Cosystoles and Cheeger Constants of the Simplex

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Table of contents

An intuitive approach

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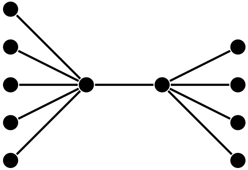

Figure: A "weakly" connected graph

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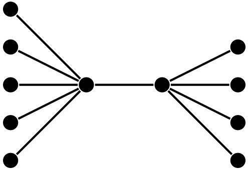

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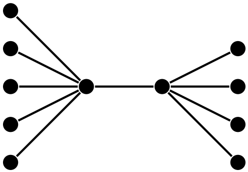

Figure: A "weakly" connected graph

Deleting the one edge in the middle will give a disconnected graph, consisting of two connected components, the smallest of them consisting of 5 vertices. The Cheeger constant of this graph is $\frac{1}{5}$.

The definition of the classical Cheeger constant

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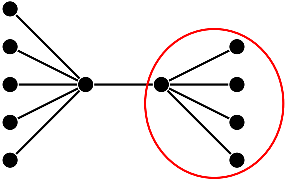
Definition

Let $G = (V, E)$ be a (simple) graph. Then the **Cheeger constant** of G is defined by

$$h(G) = \min\left\{\frac{|\delta(A)|}{|A|} : A \subset V, 1 \leq |A| \leq \frac{|V|}{2}\right\},$$

with $\delta(A) := \{e = (v, w) \in E : v \in A, w \in V \setminus A\}$.



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Then we have:

$$h(K_n) = \left\lceil \frac{n}{2} \right\rceil$$

as follows:

The classical Cheeger constant of the simplex

For any subset $A \subset [n] := \{1, \dots, n\}$ we have:

$$\frac{|\delta(A)|}{|A|} = \frac{|A|(n - |A|)}{|A|} = n - |A|,$$

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$$h(K_n) = n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$$

Cosystoles and the cosystolic norm

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Let X be an (abstract) simplicial complex. Then the **norm** of a cochain $\varphi \in C^k(X)$ is defined by:

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The **cosystolic norm** of a cochain $\varphi \in C^k(X)$ (for $k \geq 1$) is defined by:

$$\|\varphi\|_{csy} := \min\left\{\|\delta^{k-1}(\phi) + \varphi\| : \phi \in C^{k-1}(X)\right\}$$

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A cochain $\varphi \in C^k(X)$ is called a k-**cosystole**, if it satisfies $\|\varphi\|_{csy} = \|\varphi\|$.

Coboundary expansion and the k-th Cheeger constant

Definition

For a cochain $\varphi \in C^k(X) \setminus \mathrm{Im}(\delta^{k-1})$ the quotient

$$\|\varphi\|_{exp} := \frac{\|\delta^k(\varphi)\|}{\|\varphi\|_{csy}}$$

is called the **coboundary expansion** of φ and

$$h_k(X) := \min_{\substack{\varphi \in C^k(X) \\ \varphi \notin \mathrm{Im}(\delta^{k-1})}} \|\varphi\|_{exp}$$

is called the k-th **Cheeger constant** of X.
A cosystole $\varphi \in C^k(X) \setminus \mathrm{Im}(\delta^{k-1})$ satisfying $\|\varphi\|_{exp} = h_k(X)$ is called a **Cheeger cosystole**.

Example

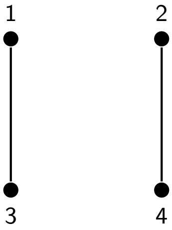

Figure: The support of a 1-cosystole

Some values for higher dimensional Cheeger constants

Theorem (Wallach, Meshulam)

Let $\Delta^{[n]}$ be the standard simplex on n vertices and $1 \leq k \leq n-2$, then we have:

$$\frac{n}{k+2} \leq h_k(\Delta^{[n]}) \leq \left\lceil \frac{n}{k+2} \right\rceil$$

If n is divided by $k+2$, then we have:

$$h_k(\Delta^{[n]}) = \frac{n}{k+2}$$

Some values for higher dimensional Cheeger constants

Theorem (Kozlov)

Let $n > 2$ not be a power of 2, then we have:

$$h_1(\Delta^{[n]}) = \frac{n}{3}$$

Relations to the classical Cheeger constant

The classical Cheeger constant of a graph can be considered as the 0-th Cheeger constant by defining the cosystolic norm of a 0-cochain $\varphi \in C^0(X)$ as

$$\|\varphi\|_{csy} := \min\{|\mathrm{supp}(\varphi)|, |X^{(0)}| - |\mathrm{supp}(\varphi)|\}$$

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The classical Cheeger constant of a graph equals 0 iff the graph is disconnected.

Relations to the classical Cheeger constant

The k-th Cheeger constant of a simplicial complex X equals 0 iff the k-th homology group $H_k(X)$ is non-trivial, as follows:

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which is equivalent to

$$H^k(X) \ncong \{0\}$$

Hitting sets and hitting numbers

Definition

Let V be some set and $\mathcal{F} \subseteq 2^V$ a family of subsets of V. A subset $P \subseteq V$ is called a **hitting set** of $\mathcal{F}$ if we have $P \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The **hitting number** of $\mathcal{F}$ is defined by

$$\tau(\mathcal{F}) := \min\{|P| : P \subseteq V \text{ is a hitting set of } \mathcal{F}\}$$

If $\mathcal{F}$ is a family of chains / cochains, the hitting sets and the hitting number of $\mathcal{F}$ are defined as the hitting sets and the hitting number of $\{\mathrm{supp}(\varphi) : \varphi \in \mathcal{F}\}$.

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Example

Let $V := \{1, 2, 3, 4, 5\}$ and $\mathcal{F} := \{\{1, 2\}, \{2, 3, 4\}, \{1, 5\}, \{2, 4, 5\}\}$, then we have $\tau(\mathcal{F}) = 2$.

The cycle detection theorem

Theorem (Kozlov)

Let X be a simplicial complex, $k \geq 1$, and $\varphi \in C^k(X)$. Let now $\mathcal{F} = \{\alpha_1, \ldots, \alpha_t\}$ be a family of k-cycles in $C_k(X)$, such that $\langle \varphi, \alpha_i \rangle = 1$ for all $1 \leq i \leq t$, then we have:

$$\|\varphi\|_{csy} \geq \tau(\mathcal{F})$$

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$$\begin{aligned}
\langle \varphi + \delta^{k-1}(\psi), \alpha_i \rangle &= \langle \varphi, \alpha_i \rangle + \langle \delta^{k-1}(\psi), \alpha_i \rangle \\
&= \langle \varphi, \alpha_i \rangle + \langle \psi, \partial_{k-1}(\alpha_i) \rangle \\
&= \langle \varphi, \alpha_i \rangle + \langle \psi, 0 \rangle \\
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This means that we have $\operatorname{supp}(\varphi + \delta^{k-1}(\psi)) \cap \operatorname{supp}(\alpha_i) \neq \emptyset$ for all $1 \leq i \leq t$, so $\operatorname{supp}(\varphi + \delta^{k-1}(\psi))$ is a hitting set of $\mathcal{F}$ and we get:

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$$\|\varphi + \delta^{k-1}(\psi)\| = |\mathrm{supp}(\varphi + \delta^{k-1}(\psi))| \geq \tau(\mathcal{F})$$

Since ψ was chosen arbitrarily we are done. $\square$

Maximal cosystoles

Definition

Let X be a simplicial complex and $1 \leq k \leq \dim(X)$, then

$$C_{max}(X, k) := \max \left\{ \|\varphi\|_{csy} : \varphi \in C^k(X) \right\}$$

is the largest norm a k-cosystole in X can attain.

The largest cosystoles of the simplex

Theorem (Renken)

$$C_{max}(\Delta^{[n]}, 1) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$$

Sketch of a proof

Cut-minimal graphs and Cheeger graphs of a simplex

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Definition

Consider a graph $G = ([n], E)$. For any subsets $A, B \subset [n]$ define

$$E_G(A, B) := \{(v, w) \in E \ : \ v \in A, \ w \in B\}$$

and

$$NE_G(A, B) := \{(v, w) \notin E \ : \ v \in A, \ w \in B\}$$

A graph $G = ([n], E)$ is called **cut-minimal**, if for every $S \subset [n]$ we have

$$|E_G(S, [n] \setminus S)| \leq |NE_G(S, [n] \setminus S)|,$$

which is equivalent to

$$|E_G(S, [n] \setminus S)| \leq \frac{|S|(n - |S|)}{2}$$

Cut-minimal graphs and Cheeger graphs of a simplex

There is a one-to-one correspondence between the cut-minimal graphs on n vertices and the 1-cosystoles in $C^1(\Delta^{[n]})$.

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Definition

A cut-minimal graph which corresponds to a Cheeger cosystole is called a **Cheeger graph**.

Maximal cut-minimal graphs

The case when n is a power of 2

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Conjecture

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Theorem (Kozlov)

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Theorem (Renken)

$$h_1(\Delta^{[16]}) > \frac{16}{3}$$

Disjoint cycle expansion

Definition

Let X be a simplicial complex, $\mathcal{F} \subset C_k(X)$ a family of cycles, such that their supports are pairwise disjoint and

$$P(\mathcal{F}) \coloneqq \{\varphi \in C^k(X) : |\mathrm{supp}(\varphi) \cap \mathrm{supp}(F)| = 1 \text{ for all } F \in \mathcal{F} \text{ and } \mathrm{supp}(\varphi) \subset \bigcup_{F \in \mathcal{F}} \mathrm{supp}(F)\}$$

Then we define

$$\gamma_{\mathcal{F}} \coloneqq \frac{\min_{\varphi \in P(\mathcal{F})} \|\delta^k(\varphi)\|}{|\mathcal{F}|}$$

and we call

$$\gamma_k(X) \coloneqq \min_{\mathcal{F} \in \mathfrak{C}} \gamma_{\mathcal{F}}$$

the k-th **disjoint cycle expansion** of X with

$$\mathfrak{C} \coloneqq \{\mathcal{F} \subset C_k(X) : F \text{ is a cycle and } \mathrm{supp}(F) \cap \mathrm{supp}(F') = \emptyset \text{ for all } F, F' \in \mathcal{F}\ (F \neq F')\}$$

Hitting expansion

Definition

Let X be a simplicial complex, $\mathcal{F} \subset C_k(X)$ a family of cycles and

$$P'(\mathcal{F}) := \{\varphi \in C^k(X) : |\mathrm{supp}(\varphi) \cap \mathrm{supp}(F)| \text{ is odd for all } F \in \mathcal{F}\}$$

Then we define

$$\rho_{\mathcal{F}} := \frac{\min\limits_{\varphi \in P'(\mathcal{F})} \|\delta^k(\varphi)\|}{\tau(\mathcal{F})}$$

and we call

$$\rho_k(X) := \min_{\mathcal{F} \in \mathfrak{C}} \rho_{\mathcal{F}}$$

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Theorem (Renken)

Let n not be a power of 2, then we have:

$$\gamma_1(\Delta^{[n]}) = \rho_1(\Delta^{[n]}) = \frac{n}{3}$$