Cosystoles and Cheeger Constants of the Simplex

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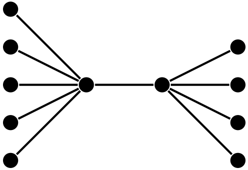

Figure: A "weakly" connected graph

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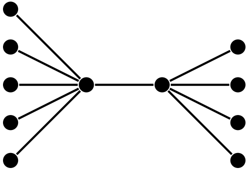

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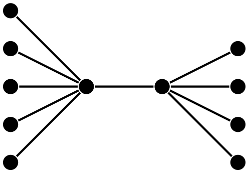

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Deleting the one edge in the middle will give a disconnected graph, consisting of two connected components, the smallest of them consisting of 5 vertices. The Cheeger constant of this graph is $\frac{1}{5}$.

The definition of the classical Cheeger constant

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Definition

Let $G = (V, E)$ be a (simple) graph. Then the **Cheeger constant** of G is defined by

$$h(G) = \min \left\{ \frac{|\delta(A)|}{|A|} : A \subset V, 1 \leq |A| \leq \frac{|V|}{2} \right\},$$

with $\delta(A) := \{e = (v, w) \in E : v \in A, w \in V \setminus A\}$.

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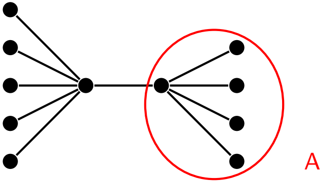

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Cosystoles and the cosystolic norm

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The **cosystolic norm** of a cochain $\varphi \in C^k(X)$ (for $k \geq 1$) is defined by:

$$\|\varphi\|_{csy} := \min\left\{\|\delta^{k-1}(\phi) + \varphi\| : \phi \in C^{k-1}(X)\right\}$$

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A cochain $\varphi \in C^k(X)$ is called a k-**cosystole**, if it satisfies $\|\varphi\|_{csy} = \|\varphi\|$.

Coboundary expansion and the k-th Cheeger constant

Definition

For a cochain $\varphi \in C^k(X) \setminus \mathrm{Im}(\delta^{k-1})$ the quotient

$$\|\varphi\|_{exp} := \frac{\|\delta^k(\varphi)\|}{\|\varphi\|_{csy}}$$

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A cosystole $\varphi \in C^k(X) \setminus \mathrm{Im}(\delta^{k-1})$ satisfying $\|\varphi\|_{exp} = h_k(X)$ is called a **Cheeger cosystole**.

Relations to the classical Cheeger constant

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The classical Cheeger constant of a graph can be considered as the 0-th Cheeger constant by defining the cosystolic norm of a 0-cochain $\varphi \in C^0(X)$ as

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The classical Cheeger constant of a graph equals 0 iff the graph is disconnected. The k-th Cheeger constant of a simplicial complex X equals 0 iff the k-th cohomology group $H^k(X)$ is non-trivial.

Example: The 1-cosystoles of $\Delta^{[6]}$

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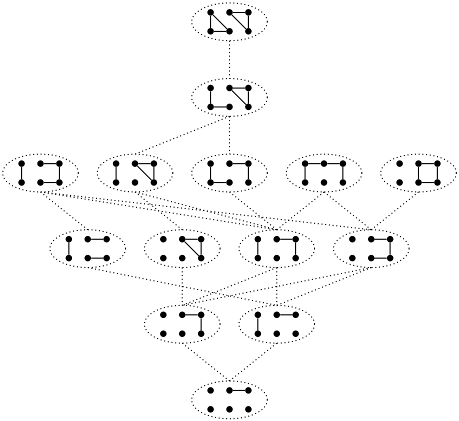

Figure: The supports of all 1-cosystoles of $\Delta^{[6]}$ (up to isomorphism)

Maximal cosystoles

Maximal cosystoles

Definition

Let X be a simplicial complex and $1 \leq k \leq \dim(X)$, then

$$C_{max}(X, k) \coloneqq \max\left\{\|\varphi\|_{csy} : \varphi \in C^k(X)\right\}$$

is the largest norm a k-cosystole in X can attain.

The largest cosystoles of the simplex

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Theorem (Renken)

$$C_{max}(\Delta^{[n]}, 1) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$$

Some values for higher dimensional Cheeger constants

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Theorem (Wallach, Meshulam)

Let $\Delta^{[n]}$ be the standard simplex on n vertices and $1 \leq k \leq n-2$, then we have:

$$\frac{n}{k+2} \leq h_k(\Delta^{[n]}) \leq \left\lceil \frac{n}{k+2} \right\rceil$$

If n is divisible by $k+2$, then we have:

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Theorem (Kozlov)

Let $n > 2$ not be a power of 2, then we have:

$$h_1(\Delta^{[n]}) = \frac{n}{3}$$

The case when n is a power of 2

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Theorem (Renken)

$$h_1(\Delta^{[16]}) > \frac{16}{3}$$

Hitting sets and hitting numbers

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Definition

Let V be some set and $\mathcal{F} \subseteq 2^V$ a family of subsets of V. A subset $P \subseteq V$ is called a **hitting set** of $\mathcal{F}$ if we have $P \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The **hitting number** of $\mathcal{F}$ is defined by

$$\tau(\mathcal{F}) := \min\{|P| : P \subseteq V \text{ is a hitting set of } \mathcal{F}\}$$

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Example

Let $V := \{1, 2, 3, 4, 5\}$ and $\mathcal{F} := \{\{1, 2\}, \{2, 3, 4\}, \{1, 5\}, \{2, 4, 5\}\}$, then we have $\tau(\mathcal{F}) = 2$.

The cycle detection theorem

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Theorem (Kozlov)

Let X be a simplicial complex, $k \geq 1$, and $\varphi \in C^k(X)$. Let now $\mathcal{F} = \{\alpha_1, \ldots, \alpha_t\}$ be a family of k-cycles in $C_k(X)$, such that $\langle \varphi, \alpha_i \rangle = 1$ for all $1 \leq i \leq t$, then we have:

$$\|\varphi\|_{csy} \geq \tau(\mathcal{F})$$

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Corollary (Kozlov)

Let $\varphi \in C^k(X)$ and $\mathcal{F} = \{\alpha_1, \ldots, \alpha_{\|\varphi\|}\} \subset C_k(X)$ be a family of k-cycles, such that their supports are pairwise disjoint and $\langle \varphi, \alpha_i \rangle = 1$ for all $1 \leq i \leq \|\varphi\|$, then φ is a cosystole.

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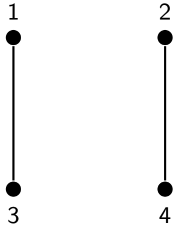

Figure: The support of a 1-cosystole, which can not be determined using disjoint cycles

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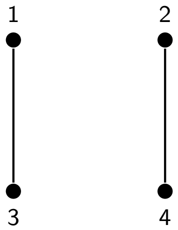

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Conjecture: For every proper n and k there is a Cheeger cosystole in $C^k(\Delta^{[n]})$ which is detectable using disjoint cycles.

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$$P(\mathcal{F}) := \{\varphi \in C^k(X) : |\mathrm{supp}(\varphi) \cap \mathrm{supp}(F)| = 1 \text{ for all } F \in \mathcal{F} \text{ and } \mathrm{supp}(\varphi) \subset \bigcup_{F \in \mathcal{F}} \mathrm{supp}(F)\}$$

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Then we define

$$\gamma_{\mathcal{F}} := \frac{\min_{\varphi \in P(\mathcal{F})} \|\delta^k(\varphi)\|}{|\mathcal{F}|}$$

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and we call

$$\gamma_k(X) := \min_{\mathcal{F} \in \mathfrak{C}} \gamma_{\mathcal{F}}$$

the k-th **disjoint cycle expansion** of X with

$$\mathfrak{C} := \{\mathcal{F} \subset C_k(X) : F \text{ is a cycle and } \mathrm{supp}(F) \cap \mathrm{supp}(F') = \emptyset \text{ for all } F, F' \in \mathcal{F}\ (F \neq F')\}$$

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Theorem (Renken)

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$$\gamma_1(\Delta^{[n]}) = \rho_1(\Delta^{[n]}) = \frac{n}{3}$$

Partitioning consecutive numbers - Introducing example

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There are four ways to display the number 15 as a sum of consecutive numbers, namely:

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We already know that for an $n \in \mathbb{N}$ there are as many ways to display n as a sum of consecutive numbers as many odd divisors n has.

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Answer: No! For example there is no way to display the number 7 as a sum of the numbers $4, 5, 6$.
But it is always possible if the longer sum's first summand is 1.

A theorem about partitioning consecutive numbers

A theorem about partitioning consecutive numbers

Theorem (Renken)

Let $n, a, b \in \mathbb{N}$ ($b \geq a$), such that $\sum_{i=1}^{n} i = \sum_{i=a}^{b} i$, then for every $a \leq t \leq b$ there exists a subset $U_t \subseteq [n]$, such that $U_i \cap U_j = \emptyset$ for all $i \neq j$ and we have $[n] = \bigcup_{a \leq t \leq b} U_t$ and $\sum_{i \in U_t} i = t$.

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