Formal Verification of Central Limit Theorem in HOL Theorem Prover [TITLE LINE 3]

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Abstract

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Draft ideas

0.0.1 Introduction

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0.0.1.1 Motivation

- Enriching the HOL4 Theory Library
- Supporting Future Applications
- Providing Analytical Tools

0.0.1.2 Contributions

- This thesis proves the Lyapunov version of the Central Limit Theorem, a fundamental result in probability theory. The CLT of Lyapunov extends the classical CLT by relaxing the identical distribution assumption to the case of independent but not necessarily identically distributed random variables.
- The proof makes use of Taylor expansions, moment-based bounds, and asymptotic error analysis to rigorously establish convergence to the normal distribution.
- The main insights involve the Lyapunov condition, which guarantees that the third absolute moments become small compared to the variance, and a new use of Big-O notation for the effective control of error terms.

0.0.1.3 Results

- **Lyapunov CLT**: For independent random variables $X_1, X_2, \ldots, X_n$ with finite means, variances, and third moments, if the Lyapunov condition is satisfied:

  $$\frac{\Gamma_n}{s_n^3} \to 0 \quad \text{as } n \to \infty,$$

  then the normalized sum $\frac{S_n - \mu}{s_n}$ converges in distribution to the standard normal distribution $N(0, 1)$.

- **Proof**: It establishes that

– Error bounds of the test function f by Taylor expansion and moment inequalities.
– Asymptotic vanishing of higher-order terms under the Lyapunov condition.
– A rigorous comparison between the distribution of $\frac{S_n}{s_n}$ and $N(0,1)$, bounded by $O\left(\frac{\Gamma_n}{s_n^3}\right)$.

These results illustrate the robustness of the Lyapunov CLT against heterogeneous distributions and, in fact, form a bridge between classical formulations and modern applications.

0.0.2 Background and Related Works

0.0.2.1 Background

Different versions of CLT Fischer (2011)

- **Classical Central Limit Theorem** Ross (2019)
  – **Proven by:** De Moivre, Laplace
  – **Statement:** If $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables with finite mean μ and variance σ^2, then:

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1).$$

  – **Significance:** Foundation of probability theory and statistics, used in inferential statistics, e.g., hypothesis testing and confidence intervals.
- **Lyapunov's Central Limit Theorem** Chung (2000)
- **Lindeberg's Central Limit Theorem**
- **The Generalized Central Limit Theorem**
  – https://en.wikipedia.org/wiki/Central_limit_theorem

0.0.2.2 Related Work

The CLT is a fundamental result in probability theory, and its formalization in theorem provers has been a highly active area of interest in the formal methods community. Several different ways of proving the CLT have been tried, and tools like Isabelle/HOL and HOL4 provide different approaches reflecting their different frameworks and mathematical libraries.

Isabelle/HOL's Formalization of the CLT In Isabelle/HOL, the CLT has been formalized in a powerful and especially elegant way using the **characteristic function approach**, which relies on the fact that characteristic functions uniquely determine probability distributions and their pointwise convergence implies convergence in distribution.

- **Methodology**
- **Characteristic Function Approach**
  - It provides a direct and compact route to proving the CLT, leveraging the algebraic simplicity of characteristic functions.
  - ...

0.0.3 Premilaires

0.0.3.1 HOL4

?

0.0.3.2 Measure Theory

?

0.0.3.3 Lebesgue Integration Theory

?

0.0.3.4 Probability Theory

?

0.0.4 Central Limit Theorem

Informal proof of Lyapounov's theorem for a single sequence by the method of Lindeberg Chung (2000).

- **Notation and Statement**

  1. $\{X_j\}_{j=1}^n$ be a sequence of independent random variables with:
     $\mathbb{E}[X_j] = 0$,
     $\mathrm{Var}(X_j) = \sigma_j^2$ (finite variances, $\sigma_j^2 < \infty$),
     $\mathbb{E}[|X_j|^3] = \gamma_j$ (finite third moments, $\gamma_j < \infty$).
  2. Define:
     $S_n = \sum_{j=1}^n X_j$,
     $s_n^2 = \sum_{j=1}^n \sigma_j^2$ (total variance),
     $\Gamma_n = \sum_{j=1}^n \gamma_j$ (total third moment).
  3. Statement: If

     $$\frac{\Gamma_n}{s_n^3} \to 0 \quad \text{as } n \to \infty$$

     then

     $$\frac{S_n}{s_n} \xrightarrow{d} \Phi$$

     where Φ denotes the standard normal distribution $\mathcal{N}(0, 1)$.

- **Step-by-Step Proof**

  1. **Normalize the Random Variables**
     Define the normalized variables:

     $$X_{n,j} = \frac{X_j}{s_n}, \quad \text{for } j = 1, \ldots, n.$$

     The sum S_n is now expressed as:

     $$\frac{S_n}{s_n} = \sum_{j=1}^n X_{n,j}.$$

  2. **Use the Lindeberg Replacement to approximate**

Replace each X_j with a corresponding normal random variable $Y_j \sim \mathcal{N}(0, \sigma_j^2)$, where $\{Y_j\}_{j=1}^n$ are independent and have the same mean and variance as X_j, then satisfy:

$$\mathbb{E}[Y_j] = 0 \quad \text{and} \quad \mathrm{Var}(Y_j) = \sigma_j^2.$$

Let all the X's and Y's be totally independent.

Since $Y_j \sim \mathcal{N}(0, \sigma_j^2)$, the normalized random variable:

$$\frac{Y_j}{s_n} \sim \mathcal{N}\left(0, \frac{\sigma_j^2}{s_n^2}\right)$$

Adding these normalized variables gives:

$$\frac{1}{s_n}\sum_{j=1}^n Y_j \sim \mathcal{N}\left(0, \sum_{j=1}^n \frac{\sigma_j^2}{s_n^2}\right).$$

Since $s_n^2 = \sum_{j=1}^n \sigma_j^2$, this simplifies to:

$$\frac{1}{s_n}\sum_{j=1}^n Y_j \sim \mathcal{N}(0, 1).$$

Now, construct the sequence Z:

$$Z_j = Y_1 + \cdots + Y_{j-1} + X_j + \cdots + X_n, \quad 1 \le j \le n$$

Thus:

$$\begin{aligned} Z_1 &= X_2 + X_3 + \cdots + X_n, \\ Z_2 &= Y_1 + X_3 + \cdots + X_n, \\ Z_n &= Y_1 + Y_2 + \cdots + Y_{n-1} \end{aligned}$$

Each Z_j represents a sum that is partially replaced. All variables before X_j are replaced by Y_j, but the variable after X_j remains as X_j.

3. **Compare Distributions**

   To show that $\frac{S_n}{s_n} \xrightarrow{d} \mathcal{N}(0, 1)$, we compare their expectations.

   Using Test Funtions

   Consider a test function f from C_B^3, the class of bounded continuous functions with three bounded derivatives.

   Consider:

$$\mathbb{E}\left[f\left(\frac{S_n}{s_n}\right)\right] \quad \text{and} \quad \mathbb{E}\left[f\left(\mathcal{N}\right)\right],$$

By introducing the replacement sequence Z_j, we rewrite:

$$\mathbb{E}\left[f\left(\frac{S_n}{s_n}\right)\right] - \mathbb{E}\left[f\left(\frac{Z_n}{s_n}\right)\right].$$

By telescoping:

$$\mathbb{E}\left[f\left(\frac{S_n}{s_n}\right)\right] - \mathbb{E}\left[f\left(\frac{Z_n}{s_n}\right)\right] = \sum_{j=1}^{n}\left[\mathbb{E}\left[f\left(\frac{X_j + Z_j}{s_n}\right)\right] - \mathbb{E}\left[f\left(\frac{Y_j + Z_{j+1}}{s_n}\right)\right]\right]$$

4. **Taylor Expansion of f**
   By Taylor's theorem for $f \in C_B^3$:

$$\left| f(x+y) - \left[f(x) + f'(x)y + \frac{f''(x)}{2}y^2\right]\right| \le \frac{M}{6}|y|^3,$$

   where $M = \sup_{x\in\mathbb{R}} |f^{(3)}(x)|$ is the maximum bound on the third derivative of f.
   Apply this to $f(\xi + \eta)$ and take Expectations, we write:

$$\left|\mathbb{E}[f(\xi+\eta)] - \mathbb{E}[f(\xi)] - \mathbb{E}[f'(\xi)]\mathbb{E}[\eta] - \frac{1}{2}\mathbb{E}[f''(\xi)]\mathbb{E}[\eta^2]\right| \le \frac{M}{6}\mathbb{E}[|\eta|^3],$$

   where ξ and η are independent random variables such that $\mathbb{E}[|\eta|^3] < \infty$.
   Then, let ζ be another independent random variable, having the same mean and variance as η. Now, consider:

$$\mathbb{E}[f(\xi+\eta)] - \mathbb{E}[f(\xi+\zeta)].$$

   Using Taylor's theorem again, this difference can be bounded as:

$$|\mathbb{E}[f(\xi+\eta)] - \mathbb{E}[f(\xi+\zeta)]| \le \frac{M}{6}(\mathbb{E}[|\eta|^3] + \mathbb{E}[|\zeta|^3]),$$

   where the sum $\mathbb{E}[|\eta|^3] + \mathbb{E}[|\zeta|^3]$ accounts for the third absolute moments of both random variables.
   Then, substitute with:
   - $\xi = Z_j/s_n$,
   - $\eta = X_j/s_n$,
   - $\zeta = Y_j/s_n$.

   We have:

$$\left|\mathbb{E}\left[f\left(\frac{Z_j}{s_n} + \frac{X_j}{s_n}\right)\right] - \mathbb{E}\left[f\left(\frac{Z_j}{s_n} + \frac{Y_j}{s_n}\right)\right]\right| \le \frac{M}{6}\left(\mathbb{E}\left[\left|\frac{X_j}{s_n}\right|^3\right] + \mathbb{E}\left[\left|\frac{Y_j}{s_n}\right|^3\right]\right) \tag{1}$$

5. **Error Bound Using Lyapunov's Condition**
From the definition of X_j/s_n and Y_j/s_n, we know:

$$\mathbb{E}\left[\left|\frac{X_j}{s_n}\right|^3\right] = \frac{\mathbb{E}[|X_j|^3]}{s_n^3}, \quad \mathbb{E}\left[\left|\frac{Y_j}{s_n}\right|^3\right] = \frac{\mathbb{E}[|Y_j|^3]}{s_n^3}.$$

For $Y_j \sim \mathcal{N}(0, \sigma_j^2)$, the third absolute moment is:

$$\mathbb{E}[|Y_j|^3] = c\sigma_j^3,$$

where $c = \sqrt{8/\pi}$ (a constant depending on the normal distribution). Since $\mathbb{E}[|X_j|^3] = \gamma_j$ and $\mathbb{E}[|Y_j|^3] = c\sigma_j^3$, we wrire:

$$\mathbb{E}\left[\left|\frac{X_j}{s_n}\right|^3\right] + \mathbb{E}\left[\left|\frac{Y_j}{s_n}\right|^3\right] = \frac{\gamma_j}{s_n^3} + \frac{c\sigma_j^3}{s_n^3}$$

By summing over $j = 1, \ldots, n$, (**??**) becomes:

$$\sum_{j=1}^{n}\left|\mathbb{E}\left[f\left(\frac{Z_j}{s_n} + \frac{X_j}{s_n}\right)\right] - \mathbb{E}\left[f\left(\frac{Z_j}{s_n} + \frac{Y_j}{s_n}\right)\right]\right| \le \frac{M}{6}\sum_{j=1}^{n}\left(\frac{\gamma_j}{s_n^3} + \frac{c\sigma_j^3}{s_n^3}\right)$$

Lyapunov's inequality states that:

$$\sigma_j^3 \le \gamma_j.$$

Thus:

$$\frac{\gamma_j}{s_n^3} + \frac{c \cdot \sigma_j^3}{s_n^3} \le \frac{\gamma_j}{s_n^3} + \frac{c \cdot \gamma_j}{s_n^3} = \frac{(1+c)\cdot\gamma_j}{s_n^3}.$$

Simplify the Sum, we have:

$$\frac{M}{6}\sum_{j=1}^{n}\left(\frac{\gamma_j}{s_n^3} + \frac{c \cdot \sigma_j^3}{s_n^3}\right) \le \frac{M}{6}\cdot\frac{1}{s_n^3}\sum_{j=1}^{n}(1+c)\gamma_j \tag{2}$$

Let $K = \frac{M}{6}(1+c)$, and the total third absolute moment of X_j:

$$\Gamma_n = \sum_{j=1}^{n}\gamma_j(asdefineabove)$$

We substitute these into the inequality (**??**):

$$\frac{M}{6}\sum_{j=1}^{n}\left(\frac{\gamma_j}{s_n^3} + \frac{c \cdot \sigma_j^3}{s_n^3}\right) \le K \cdot \frac{\Gamma_n}{s_n^3} \tag{3}$$

By the definition of Big-O Notation, we have

For a function $f(n)$, we say that $f(n) = O(g(n))$ if there exist positive constants $C > 0$ and $n_0 \geq 1$ such that:

$$|f(n)| \leq C|g(n)|, \quad \text{for all } n \geq n_0$$

Here:

- $f(n) = \frac{\Gamma_n}{s_n^3}$,
- $g(n) = \frac{\Gamma_n}{s_n^3}$

Clearly:

$$|f(n)| = \left|\frac{\Gamma_n}{s_n^3}\right| \quad \text{and} \quad |g(n)| = \left|\frac{\Gamma_n}{s_n^3}\right|$$

Since $f(n)$ and $g(n)$ are identical, we have:

$$|f(n)| \leq 1 \cdot |g(n)| \quad \text{for all } n \geq 1$$

Here, the constant $C = 1$. Then, the inequality $|f(n)| \leq C|g(n)|$ holds for all $n \geq 1$, so we can choose $n_0 = 1$.
Since we have found constants $C = 1$ and $n_0 = 1$ such that:

$$|f(n)| \leq C|g(n)| \quad \text{for all } n \geq n_0,$$

it follows by the definition of Big-O notation that:

$$\frac{\Gamma_n}{s_n^3} = O\left(\frac{\Gamma_n}{s_n^3}\right) \tag{4}$$

The Big O Notation property called **Multiplication by a constant** states that: Let k be a nonzero constant. Then:

$$O(|k| \cdot g) = O(g)$$

Apply this to (**??**), we have:

$$K \cdot \frac{\Gamma_n}{s_n^3} = O\left(\frac{\Gamma_n}{s_n^3}\right)$$

Then (**??**) becomes:

$$\frac{M}{6} \sum_{j=1}^{n} \left(\frac{\gamma_j}{s_n^3} + \frac{c \cdot \sigma_j^3}{s_n^3}\right) \leq O\left(\frac{\Gamma_n}{s_n^3}\right)$$

We have thus obtained the following estimate:

$$\forall f \in C^3 : \left|\mathbb{E}\left[f\left(\frac{S_n}{s_n}\right)\right] - \mathbb{E}\left[f(N)\right]\right| \leq O\left(\frac{\Gamma_n}{s_n^3}\right)$$

6. **Convergence to Normal**
   By Lyapunov's condition:

   $$\frac{\Gamma_n}{s_n^3} \to 0 \quad \text{as } n \to \infty$$

   This directly implies that:

   $$\left| \mathbb{E}\left[f\left(\frac{S_n}{s_n} \right) \right] - \mathbb{E}\left[f(N) \right] \right| \to 0.$$

   Since $\mathbb{E}[f(S_n/s_n)] \to \mathbb{E}[f(N)]$ for all bounded and continuous test functions f, by the general criterion for vague convergence, we conclude that:

   $$\frac{S_n}{s_n} \xrightarrow{d} \mathcal{N}(0, 1)$$

0.0.5 Conclusions

The CLT is a fundamental result in probability theory, ranging from mathematics and statistics to finance and data science. This work described the formal proof of the CLT using Lyapunov's criterion in the HOL4 theorem prover. The proof showed that the normalized sum of independent (but not necessarily identically distributed) random variables converges in distribution to the standard normal distribution by exploiting a moment-based approach along with Lyapunov's condition.

Conclusions

- **Formalization of Lyapunov's CLT**
  - The proof formalized Lyapunov's version of the CLT by providing rigorous bounds on error terms and establishing convergence under the Lyapunov condition. The use of Big-O notation and asymptotic error analysis was instrumental in managing higher-order terms.
- **Theoretical Insights**
  - In addition, a constructive, moment-based methodology used in its proof gave a natural perspective on finite variances and higher-order moments, each playing an important part in ensuring convergence according to CLT.
- **Comparison with Previous Works**

- The approach represents a complementary alternative to existing formalizations, such as **Isabelle/HOL's characteristic function-based proof**, in that it represents a constructive proof based on explicit bounds and moment inequalities. It is also an addition to the variety of formalizations and expands the reach of theorem provers regarding probabilistic results.

Formalization of CLT in HOL4 showcases the power of theorem provers in the formal verification of fundamental results in probability theory. It fills the gap between theoretical results and their verification by computation, showing the potential of formal methods in academic and applied domains.

0.0.5.1 Future Directions

...

In closing, this thesis provides not only a sound formalization of Lyapunov's CLT but also the whole range of insights into what happens when formal methods are integrated with probability theory. It lays the ground for further investigations into probabilistic results within theorem provers, opening perspectives to theoretical advances as well as practical applications.

Chapter 1

Introduction

Formal verification has become an essential approach in modern mathematics and computer science; it provides a strong framework for proving mathematical proofs and algorithms. Traditional techniques of formal verification include the following: Model Checking Clarke et al. (2018), Testing Broy et al. (2005), and Theorem Proving Bertot and Castéran (2013).

Model Checking (MC) is a well-valued static formal verification method since, when applied correctly, it provides complete assurance that something satisfies certain specifications. Model checking is a completely automated procedure and is considered easier compared to other techniques such as theorem proving. However, because of the state explosion problem Burch et al. (1992), application of model checking in large systems is quite a challenge due to the immense state space size of the systems analyzed. Also, the success of model checking strongly depends on the quality of the analyzed models. In contrast, testing, despite being highly valued for revealing defects in systems, is inherently limited, since it can only indicate the presence of errors, but lacks the ability to verify the correctness of a system.

Theorem proving is another widely used formal analysis technique. It does not suffer from the limitations of state space size as model checking does, which allows the analysis of larger and more complex systems. Furthermore, theorem provers use very expressive logics, like first-order or higher-order logics, which enable the study of a wider range of systems without the restrictions that often come with modeling. The most prominent provers are HOL4 Slind and Norrish (2008), HOL Light Harrison (2015), Coq Bertot and Castéran (2013), and Isabelle/HOL Team (2015), all broadly used in the community.

Formalization plays a vital role in probability theory to ensure that basic results are rigorous mathematically and computable in automated reasoning systems. This project formalizes the Central Limit Theorem using the HOL4 proof assistant, hence enriching the domain of probability in formalized mathematics.

1.1 Motivation

The Central Limit Theorem is one of the most fundamental results in the Probability Theory. This theory finds broad applications in statistics, data science, finance, and engineering. The theorem underlines that under certain conditions, the sum of many independent random variables will, as the number goes large, converge in distribution to a normal distribution irrespective of the individual distributions of those variables Chung (2000). This property justifies many real-world applications and models.

Although HOL4 already contains formal proofs of some fundamental probabilistic results, such as the Law of Large Numbers, a formalized proof of the Central Limit Theorem is still absent. Completing a proof would realize several important milestones, including the following:

- **Enriching the HOL4 Theory Library**:
  - The gap in the library would be filled by adding some advanced results from probability theory.
  - A more general foundation would be laid for future processes of formal verification.
- **Supporting Future Applications**:
  - Strengthening the framework for developers and researchers to model or verify real-world problems involving probabilistic reasoning.
- **Providing Analytical Tools**:
  - Enable efficient analysis of problems dependent on the normal distribution and its characteristics.

Formalizing the Central Limit Theorem also aligns with ongoing efforts to bridge traditional mathematical insights with computational tools. As a consequence, it paves the way for applications in areas such as artificial intelligence, machine learning, and quantitative modeling, where probabilistic reasoning is increasingly critical.

This project will contribute to a stronger, more versatile foundation for probabilistic formalization, empowering researchers and developers with better tools to tackle complex, real-world challenges.

Chapter 2

Background and Related Work

The history of the Central Limit Theorem, hereinafter referred to as CLT, is quite a fascinating journey through the gradual development of probability theory. Over the years, various versions of CLT have been formulated, reflecting its adaptability and foundational importance. Each such variation addresses certain mathematical challenges and applications, therefore enriching the broader understanding of convergence in probability. In this chapter, we explore some theorem-proving tools and their methodologies to formalize the CLT, emphasizing ...

2.1 Background

Let us brief the few evolvement of CLT, mainly noticing its critical features and variants developed further. The first steps towards CLT took place during the 18th century when Abraham de Moivre showed that, under summing many independent, identically distributed random variables-large number of rolled dice or coin flips-results in the normal distribution-a distribution with the shape of a bell De Moivre (1733). De Moivre laid down the very background for some approximations in the Binomial Distribution.

In 1810, Pierre-Simon Laplace expanded de Moivre's insights by proving that the sum of independent random variables converges to a normal distribution under broader conditions Laplace (1835). His work established the CLT as a universal principle for analyzing aggregate phenomena, such as population averages and measurement errors.

In the 19th century, mathematicians such as Pafnuty Chebyshev formalized the conditions of the theorem, including those of variance and expectation, so that the theorem became more precise and mathematically sound Chebyshev (1890). At the beginning of the 20th century, Lyapunov generalized CLT by introducing specific criteria-Lyapunov's condition-which explained when the theorem was applicable Lyapunov (1895). William Feller Feller (1945) later refined it to address discrete random variables.

Developments in the modern era since World War II have broadened the scope of the CLT beyond normal distributions to stable distributions and applications involving stochastic processes and high-dimensional data. These extensions illustrate how the theorem can be adapted to a wide range of probabilistic and statistical contexts.

Today, the CLT comes in a variety of variants each suited to particular situations:

- **Classical CLT:** For sums of independent, identically distributed random variables possessing finite variance and mean.
- **Generalized CLT:** When variables are weakly dependent or not identically distributed.
- **Triangular Arrays:** For sums of random variables arranged in arrays where the conditions vary across rows.
- **Local and Integral Versions:** While the local CLT concerns pointwise probabilities, the integral version deals with cumulative distributions.

From about 1810 to 1935, most of the efforts were devoted to proving the CLT for sums of independent random variables, with more recent generalizations involving only weakly dependent variables. Modern formulations neatly distinguish between normed sums, triangular arrays, and local versus integral theorems.

Chapter 3

Preliminaries

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Chapter 4

Central Limit Theorem

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Chapter 5

Future Work

Future Work

Chapter 6

Conclusion

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“modern syntax”, *see* special syntactic forms

for scripts