Formal Verification of Central Limit Theorem in HOL Theorem Prover [TITLE LINE 3]

Thi Cam Tu Phan

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School of Computing

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Abstract

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Draft ideas

0.0.1 Introduction

0.0.1.1 Motivation

- Enriching the HOL4 Theory Library
- Supporting Future Applications
- Providing Analytical Tools

0.0.1.2 Contributions

- This thesis proves the Lyapunov version of the Central Limit Theorem, a fundamental result in probability theory. The CLT of Lyapunov extends the classical CLT by relaxing the identical distribution assumption to the case of independent but not necessarily identically distributed random variables.
- The proof makes use of Taylor expansions, moment-based bounds, and asymptotic error analysis to rigorously establish convergence to the normal distribution.
- The main insights involve the Lyapunov condition, which guarantees that the third absolute moments become small compared to the variance, and a new use of Big-O notation for the effective control of error terms.

0.0.1.3 Results

- **Lyapunov CLT**: For independent random variables $X_1, X_2, \ldots, X_n$ with finite means, variances, and third moments, if the Lyapunov condition is satisfied:

  $$\frac{\Gamma_n}{s_n^3} \to 0 \quad \text{as } n \to \infty,$$

  then the normalized sum $\frac{S_n - \mu}{s_n}$ converges in distribution to the standard normal distribution $N(0, 1)$.
- **Proof**: It establishes that
  - Error bounds of the test function f by Taylor expansion and moment inequalities.
  - Asymptotic vanishing of higher-order terms under the Lyapunov condition.

– A rigorous comparison between the distribution of $\frac{S_n}{s_n}$ and $N(0,1)$, bounded by $O\left(\frac{\Gamma_n}{s_n^3}\right)$.

These results illustrate the robustness of the Lyapunov CLT against heterogeneous distributions and, in fact, form a bridge between classical formulations and modern applications.

0.0.2 Background and Related Works

0.0.3 Background

Different versions of CLT Fischer (2011)

- **Classical Central Limit Theorem** Ross (2019)
  - **Proven by:** De Moivre, Laplace
  - **Statement:** If $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables with finite mean μ and variance σ^2, then:

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1).$$

  - **Significance:** Foundation of probability theory and statistics, used in inferential statistics, e.g., hypothesis testing and confidence intervals.

Chapter 1

Introduction

Formal verification has become an essential approach in modern mathematics and computer science; it provides a strong framework for proving mathematical proofs and algorithms. Traditional techniques of formal verification include the following: Model Checking Clarke et al. (2018), Testing Broy et al. (2005), and Theorem Proving Bertot and Castéran (2013).

Model Checking (MC) is a well-valued static formal verification method since, when applied correctly, it provides complete assurance that something satisfies certain specifications. Model checking is a completely automated procedure and is considered easier compared to other techniques such as theorem proving. However, because of the state explosion problem Burch et al. (1992), application of model checking in large systems is quite a challenge due to the immense state space size of the systems analyzed. Also, the success of model checking strongly depends on the quality of the analyzed models. In contrast, testing, despite being highly valued for revealing defects in systems, is inherently limited, since it can only indicate the presence of errors, but lacks the ability to verify the correctness of a system.

Theorem proving is another widely used formal analysis technique. It does not suffer from the limitations of state space size as model checking does, which allows the analysis of larger and more complex systems. Furthermore, theorem provers use very expressive logics, like first-order or higher-order logics, which enable the study of a wider range of systems without the restrictions that often come with modeling. The most prominent provers are HOL4 Slind and Norrish (2008), HOL Light Harrison (2015), Coq Bertot and Castéran (2013), and Isabelle/HOL Team (2015), all broadly used in the community.

Formalization plays a vital role in probability theory to ensure that basic results are rigorous mathematically and computable in automated reasoning systems. This project formalizes the Central Limit Theorem using the HOL4 proof assistant, hence enriching the domain of probability in formalized mathematics.

1.1 Motivation

The Central Limit Theorem is one of the most fundamental results in the Probability Theory. This theory finds broad applications in statistics, data science, finance, and engineering. The theorem underlines that under certain conditions, the sum of many independent random variables will, as the number goes large, converge in distribution to a normal distribution irrespective of the individual distributions of those variables Chung (2000). This property justifies many real-world applications and models.

Although HOL4 already contains formal proofs of some fundamental probabilistic results, such as the Law of Large Numbers, a formalized proof of the Central Limit Theorem is still absent. Completing a proof would realize several important milestones, including the following:

- **Enriching the HOL4 Theory Library**:
  - The gap in the library would be filled by adding some advanced results from probability theory.
  - A more general foundation would be laid for future processes of formal verification.
- **Supporting Future Applications**:
  - Strengthening the framework for developers and researchers to model or verify real-world problems involving probabilistic reasoning.
- **Providing Analytical Tools**:
  - Enable efficient analysis of problems dependent on the normal distribution and its characteristics.

Formalizing the Central Limit Theorem also aligns with ongoing efforts to bridge traditional mathematical insights with computational tools. As a consequence, it paves the way for applications in areas such as artificial intelligence, machine learning, and quantitative modeling, where probabilistic reasoning is increasingly critical.

This project will contribute to a stronger, more versatile foundation for probabilistic formalization, empowering researchers and developers with better tools to tackle complex, real-world challenges.

Chapter 2

Background and Related Work

The history of the Central Limit Theorem, hereinafter referred to as CLT, is quite a fascinating journey through the gradual development of probability theory. Over the years, various versions of CLT have been formulated, reflecting its adaptability and foundational importance. Each such variation addresses certain mathematical challenges and applications, therefore enriching the broader understanding of convergence in probability. In this chapter, we explore some theorem-proving tools and their methodologies to formalize the CLT, emphasizing ...

2.1 Background

Let us brief the few evolvement of CLT, mainly noticing its critical features and variants developed further. The first steps towards CLT took place during the 18th century when Abraham de Moivre showed that, under summing many independent, identically distributed random variables-large number of rolled dice or coin flips-results in the normal distribution-a distribution with the shape of a bell De Moivre (1733). De Moivre laid down the very background for some approximations in the Binomial Distribution.

In 1810, Pierre-Simon Laplace expanded de Moivre's insights by proving that the sum of independent random variables converges to a normal distribution under broader conditions Laplace (1835). His work established the CLT as a universal principle for analyzing aggregate phenomena, such as population averages and measurement errors.

In the 19th century, mathematicians such as Pafnuty Chebyshev formalized the conditions of the theorem, including those of variance and expectation, so that the theorem became more precise and mathematically sound Chebyshev (1890). At the beginning of the 20th century, Lyapunov generalized CLT by introducing specific criteria-Lyapunov's condition-which explained when the theorem was applicable Lyapunov (1895). William Feller Feller (1945) later refined it to address discrete random variables.

Developments in the modern era since World War II have broadened the scope of the CLT beyond normal distributions to stable distributions and applications involving stochastic processes and high-dimensional data. These extensions illustrate how the theorem can be adapted to a wide range of probabilistic and statistical contexts.

Today, the CLT comes in a variety of variants each suited to particular situations:

- **Classical CLT:** For sums of independent, identically distributed random variables possessing finite variance and mean.
- **Generalized CLT:** When variables are weakly dependent or not identically distributed.
- **Triangular Arrays:** For sums of random variables arranged in arrays where the conditions vary across rows.
- **Local and Integral Versions:** While the local CLT concerns pointwise probabilities, the integral version deals with cumulative distributions.

From about 1810 to 1935, most of the efforts were devoted to proving the CLT for sums of independent random variables, with more recent generalizations involving only weakly dependent variables. Modern formulations neatly distinguish between normed sums, triangular arrays, and local versus integral theorems.

Chapter 3

Preliminaries

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Chapter 4

Central Limit Theorem

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Chapter 5

Future Work

Future Work

Chapter 6

Conclusion

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References

Yves Bertot and Pierre Castéran. *Interactive theorem proving and program development: Coq'Art: the calculus of inductive constructions*. Springer Science & Business Media, 2013.

Manfred Broy, Bengt Jonsson, J-P Katoen, Martin Leucker, and Alexander Pretschner. Model-based testing of reactive systems. In *Volume 3472 of Springer LNCS*. Springer, 2005.

Jerry R Burch, Edmund M Clarke, Kenneth L McMillan, David L Dill, and Lain-Jinn Hwang. Symbolic model checking: 1020 states and beyond. *Information and computation*, 98 (2):142–170, 1992.

Pafnutii Lvovich Chebyshev. Sur deux théorèmes relatifs aux probabilités. *Acta math*, 14 (1):305–315, 1890.

Kai Lai Chung. *A course in probability theory*. Elsevier, 2000.

Edmund M Clarke, Thomas A Henzinger, Helmut Veith, Roderick Bloem, et al. *Handbook of model checking*, volume 10. Springer, 2018.

Abraham De Moivre. *Approximatio ad summam terminorum binomii (a+ b) n in seriem expansi*. 1733.

Willy Feller. The fundamental limit theorems in probability. *Bulletin of the American Mathematical Society*, 51:800–832, 1945. Reprinted in Adams 2009, pp. 80–113.

Hans Fischer. *A history of the central limit theorem: from classical to modern probability theory*, volume 4. Springer, 2011.

John Harrison. Hol light, 2015. URL `http://www.cl.cam.ac.uk/~jrh13/hol-light/`. Accessed: 2024-12-01.

Pierre Simon Laplace. *Oeuvres complètes de Laplace*. Gautier-Villars, 1835.

Aleksandr Mikhailovich Lyapunov. Pafnutii lvovich chebyshev, 1895.

Sheldon Ross. *First Course in Probability, A*. Pearson Higher Ed, 2019.

Konrad Slind and Michael Norrish. A brief overview of hol4. In *International Conference on Theorem Proving in Higher Order Logics*, pages 28–32. Springer, 2008.

Isabelle Team. Isabelle/hol, 2015. URL `https://isabelle.in.tum.de/`. Accessed: 2024-12-01.

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for scripts