

# Cosystoles and Cheeger Constants of the Simplex

Kai Renken

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## An intuitive approach

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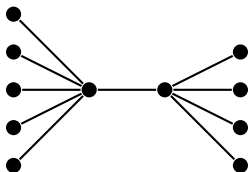


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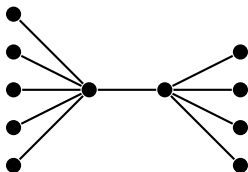


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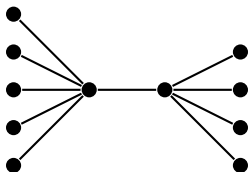


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Deleting the one edge in the middle will give a disconnected graph, consisting of two connected components, the smallest of them consisting of 5 vertices. The Cheeger constant of this graph is  $\frac{1}{5}$ .

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### Definition

Let  $G = (V, E)$  be a (simple) graph. Then the **Cheeger constant** of  $G$  is defined by

$$h(G) = \min \left\{ \frac{|\delta(A)|}{|A|} : A \subset V, 1 \leq |A| \leq \frac{|V|}{2} \right\},$$

with  $\delta(A) := \{e = (v, w) \in E : v \in A, w \in V \setminus A\}$ .

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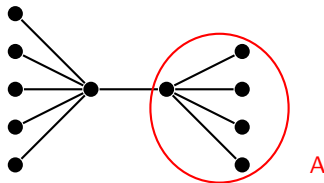


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A cochain  $\varphi \in C^k(X)$  is called a  **$k$ -cosystole**, if it satisfies  $\|\varphi\|_{\text{csy}} = \|\varphi\|$ .

## Coboundary expansion and the $k$ -th Cheeger constant

### Definition

For a cochain  $\varphi \in C^k(X) \setminus \text{Im}(\delta^{k-1})$  the quotient

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A cosystole  $\varphi \in C^k(X) \setminus \text{Im}(\delta^{k-1})$  satisfying  $\|\varphi\|_{\text{exp}} = h_k(X)$  is called a **Cheeger cosystole**.

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The classical Cheeger constant of a graph can be considered as the 0-th Cheeger constant by defining the cosystolic norm of a 0-cochain  $\varphi \in C^0(X)$  as

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The classical Cheeger constant of a graph equals 0 iff the graph is disconnected. The  $k$ -th Cheeger constant of a simplicial complex  $X$  equals 0 iff the  $k$ -th cohomology group  $H^k(X)$  is non-trivial.

## Example: The 1-cosystoles of $\Delta^{[6]}$

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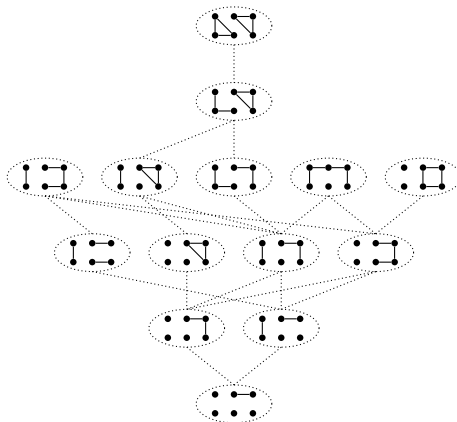


Figure: The supports of all 1-cosystoles of  $\Delta^{[6]}$  (up to isomorphism)



# Maximal cosystoles

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Let  $X$  be a simplicial complex and  $1 \leq k \leq \dim(X)$ , then

$$C_{\max}(X, k) := \max \left\{ \|\varphi\|_{\text{csy}} : \varphi \in C^k(X) \right\}$$

is the largest norm a  $k$ -cosystole in  $X$  can attain.

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Theorem (Renken)

$$C_{\max}(\Delta^{[n]}, 1) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$$

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### Theorem (Wallach, Meshulam)

*Let  $\Delta^{[n]}$  be the standard simplex on  $n$  vertices and  $1 \leq k \leq n-2$ , then we have:*

$$\frac{n}{k+2} \leq h_k(\Delta^{[n]}) \leq \left\lceil \frac{n}{k+2} \right\rceil$$

*If  $n$  is divisible by  $k+2$ , then we have:*

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### Theorem (Kozlov)

Let  $n > 2$  not be a power of 2, then we have:

$$h_1(\Delta^{[n]}) = \frac{n}{3}$$

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# Hitting sets and hitting numbers

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### Definition

Let  $V$  be some set and  $\mathcal{F} \subseteq 2^V$  a family of subsets of  $V$ . A subset  $P \subseteq V$  is called a **hitting set** of  $\mathcal{F}$  if we have  $P \cap F \neq \emptyset$  for all  $F \in \mathcal{F}$ . The **hitting number** of  $\mathcal{F}$  is defined by

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### Example

Let  $V := \{1, 2, 3, 4, 5\}$  and  $\mathcal{F} := \{\{1, 2\}, \{2, 3, 4\}, \{1, 5\}, \{2, 4, 5\}\}$ , then we have  $\tau(\mathcal{F}) = 2$ .

# The cycle detection theorem



## The cycle detection theorem

### Theorem (Kozlov)

*Let  $X$  be a simplicial complex,  $k \geq 1$ , and  $\varphi \in C^k(X)$ . Let now  $\mathcal{F} = \{\alpha_1, \dots, \alpha_t\}$  be a family of  $k$ -cycles in  $C_k(X)$ , such that  $\langle \varphi, \alpha_i \rangle = 1$  for all  $1 \leq i \leq t$ , then we have:*

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### Corollary (Kozlov)

*Let  $\varphi \in C^k(X)$  and  $\mathcal{F} = \{\alpha_1, \dots, \alpha_{\|\varphi\|}\} \subset C_k(X)$  be a family of  $k$ -cycles, such that their supports are pairwise disjoint and  $\langle \varphi, \alpha_i \rangle = 1$  for all  $1 \leq i \leq \|\varphi\|$ , then  $\varphi$  is a cosystole.*

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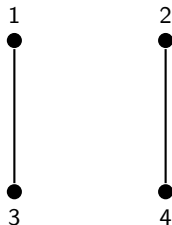


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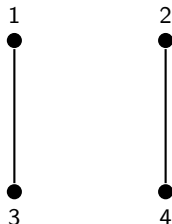


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Conjecture: For every proper  $n$  and  $k$  there is a Cheeger cosystole in  $C^k(\Delta^{[n]})$  which is detectable using disjoint cycles.



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$$\gamma_k(X) := \min_{\mathcal{F} \in \mathfrak{C}} \gamma_{\mathcal{F}}$$

the  $k$ -th **disjoint cycle expansion** of  $X$  with

$$\mathfrak{C} := \{\mathcal{F} \subset C_k(X) : F \text{ is a cycle and } \text{supp}(F) \cap \text{supp}(F') = \emptyset \text{ for all } F, F' \in \mathcal{F} (F \neq F')\}$$

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But it is always possible if the longer sum's first summand is 1.

## A theorem about partitioning consecutive numbers

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### Theorem (Renken)

*Let  $n, a, b \in \mathbb{N}$  ( $b \geq a$ ), such that  $\sum_{i=1}^n i = \sum_{i=a}^b i$ , then for every  $a \leq t \leq b$  there exists a subset  $U_t \subseteq [n]$ , such that  $U_i \cap U_j = \emptyset$  for all  $i \neq j$  and we have  $[n] = \bigcup_{a \leq t \leq b} U_t$  and  $\sum_{i \in U_t} i = t$ .*

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