I \in \mathcal{S}_I} g^*(b|I) \cdot q_{I^*|I} \cdot q_{IZ}, \\ q_{I^*Z} &= \sum_{I \in \mathcal{S}_I} \Pr(I^*|I, Z) \Pr(I, Z) \equiv \sum_{I \in \mathcal{S}_I} q_{I^*|I} \cdot q_{IZ}, \end{aligned}$$

where $g^*(b, I^*, Z)$ and q_{I^*Z} are the joint densities of (B, I^*, Z) and (I^*, Z), respectively. Varying the values of $(I^*, Z) \in \mathcal{S}_I^2$, both equations can be written in matrix form as $G^*_{bI^*Z} = Q_{I^*|I} G^*_{b|I} Q_{IZ}$ and $Q_{I^*Z} = Q_{I^*|I} Q_{IZ}$ where $G^*_{b|I}$ is a diagonal matrix whose diagonals are $[g^*(b|2), \ldots, g^*(b|\overline{I})]$. Provided Q_{I^*Z} is invertible, this gives the key equation

$$G^*_{bI^*Z} Q^{-1}_{I^*Z} = Q_{I^*|I} G^*_{b|I} Q^{-1}_{I^*|I},$$

which is an eigenvalue-eigenvector decompostion of the observed matrix $G^*_{bI^*Z} Q^{-1}_{I^*Z}$. Provided the eigenvalues $g^*(b|I)$ for $I \in \mathcal{S}_I$ are distinct, this decomposition is unique because $\sum_{I^* \in \mathcal{S}_I} q_{I^*|I} = 1$ and $q_{I^*|I} = 0$ for $I^* > I$ so that $Q_{I^*|I}$ is upper-triangular. Hence, $Q_{I^*|I}$ and $G^*_{b|I}$ are identified from observations on bids, number of actual bidders and instrument.[12] Moreover, when I is exogenous, the probability $F(p_0)$ is identified from $Q_{I^*|I}$ because I^* is distributed as a truncated Binomial $\mathcal{B}[I, (1 - F(p_0)]$ given $I^* \geq 2$. Hence, the truncated value distribution $F^*(\cdot) \equiv [F(\cdot) - F(p_0)]/[1 - F(p_0)]$ is identified on $[p_0, \overline{v}]$ from (4.2).

Regarding estimation, the main difference is that the authors consider the expectation $\int b g^*(b, I^*, Z) db$ which leads to the matrix equation $G^*_{\mathrm{E}BI^*Z} Q^{-1}_{I^*Z} = Q_{I^*|I} G^*_{\mathrm{E}B|I} Q^{-1}_{I^*|I}$ by

transformation $\sqrt{B_{i\ell} - p_{0\ell}}$ of the bids with appropriate transformation of the inverse equilibrium strategy before applying the GPV procedure.

[12]This implicitly requires that $\overline{I}$ is known. The latter is identified by the maximum number of actual bidders across auctions.

the same algebra, where $G^*_{\mathrm{E}BI^*Z}$ and $G^*_{\mathrm{E}B|I}$ are the square and diagonal matrices whose elements are $\mathrm{E}[B|I^*,Z]\cdot q_{I^*Z}$ and $\mathrm{E}[B|I]$, respectively. In the first step, they estimate the eigenvector matrix $Q_{I^*|I}=[q_{I^*|I}]$ using the empirical matrices

$$\hat{G}_{\mathrm{E}BI^*Z}=\left[\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I^*}\sum_{i=1}^{I^*}B_{i\ell}\,\mathbb{1}(I^*_\ell=I^*)\,\mathbb{1}(Z_\ell=Z)\right] \text{ and } \hat{Q}_{I^*Z}=\left[\frac{1}{L}\sum_{\ell=1}^{L}\mathbb{1}(I^*_\ell=I^*)\,\mathbb{1}(Z_\ell=Z)\right],$$

where $(I^*,Z)\in\mathcal{S}_I^2$. The probability $F(p_0)$ is also estimated from $\hat{Q}_{I^*|I}$. In the second step, they estimate the truncated conditional distribution $G^*(\cdot|I)$ by inverting the linear system $\{\hat{G}^*(b|I^*)=\sum_{I\in\mathcal{S}_I}\hat{q}_{I|I^*}\cdot\hat{G}^*(b|I);I^*\in\mathcal{S}_I\}$ for each $b\geq p_0$, where $\hat{G}^*(b|I^*)=\frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{I^*}\sum_{i=1}^{I^*}\mathbb{1}(B_{i\ell}\leq b)\,\mathbb{1}(I^*_\ell=I^*)/\hat{q}_{I^*}$ with $\hat{q}_{I^*}$ the empirical frequency of auctions with I^* actual bidders. By Bayes rule, $\hat{q}_{I|I^*}=\hat{q}_{I^*|I}\hat{q}_I/\hat{q}_{I^*}$ where $\{\hat{q}_I;I^*\in\mathcal{S}_I\}$ solves the linear system $\{\hat{q}_{I^*}=\sum_{I\in\mathcal{S}_I}\hat{q}_{I^*|I}\cdot\hat{q}_I;I^*\in\mathcal{S}_I\}$ whereas $[\hat{q}_{I^*|I}]$ is the matrix estimated in the first step. Similarly, the density $g^*(\cdot|I)$ is estimated by inverting the linear system $\{\hat{g}^*(b|I^*)=\sum_{I\in\mathcal{S}_I}\hat{q}_{I|I^*}\cdot\hat{g}^*(b|I);I^*\in\mathcal{S}_I\}$ where $\hat{g}^*(\cdot|I^*)$ is a kernel estimator of the conditional density of B given I^*. The authors establish the $\sqrt{L}$-asymptotic normality of $\hat{Q}_{I^*|I}$ and $\hat{G}^*(b|I)$ as well as the uniform consistency of $\hat{g}^*(\cdot|I)$ on inner compact subsets of its support $[p_0,\overline{b}(i)]$. In the third step, because I_ℓ remains unknown, S random bids B_s are drawn from $\hat{G}(\cdot|I)$ for a chosen I to recover estimated private values $\hat{V}_s$ from (4.2), where $F(p_0)$, $G(\cdot|I)$ and $g(\cdot|I)$ are estimated from the previous steps. Using the pseudo sample $\hat{V}_s$, $s=1,\ldots,S$, the truncated value density $f^*(\cdot)$ is estimated by a kernel estimator.

Secret Reserve Price

Though not as frequent as when it is announced, a secret reserve price is prevalent in online auctions, auctions of collectibles, some timber auctions, construction procurements to name a few. The probability of winning now contains an additional feature as a bidder's bid needs to be larger than his opponents' bids but also larger than the unknown secret reserve price. Let V_0 denote the seller's value of the good. Following the IPV paradigm, V_0 is private information to the seller and is drawn from some distribution $H(\cdot|I)$ independently of bidders' values V_i, where the dependence of $H(\cdot|I)$ on I captures some unobserved heterogeneity as for $F(\cdot|I)$. The seller's value distribution $H(\cdot|I)$ is common knowledge. It need not be equal to the bidders' distribution $F(\cdot|I)$ though it has the same support $[\underline{v}(I),\overline{v}(I)]$ to simplify with a positive density $h(\cdot|I)$. Under a first-price auction mechanism, Elyakime, Laffont, Loisel and Vuong (1994) show that the seller's dominant strategy is to set the secret reserve price p_0 at his value V_0. Thus $H(\cdot|I)$ is the distribution of the secret reserve price. Moreover, because the seller's and bidders' values are independent implying that the reserve price and bids are independent, bidder i's winning probability becomes $G^{I-1}(B_i|I)H(B_i|I)$. Hence, maximizing the expected profit $(V_i - B_i)G^{I-1}(B_i|I)H(B_i|I)$ with respect to B_i gives

$$V_i = B_i + \frac{1}{(I-1)\frac{g(B_i|I)}{G(B_i|I)} + \frac{h(B_i|I)}{H(B_i|I)}} \equiv \xi(B_i; F, H, I), \tag{4.3}$$

in the spirit of (3.6). It is worthnoting that this equation would arise if the seller was setting the reserve price randomly as $H(\cdot|I)$ thereby justifying the term of random reserve price in the literature. More importantly and in constrast to an announced reserve price, there is no closed form solution for the equilibrium strategy $B_i = s(V_i; F, H, I)$ when the reserve price is kept secret. Such a strategy solves a differential equation that is also hard to solve numerically as it has a singularity at the boundary condition $s(\underline{v}(I); F, H, I) = \underline{v}(I)$. This constitutes a first example when the indirect approach is the practical option. More examples follow in this chapter.

Turning to identification, suppose that all bids are observed. If the reserve price is observed, one recovers the seller's value through $V_0 = p_0$ and thus identifies its distribution $H(\cdot|I)$. Thus, as in Guerre, Perrigne and Vuong (2000), (4.3) identifies $F(\cdot|I)$ on its full support. Indeed, a random reserve price does not prevent any bidder from participating and therefore I is observed. The inverse equilibrium strategy $\xi(\cdot; G, H, I)$ should be increasing with a differentiable inverse whereas p_0 should be independent of the bids which are i.i.d.. These constitute the model restrictions. Regarding estimation, the ratio $h(B_i|X, I)/H(B_i|X, I)$ can be estimated by kernel methods using the observed secret prices as in (3.8). The GPV procedure applies. Specifically, assuming that $h(\cdot|I)$ is $R+1$ continuously differentiable maintains the $R+1$ differentiability of the bid density $g(\cdot|I)$. Consequently, the vanishing rates of the bandwidths for estimating $H(\cdot|X, I)$ and $h(\cdot|X, I)$ are similar to those for estimating $G(\cdot|X, I)$ and $g(\cdot|X, I)$. The resulting estimator achieves the same optimal uniform convergence rate for estimating $f(\cdot|X, I)$ as in Section 3.3. See Li and Perrigne (2003). If reserve price information is not available as in online auctions, one typically assumes that the seller's value distribution is the same as bidders' value distribution. Thus $F(\cdot|I)$ is identified from (4.3) with $H(\cdot|I) = G(\cdot|I)$. This is similar to having an additional bidder in the competitive bidding process since (4.3) reduces to (3.6) with I instead of $I-1$ bidders. Two-step estimation of $F(\cdot|X, I)$ proceeds using the I_ℓ bids with the same bandwidth rates as above. Extension when only the winning bids B_ℓ^w are observed is straightforward upon using $g^w(\cdot|X, I)/G^w(\cdot|X, I) = Ig(\cdot|X, I)/G(\cdot|X, I)$ in (4.3).

Secret Reserve Price and Bargaining

When an auctioned object is unsold due to the presence of an announced or secret reserve price, some gains from trade often remains. It then happens that the seller and the highest bidder enter a post-auction bargaining which may lead to a transaction. Elyakime, Laffont, Loisel and Vuong (1997) study such a situation where the mechanism is a first-price sealed-bid auction with a secret reserve price p_0 followed by a bargaining stage under complete information. They assume that the latter leads to a transaction price that splits equally the surplus between the seller's value V_0 and the highest bidder's value $V^{\max}$ whenever $V^{\max} > V_0$. They characterize the equilibrium strategies $s_0(\cdot; F, H, I)$ and $s(\cdot; F, H, I)$ for the seller and the bidders where V_0 and the bidders' values V_i are mutually independent and distributed as $H(\cdot|I)$ and $F(\cdot|I)$, respectively. In particular, they show that the inverse equilibrium strategy for the seller is

$$\xi_0(p_0; G, K, I) = 2p_0 - \xi(p_0; G, K, I),$$

where $\xi_0(\cdot; G, K, I)$ is the seller's inverse equilibrium strategy while $G(\cdot|I)$ and $K(\cdot|I)$ are the bid and reserve price distributions, respectively. In particular, because $\xi(p_0; G, K, I) \geq p_0$, the seller sets a secret reserve price that is larger than his valuation V_0. Moreover, the bidders' equilibrium strategy written in terms of $G(\cdot|I)$ and $K(\cdot|I)$ satisfies

$$\int_{\xi_0(B_i;G,K,I)}^{V_i} K[\xi_0^{-1}(p_0; G, K, I)|I]\ dp_0 = \frac{2G(B_i|I)K(B_i|I)}{(I-1)g(B_i|I)},$$

which implicitly defines $V_i = \xi(B_i; G, K, I)$ given $\xi_0(\cdot; G, K, I)$. The seller's and bidders' equlibrium strategies are more involved than before because both sides of the market integrate the possibility of an ex post bargaining round. The authors establish the identification of the model primitives $[F(\cdot|I), H(\cdot|I)]$ from observations on reserve prices and bids. The authors do not attempt to develop an indirect multistep nonparametric procedure. Instead, they use a direct approach upon parameterizing the latent distributions $[F(\cdot|I), H(\cdot|I)]$ as Weibull with means linear in the covariates X and constant variances. They estimate the parameters by NLLS using the means of bids, reserve prices and transaction prices conditional on X. The method is computationally demanding as it requires solving numerically a nonlinear differential equation for $s(\cdot; F, H, I)$ for each auction and each trial value of the parameters in an iterative minimization routine.

Optimal Reserve Price

The reserve price constitutes a useful tool for the seller to maximize his profit. Indeed, p_0 can optimally trade off the extraction of bidders' information rents and a foregone transaction when no bidder meets the reserve price. Within the symmetric IPV paradigm, the optimal announced reserve price p_0^* in a standard auction, i.e., an auction that allocates the good to the highest bidder, satisfies

$$p_0^* = V_0 + \frac{1 - F(p_0^*|X, I)}{f(p_0^*|X, I)}, \tag{4.4}$$

where V_0 is the seller's value for the auctioned good. See Laffont and Maskin (1980) and Riley and Samuelson (1981). As a matter of fact, within this paradigm Myerson (1981) shows that a standard auction with announced reserve price p_0^* is optimal among all possible selling mechanisms. Thus a first-price sealed-bid or ascending auction with a reserve price p_0^* dominates a first-price sealed-bid auction with secret reserve price. In many auctions, the reserve price may not be optimal. Thus a common counterfactual is to estimate the optimal reserve price and simulate the outcomes, i.e., the corresponding bids and revenue under an optimal standard auction. Given a seller' value V_0, a natural method is to plug-in the estimated distribution and density $\hat{F}(\cdot|X, I)$ and $\hat{f}(\cdot|X, I)$ in (4.4) to obtain an estimator of the optimal reserve price p_0^* for each auction provided p_0^* belongs to the support on which $F(\cdot|X, I)$ is identified. The latter condition is automatically satisfied when the reserve price is nonbinding or secret. See e.g. Li and Perrigne (2003).

An alternative method is to estimate the optimal reserve price by maximizing an estimator of the seller's expected profit. Li, Perrigne and Vuong (2003) develop such an approach in the broader context of affiliated private values. The expected profit for the seller with value V_0 when there is an announced reserve price p_0 is

$$\mathrm{E}\{V_0 1\!\!1[V^{\max} \le p_0)] + Is(V_1; F, I, p_0) 1\!\!1[V_1 \ge \max(V_{-1}^{\max}, p_0)]\},$$

where the expectation is taken with respect to $(V_1, V_{-1}^{\max})$, $s(\cdot; F, I, p_0)$ is the equilibrium strategy (4.1), $V^{\max} = \max(V_1, V_{-1}^{\max})$, $V_{-1}^{\max} = \max_{j \neq 1} V_j$ and V_1 is the private value of (say) bidder 1 without loss of generality as bidders are symmetric. The first term in the expectation accounts for the possibility that the object remains unsold whereas the second term is the seller's expected revenue when bidders face a reserve price p_0. The idea is to rewrite this expected profit using the distribution of observed bids. For the sake of simplicity, we omit the object characteristics X and assume that bids are coming from first-price sealed-bid auctions with no reserve price. Thus the bidders' value distribution $F(\cdot|I)$ is identified on its full support from these observed bids as shown in Section 3.2. Applying the authors' result to the IPV case shows that the seller's expected profit can be written as

$$\mathrm{E}\left[V_0 1\!\!1[B^w \le b_0] + I\left(B_1 + \frac{1}{I-1}\frac{G(b_0|I)}{g(b_0|I)}\left(\frac{G(b_0|I)}{G(B_1|I)}\right)^{I-1}\right) 1\!\!1[B_1 \ge \max(B_{-1}^{\max}, b_0)] | I\right], \quad (4.5)$$

where $b_0 \equiv s(p_0; F, I)$, $G(\cdot|I)$ is the bid distribution given I and $B^w = \max(B_1, B_{-1}^{\max})$ is the winning bid. Because $s(\cdot; F, I)$ is increasing, the optimal reserve price p_0^* satisfies $p_0^* = \xi(b_0^*; G, I)$ where b_0^* maximizes (4.5) with $\xi(\cdot; G, I)$ given by (3.6).

To estimate the optimal reserve price, one first estimates b_0^*. In particular, (4.5) is the expectation of a function $\Psi(\cdot)$ of the observables $(B_1, B_{-1}^{\max}, I)$, the bid distribution and density $G(\cdot|I)$ and $g(\cdot|I)$ as well as b_0. Thus, given a value V_0 and following Andrews (1994), one can use a semiparametric extremum estimator of b_0^* as

$$\hat{b}_0^* = \text{argmax}_{b_0 \in [\hat{\underline{b}}(I), \hat{\overline{b}}(I)]} \frac{1}{IL_I} \sum_{\ell: I_\ell = I}^{L_I} \sum_{i=1}^{I} \Psi(B_{i\ell}, B_{-i\ell}^{\max}, I; \hat{G}, \hat{g}, b_0),$$

where $\hat{G}(\cdot|I)$, $\hat{g}(\cdot|I)$, $\hat{\underline{b}}(I)$ and $\hat{\overline{b}}(I)$ are the kernel and boundary estimators of Section 3.3. The estimator of the optimal reserve price $\hat{p}_0^* = \xi(\hat{b}_0^*; \hat{G}, I)$ is consistent. This method does not require estimating the value density $f(\cdot|I)$ or computing the equilibrium strategy $s(\cdot; F, I)$.

Section 4.2: Asymmetry

Bidders can be asymmetric in the sense that they draw their values from different distributions. For instance, in construction procurements, firms close to the construction site may benefit from reduced transportation costs, large firms may benefit from scale economies and firms' backlogs may increase their opportunity costs. See e.g., Bajari (1997), Jofre-Bonet and Pesendorfer (2003) and Flambard and Perrigne (2006). To simplify the presentation, we assume that there are only two types of bidders denoted 0 and 1. The analysis can be easily extended to as many groups as bidders, i.e., to full asymmetry. The I_0 bidders of type 0 draw their values from $F_0(\cdot|I_0, I_1)$, whereas the I_1 bidders of type 1 draw their values from $F_1(\cdot|I_0, I_1)$ with $I_0 + I_1 \geq 2$. To fix ideas, $F_1(\cdot|I_0, I_1)$ stochastically dominates $F_0(\cdot|I_0, I_1)$ in which case the type 0 and type 1 bidders are called the weak and strong bidders, respectively. As before, we assume independence among the $I_0 + I_1$ values. Bidders know the types of their opponents and the game structure $[F_1(\cdot|I_0, I_1), F_0(\cdot|I_0, I_1), I_0, I_1]$ is common knowledge. To simplify, we assume that both distributions have the same support $[\underline{v}(I_0, I_1), \overline{v}(I_0, I_1)]$. The theoretical analysis of asymmetric auctions has been delayed by the complexity of the Bayesian Nash equilibrium strategies. Indeed, the strategies $s_0(\cdot; F_0, F_1, I_0, I_1)$ and $s_1(\cdot; F_0, F_1, I_0, I_1)$ for bidders of type 0 and 1, respectively, solve a system of two differential equations with no closed form solutions. Maskin and Riley (2000a,b) derive important properties of the equilibrium strategies. For instance, despite the strategies being monotonic, the allocation might be inefficient since the winner might not be the bidder with the largest value. Intuitively, a weak bidder needs to bid more aggressively than his strong opponents to have a chance of winning the object. As a result, he might win the auction though his value is lower than the largest value of the strong bidders. The equilibrium is subject to different boundary conditions which are $s_0(\underline{v}(I_0, I_1); F_0, F_1, I_0, I_1) = s_1(\underline{v}(I_0, I_1); F_0, F_1, I_0, I_1) = \underline{v}(I_0, I_1)$ and

$s_0(\overline{v}(I_0, I_1); F_0, F_1, I_0, I_1) = s_1(\overline{v}(I_0, I_1); F_0, F_1, I_0, I_1)$.[13]

Known Bidders' Identities

We assume that the analyst observes the bidders' identities and can classify the bidders within the two groups according for instance to some exogenous bidders' characteristics. As before, we consider the bidders' optimization problem in terms of the bid distributions $G_0(\cdot|I_0, I_1)$ and $G_1(\cdot|I_0, I_1)$ for bidders of type 0 and 1, respectively. The probability that a bidder i of type 0 wins with a bid B_i is $G_0^{I_0-1}(B_i|I_0, I_1)\ G_1^{I_1}(B_i|I_0, I_1)$ because of the independence of private values and hence of bids. Indeed, a bidder of type 0 faces $I_0 - 1$ opponents of his own type and I_1 opponents of his competing type. Similarly, the winning probability for a bidder of type 1 is $G_0^{I_0}(B_i|I_0, I_1)\ G_1^{I_1-1}(B_i|I_0, I_1)$. Maximizing the expected profit $(V_i - B_i)\Pr(B_i > B_{-i}^{\max}|I_0, I_1)$ for a member of each group gives the pair of equations

$$\begin{aligned} V_0 &= B_0 + \frac{1}{(I_0-1)\frac{g_0(B_0|I_0,I_1)}{G_0(B_0|I_0,I_1)} + I_1\frac{g_1(B_0|I_0,I_1)}{G_1(B_0|I_0,I_1)}} \equiv \xi_0(B_0; G_0, G_1, I_0, I_1), \\ V_1 &= B_1 + \frac{1}{I_0\frac{g_0(B_1|I_0,I_1)}{G_0(B_1|I_0,I_1)} + (I_1-1)\frac{g_1(B_1|I_0,I_1)}{G_1(B_1|I_0,I_1)}} \equiv \xi_1(B_1; G_0, G_1, I_0, I_1), \end{aligned} \tag{4.6}$$

where (V_j, B_j) are the value and bid for a generic bidder in group $j = 0, 1$ and $g_0(\cdot|I_0, I_1)$ and $g_1(\cdot|I_0, I_1)$ are the bid densities for groups 0 and 1, respectively.

Identification is a direct extension of Guerre, Perrigne and Vuong (2000). Since the bid distributions and densities are identified from the bid data, the inverse equilibrium strategies (4.6) identify the private value distributions $F_0(\cdot|I_0, I_1)$ and $F_1(\cdot|I_0, I_1)$ on $[\underline{v}(I_0, I_1), \overline{v}(I_0, I_1)]$. Regarding the model restrictions, they are similar as those of Section 3.2. Condition C1 requires that the bids within each type are i.i.d. and independent across types. Condition C2 requires that the inverse equilibrium strategies $\xi_j(\cdot; G_0, G_1, I_0, I_1)$ for $j = 0, 1$ are increasing with differentiable inverses. Estimation is a direct extension of the GPV procedure. This requires estimating the two bid distributions and their densities to estimate the value densities $f_0(\cdot|I_0, I_1)$ and $f_1(\cdot|I_0, I_1)$. The latter two densities are consistently estimated at the optimal uniform convergence rate upon choosing the optimal rates of the bandwidths as given in Section 3.3. See Campo, Perrigne and Vuong (2003) and Flambard and Perrigne (2006) in the case in the case of affiliated and independent private values, respectively. As for secret reserve prices, the absence of closed form solutions for

[13]Considering different supports for $F_0(\cdot|I_0, I_1)$ and $F_1(\cdot|I_0, I_1)$ imply more involved boundary conditions with some potential exclusion of bidders as in the case of an announced reserve price. See Section 5.1 for an illustration of such a phenomenon in the case of risk aversion and value uncertainty.

the equilibrium strategies would require the computation of the latter through numerical procedures as in Marshall, Meurer, Richard and Stromquist (1994) and Bajari (1997). Despite the complexity of the equilibrium, the indirect approach offers a computationally simple and elegant estimation method.

Bidding Preference

Public institutions often grant a bid preference under the form of a discount or credit to some firms. These firms are usually minority-owned, of small size or domestic in the case of products containing confidential information as in defense contracts. In a procurement auction setting, these firms are usually in a disadvanteageous position because of larger costs. The idea of a bidding preference is to render them more competitive in the bidding process. Marion (2007) studies procurement auctions of road construction projects in which small firms benefit from a bid preference. We use the same notations as above with types 0 and 1 referring to the favored and unfavored firms. The setting is similar as for an asymmetric auction with $I_0 + I_1 \geq 2$ and cost distributions $F_0(\cdot|I_0, I_1)$ and $F_1(\cdot|I_0, I_1)$. The only difference with an asymmetric auction is the allocation rule. The bids of all unfavored firms, i.e., of type 1, are inflated by $\delta > 1$. The winner is the firm with the lowest bid among all the bids possibly adjusted. The winner irrespective of type is paid his bid. In addition to the cost asymmetry among firms, the bid preference introduces an additional layer of asymmetry through the allocation rule. The probability that a favored bidder of type 0 wins with a bid B_i is $[1 - G_0(B_i|I_0, I_1)]^{I_0-1}[1 - G_1(B_i/\delta|I_0, I_1)]^{I_1}$, whereas this probability for a unfavored bidder of type 1 is $[1 - G_0(\delta B_i|I_0, I_1)]^{I_0}[1 - G_1(B_i|I_0, I_1)]^{I_1-1}$, where $G_j(\cdot|i_0, I_1)$, $j = 0, 1$ is the distribution of the unadjusted bids in group j. Maximizing the expected profit for a member of each group gives the system

$$\begin{aligned} C_0 &= B_0 - \frac{1}{(I_0 - 1)\,\frac{g_0(B_0|I_0,I_1)}{1-G_0(B_0|I_0,I_1)} + \frac{I_1}{\delta}\,\frac{g_1(B_0/\delta|I_0,I_1)}{1-G_1(B_0/\delta|I_0,I_1)}} \equiv \xi_0(B_0; G_0, G_1, I_0, I_1, \delta), \\ C_1 &= B_1 - \frac{1}{I_0\,\frac{\delta g_0(\delta B_1|I_0,I_1)}{1-G_0(\delta B_1|I_0,I_1)} + (I_1 - 1)\,\frac{g_1(B_1|I_0,I_1)}{1-G_1(B_1|I_0,I_1)}} \equiv \xi_1(B_1; G_0, G_1, I_0, I_1, \delta), \end{aligned}$$

where (C_j, B_j) are the cost and bid for a generic bidder in group $j = 0, 1$. Since δ is known to the analyst, identification and estimation proceed as above.

Anonymous Bidding

It may happen that only the identity of the winner is recorded or that all bidders' identities remain unknown for confidentiality issues. Such anonymity is not a problem when bidders are symmetric as none of the estimation methods of Section 3 requires assigning a bidder's identity to a given bid. It invalidates, however, the above estimation method which relies on knowing to which group a bidder belongs. Lamy (2012a) addresses identification and estimation when bidders are anonymous, i.e., when the analyst observes the bid vector in each auction but not the bidders' identities or know them only partially. Let $\mathcal{I}$ be the set of bidders with cardinality I. The question is to recover the value distributions $F_j(\cdot|\mathcal{I})$ for $j \in \mathcal{J}$ from the observed bids where the set $\mathcal{J}$ is unknown and indexes the distributions $F_i(\cdot|\mathcal{I})$, $i \in \mathcal{I}$ that are distinct. For instance, the index set $\mathcal{J}$ equals $\{1, 2\}$ in the case of weak and strong bidders but can also be $\{1, \ldots, I\}$ in the full asymmetric case or reduced to a singleton in the symmetric case. In other words, the setting may be asymmetric but the structure of asymmetry is unknown.

When bidders' identities are unknown, one only observes the I order statistics of the bids. Lamy (2012a) first shows that the distinct bid distributions $G_j(\cdot|\mathcal{I})$, $j \in \mathcal{J}$ are identified as well as the cardinality J of $\mathcal{J}$, which is the number of groups, and the number of bidders in each group. This is the main step of the identification argument which relies on the independence of bids to ensure that the I bid distributions $G_i(\cdot|\mathcal{I})$ are identified up to some permutation from the I bid order statistics. Specifically, for every $b \in [\underline{b}(\mathcal{I}), \overline{b}(\mathcal{I})]$, $\{G_i(b|\mathcal{I}); i = 1, \ldots, I\}$ are the roots of a polynomial of degree I whose coefficients are known linear combinations of the distribution of the bid order statistics evaluated at b.[14] The second step is as above and identifies the distinct value distributions $F_j(\cdot|\mathcal{I})$, $j \in \mathcal{J}$ through a system of J equations of the form (4.6) where $B_j \sim G_j(\cdot|\mathcal{I})$.

Estimation follows the identification argument. In a first step, one estimates the distribution and density of the bid order statistics conditional on $(X, \mathcal{I})$ using kernel estimators. In the next two steps one obtains estimates of the bid distributions and densities $G_i(\cdot|X, \mathcal{I})$ and $g_i(\cdot|X, \mathcal{I})$ for $i = 1, \ldots, I$ from the estimated identifying polynomial and especially its roots and the derivatives of its coefficients. Because bids are anonymous, the second step of the GPV procedure is also more involved. In particular, one estimates the probability that the pth-highest bid in the ℓth auction belongs to the ith bidder using the I estimated bid densities $\hat{g}_i(\cdot|X_\ell, \mathcal{I})$. Using these estimated probabilities as weights $\hat{w}_{ip\ell}$, one estimates bidder i's private value in the ℓth auction as $\sum_{p=1}^{I} \hat{w}_{ip\ell} \hat{\xi}_i(B_\ell^{(p:I)}; X_\ell, \mathcal{I})$ where $\hat{\xi}_i(b; X, \mathcal{I}) = b + \left(\sum_{j \neq i}^{I} [\hat{g}_j(b|X, \mathcal{I}) / \hat{G}_j(b|X, \mathcal{I})]\right)^{-1}$ which corresponds to (4.6) written in the full asymmetric case. Estimation of bidder i's value density follows as in (3.10). Under some assumptions, the author shows that the resulting estimators of the J distinct private value densities are uniformly consistent using the optimal vanishing rates of

[14] Anonymous bidding could be viewed as unobserved heterogeneity. When the number of bidders is unknown, An, Hu and Shum's (2010) identification argument consists in finding the eigenvalues of a matrix whereas Lamy (2012a) solves the roots of a polynomial when bidders are anonymous. We thank Laurent Lamy for this analogy.

Section 3.3. The assumptions include knowledge of the asymmetry structure and strict stochastic dominance of the distinct value distributions.[15] Moreover, the author shows that his estimator attains the optimal uniform convergence of Section 3.3. This rate is also the optimal rate when bidders' identities are known.

Section 4.3: Affiliation

Independence of private values can be a restrictive assumption as it implies that bids are also independent (conditional on auction covariates). One can relax independence of private values by allowing affiliation within the private value paradigm.

Identification and Estimation

Following the general model of Section 2.1 we have $U(\sigma_i, C) = \sigma_i \equiv V_i$ where the values $(V_1, \ldots, V_I)$ are drawn from an affiliated joint distribution $F(\cdot, \ldots, \cdot|I)$ on $[\underline{v}(I), \overline{v}(I)]^I$. Moreover, we assume that bidders are symmetric so that $F(\cdot, \ldots, \cdot|I)$ is exchangeable. This defines the symmetric Affiliated Private Value (APV) model. See Campo, Perrigne and Vuong (2003) for the identification and estimation of asymmetric APV models. From Milgrom and Weber (1982), the symmetric Bayesian Nash equilibrium strategy $s(V_i; F, I)$ in a first-price sealed-bid auction has an explicit form given by (2.5) upon letting $v(a, a) = a$, $\sigma_1 = V_1$ and $V_{-1}^{\max} = \max_{j \neq 1} V_j$. Laffont and Vuong (1996) and Li, Perrigne and Vuong (2002, 2003) study the identification and estimation of $F(\cdot, \ldots, \cdot|I)$ and the optimal APV reserve price from observed bids. In particular, the symmetric APV model is important not only because it allows for correlation among observed bids, but also because any structure $[U, F]$ of the general model of Section 2.1 is observationally equivalent to an APV model in first-price sealed-bid auctions. See Section 8.2 for details.

Because the equilibrium strategy is increasing, bids $(B_1, \ldots, B_I)$ are affiliated with some joint distribution $G(\cdot, \ldots, \cdot|I)$. Let $G_{B_{-i}^{\max}|V_i}(\cdot|V_i, I)$ be the distribution of $B_{-i}^{\max}$ given (V_i, I) with density $g_{B_{-i}^{\max}|V_i}(\cdot|V_i, I)$. Since the probability of winning is $G_{B_{-i}^{\max}|V_i}(B_i|V_i, I)$, maximizing bidder i's expected profit $(V_i - B_i)G_{B_{-i}^{\max}|V_i}(B_i|V_i, I)$ with respect to B_i gives

$$V_i = B_i + \frac{G_{B_{-i}^{\max}|B_i}(B_i|B_i, I)}{g_{B_{-i}^{\max}|B_i}(B_i|B_i, I)} \equiv \xi(B_i; G, I), \tag{4.7}$$

upon noting that $G_{B_{-i}^{\max}|V_i}(\cdot|V_i, I) = G_{B_{-i}^{\max}|B_i}(\cdot|B_i, I)$ in equilibrium since $B_i = s(V_i; F, I)$. Similarly to the benchmark model, (4.7) shows that the symmetric APV model is iden-

[15] Lamy (2008) indicates that the asymmetry structure can be estimated with probability one asymptotically. As noted by the author, when there are covariates X, identification of the private distributions under anonymous bidding is not as straightforward as with known identities. This is so because identification up to a permutation for every given value of X does not imply identification up to a permutation of the value distributions $F_i(\cdot|\cdot, \mathcal{I})$, $i = 1, \ldots, I$. The stochastic dominance assumption avoids this difficulty.

tified. Moreover, following Section 3.2, a first restriction is that $G(\cdot, \ldots, \cdot|I)$ is affiliated and exchangeable, which results from the affiliation and exchangeability of $F(\cdot, \ldots, \cdot|I)$. A second restriction is that the inverse equilibrium strategy $\xi(\cdot; G, I)$ is increasing in its argument with a differentiable inverse.

Regarding estimation and omitting covariates to simplify, Li, Perrigne and Vuong (2002) propose a two-step kernel-based estimator similar to the GPV procedure. In the first step, $G_{B_{-i}^{\max}|B_i}(\cdot|\cdot, I)$ and $g_{B_{-i}^{\max}|B_i}(\cdot|\cdot, I)$ are estimated by kernel estimators. In the second step, estimated values are obtained from (4.7) upon replacing $G_{B_{-i}^{\max}|B_i}(\cdot|\cdot, I)$ and $g_{B_{-i}^{\max}|B_i}(\cdot|\cdot, I)$ by their estimators in order to estimate the joint value density $f(\cdot, \ldots, \cdot|I)$. Exchangeability of the estimator is imposed by averaging over the $I!$ permutations of the estimated values for each auction. Assuming that $F(\cdot, \ldots, \cdot|I)$ is $R + I$ continuously differentiable, the authors show that this estimator converges uniformly to $f(\cdot, \ldots, \cdot|I)$ at the rate $(L/\log L)^{R(R+I-2)/[(R+I+1)(2R+2I-2)]}$. This rate is achieved when the bandwidths h_G, h_g and h_f are proportional to $(\log L/L)^{1/(2R+2I-3)}$, $(\log L/L)^{1/(2R+2I-2)}$ and $(\log L/L)^{(R+I-2)/(R+I+1)(2R+2I-2)}$, respectively. The convergence rate of this two-step estimator decreases exponentially with I. As expected, it is slower than the optimal rate for estimating $f(\cdot, \ldots, \cdot|I)$ if private values were observed, namely $(L/\log L)^{R/(2R+I)}$ from Stone (1982). The authors also estimate the marginal value density and conduct some Monte Carlo experiments. The latter show that estimating an IPV model when the true model is an APV model overestimates bidders' values and hence bidders' rents.

Semiparametric Estimation

To eliminate the curse of dimensionality associated with the number of bidders, Hubbard, Li and Paarsch (2012) propose a semiparametric method based on parameterizing the copula of $F(\cdot, \ldots, \cdot|I)$. The latter is a parametric distribution $\mathcal{C}(\cdot, \ldots, \cdot|I; \theta)$ on $[0, 1]^I$ such that $F(V_1, \ldots, V_I|I) = \mathcal{C}[F(V_1|I), \ldots, F(V_I|I)|I; \theta]$, where θ is a parameter vector and $F(\cdot|I)$ is the common marginal distribution of V_i which is unrestricted. One can use single-parameter members of the Archimedean family such as the Clayton, Frank or Gumbel copulas to impose affiliation. See Nelsen (1999). Exploiting the invariance of the copula under increasing transformations, which gives $\mathcal{C}[G(B_1|I), \ldots, G(B_I|I)|I; \theta] = \mathcal{C}[F(V_1|I), \ldots, F(V_I|I)|I; \theta]$, they show that (4.7) becomes

$$V_i = B_i + \frac{\mathcal{C}_1[G(B_1|I), \ldots, G(B_I|I)|I; \theta]}{(I-1)g(B_i|I)\mathcal{C}_{12}[G(B_1|I), \ldots, G(B_I|I)|I; \theta]}, \tag{4.8}$$

where the subscripts 1 and 2 indicate the derivatives with respect to the first and second argument of $\mathcal{C}(\cdot, \ldots, \cdot|I; \theta)$, and $G(\cdot|I)$ is the marginal distribution of bids given I with density $g(\cdot|I)$.

Hubbard, Li and Paarsch (2012) consider a two-step procedure in which the first step is semiparametric. Specifically, omitting the covariates X they propose a Pseudo Maximum Likelihood (PML) estimator to estimate θ as

$$\hat{\theta}_I = \text{argmax}_\theta \frac{1}{L_I} \sum_{\ell: I_\ell = I}^{L_I} \log c[\hat{G}(B_{1\ell}|I), \ldots, \hat{G}(B_{I\ell}|I)] I; \theta],$$

for each value of I where $c(\cdot, \ldots, \cdot|I;\theta)$ is the density of the copula $\mathcal{C}(\cdot, \ldots, \cdot|I;\theta)$ and $\hat{G}(\cdot|I)$ is the empirical cdf of bids given I. An estimator of θ is obtained by averaging over different values of I. Following Chen and Fan (2006), the authors derive the $\sqrt{L}$-asymptotic normal distribution of the PML estimator $\hat{\theta}_I$. The second step of the procedure follows Li, Perrigne and Vuong (2002) using (4.8) with θ, $G(\cdot|I)$ and $g(\cdot|I)$ replaced by $\hat{\theta}_I$, $\hat{G}(\cdot|I)$ and $\hat{g}(\cdot|I)$ where the latter is a kernel estimator as in (3.8). Because $\hat{\theta}_I$ is $\sqrt{L}$-consistent, they show that the resulting two-step estimators of the joint value density $f(V_1, \ldots, V_I|I) = c[F(V_1|I), \ldots, F(V_I|I)|I;\theta] \prod_{i=1}^I f(V_i|I)$ and its marginal $f(\cdot|I)$ converge uniformly at the rate $(L/\log L)^{R/(2R+3)}$ when $f(\cdot|I)$ is R continuously differentiable and the bandwidths h_g and h_f are proportional to $(\log L/L)^{1/(2R+3)}$. This convergence rate remains slower than the optimal rate for estimating $f(\cdot|I)$ which is $(\log L/L)^{R/(2R+1)}$ from Stone (1982) if private values were observed.

Conditionally Independent Private Values

As a special case of the symmetric APV model, Li, Perrigne and Vuong (2000) consider the Conditionally Independent Private Value (CIPV) model. An interesting feature of this model is that it belongs to the class of models in which private information σ_is are conditionally independent given a common component C that is unknown to all bidders.[16] Specifically, in the CIPV model, $U(\sigma_i, C) = \sigma_i$ and $\sigma_i = V_i$ where the V_is are i.i.d. given C. Hereafter, we use the notation Y instead of C to avoid confusion with the pure common value model reviewed in Section 8.1. The inverse equilibrium strategy $\xi(\cdot; G, I)$ is still given by (4.7). The restrictions of the CIPV model are that (i) the observed bids are conditionally i.i.d. and (ii) the function $\xi(\cdot; G, I)$ is increasing with a differentiable inverse. The CIPV model, however, is not identified because an increasing transformation of Y can preserve the joint distribution of $(V_1, \ldots, V_I)$. To restore identification, the authors add some structure on how private values are affiliated. Namely, they assume that the private value V_i can be decomposed as the product of the common component Y and an idiosyncratic component ϵ_i, i.e., $V_i = Y\epsilon_i$, the two components being nonnegative and independent conditional on I. The ϵ_is are i.i.d. as $F_\epsilon(\cdot|I)$ on $[\underline{\epsilon}(I), \overline{\epsilon}(I)]$ with $\mathrm{E}(\epsilon_i|I) = 1$.

[16]See also de Finetti's Theorem in Kingman (1978) for the relationship between exchangeability and conditional independence as $I \to \infty$.

The common component Y is distributed as $F_Y(\cdot|I)$ on $[\underline{y}(I), \overline{y}(I)]$. Neither Y nor ϵ_i are known to bidders. Thus the primitives of the CIPV model are $[F_Y(\cdot|I), F_\epsilon(\cdot|I)]$.

A first question is whether we can identify $[F_Y(\cdot|I), F_\epsilon(\cdot|I)]$ from observed bids. Since the joint value distribution $F(\cdot, \ldots, \cdot|I)$ is identified in an APV model, the question reduces to whether $F(\cdot, \ldots, \cdot|I)$ identifies $[F_Y(\cdot|I), F_\epsilon(\cdot|I)]$. Taking logarithms gives

$$\log V_i = \tilde{Y} + \tilde{\epsilon}_i \quad \text{where } \tilde{Y} \equiv \log Y + \mathrm{E}[\log \epsilon_i|I] \text{ and } \tilde{\epsilon}_i \equiv \log \epsilon_i - \mathrm{E}[\log \epsilon_i|I]. \tag{4.9}$$

We can view $\log V_i$, $i = 1, \ldots, I$ as multiple measurements of $\tilde{Y}$ with measurement errors $\tilde{\epsilon}_i$ that are i.i.d. and independent of $\tilde{Y}$. With $I \geq 2$, Kotlarski (1967) shows that the distributions $F_{\tilde{Y}}(\cdot|I)$ and $F_{\tilde{\epsilon}}(\cdot|I)$ of $\tilde{Y}$ and $\tilde{\epsilon}$ are identified from observing $(V_1, \ldots, V_I)$. See Prakasa Rao (1992) for a statement and Evdokimov and White (2012) for extensions. Specifically, $F_{\tilde{Y}}(\cdot|I)$ and $F_{\tilde{\epsilon}}(\cdot|I)$ are identified from their characteristic functions

$$\phi_{\tilde{Y}}(t|I) = \exp \int_0^t \frac{\partial \psi(0, u_2|I)/\partial u_1}{\psi(0, u_2|I)} du_2, \quad \phi_{\tilde{\epsilon}}(t|I) = \frac{\psi(0, t|I)}{\phi_{\tilde{Y}}(t|I)} = \frac{\psi(t, 0|I)}{\phi_{\tilde{Y}}(|It)}, \tag{4.10}$$

where $\psi(u_1, u_2|I)$ is the characteristic function of $(\log V_1, \log V_2)$ given I. Identification of the distributions $F_Y(\cdot|I)$ and $F_\epsilon(\cdot|I)$ of Y and ϵ follows from $\mathrm{E}[\log \epsilon_i|I] = -\log(\mathrm{E}[e^{\tilde{\epsilon}_i}|I])$ which results from the normalization $\mathrm{E}[\epsilon_i|I] = 1$.[17]

Using this result and omitting the covariates X to simplify, Li, Perrigne and Vuong (2000) develop a two-step estimation procedure in which the first step is the same as in the APV model. The second step follows Li and Vuong (1998) with the difference that the measurements $\log V_{i\ell}$ are estimated from the first step. Thus the estimated characteristic function of any two bidders' log-values is

$$\hat{\psi}(u_1, u_2|I) = \frac{1}{I(I-1)L_I} \sum_{\ell: I_\ell = I}^{L_I} \sum_{1 \leq i \neq j \leq I} \exp[\iota u_1 \log \hat{V}_{i\ell} + \iota u_2 \log \hat{V}_{j\ell}], \tag{4.11}$$

where $\iota^2 = -1$ and L_I is the number of auctions with I bidders (up to trimming). Plugging this estimator in (4.10) gives the estimators of the densities of $\log \tilde{Y}$ and $\log \tilde{\epsilon}$ by truncated Fourier inversion of their estimated characteristic functions

$$\hat{f}_{\tilde{Y}}(u|I) = \frac{1}{2\pi} \int_{-T}^{T} d(t) e^{-\iota t u} \hat{\phi}_{\tilde{Y}}(t|I) dt, \quad \hat{f}_{\tilde{\epsilon}}(u|I) = \frac{1}{2\pi} \int_{-T}^{T} d(t) e^{-\iota t u} \hat{\phi}_{\tilde{\epsilon}}(t|I) dt. \tag{4.12}$$

These include a damping factor $d(t) = 1 - |t|/T$ if $|t| \leq T$ or 0 otherwise to minimize fluctuating tails following Diggle and Hall (1993). Estimators of the densities of $\log Y$ and $\log \epsilon$

[17] The literature on measurement error models is abundant. See Chen, Hong and Nekipelov (2011) and Schennach (2013) for recent surveys. Li and Vuong (1998) seem to be the first to exploit Kotlarski's result in economics. Besides its use in empirical auctions, Kotlarski's result has been applied in labor economics by Cunha, Heckman and Schennach (2010) and Bonhomme and Robin (2010) among others.

are then obtained as $\hat{f}_{\log Y}(u|I) = \hat{f}_{\tilde{Y}}[u + \hat{\mathrm{E}}(\log \epsilon|I)|I]$ and $\hat{f}_{\log \epsilon}(u|I) = \hat{f}_{\tilde{\epsilon}}[u - \hat{\mathrm{E}}(\log \epsilon|I)|I]$, where $\hat{\mathrm{E}}(\log \epsilon|I) = -\log \hat{\mathrm{E}}(e^{\tilde{\epsilon}}|I) = -\log[\int e^u \hat{f}_{\tilde{\epsilon}}(u|I)du]$. Adapting Li and Vuong (1998) to account for estimation of $\log V_{i\ell}$, the authors establish the uniform consistency of $\hat{f}_{\log Y}(\cdot|I)$ and $\hat{f}_{\log \epsilon}(\cdot|I)$ with convergence rates depending on whether $\log Y$ and $\log \epsilon_i$ are ordinary smooth or supersmooth as in Fan (1991) when the smoothing parameter T suitably diverges as $L \to \infty$. They also consider a MSE criterion for T based on comparing estimates of the means and variances of $\tilde{Y}$ and $\tilde{\epsilon}$ obtained in the first and second steps.

Section 4.4: Unobserved Heterogeneity

Unobserved heterogeneity is not *per se* part of the theoretical model but rather an important econometric issue. It refers to characteristics of the auctioned objects that are known to bidders but unobserved by the analyst. Up to now, we have considered that the number of bidders captures this unobserved heterogeneity since some goods may attract more bidders even after controlling for observed characteristics. In the same vein, Roberts (2013) uses the reserve price as an instrument for the unobserved heterogeneity as the seller sets his reserve price accounting for the unobserved characteristics such as quality, etc. We review here solutions to this problem when unobserved heterogeneity is not captured sufficiently by the number of bidders or the reserve price. We can expect this to be the case when auctioned goods are too complex to characterize. For instance, in construction procurements bidders know all the project specifications which may be too costly to collect for the analyst. Some characteristics are also too difficult to quantify such as quality for used goods, etc. We focus on nonparametric approaches. For parametric ones, see e.g. Laffont, Ossard and Vuong (1995) and Athey, Levin and Seira (2011) for direct and indirect methods, respectively.

A Deconvolution Approach

Krasnokutskaya (2011) is the first to address nonparametrically the problem of unobserved heterogeneity in auctions. She considers an asymmetric IPV setting and assumes that each value V_i satisfies the multiplicative decomposition $V_i = Y\epsilon_i$, $i = 1, \ldots, I$. In contrast to Li, Perrigne and Vuong (2000), the common component Y is known to all bidders so that they can condition on it as any other covariates X when bidding, while ϵ_i is bidder i's private information. Because it is unobserved to the analyst, Y represents the unobserved heterogeneity. Omitting the conditioning on the observed covariates X and considering the symmetric case to simplify, the unobserved heterogeneity Y is distributed as $F_Y(\cdot|I)$ independently of the idiosyncratic components ϵ_i that are i.i.d. as $F_\epsilon(\cdot|I)$ with $\mathrm{E}(\epsilon_i|I) = 1$. Thus the primitives of the model are $[F_Y(\cdot|I), F_\epsilon(\cdot|I)]$ as in the CIPV model.

Let $F_{V|Y}(\cdot|I) = F_\epsilon(\cdot/Y|I)$ denote the distribution of V_i given (Y, I). The multiplicative decomposition of V_i translates to bids by the location-scale invariance of the bidding strategy mentioned in Section 2.2. Specifically, given (Y, I) the equilibrium strategy (2.7) satisfies $s(V_i; F_{V|Y}, I) = s(Y\epsilon_i; F_{V|Y}, I) = Ys(\epsilon_i; F_\epsilon, I) \equiv Y\eta_i$ for $i = 1, \ldots, I$. That is, η_i is bidder i's bid if his private value is ϵ_i or equivalently if $Y = 1$. Since $B_i = s(V_i; F_{V|Y}, I)$, taking logarithms gives

$$\log B_i = \tilde{Y} + \tilde{\epsilon}_i \text{ where } \tilde{Y} \equiv \log Y + \mathrm{E}[\log \eta_i|I] \text{ and } \tilde{\epsilon}_i \equiv \log \eta_i - \mathrm{E}[\log \eta_i|I], \tag{4.13}$$

which is similar to (4.9) where V_i and ϵ_i are replaced by B_i and η_i. Thus, viewing $\log B_i$, $i = 1, \ldots, I$ as multiple measurements of $\tilde{Y}$ shows that the distributions $F_{\tilde{Y}}(\cdot|I)$ and $F_{\tilde{\epsilon}}(\cdot|I)$ are identified by Kotlarski (1967). Hence the distributions $F_Y(\cdot|I)$ and $F_\eta(\cdot|I)$ are identified up to the multiplicative constants $e^{-\mathrm{E}[\log \eta_i|I]}$ and $e^{\mathrm{E}[\log \eta_i|I]}$, respectively. Because $\eta_i = s(\epsilon_i; F_\epsilon, I)$ or equivalently $\epsilon_i = \xi(\eta_i; F_\eta, I)$, it follows from the location-scale invariance of $s(\epsilon_i; F_\epsilon, I)$ that the distribution $F_\epsilon(\cdot|I)$ is identified up to the multiplicative constant $e^{\mathrm{E}[\log \eta_i|I]}$. Using the normalization $\mathrm{E}[\epsilon_i|I] = 1$ then identifies $e^{\mathrm{E}[\log \eta_i|I]}$. Thus, the distributions $F_Y(\cdot|I)$ and $F_\epsilon(\cdot|I)$ are identified.

In terms of restrictions imposed by unobserved heterogeneity, Krasnokutskaya (2011) indicates a number of them. For instance, the bids B_i, $i = 1, \ldots, I$ should be exchangeable and the inverse equilibrium sratategy $\xi(\eta; F_\eta, I) = \eta + F_\eta(\eta|I)/[(I-1)f_\eta(\eta|I)]$ should be increasing in η with a differentiable inverse. Moreover, for any triplet of distinct indices (i_1, i_2, i_3), the ratios B_{i_1}/B_{i_3} and B_{i_2}/B_{i_3} satisfy $\log(B_{i_1}/B_{i_3}) = \log \eta_{i_1} - \log \eta_{i_3}$ and $\log(B_{i_2}/B_{i_3}) = \log \eta_{i_2} - \log \eta_{i_3}$ from (4.13). This implies some restrictions on the moments of the observed bid ratios since the η_is are i.i.d. conditional on (X, I). Similarly, for any quadruplet of distinct indices (i_1, i_2, i_3, i_4) the ratios B_{i_1}/B_{i_2} and B_{i_3}/B_{i_4} should be independent conditional on (X, I). Because unobserved heterogeneity introduces some dependence among bids given (X, I) through the unobserved component Y, an interesting question is whether one can distinguish it from the CIPV model or more generally from the APV model of Section 4.3. The author shows that the above restrictions on the bid ratios are not necessarily satisfied by the latter two models.

Estimation follows the identification argument. Essentially, it is similar to the estimation of the CIPV model but switches its two steps. In the first step, one invokes (4.10) to estimate the densities $\hat{f}_{\tilde{Y}}(\cdot|I)$ and $\hat{f}_{\tilde{\epsilon}}(\cdot|I)$ by (4.12) using the empirical characteristic function (4.11) where the estimated values $\hat{V}_{i\ell}$ are replaced by the observed bids $B_{i\ell}$ for each given number I of bidders. Thus one obtains the estimated densities for Y and η as

$$\hat{f}_Y(Y|I) = \frac{f_{\tilde{Y}}[\log Y + \hat{\mathrm{E}}(\log \eta|I)|I]}{Y} \quad \text{and} \quad \hat{f}_\eta(\eta|I) = \frac{f_{\tilde{\epsilon}}[\log \eta - \hat{\mathrm{E}}(\log \eta|I)|I]}{\eta},$$

up to an estimate $\hat{\mathrm{E}}(\log \eta|I)]$. In the second step, because $\tilde{\epsilon}$ and hence η cannot be recovered for each observation, one draw a large number of η values from $\hat{f}_\eta(\cdot|I)$ and use the estimated inverse equilibrium strategy $\epsilon = \xi(\eta; \hat{F}_\eta, I)$ to estimate $f_\epsilon(\cdot|I)$. The constant $\hat{\mathrm{E}}(\log \eta|I)]$ is adjusted iteratively so that $\hat{f}_\epsilon(\cdot|I)$ satisfies the normalization constraint $\int x\hat{f}(x|I)dx = 1$. Lastly, one estimates the density of private values by $\hat{f}(v|I) = \int (1/y)\hat{f}_\epsilon(v/y|I)\hat{f}_Y(y|I)dy$. Following Li, Perrigne and Vuong (2000), the resulting estimator is uniformly consistent as the truncation parameter T increases with L.

Krasnokutskaya (2011) actually considers the asymmetric case. Indeed, Kotlarski's result does not require the errors $\tilde{\epsilon}_i$ in (4.13) to be identically distributed provided one error $\tilde{\epsilon}_1$ (say) has zero mean. Thus, let $\tilde{Y} \equiv \log Y + \mathrm{E}(\log \eta_1|I)$ and $\tilde{\epsilon}_i \equiv \log \eta_i - \mathrm{E}(\log \eta_1|I)$ where $\eta_i \equiv s_i(\epsilon_i; F_{\epsilon_1}, \ldots, F_{\epsilon_I}, I)$. The subscript i indicates that the equilibrium strategies can differ across bidders as $\epsilon_i \sim F_{\epsilon_i}(\cdot|I)$ with the normalization $\mathrm{E}(\epsilon_1|I) = 1$. Because $B_i = Y\eta_i$ by location-scale invariance, then $F_{\tilde{Y}}(\cdot|I)$ and $F_{\tilde{\epsilon}_i}(\cdot|I)]$ are identified by their characteristic functions

$$\phi_{\tilde{Y}}(t|I) = \exp \int_0^t \frac{\partial \psi(0, u_2|I)/\partial u_1}{\psi(0, u_2|I)} du_2, \quad \phi_{\tilde{\epsilon}_i}(t|I) = \frac{\phi_i(t|I)}{\phi_{\tilde{Y}}(t|I)},$$

where $\psi(u_1, u_2|I)$ and $\phi_i(\cdot|I)$ are the characteristic functions of $(\log B_1, \log B_2)$ and $\log B_i$ given I. As above, the mean $\mathrm{E}(\log \eta_1|I)$ is identified from the normalization $\mathrm{E}(\epsilon_1|I) = 1$. The estimation procedure follows straightforwardly.

Extensions

Krasnokutskaya (2012) extends Krasnokutskaya (2011) to allow for bidimensional unobserved heterogeneity. Private values satisfy $V_i = Y_1 + Y_2\epsilon_i$, where Y_1, Y_2 and ϵ_i are mutually independent given I with $\mathrm{E}(Y_2|I) = 1$ and $\mathrm{E}(\epsilon_i|I) = 1$. The variables (Y_1, Y_2) are known to all bidders but unknown to the analyst. Thus, (Y_1, Y_2) capture auction unobserved heterogeneity. To simplify, we consider the symmetric case so that the ϵ_is are identically distributed. Thus the model primitives are $[F_{Y_1}(\cdot|I), F_{Y_2}(\cdot|I), F_\epsilon(\cdot|I)]$. The author then exploits the location-scale invariance of the equilibrium strategy, which gives $B_i = Y_1 + Y_2\eta_i$ where $\eta_i = s(\epsilon_i; F_\epsilon, I)$. The identification argument requires having at least four bidders. First, considering the differences $B_1 - B_2 = Y_2(\eta_1 - \eta_2)$ and $B_3 - B_4 = Y_2(\eta_3 - \eta_4)$ identifies the density $f_{Y_2}(\cdot|I)$ by Kotlarski (1967) using the normalization $\mathrm{E}(Y_2|I) = 1$. Second, considering the differences $B_1 - B_3 = Y_2(\eta_1 - \eta_3)$ and $B_2 - B_3 = Y_2(\eta_2 - \eta_3)$ identifies the joint density of $(\eta_1 - \eta_3, \eta_2 - \eta_3)$ given the identification of $f_{Y_2}(\cdot|I)$. Third, knowledge of the latter bivariate density identifies $f_\eta(\cdot|I)$ by Kotlarski (1967) up to $\mathrm{E}(\eta_i|I)$. Since $\epsilon_i = \xi(\eta_i; F_\eta, I)$, this identifies $f_\epsilon(\cdot|I)$ and $\mathrm{E}(\eta_i|I)$ using the normalization $\mathrm{E}(\epsilon_i|I) = 1$. Lastly, because $B_i = Y_1 + Y_2\eta_i$ where the distribution of $Y_2\eta_i$ is identified since Y_2 and

η_i are independent with known densities, it follows that the distribution of Y_1 given I is identified. Estimation essentially proceeds based on Krasnokutskaya (2011).

Hu, McAdams and Shum (2013) relax the separability assumption in Krasnokutskaya (2011) where unobserved heterogeneity Y affects bidders' values multiplicatively. They, however, require that the unobserved heterogeneity Y be discrete with a finite number of values to apply Hu's (2008) result on nonclassical measurement errors. Thus one needs bidding data from three bidders as their bids play the role of the dependent variable, a proxy and an instrument. When the distribution $F(\cdot|Y,I)$ of i.i.d. private values conditional on (Y,I) is increasing in Y in terms of first-order stochastic dominance, the authors establish the identification of the bid density $g(\cdot|Y,I)$ from which one identifies the value distribution $F(\cdot|Y,I)$ from $v=\xi(b;G,I)$. Estimation follows the identification argument and is similar to that described in Section 4.1 when the potential number I of bidders varies and is unobserved. As a matter of fact, the authors provide high-level assumptions for identification of the benchmark model with nonseparable unobserved heterogeneity. These assumptions are a monotonicity condition and a full-rank condition that are shown to be satisfied in other situations where unobserved heterogeneity arises from not observing the reserve price or the bidding cost.[18]

Armstrong (2013) relaxes the discreteness of the unobserved heterogeneity Y by considering an arbitrary value distribution $F(\cdot|Y,I)$ where Y is continuous. As above, we omit the conditioning on the observed auction characteristics X to simplify. Moreover, he assumes that only the winning bid B^w is observed thereby dropping the availability of multiple measurements used in previous papers. Because the model is not identified, he focuses on some features of $F(\cdot|Y,I)$ that are important in counterfactuals, namely the unconditional expectations of values $\mathrm{E}(V_i|I)=\int\left[\int v\ dF(v|y,I)\right]dF_Y(y|I)$ and of the winner's value $\mathrm{E}(V^w|I)=\int\left[\int v\ dF^I(v|y,I)\right]dF_Y(y|I)$ where $F_Y(\cdot|I)$ is the distribution of the unobserved heterogeneity Y. The author establishes the following sharp bounds

$$\begin{aligned}\int b\ d[G^w(b|I)]^{1/I} &\le \mathrm{E}(V_i|I) \le \frac{I-2}{I-1}\mathrm{E}(B^w|I)+\frac{1}{I-1}\bar{b}(I),\\ \mathrm{E}(B^w|I) &\le \mathrm{E}(V^w|I) \le \frac{I}{I-1}\bar{b}(I)-\frac{1}{I-1}\mathrm{E}(B^w|I),\end{aligned}$$

where $[G^w(\cdot|I)]^{1/I}$ is not equal to $G(\cdot|I)$ because bids are not mutually independent though they are independent conditional on Y. As usual, the sample analogs $(1/L_I)\sum_{\ell:I_\ell=I}^{L_I}B_\ell^w$ and $\max_{\{\ell:I_\ell=I\}}B_\ell^w$ converge to $\mathrm{E}(B^w|I)$ and $\bar{b}(I)$ at rates equal and faster than $\sqrt{L}$, respectively. In contrast, the convergence rate and asymptotic distribution of the estimator

[18]Following a similar identification argument, the authors also relax the multiplicative separability assumption in the CIPV model of Section 4.3.

$\int b \, d[\hat{G}^w(b|I)]^{1/I}$, where $\hat{G}^w(\cdot|I)$ is the empirical winning bid distribution, are nonstandard and depend on I. See also Menzel and Morganti (2013). Indeed,

$$\int b \, d[\hat{G}^w(b|I)]^{1/I} \quad = \quad \sum_{\ell:I_\ell=I}^{L_I} B^w_{(\ell:L_I)} \left[\left(\frac{\ell}{L_I} \right)^{1/I} - \left(\frac{\ell-1}{L_I} \right)^{1/I} \right],$$

where $B^w_{(1:L_I)} \leq \ldots \leq B^w_{(L_I:L_I)}$ are the L_I order statistics of the winning bids in the L_I auctions. That is, $\int b \, d[\hat{G}^w(b|I)]^{1/I}$ is an L-statistic. See e.g., van der Vaart (2000). Armstrong (2013) characterizes the rate of convergence of such an estimator which depends on the rate at which $G^w(\cdot|I) = \int G(\cdot|y,I)^I dF_Y(y|I)$ vanishes at the lower boundary $\underline{b}(I)$. When the densities of $F(\cdot|Y,I)$ and $F_Y(\cdot|I)$ are bounded away from zero on their supports, it follows from Armstrong (2013) that $\int b \, d[\hat{G}^w(b|I)]^{1/I}$ converges at the $\sqrt{L_I}$-rate to a normal distribution if $I \leq 3$ and at the slower rate $L_I^{-(2I+1)/[I(I+1)]}$ to a non-normal distribution if $I \geq 4$.

The previous contributions deal with unobserved heterogeneity in the benchmark model. Of course unobserved heterogeneity can arise with other mechanisms than first-price sealed-bid auctions as well as in other paradigms than the symmetric IPV paradigm. For a recent survey on unobserved heterogeneity in auctions, see Haile and Kitamura (2019). Their survey also includes alternative approaches from the econometric literature. In particular, approaches based on mixture models as in Kitamura and Laage (2018) apply to auction settings as well.

Section 5: Advanced Extensions

This section reviews advanced extensions of the GPV methodology, while still considering first-price sealed-bid auctions within the IPV paradigm. The underlying models are more involved and there are more primitives to identify. Since we only observe the bid distribution, identifying the latter constitutes a challenge. This calls for different identification strategies. On economic grounds, these extensions are also important. Up to now, we have assumed risk neutral bidders. The first extension allows for risk averse bidders as risk aversion is a natural component of individual behavior. Indeed, a large experimental literature on auctions shows that individuals tend to be risk averse when facing uncertainty. In an auction, uncertainty arises from several sources which include the opponents' values and the likelihood of winning. Though bids may be associated to firms or large institutions which are generally viewed as risk neutral, bids are submitted by individuals. A second extension deals with bidders' entry. Our benchmark model of Section 2.3 assumes that the number of bidders is exogenous. In reality, bidding can be costly to a bidder as it requires an analysis of documents or other supports to assess the auctioned item. In particular, bidders must decide whether it is worth bidding given the sunk cost of participation. This leads to an entry model taking place before the auction thereby rendering the number of actual bidders endogenous. A third extension considers multi-object auctions. Up to now, we have viewed each auction as a one shot game by assuming that auctions are mutually independent. It often happens that several heterogeneous goods are sold through sequential or simultaneous first-price sealed-bid auctions. Because each bidder might be interested in acquiring more than one of the items, a bidder has now several private values. The theoretical auction literature shows that this greatly complicates the model since the informational structure for each bidder becomes multidimensional. A related issue arises with synergies. This occurs when a bidder's gain for winning several objects is larger or smaller than the sum of his gains from acquiring the objects separately. For instance, by winning several service procurements a firm can save on transportation costs or benefit from scale economies though this can also increase its opportunity costs.

Section 5.1: Risk Averse Bidders

Several experimental studies show individuals' aversion toward risk. In their survey, Kagel and Levin (2016) report persistent and significant overbidding relative to the risk neutral Bayesian Nash equilibrium strategy in numerous controlled experiments. A common explanation is bidders' risk aversion, which renders bidding more aggressive. Using a quantal response equilibrium concept, Goeree, Holt and Palfrey (2002) also find significant risk aversion. Given recent developments in the structural approach, Bajari and Hortacsu (2005) use GPV style methods to estimate various models including learning, bidders' risk aversion and quantal response equilibrium with experimental auction data. Their analysis shows that bidders' risk aversion provides the best fit thereby confirming that risk aversion can explain the overbidding phenomenon. Using eBay auction data, Ackerberg, Hirano and Shahariar (2006) also justify the success of the buy-it-now option by bidders' risk aversion, whereas Athey and Levin (2001) observe risk diversification in timber scale auctions which is consistent with bidders' risk aversion. We briefly present the benchmark model with risk aversion and discuss different identifying strategies for recovering the bidders' utility function with their corresponding semiparametric and nonparametric estimation methods. Lastly, we review recent developments involving uncertainty in bidders' values.

The Benchmark Model with Risk Averse Bidders

In the benchmark model, bidders are risk neutral so that their utility function is the identity function. Under risk aversion, bidders have a von Neuman-Morgenstern utility $U(\cdot)$ which is twice continuousy differentiable, increasing $U'(\cdot) > 0$ and weakly concave $U''(\cdot) \leq 0$. Hereafter, we set $U(0) = 0$ as a location normalization. Maskin and Riley (1984, 2000b, 2003) show that the Bayesian Nash equilibrium strategy $s(\cdot) = s(\cdot; U, F, I)$ exists, is symmetric, unique, increasing and differentiable. Thus, bidder i's expected profit is $U(V_i - B_i)\Pr(B_i \geq B_{-i}^{\max}|I) = U(V_i - B_i)F^{I-1}[s^{-1}(B_i)|I]$. Maximizing the latter with respect to B_i and using $B_i = s(V_i)$ shows that $s(\cdot)$ solves the differential equation

$$1 = (I-1)\frac{f(V_i|I)}{F(V_i|I)}\frac{1}{s'(V_i)}\lambda(V_i - B_i), \tag{5.1}$$

for $V_i \in [\underline{v}(I), \overline{v}(I)]$ where $\lambda(\cdot) \equiv U(\cdot)/U'(\cdot)$. The function $\lambda(\cdot)$ is interpreted as the fear of ruin. The equilibrium satisfies the boundary condition $s[\underline{v}(I)] = \underline{v}(I)$ since $U(0) = 0$. In particular, bidding is more aggressive than under risk neutrality leading to overbidding as widely found in auction experiments. In general, the differential equation (5.1) does not have a closed form solution except in special cases. For instance, when the utility function exhibits Constant Relative Risk Aversion (CRRA), i.e., $U(x) = x^{1-\gamma}$, where $0 \leq \gamma < 1$ captures the degree of risk aversion, the equilibrium strategy is

$$B_i = V_i - \frac{1}{F(V_i|I)^{\frac{I-1}{1-\gamma}}}\int_{\underline{v}(I)}^{V_i} F(v|I)^{\frac{I-1}{1-\gamma}} dv \equiv s(V_i; U, F, I),$$

which is similar to the risk neutral equilibrium strategy (2.7) with $I-1$ replaced by $(I-1)/(1-\gamma)$. In the general case, the model primitives are the bidders' utility function and the bidders' value distribution $[U(\cdot), F(\cdot|I)]$ where $U(\cdot)$ and $F(\cdot|I)$ are $R+2$ and $R+1$ continuously differentiable on $(0,\infty)$ and $[\underline{v}(I), \overline{v}(I)]$, respectively. Equivalently, the primitives are $[\lambda(\cdot), F(\cdot|I)]$ with a scale normalization such as $U(1) = 1$.[19]

Similarly to (3.6), Guerre, Perrigne and Vuong (2009) note that the differential equation (5.1) can be rewritten in terms of the bid distribution $G(\cdot|I)$ and density $g(\cdot|I)$ as

$$V_i = B_i + \lambda^{-1}\left(\frac{1}{I-1}\frac{G(B_i|I)}{g(B_i|I)}\right) \equiv \xi(B_i; U, G, I) \text{ for } i = 1, \ldots, I \tag{5.2}$$

since $\lambda(\cdot)$ is increasing as $\lambda'(\cdot) \geq 1$. Intuitively, without additional restriction or data, $[U(\cdot), F(\cdot|I)]$ are unidentified because one observes the bid distribution $G(\cdot|I)$ only. Formally, they consider the alternative structure $[\tilde{U}, \tilde{F}]$ with $\tilde{U}(\cdot) = [U(\cdot/\delta)/U(1/\delta)]^{\delta}$ with

[19]One also assumes that $\lim_{x\downarrow 0} d\lambda^r(x)/dx^r$ is finite for $r \leq R+1$. Thus, knowing $\lambda(\cdot)$ recovers $U(\cdot)$ as $U(x) = \int_1^x [1/\lambda(t)]dt$ where $\lim_{x\downarrow 0} U(x) = 0$. This technical assumption also implies a restriction (omitted hereafter) on the marginal bid distribution $G(\cdot|I)$, namely $\lim_{b\downarrow \underline{b}(I)} dG^r(b|I)/db^r$ is finite for $r \leq R+1$.

$\delta \in (0,1)$ and $\tilde{F}(\cdot|I)$ being the distribution of $\tilde{V} = \xi(B;\tilde{U},G,I)$. Elementary algebra gives $\xi(B;\tilde{U},G,I) = (1-\delta)B + \delta\xi(B;U,G,I)$ showing that $\xi(\cdot;\tilde{U},G,I)$ is increasing so that $[\tilde{U}(\cdot),\tilde{F}(\cdot|I)]$ also rationalizes $G(\cdot|I)$. Thus, the two structures $[U(\cdot),F(\cdot|I)]$ and $[\tilde{U}(\cdot),\tilde{F}(\cdot|I)]$ are observationally equivalent. As a matter of fact, one can always find a function $\lambda(\cdot)$ with $\lambda(0) = 0$ and $\lambda'(\cdot) \geq 1$ such that $\xi(\cdot;U,G,I)$ is increasing, thereby rendering the monotonicity restriction redundant. Specifically, the authors show that a bid distribution is rationalized if and and only if bids are independent with an $R+1$ continuously differentiable marginal density $g(\cdot|I)$ on $(\underline{b}(I),\overline{b}(I)]$. As with risk neutral bidders, the bid density $g(\cdot|I)$ is smoother than the value density $f(\cdot|I)$. In contrast to the risk neutral case, however, up to some smoothness conditions, any bid distribution can be rationalized by the benchmark model with risk aversion. In other words, given I, risk aversion is not testable from bids without additional restrictions or data. We present next three strategies to identify and estimate the benchmark model with risk aversion.

Identification and Estimation Using Exclusion Restrictions

Exclusion restrictions have a long tradition in the econometric literature since the work of the Cowles Commission on estimation of supply and demand models. Within a nonparametric context, Matzkin (1994) surveys the use of exclusion restrictions in the identification of such models. In auctions, a natural exclusion restriction is that the number I of bidders is exogenous in the sense that $F(\cdot|I) = F(\cdot)$, i.e., bidders' values are independent of I (given auction characteristics X). Haile, Hong and Shum (2006) exploit such an exclusion to test for common value. Guerre, Perrigne and Vuong (2009) use it to identify the benchmark model with risk averse bidders. The basic idea relies on exploiting variations of the bid quantiles $b(\alpha;I)$ with I while the value quantiles $v(\alpha)$ remain invariant because $F(\cdot|I) = F(\cdot)$. Specifically, expressing (5.2) in quantiles gives

$$v(\alpha;I) = b(\alpha;I) + \lambda^{-1}\left(\frac{1}{I-1}\frac{\alpha}{g[b(\alpha;I)|I]}\right), \tag{5.3}$$

for $\alpha \in [0,1]$, which extends (3.13) for risk neutral bidders where $\lambda^{-1}(u) = u$. Because $v(\cdot;I) = v(\cdot)$, using (5.3) for two levels of competition I_1 and I_2 with $I_2 > I_1 \geq 2$ gives

$$b(\alpha;I_1)+\lambda^{-1}\left(\frac{1}{I_1-1}\frac{\alpha}{g[b(\alpha;I_1)|I_1]}\right) = b(\alpha;I_2)+\lambda^{-1}\left(\frac{1}{I_2-1}\frac{\alpha}{g[b(\alpha;I_2)|I_2]}\right), \tag{5.4}$$

for any $\alpha \in [0,1]$, where $b(\cdot;I_1) < b(\cdot;I_2)$ on $(0,1]$. In particular, by letting $\alpha = 0$ we have $\underline{v} = \underline{b}(I_1) = \underline{b}(I_2) \equiv \underline{b}$. Equation (5.4) is referred as the compatibility condition for the pair (I_1,I_2) since it is part of the model restrictions as indicated below.

The authors' identification argument uses (5.4) iteratively. Let $\rho_I(\alpha)$ for $I = I_1, I_2$ be the arguments of $\lambda^{-1}(\cdot)$ in (5.4). Note that the $\rho_I(\cdot)$s are identified with $\rho_I(0) = 0$

and $\rho_{I_1}(\cdot) > \rho_{I_2}(\cdot)$ on $(0,1]$. Let α_0 be arbitrary in $(0,1]$. From (5.3)-(5.4) we have $v(\alpha_0) = b(\alpha_0; I_1) + \lambda^{-1}[\rho_{I_1}(\alpha_0)] = b(\alpha_0; I_1) + \Delta b(\alpha_0) + \lambda^{-1}[\rho_{I_2}(\alpha_0)]$ where $\Delta b(\cdot) \equiv b(\cdot; I_2) - b(\cdot; I_1) > 0$ on $(0,1]$. Using the continuity of $\rho_{I_1}(\cdot)$, let α_1 be the smallest solution in $(0,\alpha_0)$ of $\rho_{I_1}(\alpha_1) = \rho_{I_2}(\alpha_0)$ and hence of $\lambda^{-1}[\rho_{I_1}(\alpha_1)] = \lambda^{-1}[\rho_{I_2}(\alpha_0)]$. Since $\lambda^{-1}[\rho_{I_1}(\alpha_1)] = \Delta b(\alpha_1) + \lambda^{-1}[\rho_{I_2}(\alpha_1)]$ by (5.4), we obtain $v(\alpha_0) = b(\alpha_0; I_1) + \Delta b(\alpha_0) + \Delta b(\alpha_1) + \lambda^{-1}[\rho_{I_2}(\alpha_1)]$. Applying this argument iteratively shows that $v(\alpha_0) = b(\alpha_0; I_1) + \sum_{t=0}^{\infty} \Delta b(\alpha_t)$ so that $v(\alpha_0)$ and hence $F(\cdot)$ is identified. Moreover, $\lambda^{-1}[\rho_{I_1}(\alpha_0)] = \sum_{t=0}^{\infty} \Delta b(\alpha_t)$ showing that $\lambda^{-1}(\cdot)$ is identified on the range of $\rho_{I_1}(\cdot)$ which is the largest interval on which $\lambda^{-1}(\cdot)$ can be identified from equilibrium bids.

The authors characterize all the restrictions of the benchmark model with risk aversion and an exogenous number of bidders. These are (i) bids are independent given I with an $R+1$ continuously differentiable density $g(\cdot|I)$ on $(\underline{b}, \overline{b}(I)]$, (ii) the bid quantiles satisfy $b(\cdot|I_2) > b(\cdot|I_1)$ on $(0,1]$ whenever $I_2 > I_1$, and (iii) the compatibility condition (5.4) holds for any pair (I_1, I_2) where the function $\xi(\cdot; U, G, I)$ in (5.2) is increasing on $[\underline{b}, \overline{b}(I)]$ for some function $\lambda(\cdot)$ that is $R+1$ continuously differentiable with $\lambda(0) = 0$ and $\lambda'(\cdot) \geq 1$ for every I. The authors also indicate how their identification argument applies when participation is endogenous as in $I = I(X, Z, \epsilon)$ under the exclusion restriction $F(V|X, Z, \epsilon) = F(V|X, \epsilon)$. That is, Z plays the role of an instrumental variable as it affects the number of bidders but not the distribution of values, whereas ϵ represents unobserved auction heterogeneity. For examples of variables Z, see Bajari and Hortacsu (2003), Athey, Levin and Seira (2011) and Krasnokutskaya and Seim (2011). They extend their results to a reserve price, affiliated private values, asymmetry in values and/or in preferences where the latter allows for different utilities across bidders. See Guerre, Perrigne and Vuong (2009).

Though they mainly study identification, the authors suggest two estimation strategies. One strategy relies on estimating the sequence $\{\alpha_t\}$ using the recursive equation $\rho_{I_1}(\alpha_{t+1}) = \rho_{I_2}(\alpha_t)$ starting from some arbirtray $\alpha_0 \in (0,1)$ to estimate the quantile differences $\{\Delta b(\alpha_t)\}$ and hence $\lambda^{-1}(\cdot)$. This is achieved by solving recursively $\hat{\rho}_{I_1}(\alpha_{t+1}) = \hat{\rho}_{I_2}(\hat{\alpha}_t)$ where $\hat{\rho}_I(\alpha) \equiv \alpha/[(I-1)\hat{g}(\hat{b}(\alpha; I)|I)]$ and $\hat{g}(\cdot|I)$ and $\hat{b}(\cdot; I)$ are nonparametric bid density and quantile estimators for $I = I_1, I_2$. A second method consists in exploiting the compatibility condition (5.4) to estimate $\lambda^{-1}(\cdot)$ by minimizing the sum of squares $\sum_{n=1}^{n_L}[\hat{b}(\alpha_n; I_1) + \lambda^{-1}(\hat{\rho}_{I_1}(\alpha_n)) - \hat{b}(\alpha_n; I_2) - \lambda^{-1}(\hat{\rho}_{I_2}(\alpha_n))]^2$ where $\{\alpha_n\}$ are n_L arbitrary values in $(0,1)$ subject to $\lambda^{-1}(\cdot)$ belonging to a sieve $\{\Lambda_L^{-1}\}$. In the spirit of the first method, Kim (2015) proposes an iterative estimator of $\lambda^{-1}(\cdot)$ upon showing that the latter is a fixed point of the functional $\phi(\cdot) \to \mathcal{T}[\phi(\cdot)]$ defined implicitly as

$\mathcal{T}[\phi(\rho_{I_1}(\alpha))] = b(\alpha; I_2) - b(\alpha; I_1) + \phi(\rho_{I_2}(\alpha))$ for $\alpha \in [0,1]$.

Zincenko (2018) adopts the second estimation strategy upon replacing the α_n and the L_2-norm by the sup-norm over an interval $[\epsilon_L, 1-\epsilon_L]$ of $(0,1)$. With auction covariates $X \in I\!R^d$ and level of competition $I \in \mathcal{S}_I \subset \{2, 3, \ldots\}$, his objective function is

$$\hat{Q}_L(\lambda^{-1}) \equiv \sum_{I \neq \underline{I}} \max_{\alpha \in [\epsilon_L, 1-\epsilon_L]} \left| \hat{b}(\alpha; x, I) - \hat{b}(\alpha; x, \underline{I}) + \lambda^{-1}[\hat{\rho}_I(\alpha, x)] - \lambda^{-1}[\hat{\rho}_{\underline{I}}(\alpha, x)] \right|,$$

where $\underline{I} \equiv \min_{I \in \mathcal{S}_I} I$, x is a value in $\mathcal{S}_X$, $\hat{\rho}_I(\alpha; x) \equiv \alpha/[(I-1)\hat{g}(\hat{b}(\alpha; x, I)|x, I)]$ and $\epsilon_L \propto L^{-(R+1)/(2R+d+3)}$. He uses the standard bid quantile estimator $\hat{b}(\cdot; x, I) = \inf_b\{b : \hat{G}(b|x, I) \geq \cdot\}$ where $\hat{G}(\cdot|x, I)$ is the kernel estimator of Section 3.3. To obtain an estimator of the density $g(\cdot|x, I)$ that is uniformly consistent on its full support $[\underline{b}(x), \overline{b}(x, I)]$, the author extends Barron and Sheu (1991)'s exponential series estimator allowing for covariates X. Bids are first normalized to be in $[0,1]$ by the transformation $\hat{B}_i^* \equiv [B_i - \hat{\underline{b}}(x)]/[\hat{\overline{b}}(x, I) - \hat{\underline{b}}(x)]$ where the boundary estimators $\hat{\overline{b}}(x, I)$ and $\hat{\underline{b}}(x)$ are those of Section 3.3. The density of the normalized bids B_i^* is then estimated by

$$\hat{g}^*(b^*|x, I) \equiv \frac{\exp[\sum_{j=1}^{J_L} \hat{\delta}_j(x, I)\mathcal{L}_j(b^*)]}{\int_0^1 \exp[\sum_{j=1}^{J_L} \hat{\delta}_j(x, I)\mathcal{L}_j(u)]du},$$

where the $\mathcal{L}_j(\cdot)$s are the orthonormal Legendre polynomials on $[0,1]$. The J_L estimated coefficients $\hat{\delta}_j(x, I)$ solve the J_L equations

$$\int_0^1 \mathcal{L}_j(u)\hat{g}^*(u|x, I)du = \hat{\mathrm{E}}[\mathcal{L}_j(\hat{B}_i^*)|x, I],$$

for $j = 1, \ldots, J_L$, where $\hat{\mathrm{E}}[\mathcal{L}_j(\hat{B}_i^*)|x, I]$ is the kernel regression with bandwidth h_μ of $\{\mathcal{L}(\hat{B}_{i\ell}^*), i = 1, \ldots, I\}$ on X_ℓ for the L_I auctions with I bidders. Thus $\hat{g}(\cdot|x, I) = [\hat{\overline{b}}(x, I) - \hat{\underline{b}}(x)]^{-1}\hat{g}^*[(\cdot - \hat{\underline{b}}(x))/(\hat{\overline{b}}(x, I) - \hat{\underline{b}}(x))]$. The author shows that $\hat{g}(\cdot|x, I)$ converges uniformly on $[\underline{b}(x), \overline{b}(x, I)]$ at the rate L^τ where $\tau \equiv 2R(R+1)/[(2R+3)(2R+d+2)]$, when $J_L \propto L^{\tau/R}$ and $h_\mu \propto (\log L/L)^{1/(2R+d+2)}$.

The estimator of $\lambda^{-1}(\cdot)$ is obtained by minimizing $\hat{Q}_L(\lambda^{-1})$ subject to $\lambda^{-1}(\cdot)$ belonging to a sieve $\{\Lambda_L^{-1}\}$. The finite dimensional space Λ_L^{-1} is the linear space of functions of the form $\sum_{m=1}^{M_L} \delta_m \mathcal{B}_{M_L,m}(\cdot/\overline{\rho})$ where $\overline{\rho}$ is chosen so that the range of $\rho_{\underline{I}}(\cdot)$ is included in $[0, \overline{\rho}]$ whereas $\{\mathcal{B}_{M_L,m}(\cdot); m = 1, \ldots, M_L\}$ are the Bernstein polynomials of degree M_L on $[0,1]$. In addition, the coefficients δ_m satisfy some linear inequality constraints to ensure that such functions satisfy the properties of $\lambda^{-1}(\cdot)$, namely that it is increasing with uniformly bounded derivatives up to order $R+1$ on $[0, \overline{\rho}]$. See Zincenko (2018) for details and Chen (2007) for a survey on sieve estimation. When $M_L \propto L^\tau$, the author shows that

$\hat{\lambda}^{-1}(\cdot)$ converges at the rate $L^\tau/\log L$ uniformly on $[0,\overline{\rho}]$.[20] As in the GPV procedure, the value density $f(\cdot|\cdot)$ is then estimated by kernel using the estimated values from (5.2) upon plugging in $\hat{\lambda}^{-1}(\cdot)$, $\hat{G}(\cdot|I,x)$ and $\hat{g}(\cdot|I,x)$. Letting the bandwidth of this second step $h_f \propto (\log L/L^\tau)^{1/(R+2)}$ leads to a density estimator $\hat{f}(\cdot|\cdot)$ that converges uniformly at the rate $(L^\tau/\log L)^{(R+1)/(R+2)}$, where τ is defined above.

Identification and Estimation Using Ascending Auctions

A second nonparametric identification strategy exploits additional data. Lu and Perrigne (2008) combine ascending and first-price sealed-bid auctions provided there is no bidders' selection across the two mechanisms so that the distribution of bidders' values is the same across the two auction formats. From Athey and Haile (2002), the winning bid in an ascending auction, which is the second-highest private value, identifies $F(\cdot|x,I)$ and hence its quantile function $v(\cdot;x,I)$ on $[0,1]$. See Section 7.1. Since (5.3) gives

$$v(\alpha;x,I) = b(\alpha;x,I) + \lambda^{-1}\left(\frac{1}{I-1}\frac{\alpha}{g[b(\alpha;x,I)|x,I]}\right) \equiv b(\alpha;x,I) + \lambda^{-1}[\rho_I(\alpha;x)],$$

for $\alpha \in [0,1]$, it follows that $\lambda^{-1}(\cdot)$ is identified on the range of $\rho_I(\alpha;x)$ when $(\alpha,x,I) \in [0,1]\times\mathcal{S}_X\times\mathcal{S}_I$.

Following the identification argument, one estimates the value distribution by $\hat{F}(\cdot|X,I)$ using the winning bids from the ascending auctions. Because they are based on observed second-highest order statistics, $\hat{F}(\cdot|\cdot,I)$ and its value density estimator $\hat{f}(\cdot|\cdot,I)$ converge uniformly at Stone (1982) optimal rates $(L/\log L)^{(R+1)/(2R+d+2)}$ and $(L/\log L)^{R/(2R+d+1)}$ with bandwidths proportional to $(L/\log L)^{1/(2R+d+2)}$ and $(L/\log L)^{1/(2R+d+1)}$, respectively on inner compact subsets of their supports. See Section 7.1. Exploiting $F(v|X,I) = G(b|X,I)$, one then obtains estimated private values $\hat{V}_{i\ell}$ from observed bids $B_{i\ell}$ in the first-price sealed-bid auctions as $\hat{V}_{i\ell} = \hat{F}^{-1}[\hat{G}(B_{i\ell}|X_\ell,I_\ell)|X_\ell,I_\ell]$, where $\hat{G}(\cdot|X,I)$ is the kernel estimator of Section 3.3. Using (5.2), the estimator $\hat{\lambda}^{-1}(\cdot)$ is defined as an $R+1$ continuously differentiable function with uniformly bounded derivatives satisfying

$$\hat{\lambda}^{-1}\left(\frac{1}{(I_\ell-1)\hat{g}(B_{i\ell}|X_\ell,I_\ell)}\right) = \hat{V}_{i\ell} - B_{i\ell},$$

for $i=1,\ldots,I_\ell$ and $\ell = 1,\ldots L$ where $\hat{g}(B_{i\ell}|X_\ell,I_\ell)$ is the estimated bid density of Section 3.3 from the first-price sealed-bid auctions. An interesting property of this estimator is

[20] As Zincenko (2018) points out, this rate can be improved. For instance, upon noting that $\rho_I(\alpha;x) = \alpha b'(\alpha;x,I)/(I-1)$, we can use the AQR estimators $\hat{b}(\cdot;x,I)$ and $\hat{b}'(\cdot;x,I)$ of Gimenes and Guerre (2021) in $\hat{Q}_L(\lambda^{-1})$ which converge at the fast rate $(L/\log L)^{(R+1)/(2R+d+3)}$ uniformly on $[0,1]$. See Section 3.4. Moreover, $\hat{Q}_L(\lambda^{-1})$ and hence $\hat{\lambda}^{-1}(\cdot)$ actually depend on x so that τ decreases with the dimension d of X. Averaging over X_ℓ can improve the convergence rate of $\hat{\lambda}^{-1}(\cdot)$ thereby eliminating the curse of dimensionality for estimating $\lambda^{-1}(\cdot)$ and hence $U(\cdot)$.

its fast convergence rate. Lu and Perrigne (2008) show that $\hat{\lambda}^{-1}(\cdot)$ converges uniformly at the rate $(L/\log L)^{(R+1)/(2R+d+3)}$, i.e., the rate for estimating the bid density.

Semiparametric Identification and Estimation

Another identification strategy consists in looking for a mininal parameterization of the structure $[U(\cdot), F(\cdot)]$. Campo, Guerre, Perrigne and Vuong (2011) show that parameterizing either the value distribution as $F(\cdot|I;\beta)$ or the utility function as $U(\cdot;\gamma)$ is not sufficient to identify the model structure. Moreover, they show that up to some smoothness conditions any distribution $G(\cdot|I)$ can be rationalized by either some CRRA utility or Constant Absolute Risk Aversion (CARA) utility despite their different economic implications thereby strengthening Guerre, Perrigne and Vuong (2009) rationalization result. For instance, using an argument as in the general proof of nonidentification, they show that two structures $[U(\cdot), F(\cdot)]$ and $[\tilde{U}(\cdot), \tilde{F}(\cdot)]$ with CRRA utility functions $U(\cdot;\gamma)$ and $U(\cdot;\tilde{\gamma})$, $\tilde{\gamma} > \gamma$, are observationally equivalent provided the value quantile functions satisfy $\tilde{v}(\cdot;I) < v(\cdot;I)$. That is, an increase in risk aversion can be compensated by a decrease in all the value quantiles. This suggests to pin down a quantile of $F(\cdot|I)$. Semiparametric identification is then achieved by parameterizing (i) the utility function as $U(\cdot) = U(\cdot;\gamma)$ with $\gamma \in I\!\!R^p$ and (ii) a conditional quantile as $v(\alpha_0;x,I) = v(\alpha_0;x,I,\beta)$ with $\beta \in I\!\!R^q$ for some $\alpha_0 \in (0,1]$. One can choose a high degree polynomial for $v(\alpha_0;\cdot,\cdot,\beta)$ to allow for flexibility. Thus, expressing (5.3) at the α_0-quantile leads to

$$g[b(\alpha_0;x,I)|x,I] = \frac{1}{I-1}\frac{\alpha_0}{\lambda[v(\alpha_0;x,I,\beta) - b(\alpha_0;x,I);\gamma]} \equiv m(x,I;\beta,\gamma), \tag{5.5}$$

for $(x,I) \in \mathcal{S}_X \times \mathcal{S}_I$. Identification of (β,γ) is obtained from variations in (x,I) as the LHS of (5.5) is estimable from observed bids.

This semiparametric identification argument extends to reserve price, affiliated private values and asymmetry in values and/or preferences. As a matter of fact, asymmetry in preferences does not require parameterizing a conditional value quantile. Indeed, as first shown by Campo (2012), (5.3) implies that asymmetry in preferences with (say) two groups 0 and 1 leads to

$$b_0(\alpha;I)+\lambda_0^{-1}\left(\frac{1}{I-1}\frac{\alpha}{g_0[b_0(\alpha;I)|I]}\right) = b_1(\alpha;I)+\lambda_1^{-1}\left(\frac{1}{I-1}\frac{\alpha}{g_1[b_1(\alpha;I)|I]}\right), \tag{5.6}$$

for any $(\alpha,I) \in [0,1] \times \mathcal{S}_I$ using obvious notations, since both sides are equal to $v(\alpha;I)$ under symmetry in values. Though similar in spirit to the compatibility condition (5.4), which holds 'between' auctions as the competition level varies, the compatibility condition (5.6) holds 'within' each auction due to asymmetry in preferences. The author then shows that the model primitives $[U_0, U_1, F]$ are identified from observed bids when preferences (U_0, U_1) are both CRRA or both CARA without parameterizing any value quantile.

Regarding estimation, Campo, Guerre, Perrigne and Vuong (2011) propose a three-step procedure that relies on (5.2) and (5.5). The semiparametric model is given by $U(\cdot;\gamma)$ whereas a quantile of $F(\cdot|X, I)$ is parameterized as $v(\alpha_0; X, I, \beta)$ for some chosen $\alpha_0 \in (0, 1]$. The first step estimates nonparametrically the bid quantile $b(\alpha_0; x, I)$ and bid density $g[b(\alpha_0|x, I); x, I)]$ for $(x, I) = (X_\ell, I_\ell)$, $\ell = 1, \ldots, L$. The second step uses (5.5) upon plugging in the estimators of $b(\alpha_0; X_\ell, I_\ell)$ and bid density $g[b(\alpha_0|X_\ell, I_\ell); X_\ell, I_\ell)]$ to estimate (β, γ) by Weighted Nonlinear Least Squares (WNLLS). The third step estimates the value density using the pseudo values from (5.2) allowing for the covariates X and replacing $\lambda^{-1}(\cdot)$ by $\lambda^{-1}(\cdot; \hat{\gamma})$ with $\hat{\gamma}$ being the estimator of the utility parameters from the second step. The authors consider the case $\alpha_0 = 1$ and parameterize the upper value boundary as a constant, i.e., $\overline{v}(x, I; \beta) = \beta$.[21] Thus, the first step estimates $\overline{b}(x, I)$ and bid density $g[\overline{b}(x, I); x, I)]$ at the upper boundary. For the former, they use the boundary estimator proposed by Korostelev and Tsybakov (1993) and presented in Section 3.3. For the latter, they use a one-sided kernel density estimator $K_0(u) = (6u + 4)\,\mathbb{1}(-1 \leq u \leq 0)$ because of boundary effects.

Turning to the second step, (5.5) and the form of the kernel density estimator of $g[\overline{b}(X_\ell, I_\ell); x_\ell, I_\ell)]$ suggest to consider the nonlinear regression

$$Y_{i\ell} = m(X_\ell, I_\ell; \beta, \gamma) + e_{i\ell} + \epsilon_{i\ell}, \tag{5.7}$$

where $Y_{i\ell} = (1/h_L)K_0[(B_{i\ell} - \overline{b}(X_\ell, I_\ell))/h_L]$ with h_L the bandwidth in the first step, $m(x, I; \beta, \gamma) = [1/(I-1)] \times [1/\lambda(\beta - \overline{b}(x, I); \gamma)]$, $e_{i\ell} \equiv \mathrm{E}[Y_{i\ell}|X_\ell, I_\ell]$ a vanishing bias and $\epsilon_{i\ell} \equiv Y_{i\ell} - \mathrm{E}[Y_{i\ell}|X_\ell, I_\ell]$ an error term. The parameters (β, γ) are then estimated by weighted NLLS upon replacing $\overline{b}(X_\ell, I_\ell)$ by $\hat{\overline{b}}(X_\ell, I_\ell)$ in $Y_{i\ell}$ and $m(X_\ell, I_\ell; \beta, \gamma)$. Unlike standard nonlinear regression, however, the variance of $\epsilon_{i\ell}$ diverges with L thereby affecting the convergence rate of $(\hat{\beta}, \hat{\gamma})$. Indeed, the authors show that $(\hat{\beta}, \hat{\gamma})$ is asymptotically normal with convergence rate $L^{(R+1)/(2R+3)}$ when $h_L \propto (1/L)^{1/(2R+3)}$. This contrasts with semiparametric estimators converging at the $\sqrt{L}$-rate surveyed by Newey and McFadden (1994) and Powell (1994). It is worthnoting that the rate $L^{(R+1)/(2R+3)}$ is independent of the dimension d of the covariates X thereby avoiding the curse of dimensionality. This follows because WNLLS on (5.7) averages over the covariates X thereby eliminating their dimension. Moreover, the authors show that the optimal minimax rate for estimating

[21] They also indicate how their asymptotic results can extend to the case when $\alpha_0 \in (0, 1)$ with a more flexible value quantile specification $v(\alpha_0; x, I, \beta)$.

(β, γ) is $L^{(R+1)/(2R+3)}$, which is that for estimating a univariate density that is $R+1$ continuously differentiable. Thus, the estimators $(\hat{\beta}, \hat{\gamma})$ achieve the optimal rate. The third step uses (5.2) as in the GPV procedure with $\lambda^{-1}(\cdot)$ replaced by $\lambda^{-1}(\cdot; \hat{\gamma})$. The resulting estimator of the value density $\hat{f}(\cdot|\cdot, I)$ converges uniformly at the rate $(L/\log L)^{R/(2R+d+3)}$ when $h \propto (\log L/L)^{1/(2R+d+3)}$ as in Section 3.3.

Auctions with Ex Post Uncertainty

Uncertainty ex post the auction is an inherent feature in many auction situations. This uncertainty has been integrated in auctions through a common value unknown at the time of the auction and affecting equally all bidders. See Sections 2.1 and 8. Alternatively, one can view uncertainty affecting either bidders' expost payments or private values. See Section 6.3 for the former with scale auctions. Here, we consider ex post uncertainty in private values. Up to now, each bidder perfectly knows his private value. Eso and White (2004) introduce a random noise affecting bidders' values ex post, i.e., after the auction takes place. This random noise is a pure risk on which bidders do not have ex ante information in contrast to common value models. Though it has no effect and can be omitted with risk neutral bidders, this additional uncertainty affects bidding when bidders are risk averse. Luo, Perrigne and Vuong (2018a) define the ex post value as $V_i + \epsilon_i$, where the ex ante private values V_i are i.i.d. and ϵ_i is a zero mean shock realized ex post distributed as $H(\cdot)$ independently of V_i. To simplify, bidders are symmetric in their values but have heterogeneous preferences $[U_0(\cdot), U_1(\cdot)]$ with (say) two groups 0 and 1. See Campo (2012) and Luo, Perrigne and Vuong (2021) for empirical justifications for such an heterogeneity in terms of bidders' financial constraints and experience. Thus the model primitives are $[U_0(\cdot), U_1(\cdot), F(\cdot), H(\cdot), I_0, I_1]$ with $U_j(\cdot)$ the utility of group j, $F(\cdot|I_0, I_1)$ the value distribution and $I_j \geq 1$ the number of type j bidders for $j = 1, 2$.[22]

The ex post uncertainty introduces a risk premium π_j for each group $j = 1, 2$ defined as $\mathrm{E}[U_j(\pi_j + \epsilon_i)] = 0$. Since $U_j(0) = 0$, the risk premium can be interpreted as the minimum markup ensuring participation of bidders in group j. Hereafter, $\pi_1 > \pi_0 \geq 0$ so that type 1 bidders are more risk averse than type 0 bidders. The risk premia (π_0, π_1) lead to special boundary conditions given by the authors using results by Maskin and Riley (2003). In particular, $\underline{v}(I_0, I_1) + \pi_0 \leq s_0[\underline{v}(I_0, I_1)] = s_1[v(\underline{\alpha}; I_0, I_1)] = v(\underline{\alpha}; I_0, I_1) + \pi_1$ for some percentile $\underline{\alpha} \equiv \underline{\alpha}(I_0, I_1) \in (0, 1)$, where $s_j(\cdot) \equiv s_j(\cdot; U_0, U_1, F, H, I_0, I_1)$ denotes

[22] When the model is fully symmetric, i.e., when bidders share the same utility function $U(\cdot)$, the model is not identified without addtional observations on realized ex post values as shown by Lu (2004). Luo, Perrigne and Vuong (2018a) also discuss the case where asymmetry can arise from different value distributions $[F_0(\cdot), F_1(\cdot)]$ as well as when there is endogenous entry *a la* Levin and Smith (1994). Moreover, the shock distribution can differ across groups leading to $[H_0(\cdot), H_1(\cdot)]$.

the symmetric Bayesian Nash equilibrium strategy for bidders in group j and $\underline{v}(I_0, I_1)$ is the lower boundary of the support of $F(\cdot|I_0, I_1)$. Whereas bidders in group 0 always participate, bidders in group 1 are screened out whenever their values fall below the threshold $v(\underline{\alpha}; I_0, I_1)$. Hence, $v(\underline{\alpha}; I_0, I_1)$ acts as a reserve price for type 1 bidders. See Section 4.1. Thus the number of actual participants I_1^* follows a Binomial distribution with parameters $(I_1, 1-\underline{\alpha})$ where $\underline{\alpha} = \underline{\alpha}(I_0, I_1)$ typically depends on (I_0, I_1).

Bidder i's expected profit is $\Pi_i = \overline{U}_{j(i)}(V_i - B_i)\Pr(B_i \geq B_{-i}^{\max}|I_0, I_1)$ where $j(i)$ refers to bidder i's group, $\overline{U}_j(\cdot) \equiv \mathrm{E}_\epsilon[U_j(\cdot + \epsilon_i)]$ and $\mathrm{E}_\epsilon[\cdot]$ indicates expectation with respect to ϵ_i. We can view the expected utility $\overline{U}_j(\cdot)$ as a utility function since it is increasing and concave. Thus from Maskin and Riley (2000a, 2000b), the equilibrium strategies $[s_0(\cdot), s_1(\cdot)]$ exist and solve a pair of differential equations. In general, the latter has no explicit solutions as is expected in asymmetric auctions. In contrast, using a similar argument as in (4.2), (4.6) and (5.2), these equations can be rewritten in indirect form. Let $G_0(\cdot|I_0, I_1)$ and $G_1^*(\cdot|I_0, I_1) \equiv \Pr[B_1 \leq \cdot|V_1 \geq v(\underline{\alpha}; I_0, I_1)]$ be the bid distributions for a type 0 bidder and a type 1 bidder conditional on participating with respective densities $g_0(\cdot|I_0, I_1)$ and $g_1^*(\cdot|I_0, I_1)$. Adapting Luo, Perrigne and Vuong (2018a) who consider procurements gives

$$V_0 = B_0 + \overline{\lambda}_0^{-1}\left(\frac{1}{(I_0-1)\frac{g_0(B_0|I_0,I_1)}{G_0(B_0|I_0,I_1)} + I_1\frac{(1-\underline{\alpha})g_1^*(B_0|I_0,I_1)}{\underline{\alpha}+(1-\underline{\alpha})G_1^*(B_0|I_0,I_1)}}\right) \equiv \xi_0(B_0; \overline{U}_0, G_0, G_1^*, I_0, I_1, \underline{\alpha}), \tag{5.8}$$

$$V_1 = B_1 + \overline{\lambda}_1^{-1}\left(\frac{1}{I_0\frac{g_0(B_1|I_0,I_1)}{G_0(B_1|I_0,I_1)} + (I_1-1)\frac{(1-\underline{\alpha})g_1^*(B_1|I_0,I_1)}{\underline{\alpha}+(1-\underline{\alpha})G_1^*(B_1|I_0,I_1)}}\right) \equiv \xi_1(B_1; \overline{U}_1, G_0, G_1^*, I_0, I_1, \underline{\alpha}),$$

where (V_j, B_j) are the value and bid for a generic participating bidder from group j and $\overline{\lambda}_j(\cdot) \equiv \overline{U}_j(\cdot)/\overline{U}_j'(\cdot)$ for $j = 0, 1$.

The system (5.8) leads to compatibility conditions within auctions in the spirit of (5.6) arising from heterogeneous preferences. Let $b_0(\cdot; I_0, I_1)$ and $b_1^*(\cdot; I_0, I_1)$ be the quantiles of $G_0(\cdot|I_0, I_1)$ and $G_1^*(\cdot|I_0, I_1)$, respectively. Because $v(\alpha; I_0, I_1) = s_1^{-1}[b_1^*(\alpha^*; I_0, I_1)]$ for $\alpha^* \equiv (\alpha - \underline{\alpha})/(1-\underline{\alpha})$, we have

$$b_0(\alpha; I_0, I_1) + \overline{\lambda}_0^{-1}[\rho_{0I_0I_1}(\alpha)] = b_1^*(\alpha^*; I_0, I_1) + \overline{\lambda}_1^{-1}[\rho_{1I_0I_1}(\alpha^*)], \tag{5.9}$$

for every $\alpha \in [\underline{\alpha}, 1]$, where $\rho_{0I_0I_1}(\alpha)$ and $\rho_{1I_0,I_1}(\alpha^*)$ are the arguments of $\overline{\lambda}_0^{-1}(\cdot)$ and $\overline{\lambda}_1^{-1}(\cdot)$ in (5.8) written in quantile forms, i.e., with $[B_0, B_1]$ replaced by $[b_0(\alpha; I_0, I_1), b_1^*(\alpha^*; I_0, I_1)]$. As a matter of fact, (5.9) is part of the restrictions imposed by the model with heterogeneous preferences and ex post uncertainty. See Luo, Perrigne and Vuong (2018a) for the characterization of the testable restrictions imposed by such a model.

Regarding identification and estimation, the observations are the bids of participants, the number of participants from each type, and the auction characteristics X for each auction, i.e., $\{\{B_{0i\ell}\}_{i=1}^{I_{0\ell}}, \{B_{1i\ell}\}_{i=1}^{I_{1\ell}^*}, I_{0\ell}, I_{1\ell}^*, X_\ell, \ell = 1, \ldots, L\}$. Since $I_{1\ell}^* \sim \mathcal{B}(I_{1\ell}, 1 - \underline{\alpha}_\ell)$ where $\underline{\alpha}_\ell \equiv \alpha(X_\ell, I_{0\ell}, I_{1\ell})$, one can identify and estimate $I_{1\ell}$ and $\underline{\alpha}_\ell$ by an argument similar to that in Section 4.1 upon assuming that $I_{1\ell}$ is constant within subsets of auctions. The authors then consider two cases depending on whether the level of potential competition (I_0, I_1) is exogenous or endogenous. When (I_0, I_1) is exogenous, i.e., $F(\cdot|I_0, I_1) = F(\cdot)$, they follow Guerre, Perrigne and Vuong (2009) and establish the identification of the expected utility functions $[\overline{U}_0(\cdot), \overline{U}_1(\cdot)]$ and the value distribution $F(\cdot)$ up to location given by the risk premium π_0 of the least averse group. As in (5.4) the argument exploits variations from (I_0, I_1) to $(\tilde{I}_0, \tilde{I}_1)$ whereas the quantiles of $F(\cdot)$ remain constant thereby leading to the compatibility conditions between auctions

$$b_0(\alpha; I_0, I_1) + \overline{\lambda}_0^{-1}[\rho_{0I_0I_1}(\alpha)] = b_0(\alpha; \tilde{I}_0, \tilde{I}_1) + \overline{\lambda}_0^{-1}[\rho_{0\tilde{I}_0\tilde{I}_1}(\alpha)], \tag{5.10}$$

for every $\alpha \in [0, 1]$.[23] One can then estimate nonparametrically the expected utility $\overline{U}_0(\cdot)$ up to π_0 from the between compatibility conditions (5.10) using Zincenko (2018) method or the authors' sieve minimum L_2-distance method. Identification and estimation (up to π_0) of $\overline{\lambda}_1^{-1}(\cdot)$ and $F(\cdot)$ exploit the within compatibility condition (5.9) and the inverse equilibrium strategies (5.8), respectively. In contrast, $[U_0(\cdot), U_1(\cdot), H(\cdot)]$ are not identified from the knowledge of $[\overline{U}_0(\cdot), \overline{U}_1(\cdot)]$ since $\overline{U}_j(\cdot) = \int U_j(\cdot + \epsilon) dH(\epsilon)$ for $j = 0, 1$.

When (I_0, I_1) is endogenous, a parameterization of $[U_0(\cdot), U_1(\cdot), H(\cdot)]$ achieves identification despite $F(\cdot|I_0, I_1)$ being unspecified. Intuitively, this is so because the continuum of within compatibility conditions (5.9) as α varies in $[\underline{\alpha}, 1]$ identifies the finite number of parameters in $\overline{\lambda}_j^{-1}(\cdot)$, $j = 0, 1$, arising from the parameterization of $[U_0(\cdot), U_1(\cdot), H(\cdot)]$. Once $[\overline{\lambda}_0^{-1}(\cdot), \overline{\lambda}_1^{-1}(\cdot)]$ are identified, nonparametric identification of $F(\cdot|I_0, I_1)$ follows from (5.8) as usual. Luo, Perrigne and Vuong (2018a) then propose a minimum distance estimator based on the sum of squared differences between the LHS and RHS of (5.9) evaluated at a large number of quantiles. Estimation of the value density $f(\cdot|I_0, I_1)$ is kernel-based using the estimated values from (5.8) as in the second step of the GPV procedure.

Section 5.2: Bidders' Entry

Up to now, we have considered that all potential bidders participate to the auction (though

[23] The main difference between (5.4) and (5.10) is that $\lambda^{-1}(0) = 0$ in the former while $\overline{\lambda}_0^{-1}(0) = \pi_0$, which is unknown thereby leading to the identification of $\overline{\lambda}_0^{-1}(\cdot)$ up to π_0. Also, combining (5.9) and (5.10) gives similar between compatibility conditions for group 1, namely, $b_1^*(\alpha^*; I_0, I_1) + \overline{\lambda}_1^{-1}[\rho_{1I_0I_1}(\alpha^*)] = b_1^*(\tilde{\alpha}^*; \tilde{I}_0, \tilde{I}_1) + \overline{\lambda}_1^{-1}[\rho_{1\tilde{I}_0\tilde{I}_1}(\tilde{\alpha}^*)]$ for every $\alpha \in [\max\{\underline{\alpha}, \underline{\tilde{\alpha}}\}, 1]$ where $\tilde{\alpha}^* = (\alpha - \underline{\tilde{\alpha}})/(1 - \underline{\tilde{\alpha}})$ and $\underline{\tilde{\alpha}} = \underline{\alpha}(\tilde{I}_0, \tilde{I}_1)$.

some may not actually bid when the reserve price is binding as seen in Section 4.1). In reality not all potential bidders participate. In most data, we observe a large number of bidders' identities but a comparatively small number of participants in each auction. For instance, the number of planholders who register to obtain project specifications in procurement auctions is larger than the number of firms submitting bids. Auction theorists account for the discrepancy between the number of potential bidders and the number of participating bidders through an entry game taking place before the auction. Indeed, assessing the value of an auctioned object (or the cost for executing a project in the case of procurements) is a costly process requiring the analysis of documents or a potential visit to inspect the auctioned good. In addition to a monetary cost, this entry cost can include an opportunity cost. Each potential bidder then evaluates his expected net profit from participating to the auction and depending on it being positive decides whether to enter the auction. We review below the econometrics issues associated with bidders' entry in the benchmark model. Hereafter, participating bidders are the potential bidders who enter the auction, i.e., the entering bidders. The latter may not bid if there is a binding reserve price. Those who actually bid are referred as the actual bidders as in Section 4.1.

Entry Models

A general framework is the selective entry model developed by Marmer, Shneyerov and Xu (2013). The model draws its structure from Ye (2007) and involves two stages. Instead of a first stage of indicative bidding, the first stage of the selective entry model is an entry game. Each bidder i receives a private signal σ_i about his value V_i which is unknown in the entry stage. The pairs (σ_i, V_i) are i.i.d. across the $I \geq 2$ potential bidders whereas σ_i and V_i are potentially dependent. Specifically, the value V_i is distributed conditionally on the signal σ_i as $F(\cdot|\sigma_i, I)$ with support $[\underline{v}, \overline{v}]$. As in previous sections, we allow $F(\cdot|\sigma_i, I)$ to depend on the potential number of bidders I and to simplify, we let its support $[\underline{v}, \overline{v}]$ be independent of (σ_i, I). The conditional distribution $F(\cdot|\sigma_i, I)$ also satisfies a first-order stochastic dominance. Namely, $F(\cdot|\tilde{\sigma}, I) \leq F(\cdot|\sigma, I)$ whenever $\tilde{\sigma} \geq \sigma$, which is known as the 'good news' property. Bidders incur an entry cost κ and become aware of their private values V_i after entry. Only entering bidders can bid in the second stage, which is a first-price sealed-bid auction. Though they know the number I of potential bidders, entering bidders do not know how many enter the bidding stage. Marmer, Shneyerov and Xu (2013) also allow for a reserve price $p_0 \in [0, \overline{v})$. Below we present the case when $p_0 = \underline{v}$ following Kong (2018). Thus selection arises only from players choosing to enter as the number of entering/participating bidders is equal to the number of actual bidders I^*.

Moreover, because σ_i is not observed, we normalize its distribution to be $\mathcal{U}[0,1]$ following Gentry and Li (2014). The latter consider other standard auction mechanisms than a first-price sealed-bid auction in the second stage. See Section 7.2.

The two-stage game of entry and bidding is solved using the concept of perfect Bayesian equilibrium. Let σ^* denote the equilibrium signal threshold above which a potential bidder enters and does not otherwise. Thus the probability that a potential bidder enters the auction is $(1-\sigma^*)$ because $\sigma_i \sim \mathcal{U}[0,1]$. Starting from the second stage, i.e., the first-price sealed-bid auction, each entering bidder knows his private value V_i though he is uncertain about the number of potential bidders who enter. Such a situation is analyzed by Harstad, Kagel and Levin (1990) in a general setting with interdependent values. Let $s(\cdot;I)$ denote the (symmetric) increasing bidding strategy. With independent private values and entry probability $(1-\sigma^*)$, bidder i's expected profit is $(V_i - B_i)\tilde{F}^{I-1}[s^{-1}(B_i;I)|I]$ where $\tilde{F}(\cdot|I) \equiv \sigma^* + (1-\sigma^*)F(\cdot|\sigma_i \geq \sigma^*, I)$. In particular, $\tilde{F}[s^{-1}(B_i;I)|I]$ is the probability that a competing bidder does not enter or that he enters with a value V_j smaller than $s^{-1}(B_i;I)$ (or bid $B_j = s(V_j;I) \leq B_i$). The distribution $F(\cdot|\sigma_i \geq \sigma^*, I)$ is called the 'value distribution conditional on entry'. Marmer, Shneyerov and Xu (2013) derive the differential equation for the equilibrium bidding strategy $s(\cdot;I)$. Because $p_0 = \underline{v}$, the boundary condition remains $s(\underline{v};I) = \underline{v}$. Solving for $s(\cdot;I)$ gives

$$s(V_i;I) = V_i - \frac{1}{\tilde{F}(V_i|I)^{I-1}} \int_{\underline{v}}^{V_i} \tilde{F}(v|I)^{I-1}\, dv, \tag{5.11}$$

for $V_i \in [\underline{v}, \overline{v}]$. This strategy is similar to that of the benchmark model with the difference that $\tilde{F}(\cdot|I)$ has a mass point at $\underline{v}$.

In the first stage, i.e., the entry game, each bidder enters the auction if and only if his expected profit from the bidding stage net of the entry cost is positive given his signal σ_i and the number of potential bidders I. Setting this expected net profit to zero for the marginal bidder with signal σ^* defines the equilibrium threshold $\sigma^* = \sigma^*(I)$. Namely,

$$\kappa = \int_{\underline{v}}^{\overline{v}} [1 - F(v|\sigma^*, I)][\sigma^* + (1-\sigma^*)F(v|\sigma_i \geq \sigma^*, I)]^{I-1} dv. \tag{5.12}$$

See Marmer, Shneyerov and Xu (2013). Because a high signal likely corresponds to a high value, entering bidders tend to have large values thereby leading to selective entry. The model primitives are the conditional distribution $F(\cdot|\cdot, I)$ and the entry cost κ.

Though developed earlier, two models are polar cases of the selective entry model. The earliest model was proposed by Levin and Smith (1994) in which signals and private values are independent. Thus signals are uninformative and $F(\cdot|\sigma_i, I) \equiv F(\cdot|I)$. Second stage bidding follows (5.11) with $\tilde{F}(\cdot|I) = \sigma^* + (1-\sigma^*)F(\cdot|I)$.[24] In the entry game, the I potential bidders choose to enter the auction with no information about their private values. In equilibrium, the first stage defines a bidder's entry probability $(1-\sigma^*)$ from (5.12) with the conditioning on σ^* and $\sigma_i \geq \sigma^*$ removed. There is no selective entry as entering bidders have the same value distribution as non-entering bidders. Samuelson (1985) develops another model in which values and signals are perfectly dependent. Thus, each potential bidder perfectly knows his private value V_i in the entry stage. This alternative model corresponds to the limit case where $V_i = F^{-1}(\sigma_i|I)$ for some value distribution $F(\cdot|I)$. Second stage bidding proceeds as in (5.11) with $\tilde{F}(\cdot|I) = F(\cdot|I)$, v^* instead of $\underline{v}$, and an additional term $p_0 - v^*$ due to the boundary condition $s(v^*) = p_0$. The value threshold v^* solves the expected profit condition $\kappa = (v^* - p_0)F^{I-1}(v^*|I)$ where $p_0 = \underline{v}$. Because bidders who enter have valuations higher than v^*, the model exhibits a selection effect as in the selective entry model.

Identification

The analyst observes the number and bids of actual bidders as well as a vector X of exogenous characteristics. The latter play a role in the identification of the model primitives. The number of potential bidders I is observed or recovered using data-driven arguments from geographic location, etc. See Kong (2020b). Alternatively, I can be unknown but constant across auctions following Section 4.1. Adopting the indirect approach, Marmer, Shneyerov and Xu (2013) show that the inverse equilibrium strategy in the selective entry model satisfies

$$V_i = B_i + \frac{1}{I-1}\left(\frac{G^*(B_i|X,I)}{g^*(B_i|X,I)} + \frac{\sigma^*(X,I)}{1-\sigma^*(X,I)}\frac{1}{g^*(B_i|X,I)}\right), \tag{5.13}$$

where $\sigma^*(X,I)$ is the equilibrium threshold of the entry game. The authors note the similarity of (5.13) with the inverse equilibrium strategy (4.2) when there is an announced binding reserve price p_0. Indeed, $\sigma^* = \sigma^*(X,I)$ and $F(\cdot|\sigma_i \geq \sigma^*, X, I)$ play the same role as $F(p_0|X,I)$ and $F(\cdot|X,I)$, respectively, though a difference is that (5.13) holds for $V_i \in [\underline{v}, \overline{v}]$, while (4.2) holds for $V_i \in [p_0, \overline{v}]$. Thus, following Section 4.1, the threshold $\sigma^*(X,I)$ is identified as well as the value distribution conditional on entry $F(\cdot|\sigma_i \geq \sigma^*, X, I)$ on $[\underline{v}, \overline{v}]$. In general, this is insufficient to identify the primitive $F(\cdot|\cdot, X, I)$ on $[\underline{v}, \overline{v}] \times [0,1]$.

[24] As a matter of fact, Levin and Smith (1994) assume that the number of entrants is revealed to entering bidders after the entry stage and before the auction stage. Hence, second stage bidding proceeds as in the benchmark model of Section 2.3 with the number of entrants as the level of competition. The entry threshold is obtained by setting bidder's expected profit to zero where expectation is taken over all possible values of competing entrants. For empirical implementations of this model, see Li (2005), Li and Zheng (2009, 2012), Athey, Levin and Seira (2011) and Krasnokutskaya and Seim (2011) among others.

As a consequence, the expected profit for the marginal bidder with signal $\sigma^*(X,I)$ is not identified in (5.12). Thus the entry cost κ is not identified.[25]

To achieve identification of the selective entry model, Gentry and Li (2014) note that the value distribution conditional on entry satisfies $F(v|\sigma_i \geq \sigma^*, X, I) = \int_{\sigma^*}^1 F(v|s,X,I)ds/(1-\sigma^*)$ for $v \in [\underline{v}, \overline{v}]$ since $\sigma_i \sim \mathcal{U}[0,1]$. Thus, differentiating with respect to σ^* gives

$$F(v|\sigma^*, X, I) = -\frac{\partial[(1-\sigma^*)F(v|\sigma_i \geq \sigma^*, X, I)]}{\partial \sigma^*}. \tag{5.14}$$

To disentangle variations in σ^* and X, they introduce an exclusion restriction. Namely, there exists a variable Z that affects entry but not the value distribution. Specifically, the value distribution satisfies $F(\cdot|\sigma_i, X, Z, I) = F(\cdot|\sigma_i, X, I)$ while the entry cost depends on (X, Z, I), i.e., $\kappa = \kappa(X, Z, I)$. Thus, Z plays the role of an instrumental variable. Hence, the identification of $\sigma^* = \sigma^*(X, Z, I)$ and $F(\cdot|\sigma_i \geq \sigma^*, X, I)$ on $[\underline{v}, \overline{v}]$ with (X, Z) replacing X in (5.13) identifies $F(\cdot|\cdot, X, I)$ on $[\underline{v}, \overline{v}] \times [0,1]$ by (5.14) provided the following full support condition holds: The range of the threshold $\sigma^*(X, Z, I)$ is $[0,1]$ when Z varies given (X, I). This requires Z to contain some continuous variables. With such a full support condition, the entry cost $\kappa(X, Z, I)$ is also identified by

$$\kappa(X, Z, I) = \int_{\underline{v}}^{\overline{v}} [1 - F(v|\sigma^*, X, I)][\sigma^* + (1-\sigma^*)F(\cdot|\sigma_i \geq \sigma^*, X, I)]^{I-1} dv, \tag{5.15}$$

which follows from (5.12).

In the absence of the full support condition, (5.14) shows that $F(\cdot|\sigma^*, X, I)$ is identified on the interior $\mathcal{S}^o_{\sigma^*}(X, I)$ of the range $\mathcal{S}_{\sigma^*}(X, I)$ of $\sigma^*(X, \cdot, I)$ and by continuity on the closure $\overline{\mathcal{S}}^o_{\sigma^*}(X, I)$ of this interior. For signal values $\sigma \in [0,1] \backslash \overline{\mathcal{S}}^o_{\sigma^*}(X, I)$ the conditional value distribution $F(\cdot|\sigma, X, I)$ is partially identified. Specifically, by the mean value theorem and the first-order stochastic dominance of $F(\cdot|\sigma, X, I)$ in σ, one obtains bounds for $F(\cdot|\sigma, X, I)$ from the inequalities

$$\frac{1}{\sigma''_- - \sigma'_-} \int_{\sigma'_-}^{\sigma''_-} F(\cdot|u, X, I)du \geq F(\cdot|\sigma, X, I) \geq \frac{1}{\sigma''_+ - \sigma'_+} \int_{\sigma'_+}^{\sigma''_+} F(\cdot|u, X, I)du, \tag{5.16}$$

for every $\sigma \in [0,1]$, $(\sigma'_-, \sigma''_-) \in [0, \sigma]^2$ and $(\sigma'_+, \sigma''_+) \in [\sigma, 1]^2$ where $\sigma'_- < \sigma''_-$ and $\sigma'_+ < \sigma''_+$. Because $\int_{\sigma'}^{\sigma''} F(\cdot|u, X, I)du = (1-\sigma')F(\cdot|\sigma_i \geq \sigma', X, I) - (1-\sigma'')F(\cdot|\sigma_i \geq \sigma'', X, I)$ by (5.14), it follows that $\int_{\sigma'}^{\sigma''} F(\cdot|u, X, I)du/(\sigma'' - \sigma')$ is identified whenever $(\sigma', \sigma'') \in \mathcal{S}^2_{\sigma^*}(X, I)$. Thus, taking the infinimum of the LHS and the supremum of the RHS in (5.16)

[25] Regarding restrictions, when there is a reserve price $p_0 > \underline{v}$, Marmer, Shneyerov and Xu (2013) show that the selective entry model and the benchmark model with a binding reserve price p_0 are observationally equivalent. Thus, both models impose the same restrictions on observables. See their Proposition 4 which also characterizes the restrictions imposed by the Samuelson's (1985) entry model.

over $\{(s'_-, s''_-) \in \mathcal{S}^2_{\sigma^*}(X,I) : s'_- < s''_- \leq s\}$, and $\{(s'_+, s''_+) \in \mathcal{S}^2_{\sigma^*}(X,I) : s \leq s'_+ < s''_+\}$, respectively, provides bounds for $F(\cdot|\sigma, X, I)$ (with the convention that the infinimum and supremum are equal to 1 and zero when these sets are empty). Gentry and Li (2014) derive such bounds by defining upper and lower neighbors of σ and taking limits. These bounds collapse to $F(\cdot|\sigma, X, I)$ when the latter is identified. The bounds on the entry cost $\kappa(X, Z, I)$ are then obtained from the zero expected profit condition (5.15) with the upper and lower bounds for $F(\cdot|\sigma^*, X, I)$ in lieu of $F(\cdot|\sigma^*, X, I)$.

Regarding Levin and Smith (1994) and Samuelson (1985) models, these polar cases of the selective entry model are identified without covariates X and hence without an exclusion restriction and a full support condition. In both models the primitive is the value distribution $F(\cdot|X, I)$. In the former, $F(\cdot|X, I)$ is identified as it is equal to the value distribution conditional on entry $F(\cdot|\sigma_i \geq \sigma^*, X, I)$ which is identified on $[\underline{v}, \overline{v}]$ in the selective entry model. Moreover, since σ^* is identified, the entry cost $\kappa(X, I)$ is identified from the zero expected profit condition (5.12). A similar argument applies to Samuelson (1984) model thereby showing that $F(\cdot|X, I)$ is identified on $[v^*, \overline{v}]$ since $F(\cdot|V_i \geq v^*, X, I) = [F(\cdot|X, I) - F(v^*|X, I)]/[1 - F(v^*|X, I)]$, where the probability $\sigma^*(X, I) = F(v^*|X, I)$ of not participating is identified. Furthermore, the entry cost $\kappa(X, I)$ is identified from the expected profit condition $\kappa(X, I) = (v^* - p_0)F^{I-1}(v^*|X, I)$ since $v^* = v^*(X, I)$ is identified from (5.13) evaluated at $B_i = p_0$.

Estimation

Empirical papers mostly consider the entry model by Levin and Smith (1994). Li (2005) adopts a direct approach and proposes a SMM estimator with a binding reserve price. Using the same approach, Li and Zheng (2009, 2012) propose a Bayesian Markov Chain Monte Carlo estimator. They also estimate the Samuelson (1985) model. Athey, Levin and Seira (2011) and Krasnokutskaya and Seim (2011) adopt an indirect approach upon parameterizing the bid distribution(s) and using ML and Generalized Method of Moments (GMM) estimation in the first step, respectively. Plugging the estimated primitives in the zero expected net profit entry conditions provides an estimate of $\kappa(X, I)$. Also adopting an indirect approach but in a nonparametric setting, Kong (2020a) considers asymmetry in bidders' values and risk aversion combining data from ascending and first-price sealed-bid auctions in the spirit of Lu and Perrigne (2008).

Marmer, Shneyerov and Xu (2013) focus on tests for discriminating among the three entry models. An estimation procedure for the selective entry model is as follows. In a first step, one estimates the density conditional on entry $f^*(\cdot|X, I) \equiv f(\cdot|\sigma_i \geq \sigma^*(x, I), x, I)$ based on (5.13). For instance, one can estimate the bid distribution and density $G^*(\cdot|x, I)$

and $g^*(\cdot|x,I)$ as in Section 3.3. One can also estimate the participation probability $1-\sigma^*(x,I)$ by a nonparametric regression as in Section 4.1. Thus one can recover private values from (5.13) and estimate $f^*(\cdot|X,I)$ by a kernel density estimator. In a second step, given Gentry and Li (2014) exclusion restriction and full support condition, one can use a plug-in estimator for the density $f(\cdot|\sigma,x,I)$ at $\sigma=\sigma^*(x,z,I)\in\overline{\mathcal{S}}^o_{\sigma^*}(x,I)=[0,1]$ since $f(v|\sigma^*,X,I)=-\partial[(1-\sigma^*)f^*(v|X,I)]/\partial\sigma^*$ which is obtained by differentiating (5.14) with respect to v. Regarding the entry cost, it suffices to plug the estimates in (5.15) to estimate the entry cost $\kappa(X,Z,I)$. In the absence of the full support condition, one can estimate the aformentioned bounds for $F(\cdot|\sigma,X,I)$ when $\sigma\in[0,1]\backslash\overline{\mathcal{S}}^o_{\sigma^*}(X,I)$ to bound the entry cost $\kappa(X,Z,I)$.

Extensions

The preceding identification results extend straightforwardly to a binding reserve price and asymmetry in values. A more complex extension allows for bidders' risk aversion in the selective entry model as studied by Gentry, Li and Lu (2017) and Kong (2018). With risk aversion, (5.13) in quantile form becomes

$$v^*(\alpha;x,I)=b^*(\alpha;x,I)+\lambda^{-1}\left[\frac{1}{I-1}\left(\alpha+\frac{\sigma^*(x,I)}{1-\sigma^*(x,I)}\right)\frac{1}{g^*[b^*(\alpha;x,I)|x,I]}\right], \tag{5.17}$$

for $\alpha\in[0,1]$, where $\lambda(\cdot)=U(\cdot)/U'(\cdot)$ while $v^*(\alpha;x,I)$ and $b^*(\alpha;x,I)$ are the α-quantiles of $F^*(\cdot|x,I)\equiv F(\cdot|\sigma_i\geq\sigma^*(x,I),x,I)$ and $G^*(\cdot|x,I)\equiv \Pr[B_i\leq\cdot|\sigma_i\geq\sigma^*(x,I),x,I]$, respectively. In contrast to Guerre, Perrigne and Vuong (2009), variations of I are now insufficient to identify $\lambda(\cdot)$ since the α-quantiles $v^*(\alpha;x,I)$ depend on I through the threshold $\sigma^*(x,I)$ even when I is exogenous. In addition to the latter, Gentry, Li and Lu (2017) require that some variables Z are excluded from the value distribution. Thus, $F(\cdot|\sigma_i,X,Z,I)=F(\cdot|\sigma_i,X)$ as in Gentry and Li (2014). Moreover, they assume the existence of values (I_1,I_2,z_1,z_2) with $I_1\neq I_2$ such that $\sigma^*(x,z_1,I_1)=\sigma^*(x,z_2,I_2)$. Thus $v^*(\alpha;x,z_1,I_1)=v^*(\alpha;x,z_2,I_2)$ for $\alpha\in[0,1]$ thereby restoring a compatibility condition similar to (5.4). Because $\sigma^*(x,z,I)$ is identified, a recursive argument *a la* Guerre, Perrigne and Vuong (2009) establishes the identification of $\lambda(\cdot)$ and hence of $F(\cdot|\sigma_i\geq\sigma^*(x,z,I),x)$. Point and/or partial identification of $F(\cdot|\sigma_i,X)$ and $\kappa(X,Z,I)$ then follows from the argument in Gentry and Li (2014).

When I is endogenous, Gentry, Li and Lu (2017) propose two semiparametric identification strategies. A first approach follows Campo, Guerre, Perrigne, and Vuong (2011) and parameterizes the utility function as $U(\cdot)=U(\cdot;\gamma)$ with $\gamma\in I\!R^p$. Since $\overline{v}$ is constant by assumption, evaluating (5.17) at $\alpha=1$ gives

$$g^*[\overline{b}(x,I)|x,I]=\frac{1}{(I-1)[1-\sigma^*(x,I)]}\frac{1}{\lambda[\overline{v}-\overline{b}(x,I);\gamma]}, \tag{5.18}$$

for $(x, I) \in \mathcal{S}_X \times \mathcal{S}_I$, where $\sigma^*(x, I)$, $g^*(\cdot|x, I)$ and $\bar{b}(x, I)$ are identified. This equation is similar to (5.5) with $(\alpha_0, \beta) = (1, \overline{v})$ thereby identifying $(\overline{v}, \gamma)$ from variations in (x, I). Because the value distribution conditional on entry $F(\cdot|\sigma_i \geq \sigma^*(X, I), X, I)$ is identified from its quantiles by (5.17), point and/or partial identification of $F(\cdot|\sigma_i, X, I)$ and $\kappa(X, Z, I)$ follow from Gentry and Li (2014) under the exclusion $F(\cdot|\sigma_i, X, Z, , I) = F(\cdot|\sigma_i, X, I)$.

A second strategy consists in parameterizing the copula of (V_i, σ_i) given (X, I), i.e., $C(\cdot, \cdot|X, I) = C(\cdot, \cdot|X, I; \beta)$ with $\beta \in I\!R^q$, while leaving the utility $U(\cdot)$ and the marginal value distribution $F(\cdot|X, I)$ free of parameterization. Gentry, Li and Lu (2017) note that the quantiles of $F(\cdot|X, I)$ and $F(\cdot|\sigma_i \geq \sigma^*, X, I)$ are related by $v(\alpha; x, I) = v^*(\alpha^*; x, I)$ where $\alpha^* = [\alpha - C(\alpha, \sigma^*; X, I, \beta)]/(1 - \sigma^*)$. Hence, under the exclusion $F(\cdot|X, Z, I) = F(\cdot|X, I)$, the LHS of (5.17) evaluated at α^* reduces to $v(\alpha; x, I)$, while the RHS varies with z given (x, I) through $\sigma^*(x, z, I)$ thereby leading to some compatibility conditions for every $\alpha \in [0, 1]$. In particular, two values of z can identify $\lambda(\cdot)$ following an argument similar to Guerre, Perrigne and Vuong (2009), while other values of z (over)identify β. Identification of $\lambda(\cdot)$ implies identification of the quantiles $v^*(\cdot; x, I)$ from (5.15) and hence of $v(\cdot; x, I)$ from the identification of β. The value distribution $F(\cdot|X, I)$ is identified as well as the entry cost $\kappa(X, Z, I)$ from (5.15) modified to account for risk aversion.

Regarding estimation in the nonparametric case with I exogenous, one can apply Zincenko (2018) sieve estimator of $\lambda^{-1}(\cdot)$ to the compatibility condition equating the RHS of (5.17) evaluated at (x, z_1, I_1) and (x, z_2, I_2). Given $\hat{\lambda}^{-1}(\cdot)$, one obtains estimators of $v^*(\cdot; X, Z, I)$ and $F(\cdot|\sigma_i \geq \sigma^*(X, Z, I), X)$ from (5.17). Estimation of $F(\cdot|\sigma_i, X)$ and $\kappa(X, Z, I)$ or of their bounds follows as in the risk neutral case. When I is endogenous and $U(\cdot)$ is parameterized, the only difference arises from the estimation of $\lambda(\cdot; \gamma)$ which can be achieved by WNLLS as in Campo, Guerre, Perrigne and Vuong (2011) in view of the similarity between (5.5) and (5.18). In the second semiparametric case where the copula of (V_i, σ_i) is parameterized as $C(\cdot, \cdot|X, I; \beta)$, Gentry, Li and Lu (2017) propose an estimation procedure based on Chen, Tamer and Torgovitsky (2011) who study semiparametric sieve likelihood inference in partially identified models. In particular, exploiting the identification of $\lambda(\cdot)$ and $F(\cdot|X, I)$ given β, they obtain a confidence set for β by inverting a chi-square likelihood ratio test whether β is point or partially identified.

Kong (2018) allows for asymmetry in both values and risk aversion in the preceding model, but exploits the availability of ascending and sealed-bid auctions as in Luo and Perrigne (2008). For simplicity, let $j = 0, 1$ index the asymmetry reduced to 2 groups and assume that the numbers of potential bidders (I_0, I_1) are exogenous. She invokes Nowik (1990) to establish identification of the value distribution conditional on entry

$F_j(\cdot|\sigma_i \geq \sigma_j^{o*}, X)$ as well as the equilibrium thresholds $\sigma_j^{o*} = \sigma_j^{o*}(X, Z, I_0, I_1)$ in the ascending auctions. The exclusion restriction and full support condition of Gentry and Li (2014) then identify $F_j(\cdot|\sigma_i, X)$. See Section 7.2. Identification of $\lambda_j(\cdot)$ follows from the inverse equilibrium strategies in the first-price sealed-bid auctions

$$\begin{aligned} v_0^*(\alpha) &= b_0^*(\alpha) + \lambda_0^{-1}\left[\frac{1}{(I_0-1)\frac{(1-s_0^*)g_0^*[b_0^*(\alpha)|x,z,I_0,I_1]}{\sigma_0^*+(1-\sigma_0^*)\alpha} + I_1\frac{(1-\sigma_1^*)g_1^*[b_1^*(\alpha)|x,z,I_0,I_1]}{\sigma_1^*+(1-\sigma_1^*)\alpha}}\right], \\ v_1^*(\alpha) &= b_1^*(\alpha) + \lambda_1^{-1}\left[\frac{1}{I_0\frac{(1-\sigma_0^*)g_0^*[b_0^*(\alpha)|x,z,I_0,I_1]}{\sigma_0^*+(1-\sigma_0^*)\alpha} + (I_1-1)\frac{(1-\sigma_1^*)g_1^*[b_1^*(\alpha)|x,z,I_0,I_1]}{\sigma_1^*+(1-\sigma_1^*)\alpha}}\right], \end{aligned} \tag{5.19}$$

for $\alpha \in [0, 1]$, where $v_j^*(\alpha) = v_j^*(\alpha; x, I_0, I_1)$ and $b_j^*(\alpha) = b_j^*(\alpha; x, z, I_0, I_1)$ are the α-quantiles of the value distributions conditional on entry $F_j(\cdot|\sigma_i \geq \sigma_j^*, x)$ and bid distributions $G_j^*(\cdot|x, z, I_0, I_1)$, respectively, whereas $\sigma_j^* = \sigma_j^*(x, z, I_0, I_1)$ are the equilibrium thresholds in such auctions for $j = 0, 1$. As above, entry costs $\kappa_j(x, z)$ for $j = 0, 1$ are identified from the zero expected profit conditions.[26]

After estimating the entry thresholds $(\sigma_j^{o*}, \sigma_j^*) = [\sigma_j^{o*}(X, Z, I_0, I_1), \sigma_j^*(X, Z, I_0, I_1)]$ for both auction mechanisms and groups $j = 0, 1$, Kong (2018) estimates $F_j(\cdot|\sigma_i, X)$ from the transaction prices and winners' group identities in ascending auctions conditional on $(\sigma_0^{o*}, \sigma_1^{o*})$ and covariates (X, Z) by sieve maximum likelihood. See Section 7.2. Next, she obtains estimates of $\lambda_j(\cdot)$, $j = 0, 1$ from (5.19) where $b_j^*(\alpha)$ and $v_j^*(\alpha)$ are estimated by the α-quantiles of $\hat{G}_j^*(\cdot|X, Z, I_0, I_1)$ and $\hat{F}_j(\cdot|\sigma_i \geq \sigma_j^*, X) = \int_{\hat{\sigma}_j^*}^1 \hat{F}_j(\cdot|s, X)ds/(1 - \hat{\sigma}_j^*)$ with $\hat{\sigma}_j^*$ the estimate of the threshold σ_j^*. In particular, the nonparametric estimate $\hat{G}_j^*(\cdot|X, Z, I_0, I_1)$ in the first-price sealed-bid auctions is obtained by sieve approximation based on Bernstein polynomials thereby providing an estimate of its density $\hat{g}_j^*(\cdot|X, Z, I_0, I_1)$ by differentiation. Lastly, estimates of entry costs follow from the zero expected profit conditions.

Section 5.3: Sequential Auctions

Up to now, we have assumed that private values, number of (potential) bidders and exogenous variables are independent across auctions. Consequently, each auction can be viewed separately and there are no dynamic considerations. This may not hold when objects are sold one at a time in a sequence, i.e., when auctions are sequential.[27]

[26] These costs need not be the same across the two auction mechanisms. As a matter of fact, Kong (2018) interprets $\kappa(X, Z)$ as the cost of bidding incurred after entry thereby leading to some minor changes in bidders' optimization problems.

[27] This subsection heavily draws from personal communications with Yunmi Kong.

Where Auction Theory Stands

A major impediment to the development of econometric methods for analyzing sequential auctions data lies in theory. In particular, auction theory is well-developed for single-unit demand, i.e., when each bidder wants at most one item/object and once he wins disappears from the auction. For instance, consider a sequence of T first-price sealed-bid auctions in which the winning price is announced after each auction. The T auctioned items are perfectly identical with $T < I$ and each bidder draws a single value in a symmetric IPV setting. The symmetric perfect Bayesian equilibrium leads to an efficient allocation in the sense that the bidders with the T highest values win the T objects. Namely, the bidder with the highest value wins the first item, the bidder with the second highest value wins the second item and so on until the T goods are sold. In equilibrium, the winning bids do not exhibit any trend as the declining value of the winner is exactly compensated by the increase in the bidding strategy along the sequence. Formally, the winning bid follows a martingale as shown by Milgrom and Weber (1999). With affiliated private values, Laffont, Loisel and Robert (1994) and Milgrom and Weber (1999) show an upward trend in the winning bids along the sequence. These theoretical results contrast with empirical evidence showing a downward trend of winning bids known as the "declining price anomaly" as reported by Ashenfelter (1989). In an effort to explain such a phenomenon, McAfee and Vincent (1993) introduce risk aversion that needs, however, to be increasing in wealth.

Auction data seldom satisfy the restrictive assumptions of this literature, i.e., identical objects and single-unit demand. Indeed, one often observes a same bidder buying several items along the sequence. This requires bidders to have multiple values, leading to multidimensional screening. Moreover, information about competitors' private information revealed after each auction can affect a bidder's strategy in subsequent auctions. Such a phenomenon, known as the ratchet effect, provides incentives to deceive leading to pooling in equilibrium, i.e., loss of the monotonicity of equilibrium strategies. More recently, Février (2003) and Bergemann and Horner (2018) show that monotonicity can be maintained under limited information disclosure along the sequence of auctions.

A Pragmatic Assumption

Given the state of auction theory with multi-object demand, unsurprisingly only few papers have estimated sequential first-price sealed-bid auctions. See Section 7.2 for sequential ascending auctions. A pragmatic assumption combined with the indirect approach provides insights to the challenging task of analyzing sequential auctions with multi-object demand. Specifically, Kong (2021) assumes that each bidder's values are revealed one at a time before each auction instead of being all known to the bidder before the sequence of auctions. To illustrate, consider a sequence of $T \geq 2$ auctions within a symmetric IPV setting. Thus, the T private values $\{V_{1i}, \ldots, V_{Ti}\}$ are independently and identically distributed across bidders $i = 1, \ldots, I > T$. Let $G_1(\cdot)$ denote the bid distribution in the first auction in a symmetric equilibrium omitting its dependence on I. When the first auction is a first-price sealed bid auction, the expected profit for bidder i with value V_{1i} and bid B_{1i} is $\pi_i = (V_{1i} - B_{1i} + \mathcal{V}_i^w) G_1(B_{1i})^{I-1} + \mathcal{V}_i^l[1 - G_1(B_{1i})^{I-1}]$, where $\mathcal{V}_i^w$ and $\mathcal{V}_i^l$ represent the continuation values when bidder i wins and loses the first auction, respectively. Differentiating π_i with respect to B_{1i} gives the first-order condition

$$V_{1i} \;=\; B_{1i} + \frac{1}{I-1}\frac{G_1(B_{1i})}{g_1(B_{1i})} - \frac{\partial\{\mathcal{V}_i^w G_1(B_{1i})^{I-1} + \mathcal{V}_i^l[1 - G_1(B_{1i})^{I-1}]\}/\partial B_{1i}}{dG_1(B_{1i})^{I-1}/dB_{1i}}, \quad (5.20)$$

where $g_1(\cdot)$ is the density of $G_1(\cdot)$. In the absence of a dynamic link across auctions, the third term vanishes and (5.20) reduces to the first-order condition (3.6). In this case, we recover the value distribution of V_{1i} from the first auctions in several sequences with I bidders as in the benchmark model. The difficulty arises when there is a dynamic link since the exact specifications of $\mathcal{V}_i^w$ and $\mathcal{V}_i^l$ depend on several factors. These include the correlation among a bidder's values across auctions, the form of synergy (positive or negative) of auctioned items, the information revealed after each auction, and the number and format of auctions in the sequence.[28] Moreover, despite the ex ante symmetry among bidders, the game becomes asymmetric as soon as one bidder wins an auction.

Affiliation and Synergy

Kong (2021) combines these dynamic features within a symmetric IPV setting. For a sequence of two auctions, bidder i draws a value V_{1i} from $F_1(\cdot)$ with support $[\underline{v}, \overline{v}]$ at the start of the first auction, while the value V_{2i} is not known with certainty. The value V_{2i} is drawn from $F_2(\cdot|V_{1i})$ with support $[\underline{v}, \overline{v}]$ after the first auction. The conditional distribution $F_2(\cdot|\cdot)$ satisfies first-order stochastic dominance, i.e., $F_2(\cdot|v_1') \leq F_2(\cdot|v_1)$ for $v_1' > v_1$.[29] The values (V_{1i}, V_{2i}) are stand-alone values as they represent bidder i's values if he wins only one of the two auctioned items. In contrast, if he wins the first auction, then bidder i's value for the second item becomes $\delta(V_{1i}, V_{2i})$, which is increasing in its second argument. Positive (negative) synergy corresponds to $\delta(V_{1i}, V_{2i}) > (<)\ V_{2i}$. Its magnitude $\delta(V_{1i}, V_{2i}) - V_{2i}$ is private information to bidder i since (V_{1i}, V_{2i}) are private. Regarding

[28] Dynamic considerations can also arise when there are budget constraints as studied by Benoit and Krishna (2001).

[29] We follow Kong (2021) and refer to affiliation as a general form of positive correlation though affiliation is stronger than first-order stochastic dominance as she noted. See e.g. Krishna (2010).

information, as in Février (2003), she assumes that only the winner's identity is revealed after the first auction to limit the ratchet effect. Thus a symmetric perfect Bayesian equilibrium is composed of the strategies $s_1(\cdot)$ with argument V_{1i}, and $[s_2^w(\cdot;\cdot), s_2^l(\cdot;\cdot)]$ with arguments (V_{1i}, V_{2i}) where the subscripts $\{1,2\}$ refer to the first and second auctions while the superscripts $\{w,l\}$ indicate whether the bidder won or lost the first auction. The model primitives are $[F_1(\cdot), F_2(\cdot|\cdot), \delta(\cdot,\cdot)]$.

Motivated by her data, Kong (2021) analyzes a sequence composed of a first-price auction followed by an ascending auction. In this case, the information revealed to bidders at the end of the first auction could also be all the bids or only the winning bid in the first auction as they do not affect bidding in the second auction. This is because bidding the synergy-inclusive value $\delta(V_{1i}, V_{2i})$ or the second value V_{2i} depending on being a first-auction winner or loser is a dominant strategy in the ascending auction. See Section 7.1. Let $D(\cdot|V_{1i}) \equiv \Pr[\delta(V_{1i}, V_{2i}) \leq \cdot|V_{1i}]$ denote bidder i's distribution of the synergy-inclusive value in the second auction given his value V_{1i} in the first auction. To simplify, assume that its support is also $[\underline{v}, \overline{v}]$. This is also the value distribution in the second auction for the first-auction winner while a first-auction loser still draws his value from $F_2(\cdot|V_{1i})$.

To derive the equilibrium strategy in the first auction, she considers the distribution of the highest competing bid $B_{2,-i}^{\max}$ in the second auction conditional on the highest competing bid $B_{1,-i}^{\max} = b$ in the first auction when bidder i is the winner/loser in the first auction. Letting $v = s_1^{-1}(b)$, this distribution is $H^w(\cdot|b) \equiv \Pr(V_2 \leq \cdot|V_1 \leq v)^{I-2} F_2(\cdot|v)$ when i is the winner, and $H^l(\cdot|b) \equiv \Pr[V_2 \leq \cdot|V_1 \leq v]^{I-2} D(\cdot|v)$ when i is a loser. Thus bidder i's continuation values for winning/losing the first auction with a bid B_{1i} are

$$
\begin{aligned}
\mathcal{V}_i^w &\equiv \mathrm{E}\left\{[\delta(V_{1i}, V_{2i}) - B_{2,-i}^{\max}]\,1\!\!I[B_{2,-i}^{\max} \leq \delta(V_{1i}, V_{2i})]|B_{1,-i}^{\max} \leq B_{1i}, V_{1i}\right\} \\
&= \frac{1}{G_1(B_{1i})^{I-1}} \int_{\underline{b}}^{B_{1i}} \Big\{ \int_{\underline{v}}^{\overline{v}} \Big[\int_{\underline{v}}^{\delta(V_{1i}, v_2)} (\delta(V_{1i}, v_2) - x)\, dH^w(x|b) \Big] dF_2(v_2|V_{1i}) \Big\} dG_1^{I-1}(b) \\
\mathcal{V}_i^l &\equiv \mathrm{E}\left\{(V_{2i} - B_{2,-i}^{\max})\,1\!\!I(B_{2,-i}^{\max} \leq V_{2i})|B_{1,-i}^{\max} > B_{1i}, V_{1i}\right\} \\
&= \frac{1}{1 - G_1(B_{1i})^{I-1}} \int_{B_{1i}}^{\overline{b}} \Big\{ \int_{\underline{v}}^{\overline{v}} \Big[\int_{\underline{v}}^{v_2} (v_2 - x)\, dH^l(x|b) \Big] dF_2(v_2|V_{1i}) \Big\} dG_1^{I-1}(b)
\end{aligned}
$$

Plugging in (5.20) and integrating by parts, the first-order condition becomes

$$
V_{1i} = B_{1i} + \frac{1}{I-1} \frac{G_1(B_{1i})}{g_1(B_{1i})} - \int_{\underline{v}}^{\overline{v}} \left\{ \int_{\underline{v}}^{\delta(V_{1i}, v_2)} H^w(x|B_{1i})dx - \int_{\underline{v}}^{v_2} H^l(x|B_{1i})dx \right\} dF_2(v_2|V_{1i}). \quad (5.21)
$$

In particular, the third term vanishes if there is no synergy despite the dependence of V_2 on V_1. Under minor additional assumptions, the author shows that the equilibrium strategy $s_1(\cdot)$ exists, is unique and increasing. Thus, $B_{1i} = s_1(V_{1i})$.

Turning to identification, the analyst observes all the bids in the first-price auction and the winning bid in the ascending auction as well as the winner's identity in both auctions. The main difficulty is to distinguish affiliation and (positive) synergy from observed bids since both increase the probability of winning the second auction when winning the first auction.[30] Identification of the distributions $\tilde{D}(\cdot|\cdot) \equiv D[\cdot|s_1^{-1}(\cdot)]$ and $\tilde{F}_2(\cdot|\cdot) \equiv F_2[\cdot|s_1^{-1}(\cdot)]$ follows from Athey and Haile (2002) as these distributions are the value distributions in the second auction since the bid in the first auction is observed. See Section 7.1. Moreover, comparing $\tilde{D}(\cdot|b_1)$ and $\tilde{F}_2(\cdot|b_1)$ for any b_1 identifies the synergy function $\delta(\cdot,\cdot)$. Intuitively, the difference in behavior in the second auction of a first-auction winner and loser conditional on the same first-auction bid/value neutralizes affiliation and isolates the effect of synergy. Formally, the synergy function is identified by $\tilde{\delta}(b_1,\cdot) = \tilde{D}^{-1}[\tilde{F}_2(\cdot|b_1)|b_1]$ since $\Pr[V_2 \leq \cdot|b_1] = \Pr[\delta(v_1, V_2) \leq \delta(v_1,\cdot)|b_1]$ where $\tilde{\delta}(b_1,\cdot) \equiv \delta[s_1^{-1}(b_1),\cdot]$. Equivalently, $\tilde{\delta}(v_1,\cdot)$ maps the α-quantile of $\tilde{F}_2(\cdot|b_1)$ into the α-quantile of $\tilde{D}(\cdot|b_1)$ for every (α, b_1). See also Matzkin (2003). Thus, the distributions $H^w(\cdot|\cdot)$ and $H^l(\cdot|\cdot)$ are identified from observed bids.[31] Hence, replacing $\delta(V_{1i}, v_2)$ and $F_2(v_2|V_{1i})$ by $\tilde{\delta}(B_{1i}, v_2)$ and $\tilde{F}_2(v_2|B_{1i})$ in (5.21) gives $V_{1i} = \xi_1(B_{1i})$ showing that V_{1i} is recovered and that its distribution $F_1(\cdot)$ is identified. Because $s_1(\cdot) = \xi_1^{-1}(\cdot)$ is identified, the remaining primitives $F_2(\cdot|\cdot)$ and $\delta(\cdot,\cdot)$ are also identified. The identification argument extends to risk averse bidders using an exclusion restriction as in Guerre, Perrigne and Vuong (2009) or by imposing that the marginal distributions of V_{1i} and V_{2i} are equal. Kong (2021) also indicates how her argument extends to a sequence of two first-price sealed-bid auctions provided the strategy in the first auction is increasing.

Kong (2021) proposes an estimation procedure in the spirit of the identification argument. The first step consists in estimating $\tilde{D}(\cdot|\cdot)$ and $\tilde{F}_2(\cdot|\cdot)$ by sieve ML using Bernstein polynomials subject to some sign constraints on their coefficients to ensure that the estimates are proper distributions. These are used to estimate $\tilde{H}^w(\cdot|\cdot)$ and $\tilde{H}^l(\cdot|\cdot)$. The second step consists in estimating nonparametrically the synergy function $\tilde{\delta}(b_1,\cdot)$ from the map $\hat{F}_2^{-1}(\alpha|b_1)$ to $\hat{D}^{-1}(\alpha|b_1)$ on a grid of $\alpha \in [0,1]$ for every value of b_1. In the third step, she plugs the estimated function and distributions in the inverse equilibrium strategy $\xi_1(\cdot)$ to obtain pseudo values $\hat{V}_1$ from which she estimates nonparametrically the density $f_1(\cdot)$. To avoid the curse of dimensionality due to several auction covariates X, the author first homogeneizes the bids given the single index specification $F_1(\cdot|X) = F_1(\cdot|X'\beta)$ and

[30] This observational similarity is analogous to that of persistent heterogeneity and structural dependence discussed by Heckman (1981).

[31] Using the monotonicity of $s_1(\cdot)$, one has $H^w(\cdot|b) = [\int_{\underline{b}}^{b} \tilde{F}_2(\cdot|x)dG_1(x)]^{I-2}\tilde{F}_2(\cdot|b)$ and $H^l(\cdot|b) = [\int_{\underline{b}}^{b} \tilde{F}_2(\cdot|x)dG_1(x)]^{I-2}\tilde{D}(\cdot|b)$, where $\tilde{F}_2(\cdot|\cdot)$, $G_1(\cdot)$ and $\tilde{D}(\cdot|\cdot)$ are identified.

$F_2(\cdot|X) = F_2(\cdot|X'\beta)$ combined with the invariance of the copula of (V_1, V_2) with respect to X. See also Section 3.3.

DYNAMIC AUCTIONS

Jofre-Bonet and Pesendorfer (2003) consider dynamic procurement auctions over an infinite sequence ($T = \infty$) of first-price sealed-bid auctions within an IPV setting. Within the t-th auction, the I bidders's private values are independent with bidder i drawing his valuation from $F(\cdot|X_t, S_{ti}, S_{t,-i})$ where X_t are exogenous object characteristics, while S_{ti} and $S_{t,-i}$ are observed states variables pertaining to bidder i and his competitors $-i$. The characteristics X_t are i.i.d. across auctions, while the state variables $S_t \equiv (S_{t1}, \ldots, S_{tI})$ are discrete and evolve according to some known deterministic laws of motion $S_{t+1} = \omega(X_t, S_t, i) = [\omega_1(X_t, S_t, i), \ldots, \omega_I(X_t, S_t, i)]$ when i is the winner of auction t. In construction, winning a project restricts a firm's capacity to undertake future projects. Thus, each firm takes into account its backlog of remaining projects to complete when bidding thereby introducing asymmetry among firms. The authors then take S_{ti} as the vector of remaining (discretized) size and time until completion of projects won by firm i over some given period prior to auction t. This could be also the number of auctions won by firm i over that period. See Caillaud and Mezzetti (2004) where the dynamic link is built around some revealed information. The distribution $F(\cdot|X_t, S_{ti}, S_{t,-i})$ is exchangeable in $S_{t,-i} \equiv \{S_{tj}, j \neq i\}$. The game is asymmetric as $F(\cdot|X_t, S_{ti}, S_{t,-i}) \neq F(\cdot|X_t, S_{tj}, S_{t,-j})$.

Using the concept of Markov perfect equilibrium (see Maskin and Tirole (2001)), the authors show that the inverse equilibrium strategy is similar to (4.6) with an additional term involving the continuation values, i.e., firm i's cost C_{ti} satisfies

$$C_{ti} = B_{ti} - \frac{1}{\sum_{j\neq i} h_{tj}(B_{ti})} + \beta \sum_{j\neq i} \frac{h_{tj}(B_{ti})}{\sum_{k\neq i} h_{tk}(B_{ti})} \Big\{ V_i[\omega(X_t, S_t, i)] - V_i[\omega(X_t, S_t, j)] \Big\}, \tag{5.22}$$

where $h_{tj}(\cdot) \equiv g(\cdot|X_t, S_{tj}, S_{t,-j})/[1 - G(\cdot|X_t, S_{tj}, S_{t,-j})]$ and $V_i(S) \equiv V(S_i, S_{-i})$ for some ex ante value function $V(\cdot)$. The discount factor β is known as it is not identified from Rust (1994). The additional term in (5.22) captures the incremental effect on the future discounted expected profit if firm i wins instead of its competitors. It is similar to the third term in (5.20). The ex ante value function $V(\cdot)$ satisfies the functional equation

$$\begin{aligned} V_i(S_t) \;\; = \;\; & \mathrm{E}_X \Big\{ \int \frac{1}{\sum_{j\neq i} h_{tj}(b)} dG(b|X_t, S_{ti}, S_{t,-i}) + \beta \sum_{j\neq i} \Big[\Pr(j \text{ wins}|X_t, S_{ti}, S_{t,-i}) \\ & + \int \frac{h_{ti}(b)}{\sum_{k\neq i} h_{tk}(b)} dG(b|X_t, S_{tj}, S_{t,-j}) \Big] V_i[\omega(X_t, S_t, j)] \Big\}, \end{aligned}$$

where $\mathrm{E}_X\{\cdot\}$ denotes expectation with respect to X_t. Though more involved than in the two-period example, the value function $V(\cdot)$ is identified. The authors compute it via numerical methods (see Judd (1998)) based on polynomial approximations following Pakes (1994). This requires the equilibrium bid distribution $G(\cdot|X, S_i, S_{-i})$ which can be estimated by parametric or nonparametric methods. Simplification arises as $V(S_i, S_{-i})$ and $G(\cdot|X, S_i, S_{-i})$ are exchangeable in $S_{-i} \equiv \{S_j, j \neq i\}$ and independent of i. Once $V(\cdot)$ is estimated, one recovers firms' costs from (5.22) and a parametric or nonparametric estimate of the cost distribution $F(\cdot|\cdot,\cdot,\cdot)$ by pooling across bidders and auctions. The authors also consider a reserve price as well some one-time bidders in addition to the I (permanent) bidders thereby introducing another layer of asymmetry.

Section 5.4: Simultaneous Auctions

Instead of auctioning the items sequentially, the auctioneer may choose to offer them simultaneously in one let. In this case, and as in Section 5.3, the independence across auctions may be lost, especially in the presence of synergies among items. Throughout this section, we consider procurement auctions given the empirical applications in the papers we review.

Combinatorial Auctions

When bidders are allowed to bid on all possible combinations of the offered items, the mechanism is referred as a combinatorial auction. In a procurement setting, the auctioneer then chooses the allocation that minimizes total payment, charging each bidder the price he tendered for each combination (bundle) that he wins. Cantillon and Pensendorfer (2006) study auctions of bus routes where operators benefit from cost reduction by running several adjacent routes. To simplify, assume there are I symmetric buyers, each drawing independently his private cost vector $C_i \equiv \{C_{i\omega}; \omega \in \Omega\}$ from the multivariate joint distribution $F(\cdot, \ldots, \cdot)$, where Ω is the set of $2^K - 1$ nonempty combinations of the K items. Let $B_i \equiv \{B_{i\omega}; \omega \in \Omega\}$ and $H(B_i) \equiv \{H_\omega(B_i); \omega \in \Omega\}$ be the corresponding bid vector and winning probability vector, respectively, where $H_\omega(B_i)$ is the probability of winning bundle ω with bid vector B_i. Differentiating the expected profit $(B_i - C_i)'H(B_i)$ with respect to B_i gives the $2^K - 1$ first-order conditions

$$C_i = B_i - [\nabla H(B_i)]^{-1} H(B_i), \tag{5.23}$$

where $\nabla H(\cdot) \equiv \partial H(\cdot)'/\partial b$ is the Jacobian of $H(\cdot)$ provided the $(2^K - 1) \times (2^K - 1)$ matrix $\nabla H(B_i)$ is nonsingular. As shown by the authors, however, $\nabla H(B_i)$ is invertible if and only if there are no irrelevant bids, i.e., bids $B_{i\omega}$ larger than the upper boundary $\overline{b}_i$ of the support of $H_\omega(\cdot, B_{i,-\omega})$. When $\nabla H(B_i)$ is singular, the authors show that $C_{i\omega}$ can still be point identified when $B_{i\omega}$ is relevant, whereas it can be bounded from below if B_i is irrelevant. See also Athey and Haile (2007).

Regarding estimation, Cantillon and Pesendorfer (2006) propose a two-step simulation-based procedure using L combinatorial auctions of K items with I bidders. In the first step, the distribution of a bidder's bid vector is estimated parametrically to avoid the curse of dimensionality. This gives $\hat{G}(\cdot|X, I)$ where the $(2^K - 1) \times d$ matrix X collects the characteristics of the $2^K - 1$ combinations. In the second step, $I - 1$ simulated bid vectors $B^s_{-i} \equiv \{B^s_j; j \neq i\}$ drawn independently from $\hat{G}(\cdot|X, I)$ are combined with bidder i's bid vector B_i to determine numerically the allocation that minimizes the auctioneer's payment and in particular which bundle(s) bidder i wins. The empirical frequencies of such wins over a large number S of simulations estimate the vector of winning probabilities $H(B_i|X, I)$. Using numerical derivatives to obtain the Jacobian $\nabla H(B_i|X, I)$, one estimates bidder i's cost vector C_i from (5.23) and its joint distribution $F(\cdot|X, I)$ as in the GPV procedure.

As noted by Kim, Olivares and Weintraub (2014) in their study of school meals procurements, a complication arises from the dimensionality of $\nabla H(B_i)$ which increases exponentially with the number of items. In particular, the winning probabilities $H_\omega(B_i)$ can be very small when K is moderately large. Combined with a high-dimensional bid vector B_i, this prevents accurate estimation of $H(B_i)$ and its derivatives as well as the numerical inversion of its Jacobian. Provided it is an equilibrium, the authors propose to restrict the set of strategies available to bidders to be of the form $B_i = C_i + X\theta_i$ where the second term represents bidder i's markup. Thus maximizing the expected profit $(X\theta_i)'H(C_i + X\theta_i)$ gives $\theta_i = -[X'\nabla H(B_i)X]^{-1}X'H(B_i)$ and $C_i = B_i + X[X'\nabla H(B_i)X]^{-1}X'H(B_i)$. The main difference between the latter and (5.23) is that the matrix $X'\nabla H(B_i)X$ is in general of much smaller dimension than $\nabla H(B_i)$ and thus more likely to be invertible.

Simultaneous First-Price Sealed-Bid Auctions

In contrast to combinatorial auctions, bidding on combinations is not allowed in simultaneous first-price sealed-bid auctions. The number of such auctions is thus equal to the number of offered items and the winner for each item is the one with the lowest bid.[32] Based on Krishna and Rosenthal's (1996) model, Marshall, Raiff, Richard and Schulenberg

[32] Fox and Bajari (2013) study (multiple-round) simultaneous ascending spectrum auctions where telecommunication companies gain market power by winning adjacent areas. The synergy function depends parametrically on the characteristics of the items and is additive in bidder's private information. It is estimated by a Manski (1975) or Han (1987) maximum score/rank estimator using pairwise stability.

(2006) analyze simultaneous first-price sealed-bid procurement auctions of milk delivery to schools where firms save on delivery costs within the same area. The net synergy function is a constant times the number of won items minus one. Private values are i.i.d. as Weibull within and across bidders. They exploit the explicit form of the symmetric equilibrium strategy has an explicit form and estimate the parameters by ML after pre-estimating the support lower boundary to avoid the problem of a parameter-dependent support. See Section 3.1.

Gentry, Komarova and Schiraldi (2018) consider simultaneous auctions of heterogeneous road construction and maintenance contracts with a deterministic synergy depending on the characteristics of the K offered items. Though their model is quite general allowing for asymmetry among bidders, we only present a simplified version in which bidders are symmetric. Let X be the $K \times d$ matrix of items' characteristics and X_ω be the submatrix of X corresponding to the items in the bundle ω. The (net) synergy function is a $(2^K - 1)$-vector function $\delta(X) \equiv \{\delta_\omega(X_\omega); \omega \in \Omega\}$ where $\delta_\omega(X_\omega)$ captures the complementarities of the items in ω. In particular, $\delta_\omega(X_\omega) = 0$ when ω is a singleton. Each bidder $i = 1, \dots, I$ has a vector of stand-alone costs $C_i \equiv \{C_{ik},; k = 1, \dots, K\}$ for the K items which is private information and drawn from a joint distribution $F(\cdot, \dots, \cdot|X, I)$ independently across bidders. All bidders participate in the K auctions. The model primitives are $[F(\cdot, \dots, \cdot|\cdot, \cdot), \delta(\cdot)]$.

Abusing notation, let Ω be the $(2^K - 1) \times K$ matrix of 0 and 1 describing all the nonempty combinations of the K offered items so that a bundle corresponds to a row of Ω. As above, let $H(B_i|X, I) \equiv \{H_\omega(B_i|X, I); \omega \in \Omega\}$ be the $(2^K - 1)$-vector of winning probabilities for these bundles conditional on (X, I), where $B_i \equiv \{B_{ik}; k = 1, \dots, K\}$ is now the K-vector of bids in the K auctions. Because the K-vector of marginal probabilities of winning each of the K items is $\Omega' H(B_i|X, I)$, bidder i's expected profit becomes $[(B_i - C_i)'\Omega' + \delta(X)']H(B_i|X, I)$ leading to the K first-order conditions

$$C_i = B_i + [\nabla H(B_i|X, I)\Omega]^{-1} \{\Omega' H(B_i|X, I) + \nabla H(B_i|X, I)\delta(X)\}, \tag{5.24}$$

where $\nabla H(\cdot|X, I) \equiv \partial H(\cdot|X, I)'/\partial b$ provided the $K \times K$ Jacobian $\nabla H(B_i|X, I)\Omega$ is non-singular. The authors show that this is so under standard assumptions. Indeed, since each item is allocated through a first-price sealed-bid auction, the vector of marginal winning probabilities is $H(B_i|X, I)'\Omega = (\Pr[B_{i1} \le \max_{j\neq i} B_{j1}|X, I], \dots, \Pr[B_{iK} \le \max_{j\neq i} B_{jK}|X, I])$. Thus $\nabla H(B_i|X, I)\Omega$ is diagonal with kth diagonal element $\partial \Pr[B_{ik} \le \max_{j\neq i} B_{jk}|X, I]/\partial B_{ik}$. In particular, when there is no synergy, i.e., $\delta(X) = 0$, (5.24) reduces to K equations, each of the form (3.6).

The K equations (5.24) show that the cost vector C_i is recovered as soon as the synergy

$\delta(X)$ is identified. To achieve the latter, the authors impose some natural exclusion restrictions. In the symmetric case, a natural restriction is to exclude the characteristics of other items from the marginal distribution $F_k(\cdot|X, I)$ of the stand-alone cost C_{ik} for each item k, i.e., $F_k(\cdot|X, I) = F_k(\cdot|X_k, I)$ where X_k is the kth row of X. Thus the kth row of (5.24) must be independent of $X \backslash X_k$ leading to a total of $K(K-1)d$ independence restrictions. In contrast, given X there are $2^K - K - 1$ unknown elements in $\delta(X)$. Hence sufficient variations in $X \backslash X_k$ ensures identification of $\delta(\cdot)$ and $F(\cdot, \ldots, \cdot|X, I)$ from (5.24). Because of the curse of dimensionality, the authors estimate parametrically the joint bid distribution for each bidder in the first step in order to estimate the vectors of winning probabilities. They also parameterize the primitives and estimate them in the second step by GMM with moments derived from the identifying restrictions.

Section 6: Alternative Allocation Rules

In a first-price auction and other standard auction mechanisms, each bidder submits a single bid and the allocation rule designates the winner as the bidder with the highest bid. There exist more complex allocation rules in which bidders submit more than a one-dimensional bid while the allocation rule accounts for other factors.

A first situation is when the auctioned object is divisible. The seller offers a quantity for sale and each bidder is willing to buy a share of it. In such an auction, each bidder submits a negatively sloped demand or in practice, a number of price-quantity pairs. Thus, a bid is the vector of such pairs for each bidder. The seller then aggregates all the bidders' demand and allocates shares of the divisible good according to some rule. Two allocation rules are typically employed depending on whether bidders pay according to their demand (discriminatory pricing) or all bidders pay the clearing price for their demanded quantities (uniform pricing) until the divisible good is exhausted. Hence, all bidders' demand may not be satisfied. Share auctions are used by Central Banks throughout the world for the sale of Treasury bills, notes and bonds to financial institutions. They are also used in electricity markets. Given a demand for electricity, electricity generators submit a positively sloped offer or pairs of prices and quantities at which they are willing to generate power. Electricity auctions are widely spread since the recent deregulation of utilities. In these two examples, either the demand or the supply is fixed. Auctions can also involve both sides of the market as in double auctions. A typical example is the Stock Market where sellers and buyers exchange shares of firms' stocks. Though economically important, we do not cover double auctions given the limited empirical research in this area.

A second situation is when the seller/buyer accounts not only for the bid but also for other features in the bidder's proposal. This is typical in construction procurements where the buyer cares for both the project cost and its quality. In a standard procurement auction, the bidder with the lowest bid wins the auction though he might deliver a project of poor quality thereby generating future security issues and renegotiation. It is then in the buyer's interest to account for both price and quality through a score, which can take different forms depending on how the score weights quality and price. The winner is the bidder with the highest or lowest score. More than half of the states in the U.S. use scoring rules to allocate their auctioned projects. Scoring rules are also used implicitly in fast-developing online service markets as well as in online marketing campaigns looking for influencers to advertize firms' products.

A third situation is the auctions of contracts. In the latter, the actual payments commonly called contingent payments are made upon realization of the project. The project can cover several items. For instance, in scale auctions the auctioned object contains several items and bidders submit a price for each item. In construction procurements, each project involves several materials and components whereas each tract in forestry auctions contains several wood species. The auctioneer provides estimated quantities. The winner is the bidder with the lowest total cost or highest total value based on these estimated quantities. Because the winner is paid on the actual quantities exchanged at the prices he submitted, there is an incentive issue as bidders can bid higher (lower) when the quantity is underestimated (overestimated). As another example, the auction may involve a linear contract with a fixed fee that the winner pays at the time of the auction and then a royalty on future revenues. Thus a bid has two components: A fixed fee and a royalty rate. This type of auction is used for gas leases in Alaska and Louisiana where firms submit a fee and a royalty rate for the option of future oil extraction.

These alternative allocation rules introduce new challenges for the identification and estimation of model primitives as the one-to-one mapping between the observed bid and the unobserved value has to be revisited. With the exception of share auctions, most of the research in this area is recent and therefore constitutes a domain where more developments are expected in the coming years. As before, all the models presented in this section are set within the private value paradigm.

Section 6.1: Share Auctions

This section presents Wilson's (1979) share auction model, then discusses its identification and estimation partly following Hortacsu (2002). It concludes with a discussion of multi-unit auctions where each bidder has multi-dimensional private information. Our presentation focuses on a quantity of a good offered to some buyers. It can be readily adapted to the case where suppliers are bidding for shares of a demanded quantity.

THE MODEL

Wilson (1979) introduces the share auction model initially within a common value framework. See Section 8.4. On the one hand, in the case of Treasury bills, an economic rationale for common value is the possibility of a secondary market in which all financial institutions face the same unknown resale price. On the other hand, financial institutions mostly buy Treasury bills to meet liquidity requirements imposed by central banks, to satisfy customers' orders and to fulfill collateral requirements for investment funds. Altogether, these private motives underlie a large proportion of Treasury bills bought by financial institutions. Thus, following Hortacsu (2002), many empirical papers have analyzed share auctions within the private value paradigm.

Since bidders can buy several items, share auctions are part of multiple-unit auctions. Let Q be the total and known quantity of goods for sale. There are $I \geq 2$ bidders. Each bidder has a one-dimensional private information σ_i which is i.i.d. as $F(\cdot)$ with a positive density $f(\cdot)$ on $[\underline{\sigma}, \overline{\sigma}]$. Bidder i's marginal valuation from winning the qth unit is $v(q;\sigma_i)$ which is positive and decreasing in q. The model primitives are $[v(\cdot;\cdot), F(\cdot)]$ and the game is symmetric. Given private information σ_i, a bid consists of a nonnegative demand function $y(\cdot;\sigma_i)$ expressing the demanded quantity as a decreasing (and differentiable) function of price.[33] The I bidders submit their demand functions $\{y(\cdot;\sigma_1), \ldots, y(\cdot;\sigma_I)\}$ that are aggregated by the seller to give the total demand. The market clearing price P_c is the value at which the aggregate demand meets the supply Q, i.e., $\sum_{i=1}^{I} y(P_c;\sigma_i) = Q$. Equivalently, for each bidder i, the market clearing price equates bidder's i demand to his residual supply, namely

$$y(P_c;\sigma_i) = Q - \sum_{j\neq i}^{I} y(P_c;\sigma_j).$$

From bidder i's point of view, the market clearing price is random and distributed as

$$\Pr[P_c \leq p|\sigma_i] = \Pr\Big[\sum_{j\neq i}^{I} y(p;\sigma_j) \leq Q - y(p;\sigma_i)|\sigma_i\Big] \equiv H[p, y(p;\sigma_i)], \tag{6.1}$$

where the probability is taken with respect to $\sigma_j, j \neq i$ so that the conditioning on σ_i disappears by mutual independence of the signals. The interpretation of $H(\cdot,\cdot)$ is that

[33]Because of symmetry, bidders have the same demand function $y(\cdot;\sigma_i)$ though their actual demands differs due to differences in private information σ_i. Moreover, $y(\cdot;\sigma_i)$ depends in principle on Q which is common knowledge. Such a dependence as well as subsequent conditioning on Q are omitted following the theoretical literature.

$H(p,y) = \Pr[P_c \le p | y(p,\sigma_i) = y]$, i.e., the probability that $P_c \le p$ given that bidder i's demand is y at price p.

There are two main (sealed-bid) allocation mechanisms: Uniform and discriminatory pricing.[34] In uniform pricing, all bidders pay the clearing price P_c for every unit they win. Thus, bidder i's payment is $P_c \cdot y(P_c, \sigma_i)$ leading to bidder's i surplus

$$\int_0^{y(P_c;\sigma_i)} [v(q;\sigma_i) - P_c]\ dq.$$

Taking expectation with respect to the distribution (6.1) of the clearing price P_c gives bidder's i expected profit as

$$\int_0^{+\infty} \Big\{ \int_0^{y(p;\sigma_i)} [v(q;\sigma_i) - p] dq \Big\} dH[p, y(p;\sigma_i)],$$

where $dH[p, y(p;\sigma_i)] = \{H_p[p, y(p;\sigma_i)] + y_p(p;\sigma_i) H_y[p, y(p;\sigma_i)]\} dp$ is the (positive) total derivative of $H[p, y(p;\sigma_i)]$ with respect to p, and $H_p(p,y)$ and $H_y(p,y)$ are the (positive and negative) partial derivatives of $H(p,y)$ with respect to its first and second arguments. Using calculus of variations, Kastl (2011) shows that the first-order condition for $y(\cdot;\cdot)$ to be a (symmetric) Bayesian Nash equilibrium strategy is

$$v[y(p;\sigma);\sigma] = p - y(p;\sigma) \frac{H_y[p, y(p;\sigma)]}{H_p[p, y(p;\sigma)]}, \tag{6.2}$$

for all $(p,\sigma) \in [\underline{p}_c, \overline{p}_c] \times [\underline{\sigma}, \overline{\sigma}]$, where $[\underline{p}_c, \overline{p}_c]$ is the support of the market clearing price. The second term with its negative sign in the RHS represents the bid shading.

In contrast, in discriminating pricing a bidder pays his inverse demand for all units down to the market clearing price P_c leading to bidder's i surplus

$$\int_0^{y(P_c;\sigma_i)} [v(q;\sigma_i) - y^{-1}(q;\sigma_i)]\ dq$$

from winning $y(P_c;\sigma_i)$ units. That is, bidder i pays for the area below his demand as in a pay-as-bid auction. If there is a single unit for sale, this reduces to a first-price auction. Because the market clearing price P_c is random, bidder i's expected profit given σ_i is

$$\int_0^{+\infty} \Big\{ \int_0^{y(p;\sigma_i)} [v(q;\sigma_i) - y^{-1}(q;\sigma_i)] dq \Big\} dH[p, y(p;\sigma_i)],$$

[34] A third mechanism is a Vickrey auction that is seldom used in practice. See Vickrey (1961) seminal paper. In such an auction, each bidder pays for the area below his residual supply. As expected, when there is a single unit for sale, the Vickrey auction is equivalent to a second-price auction. As noted by Krishna (2010), there are several other possible pricing rules. For instance, in a variant of the uniform price auction, the price paid for each unit won equals the average of all winning bids.

where $dH[\cdot,\cdot]$ is the total derivative of $H[\cdot,\cdot]$ with respect to p as above. Using calculus of variations to solve the maximization of bidder's i expected profit, Hortacsu (2002) shows that the (symmetric) Bayesian Nash equilibrium strategy $y(\cdot;\cdot)$ solves

$$v[y(p;\sigma);\sigma] = p + \frac{H[p, y(p;\sigma)]}{H_p[p, y(p;\sigma)]}, \tag{6.3}$$

for all $(p,\sigma) \in [\underline{p}_c, \overline{p}_c] \times [\underline{\sigma}, \overline{\sigma}]$, where $H_p(\cdot,\cdot)$ is the partial derivative of $H(\cdot,\cdot)$ with respect to its first argument. Again the second term in the RHS represents the bid shading.

Equilibrium existence and monotonicity have been studied by Reny (1999), Athey (2001) and McAdams (2003) among others. An important policy question, however, is the ranking of the uniform and discriminatory price auctions in terms of generated revenues. A graphical analysis of a bidder's payments could lead to the misleading conclusion that a discriminatory auction dominates the uniform price auction since revenue is the area below the bidder's demand up to the clearing price for the former and the rectangle of quantity by clearing price for the latter. This conclusion, however, does not take into account that the equillibrium strategies differ across the two mechanisms. The question on which mechanism dominates the other is addressed empirically by Hortacsu (2002), Hortacsu and McAdams (2010) and Kastl (2011).

Identification and Estimation

We observe the offered quantity Q as well as the demand curve $Y_i(\cdot) \equiv y_I(\cdot; Q, \sigma_i)$ for each bidder $i = 1, \ldots, I$, where the equilibrium strategy $y_I(\cdot; Q, \sigma_i)$ emphasizes its dependence on the offered quantity Q and the number of bidders I. Formally, conditional on I we know the joint distribution of Q and the I independently and identically distributed random variables $Y_i(p)$ for every $p \in I\!\!R_+$. Thus, the conditional probability $H_I(p, y|Q) \equiv \Pr[\sum_{j\neq i}^{I} Y_j(p) \leq Q - y \mid Q]$ and its partial derivatives $H_{Ip}(p, y|Q)$ and $H_{Iy}(p, y|Q)$ are known. The first-order conditions (6.2) and (6.3) for uniform and discriminatory pricing are then similar to (3.6) as $Y_i(p) = y_I(p; Q, \sigma_i)$ corresponds to the observed bid while $v[y_I(p; Q, \sigma_i); \sigma_i]$ is bidder i's valuation at price p. The difference is that (6.2) and (6.3) hold for all $p \in [\underline{p}_c(Q, I), \overline{p}_c(Q, I)]$ and $Q \in [\underline{Q}, \overline{Q}]$. Hence, bidder i's marginal valuation $V_i(p) \equiv v[y_I(p; Q, \sigma_i); \sigma_i]$ is identified for such pairs (p, Q).

This does not imply, however, that the marginal valuation function $v(\cdot;\cdot)$ and the signal distribution $F(\cdot)$ are identified as bidders' signals $(\sigma_1, \ldots, \sigma_I)$ are unobserved. The problem is analogous to that studied by Matzkin (2003) where the identified bidder i's valuation is related to the unobservable σ_i by the nonseparable relation $V_i(p) = v[Y_i(p); \sigma_i]$ though a difference is that $Y_i(p)$ is correlated with σ_i. Several normalizations can be entertained. Because the unobserved signal σ_i enters nonlinearly into $v(\cdot;\cdot)$, we assume

(i) Normalization: The signal σ_i is $\mathcal{U}[0,1]$ for every i,
(ii) Monotonicity: The equilibrium strategy $y_I(\cdot;Q,\sigma_i)$ is increasing in σ_i.[35]
Instead of applying Matzkin's (2003) argument to $V_i(p)=v[Y_i(p);\sigma_i]$ because $Y_i(p)$ is correlated with σ_i, we apply it to $Y_i(p)=y_I(p;Q,\sigma_i)$. We have $\alpha = \Pr[\sigma_i \leq \alpha|Q,I] = \Pr[Y_i(p) \leq y_I(p;Q,\alpha)|Q,I] \equiv G_{Y_i(p)|Q}[y_I(p;Q,\alpha)|Q,I]$. Hence, $y_I(p;Q,\alpha) = G^{-1}_{Y_i(p)|Q}(\alpha|Q,I)$, which is the α-quantile of the distribution of $Y_i(p)$ given (Q,I). Thus, bidder i's signal σ_i is recovered as $\sigma_i = G_{Y_i(p)|Q}[Y_i(p)|Q,I]$. Since bidder i's valuation $V_i(p) = v[y_I(p;Q,\sigma_i);\sigma_i]$ is identified, the marginal valuation function $v(q;\alpha)$ is identified for all (q,α) satisfying $q = G^{-1}_{Y_i(p)|Q}(\alpha|Q,I)$ with $(p,Q,I,\alpha) \in [\underline{p}_c(Q,I),\overline{p}_c(Q,I)] \times [\underline{Q},\overline{Q}] \times \mathcal{S}_I \times [0,1]$.[36]

Turning to estimation, we observe $\ell = 1,\ldots,L$ independent auctions of Q_ℓ quantities of the same good with I_ℓ bidders. The key is to estimate the probability $H_I(p,y|Q) \equiv \Pr[\sum_{j\neq i}^I Y_j(p) \leq Q - y|Q,I]$ and its partial derivatives $H_{Ip}(p,y|Q)$ and $H_{Iy}(p,y|Q)$. This is achieved by a variety of nonparametric estimation methods from the observed demands $Y_{i\ell}(\cdot); i = 1,\ldots,I_\ell$ and supply Q_ℓ. For instance, we can use the kernel-smoothed distribution estimator as studied by Azzalini (1981), Reiss (1981) and Jones (1990) modified for the conditioning on Q

$$\hat{H}_I(p,y|Q) = \frac{\frac{1}{IL_I h_Q}\sum_{\ell=1}^{L_I}\sum_{i=1}^{I}\tilde{K}\Big(\frac{Q-y-\sum_{j\neq i}^I Y_{j\ell}(p)}{h}\Big)K\Big(\frac{Q-Q_\ell}{h_Q}\Big)}{\frac{1}{L_I h_Q}\sum_{\ell=1}^{L_I}K\Big(\frac{Q-Q_\ell}{h_Q}\Big)},$$

where L_I is the number of auctions with I bidders, and $\tilde{K}(u) = \int_{-\infty}^{u} K(v)dv$ with $K(\cdot)$, h and h_Q a kernel and bandwidths.[37] Thus, differentiating with respect to y and p gives

$$\hat{H}_{Iy}(p,y|Q) \quad = \quad -\frac{\frac{1}{IL_I h h_Q}\sum_{\ell=1}^{L_I}\sum_{i=1}^{I}K\Big(\frac{Q-y-\sum_{j\neq i}^I Y_{j\ell}(p)}{h}\Big)K\Big(\frac{Q-Q_\ell}{h_Q}\Big)}{\frac{1}{L_I h_Q}\sum_{\ell=1}^{L_I}K\Big(\frac{Q-Q_\ell}{h}\Big)},$$

[35] An alternative set of conditions is to require (i) multiplicative separability as in $v(q;\sigma_i) = \sigma_i v_o(q)$ and (ii) $\mathrm{E}[\log\sigma_i|Q] = 0$ for $i = 1,\ldots,I$. Since $V_i(p) = v[y_I(p;Q,\sigma_i);\sigma_i]$, we have $\log V_i(p) = \log v_o[Y_i(p)] + \log\sigma_i$. This corresponds to a nonparametric instrumental regression as studied by Newey and Powell (2003), Darolles, Fan, Florens and Renault (2011) and Chen and Christensen (2018) among others. Because $\mathrm{E}[\log\sigma_i|Q] = 0$, a natural instrument is Q since $Y_i(p)$ depends on Q.

[36] In general, for any such a pair (q,α), the marginal valuation $v(q;\alpha)$ is overidentified as there are many triplets (p,Q,I) satisfying $q = G^{-1}_{Y_i(p)|Q}(\alpha|Q,I)$. Similarly, σ_i is overidentified in $\sigma_i = G_{Y_i(p)|Q}[Y_i(p)|Q,I]$. This leaves some room for σ_i to be multidimensional by letting bidder i's marginal valuation for the q-th unit be $v[q,\sigma_i(q)]$ where $\sigma_i(\cdot)$ is a random process on $\mathbb{R}_+$. See the end of Section 6.1.

[37] Alternatively, from $H(p,y) = \Pr[P_c \leq p|Y_i(p) = y]$ we can estimate $H(p,y)$ by a kernel regression of $\mathbb{1}(P_c \leq p)$ or $\tilde{K}[(p-P_c)/h]$ on $Y_i(p)$.

which is up to sign the kernel density estimator of $\sum_{j\neq i}^{I} Y_j(p)$ at $Q-y$ given Q, and

$$\hat{H}_{Ip}(p,y|Q) \;=\; -\frac{\frac{1}{IL_I h h_Q}\sum_{\ell=1}^{L_I}\sum_{i=1}^{I}\left(\sum_{j\neq i}^{I} Y'_{j\ell}(p)\right) K\Big(\frac{Q-y-\sum_{j\neq i}^{I} Y_{j\ell}(p)}{h}\Big) K\left(\frac{Q-Q_\ell}{h_Q}\right)}{\frac{1}{L_I h_Q}\sum_{\ell=1}^{L_I} K\left(\frac{Q-Q_\ell}{h}\right)},$$

where $Y'_{j\ell}(p)$ is the slope of bidder j's demand $Y_{j\ell}(\cdot)$ at p in the ℓth auction. From (6.2) or (6.3) where $y(p;\sigma)$ is replaced by $Y_{i\ell}(p)$, whereas $H(\cdot,\cdot)$ and its derivatives are replaced by their estimates at $(Q,I)=(Q_\ell,I_\ell)$, we recover estimates of bidders' marginal valuations $\hat{V}_{i\ell}(p) = v[y_I(p,Q_\ell,\sigma_{i\ell});\sigma_{i\ell}]$ for every $p \in [\underline{p}_c(Q_\ell,I_\ell),\overline{p}_c(Q_\ell,I_\ell)]$, $i=1,\ldots,I$ and $\ell=1,\ldots,L$. Moreover, by the identification argument, bidders' signals are recovered as $\hat{\sigma}_{i\ell}=\hat{G}_{Y_i(p)|Q}[Y_{i\ell}(p)|Q_\ell,I_\ell]$, where $\hat{G}_{Y_i(p)|Q}(\cdot|\cdot,I)$ is (say) a kernel-smoothed distribution estimator of $Y_i(p)$ given (Q,I). Thus, one can estimate the marginal valuation function $v(q;\alpha)$ by sieve-based minimum distance on $\hat{V}_{i\ell}(p) = v[Y_{i\ell}(p);\hat{\sigma}_{i\ell}]$.

The preceding model with its identification and estimation extend to allow for asymmetry among bidders through heterogeneous marginal valuations as well as for exogenous variables X by letting bidders' marginal valuation functions be $v_i(q,X;\sigma_i)$. Allowing for heterogeneity and covariates in bidders' signal distributions through the conditional distribution $F_i(\cdot|X)$ will require a normalization other than that used above.

A Resampling Strategy

Hortacsu (2002) introduces a resampling strategy that is widely used in the analysis of Treasury bills and electricity share auctions. See Efron and Tibshirani (1993) and Horowitz (2001) for bootstrap methods. Dropping the subscript ℓ hereafter, the method estimates the key probability $H_I(p,y|Q) \equiv \Pr[\sum_{j\neq i}^{I} Y_j(p) \leq Q-y]$ by resampling the demands of bidder i's competitors within each auction. Specifically, for bidder i in each auction, one randomly draws $I-1$ demand curves $\{Y_j^s(\cdot); j\neq i\}$ with replacement from the $I-1$ observed demands $\{Y_j(\cdot); j\neq i\}$ by bidder i's competitors with equal probability. One then uses these $I-1$ resampled demands to construct the residual supply function $Q-\sum_{j\neq i}^{I_\ell} Y_j^s(\cdot)$ faced by bidder i for $s=1,\ldots,S$ replications. The intersection with the observed bidder i's demand $Y_i(\cdot)$ then gives a market clearing price P_c^s. Repeating this procedure for each bidder i in a given auction gives a large number S of bootstrapped market clearing prices holding i's demand $Y_i(\cdot)$ fixed. Since $\Pr[P_c\leq p|Q,\sigma_i] = \Pr[Y_i(p)\leq Q-\sum_{j\neq i}^{I} Y_j(p)|\sigma_i] = H_I[p,Y_i(p)|Q]$ from (6.1), the empirical distribution of the simulated clearing prices P_c^s provides an estimate denoted $\hat{H}_I^R[p,Y_i(p)|Q]$ for all p and every auction. Under some conditions, Hortacsu (2002) shows that such a resampling estimator is consistent.

In practice, one seldom observes each participant bidding a continuous demand $Y_i(\cdot)$

but instead a finite number of pairs (y_{ik}, p_{ik}), $k = 1, \ldots, K_i$ of quantities and prices for each auction, still dropping the subscript ℓ. Approximating bidder i's demand $Y_i(\cdot)$ in each auction by a step function passing through such pairs, one can apply the above resampling procedure to obtain probability estimates $\hat{H}_I^R[p, Y_i(p)|Q]$ for all p and hence $\hat{H}_I^R[p_{ik}, y_{ik}|Q]$ for $k = 1, \ldots, K_i$ since $y_{ik} = Y_i(p_{ik})$.[38] A question then arises on how to estimate the partial derivative $H_{Ip}[p_{ik}, y_{ik}|Q]$ appearing in (6.3) at $p = p_{ik}$ for a given auction. Following Hortacsu (2002), a derivative discrete approximation is

$$\hat{H}_{Ip}^R[p_{ik}, y_{ik}|Q] = \frac{\hat{H}_I^R[p_{ik}, y_{ik}|Q] - \hat{H}_I^R[p_{ik-1}, y_{ik}|Q]}{p_{ik} - p_{ik-1}}.$$

An estimate of the partial derivative $H_{Iy}[p_{ik}, y_{ik}|Q]$ appearing in (6.2) at $p = p_{ik}$ for a given auction is

$$\hat{H}_{Iy}^R[p_{ik}, y_{ik}|Q] = \frac{\hat{H}_I^R[p_{ik}, y_{ik}|Q] - \hat{H}_I^R[p_{ik}, y_{ik-1}|Q]}{y_{ik} - y_{ik-1}}.$$

Though the first term in the numerator of the preceding two equations is obtained as above, it should be noted that their second terms cannot be obtained from the empirical distribution of the simulated clearing price. Instead, $\hat{H}_I^R[p_{ik-1}, y_{ik}|Q]$ is the empirical frequency that $y_{ik} \leq Q - \sum_{j \neq i}^I Y_j^s(p_{ik-1})$, i.e., holding bidder i's demand fixed at y_{ik} irrespective of price. Similarly, $\hat{H}_{Iy}^R[p_{ik}, y_{ik-1}|Q]$ is the empirical frequency that $y_{ik-1} \leq Q - \sum_{j \neq i}^I Y_j^s(p_{ik})$, i.e., holding bidder i's demand fixed at y_{ik-1} irrespective of price.

The resampling strategy performs well when the number of bidders and number of price-quantity pairs are large in every auction. Fortunately, Treasury bills and electricity auctions tend to have a respectable number of financial institutions and firms participating to each auction. Another issue is the number of pairs each bidder submits. One needs at least two pairs. The number of pairs that a bidder can submit is usually limited by Central Banks. In practice, bidders tend to submit a lower number of bid pairs than they are allowed to. Lastly, the resampling estimator does not accommodate well exogenous variables. One usually considers a few auctions in estimation where exogenous variables are constant thereby controlling for macroeconomic conditions.

Multi-Unit Auctions

Wilson's (1979) share auction model views the offered quantity Q as perfectly divisible. An alternative approach is to view this quantity as Q identical units for sale. It is then natural to associate a marginal valuation v_{ik} to the kth unit bought, for $k = 1, \ldots, Q$. The

[38] Following Wolak (2007), Reguant (2014) uses a kernel-smoothed demand $Y_i(\cdot)$ for analyzing socalled complex bids in electricity share auctions as the number K_i of price-quantity pairs is large.

values $v_{i1} \geq, \ldots \geq v_{iQ} \geq 0$ are bidder i's private information which is multi-dimensional as $Q \geq 2$. The vector $\{v_{ik}\}_{k=1}^{Q}$ is distributed as some Q-dimensional joint distribution $F(\cdot, \ldots, \cdot)$ independently across the I bidders. Each buyer then bids a vector of unit prices $b_{i1} \geq \ldots \geq b_{iQ} \geq 0$, where b_{ik} indicates the amount bidder i is willing to pay for the additional kth unit. Thus, $b_{i1} + \ldots + b_{ik}$ is the amount bidder i is willing to pay for k units of the good. The Q units of goods are allocated to the Q winning bids, i.e., to the Q highest bids among the IQ bids tendered by all I bidders. Thus the number k_i of units won by bidder i is the largest k for which b_{ik} is a winning bid. Under uniform pricing, bidder i pays $k_i P_c$ where P_c is the clearing price equating bidders' demand and supply Q. Under discriminatory pricing, bidder i pays $b_{1i} + \ldots + b_{Ik_i}$. See e.g., Krishna (2010) and references therein.

Let $H_{ik}(\cdot)$ and $h_{ik}(\cdot)$ be the distribution and density of the $(Q-k+1)$th highest bid $B_{-i}^{(Q-k+1)}$ among the $(I-1)Q$ bids of bidder i's competitors. Thus, $H_{ik}(b)$ is the probability that bidder i wins the kth unit with a bid b. McAdams (2008) shows that the first-order condition for $\{b_{ik}\}_{k=1}^{Q}$ to be a Bayesian Nash equilibrium is

$$v_{ik} = b_{ik} + (k-1)\frac{\Pr[B_{-i}^{(Q-k+1)} > b_{ik} > B_{-i}^{(Q-k+2)}]}{h_{ik}(b_{ik})} \quad \text{for } k = 1, \ldots, Q, \tag{6.4}$$

in uniform pricing, and

$$v_{ik} = b_{ik} + \frac{H_{ik}(b_{ik})}{h_{ik}(b_{ik})} \quad \text{for } k = 1, \ldots, Q, \tag{6.5}$$

in discriminatory pricing, when the bid sequence $\{b_{ik}\}_{k=1}^{Q}$ is decreasing. If so, (6.4) and (6.5) can be viewed as discretized versions of (6.2) and (6.3), respectively. Thus, one can uniquely recover bidder i's marginal valuations $\{v_{ik}\}_{k=1}^{Q}$ as in the first step of the GPV procedure. Unfortunately, bidder i's best response $\{b_{ik}\}_{k=1}^{Q}$ can be constant over different units. McAdams (2008) then provides some lower and upper bounds around the preceding marginal valuations which are tighter than those in Hortacsu (2002) and Kastl (2011).[39] Such bounds in the discriminatory case are estimated in Hortacsu and McAdams (2010) for policy counterfactual analysis. In addition, McAdams (2008) provides a set of testable restrictions on the distribution of competitors' bids based on bidders having nonincreasing marginal valuations and observed bids being best responses.

Section 6.2: Scoring Auctions

[39]Kastl (2011) considers a somewhat different model where each bidder bids a finite number of price-quantity pairs (q_i, k_i) while incurring a cost that increases with such a number.

Scoring auctions refer to allocation rules that account for bidders' bids as well as for other relevant features such as bidders' qualities. The allocation rule is known to participants. If it is unknown, they are called beauty contests. The theoretical literature starts with Che (1993) and then develops with Asker and Cantillon (2008, 2010). In general, the literature focuses on some classical scores such as the quasilinear score, which is linear in bid and nonlinear in quality, or the inverse of the price-per-quality ratio score. The winner is the bidder with the highest score. In most procurements with scoring rules, the winner is paid his bid. Che (1993) endogeneizes the bidder's quality and bid when bidder's private information is one-dimensional. Asker and Cantillon (2008, 2010) extend this setting to bidimensional private information with a fixed cost to perform the project and a marginal cost of providing quality. The empirical analysis of scoring auctions is relatively recent. It starts with Lewis and Bajari (2011) who analyze procurement auctions where completion time matters to the buyer. Bidders' qualities can be exogenous such as their reputation, expertise and experience, or endogenous when bidders choose some inputs that render their projects of better or lower quality. We first review some papers with exogenous quality, and then present a few recent papers with endogenous quality.

The Basic Setting

Given the prevalence of scoring rules in procurement auctions, we consider the latter without loss of generality. A buyer auctions a project to $I \geq 2$ risk neutral sellers/firms. Each firm is characterized by a univariate or multivariate type σ_i which is firm i's private information. In addition to his bid B_i, firm i's proposed project quality is summarized into a measure q_i reflecting firm's experience, reputation, and expertise. To simplify, we take q_i as being univariate hereafter. We make the high level assumption that the conditional density of the type vector $(\sigma_1, \ldots, \sigma_I)$ given the quality vector $(q_1, \ldots, q_I) \in [\underline{q}, \overline{q}]^I$ satisfies $f(\sigma_1, \ldots, \sigma_I | q_1, \ldots, q_I, X) = \prod_{i=1}^I f(\sigma_i | q_i, X)$ where $f(\cdot|\cdot,\cdot)$ is the density of a conditional distribution $F(\cdot|\cdot,\cdot)$, and X denotes a vector of project characteristics. The types σ_i are independent though not identically distributed conditional on $(q_1, \ldots, q_I, X)$ since the distribution of σ_i depends on own quality q_i but not on other firms' qualities q_{-i}. Each firm incurs a positive cost $C_i = C(\sigma_i, q_i, X)$ for delivering a project with quality q_i and characteristics $X \in I\!R^d$, where $C(\cdot,\cdot,X)$ is increasing in its first argument, while it can be increasing or decreasing in quality.

The buyer enjoys a positive gross surplus $S(q, X)$ from buying a project with quality q and characteristics X. The buyer uses a scoring rule of the form $r(B_i, q_i, X)$, which is decreasing in B_i and increasing in q_i, to allocate the project to the bidder with the highest score. This includes the aforementioned quasilinear score and inverse price-quality ratio.

In particular, the scoring rule is individual specific in contrast to being interdependent when firm i's score also depends on other firms' bids B_{-i} and qualities q_{-i}. Formally, let $D_i = 1$ if firm i wins the auction, i.e.,

$$r(B_i, q_i, X) > r(B_j, q_j, X) \quad \text{for all j} \neq \text{i}. \tag{6.6}$$

The model primitives are $[F(\cdot|\cdot,\cdot), C(\cdot,\cdot,\cdot), S(\cdot,\cdot), r(\cdot,\cdot,\cdot)]$ which are common knowledge to bidders. We assume that the auctioned project is always allocated to a bidder. Thus, there is no exclusion.

Identification and Estimation with Exogenous Qualities

This part draws from Laffont, Perrigne, Simioni and Vuong (2020) where firms' signals σ_i are univariate, and qualities $(q_1, \ldots, q_I)$ are common knowledge. In the spirit of Section 4.2, the latter introduces asymmetry among bidders as a low quality bidder will bid differently from a high quality bidder. Using the subscript I to refer to the number of bidders, firm i submits a bid $B_i = s_I(\sigma_i; q_i, q_{-i}, X)$ as an increasing function of his type σ_i given his quality q_i, his competitors' qualities q_{-i}, and the project characteristics X. His expected profit is $[B_i - C(\sigma_i, q_i, X)]H_I(B_i, q_i, q_{-i}, X)$, where $H_I(B_i, q_i, q_{-i}, X) \equiv \Pr[r(B_i, q_i, X) > r(B_j, q_j, X), j \neq i | B_i, q_1, \ldots, q_I, X]$ from (6.6). That is, $H_I(B_i, q_i, q_{-i}, X)$ is the probability that firm i wins with a bid-quality pair (B_i, q_i) given his competitors qualities q_{-i} and project characteristics X. In particular, $s_I(\sigma_i; q_i, q_{-i}, X)$ and $H_I(B_i, q_i, q_{-i}, X)$ are exchangeable in the $I-1$ arguments of q_{-i}. Moreover, the joint distribution of $(B_1, \ldots, B_I)$ conditional on $(q_1, \ldots, q_I, X)$ is $\prod_{i=1}^{I} G_{B|q,q_-,X}(b_i|q_i, q_{-i}, x)$ where $G_{B|q,q_-,X}(\cdot|\cdot,\cdot,\cdot)$ is the distribution of a bidder's bid B given his quality q, his competitors' qualities q_-, and the auction covariates X. Bids $(B_1, \ldots, B_I)$ are independent but not identically distributed given $(q_1, \ldots, q_I, X)$. The distribution $G_{B|q,q_-,X}(\cdot|\cdot,\cdot,\cdot)$ is also exchangeable in the $(I-1)$ arguments of q_-.

As in Section 4, differentiating the expected profit with respect to B_i gives the inverse of the equilibrium strategy

$$C(\sigma_i, q_i, X) = B_i + \frac{H_I(B_i, q_i, q_{-i}, X)}{H_{IB}(B_i, q_i, q_{-i}, X)}, \tag{6.7}$$

where the subscript B refers to the derivative with respect to B_i. The second term in the RHS is negative as a larger bid decreases firm i's winning probability given (q_i, q_{-i}, X). It is the firm's rent up to sign. Equation (6.7) shows that the firm's cost $C_i = C(\sigma_i, q_i, X)$ is identified. Indeed, the winning probability $H_I(\cdot,\cdot,\cdot,\cdot)$ and its derivative are identified from bids and qualities data. One needs to impose further restriction(s) to identify the cost function $C(\cdot,\cdot,\cdot)$. Following Perrigne and Vuong (2011, 2012) and Luo, Perrigne and

Vuong (2018b), the cost function is multiplicatively separable in σ_i, namely $C(\sigma_i, q_i, X) = \sigma_i C_0(q_i, X)$ so that σ_i is interpreted as the firm's cost inefficiency.[40] Taking the logarithm and using the normalization $\mathrm{E}[\log \sigma_i | q_i, X] = 0$ identifies $\log C_0(q_i, X)$ from the regression $\log C_i = \log C_0(q_i, X) + \log \sigma_i$. The identification of the distribution $F(\cdot|q_i, X)$ of σ_i given (q_i, X) follows.

To identify the buyer's surplus function $S(\cdot, \cdot)$, we relate it to the scoring rule. For instance, suppose that the latter is quasilinear of the form $r(B_i, q_i, X) = S(q_i, X) - B_i$.[41] The probability $H_I(B_i, q_i, q_{-i}, X)$ that firm i wins given (B_i, q_i, q_{-i}, X) becomes

$$
\begin{aligned}
H_I(B_i, q_i, q_{-i}, X) &= \Pr[S(q_j, X) - B_j < S(q_i, X) - B_i, j \neq i | B_i, q_1, \ldots, q_I, X], \\
&= \prod_{j \neq i}^{I} \Pr[B_j > S(q_j, X) - S(q_i, X) + B_i | q_j, q_{-j}, X], \\
&= \prod_{j \neq i}^{I} \Big\{ 1 - G_{B|q,q_-,X}[\Delta S(q_j, q_i, X) + B_i | q_j, q_{-j}, X] \Big\}, \qquad (6.8)
\end{aligned}
$$

where $\Delta S(q_j, q_i, X) \equiv S(q_j, X) - S(q_i, X)$, whereas the second equality follows from the independence of $(B_1, \ldots, B_I)$ conditional on $(q_1, \ldots, q_I, X)$. Let $q_i = q_o \in [\underline{q}, \overline{q}]$, $q_{-i} = q_- \equiv (q, \ldots, q) \in [\underline{q}, \overline{q}]^{I-1}$ and $q_{o-} \equiv (q_o, q, \ldots, q) \in [\underline{q}, \overline{q}]^{I-1}$. Using the exchangeability of $G_{B|q,q_-,X}(\cdot|q_j, q_{-j}, X)$ in the arguments of q_{-j} gives

$$
H_I(b, q_o, q_-, X) = \Big\{ 1 - G_{B|q,q_-,X}[\Delta S(q, q_o, X) + b | q, q_{o-}, X] \Big\}^{I-1}, \qquad (6.9)
$$

where $B_i = b$ and $\Delta S(q, q_o, X) \equiv S(q, X) - S(q_o, X)$. Thus, $S(q, X)$ is identified up to the location $S(q_o, X)$ as the difference between the quantile of $G_{B|q,q_-,X}(\cdot|q, q_{o-}, X)$ evaluated at $1 - H_I(b, q_o, q_-, X)^{1/(I-1)}$ and a bidder's bid b.

Regarding estimation, the observables are $(D_{i\ell}, B_{i\ell}, q_{i\ell}, X_\ell), i = 1, \ldots, I_\ell, \ell = 1, \ldots, L$ where $D_{i\ell}$ is the winning indicator for firm i in auction ℓ. Estimation follows the identification argument. One first estimates the winning probability $H_I(\cdot, \cdot, \cdot, \cdot)$ from the kernel regression of $D_{i\ell}$ on $(B_{i\ell}, q_{i\ell}, q_{-i\ell}, X_\ell)$ using the subset of auctions with I bidders and averaging over i. Following Laffont, Perrigne, Simioni and Vuong (2020), one can also impose exchangeability by averaging over the $(I-1)!$ permutations of $q_{-i\ell}$ for each i. The resulting estimator $\hat{H}_I(\cdot, \cdot, \cdot, \cdot)$ and its derivative $\hat{H}_{IB}(\cdot, \cdot, \cdot, \cdot)$ provide an estimator of the

[40] This assumption is standard in theory such as in nonlinear pricing models. There are alternative identifying strategies exploiting the monotonicity of the cost in σ_i. See Matzkin (2003).

[41] Laffont, Perrigne, Simioni and Vuong (2020) show that the optimal mechanism for a private buyer is a scoring auction with individual score $S(q_i, X) - \Gamma(\sigma_i, q_i, X)$ where $\Gamma(\sigma_i, q_i, X)$ is the virtual cost defined as $C(\sigma_i, q_i, X) - C_\sigma(\sigma_i, q_i, X) F(\sigma_i|q_i, X)/f(\sigma_i|q_i, X)$. Though the scoring rule is endogenous, the identification argument is similar. They also consider the case when the buyer is a public agency with a cost of public funds.

costs $\hat{C}_{i\ell}$ from (6.7). Regressing $\log \hat{C}_{i\ell}$ on (q_ℓ, X_ℓ) estimates $\log \hat{C}_0(\cdot,\cdot)$ and the residuals $\log \hat{\sigma}_{i\ell}$. One then estimates nonparametrically the cost inefficiency density $\hat{f}(\sigma|q, X)$ from the $\hat{\sigma}_{i\ell}$s. Let the primitives $[C_0(\cdot,\cdot), S(\cdot,\cdot), F(\cdot|\cdot,\cdot)]$ and the marginal density $f_X(\cdot)$ of X be $R+1$ continuously differentiable. Thus, $G_{B|q,q_-,X}(\cdot|\cdot,\cdot,\cdot)$ is $R+1$ continuously differentiable as is $H_I(\cdot,\cdot,\cdot,\cdot)$ from (6.8). Hence, following Section 3.3, the resulting estimators $\hat{C}_0(\cdot,\cdot)$ and $\hat{f}(\cdot|\cdot,\cdot)$ converge at the rate $(L/\log L)^{R/(2R+4+d)}$ after smoothing (6.7) over q_- since $C(q_i, X)$ and $f(\cdot|q_i, X)$ do not depend on q_{-i}. Estimation of $\Delta S(q, q_o, X)$ from (6.9) can be achieved at the rate $(L/\log L)^{(R+1)/(2R+3+\underline{I}+d)}$ which is given by the rate for estimating $H_I(\cdot,\cdot,\cdot,\cdot)$, where $\underline{I}$ is the lowest observed value of I.

Extensions with Exogenous Qualities

Andreyanov (2018) considers a model in which a firm's private information is bidimensional, namely $\sigma_i = (C_i, q_i)$ where C_i denotes cost and q_i is quality. Firms' private information are i.i.d. across i as $F(\cdot,\cdot|X)$. The buyer observes bid-quality pairs (B_i, q_i), and set a reserve price p_0. The scoring rule belongs to the affine class, i.e., is of the form $r(B_i, q_i, X) = \alpha(q_i, X) + \beta(q_i, X)(p_0 - B_i)$ with $\alpha(\cdot, X)$ increasing and $\beta(\cdot,\cdot)$ positive as $B_i \le p_0$ for bidder i's bid to be competitive. This includes linear, quasilinear and log-linear scoring rules. The number of actual bidders is unknown to firms at the time of bidding whereas the number of potential bidders is assumed constant across procurement auctions. Let $R_i \equiv r(B_i, q_i, X)$ be firm i's score. Following Asker and Cantillon (2008), let $\theta_i \equiv r(C_i, q_i, X)$ be the one-dimensional private type aggregating the bidimensional private information σ_i. Because $B_i = p_0 - [R_i - \alpha(q_i, X)]/\beta(q_i, X)$ and $C_i = p_0 - [\theta_i - \alpha(q_i, X)]/\beta(q_i, X)$, the equilibrium bidding strategy can be equivalently characterized by the equilibrium scoring strategy $R_i \equiv \rho(\theta_i, q_i, X)$. The author shows that $\rho(\theta_i, q_i; X) = \max\{R_{0i}, \alpha(q_i, X)\}$ where $R_{0i} \equiv \rho_0(\theta_i; X)$ solves $\max_r(\theta_i - r)H(r, X)$ and $H(r, X)$ is firm i's winning probability with a score r given the covariates X. Thus

$$\theta_i = R_{0i} + \frac{H(R_{0i}, X)}{H_R(R_{0i}, X)} \equiv \xi(R_{0i}; X), \tag{6.10}$$

where $H_R(\cdot, X)$ is the derivative of $H(\cdot, X)$ with respect to its first argument.

The analyst observes $(B_{i\ell}, q_{i\ell}, X_\ell, p_{0\ell})$ for each auction $\ell = 1, \ldots, L$, and knows the scoring rule and hence the firms' scores $R_{i\ell}$ as well as the winning indicators $D_{i\ell}$. Because (6.10) resembles (6.7), the previous method applies for estimating $H(\cdot, X)$ and $H_R(\cdot, X)$ using the whole sample as I is unknown to the firms. A difference is that $H(\cdot, X)$ does not condition on (q_i, q_{-i}) as only the aggregate type θ_i matters, whereas competitors' qualities q_{-i} are unknown to firm i. A second difference arises from the truncation introduced by the reserve price $p_{0\ell}$, which puts an upper bound on bids. Thus, one can recover firms' aggregate type $\theta_{i\ell}$ from (6.10) only when $R_{i\ell} > \alpha(q_{i\ell}, X_\ell)$. Hence, $F(\cdot, \cdot|X)$ can be estimated only on $\{(c, q) : c = p_0 - [\theta - \alpha(q, X)]/\beta(q, X)$ where $\theta > \xi[\alpha(q, X); X]\}$.

The score can include an additive error ϵ_i independent of σ_i given X. The error ϵ_i captures some unobserved firm's specific attributes relevant to the buyer. In this case, the selection rule (6.6) leads to a nonparametric multinomial choice model. See Matzkin (1991, 1992, 1993). The preceding methods apply with appropriate adjustments when estimating $H_I(\cdot, \cdot, \cdot, \cdot)$ and $H(\cdot, \cdot)$ in (6.7) and (6.10), respectively.

Krasnokutskaya, Song and Tang (2020) analyze online markets for computer programming services where buyers' heterogeneous scoring rules and bidders' project qualities are unobserved by the analyst. The authors assume a linear score with an additive error, while introducing buyer specific weights leading to the score $\alpha_\ell q_i + \beta_\ell X_i - B_{i\ell} + \epsilon_{i\ell}$ for the ℓth auction. Moreover, the buyer has an outside option value $U_{0\ell}$, which is analogous to a reserve price. Following Berry, Levinsohn and Pakes (1995), the buyer's option value and weights $(U_{0\ell}, \alpha_\ell, \beta_\ell, \epsilon_{i\ell})$ are treated as random i.i.d. across firms and auctions. Bidders do not know $(U_{0\ell}, \alpha_\ell, \beta_\ell, \epsilon_{i\ell})$ and hence the scoring rule with certainty, though they know their joint distribution. Bidder i's quality q_i and attributes X_i are discrete, permanent across auctions, and common knowledge. Unlike X_i, quality q_i is unobserved by the econometrician thereby introducing (discrete) unobserved heterogeneity. Bidder i's cost $C_{i\ell}$ and entry cost $\kappa_{i\ell}$ are private information independently drawn from $F(\cdot|q_i, X_i)$ and $F_\kappa(\cdot|q_i, X_i)$. Following Levin and Smith (1994), $C_{i\ell}$ is revealed to bidder i only after entry, though the number of entrants is not revealed before bidding. See Section 5.2. The demand and suppply random vectors $(U_{0\ell}, \alpha_\ell, \beta_\ell, \epsilon_{i\ell})$ and $(C_{i\ell}, \kappa_{i\ell})$ are independent.[42]

The main econometric difficulty arises from the unobserved heterogeneity introduced by q_i. See Heckman and Singer (1984) for an early contribution with discrete unobserved heterogeneity. Krasnokutskaya, Song and Tang (2021) propose a consistent method to rank bidders according to their unobserved quality. The intuition is that a higher quality bidder has a higher probability to win than other bidders when their bids and observed attributes are equal and they face the same entrants. Specifically, let $\mathcal{I}_\ell$ be the set of potential bidders and $\mathcal{E}_\ell$ be the subset of entrants in auction ℓ. Considering the probability of winning $r_{ij}(b) \equiv \Pr[D_i = 1|B_i = b, i \in \mathcal{E}_\ell, j \notin \mathcal{E}_\ell, (i, j) \in \mathcal{I}_\ell]$, the authors show that bidders i and j with $B_i = B_j = b$ and $X_i = X_j = x$ can be ranked according to their unobserved qualities because $r_{ij}(b) > r_{ji}(b)$ if and only if $q_i > q_j$. They estimate the difference $r_{ij}(b) - r_{ji}(b)$ by kernel regression of D_i on B_i using subsamples in which bidder i enters but potential bidder j does not. They propose a bootstrap procedure based

[42]The authors also allow for some transitory bidders which we omit. See their paper for details.

on pairwise comparisons that determines the number of quality values and each bidder's quality ranking.

The model primitives are (i) the unobserved quality q_i for each bidder, (ii) the distribution of $(U_{0\ell}, \alpha_\ell, \beta_\ell, \epsilon_{i\ell})$ associated with the buyer unobserved heterogeneity, and (iii) the bidders' project and entry cost distributions $F(\cdot|q_i, X_i)$ and $F_\kappa(\cdot|q_i, X_i)$. Identification relies on the buyer's selection probability given his choice set which is the set of entrants $\mathcal{E}_\ell$.[43] For instance, conditional on $\mathcal{E}_\ell = \{i, j\}$, we have

$$\begin{aligned}
&\Pr[D_{i\ell} = 1|B_{i\ell}, X_i, B_{j\ell}, X_j] \\
&\quad = \Pr[\epsilon_{j\ell} - \epsilon_{i\ell} \leq \alpha_\ell(q_i - q_j) + \beta_\ell(X_i - X_j) + B_{j\ell} - B_{i\ell}, Y_{i\ell} - \epsilon_{i\ell} \leq -B_{i\ell}],
\end{aligned}$$

where $Y_{i\ell} \equiv U_{0\ell} - \alpha_\ell q_i - \beta_\ell X_i$ and the second inequality captures that choosing bidder i is better than the outside option. Letting $(q_i, X_i) = (q_j, X_j)$, this shows that the joint distribution of $\epsilon_{j\ell} - \epsilon_{i\ell}$ and $Y_{i\ell} - \epsilon_{i\ell}$ is identified provided the support of $B_{j\ell} - B_{i\ell}$ and $B_{i\ell}$ is sufficienty large. Thus, by Kotlarski (1967) and Evdokimov and White (2012), the distributions of $\epsilon_{i\ell}$, $\epsilon_{j\ell}$ and $Y_{j\ell}$ are identified under the independence of $(U_{0,\ell}, \alpha_\ell, \beta_\ell)$ and $(\epsilon_{i\ell}, \epsilon_{j\ell})$ with the normalization $\mathrm{E}[\epsilon_{j\ell}] = 0$ provided the bid support is large. Moreover, letting $X_i = X_j$ identifies the distribution of $\epsilon_{j\ell} - \epsilon_{i\ell} - \alpha_\ell(q_i - q_j)$ again under a large bid support condition. This identifies the distribution of $\alpha_\ell(q_i - q_j)$ from the identification of the distribution of $\epsilon_{j\ell} - \epsilon_{i\ell}$. Hence, the normalization $\mathrm{E}[\alpha_\ell] = 1$ identifies the distribution of α_ℓ as well as the difference $q_i - q_j$. Thus the normalization $\underline{q} = 0$ identifies the bidders unobserved quality levels. Similarly, identification of the distribution of β_ℓ is achieved by letting $q_i = q_j$. To identify the distribution of the buyer's option value $U_{0\ell}$, one considers auctions with only one entrant, i.e., $\mathcal{E}_\ell = \{i\}$. Thus, $\Pr[D_{i\ell} = 1|B_{i\ell}, X_i] = \Pr[U_{0\ell} - \alpha_\ell q_i - \beta_\ell X_i - \epsilon_{i\ell} \leq -B_{i\ell}]$. Letting $q_i = \underline{q} = 0$ identifies the distribution of $U_{0\ell} - \beta_\ell X_i - \epsilon_{i\ell}$ and hence of $U_{0\ell}$ under the independence of $U_{0\ell}$ and $(\beta_\ell, \epsilon_{i\ell})$ and a large bid support assumption.[44] Lastly, identification of the sellers' project cost distribution $F(\cdot|q_i, X_i)$ is obtained as in the GPV procedure from

$$C_{i\ell} = B_{i\ell} + \frac{H_{\mathcal{I}_\ell}(B_{i\ell}, q_i, X_i)}{H_{\mathcal{I}_\ell B}(B_{i\ell}, q_i, X_i)} \tag{6.11}$$

where $H_{\mathcal{I}_\ell}(B_{i\ell}, q_i, X_i)$ is bidder i's probability of winning conditional on his bid, quality, observed attributes and the set $\mathcal{I}_\ell$ of potential bidders in auction ℓ, whereas $H_{\mathcal{I}_\ell B}(\cdot, \cdot, \cdot)$ is its derivative with respect to $B_{i\ell}$. As in Section 5.3, the entry cost $\kappa_{i\ell}$ and hence its distribution $F_\kappa(\cdot|q_i, X_i)$ are identified from the zero expected profit condition.

[43]See also Berry and Haile (2014) for identification in differentiated product markets.

[44]The authors also show that the joint distribution of $(U_{0\ell}, \alpha_\ell)$ is identified under some rank conditions.

Regarding estimation, Krasnokutskaya, Song and Tang (2021) specify the distributions of the buyer's option value and weights $(U_{0\ell}, \alpha_\ell, \beta_\ell)$ parametrically with $\epsilon_{i\ell}$ i.i.d. Type I extreme value distributed. Using the above classification procedure for bidders' unobserved qualities, they apply GMM using the choice probabilities $\Pr[D_{i\ell} = 1|B_\ell, \mathcal{E}_\ell]$ where B_ℓ is the vector of bids in auction ℓ. In particular, this gives estimates of firm's unobserved qualities. The project cost distribution $F(\cdot|q_i, X_i)$ is estimated nonparametrically from (6.11). The entry cost distribution $F_\kappa(\cdot|q_i, X_i)$ is parameterized and estimated by GMM using firms' entry probabilities based on the zero expected pofit condition.

Endogenous Qualities

Up to now, quality is exogenous, similar to a seller's characteristic though it can be private instead of common knowledge. Quality can also be endogenous and part of a bidder's submission. Takahashi (2018) considers a model in which a bidder's submission is the pair (B_i, q_i), where there is uncertainty on how quality is evaluated by the buyer. Private information $\sigma_i \equiv (\kappa_i, c_i)$ is bidimensional, mutually independent, distributed as $F(\cdot,\cdot|X) = F_\kappa(\cdot|X)F_c(\cdot|X)$ and i.i.d. across the I competiting firms conditional on the project characteristics X. Specifically, bidder i's cost for a project of quality q_i is $C_i = \kappa_i + c_i q_i^\gamma$ with $\gamma > 1$, where κ_i and c_i represent fixed and variable costs, respectively. The buyer has a noisy evaluation $\tilde{q}_i \equiv q_i\epsilon_i$ of bidder i's quality where ϵ_i is an evaluation noise independent of q_i. Such a noise arises from averaging $T \geq 2$ evaluations $\tilde{q}_{it} \equiv q_i\epsilon_{it}$ with ϵ_{it} i.i.d. as $F_\epsilon(\cdot|X)$ across i and t. The buyer allocates the project to the lowest price-per-(noisy)quality ratio, i.e., the winner is the firm with the highest score $\tilde{q}_i/B_i$. The author shows that the equilibrium strategy is of the form

$$q_i = \left(\frac{R_i}{\gamma c_i}\right)^{\frac{1}{\gamma-1}} \quad \text{and} \quad B_i = R_i q_i, \tag{6.12}$$

where the first equation says that the marginal cost $\gamma c_i q_i^{\gamma-1}$ is equal to the bid-quality ratio $R_i = B_i/q_i$. This ratio maximizes a one-dimensional problem where $\theta_i \equiv \kappa_i c_i^{1/(\gamma-1)}$ aggregates the bidimensional type σ_i, and $H_{IT}(r_i|X)$ is firm i's winning probability with a price-quality ratio r_i conditional on (X, I, T) since T affects the precision of the evaluation noise ϵ_i. Specifically, R_i maximizes $[\lambda(R_i) - \theta_i]H_{IT}(R_i|X)$ where $\lambda(R_i) \equiv (\gamma - 1)(R_i/\gamma)^{\gamma/(\gamma-1)}$. This leads to the first-order condition

$$\theta_i = \lambda(R_i) + \lambda_R(R_i)\frac{H_{IT}(R_i|X)}{H_{ITR}(R_i|X)}, \tag{6.13}$$

where the subscript R denotes differentiation with respect to R.

We can establish identification of the model primitives $[F_\kappa(\cdot|\cdot), F_c(\cdot|\cdot), F_\epsilon(\cdot|\cdot), \gamma]$ from the observables $(D_{i\ell}, B_{i\ell}, \tilde{q}_{it\ell}, I_\ell, T_\ell)$, $i, \ldots, I_\ell$, $t = 1, \ldots, T_\ell$, $\ell = 1, \ldots, L$ as follows. Taking logarithm of (6.12) with the measurement equation of quality gives

$$\begin{aligned}\log \tilde{q}_{it} &= \log q_i + \log \epsilon_{it},\\ \log[B_i/\tilde{q}_{it}] &= \log R_i - \log \epsilon_{it},\\ \log[\tilde{R}_{it}/\tilde{q}_{it}^{\gamma-1}] &= \log[\gamma c_i] - \gamma \log \epsilon_{it},\end{aligned}$$

for $t = 1, \ldots, T$, where $\tilde{R}_{it} \equiv R_i/\epsilon_{it} = B_i/\tilde{q}_{it}$. Since $(B_i, \tilde{q}_{it})$ is observed, then the three LHS are observed provided γ is identified. Since $T \geq 2$, conditioning on X and applying Kotlarski (1967) and Li and Vuong (1998) on measurement errors with multiple indicators to each of the above equations establish identification and estimation of the distributions $G_q(\cdot|X)$ and $G_R(\cdot|X)$ of q_i and R_i as well as the distributions $F_c(\cdot|X)$ and $F_\epsilon(\cdot|X)$ under the normalization $\mathrm{E}[\log \epsilon_{it}] = 0$ (say). Given $G_R(\cdot|X)$, then (6.13) provides identification of the distribution $F_\theta(\cdot|X)$ of the aggregate type θ_i as in the first step of the GPV procedure. Since $F_c(\cdot|X)$ is identified, it follows from $\theta_i \equiv \kappa_i c_i^{1/(\gamma-1)}$ that $F_\kappa(\cdot|X)$ is identified by deconvolution as κ_i and c_i are mutually independent given X. It remains to show that γ is identified. Taking the expectation of the above third equation conditional on (X, I, T) implies the conditional moment restriction

$$\mathrm{E}[\log \tilde{R}_{it}|X, I, T] - (\gamma - 1)\mathrm{E}[\log \tilde{q}_{it}|X, I, T] - \log \gamma - \mathrm{E}[\log c_i|X] = 0,$$

since $\mathrm{E}[\log \epsilon_{it}|X, I, T] = 0$, where c_i is independent of (I, T) given X. Thus, fixing X, variations in (I, T) establishes identification of γ and $\mathrm{E}[\log c_i|X]$.[45]

Hanazono, Hirose, Nakabayashi and Tsuruoka (2018) extend the preceding model by allowing quality q_i and signal σ_i to be multidimensional with possibly different dimensions. They allow the components of σ_i to be correlated so that the signal vector σ_i is i.i.d. as $F(\cdot, \ldots, \cdot|X)$ across competing firms. The cost function is of the general form $C(q_i, \sigma_i, X) = \kappa_i + C_0(q_i, \sigma_{0i}, X)$ where $\sigma_i = (\kappa_i, \sigma_{0i})$ and the partial derivatives of $C_0(\cdot, \cdot, X)$ with respect to each component of q_i and σ_{0i} are positive. They consider an announced general scoring rule $r(B_i, q_i, X)$ for which the partial derivative with respect to B_i and each component of q_i are positive and negative, respectively. Thus, the winner is the firm with the lowest score, where quality q_i is perfectly observed by the buyer as well as the analyst.[46] We only present the case where $\dim q_i = \dim \sigma_{0i} \geq 1$.

[45]Following Krasnokutskaya (2011), Takahashi (2018) adds some random effects for quality, fixed and variable costs as well as for reviewers to account for unobserved heterogeneity. These random effects are assumed to be independently and normally distributed. Moreover, he restricts $[F_\kappa(\cdot|\cdot), F_c(\cdot|\cdot), F_\epsilon(\cdot|\cdot)]$ to be location invariant with linear means conditional on X. Estimation is parametric.

[46]The authors also consider a second-score auction mechanism where the winner has the lowest score but is paid B for a project of quality q satisfying $r(B, q, X)$ equal to the second-highest score.

Bidder i's expected profit is $[B_i - \kappa_i - C_0(q_i, \sigma_{0i}, X)]H_I(B_i, q_i, X)$ where $H_I(B_i, q_i, X) = \Pr[r(B_i, q_i, X) \leq \max_{j \neq i} r(B_j, q_j, X)|X, I]$ is his winning probability for a bid (B_i, q_i) given (X, I). By symmetry and independence, we have $H_I(B_i, q_i, X) = G_R^{I-1}(R_i|X, I)$ where $R_i \equiv r(B_i, q_i, X)$ is bidder i's score and $G_R(\cdot|X, I)$ its distribution given (X, I) with density $g_R(\cdot|X, I)$. Moreover, because the scoring rule is increasing in B_i, let $b(R, q_i, X)$ be the bid level that ensures a score R given (q_i, X), i.e., $R = r[b(R, q_i, X), q_i, X]$. Since maximization with respect to (B_i, q_i) is equivalent to maximization with respect to (R_i, q_i), the first-order conditions for bidder i's best response are

$$\begin{aligned} \kappa_i + C_0(q_i, \sigma_{0i}, X) &= b(R_i, q_i, X) - b_R(R_i, q_i, X)\frac{G_R(R_i|X, I)}{(I-1)g_R(R_i|X, I)}, && (6.14) \\ C_{0q}(q_i, \sigma_{0i}, X) &= b_q(R_i, q_i, X), && (6.15) \end{aligned}$$

where the subscripts R and q denote differentiation of $b(\cdot, \cdot, X)$ with respect to these variables. Equation (6.15) determines firm i's optimal quality vector q_i given R_i, while (6.14) determines the optimal score R_i and hence optimal bid B_i. Following Asker and Cantillon (2008), these are referred as the inner and outer loops. Under some assumptions, the authors show that the resulting vector (B_i, q_i) and hence $B_i = b(R_i, q_i, X)$ as functions of $\sigma_i = (\kappa_i, \sigma_{0i})$ and X constitute a Bayesian Nash equilibrium following Reny (2011).

The model primitive is the joint distribution $F(\cdot, \ldots, \cdot|X)$ of σ_i given X as the functional form of the variable cost $C_0(\cdot, \cdot, X)$ is known. The analyst knows the scoring rule $r(B_i, q_i, X)$ and observes the bid-quality tuples $(B_i, q_i), i = 1, \ldots, I$ and the covariates X. Thus, the functions $b(\cdot, \cdot, X)$, $b_R(\cdot, \cdot, X)$, $b_q(\cdot, \cdot, X)$ and $C_{0q}(\cdot, \cdot, X)$ are known. Hence, provided $C_{0q}(q_i, \cdot, X)$ is invertible, (6.15) identifies σ_{0i}. Identification of κ_i follows from (6.14) where $B_i = b(R_i, q_i, X)$, and $R_i = r(B_i, q_i, X)$. This establishes the identification of $F(\cdot, \ldots, \cdot|X)$. Regarding estimation, one needs only to estimate the score distribution $G_R(\cdot|X, I)$ in (6.14) to estimate κ_i since σ_{0i} can be exactly recovered from (6.15).

Sant' Anna (2018) considers an auction of oil lease rights in which a bidder's submission is a pair (B_i, q_i), while private information is bidimensional $\sigma_i \equiv (V_i, c_i)$ i.i.d. as $F(\cdot, \cdot|X)$ across the I competiting firms conditional on the project characteristics X. The non-monetary dimension q_i represents the exploratory effort with costs $c_i q_i$, whereas V_i is firm i's gross private value for the oil tract. The seller allocates the project to the highest score according to the interdependent scoring rule

$$r(B_i, q_i, B_{-i}, q_{-i}) \equiv \gamma_b \frac{B_i}{\max_{j \neq i} B_j} + \gamma_q \frac{q_i}{\max_{j \neq i} q_j}, \tag{6.16}$$

where γ_b and γ_q are known positive weights summing to one. There is a reserve price p_0 as bids smaller than p_0 are discarded. Bidders know (p_0, X) and the scoring rule but do not know the number I^* of actual bidders. Let $H(B_i, q_i|X, p_0)$ be the probability of winning when submitting (B_i, q_i) given (X, p_0) as the number of potential bidders is assumed constant across auctions. Maximizing firm i's expected profit $(V_i - c_i q_i - B_i)H(B_i, q_i|X, p_0)$ with respect to (B_i, q_i) gives

$$\begin{aligned} V_i &= B_i + \frac{H(B_i, q_i|X, p_0)}{H_B(B_i, q_i|X, p_0)} + q_i \frac{H_q(B_i, q_i|X, p_0)}{H_B(B_i, q_i|X, p_0)}, \\ c_i &= \frac{H_q(B_i, q_i|X, p_0)}{H_B(B_i, q_i|X, p_0)}, \end{aligned}$$

for $B_i \geq p_0$ where the subscripts q and B refer to the partial derivatives of $H(\cdot, \cdot|X)$. The first equation is similar to (3.6) except for an additional term due to the exploratory cost $c_i q_i$. As the RHS in both equations are observed, identification and estimation of the joint distribution $F(\cdot, \cdot|X)$ of (V_i, c_i) follows subject to a support assumption on p_0. In particular, the author exploits the knowledge of the scoring rule (6.16) to estimate $H(\cdot, \cdot|X, p_0)$. One can estimate the latter by

$$\hat{H}(b, q|x, p_0) = \frac{\sum_{\ell=1}^{L} \frac{1}{I_\ell^*} \tilde{K}\left(\frac{R_i(b,q)}{h_R}\right) K_X\left(\frac{x - X_\ell}{h_X}\right) K_0\left(\frac{p_0 - P_{0\ell}}{h_0}\right)}{\sum_{\ell=1}^{L} K_X\left(\frac{x - X_\ell}{h_X}\right) K_0\left(\frac{p_0 - P_{0\ell}}{h_0}\right)},$$

where $R_i(b, q) \equiv r(b, q, B_{-i\ell}, q_{-i\ell}) - \max_{j \neq i} r(B_{j\ell}, q_{j\ell}, b, q, B_{-\{ij\}\ell}, q_{-\{ij\}\ell})$, $\tilde{K}(u) = \int_{-\infty}^{u} K(v)\, dv$, $K(\cdot)$, $K_X(\cdot)$, $K_0(\cdot)$ are kernel functions, and h_R, h_X, h_0 are bandwidths. To minimize the curse of dimensionality, the author applies Haile, Hong and Shum (2005) homogeneization of the bids. See Section 3.3.

Section 6.3: Auctions of Contracts

A contract is an economic term designating an arrangement between two individuals. A standard procurement auction can be viewed as a fixed-price contract where the auction aspect simply allows the buyer to select the least expensive firm through a competitive process. When the contract covers several features, it is typicallly allocated through a scoring rule to the bidder with the highest/lowest score. Thus, auctions of contracts can be viewed as special cases of scoring auctions reviewed in Section 6.2. A difference is that the total payment is contingent to some ex post realizations so that it is not necessarily equal to the winner's bid at the time of the auction. We review here recent papers that fall within such a definition.

Scale Auctions

Scale or unit-price auctions are frequently used for selling timber or procuring construction projects. The seller/buyer provides information on some estimated quantities associated with the project to the participants. For instance, in timber auctions, the US Forest Service gives estimated quantities of the various timber species on a tract. In construction procurements, the buyer provides estimated quantities of various materials such as concrete necessary for the construction project. Bidders submit a unit-bid for every item and the seller/buyer computes a score by summing the unit-bids multiplied by the estimated quantities. The winner is the firm with the lowest (highest) score in a procurement auction (auction). An interesting feature is the socalled skewed/unbalanced bidding as shown by Athey and Levin (2001) in a common value setting. That is, sellers tend to bid higher when quantities are underestimated and lower when they are overestimated. Thus, bidders skew their bids to benefit from additional rent extraction. Moreover, when bidders are risk averse, bid skewing is a tool for risk diversification.

Bajari, Houghton and Tadelis (2014) develop a model of scale auctions in which the procured project is composed of K items with a quantity q_k^e estimated by the buyer for each item $k = 1, \ldots, K$. Each bidder $i = 1, \ldots, I$, submits a bid vector $B_i = (B_{i1}, \ldots, B_{iK})$ where B_{ik} is the unit-bid on item k. The score for bidder i is computed as $R_i = \sum_{k=1}^K B_{ik} q_k^e$, and the project is allocated to the bidder with the lowest score. In contrast, the quantities actually used for the project are $q_k, k = 1, \ldots, K$, and the winner is paid $\sum_{k=1}^K B_{ik} q_k - \lambda(B_i|X)$ where $\lambda(\cdot|X)$ is a known convex penalty increasing with the skewness of a seller's bid B_i relative to some engineer unit-bid estimates $b^e = (b_1^e, \ldots, b_K^e)$.[47] The actual quantities q_k are unknown at the time of procurement, common to all bidders, and independent of bidders' private information. Bidder i's unit-cost for item k is c_{ik} which is private information so that his actual total private cost is $\sum_{k=1}^K c_{ik} q_k$. In particular, private information is multidimensional with $\sigma_i = (c_{i1}, \ldots, c_{iK})$ and independently distributed across bidders as $F_i(\cdot, \ldots, \cdot|X)$ thereby allowing for full asymmetry. See Section 4.2. The covariates X include $(q_k^e, b_k^e), k = 1, \ldots, K$. Bidders are risk neutral and the set $\mathcal{I}$ of different bidders is known at the time of each procurement.

Bidder i's expected profit is $\mathrm{E}[\sum_{k=1}^K (B_{ik} - c_{ik}) q_k - \lambda(B_i|X)|X] \prod_{j \neq i} [1 - G_j(R_i|X, \mathcal{I})]$ where the expectation is with respect to $(q_1, \ldots, q_k)$ and $G_j(\cdot|X, \mathcal{I})$ is the equilibrium distribution of the score R_j of bidder j conditional on $(X, \mathcal{I})$. As in Asker and Cantillon (2008), maximization of expected profit is decomposed as a maximization of the expectation term (or expected profit upon winning) with respect to $B_i = (B_{i1}, \ldots, B_{iK})$ subject to a score $R_i = \sum_{k=1}^K B_{ik} q_k^e$ (inner loop) followed by a maximization with respect to R_i (outer loop). Let the solution of the first problem be $b(R_i, X) \in I\!R_+^K$, which depends on

[47] As a matter of fact, Bajari, Houghton and Tadelis (2014) include some additional revenues net of their costs due to project adjustments, extra works and deductions. We abstract from these features.

X through the conditional means $\mu_k \equiv E[q_k|X]$, $k = 1\ldots, K$ and the penalty $\lambda(\cdot|X)$.[48] The first-order condition for the second problem is

$$\theta_i = \sum_{k=1}^{K} B_{ik}\mu_k - \frac{\sum_{k=1}^{K} b_R(R_i, X)[\mu_k - \lambda_{B_k}(B_i|X)]}{\sum_{j\neq i} \frac{g_j(R_i|X,\mathcal{I})}{1-G_j(R_i|X,\mathcal{I})}} - \lambda(B_i|X), \tag{6.17}$$

where θ_i is the private expected total cost $\sum_{k=1}^{K} c_{ik}\mu_k$, whereas the subscripts R and B_k indicate partial derivatives and $g_j(\cdot|X,\mathcal{I})$ is the density of $G_j(\cdot|X,\mathcal{I})$. When $\mu_k = q_k^e$ and $\lambda(b^e|X) = \lambda_{B_k}(b^e|X) = 0$, (6.17) reduces to (4.6) for a fully asymmetric procurement with the private cost θ_i and the score R_i as bidder i's univariate private information and bid, respectively. Estimation is based on (6.17). To estimate $G_j(\cdot|X,\mathcal{I})$ and its density in a first step, the authors assume that the asymmetry reduces to a location-scale family, namely, $G_j(\cdot|X,\mathcal{I}) = G_{\mathcal{I}}[(\cdot/\sum_{k=1}^{K} b_k^e q_k^e) - X_j'\beta - \eta]$ where X_j include firm j's characteristics and observed project covariates. In a second step, the authors rewrite the aggregate type as $\theta_i = (1+\epsilon_i)\sum_{k=1}^{K} b_k^e\mu_k$ for some scalar random variable ϵ_i independent across i. Upon parameterizing the conditional mean $\mathrm{E}[\epsilon_i|X]$, the conditional means $\mu_k = E[q_k|X]$, $k = 1,\ldots,K$, and the skewing penalty $\lambda(\cdot|X)$, estimation proceeds by GMM based on (6.17) with the score distributions and densities estimated in a first step.

Bolotnyy and Vasserman (2020) draw from Bajari, Houghton and Tadelis (2014). They drop the skewing penalty and let bidders be risk averse with von Neuman-Morgenstern utility function $U(\cdot)$. See Section 5.1. They restrict bidders' private information σ_i to be one-dimensional by assuming bidder i's unit-cost for item k to be $\sigma_i c_k$ where c_k is the market unit-price for item k. Thus σ_i captures bidder i's cost (in)efficiency which is distributed as $F_i(\cdot|X)$ independently across i. The covariates X include $(q_k^e, c_k), k = 1,\ldots,K$. As above, bidder i's score is $R_i = \sum_{k=1}^{K} B_{ik}q_k^e$. Upon winning, bidder i is paid $\sum_{k=1}^{K} B_{ik}q_k$ where the actual quantity q_k is distributed as $F_k(\cdot|X)$ independently across k. Hence, bidder i's expected profit is $\mathrm{E}\{U[\sum_{k=1}^{K}(B_{ik} - \sigma_i c_k)q_k]|X\}\prod_{j\neq i}[1 - G_j(R_i|X,\mathcal{I})]$ where the expectation is taken with respect to $(q_1,\ldots,q_k)$ and $G_j(\cdot|X,\mathcal{I})$ is bidder j's equilibrium score distribution. A difference with Luo, Perrigne and Vuong (2018a) is that ex post uncertainty affects both bids and costs through the uncertainty in q_k.

The authors parameterize $F_k(\cdot|X)$ as normal $\mathcal{N}(\mu_k, \omega_k^2)$ with μ_k and $\log\omega_k$ linear in X, while taking $U(\cdot)$ to be CARA, i.e., $U(x) = 1 - \exp(-\gamma x)$. As above, maximization of expected profit is in two steps: One maximizing expected profit upon win-

[48]The solution $b(R_i, X)$ does not depend on $(c_{i1},\ldots,c_{iK})$. In contrast, by (6.17) the score R_i depends on the latter through the expected total cost $\theta_i \equiv \sum_{k=1}^{K} c_{ik}\mu_k$. Thus the unit-bid vector $B_i = b(R_i, X)$ depends on $\sigma_i = (c_{i1},\ldots,c_{iK})$ only through the aggregate type θ_i. As a consequence, the joint distribution of $(c_{i1},\ldots,c_{iK})$ is not identified without further assumptions.

ning $\mathrm{E}\{U[\sum_{k=1}^K(B_{ik}-\sigma_i c_k)q_k]|X\}$ with respect to $(B_{i1},\ldots,B_{iK})$ subject to a given score $R_i=\sum_{k=1}^K B_{ik}q_k^e$, followed by a maximization with respect to R_i. The first part gives

$$B_{ik}=\sigma_i c_k+\frac{\mu_k}{\gamma\omega_k^2}+\frac{q_k^e/\omega_k^2}{\sum_{k=1}^K q_k^{e2}/\omega_k^2}\left[R_i-\sum_{k=1}^K\left(\sigma_i c_k q_k^e+\frac{\mu_k q_k^e}{\gamma\omega_k^2}\right)\right]\equiv b_k(R_i,\sigma_i,X), \quad (6.18)$$

which is linear in R_i and σ_i. In contrast to Bajari, Houghton, Tadelis (2014), $b_k(R_i,\sigma_i,X)$ also depends on σ_i. Inserting $b_k(R_i,\sigma_i,X)$ for B_{ik} in bidder i's expected profit gives $\overline{U}(R_i,\sigma_i)\prod_{j\neq i}[1-G_j(R_i|X,\mathcal{I})]$. Maximizing the latter with respect to R_i gives the optimal score $\rho_{\mathcal{I}}(\sigma_i,X)$ and hence unit-bids $B_{ik}=b_k(R_i,\sigma_i,X)$ as functions of σ_i. For each bidder i, the only source of randomness in the K bidding equations (6.18) comes from the one-dimensional private information σ_i. The authors then introduce K zero-mean measurement errors η_{ik} leading to $\tilde{B}_i=B_{ik}+\eta_{ik}$, while treating σ_i as essentially fixed through the specification $\sigma_i=\delta_i+\delta' X$ where δ_i is bidder i's fixed effect. The data are $(\tilde{B}_{ik\ell},q_{k\ell},X_\ell,I_\ell)$, $i=1,\ldots,I_\ell,k=1,\ldots,K_\ell,\ell=1,\ldots,L$, where X_ℓ includes $(q_{k\ell}^e,c_{k\ell}),k=1,\ldots,K_\ell$. The (ex post auction) quantities $q_{k\ell}$ are used to estimate the mean $\mu_{k\ell}$ and variance $\omega_{k\ell}^2$ conditional on X_ℓ. Replacing $\mu_{k\ell}$ and $\omega_{k\ell}^2$ by such estimates and the true score $R_{i\ell}$ by $\tilde{R}_{i\ell}\equiv\sum_{k=1}^K\tilde{B}_{ik\ell}q_{k\ell}^e$, the bidding equations (6.18) are estimated by GMM based on the errors $\eta_{ik\ell}$ being uncorrelated with X_ℓ.

Luo and Takahashi (2019) build a scale auction model inspired by Ewerhart and Fieseler (2003) that allows for an indivisible item and several variable items.[49] The quantity of the indivisible item is normalized to one, while the estimated quantities of the K variable items are $(q_1^e,\ldots,q_K^e)$ with unit-costs $(c_1,\ldots,c_K)$ as above. Each bidder must bid on every item and bidder i's score is $R_i=B_{i0}+\sum_{k=1}^K B_{ik}q_k^e$. Upon winning, bidder i is paid $B_{i0}+\sum_{k=1}^K B_{ik}q_{ik}$ where $(q_{i1},\ldots,q_{iK})$ are the quantities for the K variable items actually used by bidder i. In contrast to Bolotnyy and Vasserman (2020), each bidder's private information is multi-dimensional with $\sigma_i=(C_{i0},\sigma_{i1},\ldots,\sigma_{iK})\in I\!R^{K+1}$ i.i.d. as $F(\cdot,\ldots,\cdot|X)$ across i where C_{i0} is bidder i's private cost for the indivisible item. Bidder i's actual quantities for the K variable items satisfy $q_{ik}=q_k^e(\sigma_{ik}+\epsilon_{ik})$ where $(\epsilon_{i1},\ldots,\epsilon_{iK})$ are expost shocks jointly distributed as $\mathcal{N}_K(0,\Omega)$ independently of $(\sigma_1,\ldots,\sigma_I)$. Bidders have a CARA utility function $U(x)=1-\exp(-\gamma x)$. Hence, bidder i's expected profit is $\mathrm{E}\{U[B_{i0}-C_{i0}+\sum_{k=1}^K(B_{ik}-c_k)q_k^e(\sigma_{ik}+\epsilon_{ik})]|X\}[1-G(R_i|X,I)]^{I-1}$ where the expectation is taken with respect to $(\epsilon_{i1},\ldots,\epsilon_{iK})$, I is the known number of bidders and $G(\cdot|X,I)$ is an arbitrary bidder's equilibrium score distribution since the game is symmetric.[50]

[49] Luo and Takahashi (2019) add an entry model *a la* Levin and Smith (1994) where the number of entrants is revealed before the bidding stage. See Section 5.2. They also account for project unobserved discrete heterogeneity through a finite mixture. See Section 4.4. To simplify, we omit such features.

[50] Asymmetry could be easily accounted for. See Section 4.2. An alternative empirically richer model

As in Asker and Cantillon (2008), the inner loop gives the first-order conditions

$$(B_{i1}q_1^e, \ldots, B_{iK}q_K^e) = (c_1q_1^e, \ldots, c_Kq_K^e) + \frac{1}{\gamma}(\sigma_{i1}-1, \ldots, \sigma_{iK}-1)\Omega^{-1} \tag{6.19}$$

with the constraint $R_i = B_{i0} + \sum_{k=1}^K B_{ik}q_k^e$. The outer loop reduces to $\max_{R_i} U(R_i - \theta_i)[1 - G(R_i|X,I)]^{I-1}$ given the one-dimensional aggregate type

$$\theta_i \equiv C_{i0} + \sum_{k=1}^K c_kq_k^e - \frac{1}{2\gamma}(\sigma_{i1}-1, \ldots, \sigma_{iK}-1)\Omega^{-1}(\sigma_{i1}-1, \ldots, \sigma_{iK}-1)'. \tag{6.20}$$

Similarly to (5.2), the first-order condition for the score bidding step is

$$\theta_i = R_i - \lambda^{-1}\left[\frac{1 - G(R_i|X,I)}{(I-1)g(R_i|X,I)}\right], \tag{6.21}$$

where $\lambda(x) \equiv U(x)/U'(x) = [\exp(-\gamma x) - 1]/\gamma$ is the CARA fear-of-ruin function.

The observations are $(B_{ik\ell}, q_{ik\ell}, X_\ell, I_\ell)$, $i = 1, \ldots, I_\ell, k = 1, \ldots, K_\ell, \ell = 1, \ldots, L$ where X_ℓ includes $(q_{k\ell}^e, c_{k\ell}), k = 1, \ldots, K_\ell$ as in Bolotnyy and Vasserman (2020). From Guerre, Perrigne and Vuong (2009), (6.21) identifies nonparametrically the distribution of the aggregate type $F_\theta(\cdot|X)$ as well as the function $\lambda(\cdot)$. Alternatively a parametric specification of a quantile of $F_\theta(\cdot|X)$ identifies $F_\theta(\cdot|X)$ and γ by Campo, Guerre, Perrigne and Vuong (2011). See Section 5.1. Identification of the distribution $F(\cdot, \ldots, \cdot|X)$ of $(C_{i0}, \sigma_{i1}, \ldots, \sigma_{iK})$ and the variance Ω is as follows. For any given value of Ω, (6.19) recovers $(\sigma_{i1}, \ldots, \sigma_{iK})$ and hence their joint distribution. Such distribution is also equal to that obtained by deconvoluting the measurement equations $q_{ik}/q_{ik}^e = \sigma_{ik} + \epsilon_{ik}$ as $(\epsilon_{i1}, \ldots, \epsilon_{iK}) \sim \mathcal{N}_K(0, \Omega)$. This determines Ω. Thus, (6.20) recovers C_{i0} given θ_i from (6.21) and $(\sigma_{i1}, \ldots, \sigma_{iK})$, thereby identifying $F(\cdot, \ldots, \cdot|X)$.[51] Upon parameterizing the distribution $F(\cdot, \ldots, \cdot|X)$ they estimate the model by ML on (6.19) and SMM on (6.21) using homogeneized bids. See Sections 3.1 and 3.3.

Royalty Auctions

A typical contract involves a fixed payment and a sharing rule. Following the contract literature with adverse selection and moral hazard, McAfee and McMillan (1987b) and Laffont and Tirole (1987) develop models of auctioning contracts where each bidder's private information is one-dimensional. Building on this literature, Kong, Perrigne and

would consider a pure risk on the actual quantities as $q_{ik} \equiv q_k^e \epsilon_{ik}$ and private cost inefficiency for each item as $\sigma_{ik}c_k$. This would lead to the ex post profit as $B_{i0} - C_{i0} + \sum_{k=1}^K (B_{ik} - \sigma_{ik}c_k)q_k^e\epsilon_{ik}$, which is comparable to Bolotnyy and Vasserman (2020)'s expost profit $\sum_{k=1}^K (B_{ik} - \sigma_i c_k)q_k$ but with multidimensional private information on firm i's cost.

[51] Luo and Takahashi (2019) show how to identify the model primitives when one observes $(B_{i0}, B_{i1}q_1^e, \ldots, B_{iK}q_K^e)$ but not (c_k, q_k^e).

Vuong (2020) study auctions of oil lease contracts with bidimensional private information. Each bidder proposes a royalty rate A_i and a cash/bonus payment B_i to the principal. Bidder i's contract value is $V(A_i;\sigma_i,X)$ where $\sigma_i \equiv (\sigma_{i0},\sigma_{i1}) \in I\!R^2$ is bidder i's private information. An example is $V(A_i;\sigma_i) = (1-A_i)\sigma_{i1} - \sigma_{i0}$ where σ_{i1} and σ_{i0} represent bidder i's expected revenue and cost, respectively.[52] In oil lease auctions, bidders bid on the option to exercise drilling and extraction. Following Black and Scholes (1973) and Merton (1973), the authors view $V(A_i;\sigma_i,X)$ as an option value since the winner may not exercise the option by not developing the tract. In this case, X also includes macroeconomic variables such as oil price, price volatility and interest rate. Bidders' private information σ_i are i.i.d. as $F(\cdot|X)$. The principal's allocation rule can be deterministic or probabilistic, announced or not, or even interdependent. In equilibrium it is subsumed in the winning probability $H_I(a,b|\sigma_i,X)$ when bidder i submits (a,b) given his private information σ_i, covariates X and the known number of bidders I. Since signals are independent, $H_I(a,b|\sigma_i,X) = H_I(a,b|X)$, which is increasing in (a,b).[53]

Maximizing bidder i's expected profit $[V(A_i;\sigma_i,X) - B_i]H_I(A_i,B_i|X)$ with respect to (A_i,B_i) conditional on (σ_i,X) gives the first-order conditions

$$V_A(A_i;\sigma_i,X) = -\frac{H_{IA}(A_i,B_i|X)}{H_{IB}(A_i,B_i|X)}, \tag{6.22}$$

$$V(A_i;\sigma_i,X) = B_i + \frac{H_I(A_i,B_i|X)}{H_{IB}(A_i,B_i|X)}, \tag{6.23}$$

where the subcripts A and B indicate partial derivatives with respect to the corresponding variables. These conditions can be obtained in two steps. The inner loop gives (6.22) by maximizing the expected profit upon winning $V(A_i;\sigma_i,X) - B_i$ subject to a given winning probability $H_I(A_i,B_i;X) = h$. The outer loop maximizes the expected profit with respect to h. In contrast to Asker and Cantillon (2008), A_i depends on h because the allocation rule is not necessarily quasi-linear. Relative to scoring auctions where the principal's payoff depends only on bid and qualities, a difference is that such a payoff is $B_i + A_i\sigma_{i1}$ and thus depends directly on the winner's private information. Bidders

[52] Their framework extends to the multi-dimensional case by letting $\dim A_i = \dim \sigma_i \equiv K \geq 2$. Thus, (6.22) becomes a K-dimensional vector equation where $H_{IA}(A_i,B_i;\sigma_i,X)$ is the gradient of $H_I(A_i,B_i;\sigma_i,X)$ with respect to A_i.

[53] Kong, Perrigne and Vuong (2020) actually consider symmetric affiliated private information where $(\sigma_1,\ldots,\sigma_I)$ is distributed as $F(\cdot,\ldots,\cdot|X)$, which is exchangeable in σ_i. See Section 4.3. Hence, σ_i appears in the RHS of (6.22)–(6.23) through $H_I(\cdot,\cdot|\sigma_i,X)$ and its partial derivatives. To achieve identification, they assume that the best response mapping $\sigma_i = (\sigma_{10},\sigma_{i1}) \to [a_I(\sigma_i;X), b_I(\sigma_i,X)]$, which is the solution of this revised system, is also invertible. Thus, conditioning on $\sigma_i = (\sigma_{i0},\sigma_{i1})$ is equivalent to conditioning on (A_i,B_i) so that $H_I(\cdot,\cdot|\sigma_i,X) = \tilde{H}_I(\cdot,\cdot|A_i,B_i,X)$.

exploit this incomplete information by trading off cash and royalty to the detriment of the principal. This relates to adverse selection. Bidding on royalty also introduces a moral hazard aspect since a high royalty reduces the winner's incentives to execute the option.

The observations are $(D_{i\ell}, A_{i\ell}, B_{i\ell}, X_\ell, I_\ell), i = 1, \ldots, I_\ell, \ell = 1, \ldots, L$. The value function $V(\cdot;\cdot,\cdot)$ is known. Assuming that the mapping $\sigma_i = (\sigma_{i0}, \sigma_{i1}) \to [V(A;\sigma_i;X), V_A(A;\sigma_i,X)]$ is invertible, (6.22)-(6.23) show that the bidimensional distribution $F(\cdot,\cdot|X)$ of $\sigma_i = (\sigma_{10}, \sigma_{i1})$ is identified because $H_I(\cdot,\cdot|X)$ and its partial derivatives are identified from the data. To see the latter, $H_I(a,b|X)$ can be decomposed as

$$H_I(a,b|X) = \int C_I(a,b,a_-,b_-,X)dG_{A_-,B_-|X,I}(a_-,b_-|X,I), \tag{6.24}$$

where $C_I(a,b,a_-,b_-,X)$ is the choice probability that a bidder with royalty-bonus (a,b) wins when his opponents bid (a_-,b_-) conditional on (X,I), whereas $G_{A_-,B_-|X,I}(\cdot,\cdot|X,I)$ is the $2(I-1)$-dimensional conditional distribution of the opponents' bids (A_-,B_-) given (X,I). These two functions are identified from the data. Thus taking derivatives of (6.24) with respect to a and b identifies the partial derivatives of $H_I(a,b|X)$ with respect to the first two arguments. Estimation follows the identification argument. By independence of bidders' signals and hence of their bids, we have $g_{A_-,B_-|X,I}(\cdot,\cdot|X,I) = \prod_{i=1}^{I-1} g_{A,B|X,I}(\cdot,\cdot|X,I)$ where $g_{A,B|X,I}(\cdot,\cdot|X,I)$ is the joint density of a royalty-bonus bid given (X,I). The latter can be estimated nonparametrically or semiparametrically using a parametric copula with marginal densities $g_A(\cdot|X,I)$ and $g_B(\cdot|X,I)$ estimated nonparametrically. Regarding the choice probability $C_I(a,b,a_-,b_-,X)$ in (6.24), the authors estimate it by sieve ML using Bernstein polynomials upon imposing monotonicity, exchangeability and anonymity. To minimize the curse of dimensionality induced by X, they apply Haile, Hong and Shum (2006) demeaning method. See Section 3.3. Inversion of (6.22)-(6.23) gives estimates of $\sigma_i = (\sigma_{i0}, \sigma_{i1})$, $i = 1, \ldots, I$ from which one estimates the joint distribution $F(\cdot,\cdot|X)$ of a bidders' signals $(\sigma_{i0}, \sigma_{i1})$ as in the GPV procedure.

Section 7: Ascending Auctions

Up to now, we have considered first-price sealed-bid auctions and alternative auction mechanisms that are analyzed using the indirect approach. In this section, we turn to another important class of auctions, namely ascending auctions. Except in Section 7.3, we view such auctions as button auctions in which the price continuously increases. In practice, ascending auctions may be different from button auctions. For instance, bidders bid several times along the bidding process and there is a minimum bid increment between two successive bids imposed by the seller. In the same vein, online auctions use a variety of ascending mechanisms. See Bajari and Hortacsu (2003).[54] Part of the empirical literature has abstracted away from these idiosyncratic features by focusing on the transaction price. The other bids are then used when they can be reasonably approximated as the outcomes of a button auction. We adopt such a point of view in Sections 7.1 and 7.2 by presenting methods using mainly the transaction prices. See Athey and Haile (2002) for the first contribution on the identification of ascending auctions.

As in previous sections, we consider the private value paradigm. In a button ascending auctions, bidders' dominant strategy is to withdraw when the ascending price reaches their private values. See Section 2.3. Thus, the winner is the remaining bidder who pays the value at which the bidder before him withdraws. This result holds when bidders are symmetric, asymmetric, risk averse as well as when private values are affiliated. On one hand, the fact that participants bid their private values in ascending auctions renders their bid analysis closer to a statistical problem than for first-price auctions. In particular, it involves a different methodology based on order statistics instead of first-order conditions. The main statistical challenge is that not all bids are observed since the winner's bid is never observed and often only the transaction price is available. On the other hand, more complex situations such as sequential auctions and auctions with bargaining can be analyzed as reviewed in Sections 7.2 and 7.3.

In Section 7.1, we review the identification and estimation of button ascending auctions within the private value paradigm whether bidders are symmetric or asymmetric and whether private values are independent or affiliated. The main idea relies on order statistics as the transaction price is the second highest private value. In particular, we address the issue of an unobserved number of bidders as data might not contain this information which is crucial for order-statistics. We also address the issue of auction unobserved heterogeneity. In Section 7.2, we review extensions such as bidders' risk aversion, entry and sequential auctions. Lastly, in Section 7.3 we review an important approach initiated by Haile and Tamer (2003) who deviate from the dominant strategy approximation by proposing an incomplete model based on minimal assumptions of rational behavior. Their approach allows to exploit all the observed bids and leads to set identification. We then discuss ascending auctions with bargaining. Throughout, we consider auctions instead of procurements. The results can be straightforwardly adapted to the latter.

Section 7.1: Identification and Estimation

We first consider the case of symmetric independent private values with and without a re-

[54] An interesting feature of online ascending auctions is the important bidding activity just before the auction stops known as sniping as noted by Bajari and Hortacsu (2004) in their survey of online auctions.

serve price whether the number of actual bidders is known or unknown to the analyst. We then consider asymmetric bidders, affiliated private values and unobserved heterogneity.

Symmetric Independent Private Values

Following Section 3, let bidders' private values $V_i, i = 1, \ldots, I$ be i.i.d. as $F(\cdot|X, I)$ on $[\underline{v}(X, I), \overline{v}(X, I)]$. In a button ascending auction with no reserve price, the transaction price P is the second-highest value $V^{(I-1:I)}$ among the I bidders' private values. The second-highest value is distributed as $F^{(I-1:I)}(\cdot|X, I)$ on $[\underline{v}(X, I), \overline{v}(X, I)]$. Thus, assuming that we observe the transaction price P and the number I of bidders in each auction, it suffices to show that we can recover uniquely the value distribution $F(\cdot|X, I)$ from the distribution $F^{(I-1:I)}(\cdot|X, I)$. As noted by Athey and Haile (2002), the distributions $F^{(i:I)}(\cdot|X, I)$ of the i-th lowest order statistic $V^{(i:I)}$ is related to $F(\cdot|X, I)$ by

$$F^{(i:I)}(v|X, I) = \frac{I!}{(i-1)!(I-i)!} \int_0^{F(v|X,I)} u^{i-1}(1-u)^{I-i} du \equiv \phi_{(i:I)}[F(v|X, I)], \tag{7.1}$$

for $v \in [\underline{v}(X, I), \overline{v}(X, I)]$, where $\phi_{(i:I)}(\cdot)$ is increasing on $[0, 1]$ for every $i = 1, \ldots, I$. See David and Nagaraja (2003). Thus, $\phi_{(I-1:I)}[F(v|X, I)] = IF^{I-1}(v|X, I) - (I-1)F^I(v|X, I)$ so that the distribution $F^{(I-1:I)}(\cdot|X, I)$ uniquely determines $F(\cdot|X, I)$ thereby establishing its identification from transaction prices and the number of bidders.

Regarding estimation, one can parameterize $F(\cdot|X, I)$ as $F(\cdot|X, I; \theta)$. Provided θ is identified from the first conditional moment of the transaction price, Laffont, Ossard and Vuong (1995) SNLL method applies to estimate θ using transaction prices in ascending auctions. See Section 3.1. Donald and Paarsch (1996) and Paarsch (1997) apply ML estimation. Because the transaction price P is equal to the second-highest statistic $V^{(I-1,I)}$, differentiating (7.1) for $i = I-1$ gives the likelihood of observing P in a generic ascending auction as

$$g_P(P|X, I; \theta) = I(I-1)F(P|X, I; \theta)^{I-2}[1 - F(P|X, I; \theta)]f(P|X, I; \theta), \tag{7.2}$$

where $f(\cdot|X, I; \theta)$ is the density of $F(\cdot|X, I; \theta)$. If all bids are observed (except for the winner's bid which is unobserved because the auction stops before reaching his valuation), they also show that the likelihood in a generic ascending auction becomes

$$g(B^{(1:I)}, \ldots, B^{(I-1:I)}|X, I; \theta) = I![1 - F(B^{(I-1:I)}|X, I; \theta)] \prod_{i=1}^{I-1} f(B^{(i:I)}|X, I; \theta), \tag{7.3}$$

where $B^{(1:I)} \leq, \ldots \leq B^{(I-1:I)} = P$ are the ordered observed bids, while the term $[1 - F(B^{(I-1:I)}|X, I; \theta)]$ reflects that the winner's unobserved bid is above the transaction price. Under standard regularity conditions on $F(\cdot|X, I, \theta)$, maximizing the likelihood obtained by taking the product of either (7.2) or (7.3) over L independent auctions provides a consistent and $\sqrt{L}$-asymptotically normal estimator of θ.[55]

Alternatively, we can readily develop a nonparametric estimator of $F(\cdot|X,I)$ by exploiting (7.1). Let $\hat{G}_P(\cdot|X,I)$ be a nonparametric estimator of the distribution of the transaction price P given (X,I). For instance,

$$\hat{G}_P(\cdot|x,I) = \frac{\sum_{\ell:I_\ell=I}^{L_I} 1\!\!I(P_\ell \le \cdot) K\left(\frac{x-X_\ell}{h}\right)}{\sum_{\ell:I_\ell=I}^{L_I} K\left(\frac{x-X_\ell}{h}\right)},$$

where L_I is the number of observations with I bidders, $K(\cdot)$ a kernel function, h a bandwidth and (P_ℓ, X_ℓ) are the observations on (P,X). Thus, a nonparametric estimator of $F(\cdot|X,I)$ is $\hat{F}(\cdot|X,I) \equiv \phi^{-1}_{(I-1:I)}[\hat{G}_P(\cdot|X,I)]$ since $\phi_{(I-1:I)}(\cdot)$ is invertible. When there are no covariates X so that $\hat{G}_P(\cdot|I) = \frac{1}{L_I}\sum_{\ell:I_\ell=I}^{L_I} 1\!\!I[P_\ell \le \cdot]$, such an estimator is consistent at the parametric rate $\sqrt{L_I}$ uniformly on any inner compact subset of the private value distribution $F(\cdot|X,I)$ by the Glivenko-Cantelli Theorem. Menzel and Morganti (2013), however, note that such a parametric rate does not necessarily extend to the whole support $[\underline{v}(I), \overline{v}(I)]$ of $F(\cdot|I)$ because the inverse function $\phi^{-1}_{(I-1:I)}(\cdot)$, though uniformly continuous, is not Lipschitz continuous on $[0,1]$.[56] Specifically, they show that the optimal uniform rate for estimating $F(\cdot|I)$ from the transaction price is $L_I^{\min\{\frac{1}{I-1},\frac{1}{2}\}}$. Such a rate is achieved by trimming the preceding estimator appropriately around 0 and 1 as

$$\tilde{F}(\cdot|I) \equiv \begin{cases} 0 & \text{if } \hat{G}_P(\cdot|I) < b_{n,0} \\ \phi^{-1}_{(I-1:I)}[\hat{G}_P(\cdot|I)] & \text{otherwise} \\ 1 & \text{if } 1 - b_{n,1} < \hat{G}_P(\cdot|I) \end{cases}$$

with $b_{n,0} = b_{n,1} \propto L_I^{-1}$.

Semiparametric Quantile Estimation

As for first-price sealed-bid auctions (see Section 3.4), quantile methods provide an alternative estimation approach to kernel methods. Following Koenker and Bassett (1978), Gimenes (2017) specifies the α-quantile $v(\alpha;X,I)$ of $F(\cdot|X,I)$ as

$$v(\alpha;X,I) = \gamma_0(\alpha;I) + X'\gamma_1(\alpha;I)$$

[55] Because the actual mechanism often differs from the button model, considering that the ordered bids beyond the transaction price reflect the order statistics of private values is a strong assumption.

[56] In particular, $\phi^{-1}_{(I-1:I)}(\cdot)$ is not Hadamard differentiable at 0 and 1 so that one cannot apply the functional delta method. See e.g. van der Vaart (1998). Moreover, Menzel and Morganti (2013) show that the expected revenue and optimal reserve price cannot be estimated at the parametric rate from transaction prices only.

for each $\alpha \in (0,1)$. Let $p(\alpha; X, I)$ be the α-quantile of the distribution of the transaction price $P = V^{(I-1:I)}$ given (X, I). Because $F^{(I-1:I)}(\cdot|X,I) = \phi_{(I-1:I)}[F(\cdot|X,I)]$ from (7.1) where $\phi_{(I-1:I)}(\cdot)$ is increasing on $[0,1]$, it follows that $v(\alpha; X, I) = p[\phi_{(I-1:I)}(\alpha); X, I]$, i.e., the α-quantile of the value distribution is equal to the $\phi_{(I-1:I)}(\alpha)$-quantile of the transaction price distribution. For each $I \in \mathcal{S}_I$ and $\alpha \in (0,1)$, this suggests to estimate the parameters $\gamma(\alpha; I) \equiv [\gamma_0(\alpha; I), \gamma(\alpha; I)]$ by the quantile regression

$$\min_{\gamma_0, \gamma_1} \sum_{\ell=1}^{L_I} \rho_{\phi_{(I-1:I)}(\alpha)}[P_\ell - \gamma_0 - X'\gamma_1],$$

where $\rho_{\phi_{(I-1:I)}(\alpha)}(\cdot)$ is the $\phi_{(I-1:I)}(\alpha)$-check function and L_I is the number of auctions with I bidders. The resulting estimator $\hat{\gamma}(\alpha; I)$ is consistent and $\sqrt{L_I}$-asymptotically normal. Similarly to Li, Perrigne and Vuong's (2003) method for estimating the optimal reserve price (4.4) from first-price seled-bid auctions, Gimenes (2017) estimates the latter from ascending auctions by maximizing a suitable quantile-based estimator of the seller's expected revenue. See her paper for details.

Reserve Price and Unknown Number of Bidders

As in Section 4.1, a binding announced reserve price $p_0 \in (\underline{v}(X,I), \overline{v}(X,I))$ introduces a truncation since potential bidders with valuations smaller than p_0 do not bid. Thus, the number I^* of actual bidders is random and distributed as a Binomial $\mathcal{B}[I, 1 - F(p_0|X,I)]$. When no bidder is participating ($I^* = 0$), the transaction price P is set at zero by convention. When there is at least one participating bidder ($I^* \geq 1$), then $P = \max\{p_0, V^{(I-1:I)}\}$. Hence, the distribution of the transaction price has two mass points: One at zero with probability $F(p_0|X,I)^I$ and one at p_0 with probability $\Pr[V^{(I-1:I)} \leq p_0 < V^{(I:I)}] = IF(p_0|X,I)^{I-1}[1 - F(p_0|X,I)]$ corresponding to $I^* = 1$. Moreover, conditional on $P > p_0$ or equivalently $I^* \geq 2$, the truncated distribution of P is

$$G_P^*(\cdot|p_0, X, I) \equiv \frac{F^{(I-1:I)}(\cdot|X,I) - F^{(I-1:I)}(p_0|X,I)}{[1 - F^{(I-1:I)}(p_0|X,I)} \tag{7.4}$$

on $[p_0, \overline{v}(X,I)]$ where $F^{(I-1:I)}(\cdot|X,I)$ is the distribution of $V^{(I-1:I)}$.

We now discuss four scenarios depending on information about the number I of potential bidders and the number I^* of actual bidders when the transaction price, the reserve price and the auction covariates are observed. Assume that (I, I^*) are observed. The probability $F(p_0|X,I)$ is identified from the Binomial distribution of I^*. Thus by (7.1), $F^{(I-1:I)}(p_0|X,I) = \phi_{(I-1:I)}[F(p_0|X,I)]$ is identified. Because $G_P^*(\cdot|p_0,X,I)$ is observed, (7.4) identifies $F^{(I-1:I)}(\cdot|X,I)$ and hence $F(\cdot|X,I) = \phi_{(I-1:I)}^{-1}[F^{(I-1:I)}(\cdot|X,I)]$ on $[p_0, \overline{v}(X,I)]$. Sufficient variations in p_0 identifies $F(\cdot|X,I)$ on its support. Upon parameterizing the value distribution as $F(\cdot|, I; \theta)$, Donald and Paarsch (1996) apply ML estimation. The likelihood of observing P given (p_0, X, I) in a generic auction is

$$g_P(P|p_0, X, I; \theta) = F(p_0|X, I; \theta)^{ID_0} \times \{IF(p_0|X, I; \theta)^{I-1}[1 - F(p_0|X, I; \theta)]\}^{D_1}$$
$$\times \{I(I-1)F(P|X, I; \theta)^{I-2}[1 - F(P|X, I; \theta)]f(P|X, I; \theta)\}^{1-D_0-D_1}; \quad (7.5)$$

where $D_0 \equiv \mathbb{1}(I^* = 0)$ and $D_1 \equiv \mathbb{1}(I^* = 1)$. Taking the product over L auctions and maximizing the resulting likelihood provide a consistent and $\sqrt{L}$-asymptotically normal estimator of θ under standard conditions. Alternatively, following Gallant and Nychka (1987), a nonparametric ML estimator can be used with the density $f(\cdot|X, I)$ approximated by a Hermite form.

A second data scenario consists in observing I^* but not I while assuming that I is exogenous, i.e., $F(\cdot|X, I) = F(\cdot|X)$. It can be shown that the conditional density of P given (p_0, X, I^*) is

$$g_P^*(P|p_0, X, I^*) = I^*(I^* - 1)F^*(P|p_0, X)^{I^*-2}[1 - F^*(P|p_0, X)]f^*(P|p_0, X) \quad (7.6)$$

for $I^* \geq 2$, where $F^*(\cdot|p_0, X) = [F(\cdot|X) - F(p_0|X)]/[1 - F(p_0|X)]$. Equation (7.6) resembles (7.2) with I, $F(\cdot|X, I; \theta)$ and $f(\cdot|X, I; \theta)$ replaced by I^*, $F^*(\cdot|p_0, X)$ and $f^*(\cdot|p_0, X)$, respectively. Thus, by considering (7.6) the analyst can proceed as if the number I^* of actual bidders is exogenous provided the transaction price P is viewed as the second highest bid $B^{(I^*-1:I^*)}$ in I^* bids independently drawn from the truncated distribution $F^*(\cdot|p_0, X)$. Adapting Gallant and Nychka (1987) and maximizing the resulting conditional likelihood obtained by taking the product of (7.6) over independent auctions with at least two actual bidders provide a consistent nonparametric ML estimator of $f^*(\cdot|\cdot, \cdot)$ on its support. In particular, if the lower bound of the support of p_0 given X is $\underline{v}(X)$, one can estimate consistently $f(\cdot|X)$ on $[\underline{v}(X), \overline{v}(X)]$. As a matter of fact, Paarsch (1997) proposes a parametric version of this conditional ML estimator using all the ordered bids $B^{(1:I^*)}, \ldots, B^{(I^*-1:I^*)}$.

In a third data scenario, one observes neither I nor I^* as in Song's (2014) study of online ascending auctions. Instead she exploits the availability of the transaction price and the second highest losing bid, i.e., $B^{(I^*-2,I^*)}$. Instead of conditioning on the number of actual bidders I^*, she conditions the likelihood on $B^{(I^*-2:I^*)}$. Indeed, assuming that I is exogenous so that $F(\cdot|X, I) = F(\cdot|X)$, she shows that the conditional density of the transaction price $P = B^{(I^*-1:I^*)}$ given $(B^{(I^*-2:I^*)}, p_0, X)$ is

$$g_P^*(P|B^{(I^*-2:I^*)}, p_0, X) = \frac{2[1 - F(P|X)]f(P|X)}{[1 - F(B^{(I^*-2:I^*)}|X)]^2} \quad \text{for } B^{(I^*-2:I^*)} \geq p_0, \quad (7.7)$$

which does not depend on the number of potential bidders I.[57] The RHS is the conditional density of the lowest order statistic in two independent draws from the truncated density $f(\cdot|X)/[1-F(B^{(I^*-2:I^*)}|X)$. Because the support of $B^{(I^*-2:I^*)}$ is $[p_0, \overline{v}(X)]$, then $f(\cdot|X)$ is identified on $[p_0, \overline{v}(X)]$ and hence on $[\underline{v}(X), \overline{v}(X)]$ with sufficient variations in p_0. Moreover, adapting Gallant and Nychka (1987) to approximate this truncated conditional density by a Hermite form, maximization of the conditional likelihood obtained by taking the product of (7.7) over independent auctions with three or more actual bidders provides a consistent estimator of $f(\cdot|X)$.

In a fourth data scenario, (I, I^*) are not observed but I is constant across auctions or on known subsets of auctions. We exploit the fact that the distribution of the transaction price has two mass points at zero and p_0 with probabilities $\Pr(P=0|p_0, X, I) = F(p_0|X, I)^I$ and $\Pr(P=p_0|p_0, X, I) = IF(p_0|X, I)^{I-1}[1-F(p_0|X, I)]$. This gives a system of two equations with a unique solution in $[I, F(p_0|X, I)]$ thereby identifying the latter.[58] Once these are known, the argument of the first data scenario applies to identify $F(\cdot|X, I)$ on $[p_0, \overline{v}(X, I)]$ and on $[\underline{v}(X, I), \overline{v}(X, I)]$ with sufficient variations in p_0. Regarding estimation, we first estimate the probabilities $\Pr(P=0|p_0, X, I)$ and $\Pr(P=p_0|p_0, X, I)$ to obtain estimates of I and $F(p_0|X, I)$. We then apply parametric or nonparametric ML to (7.5) by noting that $D_0 = \mathbb{1}(P=0)$ and $D_1 = \mathbb{1}(P=p_0)$.

Asymmetric Bidders

We now assume that private values are independent but not identically distributed, namely $V_i \sim F_i(\cdot|X, \mathcal{I})$ for $i \in \mathcal{I}$, where $\mathcal{I}$ is the set of potential bidders with cardinality I. Athey and Haile (2002) are the first to note the relationship between ascending auctions and general competing risk models. To establish the identification of $[F_1(\cdot|\cdot,\cdot), \ldots, F_I(\cdot|\cdot,\cdot)]$ from observations on the transaction price, the winner's identity and the set of potential bidders $\mathcal{I}$, they rely on Meilijson (1981) result. This says that the lifetime of a machine, which breaks down when k of its I independent components fail, combined with the knowledge of which components fail, identifies the lifetime distributions of all I components. Thus, when there is no reserve price, setting $k = I-1$ with P and B_i being the lifetimes of the machine and the i-th component gives the desired result.[59]

[57] Song (2004) actually establishes such a result for any pair of order statistics $(B^{(i:I^*)}, B^{(j:I^*)})$.

[58] Specifically, $I = \log\Pr(P=0|p_0, X, I)/\log F(p_0|X, I)$ where $F(p_0|X, I)$ uniquely solves $[F(p_0|X, I)\log F(p_0|X, I)]/[1-F(p_0|X, I)] = [\Pr(P=0|p_0, X, I)\log\Pr(P=0|p_0, X, I)]/\Pr(P=p_0|p_0, X, I)$.

[59] Komarova (2013a) allows the densities $f_i(\cdot|X, \mathcal{I})$, $i \in \mathcal{I}$, not to be bounded away from zero on their common support. Her identification analysis relies on showing that the system of I differential equations (7.8) with $W = i \in \mathcal{I}$ has a unique solution in $\{F_i(\cdot|\cdot,\cdot); i \in \mathcal{I}\}$. She provides a set of necessary and sufficient conditions for $\{g(\cdot, i|\cdot,\cdot); i \in \mathcal{I}\}$ to be rationalized as well as a sufficient condition for

When there is a reserve price p_0, Meilijson's (1981) result no longer applies as it requires the lifetimes of all components to be absolutely continuous. Following Kong (2018), we rely on Nowik (1990) to establish the identification of the distributions conditional on bidding $F_i(\cdot|V_i > p_0, X, \mathcal{I})$ as well as the probabilities of not bidding $\Pr(V_i \leq p_0|X, \mathcal{I})$. Identification of $F_i(\cdot|X, \mathcal{I})$ follows with sufficient variations in p_0 as above.

Regarding estimation, let W take its values in $\mathcal{I}$ to indicate the winner's identity. Assuming no reserve price to simplify, the joint density of (P, W) given $(X, \mathcal{I})$ is

$$g_P(P, W|X, \mathcal{I}) = [1 - F_W(P|X, \mathcal{I})] \frac{d}{dP} \left[\prod_{j \in \mathcal{I} \setminus W} F_j(P|X, \mathcal{I}) \right]. \tag{7.8}$$

Assuming that $F_i(P|X, \mathcal{I}) = F_i(P|X)$ for every i so that $\mathcal{I}$ is exogenous, Brendstrup and Paarsch (2006) develop a semi-nonparametric ML estimator following Gallant and Nychka (1987). Specifically, they let bidder i's private value density $f_i(\cdot|X)$ belong to the location family $f_i(\cdot - X'\beta_i)$ with $f_i(\cdot)$ taking the Hermite form

$$f_i(u) = \left[\sum_{k=1}^{K} \gamma_{ik} H_k(u) \right]^2 \exp^{-u^2/2} + \epsilon \exp^{-u^2/2},$$

where $H_k(\cdot)$ is the Hermite polynomial of degree k and ϵ is a small positive constant so as to prevent the density $f_i(\cdot)$ to vanish. Taking the product of (7.8) over L independent auctions and maximizing the resulting likelihood with respect to $(\beta_i, \gamma_{ik}), k = 1, \ldots, K, i \in \mathcal{I}$ subject to $f_i(\cdot)$ being a density provide consistent estimators of the private value densities $f_i(\cdot|X)$ when K diverges at a slower rate than L.

Affiliated Private Values

Athey and Haile (2002) show that the ascending auction model with affiliated private values (even symmetric) is not identified as soon as one bid is not observed. The argument is intuitive. For instance, when $I = 3$ one is able to capture at most the positive dependence between two out of three bids but not the full dependence necessary to identify the joint distribution $F(\cdot, \cdot, \cdot)$. In view of this non-identification result, Aradillas-Lopez, Gandhi and Quint (2013) adopt a partial identification approach while focusing on some functionals of interest such as the seller's expected profit and bidder's expected surplus. Both quantities are important in counterfactual analyses. Moreover, these only depend on the marginal distributions of the highest and second-highest order statistics. Indeed, with symmetric affiliated private values and I exogenous, the seller's expected profit and

identification. These conditions are satisfied when $f_i(\cdot|X, \mathcal{I})$ is continuous and bounded away from zero so that $0 < \lim_{p \downarrow \underline{v}(X, \mathcal{I})} g(p, i|X, \mathcal{I})/[p - \underline{v}(X, \mathcal{I})] < \infty$ for every $i \in \mathcal{I}$.

bidder's expected surplus given (p_0, X, I) are

$$\begin{aligned}\pi(p_0,X,I) &= \int_0^\infty \max(p_0,v)dF^{(I-1:I)}(v|X) - V_0 - F^{(I:I)}(p_0|X)(p_0-V_0), \quad (7.9)\\ S(p_0,X,I) &= \int_0^\infty \max(p_0,v)dF^{(I:I)}(v|X) - \int_0^\infty \max(p_0,v)dF^{(I-1:I)}(v|X), \quad (7.10)\end{aligned}$$

where V_0 denotes the seller's value and p_0 the reserve price. The analyst observes the transaction price P and (p_0, X, I) over L independent ascending auctions. The truncated distribution $G_P^*(\cdot|p_0, X, I) = [F^{(I-1:I)}(\cdot|X) - F^{(I-1:I)}(p_0|X)]/[1 - F^{I-1:I)}(p_0|X)]$ of the transaction price $P = V^{(I-1:I)}$ given $(P \geq p_0, X, I)$ is identified from the data. Hence, when the lower bound of the support of p_0 given X is $\underline{v}(X)$, the distribution $F^{(I-1:I)}(\cdot|X)$ of $V^{(I-1:I)}$ given X is identified. In contrast, the distribution $F^{(I:I)}(\cdot|X)$ of the highest order statistic $V^{(I:I)}$ given X is not identified because the winner's bid is unobserved and $(V_1, \ldots, V_I)$ are dependent so that (7.1) no longer holds.

When private values are symmetric and affiliated, Quint (2008) shows that

$$\left\{\phi^{-1}_{(I-1:I)}[F^{(I-1:I)}(\cdot|X)]\right\}^I \leq F^{(I:I)}(\cdot|X) \leq F^{(I-1:I)}(\cdot|X), \quad (7.11)$$

where $\phi_{(I-1:I)}(\cdot)$ is the increasing mapping in (7.1). The lower and upper bounds in (7.11) are sharp as they correspond to independent and perfectly correlated private values, respectively. Though such bounds can be used to bound the seller's profit and bidder's surplus in (7.9)-(7.10), the resulting bounds can be wide. To tighten them, Aradillas-Lopez, Gandhi and Quint (2013) exploit the exogeneity of I. From David and Nagaraja (2003), the authors iteratively apply the recurrence relation $(I+1)F^{(I:I)}(\cdot|X) = F^{(I:I+1)}(\cdot|X) + IF^{(I+1:I+1)}(\cdot|X)$ for every I, which holds whether private values are independent or not. When I varies in $\{2, 3, \ldots, \overline{I}\}$, they obtain

$$F^{(I:I)}(\cdot|X) = \sum_{m=I+1}^{\overline{I}} \frac{I}{(m-1)m} F^{(m-1:m)}(\cdot|X) + \frac{I}{\overline{I}} F^{(\overline{I}:\overline{I})}(\cdot|X). \quad (7.12)$$

Thus applying (7.11) with $I = \overline{I}$ to (7.12) gives

$$\begin{aligned}F_L^{(I:I)}(\cdot|X) &\equiv \sum_{m=I+1}^{\overline{I}} \frac{I}{(m-1)m} F^{(m-1:m)}(\cdot|X) + \frac{I}{\overline{I}} \left\{\phi^{-1}_{(\overline{I}-1:\overline{I})}[F^{(\overline{I}-1:\overline{I})}(\cdot|X)]\right\}^{\overline{I}}\\ &\leq F^{(I:I)}(\cdot|X) \leq \sum_{m=I+1}^{\overline{I}} \frac{I}{(m-1)m} F^{(m-1:m)}(\cdot|X) + \frac{I}{\overline{I}} F^{(\overline{I}-1:\overline{I})}(\cdot|X) \equiv F_U^{(I:I)}(\cdot|X).\end{aligned}$$

The bounds $[F_L^{(I:I)}(\cdot|X), F_U^{(I:I)}(\cdot|X)]$ are tighter than those in (7.11).[60] They depend on

[60] The authors show that such bounds also hold when private values are conditionally independent given some variable whether the latter is unobserved by bidders as in Li, Perrigne and Vuong (2000) or known to bidders but unobserved by the analyst in models with unobserved heterogeneity as in Krasnokutskaya (2011). Moreover, they show that the lower bound $F_L^{I:I}(\cdot|X, I)$ still holds when $F(\cdot|X, I)$ stochastically increases with I as in entry models such as in Marmer, Shneyerov and Xu (2010).

the distributions $F^{(m-1:m)}(\cdot|X)$, $m = I, \ldots, \overline{I}$ of the transaction price $P = V^{(m-1:m)}$ and hence are identified from data on (P, p_0, X, I) under the above support assumption on p_0. Using such bounds in (7.9)-(7.10) provides bounds $[\pi_L(p_0, X, I), \pi_U(p_0, X, I)]$ for the seller's expected profit and bounds $[S_L(p_0, X, I), S_U(p_0, X, I)]$ for bidder's expected surplus. In particular, the authors show that the reserve price p_0^* that maximizes the seller's expected profit $\pi(\cdot, X, I)$ satisfies $\pi_U(p_0^*, X, I) \geq \pi_L(p_0^*, X, I)$ thereby providing bounds for p_0^*. All these bounds are then estimated by plugging kernel estimators of $F^{(m-1:m)}(\cdot|X)$ for $m = I, \ldots, \overline{I}$ in the resulting expressions. They rely on Imbens and Manski (2004) and Stoye (2009) to derive confidence intervals for the functions of interest.

Coey, Larsen, Sweeney and Waisman (2017) extend these bounds to asymmetric affiliated private values by showing that (7.11)-(7.12) still hold under some conditions. Let $\mathcal{I}_o$ collects all potential bidders across auctions, and $\mathcal{I}$ be a random subset of $\mathcal{I}_o$.[61] The marginal distribution of the i-th highest order statistic among m participating bidders is

$$F^{(i:m)}(\cdot|X) \equiv \sum_{\mathcal{I} \subseteq \mathcal{I}_o : I=m} \Pr(\mathcal{I}|I\!=\!m, X)\ F^{(i:m)}(\cdot|X, \mathcal{I}),$$

where $F^{(i:m)}(\cdot|X, \mathcal{I})$ is the distribution of the i-th highest value in $\{V_i; i \in \mathcal{I}\}$ drawn from the multivariate distribution $F(\cdot, \ldots, \cdot|X, \mathcal{I})$ of dimension $I = m$. The authors show that

$$\mathrm{E}\left[\left\{\phi^{-1}_{(m-1:m)}[F^{(m-1:m)}(\cdot|X, \mathcal{I})]\right\}^m |X, I = m\right] \leq F^{(m:m)}(\cdot|X) \leq F^{(m-1:m)}(\cdot|X), \quad (7.13)$$

where the expectation is with respect to $\mathcal{I}$ given $(X, I = m)$, and $\phi_{(m-1:m)}(\cdot)$ is the mapping defined by (7.1). When $\mathcal{I}$ is observed, (7.13) plays the same role as (7.11). By Jensen's inequality, $\left\{\phi^{-1}_{(m-1:m)}[F^{(m-1:m)}(\cdot|X)]\right\}^m \leq \mathrm{E}\left[\left\{\phi^{-1}_{(m-1:m)}[F^{(m-1:m)}(\cdot|X, \mathcal{I})]\right\}^m\right]$ so that exploiting asymmetry when $\mathcal{I}$ is observed improves the lower bound in (7.11). Next, the authors show that (7.12) holds provided

$$\begin{aligned} &(i) && F(\cdot, \ldots, \cdot|X, \mathcal{I}) = F(\cdot, \ldots, \cdot|X, \mathcal{I}') \text{ for all } \mathcal{I} \subset \mathcal{I}', \\ &(ii) && \Pr(\mathcal{I}|I\!=\!m) = \sum_{\mathcal{I}' \supset \mathcal{I} : I'=m+1} \Pr_o(\mathcal{I}|\mathcal{I}') \Pr(\mathcal{I}'|I'\!=\!m+1) \text{ for all } (\mathcal{I}, \mathrm{m}), \end{aligned}$$

where $\Pr_o(\mathcal{I}|\mathcal{I}')$ is the probability that $\mathcal{I}$ is obtained by randomly dropping one bidder from $\mathcal{I}'$. With full asymmetry, $\Pr_o(\mathcal{I}|\mathcal{I}') = \frac{1}{m+1}$ since $I' = m+1$. Condition (i) is a natural extension of exogeneity of I to the asymmetric case following Athey and Haile (2002). Condition (ii) restricts how the set of participating bidders $\mathcal{I}$ is obtained from $\mathcal{I}_o$, or how potential bidders in $\mathcal{I}_o$ become bidders in $\mathcal{I}$. Combining (7.12) and (7.13) for $m = \overline{I}$, the

[61] In entry models as reviewed in Sections 5.2 and 7.2, we can view $\mathcal{I}_o$ as a fixed set of potential bidders and $\mathcal{I}$ as the set of participating/entering bidders, respectively.

authors derive bounds $[F_L^{(I:I)}(\cdot|X), F_U^{(I:I)}(\cdot|X)]$ for $F^{(I:I)}(\cdot|X)$, where $F_U^{(I:I)}(\cdot|X)]$ is as in Aradillas-Lopez, Gandhi and Quint (2013) whereas

$$F_L^{(I:I)}(\cdot|X) = \sum_{m=I+1}^{\overline{I}} \frac{I}{(m-1)m} F^{(m-1:m)}(\cdot|X) + \frac{I}{\overline{I}}\, \mathrm{E}\Big[\Big\{\phi^{-1}_{(\overline{I}-1:\overline{I})}[F^{\overline{I}-1:\overline{I}}(\cdot|X,\mathcal{I})]\Big\}^{\overline{I}} |X, I=\overline{I}\Big].$$

Bounds for the seller's expected profit $\pi(p_0, X, I)$, bidder's expected surplus $S(p_0, X, I)$, and optimal reservation price p_0^* follow as above from (7.9)-(7.10). Regarding estimation, $F^{(m-1:m)}(\cdot|X)$ is estimated by a standard kernel estimator $\hat{F}^{(m-1:m)}(\cdot|X)$ on auctions with m participating bidders suitably adjusted for truncation if there is a reserve price. The bound $\mathrm{E}\Big[\Big\{\phi^{-1}_{(\overline{I}-1:\overline{I})}[F^{\overline{I}-1:\overline{I}}(\cdot|X,\mathcal{I})]\Big\}^{I} |X, I=\overline{I}\Big]$ is estimated by the average of $\Big\{\phi^{-1}_{(\overline{I}-1:\overline{I})}[\hat{F}^{\overline{I}-1:\overline{I}}(\cdot|X,\mathcal{I})]\Big\}^{\overline{I}}$ over sets $\mathcal{I}$ with $\overline{I}$ participating bidders weighted by their empirical frequencies in auctions with $\overline{I}$ participating bidders.

Komarova (2013b) directly investigates partial identification of the joint distribution of bidders' private values when the latter are asymmetric and dependent. She considers a fixed set of potential bidders with no reserve price and no covariates. She then studies various data scenarios from observing only the transaction price and the winner's identity to observing all bids and identities (except for the winner's bid). In the former case, she obtains a tight upper bound for $F(v_1, \ldots, v_I)$ as $\sum_{i=1}^{I} G_i[\min\{v_i, \max_{j\neq i} v_j\}]$ where $G_i(\cdot) \equiv \Pr[P \leq \cdot, W = i]$. To obtain nontrivial lower bounds, she then exploits various forms of positive dependence that are weaker than affiliation.

Unobserved Heterogeneity

As in Section 4.4, unobserved heterogeneity is an important feature in analyzing bidding data. In ascending auctions within the IPV paradigm, Roberts (2013) considers the reserve price and the transaction price as noisy observations of the unobserved heterogeneity upon assuming that the reserve price is monotonic in the latter. Mbakop (2017) and Freyberger and Larsen (2021) exploit at least one losing bid in addition to the transaction price and the reserve price. In contrast, Hernandez, Larsen and Turansick (2020) use the transaction price only combined with variations in the exogenous number of bidders. They assume additive separable unobserved heterogeneity $V_i = Y + \epsilon_i$ as in Krasnokutskaya (2011), where Y and ϵ_i are mutually independent and distributed as $F_Y(\cdot)$ and $F_\epsilon(\cdot)$, respectively under some smoothness conditions (continuous and piecewise real analytic). Upon a normalization such as $\mathrm{E}(\epsilon_i) = 0$, the authors show that $[F_Y(\cdot), F_\epsilon(\cdot)]$ are identified from observations on (P, I). Because they exploit auctions with two different number of bidders (say) I_1 and I_2, we have $P_1 = Y_1 + \epsilon^{(I_1-1:I_1)}$ and $P_2 = Y_2 + \epsilon^{(I_2-1:I_2)}$ where the indices 1,2 refer to the auction with I_1 and I_2 bidders. Because $Y_1 \neq Y_2$, we are no longer in a measurement error setting and Kotlarski (1967) no longer applies. The authors establish identification using a proof by contradiction exploiting the piecewise real analyticity assumption. They extend their identification result when (i) I is unobserved but there exists an instrumental variable Z affecting its distribution but not the value distribution, and (ii) there is an entry model determining the distribution of the number of participating bidders I^*. Homogeneizing transaction prices as in Section 3.3, their application considers case (ii) with a parametric specification for the entry probabilities and a nonparametric ML estimator for $[F_Y(\cdot), F_\epsilon(\cdot)]$ following Gallant and Nychka (1987).

Section 7.2: Extensions

We review here important extensions for ascending auctions mainly following those discussed in Section 5 for first-price auctions. When bidders are risk averse, bidding private values remains a dominant strategy in button ascending auctions. Thus, because risk aversion is not identified, additional information is needed to identify risk aversion. Other important extensions are bidders' and seller's endogenous entry and sequential auctions.

Risk Averse Bidders

Without additional data, risk aversion in button ascending auctions is not identified. Specifically, bidders' private value distributions are identified but risk aversion is not since bidders' dominant strategy is to bid their private values irrespective of their level of risk aversion. Lu and Perrigne (2008) then combine ascending auctions and first-price auctions to identify risk aversion. See Section 5.1.

Ackerberg, Hirano and Shahriar (2017) exploit instead the buy-it-now option found in some online auctions such as eBay. Each potential bidder arriving in such an auction has three choices: (i) To buy the object at the seller's posted price $p_1 \geq p_0$ (if the buy-it-now option is still available), (ii) to tender a bid above the seller's reserve price p_0, in which case the buy-it-now option disappears for every bidder coming afterward, or (iii) to do nothing. Thus each auction consists of two phases: A buying phase that ends as soon as a bidder exercises the buy-it-now option or tenders a bid, followed by a bidding phase (if any) of length τ where the winner is the highest bidder who pays the maximum of the second-highest bid and the reserve price.[62] Potential bidders arrive at the auction according to a Poisson process of varying rate $\lambda(t) > 0$ where t indexes time starting from the beginning of the sale. Potential bidders' private values are independently drawn from

[62]The overall length of the auction is equal to the length of the buying phase plus τ. The authors accommodate other variants where the overall length of the auction is fixed (unless the buy-it-now option is exercised) as in eBay auctions, or where potential buyers can acquire the object at the posted price in a preliminary phase of fixed length as in General Motors Assistance Corporation's car auctions.

$F(\cdot)$. Bidders are risk averse with a concave utility function $U(\cdot)$ and decreasing time preference $\delta(\cdot) > 0$ that captures bidders' impatience. These are normalized by $U(0) = 0$, $U'(0) = 1$ and $\delta(0) = 1$. The analysis is conditional on auction covariates which are suppressed to simplify.

In a perfect Bayesian Nash equilibrium, the authors show that potential bidders with valuations smaller than p_0 do nothing. Potential bidders with valuations larger than p_0 arriving during the bidding phase bid their private values. In contrast, the first potential bidder arriving during the buying phase with valuation V larger than p_0 immediately either takes the buy-it-now option ($D = 1$) or bids his private value ($D = 0$) depending on whether $V \geq v^*$ or $V < v^*$, respectively, for some threshold $v^* > p_1$. The value v^* equates this bidder's utility $U(V - p_1)$ of taking the buy-it-now option to his expected utility of winning the bidding phase with a bid $B = V$. Letting T denote this bidder's arrival time, which is also the length of the buying phase, the authors show that his expected utility is

$$\overline{U}(V, p_0, \tau, T) = \delta(\tau) \left[\alpha(p_0, \tau, T) U(V - p_0) + \int_{p_0}^{V} U(V - y) h(y, \tau, T) dy \right], \qquad (7.14)$$

where $\alpha(p_0, \tau, T) \equiv \exp[\gamma F(p_0) - \gamma]$ is the probability that no other potential bidder arriving during the bidding phase $[T, T+\tau]$ has a value above p_0, $h(y, \tau, T) \equiv \gamma f(y) \exp[\gamma F(y) - \gamma]$ is the density of the maximum Y of other potential bidders' values arriving during the bidding phase, and $\gamma \equiv \int_T^{T+\tau} \lambda(t) dt$ is the expected number of such potential bidders. In particular, $\alpha(p_0, \tau, T) + \int_{p_0}^{\infty} h(y, \tau, T) dy = 1$ and $\partial \alpha(p_0, \tau, T)/\partial p_0 = h(p_0, \tau, T)$. Because of impatience and incoming competition, bidders take a decision upon their arrivals.

The authors show that the model primitives $[F(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)]$ are identified from observations on the reserve price p_0, the buy-it-now price p_1, the bidding length τ, the buying length T and the outcome D at T. They require some full support assumption on the distribution of (p_0, p_1, τ) but do not require observations on actual bidders' bids. In particular, they use neither the transaction price nor the number of potential or actual bidders. This differs from other papers in ascending auctions. Note that T given p_0 has conditional hazard rate $\lambda(t)[1 - F(p_0)]$ at t. Thus, provided the support of p_0 contains the support $[\underline{v}, \overline{v}]$ of $F(\cdot)$, setting $p_0 = \underline{v}$ identifies $\lambda(\cdot)$ from which $F(\cdot)$ is identified from variations in p_0 upon fixing t. This identifies γ, $\alpha(p_0, \tau, T)$ and $h(p_0, \tau, T)$ from their definitions. Moreover, the observed probability $\Pr[D = 1 | p_0, p_1, \tau, T] = [1 - F(v^*)]/[1 - F(p_0)]$ identifies the threshold $v^* = v^*(p_0, p_1, \tau, T)$.

To identify $[U(\cdot), \delta(\cdot)]$, the authors exploit variations in (p_0, p_1). They remark that the impatience $\delta(\cdot)$ enters multiplicatively in (7.14) so that it can be eliminated by taking the ratio of the derivatives of the indifference equation $U(v^* - p_1) = \overline{U}(v^*, p_0, \tau, T)$ with respect to (p_0, p_1). Specifically, because $v^* = v^*(p_0, p_1, \tau, T)$ increases in p_1 so that $p_1 = p_1(p_0, v^*, \tau, T)$, differentiating the indifference equation with respect to (p_0, v^*) and taking the ratio of the resulting derivatives gives the integral equation for $U'(\cdot)$

$$\alpha(p_0, \tau, T)\left[\frac{1 - \partial p_1(p_0, v^*, \tau, T)/\partial p_0}{\partial p_1(p_0, v^*, \tau, T)/\partial v^*} - 1\right] U'(v^* - p_0) = \int_{p_0}^{v^*} U'(v^* - y) h(y, \tau, T) dy, \quad (7.15)$$

where the function $p_1(p_0, v^*, \tau, T)$ is also identified as being the inverse of $v^*(p_0, p_1, \tau, T)$ with respect to p_1. To solve (7.15), one differentiates it with respect to p_0. This gives a differential equation in $U''(v^* - p_0)/U'(v^* - p_0)$, which has a unique solution given the normalizations $U(0) = 0$ and $U'(0) = 1$. Namely, upon letting $u = v^* - p_0$,

$$U(x) = \int_0^x \exp\left[\int_0^v \frac{h(p_0, \tau, T) + \partial\Psi(p_0, u+p_0, \tau, T)/\partial p_0}{\Psi(p_0, u+p_0, \tau, T)} du\right] dv$$

for $x \geq 0$, where $\Psi(p_0, v^*, \tau, T)$ denote the factor multiplying $U'(v^* - p_0)$ in the LHS of (7.15). Identification of $\delta(\cdot)$ follows from the indifference equation and (7.14). This gives

$$\delta(\tau) = \frac{U(v^* - p_1)}{\left[\alpha(p_0, \tau, T) U(v^* - p_0) + \int_{p_0}^{v^*} U(v^* - y) h(y, \tau, T) dy\right]}$$

for $\tau \geq 0$, where every term in the RHS is identified. Estimation of the model primitives $[F(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)]$ follows from their explicit expressions. The authors also derive a number of testable restrictions imposed by the model on observables. For instance, the conditional hazard of T conditional on (p_0, p_1, τ) does not depend on (p_1, τ).

Bidders' Entry

Gentry and Li (2014) consider Marmer, Shneyerov and Xu (2013) selective entry model with the auction stage consisting of an ascending auction instead of a first-price auction. See Section 5.2. Each bidder i receives a private signal σ_i about his value V_i which is unknown in the entry stage. The pairs (σ_i, V_i) are i.i.d. across the $I \geq 2$ potential bidders. The value V_i is distributed conditionally on σ_i as $F(\cdot|\sigma_i, X, I)$ with support $[\underline{v}, \overline{v}]$ and first-order stochastic dominance property $F(\cdot|\tilde{\sigma}, X, I) \leq F(\cdot|\sigma, X, I)$ whenever $\tilde{\sigma} \geq \sigma$. The marginal distribution of σ_i is normalized to be uniform $\mathcal{U}[0, 1]$. Bidders incur an entry cost $\kappa = \kappa(X, I)$ to enter and become aware of their private values after entry. Only entering bidders can bid in the second stage. As in Section 5.2, we let the reserve price $p_0 = \underline{v}$ so that selection arises only from players choosing to enter.

Let $\sigma^* = \sigma^*(X, I)$ denote the equilibrium signal threshold above which a potential bidder enters $(\sigma_i > \sigma^*)$ and does not otherwise. Starting from the second stage, i.e., the ascending auction, each entering bidder's dominant strategy is to bid his private value whether the number I^* of entering bidders is known or unknown to bidders. Turning to the entry stage, let $F(\cdot|\sigma \geq \sigma^*, X, I) \equiv \int_{\sigma^*}^1 F(\cdot|s, X, I)ds/(1-\sigma^*)$ be the value distribution conditional on entry and $\tilde{F}(y|X, I) \equiv \sigma^* + (1-\sigma^*)F(y|\sigma \geq \sigma^*, X, I)$ be the probability that a bidder does not enter or that he enters with a value smaller than y as in Section 5.2. Hence the expected gross payoff for a potential bidder with signal $\sigma_i \geq \sigma^*$ is $\int_{\underline{v}}^{\overline{v}} \pi(V_i; \sigma^*, X, I)dF(V_i|\sigma_i, X, I)$, where

$$\pi(V_i; \sigma^*, X, I) = (V_i - p_0)\sigma^{*I-1} + \int_{\underline{v}}^{V_i} (V_i - y)d\tilde{F}^{I-1}(y|X, I) = \int_{\underline{v}}^{V_i} \tilde{F}^{I-1}(y|X, I)dy,$$

using integration by parts and $p_0 = \underline{v}$. In the sum the first and second term correspond to the cases when none of bidder i's competitors enters and at least one enters, respectively. The equilibrium threshold is determined by setting the expected net profit equal to zero when bidder i receives the signal σ^*, i.e., $\kappa(X, I) = \int_{\underline{v}}^{\overline{v}} \pi(V_i; \sigma^*, X, I)dF(V_i|\sigma^*, X, I)$. This gives

$$\kappa(X, I) = \int_{\underline{v}}^{\overline{v}} [1 - F(v|\sigma^*, X, I)][\sigma^* + (1-\sigma^*)F(v|\sigma \geq \sigma^*, X, I)]^{I-1}dv \tag{7.16}$$

using integration by parts. In particular (7.16) is identical to (5.12).

The analyst observes the number I of potential bidders and the transaction price P, which is set to zero if no bidder enters, equal to $\underline{v}$ if there is only one entering bidder, and equal to $V^{(I-1:I)}$ otherwise since $p_0 = \underline{v}$. Thus the transaction price has two mass points: One at zero with probability σ^{*I} and one at $p_0 = \underline{v}$ with probability $I\sigma^{*I-1}(1-\sigma^*)$. Moreover, from Kong (2018) the distribution of P conditional on $P > p_0$ is

$$G_P^*(\cdot|p_0, X, I) \equiv \frac{\tilde{F}^{(I-1:I)}(\cdot|X, I) - \tilde{F}^{(I-1:I)}(p_0|X, I)}{1 - \tilde{F}^{(I-1:I)}(p_0|X, I)} \tag{7.17}$$

on $[\underline{v}, \overline{v}]$, where $\tilde{F}^{(I-1:I)}(\cdot|X, I)$ is the distribution of the second-highest statistic from I independent draws from $\tilde{F}(\cdot|X, I)$. In particular, $\tilde{F}^{(I-1:I)}(p_0|X, I) = \sigma^{*I} + I\sigma^{*I-1}(1 - \sigma^*)$ since $p_0 = \underline{v}$. Identification of the threshold $\sigma^* = \sigma^*(X, I)$ follows from $\Pr(P = 0|X, I) = \sigma^{*I}$.[63] Thus, $\tilde{F}^{(I-1:I)}(p_0|X, I)$ is identified. Because (7.17) resembles (7.4), the same argument based on (7.1) identifies $\tilde{F}(\cdot|X, I)$ and hence the value distribution conditional on entry $F(\cdot|\sigma \geq \sigma^*, X, I)$ on $[\underline{v}, \overline{v}]$.[64] This is insufficient to identify the primitive $F(\cdot|\cdot, X, I)$ on $[\underline{v}, \overline{v}] \times [0, 1]$ and the entry cost $\kappa = \kappa(X, I)$ from (7.16). To

[63] As a matter of fact, Gentry and Li (2014) assume that the analyst also observes I^*, in which case σ^* is identified by $1 - [\mathrm{E}(I^*|X, I)/I]$.

[64] As indicated in the fourth data scenario when there is a reserve price in Section 7.1, such identification results still hold if the analyst does not know I because (I, σ^*) is identified from $\Pr(P = 0|X, I)$ and $\Pr(P=1|X, I)$ when I is constant across known subsets of auctions.

achieve identification of $[F(\cdot|\cdot, X, I), \kappa(\cdot,\cdot)]$, Gentry and Li (2014) introduce an exclusion restriction, namely $F(\cdot|X, Z, I) = F(\cdot|X, I)$, and a full support condition on the threshold $\sigma^* = \sigma^*(X, Z, I)$ as Z varies. In the absence of the latter, they provide bounds for $F(\cdot|\sigma, X, I)$ and the entry cost $\kappa(X, I)$. See Section 5.2 for the argument.

Kong (2018) allows bidders to be asymmetric.[65] With two groups $j = 0, 1$ and $I_0+I_1 = I$, private values within each group are i.i.d. as $F_j(\cdot|\cdot, X, I_0, I_1)$ conditional on private signals which are independent and normalized to be uniform $\mathcal{U}[0, 1]$. Using $p_0 = \underline{v}$, she shows that the entry thresholds satisfy

$$\sigma_j^* = \frac{\Pr(P = 0|X, I_0, I_1)}{\Pr(P = 0|X, I_0, I_1) + \Pr(P = p_0 \text{ and Winner is from group } j|X, I_0, I_1)/I_j}, \tag{7.18}$$

for $j = 0, 1$. Thus, (σ_0^*, σ_1^*) are identified. Because Meilijson (1981) no longer applies due to the probability mass at $\underline{v}$ of $\tilde{F}_j(\cdot|X, I_0, I_1) \equiv \sigma_j^* + (1 - \sigma_j^*)F_j(\cdot|\sigma \geq \sigma_j^*, X, I_0, I_1)$, she relies on Nowik (1990) to identify the value distributions conditional on entry $F_j(\cdot|\sigma \geq \sigma_j^*, X, I_0, I_1)$ on $[\underline{v}, \overline{v}]$. As above, to achieve identification of $[F_j(\cdot|\cdot, X, I_0, I_1), \kappa_j(X, I_0, I_1)]$ for $j = 0, 1$, she introduces an exclusion restriction $F_j(\cdot|X, Z, I_0, I_1) = F_j(\cdot|X, I_0, I_1)$, and a full support condition on $\sigma_j^* = \sigma_j^*(X, Z, I_0, I_1)$ as Z varies. Kong (2020b) generalizes this setting to full asymmetry $F_i(\cdot|X, \mathcal{I})$, $i = 1, \ldots, I$. She shows that the thresholds σ_i^* and the distributions $F_i(\cdot|\sigma_i \geq \sigma_i, X, \mathcal{I})$ are identified by Nowik (1990) from observed transaction price, winner's identity and the set of potential bidders $\mathcal{I}$.

Regarding estimation, the entry thresholds (σ_0^*, σ_1^*) are estimated from (7.18). Kong (2018) then estimates $F_j(\cdot|\sigma_j, X, I_0, I_1)$ from the transaction prices and winners' group identities by sieve ML. For instance, the likelihood of observing a bidder of group 0 winning at a transaction price P is

$$\begin{aligned} & I_0[1 - \tilde{F}_0(P)]\frac{d\{\tilde{F}_0^{I_0-1}(P)\tilde{F}_1^{I_1}(P)\}}{dP} \\ = \;\; & I_0[1 - \tilde{F}_0(P)]\left\{(I_0 - 1)\tilde{f}_0(P)\tilde{F}_0^{I_0-2}(P)\tilde{F}_1^{I_1}(P) + I_1\tilde{f}_1(P)\tilde{F}_1^{I_1-1}(P)\tilde{F}_0^{I_0-1}(P)\right\}, \end{aligned}$$

where $\tilde{F}_j(\cdot) \equiv \sigma_j^* + (1 - \sigma_j^*)F_j(\cdot|\sigma_j \geq \sigma_j^*, X, I_0, I_1) = \sigma_j^* + \int_{\sigma_j^*}^1 F_j(\cdot|s, X, I_0, I_1)ds$. Upon approximating $F_j(\cdot|\cdot, X, I_0, I_1)$ by Bernstein polynomials whose degrees are chosen by minimizing a mean square criterion, she estimates $\hat{F}_j(\cdot|\cdot, X, I_0, I_1)$ for $j = 0, 1$ by maximizing the likelihood where the thresholds are replaced by their estimates $\hat{\sigma}_j^*$. Estimates of the entry costs $\kappa_j(X, Z, I_0, I_1)$ for $j = 0, 1$ follow from the zero expected profit conditions

[65] She also allows for bidders' risk aversion. Because the latter is not identified from ascending auctions, she exploits the availability of first-price auctions to identify the risk aversion function and the entry cost from the zero expected profit condition for each group. See Section 5.2.

$\kappa_j(X,Z,I_0,I_1) = \int_{\underline{v}}^{\overline{v}} \pi_j(V_i;\sigma_0^*,\sigma_1^*,X,I_0,I_1) dF_j(V_i|\sigma_j^*,X,I_0,I_1)$ where $\pi_j(V_i;\sigma_0^*,\sigma_1^*,X,I_0,I_1)$ $= \int_{\underline{v}}^{V_i} \tilde{F}_j(y)^{I_j-1}\tilde{F}_{j'}(y)^{I_{j'}}dy$ using integration by parts and $p_0 = \underline{v}$. This gives

$$\kappa_j(X,I_0,I_1) = \int_{\underline{v}}^{\overline{v}} [1 - F_j(v|\sigma_j^*,W,I_0,I_1)]\tilde{F}_j(v)^{I_j-1}\tilde{F}_{j'}(v)^{I_{j'}}dv,$$

thereby extending (7.16) to account for group asymmetry.

Bidders' and Sellers' Entry

Up to now, we have considered one-sided entry. Marra (2021) develops an equilibrium model of two-sided entry for online auction platforms where fees are charged to both buyers and sellers. Such platforms attract more buyers when the number of sellers is large and vice versa. There are several potential (single-unit) buyers and sellers, each having respective private values V_i and V_0 independently distributed as $F(\cdot|X) = \tilde{F}[\log(\cdot) - m(X)]$ and $H(\cdot|X) = \tilde{H}[\log(\cdot) - m(X)]$ where $\tilde{F}(\cdot)$ and $\tilde{H}(\cdot)$ are two distributions while $m(\cdot)$ is a function of the characteristics X of the seller's item to mitigate the curse of dimensionality. The platform levies some commission fees $c_B = c_B(X)$ and $c_S = c_S(X)$ in percent of the transaction price P from both the buyer and seller. Thus, a buyer's profit is $V_i - (1+c_B)P - \kappa_B$ if winning the object and $-\kappa_B$ if loosing, while the seller's profit is $(1-c_S)P - V_0 - \kappa_S$ if selling the object and $-\kappa_S$ if failing to sell it. As above, $\kappa_B = \kappa_B(X)$ and $\kappa_S = \kappa_S(X)$ denote the entry costs for an arbitrary buyer and seller, which include the buyers/sellers opportunity costs as well as entry fees imposed by the platform. The seller can set a secret reserve price p_0 at an extra fee $e_R = e_R(X)$ or choose not to have one in which case $p_0 = 0$ by convention. Potential bidders know whether there is a secret reserve price. The allocation mechanism is an ascending auction. Thus the transaction price satisfies $P = \max\{B^{(I^*-1:I^*)}, p_0\}$ where $B^{(I^*-1:I^*)}$ is the second-highest bid among I^* actual bidders for the seller's item.

The model consists of two stages. In the entry stage, each seller learns his valuation V_0 and chooses whether to enter, i.e., to list his item, as in Samuelson (1985). Each buyer then chooses whether to enter not knowing his valuation as in Levin and Smith (1994). This is justified since bidders need to pay an entry fee before accessing to the platform and knowing the items on sale. In the auction stage, each entering seller may set a reserve secret price, then each entering bidder learns his valuation V_i and bids. The optimal bid is $B_i = V_i/(1+c_B)$ because, up to the commission fee c_B, bidding his valuation remains a bidder's dominant strategy. The optimal secret reserve price p_0^* satisfies

$$p_0^* = \frac{V_0}{1-c_S} + \frac{1 - F[(1+c_B)p_0^*|X]}{(1+c_B)f[(1+c_B)p_0^*|X]}, \tag{7.19}$$

where $f(\cdot|X)$ is the bidders' value density. Indeed, up to the commission fees (c_B, c_S), p_0^* is equal to the optimal announced reserve price (4.4) in ascending and first-price auctions.

Let $I_r^* = I_r^*(X)$ denote the number of actual bidders for a listing with characteristics X where $r = 1$ or 0 indicates the presence or absence of a secret reserve price. In either case, I_r^* is also the number of bidders entering the auction since the reserve price (if any) is secret. In equilibrium, entering bidders spread over the listed items until their expected net profits from every auction vanish. When the number of potential bidders is large, Marra (2021) shows that the equilibrium Binomial distribution of I_r^* can be approximated by a Poisson distribution $\mathcal{P}(\lambda_r^*)$ for $r = 1, 0$. In particular, $\lambda_r^* = \lambda_r^*(X)$ equates the expected gross profit of an entering bidder to the entry cost $\kappa_{Br} = \kappa_{Br}(X)$ which is allowed to depend on r as well. Regarding sellers' entry, a seller with valuation V_0 chooses to list his item if $V_0 \leq v_0^*$, where $v_0^* = v_0^*(X)$ is the equilibrium value threshold that equates his expected gross profit to the entry cost κ_S. Moreover, as in Jehiel and Lamy (2015), the seller sets a secret reserve price if $v_r < V_0$ where $v_r = v_r(X)$ is a threshold satisfying $v_r \leq v_0^*$.[66]

The model primitives are $[\tilde{F}(\cdot), \tilde{H}(\cdot), m(\cdot), \kappa_{B0}(\cdot), \kappa_{B1}(\cdot), \kappa_S(\cdot)]$ since the platform fees (c_S, c_B, e_R) are observed. The analyst observes the transaction price P, the secret reserve price p_0^* and the number of actual bidders I_r^* for each listed item. Because $\log(1 + c_B)B_i = \log V_i \equiv m(X) + \tilde{V}_i$, we have $\log(1 + c_B)P = m(X) + \tilde{V}^{(I^*-1:I^*)}$ when $I^* \geq 2$. Moreover, because $\tilde{V}_i$ is independent of X (conditional on I^*), the "residual" $\tilde{V}^{(I^*-1:I^*)}$ has conditional mean $\mathrm{E}[\tilde{V}^{(I^*-1:I^*)}|X, I^*] = \mathrm{E}[\tilde{V}^{(I^*-1:I^*)}|I^*] \neq 0$. Hence, for any fixed value $I^* \geq 2$, the regression of $\log(1+c_B)P$ on (X, I^*) identifies $m(\cdot)$ as well as the distribution of $\tilde{V}^{(I^*-1:I^*)}$ given I^* up to the additive constant $\mathrm{E}[\tilde{V}^{(I^*-1:I^*)}|I^*]$.[67] Thus, an argument as in Section 7.1 shows that the distribution $\tilde{F}(\cdot)$ of $\tilde{V}_i$ is identified on its support up to location. A normalization such as $\mathrm{E}[\tilde{V}_i] = 0$ then identifies the location so that $m(\cdot)$ and $\tilde{F}(\cdot)$ are identified. Turning to the seller's distribution $\tilde{H}(\cdot)$, (7.19) gives

$$\tilde{V}_0 = \log\left\{(1 - c_S)p_0^*\left[1 - \frac{1 - \tilde{F}[\log(1 + c_B)p_0^* - m(X)]}{\tilde{f}[\log(1 + c_B)p_0^* - m(X)]}\right]\right\} - \log m(X) \equiv \xi_0(p_0^*, X),$$

where $\log V_0 \equiv m(X) + \tilde{V}_0$. Because $\xi_0(\cdot, X)$ is identified, the seller's homogeneized valuation $\tilde{V}_0$ can be recovered whenever the seller sets a reserve price p_0^*, i.e., whenever $\tilde{v}_r \equiv \log v_r - m(X) < \tilde{V}_0 \leq \tilde{v}_0^* \equiv \log v_0^* - m(X)$. That is, the seller's conditional distribution $[\tilde{H}(\cdot) - \tilde{H}(\tilde{v}_r)]/[\tilde{H}(\tilde{v}_0^*) - \tilde{H}(\tilde{v}_r)]$ is identified on $[\tilde{v}_r, \tilde{v}_0^*]$. In particular, when $\xi_0(\cdot, X)$ is increasing, the thresholds $\tilde{v}_r$ and $\tilde{v}_0^*$ are identified as $\xi_0(\underline{p}_0^*, X)$ and $\xi_0(\overline{p}_0^*, X)$ where $[\underline{p}_0^*, \overline{p}_0^*] = [\underline{p}_0^*(X), \overline{p}_0^*(X)]$ is the support of p_0^* given X. Identification of the entry costs $[\kappa_S(\cdot), \kappa_{B1}(\cdot), \kappa_{B0}(\cdot)]$ follows from the zero expected net profits for entering sellers

[66] Unlike v_0^*, Marra (2021) does not impose the threshold v_r to be in equilibrium.

[67] This is reminiscent of Rezende (2008), though a key difference is that (3.5) holds only in expectation.

and bidders who bid on auctions with and without reserve prices. See Marra (2021) for details. To this end, identification of $\lambda_r^* = \lambda_r^*(X)$ follows from $\lambda_r^*(X) = \mathrm{E}[I_r^*|X]$. Clearly, one can develop a nonparametric estimation procedure from the identification argument.[68]

Sequential Ascending Auctions with Multi-Unit Demand

As indicated in Section 5.3, the major impediment to the structural analysis of sequential auctions arises from limited theoretical results when bidders have multi-unit or multi-object demand. Most of existing results are obtained when the auction sequence is composed of only two ascending auctions as bidders' dominant strategy in the second auction is to bid their private values. To simplify, we assume hereafter that there is no reserve price and that the number of potential bidders I is exogenous.

With two units of an homogenous good for sale in a sequence of ascending auctions, Katzman (1999) exhibits a symmetric equilibrium strategy, whereas Lamy (2012b) characterizes the full set of such strategies. Each bidder draws two private values independently from a univariate distribution $F(\cdot)$ with support $[\underline{v}, \overline{v}]$. Let V_{1i} and V_{2i} denote the highest and lowest of these two draws for bidder i. Viewing $V_{1i} \geq V_{2i}$ as bidder i's marginal valuations for the first and second units, his payoff is $V_{1i} + V_{2i}$ if he wins both units and V_{1i} if he wins only one unit.[69] Because the second auction is ascending, bidder i's dominant strategy is to bid his marginal valuation, i.e., V_{2i} if he wins the first auction and V_{1i} if he looses the first auction. For the first ascending auction, consider the following increasing strategy that depends only on V_{1i}

$$s_1(V_{1i}) = v_{1i} - \frac{1}{F(V_{1i})^{2I-3}} \int_{\underline{v}}^{V_{1i}} F(v)^{2I-3} dv. \tag{7.20}$$

Katzman (1999) shows that $s_1(\cdot)$ is a symmetric equilibrium strategy, whereas Lamy (2012b) shows that it is also unique when $I \geq 3$. An important property is that the transaction prices (P_1, P_2) follow a submartingale, i.e., $\mathrm{E}(P_2|P_1, I) \geq P_1$ so that the transaction price increases in expectation. See Lamy (2012b) for additional model restrictions.

Turning to identification of the model primitive $F(\cdot)$, the analyst observes (P_1, P_2, I). Because the game is symmetric, the winner's identity in either auction is not required. Thus, $P_1 = s_1(V_1^{(I-1:I)})$ where $V_1^{(I-1:I)}$ is the second-highest value among $\{V_{1i}; i =$

[68]To conduct some counterfactuals, Marra (2021) parameterizes $\tilde{H}(\cdot)$ and applies ML to estimate $\tilde{H}(\cdot)$ as the latter is not identified outside $[\tilde{v}_r, \tilde{v}_0^*]$.

[69]Lamy (2012b) allows for a discount factor $\delta \leq 1$ so that bidder's payoff is $V_{1i} + \delta V_{2i}$. When $\delta < 1$ the symmetric equilibrium strategy (7.20) is unique. A more general case is when (V_{1i}, V_{2i}) are drawn from a bivariate distribution $F(\cdot, \cdot)$ with support $\{(v_1, v_2) : v_1 \in [\underline{v}, \overline{v}], \underline{v} \leq v_2 \leq v_1\}$. He then shows that $F(\cdot, \cdot)$ is identified from observations on the transaction prices in both ascending auctions with known (δ, I).

$1, \dots, I\}$, while P_2 is the second-highest value among $\{V_{2w}, V_{1j}; j \neq w\}$ where w indicates the winner of the first ascending auction. An interesting question is whether $F(\cdot)$ is identified from the second ascending auction which is now asymmetric since one bidder has won the first unit. Lamy (2012b) shows that the distribution of P_2 given I is $G_{P_2|I}(\cdot|I) = \psi_I[F(\cdot)]$ where

$$\psi_I(u) = \frac{2I(I-1)}{3} u^{2I-3} + I u^{2I-2} - 2I(I-1) u^{2I-1} + \frac{(I-1)(4I-3)}{3} u^{2I},$$

which is increasing in $u \in [0,1]$. Thus, the private value distribution $F(\cdot)$ is identified from observations on (P_2, I) alone as $F(\cdot) = \psi_I^{-1}[G_{P_2|I}(\cdot|I)]$.[70]

As a matter of fact, Lamy (2012b) also shows that $F(\cdot)$ is identified from observations on (P_1, I) alone. Indeed, similarly to Guerre, Perrigne and Vuong (2000), by differentiating (7.20) he shows that it can be written as

$$V_{1i} = B_{1i} + \frac{2}{2I-3} \frac{G_1(B_{1i}|I)}{g_1(B_{1i}|I)} \equiv \xi_1(B_{1i}; G_1, I),$$

where $B_{1i} = s_1(V_{1i})$ is bidder i's bid in the first auction distributed as $G_1(\cdot|I)$ with density $g_1(\cdot|I)$ conditional on I. Thus $V_1^{(I-1:I)} = \xi_1(P_1; G_1, I)$. Since $V_{1i} \sim F(\cdot)^2$, $V_1^{(I-1:I)} \sim \phi_{(I-1:I)}[F(\cdot)^2]$ and $P_1 \sim \phi_{(I-1:I)}[G_1(\cdot|I)]$ where $\phi_{(I-1:I)}(\cdot)$ is defined by (7.1), it follows that $G_1(\cdot|I)$, $F(\cdot)^2$ and hence $F(\cdot)$ are identified from observations on (P_1, I) alone. Specifically, $F(\cdot) = \left(\phi^{-1}_{(I-1:I)}\left\{ \Pr[\xi_1(P_1; G_1, I) \leq \cdot|I]\right\}\right)^{1/2}$ where $G_1(\cdot|I) = \phi^{-1}_{(I-1:I)}[G_{P_1|I}(\cdot|I)]$ and $G_{P_1|I}(\cdot|I)$ is the distribution of P_1 conditional on I.

Estimation follows the identification argument using L pairs of sequential ascending auctions. Though two units of the same good are sold within a pair, the auctioned goods may differ across pairs through some exogenous characteristics X. The private value distribution becomes $F(\cdot|X, I)$ where I controls for some unobserved heterogeneity as before. From nonparametric estimators $\hat{G}_{P_2|X,I}(\cdot|\cdot,\cdot)$ and $\hat{G}_{P_1|X,I}(\cdot|\cdot, I)$, one obtains the estimators $\hat{F}(\cdot|X, I) = \psi_I^{-1}[\hat{G}_{P_2|X,I}(\cdot|X, I)]$ from the second auctions and $\tilde{F}(\cdot|X, I) = \left(\phi^{-1}_{(I-1:I)}\left\{ \Pr[\tilde{V}_1 \leq \cdot|X, I]\right\}\right)^{1/2}$ from the first auctions, where $\tilde{V}_1 \equiv \xi_1(P_1; \hat{G}_1, I)$ is the pseudo-true value as in the GPV procedure with $\hat{G}_1(\cdot|X, I) = \phi^{-1}_{(I-1:I)}[\hat{G}_{P_1|X,I}(\cdot|X, I)]$.

[70] Though V_{2w} and $V_{1j}; j \neq w$ are independent conditionally on the highest valuation $V_1^{(I:I)}$ in the first auction, a property used to derive $G_{P_2|I}(\cdot|\cdot)$, they are not mutually independent. Intuitively, a bidder who wins the first auction is likely to have a higher second value V_{2i} than drawn from $F^{(1:2)}(\cdot)$, while a bidder who looses the first auction is likely to have a lower first value V_{1i} than drawn from $F^{(2:2)}(\cdot)$. Because of this selection bias, $G_{P_2|I}(\cdot|\cdot)$ is not equal to the distribution of the transaction price in an asymmetric ascending auction with I independent draws, one from $F_{(1:2)}(\cdot)$ and $I-1$ from $F_{(2:2)}(\cdot)$ as considered by Brendstrup and Paarsch (2006) and Brendstrup (2007).

Moreover, one can take an optimally weighted average of these two estimators of $F(\cdot|X, I)$ to improve efficiency.

An empirically useful contribution with $K \geq 2$ sequential ascending auctions and multi-unit demand is Donald, Paarsch and Robert (2006). Each potential bidder $i = 1, \ldots, I$ independently draws M_i valuations from a distribution $F(\cdot)$ with support $[\underline{v}, \overline{v}]$. This gives the descending marginal values $V_i^{(1:M_i)} \geq \ldots \geq V_i^{(M_i:M_i)}$. The number of valuations M_i is random and distributed as Poisson $\mathcal{P}(\lambda)$. Thus, the model allows for a random number I^* of participants as $I^* \sim \mathcal{B}(I, 1 - e^{-\lambda})$ where $e^{-\lambda} = \Pr(M_i = 0)$ is the probability that bidder i does not participate. As the authors show, a key property of such a demand generating scheme is to ensure some symmetry in the number of units K_i won by each bidder i. Namely, $\Pr(V_i^{(K_i+k:M_i)} \leq \cdot | V_i^{(K_i+1:M_i)} = \cdot, M_i > 0)$ is independent of i for $k \geq 1$. The authors then exhibit a symmetric equilibrium strategy that recursively specifies the price at which each participating bidder should drop out from each auction. Bidder i's strategy for the $(K_i + 1)$-th unit depends on $V_i^{(K_i+1:M_i)}$ as well as the number of units won by his competitors in previous auctions and the prices at which they are dropping from the current auction. Moreover, the expected price paid by bidder i for the $(K_i + 1)$-th unit is the expectation of the $(K - K_i)$-th highest valuation $V_{-i}^{(K-K_i)}$ among all valuations of the other participants, i.e., $\mathrm{E}[V_{-i}^{(K-K_i)}]$. This is also the expected price that bidder i would pay in a generalized Vickrey auction where the K units are allocated to the K highest valuations among all participating bidders who truthfully reveal their willingnesses-to-pay $V_i^{(k+1:M_i)}$ and pay $V_{-i}^{(K-k)}$ when winning their $(k + 1)$-th unit, $k = 0, \ldots, M_i - 1$. See Krishna (2010). Similarly to Laffont, Ossard and Vuong (1995) who rely on the RET to simulate the expected winning price in a first-price auction, the authors exploit the preceding property to simulate the expected transaction prices in the sequential ascending auctions from the generalized Vickrey auction. They parameterize $F(\cdot)$ as Weibull $\mathcal{W}(\alpha_1, \alpha_2)$. The model parameters are $(I, \lambda, \alpha_1, \alpha_2)$. Following McFadden (1989) and Pakes and Pollard (1989), they apply SMM on the K conditional moments $\mathrm{E}(P_1|I^*, K), \mathrm{E}(P_2|P_1, I^*, K), \ldots, \mathrm{E}(P_K|P_{K-1}, \ldots, P_1, I^*, K)$ and $\mathrm{E}(I^*)$.

Sequential Ascending Auctions with Multi-Object Demand

We now turn to the situation when different objects are for sale to bidders with multi-object demand in a sequence of ascending auctions. To analyze such auctions, Brendstrup and Paarsch (1997) follow Chakraborty (1999) theoretical model with two objects and $I \geq 2$ potential bidders. Each bidder draws a pair of values (V_{1i}, V_{2i}) independently from a bivariate distribution $F(\cdot, \cdot)$, where V_{ki} represents bidder i's private value for the k-th object with $k = 1, 2$. A bidder's payoff is the sum of his valuations for the objects that he wins minus the price paid for each. Because the payoff is additively separable, bidding V_{ki} in the k-th ascending auction is a dominant strategy. The econometric problem is to identify and estimate the model primitive $F(\cdot,\cdot)$ from L independent pairs of ascending auctions when the analyst observes the transaction prices and the number of bidders. The authors assume that the auctioned objects are the same across the L sequences. There is no reserve price. Thus, $P_1 = V_1^{(I-1:I)}$ and $P_2 = V_2^{(I-1:I)}$. The main difficulty is that these values may not belong to the same bidder in the two auctions. Upon deriving the joint density of $(V_1^{(I-1:I)}, V_2^{(I-1:I)})$, Brendstrup and Paarsch (1997) show that $F(\cdot,\cdot)$ is identified from (P_1, P_2, I) when the copula $C(\cdot,\cdot)$ of $F(\cdot,\cdot)$ belongs to the Archimedean family, i.e., $C(u_1, u_2) = \zeta^{-1}[\zeta(u_1) + \zeta(u_2)]$ for $(u_1, u_2) \in [0,1]^2$. The function $\zeta(\cdot)$ is decreasing and convex with $\zeta(1) = 0$. See Nelsen (1999). Thus $\zeta(\cdot)$ is identified up to a scale normalization. Regarding estimation, the authors parameterize $C(\cdot,\cdot)$ and employ a semiparametric ML method where the marginal distributions of P_1 and P_2 are estimated by their empirical distributions following Genest, Ghoudi and Rivest (1995). Because the joint density of (P_1, P_2) is a $2(I-1)$-multivariate integral, they approximate it by a simulation-based method. See Section 3.1.

By allowing for possible synergy between the auctioned objects, Kong (2021) introduces an economically important link between the ascending auctions. Her setting is that of Section 5.3 in which bidder i learns his (stand-alone) valuation V_{ki} at the beginning of the k-th auction, whereas $\delta(V_{1i}, V_{2i})$ denotes the synergy-inclusive value of the second object when bidder i has won the first object. With a bid B_{1i} in the first ascending auction, bidder i's expected profit is

$$\pi_i = \left[V_{1i} - \mathrm{E}(B_{1,-i}^{\max} | B_{1,-i}^{\max} \le B_{1i}) + \mathcal{V}_i^w\right] \Pr(B_{1,-i}^{\max} \le B_{1i}) + \mathcal{V}_i^l \Pr(B_{1,-i}^{\max} > B_{1i}),$$

where $B_{1,-i}^{\max}$ is the highest competitor's bid in the first auction, and $(\mathcal{V}_i^w, \mathcal{V}_i^l)$ are bidder i's continuation values for winning and losing the first auction. These continuation values are the same as those in Section 5.3 since the second auction is ascending where bidder i's dominant strategy is to bid $\delta(V_{1i}, V_{2i})$ or V_{2i} if he wins or loses the first auction, respectively. Maximizing π_i with respect to B_{1i} shows that the symmetric equilibrium strategy $s_1(\cdot)$ in the first auction is no longer to bid the first valuation but satisfies

$$V_{1i} = B_{1i} - \int_{\underline{v}}^{\overline{v}} \left[\int_{\underline{v}}^{\tilde{\delta}(B_{1i}, v_2)} H^w(x|B_{1i}) du - \int_{\underline{v}}^{v_2} H^l(x|B_{1i})\right] d\tilde{F}_2(v_2|B_{1i}) \equiv \xi_1(B_{1i}), \quad (7.21)$$

where $\tilde{\delta}(b_1, v_2) \equiv \delta[s_1^{-1}(b_1), v_2]$ and $\tilde{F}_2(v_2|b_1) \equiv F[v_2|s_1^{-1}(b_1)]$ for all (b_1, v_2). In particular, (7.21) is identical to (5.21) with no second term in the latter. The second term in the RHS of (7.21) captures the expected benefit in the second auction from winning the first auction. Her identification argument in Section 5.3 applies here as well. The only difference is to use (7.21) instead of (5.21) to identify $F_1(\cdot)$. Thus, the model primitives $[F(\cdot,\cdot), \delta(\cdot,\cdot)]$ are nonparametrically identified from observations on all bids in the first ascending auction, the transaction price P_2 in the second ascending auction and the winners' identities in both auctions. We also note that if one observes the number of bidders I and the transaction price P_1 instead of all bids in the first auction, then identification of the model primitives still holds by recovering $V_1^{(I-1:I)} = \xi_1(P_1)$ by (7.21). Thus, by (7.1) the distribution $F_1(\cdot) = \phi^{-1}_{(I-1:I)}[F_1^{(I-1:I)}(\cdot)]$ is identified.

Section 7.3: Incomplete Models

Ascending auctions may differ from the button auction in several dimensions such as a minimum bid increment, jump bidding, multiple bids, etc. As a consequence, characterizing the Bayesian Nash equilibrium strategies or dominant strategies in such auctions can be infeasible even within an IPV paradigm.

An Incomplete Model for Ascending Auctions

In an influential paper, Haile and Tamer (2003) relax the equilibrium link between bids and private values by developing an incomplete approach for exploiting all (final) bids in open outcry ascending (English) auctions with minimum bid increments. In such auctions, the highest bidder is the winner who pays his bid. Instead of relying on an equilibrium model to explain the transaction price, they consider two minimal assumptions of bidders' rational behavior. Namely,

(i) Bidders do not bid more than they are willing to pay,

(ii) Bidders do not let opponents win at a price they can beat.

Within a symmetric IPV paradigm, these natural assumptions lead to some bounds for the distribution $F(\cdot)$ of private values and hence to partial identification. Hereafter, we assume that the number I of potential bidders is exogenous, and to simplify, that there are no auction covariates X and no reserve price p_0. We also assume that final bids for all potential bidders are available.

Condition (i) implies that the final bid is less than the private value for each bidder, i.e., $B_i \leq V_i$. Moreover, a proof by contradiction implies that $B^{(k:I)} \leq V^{(k:I)}$ for $k = 1, \ldots, I$ whether these quantities belong to the same bidder. Since $F^{(k:I)}(\cdot) \leq G^{(k:I)}(\cdot)$, applying the increasing transformation $\phi^{-1}_{(k:I)}(\cdot)$ on (7.1) gives the upper bound

$$F(\cdot) \leq F_U(\cdot) \equiv \min_{k\in\{1,\ldots,I\}, I\in\mathcal{S}_I} \phi^{-1}_{(k:I)}[G^{(k:I)}(\cdot)], \tag{7.22}$$

on $[\underline{v}, \overline{v}]$, where $\mathcal{S}_I$ is the support of $I \geq 2$. Turning to the lower bound, let $\Delta \geq 0$ be the minimum bid increment. Condition (ii) says that $V_i < B^{(I:I)} + \Delta$ when bidder i does not win, i.e., when $B_i < B^{(I:I)}$. This implies $V^{(I-1:I)} \leq B^{(I:I)} + \Delta$ whose distribution is denoted $G_\Delta^{(I:I)}(\cdot)$. Using $F(\cdot) = \phi^{-1}_{(I-1:I)}[F^{(I-1:I)}(\cdot)]$ gives the lower bound

$$F_L(\cdot) \equiv \max_{I \in \mathcal{S}_I} \phi^{-1}_{(I-1:I)}[G_\Delta^{(I:I)}(\cdot)] \leq F(\cdot), \tag{7.23}$$

on $[\underline{v}, \overline{v}]$. As expected, they collapse to $F(\cdot)$ in the button ascending auction upon setting the unobserved winner's bid $B^{(I:I)} = B^{(I-1:I)}$. Haile and Tamer (2003) also show how to bound the optimal reserve price p_0^* in (4.4). They rely on the property that p_0^* solves the maximization problem $\max_p (p - V_0)[1 - F(p)]$. See, e.g., Laffont and Maskin (1980). Letting $\pi_L(p) \equiv (p - V_0)[1 - F_U(p)]$ and $\pi_U(p) \equiv (p - V_0)[1 - F_L(p)]$, they show that bounds for p_0^* are $[p_L, p_U]$ where

$$p_L \equiv \sup\{p < p* : \pi_U(p) \leq \pi_L^*\} \quad \text{and} \quad p_U \equiv \inf\{p > p* : \pi_U(p) \leq \pi_L^*\},$$

with $p^* = \arg\sup_p \pi_L(p)$ and $\pi_L^* = \sup_p \pi_L(p)$.

Regarding estimation, the lower and upper bounds $[F_L(\cdot), F_U(\cdot)]$ are estimated by plugging the empirical cdf's of $G_\Delta^{(I:I)}(\cdot)$ and $G^{(k:I)}(\cdot)$, $k = 1, \ldots, I \in \mathcal{S}_I$ in (7.22)-(7.23). The resulting estimators converge uniformly at the parameteric rate on inner compact subsets of $[\underline{v}, \overline{v}]$. See also Menzel and Morganti (2013) for uniform convergence rates on $[\underline{v}, \overline{v}]$. Moreover, in small samples, these estimators can be biased and cross each other. To avoid this issue, Haile and Tamer (2003) propose the smooth weighted average

$$\tilde{F}_L(\cdot) = \sum_{I \in \mathcal{S}_I} \hat{F}_I(\cdot) \left(\frac{\exp[\rho \hat{F}_I(\cdot)]}{\sum_{\tilde{I} \in \mathcal{S}_I} \exp[\rho \hat{F}_{\tilde{I}}(\cdot)]} \right),$$

where $\hat{F}_I(\cdot) \equiv \phi^{-1}_{I-1:I}[\hat{G}_\Delta^{(I:I)}(\cdot)]$ and the parameter $\rho \uparrow +\infty$. Its motivation comes from $\lim_{\rho \uparrow +\infty} \sum_i y_i \exp(\rho y_i)/[\sum_{\tilde{i}} \exp(\rho y_{\tilde{i}})] = \max_i y_i$ to approximate the maximum in (7.23). The smooth weighted average $\tilde{F}_U(\cdot)$ is similarly defined with $\rho \downarrow -\infty$ to approximate the minimum in (7.22). Estimation of the bounds $[p_L, p_U]$ for the optimal reserve price p_0^* are obtained by plugging-in $\tilde{F}_L(\cdot)$ and $\tilde{F}_U(\cdot)$ and essentially solving the resulting optimization problems. With auction covariates X, the authors parameterize the conditional mean $\mathrm{E}(V_i|X)$ and estimate the identifed set of the parameters using $[\tilde{F}_L(\cdot|X), \tilde{F}_U(\cdot|X)]$ following Manski and Tamer (2002). See Haile and Tamer (2003) for details.

In general, the bounds $[F_L(\cdot), F_U(\cdot)]$ for the private value distribution $F(\cdot)$ are not sharp. Using the theory of random sets and Artstein (1983) inequality– see e.g., Molchanov (2005), Chesher and Rosen (2017a) propose a general method for studying point and partial identification in "General Instrumental Variable models" that are completely or incompletely specified. Applying it to the above English auction with minimal rational conditions (and $\Delta = 0$ to simplify), they show that the identified set for $F(\cdot)$ is

$$\mathcal{F}_o \equiv \left\{ F(\cdot) : \forall \text{ closed set } \mathcal{C} \subseteq \mathcal{R}_u, \Pr\left[\mathcal{U}(B^{(\cdot:I)}; F) \subseteq \mathcal{C}\right] \leq I! \int_{\mathcal{C}} du \right\}, \tag{7.24}$$

where $\mathcal{R}_u \equiv \{u = (u_1, \ldots, u_I) : 0 \leq u_1 \leq \ldots \leq u_I \leq 1\}$ and $\mathcal{U}(B^{(\cdot:I)}; F)$ is a "U-level set" that is random with realization

$$\mathcal{U}(b^{(\cdot:I)}; F) \equiv \{u \in \mathcal{R}_u : F(b^{(1:I)}) \leq u_1, \ldots, F(b^{(I:I)}) \leq u_I, u_{I-1} \leq F(b^{(I:I)})\},$$

where $b^{(\cdot:I)} \equiv (b^{(1:I)}, \ldots, b^{(I:I)})$. They further show that the closed sets $\mathcal{C}$ in (7.24) can be replaced by the "core-determining sets" composed of arbitrary unions of the sets $\mathcal{U}(\tilde{b}^{(\cdot:I)}; F)$ where $\tilde{b}^{(\cdot:I)} \equiv (\tilde{b}^{(1:I)}, \ldots, \tilde{b}^{(I:I)})$ with $\underline{v} \leq \tilde{b}^{(1:I)} \leq \ldots \leq \tilde{b}^{(I:I)} \leq \overline{v}$. Because the cardinality of such unions is uncountable so that the identified set $\mathcal{F}_o$ is difficult if not impossible to compute, the authors consider a selection of such unions, each union leading to a restriction on $F(\cdot)$. Specifically, for $\check{b}^{(I:I)} \geq \tilde{b}^{(I:I)}$ they consider the "contiguous union"

$$\begin{aligned} \mathcal{U}(\tilde{b}^{(\cdot:I)}, \check{b}^{(I:I)}; F) &\equiv \cup_{b \in [\tilde{b}^{(I:I)}, \check{b}^{(I:I)}]} \mathcal{U}(\tilde{b}^{(1:I)}, \ldots, \tilde{b}^{(I-1:I)}, b; F) \\ &= \{u \in \mathcal{R}_u : F(\tilde{b}^{(1:I)}) \leq u_1, \ldots, F(\tilde{b}^{(I:I)}) \leq u_I, u_{I-1} \leq F(\check{b}^{(I:I)})\}, \end{aligned}$$

as a set $\mathcal{C}$ in (7.24). In particular, letting $(\tilde{b}^{(1:I)}, \ldots, \tilde{b}^{(k-1:I)}, \tilde{b}^{(k:I)}, \ldots, \tilde{b}^{(I:I)}) = (\underline{v}, \ldots, \underline{v}, v, \ldots, v)$ for $k = 1, \ldots, I$ and $\check{b}^{(I:I)} = v$ gives Haile and Tamer (2003) upper bound inequality (7.22) when (v, I) runs in $[\underline{v}, \overline{v}] \times \mathcal{S}_I$. Similarly, letting $(\tilde{b}^{(1:I)}, \ldots, \tilde{b}^{(I:I)}) = (\underline{v}, \ldots, \underline{v})$ and $\check{b}^{(I:I)} = v$ gives Haile and Tamer (2003) lower bound inequality (7.23) when (v, I) runs in $[\underline{v}, \overline{v}] \times \mathcal{S}_I$ and $\Delta = 0$. These bounds only involve the marginal distributions of $(B^{(1:I)}, \ldots, B^{(I:I)})$. Chesher and Rosen (2017a) point out that there are other inequalities involving the joint distribution of two or more of the bid order statistics. For instance, choosing $(\tilde{b}^{(1:I)}, \ldots, \tilde{b}^{(I-1:I)}, \tilde{b}^{(I:I)}) = (\underline{v}, \ldots, \underline{v}, v_1, v_2)$ and $\check{b}^{(I:I)} = v_3$ with $v_1 \leq v_2 \leq v_3$ gives the additional inequality

$$\begin{aligned} \Pr[B^{(I-1:I)} \geq v_1, v_3 \geq B^{(I:I)} \geq v_2] &\leq I[1 - F(v_3)][F(v_3)^{I-1} - F(v_1)^{I-1}] \\ &\quad - IF(v_1)^{I-1}[F(v_3) - F(v_2)] + F(v_3)^I - F(v_2)^I. \end{aligned}$$

Further inequalities are obtained by choosing other contiguous unions $\mathcal{U}(\tilde{b}^{(\cdot:I)}, \check{b}^{(I:I)}; F)$ thereby improving the sharpness of the bounds although determining the sharp bounds in practice will involve using the uncountable number of core-determining sets.

Chesher and Rosen (2017b) extend these results to symmetric affiliated private values jointly distributed as $F(\cdot, \ldots, \cdot)$.[71] The latter is decomposed as $[F_m(\cdot), C(\cdot, \ldots, \cdot)]$ where

[71] See also Komarova (2013b) who indicates that some of her bounds in the asymmetric affiliated case of the button auction still hold under Haile and Tamer (2003) mimimal rationality conditions.

$F_m(\cdot)$ is the common marginal distribution of $F(\cdot)$ and $C(\cdot,\ldots,\cdot)$ its copula. The U-level sets are $\mathcal{U}(B^{(\cdot:I)};F_m)$, whereas the identified set for $[F_m(\cdot),C(\cdot,\ldots,\cdot)]$ becomes

$$\mathcal{F}_o \equiv \left\{[F_m(\cdot),C(\cdot,\ldots,\cdot)] : \forall \text{ closed set } \mathcal{C} \subseteq \mathcal{R}_u, \Pr\Big[\mathcal{U}(B^{(\cdot:I)};F_m) \subseteq \mathcal{C}\Big] \leq \int_{\mathcal{C}} dC^{(\cdot:I)}\right\},$$

where $C^{(\cdot:I)}(\cdot,\ldots,\cdot)$ is the copula of $(V^{(1:I)},\ldots,V^{(I:I)})$ induced by the copula $C(\cdot,\ldots,\cdot)$ of $(V_1,\ldots,V_I)$. As before, the closed sets $\mathcal{C}$ can be replaced by the core-determining sets composed of arbitrary unions of the sets $\mathcal{U}(\tilde{b}^{(\cdot:I)};F_m)$. The authors also consider observed auction characteristics X though a parametric index restriction (see Section 3.3) as well as additive unobserved heterogeneity with zero mean (see Section 4.4).

Ascending Auctions and Bargaining

Auctions with a secret reserve price may not conclude to a sale. When the auction fails, the seller and the highest bidder sometimes enter into some bargaining. See Elyakime, Laffont, Loisel and Vuong (1997) for first-price sealed-bid auctions as reviewed in Section 4.1. Analyzing wholesale used-car auctions, Larsen (2021) considers button ascending auctions followed by bargaining. In contrast to Elyakime, Laffont, Loisel and Vuong (1997) who consider bargaining under complete information, Larsen (2021) tackles bargaining with two-sided incomplete information where characterization of equilibria is known to be difficult. See Fudenberg and Tirole (1991). The author exploits the tractability of the effects of the ascending auction outcome on bargaining and vice versa.

We assume that there are no auction covariates X. The I potential buyers have private values V_i which are i.i.d. as $F(\cdot)$, whereas the seller has a private value V_0 drawn independently from $H(\cdot)$. The seller sets a secret reserve price P_0. If the auction concludes, i.e., if $V^{(I-1:I)} \geq P_0$, the bidder with the highest valuation $V^{(I:I)}$ wins and pays the auction price $P_a = V^{(I-1:I)}$. If the auction fails, i.e., if $V^{(I-1:I)} < P_0$, the remaining bidder may quit ($D_0^B = 0$) or enter ($D_0^B = 1$) with P_a as his first offer into an alternating-offer bargaining process in up to T rounds with the seller. In the t-th round, depending on which turn it is, the seller or buyer decides whether to accept the offer (D_t^S or $D_t^B = A$), to quit (D_t^S or $D_t^B = Q$) or to make a counteroffer (D_t^S or $D_t^B = C$) in which case P_t^S or P_t^B denotes the counteroffers. The auction price and preceding counteroffers are known at any round though the reserve price is not. Each party incurs a cost for each offer.

Clearly, there are many Bayesian Nash equilibrium strategies for a seller's reserve price, bidders' bids and decisions to enter into bargaining, as well as bidders' and seller's decisions to quit, accept or counter with a specific offer in each round. In general, they are history dependent and can be very complicated. Under some regularity conditions, Larsen (2021) shows that bidding truthfully, i.e., dropping at a price equal to his private value, and entering into bargaining only if the expected payoff for doing so is nonnegative is a bidder's best response in any Bayesian Nash equilibrium. Thus, the author considers that bidders bid according to the usual dominant strategy in the button ascending auction. Identification of the bidders' valuation distribution $F(\cdot)$ then follows from the auction price $P_a = V^{(I-1:I)}$ and the knowledge of I. See Section 7.1.

Turning to the seller's valuation distribution $H(\cdot)$, there are still several Bayesian Nash equilibria in the bargaining game, and hence many equilibrium strategies for the seller's reserve price. To avoid characterizing such strategies, Larsen (2021) introduces minimal assumptions of rational behavior in the spirit of Haile and Tamer (2003). Namely,

(i) The seller never accepts a price below his value,

(ii) The seller never quits at a price above or equal to his value.

These conditions are satisfied by any Bayesian Nash equilibrium. The author focuses on the auction and the first round of bargaining. Applied to the auction price $P_a = p_a$, conditions (i) and (ii) give the inclusions $\{p_a \geq P_0\} \cup \{D_1^S = A\} \subseteq \{V_0 \leq p_a\}$ and $\{D_1^S = Q\} \subseteq \{p_a < V_0\}$, respectively. These lead to the bounds

$$H_L(p_a) \equiv \Pr(P_0 \leq p_a) + \Pr(D_1^S = A | P_a = p_a) \leq H(p_a) \leq \Pr(D_1^S \neq Q | P_a = p_a) \equiv H_U(p_a), \quad (7.25)$$

for $p_a \in [\underline{v}, \overline{v}]$ using the independence of V_0 (and hence P_0) from $P_a = V^{(I-1:I)}$. Since

$$\begin{aligned} 1 &= \Pr(P_0 \leq p_a) + \Pr(P_0 > p_a, D_0^B = 0 | P_a = p_a) \\ &\quad + \Pr(D_1^S = A | P_a = p_a) + \Pr(D_1^S = Q | P_a = p_a) + \Pr(D_1^S = C | P_a = p_a), \end{aligned}$$

the RHS of (7.25) is equal to the LHS plus the sum $\Pr(P_0 > p_a, D_0^B = 0 | P_a = p_a) + \Pr(D_1^S = C | P_a = p_a)$, which is the probability that the highest bidder opts out from bargaining or that the seller makes a counteroffer to p_a.

Regarding estimation, Larsen (2021) accounts for observed heterogeneity through the index $X'\gamma$ and an independent additive unobserved heterogeneity Y with zero mean. The buyers and sellers' valuations distributions are of the form $F(\cdot|X,Y) = \tilde{F}(\cdot - X'\gamma - Y)$ and $H(\cdot|X,Y) = \tilde{H}(\cdot - X'\gamma - Y)$. The author first homogeneizes the auction and reserve prices by linear regressions of $(P_a, P_0)' \equiv (X'\gamma + Y + \tilde{P}_a, X'\gamma + Y + \tilde{P}_0)'$ on X and a set of dummies for the observed number of bidders. See Section 3.3. Using the residuals for given I, he then deconvolutes $(Y + \tilde{P}_a, Y + \tilde{P}_0)'$ using Kotlarski (1967) and Li and Vuong (1998) to estimate the distribution $\tilde{F}^{(I-1:I)}(\cdot)$ of the second highest demeaned valuation $\tilde{V}^{(I-1:I)} = \tilde{P}_a$. Estimation of $\tilde{F}(\cdot)$ follows from inversion of (7.1) and pooling over I. Conditional on I, the estimation of lower and upper bounds $[\tilde{H}_{LI}(\cdot), \tilde{H}_{UI}(\cdot)]$ for $\tilde{H}(\cdot)$ are obtained by plugging nonparametric estimators in (7.25) which holds with $\tilde{H}(\cdot)$, $\tilde{P}_a$,

p_a, $\tilde{p}_0$ replacing $H(\cdot)$, P_a, p_a, p_0. Thus $[\tilde{H}_L(\cdot), \tilde{H}_U(\cdot)] \equiv [\max_{I\in\mathcal{S}_I} \tilde{H}_{LI}(\cdot), \min_{I\in\mathcal{S}_I} \tilde{H}_{UI}(\cdot)]$. See Larsen (2021) for an alternative multi-step procedure based on Gallant and Nychka's (1987) nonparametric ML method.

Besides observations on the number of bidders and auction covariates, the preceding results only use the reserve price, the auction price and the seller's decision in the first bargaining round. Recently, Larsen and Zhang (2018) develop a general methodology for analyzing trading games with quasi-linear utilities. Based on revealed preferences, their approach relies on the final allocation and payment for each agent. In their application to ascending auctions followed by bargaining, the authors then use the secret reserve price P_0, whether trade occurs ($D = 1$) and the trading price P at the end of the game to identify the seller's homogeneized value distribution $\tilde{H}(\cdot)$. In particular, under some assumptions such as P_0 is an increasing function of V_0, they show that

$$V_0 = \frac{\partial \mathrm{E}(P|P_0, X, Y, I)/\partial P_0}{\partial \Pr(D = 1|P_0, X, Y, I)/\partial P_0},$$

which generalizes the optimal price equation (4.4) to trading games. Thus, if Y was observed, the seller's valuation distribution $H(\cdot|X, Y)$ would be identified and estimation would follow as in Guerre, Perrigne and Vuong (2000) from the resemblance of the above equation to (3.6). Since Y cannot be recovered from Kotlarski's (1967) deconvolution, Larsen and Zhang (2018) propose a minimum distance method to estimate $\tilde{H}(\cdot)$ upon exploiting the additively separable form of the observed and unobserved heterogeneity. See their paper for details.

Section 8: Auctions with Common Value

Common value has played an important role in the early development of the auction literature. Pathbreaking contributions are the theoretical work by Wilson (1967, 1969, 1977) and the pioneering empirical papers by Hendricks and Porter (1988, 1992) surveyed by Porter (1995) establishing the relevance of game theoretical models in analyzing auction data. Most of this literature originates from the study of gas lease auctions where the amount of oil, though uncertain, is common to any bidder winning the auction. The pure common value (PCV) model is often called the mineral rights model.

Since then, much progress has been made with the introduction of more general models, the availability of several new data sets and the development of the structural approach. Though common and private components are likely to be present in most cases, common value models have become less popular in favor of private value models. Indeed, the former require that competitors' information is valuable to a bidder. Thus, information about the value of the auctioned object must differ across bidders. Moreover, common value models are more difficult to identify from observed bids. Despite leading to quite different policy recommendations, they are observationally equivalent to private value models as shown by Laffont and Vuong (1996). As a result, the econometric literature with common value is not as developed as that covered in previous sections with private values. In contrast, there is an important literature on testing for the presence of common value. See Hendricks, Porter and Wilson (1994), Hendricks, Pinske and Porter (2003), Haile, Hong and Shum (2006) and Hortacsu and Kastl (2012) for key contributions. As testing is outside the scope of this chapter, we focus below on estimation methods.

In Section 8.1, we review papers that employ parametric direct methods for estimating first-price sealed-bid auctions with a common value component. In Section 8.2, we address the issue of nonindentification in such auction models. We then discuss several contributions that use nonparametric indirect methods. In particular, we indicate how progress is made based on either functional form or exclusion restrictions. In Section 8.3, we turn to ascending auctions, where estimation is mostly parametric. In Section 8.4, we consider share auctions in a common value framework.

Section 8.1: Direct Estimation Methods for First-Price Auctions

As for the benchmark IPV model, the econometrics literature started with parametric direct estimation methods of the PCV model. In this model, every bidder has the same utility C for the auctioned object which is uncertain at the time of bidding and distributed as $F_C(\cdot)$. Each bidder has some private information σ_i independently drawn given C from some distribution $F_{\sigma|C}(\cdot|\cdot)$ with possibly unbiased estimate, i.e., $\mathrm{E}(\sigma_i|C) = C$. See Section 2.1. Paarsch (1992) is the first to estimate structurally PCV models.[72] He considers Rothkopf's (1969) model where the common value density is $f_C(c) \propto 1/c^2$ and $F_{\sigma|C}(\lambda\sigma|\lambda c) = F_{\sigma|C}(\sigma|c)$ for any $\lambda > 0$. As noted by Smiley (1979), several distributions satisfy the latter condition such as the Weibull distribution which Paarsch (1992) uses in his empirical study. The equilibrium strategy is then proportional to bidder's signal, namely $B_i = \beta_I \sigma_i$, with β_I a positive constant depending on the number I of bidders. He also considers Thiel's (1988) model as revisited by Levin and Smith (1991) where $F_{\sigma|C}(\cdot|C)$ and $F_C(\cdot)$ are normal $\mathcal{N}(C, \omega^2)$ and uniform $\mathcal{U}[\underline{c}, \overline{c}]$, respectively. The equilibrium strategy is then

$$B_i = \sigma_i - \omega\alpha_I + \beta_I \exp[-\sigma_i \mathrm{E}(\tilde{\sigma}^{(I:I)})/\omega],$$

[72] He studies low-bid auctions (procurements) whereas we present the high-bid version below.

with $\beta_I \leq 0$, $\alpha_I \equiv \mathrm{E}[(\tilde{\sigma}^{(I:I)})^2]/\mathrm{E}[\tilde{\sigma}^{(I:I)}]$ and $\tilde{\sigma}^{(I:I)} \equiv (\sigma^{(I:I)} - C)/\omega$. After introducing the structural approach, Paarsch (1992) estimates both models by deriving some moments or the likelihood induced by the distribution $F_{\sigma|C}(\cdot|C)$ of the unobserved signals σ_i. In contrast to the model, he treats the common value C as an unknown parameter constant across auctions.

Laffont and Vuong (1993) do not restrict the strategies to have some specific closed forms. Moreover, they consider the symmetric Affiliated Value (AV) model of Section 2.1, where bidder i's utility is $U_i = U(\sigma_i, C)$ with $(\sigma_1, \ldots, \sigma_I, C)$ distributed as $F(\cdot, \ldots, \cdot)$ which is exchangeable in its first I arguments and affiliated.[73] The equilibrium strategy $s(\cdot; I)$ is given by (2.5). Integrating by parts, the latter can be rewritten as

$$B_i = s(\sigma_i; I) = \mathrm{E}[\nu(A; I)|\sigma_i, I], \tag{8.1}$$

where $\nu(a; I) \equiv v(a, a; I) = \mathrm{E}[U_i|\sigma_i{=}a, \sigma_{-i}^{\max}{=}a, I]$ is called the (expected) pivotal value, $\sigma_{-i}^{\max} \equiv \max_{j\neq i} \sigma_j$ is distributed as $F_{\underline{\sigma}^{\max}|\sigma}(\cdot|\sigma_i, I)$ with density $f_{\underline{\sigma}^{\max}|\sigma}(\cdot|\sigma_i, I)$ and A is a random variable distributed as $L_{A|\sigma}(\cdot|\sigma_i, I) = \exp[-\int_{\cdot}^{\sigma_i} f_{\underline{\sigma}^{\max}|\sigma}(\sigma|\sigma, I)/F_{\underline{\sigma}^{\max}|\sigma}(\sigma|\sigma, I)d\sigma]$ on $[\underline{\sigma}, \sigma_i]$ conditional on (σ_i, I). See Milgrom and Weber (1982). In particular, $L_{A|\sigma}(\cdot|\sigma_i, I)$ is absolutely continuous on $[\underline{\sigma}, \sigma_i]$.[74]

The structural econometric model is obtained from (8.1) by exploiting that the unobserved signal σ_i is distributed as $F_\sigma(\cdot) \equiv \int_{\underline{c}}^{\overline{c}} F_{\sigma|C}(\cdot|c)dF_C(c)$. In particular, the distribution of B_i only depends on $[\nu(\cdot; I), F_{\underline{\sigma}^{\max}|\sigma}(\cdot|\cdot, I), F_\sigma(\cdot)]$ so that one can at most identify the latter three functions from observed bids. A parametric model is obtained by parameterizing these functions with $\theta \in I\!R^k$ and conditioning on auction covariates X. Because the resulting equilibrium strategy (8.1) depends on (θ, X, I) in a complex way, the authors recommend simulation-based methods for estimation. For instance, (8.1) gives

$$\mathrm{E}[B_i|X, I] = \iint_{\mathcal{S}} \left[\nu(a; X, I, \theta)\frac{f_{\underline{\sigma}^{\max}|\sigma,X}(a|a, X, I; \theta)}{F_{\underline{\sigma}^{\max}|\sigma,X}(a|a, X, I; \theta)} L_{A|\sigma,X}(a|\sigma, X, I; \theta) f_{\sigma|X}(\sigma|X; \theta)\right] da d\sigma,$$

where $\mathcal{S} \equiv \{(a, \sigma) : \underline{\sigma} \leq a \leq \sigma, \underline{\sigma} \leq \sigma \leq \overline{\sigma}\}$ and $f_{\sigma|X}(\cdot|X; \theta)$ is the density of a bidder's signal given X. Thus, one can apply Laffont, Ossard and Vuong's (1995) SNLLS estimator

[73] As a matter of fact, they consider Milgrom and Weber's (1982) extension of Wilson's (1977) model. In this case, $U_i = U(\sigma_i, \sigma_{-i}, C)$ and the rest follows.

[74] We assume that $F_{\sigma|C}(\cdot|c)$ is I-continuously differentiable on $[\underline{\sigma}, \overline{\sigma}]$ with I-th derivative bounded away from zero uniformly in $c \in [\underline{c}, \overline{c}]$. Thus, using $F_{\underline{\sigma}^{\max}|\sigma}(y|\sigma, I) = \int_{\underline{c}}^{\overline{c}} \Pr(\underline{\sigma}^{\max} \leq y|\sigma, c, I)dF_{C|\sigma}(c|\sigma) = \int_{\underline{c}}^{\overline{c}} F_{\sigma|C}^{I-1}(y|c)dF_{C|\sigma}(c|\sigma)$, it can be shown that $f_{\underline{\sigma}^{\max}|\sigma}(\underline{\sigma}|\sigma, I) = \ldots = f_{\underline{\sigma}^{\max}|\sigma}^{(I-3)}(\underline{\sigma}|\sigma, I) = 0$ and $f_{\underline{\sigma}^{\max}|\sigma}^{(I-2)}(\underline{\sigma}|\sigma, I) = (I-1)! \int_{\underline{c}}^{\overline{c}} f_{\sigma|C}^{(I-1)}(\underline{\sigma}|c)dF_{C|\sigma}(c|\sigma) > 0$. Thus, Taylor expansions of order $(I-2)$ and $(I-1)$ of $f_{\underline{\sigma}^{\max}|\sigma}(\cdot|\sigma, I)$ and $F_{\underline{\sigma}^{\max}|\sigma}(\cdot|\sigma, I)$ around $\underline{\sigma}$ show that $f_{\underline{\sigma}^{\max}|\sigma}(\sigma|\sigma, I)/F_{\underline{\sigma}^{\max}|\sigma}(\sigma|\sigma, I)$ behaves as $1/(\sigma - \underline{\sigma})$ in the neighborhood of $\underline{\sigma}$. Hence, $\lim_{a\downarrow\underline{\sigma}} \int_a^{\sigma_i} f_{\underline{\sigma}^{\max}|\sigma}(\sigma|\sigma, I)/F_{\underline{\sigma}^{\max}|\sigma}(\sigma|\sigma, I)d\sigma = +\infty$ implying $\lim_{a\downarrow\underline{\sigma}} L_{A|\sigma}(a|\sigma_i, I) = 0$.

by dividing the integrand by $\phi(a,\sigma|X)$ and averaging over random draws from a known importance sampling joint density $\phi(\cdot,\cdot|X)$ with support $\mathcal{S}$ for each bidder and each auction. As a matter of fact, Laffont and Vuong (1993) consider only the winning bid B^w since they are studying descending auctions. Because $B^w = s(\sigma^{(I:I)};I)$ where the highest signal $\sigma^{(I:I)}$ is distributed as $F^{(I:I)}(\sigma|I) = \int_{\underline{\sigma}}^{\sigma} F_{\underline{\sigma}^{\max}|\sigma,I}(\sigma|u,I) f_\sigma(u)du$, (8.1) gives

$$\begin{aligned}\mathrm{E}[B^w|X,I] = \iiint_{\mathcal{S}^w} \Bigg[\nu(a|X,I;\theta) \frac{f_{\underline{\sigma}^{\max}|\sigma,X}(a|a,X,I;\theta)}{F_{\underline{\sigma}^{\max}|\sigma,X}(a|a,X,I;\theta)} L_{A|\sigma,X}(a|\sigma,X,I;\theta) \\ \times \Big[f_{\underline{\sigma}^{\max}|\sigma,X}(u|\sigma,X,I;\theta) f_{\sigma|X}(\sigma|X;\theta) + f_{\underline{\sigma}^{\max}|\sigma,X}(\sigma|u,X,I;\theta) f_{\sigma|X}(u|X;\theta) \Big] \Bigg] du\,da\,d\sigma,\end{aligned}$$

where $\mathcal{S}^w \equiv \{(u,a,\sigma) : \underline{\sigma} \le u \le \sigma, \underline{\sigma} \le a \le \sigma, \underline{\sigma} \le \sigma \le \overline{\sigma}\}$. As above, an unbiased estimator of $\mathrm{E}[B^w|X,I]$ is obtained by simulating the integrand divided by $\phi^w(u,a,\sigma|X)$ using random draws from an importance sampling density $\phi^w(\cdot,\cdot,\cdot|X)$ with support $\mathcal{S}^w$ conditional on X. As in Section 3.1, one can also entertain higher moments based on (8.1) to ensure identification of θ. One then applies a SMM estimator following McFadden (1989) and Pakes and Pollard (1989).

Hong and Shum (2002) also consider the symmetric AV model but start from the affiliated joint distribution of bidders' utilities and signals, which they parameterize as $F(u_1,\ldots,u_I,\sigma_1,\ldots,\sigma_I;\theta)$.[75] Specifically, following Wilson (1998) they specify bidder i's utility and signal as $U_i \equiv C\eta_i$ and $\sigma_i \equiv U_i\epsilon_i = C\eta_i\epsilon_i$ with (C,ϵ_i,η_i) mutually independent and distributed as log-normals $\mathcal{LN}(\mu_C,\omega_C^2)$, $\mathcal{LN}(0,\omega_\epsilon^2)$ and $\mathcal{LN}(0,\omega_\eta^2)$, respectively. Bidders only know their own signal σ_i. With this specification, $F(u_1,\ldots,u_I,\sigma_1,\ldots,\sigma_I;\theta)$ is exchangeable in the subscript i and affiliated with $\theta = (\mu_C,\omega_C^2,\omega_\epsilon^2,\omega_\eta^2)$. This specification has the advantage of encompassing a large class of models. In particular, the model is PCV when ω_η^2 approaches 0 whereas it is APV when ω_ϵ^2 approaches 0 and IPV when ω_C^2 also approaches 0. Regarding estimation, the authors are the first to propose a quantile-based method. From the monotonicity of the equilibrium strategy (2.5), the α-quantile of the bid distribution is $s[\sigma(\alpha;\theta);I,\theta]$ where $\sigma(\alpha;\theta)$ is the α-quantile of the signal distribution $F_\sigma(\cdot;\theta) = \int_0^{+\infty} F_{\sigma|C}(\cdot|c;\theta) f_C(\cdot;\theta)dc$. Thus, using a grid of K quantiles $(\alpha_1,\ldots,\alpha_K)$ in $(0,1)$, they estimate θ by K nonlinear quantile regressions, i.e., by solving

$$\min_\theta \sum_{k=1}^K \sum_{\ell=1}^L \sum_{i=1}^{I_\ell} \rho_{\alpha_k}\Big(B_{i\ell} - s[\sigma(\alpha_k;\theta);I_\ell,\theta]\Big),$$

where $\rho_{\alpha_k}(x) = [\alpha_k - \mathbb{1}(x<0)]x$ is the check function. To evaluate $s(\cdot;I,\theta)$ from (2.5), they use $F_{\underline{\sigma}^{\max}|\sigma}(\cdot|\sigma,I;\theta) = \int_0^{+\infty} F_{\sigma|C}(\cdot|c;\theta)^{I-1} f_{C|\sigma}(c|\sigma;\theta)dc$ where $f_{C|\sigma}(c|\sigma;\theta) =$

[75] They study construction procurement auction data. We present the high-bid version here.

$f_{\sigma|C}(\sigma|c;\theta)\ f_C(c;\theta)/f_\sigma(\sigma;\theta)$ and $F_{\sigma|C}(\cdot|c;\theta)$ is $\mathcal{LN}(\log c, \omega_\epsilon^2 + \omega_\eta^2)$. They also approximate the pivotal value $v(\sigma,\sigma;I) = \mathrm{E}[U_1|\sigma_1 = \sigma, \sigma_2 = \sigma, \sigma_3 \le \sigma, \ldots, \sigma_I \le \sigma)$ using a large number of simulations from the log-normal distribution of $(\log\sigma_3, \ldots, \log\sigma_I)$ given $(\log\sigma_1, \log\sigma_2)$ upon exploiting the explicit form of $\mathrm{E}(U_i|\sigma_1, \ldots, \sigma_I)$ from the log-normal specification. The resulting estimator of θ is $\sqrt{L}$-asymptotically normal. By exploiting the log-normal specification, an advantage of this method is to reduce the computational burden associated with the preceding SNLLS or SMM methods.

Section 8.2: Indirect Estimation Methods for First-Price Auctions

Identification in nonlinear parametric models is not entirely satisfactory as it is often local and dependent on functional forms. Laffont and Vuong (1996) are the first to address the global nonparametric identification of the general AV model. In view of their negative results, researchers have introduced some functional forms or exclusion restrictions to achieve identification, sometimes focusing on a few features of interest. In this subsection, we review these results and the proposed indirect estimation methods. Throughout, there is no reserve price and all bids are observed.

Nonidentification of Affiliated Value Models

Laffont and Vuong (1996) consider Wilson's (1977) model where bidder i's utility is $U_i = U(\sigma_i, C)$ where $(\sigma_1, \ldots, \sigma_I, C)$ is distributed as $F(\cdot, \ldots, \cdot|I)$ which is exchangeable in its first I arguments and affiliated.[76] They show that the symmetric AV model is observationally equivalent to some symmetric APV model and thus cannot be distinguished from the latter based on observed bids in first-price auctions. Intuitively, any dependence among bidder's utilities arising from the common component can be mimicked by an appropriate dependence among private values. Formally, let $B_{-i}^{\max} \equiv \max_{j\neq i} B_j$ with distribution and density $G_{B_{-}^{\max}|B}(\cdot|\cdot, I)$ and $g_{B_{-}^{\max}|B}(\cdot|\cdot, I)$ given (B_i, I) upon dropping the subscript i because of symmetry. Using the monotonicity of the equilibrium strategy (2.5) so that $F_{\sigma_{-}^{\max}|\sigma}(y|\sigma, I) = G_{B_{-}^{\max}|B}[s(y;I)|s(\sigma;I), I]$ and $f_{\sigma_{-}^{\max}|\sigma}(y|\sigma, I) = s'(y;I)g_{B_{-}^{\max}|B}[s(y;I)|s(\sigma), I]$, the differential equation (2.4) becomes

$$v(\sigma_i, \sigma_i; I) = B_i + \frac{G_{B_{-}^{\max}|B}(B_i|B_i, I)}{g_{B_{-}^{\max}|B}(B_i|B_i, I)} \equiv \xi(B_i; G, I), \tag{8.2}$$

[76]Whether I is exogenous or endogeneous, this joint distribution depends on I as it has $I+1$ arguments. The authors also consider Wilson's (1967) asymmetric PCV model in which one bidder knows the common value while other bidders are completely uninformed. They show that this model is identified and they characterize its restrictions on observed bids. They show, however, that it cannot be distinguished from an asymmetric IPV model in which one bidder differs from the others.

where $B_i = s(\sigma_i; I)$ and $v(\sigma_i, \sigma_i; I) \equiv \mathrm{E}[U_i|\sigma_i, \sigma_{-i}^{\max} = \sigma_i, I]$ is bidder i's pivotal value. See Li, Perrigne and Vuong (2000).[77] Thus, the utility $U_i \equiv U(\sigma_i, C)$ can be replaced by a private value equal to the pivotal value, i.e., $V_i \equiv v(\sigma_i, \sigma_i; I)$, thereby leading to the same joint bid distribution $G(\cdot, \ldots, \cdot|I)$. Because the symmetric APV model is a special case of the symmetric AV model (see Section 2.1), it follows that the latter imposes the same restrictions on observed bids as the former as characterized in Section 4.3.[78]

Another important consequence of this observational equivalence is that the symmetric AV model is not identified since there exists another symmetric AV model, namely a symmetric APV model, that leads to the same bid distribution. Such a nonidentification result also holds for the PCV model with primitives $[F_{\sigma|C}(\cdot|\cdot), F_C(\cdot)]$. Indeed, it suffices to consider a monotonic transformation of the signals σ_i to generate the same bid distribution. This is so even if bidders' signals are unbiased estimates of the common value C. Despite being nonidentified, the PCV model imposes some restrictions on observed bids as it is observationally equivalent to an APV model.[79] In view of such negative results, the general AV model and its special case the PCV model have lost some popularity in empirical research in favor of the APV and its special case the IPV model.

Though the primitives $[U(\cdot,\cdot), F(\cdot, \ldots, \cdot|I)]$ of the symmetric AV model are not identified, one may wonder whether some economically important features are identified such as bidders' pivotal values $\nu(\sigma; I) \equiv v(\sigma, \sigma; I)$, the winner's expected value $\nu^w(\sigma; I) \equiv \mathrm{E}[U(\sigma_i, C)|\sigma_i = \sigma, \sigma_{-i}^{\max} \leq \sigma, I]$ for any $\sigma \in [\underline{\sigma}, \overline{\sigma}]$ and the distribution of seller's revenue $F^R(\cdot|p_0, I)$ for some seller's valuation and reserve price (V_0, p_0). Because the RHS of (8.2) is known from observed bids, it follows that bidder i's pivotal value $\nu(\sigma_i; I) \equiv v(\sigma_i, \sigma_i; I)$ is identified from his bid B_i in the symmetric AV model. This does not mean that the pivotal value function $\nu(\cdot; I)$ is identified since the bidding strategy $s(\cdot; I)$ linking σ_i to B_i is not. Because the distribution of signals is defined up to monotone transformations, Athey and Haile (2002) propose the normalization $\nu(\sigma; I) = \sigma$. Some researchers have

[77]As for (3.6), an alternative derivation of (8.2) considers bidder i's expected profit $\mathrm{E}[(U_i - B_i)\mathbb{1}(B_i \geq B_{-i}^{\max})|\sigma_i] = \mathrm{E}\{\mathrm{E}[(U_i - B_i)\mathbb{1}(B_i \geq B_{-i}^{\max})|\sigma_i, B_{-i}^{\max}]|\sigma_i\} = \int_{\underline{b}}^{B_i}[\mathrm{E}(U_i|\sigma_i, B_{-i}^{\max} = b) - B_i] g_{B_-^{\max}|\sigma}(b|\sigma_i)db$. Taking the FOC with respect to B_i and using $\mathrm{E}(U_i|\sigma_i, B_{-i}^{\max} = B_i) = v(\sigma_i, \sigma_i; I)$ and $G_{B_-^{\max}|\sigma}(B_i|\sigma_i)/g_{B_-^{\max}|\sigma}(B_i|\sigma_i) = G_{B_-^{\max}|B}(B_i|B_i)/g_{B_-^{\max}|B}(B_i|B_i)$ since $B_i = s(\sigma_i; I)$ give (8.2).

[78]In PV models, $v(\sigma_i, \sigma_i; I)$ is bidder i's private valuation V_i so that (8.2) generalizes (4.7) to the symmetric AV model. In particular, in PV models with I exogenous, V_i is independent of I whereas in AV models, $v(\sigma_i, \sigma_i; I)$ decreases with I as shown by Haile, Hong and Shum (2006). The latter use this property to test for the presence of common value under exogenous variations of I.

[79]More precisely, Li, Perrigne and Vuong (2000) show that a PCV model is observationally equivalent to a CIPV model as reviewed in Section 4.3.

adopted instead the normalization $s(\sigma; I) = \sigma$.[80] The latter implies that bidders' signals are known since they are equal to their observed bids. Moreover, the pivotal value function $\nu(\cdot; I)$ becomes immediately identified from (8.2).

Despite the normalization $s(\sigma; I) = \sigma$, Tang (2011) shows that the winner's expected value $\nu^w(\cdot; I)$ and the seller's revenue distribution $F^R(\cdot|p_0, I)$ are not identified.[81] He then derives some tight bounds for such features. For $\nu^w(\cdot; I)$, these are

$$\nu_L^w(\cdot; I) \equiv \int_{\underline{b}}^{\cdot} \nu(\sigma; I) \frac{g_{B_-^{\max}|B}(\sigma|\cdot, I)}{G_{B_-^{\max}|B}(\cdot|\cdot, I)} d\sigma \ \le \ \nu^w(\cdot; I) \ \le \ \nu(\cdot; I) \equiv \nu_U^w(\cdot; I), \tag{8.3}$$

where the bounds $[\nu_L^w(\cdot;I), \nu_U^w(\cdot;I)]$ are identified by (8.2). This follows from $g_{B_-^{\max}|B}(\sigma|\cdot,I)/G_{B_-^{\max}|B}(\cdot|\cdot, I) = f_{\sigma_-^{\max}|\sigma}(\sigma|\cdot)/F_{\sigma_-^{\max}|\sigma}(\cdot|\cdot)$ since $s'(\cdot; I) = 1$ whereas $\mathrm{E}[U_i|\sigma_i = \sigma, \sigma_{-i}^{\max} = \sigma, I] \le \mathrm{E}[U_i|\sigma_i = \sigma, \sigma_{-i}^{\max} = \cdot, I] \le \mathrm{E}[U_i|\sigma_i = \cdot, \sigma_{-i}^{\max} = \cdot, I]$ for $\sigma \le \cdot$ by affiliation. To bound $F^R(\cdot|p_0, I)$, he first bounds bidders' equilibrium strategy when there is a binding reserve price p_0. From Milgrom and Weber (1982), the latter is

$$s(\sigma_i; p_0, I) = p_0 L[\sigma^*(p_0; I)|\sigma_i, I] + \int_{\sigma^*(p_0,I)}^{\sigma_i} \nu(a; I)\ dL(a|\sigma_i, I), \tag{8.4}$$

for $\sigma_i \ge \sigma^*(p_0; I)$ where $L(\cdot|\sigma_i, I)$ is as above and $\sigma^*(p_0; I)$ is the signal screening level solving $\nu^w[\sigma^*(p_0; I); I] = p_0$. In particular, $s(\cdot; p_0, I)$ depends on p_0 only through $\sigma^*(p_0; I)$ suggesting bounds $[s_L(\cdot; p_0, I), s_U(\cdot; p_0, I)]$ obtained from (8.4) by replacing $\sigma^*(p_0; I)$ by its bounds $[\sigma_L^*(p_0; I), \sigma_U^*(p_0; I)]$, respectively. Because $\nu_L^w(\cdot; I)$, $\nu^w(\cdot; I)$ and $\nu_U^w(\cdot; I)$ are increasing, (8.3) gives tight bounds for the screening level $\sigma^*(p_0; I)$ as $[\sigma_L^*(p_0; I), \sigma_U^*(p_0; I)]$ where $\nu_U^w[\sigma_L^*(p_0; I); I] = p_0$ and $\nu_L^w[\sigma_U^*(p_0; I); I] = p_0$. Given a seller's valuation $V_0 \le p_0$ and the normalization $s(\sigma; I) = \sigma$, bounds for $F^R(\cdot|p_0, I)$ are

$$F_k^R(r|p_0, I) = \begin{cases} 0 & \text{if } r < V_0 \\ G_{B^{\max}}[\sigma_k^*(p_0; I)|I] & \text{if } V_0 \le r < p_0 \\ G_{B^{\max}}[s_k^{-1}(r; p_0, I)|I] & \text{if } r \ge p_0 \end{cases} \tag{8.5}$$

for $k \in \{L, U\}$ where $B^{\max} = \max\{B_1, \ldots, B_I\} \sim G_{B^{\max}}(\cdot|I)$. The author then shows that the bounds $[F_L^R(\cdot|p_0, I), F_U^R(\cdot|p_0, I)]$ are tight. Estimation of such bounds is obtained by (i) estimating $\nu_U^w = \nu(\cdot; I)$ and $\nu_L^w(\cdot; I)$ using their empirical analogs in (8.2) and (8.3), (ii) solving the resulting estimated equations $\hat{\nu}_U^w[\hat{\sigma}_L^*(p_0; I); I] = p_0$ and $\hat{\nu}_L^w[\hat{\sigma}_U^*(p_0; I), I] = p_0$ for

[80] Both normalizations imply some restrictions on the model primitives which have not been investigated. Moreover, because $\sigma_-^{\max}$ depends on I, these normalizations imply that the model primitives also depend on I.

[81] As a matter of fact, Tang (2011) considers the model $U_i = U(\sigma_i, \sigma_{-i})$ where $U(\sigma_i, \cdot)$ is exchangeable in σ_{-i} with $(\sigma_1, \ldots, \sigma_I)$ distributed as $F(\cdot, \ldots, \cdot|I)$ which is exchangeable and affiliated.

$[\hat{\sigma}^*_L(p_0; I), \hat{\sigma}^*_U(p_0; I)]$, and (iii) using (8.5) with such estimates to obtain estimated bounds $[\hat{F}^R_L(\cdot|p_0, I), \hat{F}^R_U(\cdot|p_0, I)]$. The author establishes the consistency of these estimated bounds relying on Li, Perrigne and Vuong's (2002) asymptotic results.

Common Value Models Under Functional Form Restrictions

This subsection focuses on CV models where $U_i = C$ for every $i = 1, \ldots, I$. Thus, bidder i's pivotal value is $\nu(\sigma_i; I) = \mathrm{E}[C|\sigma_i, Y_i = \sigma_i, I]$. Li, Perrigne and Vuong (2000) focus on the PCV model as an example of conditionally independent private information. To restore identification of the PCV model, they assume that bidder i's signal satisfies the multiplicative decomposition $\sigma_i = C\epsilon_i$ where the idiosyncratic component ϵ_i is i.i.d. as $F_\epsilon(\cdot)$ independently of the common value C. Thus the model primitives are $[F_C(\cdot), F_\epsilon(\cdot)]$. Moreover, they assume bidders' pivotal values to be of the form $\nu(\sigma_i; I) = \beta_{I0}\sigma_i^{\beta_{I1}}$ with $(\beta_{I0}, \beta_{I1}) > 0$. They provide examples of primitives $[F_C(\cdot), F_\epsilon(\cdot)]$ where the pivotal value is of this form. Hence, letting $\log \tilde{C} \equiv \log \beta_{I0} + \beta_{I1}\mathrm{E}(\log \epsilon_i) + \beta_{I1} \log C$ and $\log \tilde{\epsilon}_i \equiv \beta_{I1}[\log \epsilon_i - \mathrm{E}(\log \epsilon_i)]$ shows that $\log \nu(\sigma_i; I) = \log \tilde{C} + \log \tilde{\epsilon}_i$. Because $\nu(\sigma_i; I)$ is identified for every $i = 1, \ldots, I$, application of Kotlarski's (1967) result shows that the distributions of $\tilde{C}$ and $\tilde{\epsilon}$ are identified from observed bids. Thus, given a normalization such as $\mathrm{E}(\epsilon_i) = 1$, the distributions $[F_C(\cdot), F_\epsilon(\cdot)]$ are identified up to the location and scale parameters (β_{I0}, β_{I1}). In particular, the ratio $\mathrm{Var}(\log \epsilon_i)/\mathrm{Var}(\log C)$ of the variances of the idiosyncratic part and common value is identified. Regarding estimation, one proceeds as for the CIPV model reviewed in Section 4.3. In the first step, estimates $\hat{\nu}(\sigma_i|I) = \hat{v}(\sigma_i, \sigma_i|I)$ of pivotal values are obtained from (8.2) using nonparametric estimates of the bid distribution and density $[G_{\underline{B}^{\max}|B}(\cdot|\cdot, I), g_{\underline{B}^{\max}|B}(\cdot|\cdot, I)]$. In the second step, these $\hat{\nu}(\sigma_i; I)$ are used to estimate the joint characteristic function of a pair of pivotal values. Plugging the latter in (4.10) provides consistent estimates of the densities $[f_{\tilde{C}}(\cdot), f_{\tilde{\epsilon}}(\cdot)]$ and hence $[f_C(\cdot), f_\epsilon(\cdot)]$ up to (β_{I0}, β_{I1}) by Fourier inversion.

Février (2008) considers a very specific functional form restriction for the PCV model. He assumes that the distribution $F_{\sigma|C}(\cdot|\cdot)$ arises from the truncation of some distribution $H(\cdot)$ on $[\underline{s}, \overline{s}]$, namely $F_{\sigma|C}(\cdot|c) = H(\cdot)/H[\overline{s}(c)]$ on $[\underline{s}, \overline{s}(c)]$ where $\overline{s}(\underline{c}) = \underline{s}$ and $\overline{s}(\cdot)$ is increasing. Thus $\overline{s} = \overline{s}(\overline{c})$ and $f_{\sigma|C}(\cdot|C) = h(\cdot)/H[\overline{s}(C)]$ with $h(\cdot)$ the density of $H(\cdot)$. The model primitives become $[F_C(\cdot), H(\cdot), \overline{s}(\cdot)]$. Besides some smoothness assumptions, he adopts the normalization $s(\sigma; I) = \sigma$ so that bidders' signals are known since their bids are observed. Hence, $[\underline{s}, \overline{s}] = [\underline{b}, \overline{b}]$ whereas the joint distribution of two bidders' signals is equal to the bivariate distribution $G(\cdot, \cdot|I)$ of two observed bids. Letting $g(\cdot, \cdot|I)$ be its density and $G_2(\cdot,\cdot|I)$ its derivative with respect to the second argument, he shows that

$$F_{\sigma|C}(\sigma|c) = \frac{G_2[\sigma,\overline{s}(c)|I]}{G_2[\overline{s}(c),\overline{s}(c)|I]} \quad \text{and} \quad F_C(c) = G[\overline{s}(c)|I] - \frac{G_2[\overline{s}(c),\overline{s}(c)|I]}{g[\overline{s}(c),\overline{s}(c)|I]}g[\overline{s}(c)|I],$$

for all $\sigma \in [\underline{s},\overline{s}(c)]$ and $c \in [\underline{c},\overline{c}]$. Thus, $[F_{\sigma|C}(\cdot|\cdot),F_C(\cdot)]$ is identified up to $\overline{s}(\cdot)$. He then combines the FOC (2.4) with the normalization $s(\sigma;I) = \sigma$ to provide an explicit though complicated expression for $\overline{s}^{-1}(\cdot)$ in terms of $G(\cdot|I)$, $G(\cdot,\cdot|I)$, their derivatives, $G_{B^{\max}}(\cdot|I)$ and the number of bidders I. Hence, $\overline{s}(\cdot)$ is identified. Estimation follows the identification argument by plugging-in kernel estimators. The author shows uniform consistency of the estimators of $[F_{\sigma|C}(\cdot|\cdot),F_C(\cdot)]$ on inner compact subsets of their supports.

More recently, He (2015) exploits alternative functional form restrictions to identify a CV model where bidders' signals are allowed to be dependent given C. Drawing from Klemperer (1998) and Goeree and Offerman (2003), he assumes that $C = \frac{1}{I}\sum_{i=1}^{I}\sigma_i + \epsilon$ where ϵ is distributed as $F_\epsilon(\cdot)$ with density $f_\epsilon(\cdot)$ and mean $\mathrm{E}(\epsilon) = 0$ independently from $(\sigma_1,\ldots,\sigma_I)$. The latter is jointly distributed as $F_\sigma(\cdot,\ldots,\cdot|I)$ with density $f_\sigma(\cdot,\ldots,\cdot|I)$, which is exchangeable and affiliated in its I arguments with marginal distribution $F_\sigma(\cdot)$ and density $f_\sigma(\cdot)$.[82] The model primitives are $[F_\sigma(\cdot,\ldots,\cdot|I),F_\epsilon]$ which are reparameterized as $[C_\sigma(\cdot,\ldots,\cdot|I),F_\sigma(\cdot),F_\epsilon(\cdot)]$ where $C_\sigma(\cdot,\ldots,\cdot|I)$ is the copula of $F_\sigma(\cdot,\ldots,\cdot|I) = C_\sigma[F_\sigma(\cdot),\ldots,F_\sigma(\cdot)|I]$ by Sklar' theorem. See Nelsen (1999).

To identify the model, He (2015) exploits the monotonicity of the equilibrium strategy to rewrite (8.2) with $\nu(\sigma_i;I) \equiv v(\sigma_i,\sigma_i;I) = \mathrm{E}[C|\sigma_i,\sigma_{-i}^{\max} = \sigma_i, I]$ in the quantile form

$$\nu[\sigma(\alpha);I] = b(\alpha;I) + \frac{G_{B_-^{\max}|B}[b(\alpha;I)|b(\alpha;I),I]}{g_{B_-^{\max}|B}[b(\alpha;I)|b(\alpha;I),I]} \equiv \xi(\alpha;G,I), \tag{8.6}$$

for $\alpha \in [0,1]$ where $\sigma(\alpha)$ and $b(\alpha;I)$ are the α-quantiles of $F_\sigma(\cdot)$ and $G(\cdot|I)$. When $I = 2$, bidder i's pivotal value reduces to $\nu(\sigma_i;2) = \sigma_i$ since $\sigma_{-1}^{\max} = \sigma_2$ and $C = \frac{1}{2}(\sigma_1+\sigma_2)+\epsilon$. Thus, the bivariate distribution $F_\sigma(\cdot,\cdot|2)$ of the signals (σ_1,σ_2) is identified from (8.6) as in the APV model of Section 4.3. When $I \geq 3$, the author shows that

$$\nu[\sigma(\alpha);I] \;=\; \sigma(\alpha) - \frac{I-2}{I}\int_0^{\sigma(\alpha)} \frac{C_{\sigma 12}[\alpha,\alpha,F_\sigma(s),\alpha,\ldots\alpha|I]}{C_{\sigma 12}(\alpha,\ldots,\alpha|I)}ds,$$

where subscripts indicate partial derivatives of the copula $C_\sigma(\cdot,\ldots,\cdot|I)$ with respect to the corresponding arguments. Making the change-of-variable $u = F_\sigma(s)$, integrating by

[82]The density of C conditional on I is $f_C(\cdot|I) = \int f_\epsilon(\cdot - \frac{1}{I}\sum_{i=1}^{I}\sigma_i)f(\sigma_1,\ldots,\sigma_I|I)d\sigma_1\ldots\sigma_I$ whereas the joint density of $(\sigma_1,\ldots,\sigma_I)$ conditional on (C,I) is $f_{\sigma_1,\ldots,\sigma_I|C}(\sigma_1,\ldots,\sigma_I|c,I) = f_\epsilon(c - \frac{1}{I}\sum_{i=1}^{I}\sigma_i)$ $f(\sigma_1,\ldots,\sigma_I|I)/f_C(c|I)$. The latter is not equal to $\prod_{i=1}^{I} f_{\sigma_i|C}(\sigma_i|c,I)$ where $f_{\sigma_i|C}(\sigma_i|c,I)$ is the conditional density of σ_i given (C,I). Thus bidders' signals are not conditionally independent given C.

parts and substituting the resulting expression in (8.6) give for $\alpha \in [0,1]$

$$\sigma(\alpha) + \int_0^\alpha \left[\frac{I-2}{2} \frac{C_{\sigma 123}(\alpha, \alpha, u, \alpha, \ldots, \alpha|I)}{C_{\sigma 12}(\alpha, \ldots, \alpha|I)} \right] \sigma(u) du = \frac{I}{2} \xi(\alpha; G, I). \tag{8.7}$$

This is a Volterra integral equation of the second kind in the unknown signal quantile function $\sigma(\cdot)$. Indeed, the function $\xi(\cdot; G, I)$ is known from observed bids and the number of bidders I, whereas the copula $C_\sigma(\cdot, \ldots, \cdot|I)$ of the signals $(\sigma_1, \ldots, \sigma_I)$ is equal to the copula of the bids $(B_1, \ldots, B_I)$ since $B_i = s(\sigma_i; I)$ where the equilibrium strategy $s(\cdot; I)$ is increasing. Thus $C_\sigma(\cdot, \ldots, \cdot|I)$ is also identified. Because a Volterra integral equation of the second kind has at most one solution from Kress (1999), it follows that $\sigma(\cdot)$ and hence $F_\sigma(\cdot)$ are identified. Though nothing can be said about $F_\epsilon(\cdot)$, following a similar idea as in Li, Perrigne and Vuong (2003) the author shows that the seller and bidders' expected profits under any reserve price are identified as they depend on the identified primitives $[C_\sigma(\cdot, \ldots, \cdot|I), F_\sigma(\cdot)]$ only. This contrasts with Tang (2011).

Regarding estimation, He (2015) assumes that $C_\sigma(\cdot, \ldots, \cdot|I)$ belongs to a parametric Archimedean family to circumvent the curse of dimensionality. He then estimates it as the copula of the observed bids $(B_1, \ldots, B_I)$ by ML upon plugging-in the empirical marginal distribution $\hat{G}(\cdot|I)$ of bids following Genest, Goudhi and Rivest (1995). See also Chen, Fan and Tsynnerikov (2016) for a semiparametric efficient estimator. To estimate the signal quantile function $\sigma(\cdot)$, he views (8.7) as the linear functional problem $(\mathcal{I}d - \mathcal{K}_I)[\sigma(\cdot)] = \frac{I}{2}\xi(\cdot; G, I)$ where $\mathcal{I}d$ is the identity operator and $\mathcal{K}_I[\sigma(\cdot)]$ is the negative of the integral in (8.7). Because $(\mathcal{I}d - \mathcal{K}_I)$ is invertible with a linear continuous inverse, it suffices to plug-in estimates of $[C_\sigma(\cdot, \ldots, \cdot|I), \xi(\cdot : G, I)]$. See Kress (1999) and Carrasco, Florens and Renault (2007) for linear inverse problems. An estimator of $C_\sigma(\cdot, \ldots, \cdot|I)$ is the above parametric one, whereas an estimator of $\xi(\cdot; G, I)$ is obtained from Li, Perrigne and Vuong (2002). See Section 4.3. Inverting the quantile estimator $\hat{\sigma}(\cdot) = (\mathcal{I}d - \hat{\mathcal{K}}_I)^{-1}[\xi(\cdot; \hat{G}, I)]$ provides a consistent estimator of $F_\sigma(\cdot)$.

Affiliated Value Models Under Exclusion Restrictions

The preceding methods straightforwardly apply when there are observed auction covariates X by conditioning the model primitives on them. Instead of restricting functional forms, Somaini (2020) exploits some exclusion restrictions on X to identify essential features of the general AV model. Following Athey and Haile (2002), $F(u_1, \ldots, u_I, \sigma_1, \ldots, \sigma_I| X, \mathcal{I})$ denote the joint distribution of utilities $U \equiv (U_1, \ldots, U_I)$ and signals $\sigma = (\sigma_1, \ldots, \sigma_I)$ given $(X, \mathcal{I})$ where $\mathcal{I}$ is a set of bidders indexed by $i = 1, \ldots, I$. The joint distribution $F_\sigma(\cdot, \ldots, \cdot|X, \mathcal{I})$ of $\sigma = (\sigma_1, \ldots, \sigma_I)$ given $(X, \mathcal{I})$ is affiliated but not necessarily exchangeable in its I arguments. A first exclusion restriction is that there is a scalar continuous variable X_i for each bidder i that only affects his utility U_i with $X = (X_1, \ldots, X_I)$.[83] Formally, bidder i's full-information value $\mathrm{E}(U_i|\sigma_i, \sigma_{-i}, X, \mathcal{I})$ does not depend on X_{-i} so that $\mathrm{E}(U_i|\sigma_i, \sigma_{-i}, X, \mathcal{I}) = \mathrm{E}(U_i|\sigma_i, \sigma_{-i}, X_i, \mathcal{I}) \equiv v_i(\sigma_i, \sigma_{-i}; X_i, \mathcal{I})$. Thus, the game is necessarily asymmetric. A second exclusion restriction is that $F_\sigma(\cdot, \ldots, \cdot|X, \mathcal{I})$ does not depend on X so that $F_\sigma(\cdot, \ldots, \cdot|X, \mathcal{I}) = F_\sigma(\cdot, \ldots, \cdot|\mathcal{I})$. Bidder i's expected payoff is

$$\mathrm{E}[(U_i - B_i) 1\!\!1(B_i \geq B_{-i}^{\max})|\sigma_i, X, \mathcal{I}] = \mathrm{E}\{[v_i(\sigma_i, \sigma_{-i}; X_i, \mathcal{I}) - B_i] 1\!\!1(B_i \geq B_{-i}^{\max})|\sigma_i, X, \mathcal{I}\},$$

where $B_{-i}^{\max} = \max_{j \neq i} B_j$ and $B_j = s_j(\sigma_j; X, \mathcal{I})$. This shows that the essential primitives of the AV model are $[v_1(\cdot, \cdot; X_1, \mathcal{I}), \ldots, v_I(\cdot; \cdot, X_I, \mathcal{I}), F_\sigma(\cdot, \ldots, \cdot|\mathcal{I})]$ since other features of the joint distribution of utilities and signals are payoff irrelevant. Relying on Maskin and Riley (2000b) and Reny and Zamir (2004), the author establishes the existence and properties of a Bayesian Nash equilibrium in pure strategies $[s_1(\cdot; X, \mathcal{I}), \ldots, s_I(\cdot; X, \mathcal{I})]$. In particular, the distribution of the winning bid is continuous and increasing on its support $[\underline{b}, \overline{b}] = [\underline{b}(X, \mathcal{I}), \overline{b}(X, \mathcal{I})]$. Moreover, for a large set of primitives, $s_i(\cdot; X, \mathcal{I})$ is continuous and increasing on $[\sigma_i^*; \overline{\sigma}]$ with $s_i(\sigma_i^*; X, \mathcal{I}) = \underline{b}$ and $s_i(\overline{\sigma}; X, \mathcal{I}) = \overline{b}$ where $\sigma_i^* = \sigma_i^*(X, \mathcal{I}) \in [\underline{\sigma}, \overline{\sigma}]$ is a signal threshold above which bidder i participates.

Turning to identification, Somaini (2020) normalizes each signal to be uniform $\mathcal{U}[0, 1]$ since it is defined up to an increasing transformation. Thus, $[\underline{s}, \overline{s}] = [0, 1]$ and the joint signal distribution $F_\sigma(\cdot, \ldots, \cdot|\mathcal{I})$ is a copula. Because $B_i = s_i(\cdot; X, \mathcal{I})$ with $s(\cdot; X, \mathcal{I})$ increasing on $[\sigma_i^*, 1]$, then $F_\sigma(\cdot, \ldots, \cdot|\mathcal{I})$ is identified on $\times_{i=1}^I [\sigma_i^*, 1]$ as the copula of the observed bids $[B_1, \ldots, B_I]$ given $(X, \mathcal{I})$, i.e., $F_\sigma(\sigma_1, \ldots, \sigma_I|\mathcal{I}) = \mathcal{C}^*_{B_1,\ldots,B_I|X}(\sigma_1, \ldots, \sigma_I|X, \mathcal{I}) = G^*_{B_1,\ldots,B_I|X}[G^{*-1}_{B_1|X}(\sigma_1|X, \mathcal{I}), \ldots, G^{*-1}_{B_I|X}(\sigma_I|X, \mathcal{I})|X, \mathcal{I})]$, where $G^*_{B_1,\ldots,B_I|X}(\cdot, \ldots, \cdot|X, \mathcal{I})$ and $G^*_{B_i|X}(\cdot|X, \mathcal{I})$ are the (truncated) joint distribution of observed bids and i-th marginal distribution with support $[\underline{b}, \overline{b}]^I$ and $[\underline{b}, \overline{b}]$, respectively. Hence, sufficient variations in the signal thresholds $\sigma_i^* = \sigma_i^*(X, \mathcal{I})$ as X varies identify $F_\sigma(\cdot, \ldots, \cdot|\mathcal{I})$ on $[0, 1]^I$. Moreover, the equilibrium strategy $s_i(\cdot; X, \mathcal{I})$ is identified on $[\sigma_i^*, 1]$ for every i since $s_i(\cdot; X, \mathcal{I}) = G^{*-1}_{B_i|X}(\cdot|X, \mathcal{I})$ with $\sigma_i \sim \mathcal{U}[0, 1]$.

To identify bidders' full-information value functions $[v_1(\cdot, \cdot; X_1, \mathcal{I}), \ldots, v_I(\cdot, \cdot; X_I, \mathcal{I})]$, the author exploits the FOC for each bidder. Extending (8.2) to the asymmetric AV model, the FOC for bidder i is rewritten as

$$\mathrm{E}\Big[U_i|\sigma_i, \max_{j \neq i} s_j(\sigma_j; X, \mathcal{I}) = s_i(\sigma_i; X, \mathcal{I}), X_i, \mathcal{I}\Big] = B_i + \frac{G^*_{B_{-i}^{\max}|B_i,X}(B_i|B_i, X, \mathcal{I})}{g^*_{B_{-i}^{\max}|B_i,X}(B_i|B_i, X, \mathcal{I})} \tag{8.8}$$

[83]Considering $\dim X_i = 1$ simplifies the presentation. If there are auction covariates affecting all utilities, they are conditioned upon and thus omitted hereafter. Somaini (2020) studies highway procurement auction data where X_i is bidder i' distance to the project site. We present the high-bid version here.

for $B_i = s_i(\sigma_i; X, \mathcal{I})$ with $\sigma_i \in [\sigma_i^*, 1]$, where $G^*_{B^{\max}_{-i}|B_i,X}(\cdot|\cdot, X, \mathcal{I})$ and $g^*_{B^{\max}_{-i}|B_i,X}(\cdot|\cdot, X, \mathcal{I})$ are the (truncated) distribution and density of $B^{\max}_{-i} \equiv \max_{j\neq i} B_j$ given $(B_i, X, \mathcal{I})$.[84] See also Athey and Haile (2002). The RHS of (8.8) denoted $\xi_i(\cdot; X, \mathcal{I})$ is identified and depends on i because of bidders' asymmetry. The LHS is bidder i's pivotal value $\nu_i(\sigma_i; X, \mathcal{I})$ extended to the asymmetric case since $\max_{j\neq i} s_j(\sigma_i) = s_i(\sigma_i)$ is equivalent to $B^{\max}_{-i} = B_i$.[85] When $I \geq 3$, the idea is to identify first

$$\begin{aligned}\int_{\sigma^*_{-i}\leq\tilde{\sigma}_{-i}\leq\sigma_{-i}} & v_i(\sigma_i, \tilde{\sigma}_{-i}; X_i, \mathcal{I}) f_{\sigma_{-i}|\sigma_i}(\tilde{\sigma}_{-i}|\sigma_i, \mathcal{I}) d\tilde{\sigma}_{-i} \\ &= \sum_{t=1}^{T} \int_{\mathcal{L}_t} v_i(\sigma_i, \tilde{\sigma}_{-i}; X_i, \mathcal{I}) f_{\sigma_{-i}|\sigma_i}(\tilde{\sigma}_{-i}|\sigma_i, \mathcal{I}) d\tilde{\sigma}_{-i} \qquad (8.9)\end{aligned}$$

for covariates (X_i, X_{-i}) and any bidders' signals $(\sigma_i, \sigma_{-i}) \in (\sigma_i^*, 1] \times_{j\neq i} (\sigma_j^*, 1]$, where $\{\mathcal{L}_t; t = 1, \ldots, T\}$ is a suitable partition of $\{\tilde{\sigma}_{-i} : \sigma^*_{-i} \leq \tilde{\sigma}_{-i} \leq \sigma_{-i}\}$. The purpose of the partition is to approximate each integral in the RHS of (8.9) by (8.8) times the probability that $\tilde{\sigma}_{-i}$ belongs to $\mathcal{L}_t$ given σ_i. For each t, $\mathcal{L}_t \equiv \mathcal{M}_t \backslash \mathcal{M}_{t+1}$ where $\mathcal{M}_t \equiv \{\tilde{\sigma}_{-i} : \sigma^*_{-i} \leq \tilde{\sigma}_{-i} \leq \sigma^{(t)}_{-i}\}$ with $\sigma^*_{-i} < \sigma^{(T)}_{-i} < \ldots < \sigma^{(2)}_{-i} < \sigma^{(1)}_{-i} \equiv \sigma_{-i}$. The sequence $\{\sigma^{(t)}_{-i}\}_{t=1}^{T}$ is defined recursively by

$$\sigma_j^{(t+1)} \equiv G^*_{B_j|X_i,X_{-i}}[G^{*-1}_{B_i|X_i,X_{-i}}(\sigma_i - \delta|X_i, x^{(t)}_{-i}, \mathcal{I})|X_i, x^{(t)}_{-i}, \mathcal{I}] \qquad (8.10)$$

for $j \neq i$ and some small positive $\delta < \sigma_i - \sigma_i^*$, where $x^{(t)}_{-i}$ satisfies

$$G^{*-1}_{B_j|X_i,X_{-i}}(\sigma_j^{(t)}|X_i, x^{(t)}_{-i}, \mathcal{I}) = G^{*-1}_{B_i|X_i,X_{-i}}(\sigma_i|X_i, x^{(t)}_{-i}, \mathcal{I}), \qquad (8.11)$$

for $j \neq i$ and $t = 1, \ldots, T-1$. The sequence $\{x^{(t)}_{-i}\}_{t=1}^{T-1}$ exists provided the other bidders' covariates X_{-i} vary sufficiently. Thus,

$$\mathcal{L}_t = \{\tilde{\sigma}_{-i} : s_i(\sigma_i - \delta|X_i, x^{(t)}_{-i}, \mathcal{I}) < \max_{j\neq i} s_j(\tilde{\sigma}_j; X_i, x^{t}_{-i}, \mathcal{I}) \leq s_i(\sigma_i|X_i, x^{(t)}_{-i}, \mathcal{I})\},$$

using (8.10)-(8.11) and $s_k(\cdot|X_i, x^{(t)}_{-i}, \mathcal{I}) = G^{*-1}_{B_k|X_i,X_{-i}}(\cdot|X_i, x^{(t)}_{-i}, \mathcal{I})$ for $k = 1, \ldots, I$. This suggests to approximate the RHS of (8.9) by

$$\sum_{t=1}^{T} \mathrm{E}\Big[U_i|\sigma_i, \max_{j\neq i} s_j(\sigma_j; X_i, x^{(t)}_{-i}, \mathcal{I}) = s_i(\sigma_i; X_i, x^{(t)}_{-i}, \mathcal{I}), X_i, \mathcal{I}\Big] \Pr(\sigma_{-i} \in \mathcal{L}_t|\sigma_i)$$

[84] As for (8.2), (8.8) follows from maximizing bidder i's expected profit $\mathrm{E}[(U_i - B_i)\mathbb{1}(B_i \geq B^{\max}_{-i})|\sigma_i] = \int_{\underline{b}}^{B_i} [\mathrm{E}(U_i|\sigma_i, B^{\max}_{-i} = b) - B_i] g^*_{B^{\max}_{-i}|\sigma_i}(b|\sigma_i) db$ upon omitting $(X_i, X, \mathcal{I})$. The FOC with respect to B_i gives (8.8) using $\{B^{\max}_{-i} = B_i\} = \{\max_{j\neq i} s_j(\sigma_i) = s_i(\sigma_i)\}$ where $(\sigma_1, \ldots, \sigma_I) \in \times_{i=1}^{I} [\sigma_i^*, 1]$.

[85] When $I = 2$, the identification proof simplifies since $\mathrm{E}\{U_i|\sigma_i, \sigma_j = s_j^{-1}[s_i(\sigma_i; X, \mathcal{I}); X, \mathcal{I}], X_i, \mathcal{I}\} = \mathrm{E}\Big[U_i|\sigma_i, \sigma_j = G^*_{B_j|X}[G^{*-1}_{B_i|X}(\sigma_i|X, \mathcal{I})|X, \mathcal{I}], X_i, \mathcal{I}\Big] = \xi_i(B_i; X, \mathcal{I})$ by $s_k(\cdot; X, \mathcal{I}) = G^{*-1}_{B_k|X}(\cdot|X, \mathcal{I})$ for $k = 1, 2$ and (8.8). Since $X = [X_i, X_j]$, sufficient variations in $G^*_{B_j|X}[G^{*-1}_{B_i|X}(\cdot|X, \mathcal{I})|X, \mathcal{I}]$ through X_j given X_i identifies bidder i's full-information value $\mathrm{E}(U_i|\sigma_i, \sigma_j, X_i, \mathcal{I})$ on $\{(\sigma_i, \sigma_j) : \sigma_i \in [\sigma_i^*(X_i, X_j, \mathcal{I}), 1] \times [\sigma_j^*(X_i, X_j, \mathcal{I}), 1], X_j \in \mathcal{S}_{X_j|X_i}\}$ and hence eventually on $[0, 1]^2$.

$$= \sum_{t=1}^{T} \xi_i(B_i^{(t)}; X_i, x_{-i}^{(t)}, \mathcal{I}) \Pr(\sigma_{-i} \in \mathcal{L}_t | \sigma_i)$$

by (8.8), where $B_i^{(t)} \equiv s_i(\sigma_k; X_i, x_{-i}^{(t)}, \mathcal{I})$. Somaini (2020) shows that the approximation error vanishes as δ approaches 0. Moreover, since $s_k(\cdot; X_i, x_{-i}^{(t)}, \mathcal{I}) = G^{*-1}_{B_k|X}(\cdot|X_i, x_{-i}^{(t)}, \mathcal{I})$ for every $k = 1, \dots, I$, it follows that its limit and hence the LHS of (8.9) are identified. Next, taking the cross partial derivative $\partial^{I-1}(\cdot)/\prod_{j\neq i} \partial\sigma_j$ shows that bidder i's full-information value $v_i(\sigma_i, \sigma_{-i}; X_i, \mathcal{I})$ is identified on $\{(\sigma_i, \sigma_{-i}) : \sigma_i \in [\sigma_i^*(X, \mathcal{I}), 1], \sigma_{-i} \in \times_{j\neq i}[\sigma_j^*(X, \mathcal{I}), 1], X_{-i} \in \mathcal{S}_{X_{-i}|X_i}\}$ given X_i and hence eventually on $[0,1]^I$. The author discusses extensions to a reserve price and entry. As in the IPV paradigm, these features may limit the ability of X to generate sufficient variations to identify bidders' full information values on their full support.

Regarding estimation, the author extends Wilson's (1998) log-normal model by developing a Gaussian framework where signals are jointly Normal $\mathcal{N}(0, \Omega)$ with normalized diagonal elements $\omega_{ii} = 1$.[86] He then parameterizes bidder i's full information value as

$$\mathrm{E}(U_i|\sigma_i, \sigma_{-i}, X, \mathcal{I}) = \beta_{i0} + \beta_{i1}\sigma_i + \beta_{i2} \cdot \sum_{j\neq i} \tilde{\omega}_{ij}\sigma_j + \gamma_{i0}' X_0 + \gamma_{i1} X_i,$$

where $X \equiv (X_0, X_1, \dots, X_I)$, X_0 is a vector of auction covariates affecting all utilities, $(\beta_{i0}, \beta_{i1}, \beta_{i2}, \gamma_{i0}', \gamma_{i1})$ are bidder i's specific coefficients and $\Omega^{-1} \equiv -[\tilde{\omega}_{ij}]$ with $\beta_{i1} > 0$, $\beta_{i2} > 0$ and $\omega_{ij} \geq 0$ because of affiliation. The interdependence of signals is captured by β_{i2}. Using (8.8) and $\mathrm{E}[\sigma_j|\sigma_i, \max_{k\neq i} s_k(\sigma_k; X, \mathcal{I}) = B_i, X, \mathcal{I}] = \mathrm{E}[\mathrm{E}(\sigma_j|\sigma_i, \sigma_{-i}, X_i, \mathcal{I})|\sigma_i, \max_{k\neq i} s_k(\sigma_k; X, \mathcal{I}) = B_i, X, \mathcal{I}]$, it follows that bidder i's equilibrium strategy satisfies

$$\xi_i(B_i; X, \mathcal{I}) = \beta_{i0} + \beta_{i1}\sigma_i + \beta_{i2} W_i + \gamma_{i0}' X_0 + \gamma_{i1} X_i, \tag{8.12}$$

where $W_i \equiv \sum_{j\neq i} \tilde{\omega}_{ij} \mathrm{E}[\sigma_j|\sigma_i, \max_{k\neq i} s_k(\sigma_k; X, \mathcal{I}) = B_i, X, \mathcal{I}]$, which is also equal to

$$W_i = \sum_{j\neq i} \tilde{\omega}_{ij} \Big\{ \pi_{ij} s_j^{-1}(B_i; X, \mathcal{I}) + \sum_{k\neq i,j} \pi_{ik} \mathrm{E}\Big[\sigma_j \mid \sigma_i, \sigma_k = s_k^{-1}(B_i; X, \mathcal{I}), \max_{m\neq i,j} s_m(\sigma_m; X, \mathcal{I}) \leq B_i, X, \mathcal{I}\Big] \Big\},$$

where $\pi_{ij} \propto \Pr[\max_{m\neq i,j} B_m \leq B_i \mid B_i, B_j = B_i, X, \mathcal{I}] \cdot g^*_{B_j}(B_i|X, \mathcal{I})$ for $j \neq i$ are weights summing to one over $j \neq i$. The variable W_i accounts for the winner's curse and is estimable since $s_j^{-1}(\cdot; X, \mathcal{I}) = \Phi^{-1}[G^*_{B_j}(\cdot|X, \mathcal{I})]$ from the $\mathcal{N}(0,1)$-normalization of σ_j whereas the truncated Gaussian copula of $(\sigma_1, \dots, \sigma_I)$ given $\{\sigma_1 \geq \sigma_1^*, \dots, \sigma_I \geq \sigma_I^*\}$

[86] Thus, each signal is $\mathcal{N}(0,1)$ instead of $\mathcal{U}[0,1]$ distributed. This normalization does not change the identification argument.

is the copula of $(B_1, \ldots, B_I)$. The signal threshold σ_i^* is estimated from $\Phi(\sigma_i^*) = \Pr(D_i = 0|X, \mathcal{I})$ where $D_i = 1$ if bidder i participates and 0 otherwise. Thus, Ω and hence $[\tilde{\omega}_{ij}]$ are estimated by ML from observed bids given the estimated thresholds $\hat{\sigma}_i^*$. The remaining parameters $(\beta_{i0}, \beta_{i1}, \beta_{i2}, \gamma'_{i0}, \gamma_{i1})$ are estimated from (8.12) upon replacing W_i by its estimator $\hat{W}_i$. The author adapts Buchinsky and Hahn's (1998) censored quantile regression estimator since bidders participate when their signals are above their thresholds. Let $\sigma_i(\alpha) = \Phi^{-1}(\alpha)$ be some α-quantile of σ_i. Because $\pi_i(\alpha; X, \mathcal{I}) \equiv \Pr[\sigma_i \leq \sigma_i(\alpha)|\sigma_i \geq \sigma_i^*, X, \mathcal{I}] = [\alpha - \Phi(\sigma_i^*)]/[1 - \Phi(\sigma_i^*)]$ for $\alpha \geq \Phi(\sigma_i^*)$, the adjusted conditional quantile restriction is

$$\mathrm{E}\left\{ I\!\!I\Big[\xi_i(B_i; X, \mathcal{I}) - \beta_{i2}W_i - \gamma'_{i0}X_0 - \gamma_{i1}X_i \leq \beta_{i0} + \beta_{i1}\sigma_i(\alpha)\Big] \mid X, \mathcal{I}\right\} - \pi_i(\alpha; X, \mathcal{I}) = 0.$$

With $\xi_i(\cdot; X, \mathcal{I})$ estimated using nonparametric estimators of $G^*_{B_{-i}^{\max}|B_i,X}(\cdot|\cdot, X, \mathcal{I})$ and its density, W_i estimated as above and $\pi_i(\alpha; X, \mathcal{I})$ estimated using $\Phi(\hat{\sigma}_i^*)$, this leads to minimizing a quadratic form of unconditional moment restrictions with respect to $(\beta_{i0}, \beta_{i1}, \beta_{i2}, \gamma'_{i0}, \gamma_{i1})$. See Chen, Linton and Van Keilegom (2003) when the criterion function is not smooth.[87]

Affiliated Values, Unobserved Heterogeneity and Entry

In a recent paper, Compiani, Haile and Sant' Anna (2020) allow for both unobserved heterogeneity and endogenous entry in a model of affiliated values. They consider a symmetric setting where bidders' utilities $U \equiv (U_1, \ldots, U_I)$ and signals $\sigma = (\sigma_1, \ldots, \sigma_I)$ are affiliated given (X, Y, I) and exchangeable in the subscript i. Extending Krasnokutskaya (2011), the unobserved heterogeneity $Y \in I\!\!R$ enters in a multiplicatively separable single index so that $U_i \equiv \gamma(X, Y)\tilde{U}_i$ where $\gamma(X, \cdot)$ is a positive increasing function and $\tilde{U}_i$ is bidder i's rescaled utility. Bidders' rescaled utilities $\tilde{U} \equiv (\tilde{U}_1, \ldots \tilde{U}_I)$ are jointly distributed as $\tilde{F}_{\tilde{U}|\sigma}(\cdot, \ldots, \cdot|\cdot, \ldots, \cdot, I)$ whereas bidders' signals σ are distributed as $F_\sigma(\cdot, \ldots, \cdot|I)$ with $(\tilde{U}, \sigma) \perp (X, Y) \mid I$. The authors consider a reduced form model $I = I(X, Z, Y)$ for the number of bidders, where $I(X, Z, \cdot)$ is weakly increasing. Following Levin and Smith (1994), bidder i learns his signal σ_i and the number of bidders I after entry so that bidding proceeds as in Milgrom and Weber (1982). The unobserved heterogeneity Y is independent of (X, Z), i.e., $Y \perp (X, Z)$, whereas the continuous instrument Z is excluded from the joint distribution of $(\tilde{U}, \sigma)$ given I, i.e., $(\tilde{U}, \sigma) \perp Z \mid (X, Y, I)$. Because unobserved Y enters in both the entry and utility equations, entry is endogenous. The model primitives

[87] As a matter of fact, Somaini (2020) estimates these parameters using Chernozhukov and Hansen's (2006) IV quantile method. Alternatively, the parameters $(\beta_{i0}, \beta_{i1}, \beta_{i2}, \gamma'_{i0}, \gamma_{i1})$ can be estimated from (8.12) using a minimum distance estimator with $\hat{\xi}(B_i; X, \mathcal{I})$ and $\hat{W}_i$.

are $[\gamma(\cdot,\cdot), \tilde{F}_{\tilde{U}|\sigma}(\cdot,\ldots,\cdot|\cdot,\ldots,\cdot,I), F_\sigma(\cdot,\ldots,\cdot|I), I(\cdot,\cdot,\cdot), F_{Y|X}(\cdot|\cdot,I)]$. The observations are $[B_1,\ldots,B_I,X,Z,I]$ over independent auctions.

Given the nonlinearity and multiplicative separability of the single index, the authors normalize the distribution of the unobserved heterogeneity Y to be uniform $\mathcal{U}[0,1]$ and set $\gamma(x_o,0)=1$ for some value $x_o \in \mathcal{S}_X$. Similarly to d'Haultfoeuille and Février (2015), the identification argument exploits variations of the support $[\underline{I}(X,Z),\overline{I}(X,Z)]$ in the number of bidders I given (X,Z). In a first step, the authors note that the function $I(X,Z,\cdot)$ is defined by its thresholds $0=\tau_0(X,Z)<\tau_1(X,Z)<\ldots<\tau_{K+1}(X,Z)=1$ so that $I=\underline{I}(X,Z)+k$ if $\tau_k(X,Z)<Y\leq\tau_{k+1}(X,Z)$ for $k=0,1,\ldots,K\equiv\overline{I}(X,Z)-\underline{I}(X,Z)\geq 1$. Thus, $I=I(X,Z,Y)$ reduces to an ordered discrete model with error $Y\sim\mathcal{U}[0,1]$ independent of (X,Z). Hence, $I(\cdot,\cdot,\cdot)$ is identified from its thresholds $\tau_k(X,Z)=\Pr[I\leq \underline{I}(X,Z)+k-1|X,Z]$. Moreover, because the distribution $F_{Y|X,Z}(\cdot|\cdot,\cdot,I)$ of Y given (X,Z,I) is uniform $\mathcal{U}[\tau_{k_I}(X,Z),\tau_{k_I+1}(X,Z)]$ with $k_I=I-\underline{I}(X,Z)$ by $Y\perp(X,Z)$, then the distribution $F_{Y|X}(\cdot|\cdot,I)$ of Y given (X,I) is identified by integrating out Z given I.

In a second step, the authors establish the identification of the index function $\gamma(\cdot,\cdot)$ on $\mathcal{S}_X\times[0,1]$. Its proof relies on the assumption among others that any value I in $\mathcal{S}_I\equiv[\underline{I},\overline{I}]$ is either a lower bound $\underline{I}(X,Z)$ or an upper bound $\overline{I}(X,Z)$ for some $(X,Z)\in\mathcal{S}_X\times\mathcal{S}_Z$. See the paper for details. In a third step, because $\tilde{F}_{\tilde{U}|\sigma}(\cdot,\ldots,\cdot|\cdot,\ldots,\cdot,I)$ and $F_\sigma(\cdot,\ldots,\cdot|I)$ are not identified in general, the authors focus on the joint distribution of bidders' pivotal values $\nu(\sigma_i;X,Y,I)$ which satisfy (8.2). By the location-scale property of equilibrium strategies (see Section 2.2), bidder i's rescaled pivotal value $\tilde{\nu}(\sigma_i;I)\equiv \mathrm{E}[\tilde{U}_i|\sigma_i,Y_i=\sigma_i,I]=\nu(\sigma_i;X,Y,I)/\gamma(X,Y)$ satisfies a similar equation, namely,

$$\tilde{\nu}(\sigma_i;I)=\tilde{B}_i+\frac{\tilde{G}_{\tilde{B}_-^{\max}|\tilde{B}}(\tilde{B}_i|\tilde{B}_i,I)}{\tilde{g}_{\tilde{B}_-^{\max}|\tilde{B}}(\tilde{B}_i|\tilde{B}_i,I)}\equiv\xi(\tilde{B}_i;\tilde{G},I), \tag{8.13}$$

where $\tilde{B}_j\equiv B_j/\gamma(X,Y)$ for $j=1,\ldots,I$ are the rescaled bids, which are not identified because Y is unobserved. But $\log B_j=\log\tilde{B}_j+\log\gamma(X,Y)$ where $\tilde{B}_j=\tilde{s}_j(\sigma_j;I)$ and $\gamma(X,Y)$ are independent given I. From the identification of the distribution of $\gamma(X,Y)$ given I, which follows from the earlier identification of $\gamma(\cdot,\cdot)$ and $F_{Y|X}(\cdot|\cdot,I)$, a standard deconvolution argument shows that the joint distribution $\tilde{G}(\cdot,\ldots,\cdot|I)$ of the rescaled bids $(\tilde{B}_1,\ldots,\tilde{B}_I)$ given I is identified. Hence, the joint distribution of bidders' rescaled pivotal values $[\tilde{\nu}(\sigma_1;I),\ldots,\tilde{\nu}(\sigma_I;I)]$ is identified by (8.13).

Regarding estimation, the authors parameterize the thresholds as $\tau_k(X,Y)=\Phi(\beta_k-X'\beta_X-Z'\beta_Z)$ for $1\leq k\leq K\equiv K(X,Z)$, the single index as $\gamma(X,Y;\beta_\gamma)$ and the signal copula $F_\sigma(\cdot,\ldots,\cdot|I)$ as the I-dimensional Gaussian copula $\mathcal{G}_I(\cdot,\ldots,\cdot;\rho_I)$ with correlation matrix ρ_I. In contrast, they leave unrestricted $\tilde{F}_{\tilde{U}|\sigma}(\cdot,\ldots,\cdot|\cdot,\ldots,\cdot,I)$ and hence bidders' rescaled pivotal values $\tilde{\nu}(\cdot; I)$. In a first step, up to a varying support $[\underline{I}(X,Z), \overline{I}(X,Z)]$ which is estimated from the data, the threshold parameters $(\beta_k, \beta_X, \beta_Z)$ are estimated by standard ordered probit ML since the number of bidder I is equal to $\underline{I}(X,Z) + k$ whenever $X'\beta_X + Z'\beta_Z + \epsilon \in (\beta_k, \beta_{k+1}]$ with $\beta_0 \equiv -\infty$, $\beta_{K+1} = +\infty$ and $\epsilon \equiv \Phi^{-1}(Y) \sim \mathcal{N}(0,1)$. In a second step, they maximize the semiparametric joint likelihood of the logarithm of the bids $(\log B_1, \ldots, \log B_I)$ given (X, Z, I). Because the copula of $[\log \tilde{B}_1, \ldots, \log \tilde{B}_I]$ is $\mathcal{G}_I(\cdot, \ldots, \cdot|\rho_I)$ since $\log \tilde{B}_i = \log \tilde{s}(\sigma_i; I)$, it follows that the joint distribution of $(\log B_1, \ldots, \log B_I)$ given (X, Y, Z, I) is

$$\mathcal{G}_I\Big\{\tilde{G}_{\log \tilde{B}}[\cdot - \log \gamma(X, Y; \beta_\gamma)|I], \ldots, \tilde{G}_{\log \tilde{B}}[\cdot - \log \gamma(X, Y; \beta_\gamma)|I]; \rho_I\Big\}, \tag{8.14}$$

where the marginal distribution of the logarithm of a rescaled bid given I is $\tilde{G}_{\log \tilde{B}}(\cdot|I)$, which is approximated by a Bernstein polynomial sieve. The likelihood of $(\log B_1, \ldots, \log B_I)$ given (X, Z, I) is then obtained by integrating the density associated with (8.14) with respect to Y which is uniform $\mathcal{U}[\tau_{k_I}(X,Z), \tau_{k_I+1}(X,Z)]$ with $k_I = I - \underline{I}(X,Z)$ given (X, Z, I). The thresholds are replaced by their estimates from the first step. This step provides estimates $\hat{\beta}_\gamma$ and $\hat{\rho}_I$. In a third step, the joint distribution of bidders' rescaled pivotal values $[\tilde{\nu}(\sigma_1; I), \ldots, \tilde{\nu}(\sigma_I; I)]$ is simulated from (8.13) using many draws from the estimated joint distribution $(\tilde{B}_1, \ldots, \tilde{B}_I)$ given I obtained in the second step.

Section 8.3: Affiliated Values in Ascending Auctions

In contrast to the PV paradigm where each bidder knows his value for the auctioned object, with affiliated values information held by competitors is informative about each bidder's unknown valuation. As a result, in a button ascending auction a bidder no longer bids his valuation since the prices at which some bidders have withdrawn are now informative. We present below two contributions using direct parametric estimation methods though one could develop an indirect estimation procedure based on Somaini's (2020) nonparametric identification argument using exclusion restrictions as it also applies to ascending and second-price sealed-bid auctions. The first contribution estimates an asymmetric affiliated value model of ascending auctions upon extending Milgrom and Weber's (1982) equilibrium reviewed in Section 2.2. The second paper circumvents the issue of information updating by approximating the last instant of an ascending auction as a second-price sealed-bid auction with a random number of bidders.

Asymmetric Affiliated Values

Hong and Shum (2003) consider the general AV model where bidders' utilities and signals are jointly distributed as $F(u_1, \ldots, u_I, \sigma_1, \ldots, \sigma_I|\mathcal{I})$. The latter is affiliated but needs not be exchangeable in its subscipt i. To simplify, we present the case with no auction covariates and no reserve price. In a button ascending auction, each bidder needs to decide whether to remain active or drop out at the current price p given the prices at which his competitors have withdrawn if any. Let $p_1 < \ldots < p_k$ $(< p)$ be such prices and $i_1, \ldots, i_k$ be the identities of the corresponding bidders who dropped out. Let $i_{k+1}, \ldots, i_I$ index the remaining bidders including bidder i. Given $\mathcal{I}$ omitted hereafter, the authors show that a Bayesian Nash equilibrium in monotone strategies is as follows. If no competitor has withdrawn, then bidder i quits when the price reaches

$$s_{i0}(\sigma_i) = \mathrm{E}\{U_i|\sigma_i, \sigma_j = s_{j0}^{-1}[s_{i0}(\sigma_i)] \text{ for } j \neq i\}.$$

If $k = 1, \ldots, I-2$ competitors have withdrawn, bidder i quits when the price reaches

$$s_{ik}(\sigma_i) = \mathrm{E}\{U_i|\sigma_i, \sigma_{i_j} = s_{i_jk}^{-1}[s_{ik}(\sigma_i)] \text{ for } j \geq k+1, i_j \neq i, \sigma_{i_j} = s_{i_j,j-1}^{-1}(p_j) \text{ for } j \leq k\}.$$

That is, bidder i drops out when the price reaches his expected utility conditional on his signal, the remaining competitors tying with him and the bidders who dropped out having signals consistent with their withdrawals at the prices $p_1, \ldots, p_k$. Thus $s_{ik}(\cdot)$ depends on $(p_1, \ldots, p_k)$ and the order and identities of the bidders who dropped out. This constitutes the history at stage k. Bidder i's equilibrium strategy is $[s_{i0}(\cdot), \ldots, s_{i,I-2}(\cdot)]$. This extends Milgrom and Weber's (1982) equilibrium (2.1)-(2-2) to the asymmetric case. Importantly, the authors show that bidders' inverse strategies recursively solve the $(I-k)$ equations

$$b = \mathrm{E}\{U_{i_n}|\sigma_{i_j} = s_{i_jk}^{-1}(b) \text{ for } j \geq k+1, \sigma_{i_j} = s_{i_j,j-1}^{-1}(p_j) \text{ for } j \leq k\}, \tag{8.15}$$

$n = k+1, \ldots, I$ in the $(I-k)$ unknowns $s_{i_jk}^{-1}(b)$, $j = k+1, \ldots, I$, for an arbitrary b and each $k = 0, \ldots, I-2$ whenever the solutions are unique and increasing in b.

To render the estimation problem tractable, the authors choose a parametric family so that these conditional expectations have closed forms. Drawing from Wilson (1998), they let $U_i \equiv C\eta_i$ and $\sigma_i \equiv U_i\epsilon_i = C\eta_i\epsilon_i$ with (C, ϵ_i, η_i) mutually independent and distributed as log-normals $\mathcal{LN}(\mu_C, \omega_C^2)$, $\mathcal{LN}(0, \omega_{\epsilon_i}^2)$ and $\mathcal{LN}(\mu_{\eta_i}, \omega_{\eta_i}^2)$, respectively. Thus, the parameter vector is $\theta \equiv (\mu_C, \omega_C^2, \omega_{\epsilon_1}^2, \ldots, \omega_{\epsilon_I}^2, \mu_{\eta_1}, \ldots, \mu_{\eta_I}, \omega_{\eta_1}^2, \ldots, \omega_{\eta_I}^2)$. Using (8.15), they show that the equilibrium strategies are log-linear in log-signals

$$\log s_{ik}(\sigma_i) = \alpha_{ik} + \beta_{ik} \log \sigma_i + \sum_{j=1}^{k} \gamma_{ikj} \log s_{i_j,j-1}^{-1}(p_j), \tag{8.16}$$

for $i = 1, \ldots, I$ and $k = 0, \ldots, I-2$, where the coefficients $(\alpha_{ik}, \beta_{ik}, \gamma_{ik1}, \ldots, \gamma_{ikk})$ depend on the structural parameters θ in a complex way. The analyst observes the drop-out prices $p_1 < \ldots < p_{I-1}$ as well as the identities of the bidders dropping out at these prices. Estimation can be performed by ML exploiting the simplicity and recursivity of (8.16) which gives $p_k = s_{i_k,k-1}(\sigma_{i_k})$ for $k = 1, \ldots, I-1$. In addition to the usual censoring arising from the non-observability of the winner's bid, the main difficulty is that the likelihood must condition on the order and identities of the bidders who dropped out because of the asymmetry. This introduces a truncation on the signals $(\sigma_1, \ldots, \sigma_I)$ and hence an I-variate integration over this region which depends on θ through bidders' equilibrium strategies.[88] Though a simulated ML method can theoretically approximate the truncated I-variate integral as the number of simulations S grows to infinity, in pratice it is preferable to employ an estimation method that is consistent for a fixed number of simulations especially when $I \geq 4$. The authors then propose a SNLLS estimation method and a smooth version of it based on the first moments of the drop-out prices in the spirit of Laffont, Ossard and Vuong (1995). See their paper for details.

Second-Price Sealed-Bid Auctions with Entry

A striking feature of eBay auctions is that more than 50% of the final bids are submitted in the last 10% of the auction duration. Because of such a phenomenon called sniping, Bajari and Hortacsu (2003) argue that the late bidding can be approximated as a second-price sealed-bid auction with a random number of entering bidders. They consider a PCV model to which they add an entry model in the spirit of Levin and Smith (1994) with the difference that the number of entering bidders I^* is unknown at the time of bidding. Specifically, the I potential bidders have the same unknown utility C distributed as $F_C(\cdot)$ on $[\underline{c}, \overline{c}]$ whereas their private signals $(\sigma_1, \ldots, \sigma_I)$ are independently distributed as $F_{\sigma|C}(\cdot|C)$ on $[\underline{\sigma}, \overline{\sigma}]$ given C. The corresponding densities are $f_C(\cdot)$ and $f_{\sigma|C}(\cdot|C)$. Bidders incur an entry cost κ and learn their signals after entry. Thus, entry decisions are independent of signals.

In equilibrium, each bidder enters with the same probability q^*. By symmetry, considering bidder 1 (say), q^* is determined by the zero expected profit condition

$$\kappa = \mathrm{E}\Big[\mathrm{E}\Big\{[C - s(\sigma_{-1}^{\max}; I)]\, I\!I(\sigma_{-1}^{\max} \leq \sigma_1)|\sigma_1, J \geq 1\Big\} \Pr(J \geq 1) + \mathrm{E}(C|\sigma_1, J=0) \Pr(J=0)\Big],$$

where $\sigma_{-1}^{\max} \equiv \max_{j=2,\ldots,J+1} \sigma_j$, J is the number of i's entering rivals distributed as Binomial $\mathcal{B}(I-1, q^*)$ and $s(\cdot; I)$ is the bidders' symmetric increasing equilibrium strategy. Let $q_j \equiv \Pr(J=j)$ and $f_{\sigma_{-}^{\max}|\sigma}(\cdot|\sigma_1, J) = \int_{\underline{c}}^{\overline{c}} J f_{\sigma|C}(\cdot|c) F_{\sigma|C}^{J-1}(\cdot|c) f_{C|\sigma}(c|\sigma_1) dc$ be the density of $\sigma_{-1}^{\max}$ given (σ_1, J) with $f_{C|\sigma}(c|\sigma_1) = f_{\sigma|C}(\sigma_1|c) f_C(c) / \int_{\underline{c}}^{\overline{c}} f_{\sigma|C}(\sigma_1|c) f_C(c) dc$. The authors

[88]Despite this dependence, the authors show that the support of the drop-out prices is $\{0 \leq p_1 \leq \ldots \leq p_{I-1} < \infty\}$. Thus, in contrast to Donald and Paarsch (1996), the standard properties of ML apply.

show that

$$s(\sigma_1; I) = \frac{\sum_{j=1}^{I-1} q_j f_{\sigma_{-}^{\max}|\sigma}(\sigma_1|\sigma_1, j)\nu(\sigma_1; j+1)}{\sum_{j=1}^{I-1} q_j f_{\sigma_{-}^{\max}|\sigma}(\sigma_1|\sigma_1, j)}, \tag{8.17}$$

for $\sigma_1 \in [\underline{\sigma}, \overline{\sigma}]$, where $\nu(\sigma_1; j+1) \equiv \mathrm{E}[C|\sigma_1, \sigma_{-1}^{\max} = \sigma_1, I^* = j+1]$ is bidder 1's pivotal value when there are $(j+1)$ entering bidders. That is, $s(\sigma_1; I)$ is a weighted average of what bidder 1 would bid, i.e., $\nu(\sigma_1; j+1)$, in a second-price sealed-bid auction with $j+1$ bidders. See Milgrom and Weber (1982) and Section 2.2 for the latter.

In their empirical study, Bajari and Hortacsu (2003) set $\sigma_i = C + \epsilon_i$ with C and ϵ_i, $i = 1, \ldots, I$, independently normally distributed as $\mathcal{N}(\mu_C, \omega_C^2)$ and $\mathcal{N}(0, \omega_\epsilon^2)$, respectively. Thus, $f_{\sigma|C}(\cdot|c) = (1/\omega_\epsilon)\phi[(\cdot - c)/\omega_\epsilon]$. Moreover, because the number of potential bidders I is large whereas the probability of entering is small, they approximate q_j using a Poisson distribution $\mathcal{P}(\lambda)$ thereby avoiding the need to know I. They also allow for auction covariates X by specifying μ_C and $\log \lambda$ as parametric functions of X. The likelihood of observing $(B_1, \ldots, B_{I^*}, I^*)$ given X is

$$\tilde{q}_{I^*} \times \int_{-\infty}^{+\infty} \prod_{i=1}^{I^*} f_{B|C}(B_i|c) f_C(c) dc,$$

where $\tilde{q}_{I^*} \equiv e^{-\lambda}\lambda^{I^*}/(I^*!)$ and $f_{B|C}(\cdot|c) = f_{\sigma|C}[s^{-1}(\cdot; +\infty)|c] s^{-1\prime}(\cdot; +\infty)$ is the density of an arbitrary bid induced by (8.17) given $C = c$ as I is very large.[89] The authors estimate the model by Bayesian methods using a prior on the parameters. They also exploit the location-scale property of the equilibrium strategy to compute $s(\cdot; +\infty)$ only once for a specific value of $(\mu_C, \omega_C^2, \omega_\epsilon^2)$.

Section 8.4: Share Auctions

In contrast to Section 6.1 which considers the IPV paradigm, Février, Préget and Visser (2004) estimate structurally Wilson's (1979) original share auction model, which is developed within the PCV paradigm. The value C for one unit of the auctioned good is unknown and distributed as $F_C(\cdot)$ with density $f_C(\cdot)$ and support $[\underline{c}, \overline{c}]$ whereas bidders' signals $(\sigma_1, \ldots, \sigma_I)$ are independently distributed as $F_{\sigma|C}(\cdot|c)$ with density $f_{\sigma|C}(\cdot|c)$ and support $[\underline{\sigma}, \overline{\sigma}]$ given $C = c$. Restricting Bayesian Nash equilibria to be symmetric, every bidder submits a decreasing demand $y_i(\cdot; \sigma_i) = y(\cdot; \sigma_i)$ as a function of price. The market clearing price P_c equates the aggregate demand to the seller's supply of Q units of the

[89] This likelihood is substantially simpler than that given by Bajari and Hortacsu (2003) as they allow for a minimum and/or a secret reserve price. As a consquence of the latter, an entering bidder may not bid if his signal is lower than a threshold. To simplify, we abstract from these features.

good, which is assumed known to bidders. Thus $\sum_{i=1}^{I} y(P_c;\sigma_i) = Q$. Because rivals' signals are unknown to bidder i, the clearing price P_c is random and distributed as

$$\Pr[P_c \le p|\sigma_i] = \Pr\Big[\sum_{j\neq i} y(p;\sigma_j) \le Q - y(p;\sigma_i)|\sigma_i\Big] \equiv H[p, y(p;\sigma_i)|\sigma_i],$$

which is similar to (6.1) but with the additional conditioning on σ_i because $(\sigma_1,\ldots,\sigma_i)$ are affiliated.[90]

Though Wilson (1979) studies uniform pricing, the authors analyze discriminatory pricing given their empirical application to French Treasury Bill auctions. Let

$$\Pr[P_c \le p|C,\sigma_i] = \Pr\Big[\sum_{j\neq i} y(p;\sigma_j) \le Q - y(p;\sigma_i)|C,\sigma_i\Big] \equiv H[p, y(p;\sigma_i)|C],$$

where $H(p,y|C) = \Pr[P_c \le p|C, y(p,\sigma_i) = y]$ using the conditional independence of $(\sigma_1,\ldots,\sigma_I)$ given C. Thus, given σ_i bidder i's expected profit is

$$\mathrm{E}\left\{\int_{\underline{p}_c}^{\overline{p}_c} \Big[(C-p)y(p;\sigma_i) - \int_p^{+\infty} y(\tilde{p};\sigma_i)d\tilde{p}\Big]dH[p,y(p;\sigma_i)|C] \mid \sigma_i\right\},$$

where $[\underline{p}_c, \overline{p}_c]$ is the support of P_c and $dH[p,y(p;\sigma_i)|C]$ is the total derivative of $H[p,y(p;\sigma_i)|C]$ with respect to p. Maximizing with respect to $y(\cdot;\sigma_i)$, they show that the FOC is

$$0 = \mathrm{E}\Big\{(C-p)H_p[p,y(p;\sigma_i)|C] - H[p,y(p;\sigma_i)|C] \mid \sigma_i\Big\} \tag{8.18}$$

for all $(p,\sigma_i) \in [\underline{p}_c, \overline{p}_c] \times [\underline{\sigma}, \overline{\sigma}]$, where $H_p[p,y|C]$ is the (positive) partial derivative of $H[p,y|C]$ with respect to p. In particular, (8.18) is for the PCV paradigm the analog of (6.3) for the IPV paradigm in share auctions. The main difference lies in the expectation because C is unknown to bidders.

For identification and estimation, the authors integrate (8.18) with respect to σ_i. This eliminates the distribution of market clearing price $H(p,y|C)$ given C and bidder's demand at price p thereby greatly simplifying estimation. Specifically, with conditioning on I and auction covariates X, which include Q, they show that (8.18) gives

$$0 = \mathrm{E}\Big\{(I-1)\Big[\mathrm{E}(C|\sigma_1,\ldots,\sigma_I) - p\Big]\mathbb{1}(P_c \le p) \mid X, I\Big\} - \mathrm{E}\Big[(p-P_c)\mathbb{1}(P_c \le p) \mid X, I\Big], \tag{8.19}$$

for all $p \in [\underline{p}_c(X,I), \overline{p}_c(X,I)]$. They parameterize $[F_C(\cdot), F_{\sigma|C}(\cdot|\cdot)]$ by θ and let I be

[90] Following the literature, we suppress the dependence of $y(\cdot,\sigma_i)$ on Q as well as any conditioning on Q. As a matter of fact, this is justified if $(\sigma_1,\ldots,\sigma_I,C)$ are independent of Q. Indeed, because C and p are value and price per unit, we can normalize by Q and view $y(p,\sigma_i)$ as bidder i's demand for a share.

independent of $(\sigma_1, \ldots, \sigma_I, C)$ given X.[91] In principle, because the model is fully parametric, (8.19) provides a conditional moment restriction on the observables (P_c, X, I) for each $p \in [\underline{p}_c(X,I), \overline{p}_c(X,I)]$. A difficulty, however, is to determine the first expectation in (8.19) since $(\sigma_1, \ldots, \sigma_I)$ and P_c are dependent through the equality of aggregate demand and supply $\sum_{i=1}^I y_I(P_c; X, \sigma_i) = Q$. To avoid such a computation, the authors exploit the observability of bidder i's demand $Y_i(p) \equiv y_I(p; X, \sigma_i)$ at price p through $F_{\sigma|X}(\cdot|X) = \Pr(\sigma_i \leq \cdot|X,I) = \Pr[Y_i(p) \leq y_I(p; X, \cdot)|X, I] \equiv G_{Y_i(p)|X}[y_I(p; X, \cdot)|X, I]$ when $y_I(p; X, \sigma_i)$ is increasing in σ_i. This gives

$$\sigma_i = F_{\sigma|X}^{-1}\Big\{G_{Y_i(p)|X}[Y_i(p)|X, I] \mid X; \theta\Big\} \tag{8.20}$$

for $i = 1, \ldots, I$, where $F_{\sigma|X}(\cdot|X;\theta) = \int_{\underline{c}}^{\overline{c}} F_{\sigma|C}(\cdot|c;\theta) dF_{C|X}(c|X;\theta)$. Combining (8.19)-(8.20), the authors propose a semiparametric procedure where the first step estimates nonparametrically $G_{Y_i(p)|X}(\cdot|X, I)$ by

$$\hat{G}_{Y_i(p)|X}(\cdot|x, I) = \frac{\sum_{\ell=1}^{L_I} \frac{1}{I} \sum_{j=1}^{I} 1\!\!I[Y_{i\ell}(p) \leq \cdot] K[(x - X_\ell)/h]}{\sum_{\ell=1}^{L_I} K[(x - X_\ell)/h]}$$

for every (p, x) given symmetry, where L_I is the number of auctions with I bidders, $Y_{j\ell}(p) \equiv y_{I_\ell}(p; \sigma_{j\ell}, X_\ell)$ is bidder j's demand at price p in the ℓ-th auction, $K(\cdot)$ is a kernel and h is a bandwidth. By (8.20) bidders' signals are estimated by $\hat{\sigma}_{i\ell}(\theta) \equiv F_{\sigma|X}^{-1}\{\hat{G}_{Y_i(p)|X}[Y_{i\ell}(p)|X_\ell, I_\ell]|X_\ell; \theta\}$ for some p given θ. In the second step, the authors apply GMM on some unconditional moment resulting from (8.19). Let $m(\sigma_1, \ldots, \sigma_I, X; \theta) \equiv \mathrm{E}[C|\sigma_1, \ldots, \sigma_I, X; \theta]$ under θ and Z_ℓ be a vector of instruments, which are functions of (X_ℓ, I_ℓ). They minimize a quadratic form of the empirical analog

$$\frac{1}{L}\sum_{\ell=1}^{L} Z_\ell \Big\{(I_\ell - 1) m[\hat{\sigma}_{1\ell}(\theta), \ldots, \hat{\sigma}_{I_\ell \ell}(\theta), X_\ell; \theta] - I_\ell p_k + P_{c\ell}\Big\} 1\!\!I(P_{c\ell} \leq p_k),$$

where p_k belongs to a finite grid $\{p_1, \ldots, p_K\}$ of clearing prices. The authors show that the resulting estimator $\hat{\theta}$ is consistent and $\sqrt{L}$-asymptotically normal using Newey and McFadden's (1994) results on semiparametric estimation.

In a subsequent paper, Armantier and Sbaï (2006) argue that bidders do not know the supplied quantity because of the presence of non-competitive bidding. Their analysis

[91] In particular, they consider two examples. The first extends Wilson's (1979) gamma-exponential example, whereas the second is Kyle's (1989) normal-normal example. Remarkably, they show that both parametric models are globally identified from observations on (X, I) and the knowledge of $G_{Y_i(p)|X}(\cdot|X, I)$ using only the conditional moment restriction (8.19) with signals evaluated at (8.20). It should be noted, however, that signals are negatively correlated with C and hence not affiliated in their first example.

relies on Wang and Zender's (2002) theoretical model with a random quantity for sale and allows for asymmetry in both ex ante information and risk aversion. With two groups of bidders and auction covariates X, the model is parameterized with primitives $[\gamma_0, \gamma_1, F_{\sigma_0|C}(\cdot|\cdot;\tilde{\beta}), F_{\sigma_1|C}(\cdot|\cdot;\tilde{\beta}), F_C(\cdot;\tilde{\beta}), G_Q(\cdot;\tilde{\beta})]$ where γ_j and $F_{\sigma_j|C}(\cdot|\cdot;\tilde{\beta})$ are the CARA coefficient and signal distribution for group $j = 0, 1$, $F_C(\cdot;\tilde{\beta})$ and $G_Q(\cdot;\tilde{\beta})$ are the common value and supplied quantity distributions, and $\tilde{\beta} \equiv \tilde{\beta}(X;\beta)$ is a vector of known functions of X parameterized by β. Thus the parameters are $\theta \equiv (\gamma_0, \gamma_1, \beta)$. Bidders' signals $(\sigma_{01}, \ldots, \sigma_{0I_0}, \sigma_{11}, \ldots, \sigma_{1I_1})$ are mutually independent given (C, X, I_0, I_1), where I_j is the number of bidders in group j, $j = 0, 1$. Moreover, (I_0, I_1, Q) are exogenous, i.e., they are independent of $(\sigma_{01}, \ldots, \sigma_{0I_0}, \sigma_{11}, \ldots, \sigma_{1I_1}, C)$ given X.

Because X enters only through $\tilde{\beta}(X;\beta)$, bidders' equilibrium strategies are of the form $y_j(\cdot;\sigma_{ji},\tilde{\theta})$ for $i = 1, \ldots, I_j$ and $j = 0, 1$ with $\tilde{\theta} \equiv (\gamma_0, \gamma_1, \tilde{\beta})$. These strategies do not depend on Q which is unknown, but depend on (I_0, I_1) which are omitted. Under discriminatory pricing, given his signal σ_{ji}, bidder i's expected profit in group j is

$$\pi_j(\sigma_{ji};\tilde{\theta}) \equiv \mathrm{E}\left\{-\exp\left(-\gamma_j\Big[(C - P_c)y_j(P_c;\sigma_{ji},\tilde{\theta}) - \int_{P_c}^{+\infty} y_j(p;\sigma_{ji},\tilde{\theta})dp\Big]\right) \mid \sigma_{ji};\tilde{\beta}\right\},$$

where the clearing price $P_c \equiv p_c(\sigma_{01}, \ldots, \sigma_{0I_0}, \sigma_{11}, \ldots, \sigma_{1I_1}, Q;\tilde{\theta})$ satisfies $\sum_{i=1}^{I_0} y_0(P_c;\sigma_{0i},\tilde{\theta}) + \sum_{i=1}^{I_1} y_1(P_c;\sigma_{1i},\tilde{\theta}) = Q$ and expectation is taken with respect to the $\tilde{\beta}$-distribution of $(\sigma_{01}, \ldots, \sigma_{0I_0}, \sigma_{11}, \ldots, \sigma_{1I_1}, C, Q)$ given σ_{ji}. For every $\tilde{\theta}$-value, the authors approximate the equilibrium strategies $[y_0(\cdot;\cdot,\tilde{\theta}), y_1(\cdot;\cdot,\tilde{\theta})]$ given (I_0, I_1) using the Constrained Strategic Equilibrium (CSE) method developed by Armantier, Florens and Richard (2005). This gives $[\tilde{y}_0(\cdot;\cdot,\delta_0(\tilde{\theta})), \tilde{y}_1(\cdot;\cdot,\delta_1(\tilde{\theta}))]$, where $\tilde{y}_j(\cdot;\cdot,\delta_j)$ is a flexible specification parameterized by δ_j such as a polynomial in (p, σ_j) for $j = 0, 1$.[92] Letting $Y_{ji}(\cdot) \equiv y_j(\cdot;\sigma_{ji},\tilde{\theta})$ be the observed demand from bidder i's demand in group j, estimation is based on the conditional moments $\mathrm{E}[Y_{ji}(p_k) - \tilde{y}_j(p_k;\sigma_{ji},\delta_j(\tilde{\theta}))|X, I_0, I_1] \approx 0$ for $i = 1, \ldots, I_j$, $j = 0, 1$ and p_k belonging to a finite grid $\{p_1, \ldots, p_K\}$ of clearing prices. Because $\mathrm{E}[\tilde{y}_j(p_k;\sigma_{ji},\delta(\tilde{\theta}))|X, I_0, I_1]$ does not have an explicit form, SMM is used following Pakes and Pollard (1989) and Gourieroux and Monfort (1997). As for many direct methods, the estimation procedure is computationally demanding as it requires the determination/approximation of the equilibrium strategies $[y_0(\cdot;\cdot,\tilde{\theta}), y_1(\cdot;\cdot,\tilde{\theta})]$ for every value of $\tilde{\theta} = [\gamma_0, \gamma_1, \tilde{\beta}(X,\beta)]$ and hence for every value of X as well as for any trial value of $\theta = (\gamma_0, \gamma_1, \beta)$ in the minimization process.

Section 9: Concluding Remarks

[92] Given $\tilde{\theta}$, the parameters $[\delta_0(\tilde{\theta}), \delta_1(\tilde{\theta})]$ are obtained by solving a Nash equilibrium problem where strategies are constrained to be of the form $\tilde{y}_j(\cdot;\cdot,\delta_j)$. See Armantier and Sbaï (2006) for details.

This chapter offers a comprehensive overview of econometric methods for the structural analysis of auction data. We focus on methodological contributions to the identification and estimation of a wide range of auction models. We leave aside the problem of testing which constitutes an important topic by itself. Regarding empirical and policy motivations, we invite the reader to consult the companion chapter by Hortacsu and Perrigne (2021) in the recent Handbook of Industrial Organization. These authors provide an exhaustive survey of the empirical studies of auction data. The econometrics of auctions covers a large span of parametric to nonparametric methods providing a rich background for teaching microeconometrics and applied econometrics. In this respect, we avoid here technicalities as much as we can and clarify the surveyed papers' contributions to produce a self-contained chapter that hopefully colleagues will use in their graduate courses. We also hope that this chapter will be valuable to anyone who is interested in undertaking research in this exiciting field. He/she will find the necessary econometric foundations for developing his/her own research agenda in structural modeling.

The past twenty-five years show that the field is moving at a fast pace. We believe that there will be more developments in the coming years for the following reasons. First, as new data become available, new models and econometric questions will arise calling for the development of suitable econometric methods. Indeed, auction data consitute an endless source of inspiration for theoretical and empirical research. Second, Sections 5 and 6 in this chapter review some recent contributions whose common feature is the multidimensional aspect of bidders' private information. This trend is likely to expand in the coming years to analyze further multi-object and sequential auctions to name a few. Moreover, most of the literature considers only one side of the market, i.e., the bidders, taking the other side as given. We foresee more contributions involving both sides such as in double-sided auctions. Lastly, the influence of the econometrics of auctions goes beyond the field. Auctions can be viewed as a price formation with an allocation mechanism under limited competition. Some recent papers exploit this aspect to analyze of broad range of topics involving strategic interactions. Moreover, the econometrics of auctions with the indirect approach which relies on the model first-order conditions contributes to the development of the structural modeling in a variety of subjects such as nonlinear pricing, bargaining, contract and regulation under incomplete information.

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