# Cosystoles and Cheeger Constants of the Simplex

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# An intuitive approach

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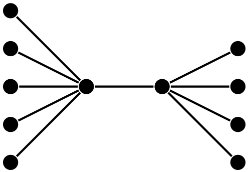

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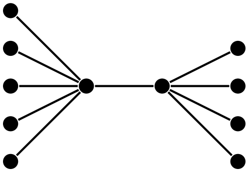

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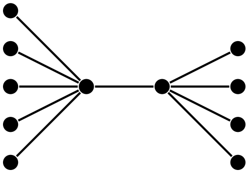

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Deleting the one edge in the middle will give a disconnected graph, consisting of two connected components, the smallest of them consisting of 5 vertices. The Cheeger constant of this graph is $\frac{1}{5}$.

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## Definition

Let $G = (V, E)$ be a (simple) graph. Then the **Cheeger constant** of $G$ is defined by

$$h(G) = \min\left\{\frac{|\delta(A)|}{|A|} : A \subset V, 1 \leq |A| \leq \frac{|V|}{2}\right\},$$

with $\delta(A) := \{e = (v, w) \in E : v \in A, w \in V \setminus A\}$.

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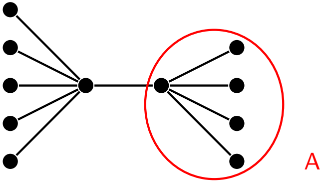


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A cochain $\varphi \in C^k(X)$ is called a $k$-**cosystole**, if it satisfies $\|\varphi\|_{csy} = \|\varphi\|$.

# Coboundary expansion and the $k$-th Cheeger constant

### Definition

For a cochain $\varphi \in C^k(X) \setminus \mathrm{Im}(\delta^{k-1})$ the quotient

$$\|\varphi\|_{exp} := \frac{\|\delta^k(\varphi)\|}{\|\varphi\|_{csy}}$$

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A cosystole $\varphi \in C^k(X) \setminus \mathrm{Im}(\delta^{k-1})$ satisfying $\|\varphi\|_{exp} = h_k(X)$ is called a **Cheeger cosystole**.

# Relations to the classical Cheeger constant

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The classical Cheeger constant of a graph can be considered as the 0-th Cheeger constant by defining the cosystolic norm of a 0-cochain $\varphi \in C^0(X)$ as

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The classical Cheeger constant of a graph equals 0 iff the graph is disconnected.

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# Example: The $1$-cosystoles of $\Delta^{[6]}$

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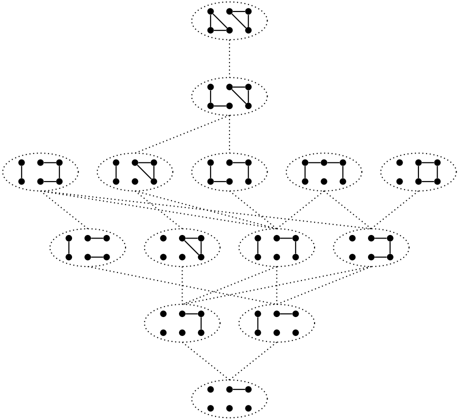

Figure: The supports of all 1-cosystoles of $\Delta^{[6]}$ (up to isomorphism)

# Maximal cosystoles

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**Definition**

Let $X$ be a simplicial complex and $1 \leq k \leq \dim(X)$, then

$$C_{max}(X, k) \coloneqq \max\left\{\|\varphi\|_{csy} : \varphi \in C^k(X)\right\}$$

is the largest norm a $k$-cosystole in $X$ can attain.

# The largest cosystoles of the simplex

## The largest cosystoles of the simplex

**Theorem (Renken)**

$$C_{max}(\Delta^{[n]}, 1) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$$

# Some values for higher dimensional Cheeger constants

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**Theorem (Wallach, Meshulam)**

*Let $\Delta^{[n]}$ be the standard simplex on $n$ vertices and $1 \leq k \leq n-2$, then we have:*

$$\frac{n}{k+2} \leq h_k(\Delta^{[n]}) \leq \left\lceil \frac{n}{k+2} \right\rceil$$

*If $n$ is divisible by $k+2$, then we have:*

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**Theorem (Kozlov)**

*Let $n > 2$ not be a power of 2, then we have:*

$$h_1(\Delta^{[n]}) = \frac{n}{3}$$

# The case when $n$ is a power of 2

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### Conjecture

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**Theorem (Renken)**

$$h_1(\Delta^{[16]}) > \frac{16}{3}$$

# Hitting sets and hitting numbers

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Let $V$ be some set and $\mathcal{F} \subseteq 2^V$ a family of subsets of $V$. A subset $P \subseteq V$ is called a **hitting set** of $\mathcal{F}$ if we have $P \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The **hitting number** of $\mathcal{F}$ is defined by

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## Example

Let $V := \{1, 2, 3, 4, 5\}$ and $\mathcal{F} := \{\{1, 2\}, \{2, 3, 4\}, \{1, 5\}, \{2, 4, 5\}\}$, then we have $\tau(\mathcal{F}) = 2$.

# The cycle detection theorem

## The cycle detection theorem

**Theorem (Kozlov)**

*Let $X$ be a simplicial complex, $k \geq 1$, and $\varphi \in C^k(X)$. Let now $\mathcal{F} = \{\alpha_1, \ldots, \alpha_t\}$ be a family of $k$-cycles in $C_k(X)$, such that $\langle \varphi, \alpha_i \rangle = 1$ for all $1 \leq i \leq t$, then we have:*

$$\|\varphi\|_{csy} \geq \tau(\mathcal{F})$$

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$$\begin{aligned}
\langle \varphi + \delta^{k-1}(\psi), \alpha_i \rangle &= \langle \varphi, \alpha_i \rangle + \langle \delta^{k-1}(\psi), \alpha_i \rangle \\
&= \langle \varphi, \alpha_i \rangle + \langle \psi, \partial_{k-1}(\alpha_i) \rangle \\
&= \langle \varphi, \alpha_i \rangle + \langle \psi, 0 \rangle \\
&= \langle \varphi, \alpha_i \rangle = 1
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This means that we have $\operatorname{supp}(\varphi + \delta^{k-1}(\psi)) \cap \operatorname{supp}(\alpha_i) \neq \emptyset$ for all $1 \leq i \leq t$, so $\operatorname{supp}(\varphi + \delta^{k-1}(\psi))$ is a hitting set of $\mathcal{F}$ and we get:

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$$\|\varphi + \delta^{k-1}(\psi)\| = |\mathrm{supp}(\varphi + \delta^{k-1}(\psi))| \geq \tau(\mathcal{F})$$

Since $\psi$ was chosen arbitrarily we are done. $\square$

# The cycle detection theorem

## The cycle detection theorem

**Corollary (Kozlov)**

*Let $\varphi \in C^k(X)$ and $\mathcal{F} = \{\alpha_1, \ldots, \alpha_{\|\varphi\|}\} \subset C_k(X)$ be a family of k-cycles, such that their supports are pairwise disjoint, then $\varphi$ is a cosystole.*

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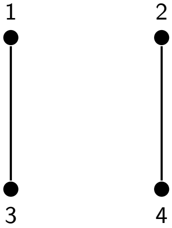


Figure: The support of a 1-cosystole, which can not be determined using disjoint cycles

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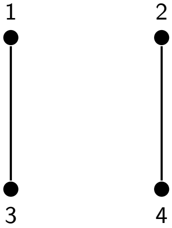


Figure: The support of a 1-cosystole, which can not be determined using disjoint cycles

Conjecture: For every proper $n$ and $k$ there is a Cheeger cosystole in $C^k(\Delta^{[n]})$ which is detectable using disjoint cycles.

# Disjoint cycle expansion

## Disjoint cycle expansion

### Definition

Let $X$ be a simplicial complex, $\mathcal{F} \subset C_k(X)$ a family of cycles, such that their supports are pairwise disjoint and

$$P(\mathcal{F}) \coloneqq \{\varphi \in C^k(X) : |\mathrm{supp}(\varphi) \cap \mathrm{supp}(F)| = 1 \text{ for all } F \in \mathcal{F} \text{ and } \mathrm{supp}(\varphi) \subset \bigcup_{F \in \mathcal{F}} \mathrm{supp}(F)\}$$

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$$\gamma_{\mathcal{F}} := \frac{\min_{\varphi \in P(\mathcal{F})} \|\delta^k(\varphi)\|}{|\mathcal{F}|}$$

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and we call

$$\gamma_k(X) := \min_{\mathcal{F} \in \mathfrak{C}} \gamma_{\mathcal{F}}$$

the $k$-th **disjoint cycle expansion** of $X$ with

$$\mathfrak{C} := \{\mathcal{F} \subset C_k(X) : F \text{ is a cycle and } \mathrm{supp}(F) \cap \mathrm{supp}(F') = \emptyset \text{ for all } F, F' \in \mathcal{F}\ (F \neq F')\}$$

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## Hitting expansion

### Definition

Let $X$ be a simplicial complex, $\mathcal{F} \subset C_k(X)$ a family of cycles and

$$P'(\mathcal{F}) := \{\varphi \in C^k(X) : |\mathrm{supp}(\varphi) \cap \mathrm{supp}(F)| \text{ is odd for all } F \in \mathcal{F}\}$$

Then we define

$$\rho_{\mathcal{F}} := \frac{\min_{\varphi \in P'(\mathcal{F})} \|\delta^k(\varphi)\|}{\tau(\mathcal{F})}$$

and we call

$$\rho_k(X) := \min_{\mathcal{F} \in \mathfrak{C}} \rho_{\mathcal{F}}$$

the $k$-th **hitting expansion** of $X$, with

$$\mathfrak{C} := \{\mathcal{F} \subset C_k(X) : F \text{ is a cycle, for all } F \in \mathcal{F}\}$$

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**Theorem (Kozlov)**

*Let $X$ be a simplicial complex and $k \geq 1$, then we have:*

$$h_k(X) \leq \rho_k(X) \leq \gamma_k(X)$$

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$$\gamma_k(\Delta^{[n]}) = \rho_k(\Delta^{[n]}) = \frac{n}{k+2}$$

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**Theorem (Renken)**

*Let $n$ not be a power of 2, then we have:*

$$\gamma_1(\Delta^{[n]}) = \rho_1(\Delta^{[n]}) = \frac{n}{3}$$

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