Universität Bremen

DISSERTATION

Cosystoles and Cheeger constants

Autor:
Kai Michael RENKEN

Betreuer:
Prof. Dr. Dmitry FEICHTNER-KOZLOV

Dissertation zur Erlangung des Doktorgrades Dr. rer. nat.

in der Forschungsgruppe

CALTOP
ALTA - Institute for Algebra, Geometry, Topology and their applications

11. Oktober 2018

Urheberrechtliche Erklärung

Ich, Kai Michael RENKEN, versichere hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe. Alle Stellen, die ich wörtlich oder sinngemäß aus anderen Werken entnommen habe, habe ich unter Angabe der Quellen als solche kenntlich gemacht.

Unterschrift: ____________________________

Datum: ____________________________

UNIVERSITÄT BREMEN

Abstract

Fachbereich 3 - Mathematik und Informatik
ALTA - Institute for Algebra, Geometry, Topology and their applications

Dr. rer. nat.

Cosystoles and Cheeger constants

by Kai Michael RENKEN

The central interest of this thesis is to develop tools to determine the cosystolicity of a cochain, a property which is important to determine the Cheeger constants of a simplicial complex. We develop a general theory about the cosystolic norm of a cochain, in which we establish an interesting connection between that norm and the piercing number of a certain set system (see Chapter 3). In Chapter 4 we restrict our research to 1-dimensional cosystoles of a simplex which are slightly easier to understand, so we can provide more explicit results for that case, including the explicit determination of the largest 1-dimensional cosystoles of a simplex and a rough insight, how all cosystoles of a simplex in a certain dimension can be arranged in a certain simplicial complex. In the appendix we solve a beautiful combinatorial ordering problem, which is not directly related to the main subject of this thesis but arose during considerations about that and is worth to be stated as well.

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

Contents

Dedicated to my uncle Eddie

Chapter 1

Introduction

The original and well-studied notion of the Cheeger constant of a graph can be considered as a measure of connectedness, measuring the relation of disconnecting relatively large connected components of a simple graph by removing a relatively small number of edges. If $G = (V, E)$ is a simple graph (undirected and no loops or double edges allowed) on the vertex set V and the edge set $E \subseteq \binom{V}{2}$ (where $\binom{V}{2}$ denotes the set of all subsets of V of cardinality 2), then the (0-th) Cheeger constant of G is defined as:

$$h_0(G) := \min \left\{ \frac{|\delta(A)|}{|A|} : A \subset V, 1 \leq |A| \leq \frac{|V|}{2} \right\},$$

with $\delta(A) := \{e = (e_1, e_2) \in E : e_1 \in A \text{ and } e_2 \in V \setminus A\}$.

Chapter 2

Setting up the board

Let us shortly recall some basic algebraic and combinatorial concepts, which we will use within this thesis.

2.1 Simplicial complexes

Let S be some set (whose elements are called **vertices**) and $X \subseteq 2^S$ a family of subsets of S (we will use the notation 2^S for the power set of S within the whole thesis), such that for all $\sigma \in X$ and all $\sigma' \subseteq \sigma$ we have $\sigma' \in X$. Then we call X an **(abstract) simplicial complex**. We just use the notation "simplicial complex" in this thesis, because we will only consider abstract simplicial complexes and are not interested in their geometric realization.

An element of a simplicial complex X is called **simplex** and a **face** of a simplex $\sigma \in X$ is a simplex $\sigma' \in X$, such that $\sigma' \subset \sigma$ and $|\sigma'| = |\sigma| - 1$. Furthermore, we denote the k-**skeleton** of a simplicial complex X by

$$X(k) := \{\sigma \in X : \ |\sigma| \leq k+1\},$$

and the **uniform** k-**skeleton** of X by

$$X^{(k)} := \{\sigma \in X : \ |\sigma| = k+1\}$$

Let $\sigma \subset S$ be a simplex and $s \in S \setminus \sigma$, then the simplex constructed by adding s to σ is denoted by $(\sigma, s) := \sigma \cup \{s\}$.
For a simplicial complex X we call $dim(X) := \max\{|\sigma| - 1 : \sigma \in X\}$ the **dimension** of X (if it exists).
A simplicial complex is called **finite** if its vertex set is finite and **finite dimensional** if its dimension is finite.
The most frequently considered simplicial complex in this thesis will be the complex induced by the standard simplex on n-vertices. It can be considered as the complete power set of $[n] := |\{i \in \mathbb{N} : 1 \leq i \leq n\}|$ and we will denote it by $\Delta^{[n]} := 2^{[n]}$.

2.2 Chain- / Cochain complexes & Homology / Cohomology

Let X be a simplicial complex and $0 \leq k \leq dim(X)$, then

$$C_k(X, \mathbb{Z}_2) := \left\{ \sum_{i \in I} c_i \sigma_i : \sigma_i \in X, c_i \in \mathbb{Z}_2 \right\}$$

is called the k-th **chain group** of X, where I is some index set. (The elements of $C_k(X, \mathbb{Z}_2)$ are called k-**chains**)
Note, that in general we have more possible coefficient systems than $\mathbb{Z}_2$ and X can be any topological space, but we will restrict ourselves to simplicial complexes in this thesis. Furthermore, since we only consider chain groups with $\mathbb{Z}_2$-coefficients in this thesis, we will use the notation $C_k(X) := C_k(X, \mathbb{Z}_2)$.
The linear map $\partial_k : C_{k+1}(X) \to C_k(X)$ defined on a simplex $\sigma = (v_0, \dots, v_{k+1}) \in X$ as

$$\partial_k(\sigma) = \sum_{i=0}^{k+1} (-1)^i (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{k+1})$$

is called the k-th **boundary map**. (Recall, that the boundary maps have the property $\partial_k \circ \partial_{k+1} = 0$)

The k-th **homology group** of X is then defined as:

$$H_k(X) := \frac{\mathrm{Ker}(\partial_{k-1})}{\mathrm{Im}(\partial_k)},$$

where the elements from $\mathrm{Ker}(\partial_{k-1})$ are called k-**cycles** and the elements from $\mathrm{Im}(\partial_k)$ are called k-**boundaries**.
Dualizing this concept, we get the k-th **cochain group** of X by

$$C^k(X) := C^k(X, \mathbb{Z}_2) := \{\varphi : C_k(X) \to \mathbb{Z}_2 : \varphi \text{ is a linear map}\},$$

whose elements are called k-**cochains**, the k-th **coboundary map** $\delta^k : C^k(X) \to C^{k+1}(X)$ by $\delta^k(\varphi) := \varphi \circ \partial_k$, and the k-th cohomology group of X by:

$$H^k(X) := \frac{\mathrm{Ker}(\delta^k)}{\mathrm{Im}(\delta^{k-1})}$$

Furthermore, the sequence

$$\cdots \xrightarrow{\partial_{k+1}} C_{k+1}(X) \xrightarrow{\partial_k} C_k(X) \xrightarrow{\partial_{k-1}} C_{k-1}(X) \xrightarrow{\partial_{k-2}} \cdots$$

is called a **chain complex** and

$$\cdots \xrightarrow{\delta^{k-2}} C^{k-1}(X) \xrightarrow{\delta^{k-1}} C^k(X) \xrightarrow{\delta^k} C^{k+1}(X) \xrightarrow{\delta^{k+1}} \cdots$$

is called a **cochain complex**.

Since, we are working with $\mathbb{Z}_2$-coeficients only, there is a very intuitive way to talk about chains (cochains, respectively). A k-chain is a linear combination of simplices with coefficients in $\mathbb{Z}_2$, so it can just be considered as a subset of the uniform k-skeleton of the underlying simplicial complex X. Furthermore, there is a one-to-one correspondence between chains and cochains, so to every chain $c \in C_k(X)$ we can associate its characteristic cochain we denote by $c^* \in C^k(X)$ and for every cochain $\varphi \in C^k(X)$ there exists a unique chain $c \in C_k(X)$, such that we have $c^* = \varphi$.
Let $c \in C_k(X)$ be some chain and $\varphi \in C^k(X)$ some cochain, then we denote the **evaluation** of φ on c as

$$\langle \varphi, c \rangle := \varphi(c) \in \mathbb{Z}_2,$$

and the **support** of φ as

$$supp(\varphi) := \{\sigma \in X : \langle \varphi, \sigma \rangle = 1\}$$

Furthermore, we define the support of a chain $c \in C_k(X)$ as $supp(c) := supp(c^*)$.

2.3 Cosystoles & Cheeger constants

Let X be a simplicial complex and $\varphi \in C^k(X)$, such that $\|\delta^{k-1}(\phi) + \varphi\| \geq \|\varphi\|$ for every $\phi \in C^{k-1}(X)$, where δ^{k-1} denotes the coboundary map $C^{k-1}(X) \to C^k(X)$, then we call φ a **cosystole**.
For general cochains $\varphi \in C^k(X)$ we define the **cosystolic norm** of φ by:

$$\|\varphi\|_{csy} := \min \left\{ \|\delta^{k-1}(\phi) + \varphi\| : \phi \in C^{k-1}(X) \right\}$$

Furthermore, any $c \in C^k(X)$, satisfying $c = \delta^{k-1}(\phi) + \varphi$ and $\|c\| = \|\varphi\|_{csy}$ is called a **cosystolic form** of φ.
The quotient

$$\|\varphi\|_{exp} := \frac{\|\delta^k(\varphi)\|}{\|\varphi\|_{csy}}$$

is called the **coboundary expansion** of φ and

$$h_k(X) := \min_{\varphi \in C^k(X),\, \delta^k(\varphi) \neq 0} \|\varphi\|_{exp}$$

is called the k-th **Cheeger constant** of X.
Furthermore, for any simplicial complex X and any $1 \leq k \leq dim(X)$, we define the following number (the largest norm, a cosystole can attain):

$$C_{max}(X,k) := \max \left\{ \|\varphi\|_{csy} : \varphi \in C^k(X) \right\}$$

2.4 Graphs & Hypergraphs

Let V be some set and $E \subseteq \binom{V}{2}$ (we will always use the notation $\binom{V}{k} := \left\{ S \in 2^V : |S| = k \right\}$ to denote the set of all subsets of cardinality k of a set V). Then the pair $G = (V, E)$ is called a **(simple) graph**, where the elements of V are called **vertices** and the elements of E are called **edges**. Since we only consider simple graphs (undirected graphs with no loops or double edges) in this thesis, we will just call them graphs. Even though we only consider undirected graphs, we want to stick to the common notation and denote an edge by $e = (v, w)$ instead of using set brackets $e = \{v, w\}$.
Note, that a simple graph can be considered as a 1-dimensional simplicial complex, where V is the 0-skeleton and E is the uniform 1-skeleton.
According to the terminology of simplicial complexes we call a graph **finite**, if the number of vertices is finite.
A graph $G = (V, E)$ is called **complete**, if $E = \binom{V}{2}$ and a graph $G' = (V', E')$ is called a **subgraph** of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.
Let $G = (V, E)$ be a graph, then $deg_G(v) := |\{w \in V : (v, w) \in E\}|$ is called the **degree** of the vertex $v \in V$.

A **hypergraph** is a pair $H = (V, E)$, where the edge set $E \subseteq 2^V$ can be any set of subsets of V. Note, that every simplicial complex is a hypergraph, but not every hypergraph is a simplicial complex, since subsets of an edge do not have to be an edge in a hypergraph. If all edges of a hypergraph have the same cardinality k, then we call it a k-**uniform hypergraph**. Analogously to the terminology of graphs, a hypergraph $H' = (V', E')$ is called a **subhypergraph** of the hypergraph $H = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

Chapter 3

Detecting cosystoles

When we want to determine whether a cochain is a cosystole or not until now we only have the original definition of cosystolicity, which does not seem to be very useful. In this chapter we want to develop tools to get hands on this problem and investigate the structure how cosystoles in certain simplicial complexes are arranged.

3.1 Piercing sets and the cycle detection theorem

The following definition is adopted from [6].

Definition 1. *Let V be some set and $\mathcal{F} \subseteq 2^V$ a family of finite subsets of V. A subset $P \subseteq V$ is called a **piercing set** of $\mathcal{F}$ if we have $P \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The minimal cardinality of a piercing set of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is called the **piercing number** of $\mathcal{F}$.*

Example 1. *Let $V := \{1, 2, 3, 4, 5\}$ and $\mathcal{F} := \{\{1, 2\}\{2, 3, 4\}, \{1, 5\}, \{2, 4, 5\}\}$, then we have $\tau(\mathcal{F}) = 2$, since for example $P := \{2, 5\}$ is a minimal piercing set of $\mathcal{F}$.*

Later we will often talk about piercing numbers and piercing sets of families of k-chains, which we define as follows:

Definition 2. *Let X be a simplicial complex and $\mathcal{F} \subseteq C_k(X)$ a family of k-chains. The **piercing sets** and the **piercing number** of $\mathcal{F}$ are defined as the piercing sets and the piercing number of the family $\{supp(F) : F \in \mathcal{F}\}$.*

In [6] Kozlov stated the following useful method to bound the cosystolic norm of a cochain.

Theorem 1 (The cycle detection theorem)**.** *Let X be a simplicial complex and $\varphi \in C^k(X)$. Let now $\mathcal{F} = \{\alpha_1, \ldots, \alpha_t\}$ be a family of k-cycles in $C_k(X)$, such that $\langle \varphi, \alpha_i \rangle = 1$ for all $1 \leq i \leq t$, then we have:*

$$\|\varphi\|_{csy} \geq \tau(\mathcal{F})$$

Proof. Let $\psi \in C^{k-1}(X)$, then for any $1 \leq i \leq t$ we have:

$$\begin{aligned}
\langle \varphi + \delta^{k-1}(\psi), \alpha_i \rangle &= \langle \varphi, \alpha_i \rangle + \langle \delta^{k-1}(\psi), \alpha_i \rangle \\
&= \langle \varphi, \alpha_i \rangle + \langle \psi, \partial_{k-1}(\alpha_i) \rangle \\
&= \langle \varphi, \alpha_i \rangle + \langle \psi, 0 \rangle \\
&= \langle \varphi, \alpha_i \rangle = 1
\end{aligned}$$

This means that we have $supp(\varphi + \delta^{k-1}(\psi)) \cap supp(\alpha_i) \neq \emptyset$ for all $1 \leq i \leq t$, so $supp(\varphi + \delta^{k-1}(\psi))$ is a piercing set of $\mathcal{F}$ and we get:

$$\|\varphi + \delta^{k-1}(\psi)\| = |supp(\varphi + \delta^{k-1}(\psi))| \geq \tau(\mathcal{F})$$

Since ψ was chosen arbitrarily we get the requested result. $\square$

The following corollary (a special case of the preceding theorem) was also stated by Kozlov in [6].

Corollary 1. *Let X be a simplicial complex and $\varphi \in C^k(X)$.*
Let now $\mathcal{F} = \{\alpha_1, \ldots, \alpha_{\|\varphi\|}\}$ be a family of k-cycles in $C_k(X)$, such that $\langle \varphi, \alpha_i \rangle = 1$ for all $1 \le i \le \|\varphi\|$ and $supp(\alpha_i) \cap supp(\alpha_j) = \emptyset$ for all $i \neq j$, then φ is a cosystole.

Proof. Since the supports of the cycles $\alpha_1, \ldots, \alpha_{\|\varphi\|}$ are pairwise disjoint, we obviously have $\tau(\mathcal{F}) = \|\varphi\|$ and using the cycle detection theorem we are done. □

To get the best possible results using the cycle detection theorem, the challenge is now for a certain cochain φ to find families of cycles such that they have a large piercing number and φ evaluates to 1 on every cycle. The following construction seems to be suited well to get hands on this problem.
For a cochain $\varphi \in C^k(X)$ we define the following set of cycles:

$$\mathcal{T}_\varphi := \left\{ \partial_k(\sigma) : \sigma \in supp(\delta^k(\varphi)) \right\}$$

Proposition 1. *Let $\varphi \in C^k(X)$, then we have:*

$$\|\varphi\|_{csy} \ge \tau(\mathcal{T}_\varphi)$$

Proof. By the definition of the coboundary map, we obviously have $\langle \varphi, \partial_k(\sigma) \rangle = 1$ for all $\partial_k(\sigma) \in \mathcal{T}_\varphi$ and so by the cycle detection theorem we are done. □

It seems to be very difficult to determine the piercing number of $\mathcal{T}_\varphi$ explicitely, but the concept of piercing complexes might be useful on the way to solve this problem.

3.2 Piercing complexes

Let V be some set, $\mathcal{F} \subset 2^V$ a family of subsets and P a piercing set of $\mathcal{F}$. Then for any $v \in V$ obviously $P \cup \{v\}$ is also a piercing set of $\mathcal{F}$. We can use this fact to construct a simplicial complex, which contains all information about the piercing sets for a given family of sets as follows:

Definition 3. *Let V be a set and $\mathcal{F} \subset 2^V$ a family of subsets. Then the* ***piercing complex*** *of $\mathcal{F}$ is defined as:*

$$\Delta_{\mathcal{F}} := \left\{ V' \subseteq V, : (V \setminus V') \cap F \neq \emptyset \text{ for all } F \in \mathcal{F} \right\}$$

So, $\Delta_{\mathcal{F}}$ consists of all subsets of V, such that their complements in V are piercing sets of $\mathcal{F}$ and indeed, $\Delta_{\mathcal{F}}$ defines a simplicial complex, since deleting an element from the complement of a piercing set is equivalent to adding an element to a piercing set, which preserves the condition of being a piercing set.

Example 2. *Let V be an arbitrary set and $\mathcal{F} := 2^V$ its power set. Then the piercing complex $\Delta_{\mathcal{F}}$ is empty, since even the complement of a single vertex $v \in V$ is not a piercing set of $\mathcal{F}$. More general for an arbitrary set V we have that $\Delta_{\mathcal{F}}$ is empty if and only if $\{v\} \in \mathcal{F}$, for all $v \in V$.*
On the other hand $\Delta_{\mathcal{F}}$ is a complete simplex on $|V|$ vertices if and only if $\mathcal{F}$ is empty, since only in this case even the empty set is a piercing set of $\mathcal{F}$.

We can now reformulate the question of determining the piercing number $\tau(\mathcal{F})$ by asking for the dimension of $\Delta_{\mathcal{F}}$, since we have the equality:

$$\tau(\mathcal{F}) = |V| - dim(\Delta_{\mathcal{F}}) - 1$$

Since our main interest in this section will be to investigate the piercing complex of $\mathcal{T}_\varphi$ for a given cochain $\varphi \in C^k(X)$ (where X is some simplicial complex) we will use a shorter notation for this piercing complex and set $\Delta_\varphi := \Delta_{\mathcal{T}_\varphi}$. Then the preceding formula turns to:

$$\tau(\mathcal{T}_\varphi) = |X^{(k)}| - dim(\Delta_\varphi) - 1$$

Theorem 2. *Let X be a simplicial complex and $\varphi \in C^k(X)$, then we have:*

$$\tilde{H}_i(\Delta_\varphi) \cong 0 \quad \textit{for all } i \leq k-1$$

Proof.

$$\begin{aligned}
& \varphi \in C^k(X) \\
\implies \ & \text{For all } \sigma \in supp(\delta^k(\varphi)) \text{ we have } |supp(\partial_k(\sigma))| = k+2 \\
\implies \ & \text{For all } S \subset X^{(k)} \text{ such that } |S| \leq k+1 \text{ we have that} \\
& X^{(k)} \setminus S \text{ is a piercing set of } \mathcal{T}_\varphi \\
\implies \ & \Delta_\varphi \text{ has a full } k\text{-skeleton} \\
\implies \ & \tilde{H}_i(\Delta_\varphi) \cong 0 \text{ for all } i \leq k-1
\end{aligned}$$

□

Definition 4. *Let X be a simplicial complex on the vertex set V. Then the simplicial complex*

$$X^\vee := \{\sigma \subseteq V : V \setminus \sigma \notin X\}$$

is called the ***Alexander dual*** *of X.*

The following theorem can be found in [8].

Theorem 3 (The Alexander duality theorem)**.** *Let X be a simplicial complex on n vertices and $X^\vee$ its Alexander dual. Then we have:*

$$\tilde{H}_i(X) \cong \tilde{H}^{n-i-3}(X)$$

Definition 5. *Let V be some set and $\mathcal{F} \subseteq 2^V$ a family of subsets of V. Then the simplicial complex*

$$\Delta[\mathcal{F}] := \{\sigma \subseteq V : \textit{there exists an } F \in \mathcal{F} \textit{ such that } \sigma \subseteq F\}$$

is called the ***induced complex*** *of $\mathcal{F}$.*

The following statement was developed in [9].

Proposition 2. *Let V be some set and $\mathcal{F} \subseteq 2^V$ a family of subsets of V. Then we have:*

$$\Delta\left[\bar{\mathcal{F}}\right]^\vee = \Delta_{\mathcal{F}}$$

where we set $\bar{\mathcal{F}} := \{V \setminus F : F \in \mathcal{F}\}$.

Proof. We have:

$$
\begin{aligned}
& \sigma \in \Delta\left[\bar{\mathcal{F}}\right]^{\vee} \\
\iff\ & V \setminus \sigma \notin \Delta\left[\bar{\mathcal{F}}\right] \\
\iff\ & \nexists F \in \bar{\mathcal{F}} : V \setminus \sigma \subseteq F \\
\iff\ & \nexists F \in \bar{\mathcal{F}} : (V \setminus \sigma) \cap (V \setminus F) = \emptyset \\
\iff\ & \nexists F' \in \mathcal{F} : (V \setminus \sigma) \cap F' = \emptyset \\
\iff\ & V \setminus \sigma \text{ is a piercing set of } \mathcal{F} \\
\iff\ & \sigma \in \Delta_{\mathcal{F}}
\end{aligned}
$$

□

Theorem 4. *Let X be a finite simplicial complex and $\varphi \in C^k(X)$, then we have:*

$$\tilde{H}_k(\Delta_\varphi) \cong 0$$

Proof. For all $\partial_k(\sigma) \in \mathcal{T}_\varphi$ we have $|supp(\partial_k(\sigma))| = k+2$, so we get:

$$dim(\Delta[\bar{\mathcal{T}}_\varphi]) = |X^{(k)}| - (k+2) - 1 = |X^{(k)}| - k - 3$$

Now, there exist no two simplices of dimension $|X^{(k)}| - k - 3$ in $\Delta[\bar{\mathcal{T}}_\varphi]$, such that they have a face in common, so we have:

$$\tilde{H}_{|X^{(k)}|-k-3}(\Delta[\bar{\mathcal{T}}_\varphi]) \cong 0$$

By the Alexander duality theorem and Proposition 2 we get:

$$\tilde{H}_k(\Delta_\varphi) \cong \tilde{H}^k(\Delta_\varphi) = \tilde{H}^{|X^{(k)}|-(|X^{(k)}|-k-3)-3}(\Delta_\varphi) \cong 0,$$

where the first isomorphy is true, because we consider homology / cohomology over a field and Δ_φ is finite. □

3.3 Large cosystoles of a simplex

In this section we focus our investigations onto the question, what is the largest norm, a cosystole can attain when the underlying simplicial complex is a complete simplex, where the first statement we make is still valid for general simplicial complexes.

Lemma 1. *If k is odd, then we have:*

$$C_{max}(X,k) \geq \tau(\mathcal{T}_\varphi),$$

with $\varphi = \left(\sum_{\sigma \in X^{(k)}} \sigma\right)^*$.

Proof. Since k is odd we have $\langle \varphi, c \rangle = 1$ for all $c \in \mathcal{T}_\varphi$, so by the cycle detection theorem we get $\|\varphi\|_{csy} \geq \tau(\mathcal{T}_\varphi)$ and we are done. □

A direct consequence of the preceding lemma is the following estimate. For simplicity we want to introduce the notation $\mathcal{T}_n^k := \mathcal{T}_\varphi$ for $\varphi := \left(\sum_{\sigma \in \binom{[n]}{k+1}} \sigma\right)^*$.

Proposition 3. *Let k be odd, then we have:*

$$C_{max}(\Delta^{[n]}, k) \geq \left\lceil \frac{\binom{n}{k+2}}{n-k-1} \right\rceil$$

Proof. Obviously, we have $|\mathcal{T}_n^k| = \binom{n}{k+2}$, so since any simplex $\sigma \in \binom{[n]}{k+1}$ intersects the support of exactly $n-k-1$ cycles from $\mathcal{T}_n^k$, any piercing set of $\mathcal{T}_n^k$ must contain at least $\left\lceil \frac{\binom{n}{k+2}}{n-k-1} \right\rceil$ elements and by Lemma 1 we are done. □

Using the preceding lemma we can determine a lower bound of the maximal size of the 1-dimensional cosystoles in a simplex. A more elementary proof of this estimate is given as Proposition 8) in the following chapter, where we will even see, that equality can be reached, shown in Theorem 8.

Theorem 5. $C_{max}(\Delta^{[n]}, 1) \geq \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor$

Proof. Asking for the smallest piercing set of $\mathcal{T}_n^1$ is equivalent to asking for the largest triangle-free graph (i.e. a graph on n vertices, containing as many edges as posssible, but no complete graph on 3 vertices as a subgraph) and taking the complement. Mantel's theorem (see [7]) says, that a triangle-free graph on n vertices has at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, so we immediately get:

$$\tau(\mathcal{T}_n^1) = \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor$$

and by Lemma 1 we are done. □

Unfortunately, determining the piercing number of $\mathcal{T}_n^k$ for $k \geq 2$, or equivalently, determining the largest k-uniform hypergraph on n-vertices, containing no complete k-uniform hypergraph on $k+2$ vertices as a subhypergraph, seems to be very difficult (see [7]), so we can not use the preceding procedure to say something about $C_{max}(\Delta^{[n]}, k)$ for larger k's in general.
Eventhough, we can exactly determine this number for the ultimate and the penultimate proper dimension as follows.

Theorem 6. $C_{max}(\Delta^{[n]}, n-2) = 1$*, for all* $n \geq 3$

Proof. Let $S \in \binom{[n]}{n-1}$ be chosen arbitrarily, $\varphi := S^* \in C^{n-2}(\Delta^{[n]})$ and $\mathcal{F} := \{\alpha\}$, where α is the boundary of the single $(n-1)$-dimensional simplex in $\Delta^{[n]}$. Obviously, we have $\langle \varphi, \alpha \rangle = 1$, since $supp(\varphi) \cap supp(\alpha) = supp(\varphi)$ and $\tau(\mathcal{F}) = 1$, so by the cycle detection theorem we have $\|\varphi\|_{csy} \geq 1$.
Now, let $S_1, S_2 \in \binom{[n]}{n-1}$ be chosen arbitrarily again ($S_1 \neq S_2$) and $C := S_1 \cap S_2$. Then we have $\delta^{n-3}(C^*) + S_1^* + S_2^* = 0$, so there exists no $(n-2)$-cosystole attaining norm 2 and we are done. □

Lemma 2. *For* $n \geq 4$ *we have:*

$$\tau(\mathcal{T}_n^{n-3}) = \left\lceil \frac{n}{2} \right\rceil$$

Proof. For each $\sigma \in \binom{[n]}{n-2}$ there exist exactly two cycles $\alpha_1, \alpha_2 \in \mathcal{T}_n^{n-3}$ ($\alpha_1 \neq \alpha_2$), such that $\sigma \in supp(\alpha_1) \cap supp(\alpha_2)$, so the largest possible number of cycles from $\mathcal{T}_n^{n-3}$ that can be pierced by one simplex is two. Furthermore, we have $\left|\mathcal{T}_n^{n-3}\right| = \binom{n}{n-1} = n$, so we get $\tau(\mathcal{T}_n^{n-3}) \geq \lceil \frac{n}{2} \rceil$.
On the other hand, for all $\alpha_1, \alpha_2 \in \mathcal{T}_n^{n-3}$ there exists a $\sigma \in \binom{[n]}{n-2}$, such that $\sigma \in supp(\alpha_1) \cap supp(\alpha_2)$, so we get $\tau(\mathcal{T}_n^{n-3}) \leq \lceil \frac{n}{2} \rceil$. □

Lemma 3. *Let $S \subset \binom{[n]}{n-2}$, such that $|S| \geq \lfloor \frac{n}{2} \rfloor + 1$, then there exist $\sigma, \sigma' \in S$ ($\sigma \neq \sigma'$), such that $|\sigma \cap \sigma'| = n - 3$.*

Proof. For $\sigma, \sigma' \in \binom{[n]}{n-2}$ the condition $|\sigma \cap \sigma'| < n - 3$ is equivalent to the condition $([n] \setminus \sigma) \cap ([n] \setminus \sigma') = \emptyset$. Since we obviously have $|[n] \setminus \sigma| = 2$ for all $\sigma \in \binom{[n]}{n-2}$ we can find at most $\lfloor \frac{n}{2} \rfloor$ simplices $\sigma_1, \ldots, \sigma_{\lfloor \frac{n}{2} \rfloor} \in \binom{[n]}{n-2}$, such that the sets $[n] \setminus \sigma_1, \ldots, [n] \setminus \sigma_{\lfloor \frac{n}{2} \rfloor}$ are pairwise disjoint and we are done. □

Theorem 7. $C_{max}(\Delta^{[n]}, n-3) = \lfloor \frac{n}{2} \rfloor$*, for all* $n \geq 4$

Proof. Let $S \subset \binom{[n]}{n-2}$ be a minimal piercing set of $\mathcal{T}_n^{n-3}$ as constructed in the proof of Lemma 2 and $\varphi := S^* \in C^{n-3}(\Delta^{[n]})$.
If n is even we have $\langle \varphi, \alpha \rangle = 1$ for all $\alpha \in \mathcal{T}_n^{n-3}$, so we immediately get $C_{max}(\Delta^{[n]}, n - 3) \geq \tau(\mathcal{T}_n^{n-3}) = \frac{n}{2}$ by Lemma 2 and the cycle detection theorem.
If n is odd there exists exactly one $\alpha \in \mathcal{T}_n^{n-3}$, such that $\langle \varphi, \alpha \rangle = 0$, since $|supp(\alpha) \cap supp(\varphi)| = 2$. Let $\sigma \in supp(\alpha) \cap supp(\varphi)$ be one of the two simplices in $supp(\alpha) \cap supp(\varphi)$. Now set $\varphi' := (S \setminus \sigma)^* \in C^{n-3}(\Delta^{[n]})$, then we have $\langle \varphi', \alpha \rangle = 1$, but there exists exactly one $\alpha' \in \mathcal{T}_n^{n-3}$, such that $\langle \varphi', \alpha' \rangle = 0$, since $|supp(\varphi') \cap supp(\alpha')| = 0$. Set $\mathcal{F} := \mathcal{T}_n^{n-3} \setminus \alpha'$, then we have $\langle \varphi', \alpha \rangle = 1$ for all $\alpha \in \mathcal{F}$ and $\tau(\mathcal{F}) = \tau(\mathcal{T}_n^{n-3}) - 1 = \lceil \frac{n}{2} \rceil - 1 = \lfloor \frac{n}{2} \rfloor$.
by Lemma 2 and by the cycle detection theorem we have $C_{max}(\Delta^{[n]}, n - 3) \geq \lfloor \frac{n}{2} \rfloor$.
One the other hand let $\varphi \in C^{n-3}(\Delta^{[n]})$, such that $\|\varphi\| = \lfloor \frac{n}{2} \rfloor + 1$, then by Lemma 3 there exist $\sigma_1, \sigma_2 \in supp(\varphi)$, such that $|\sigma_1 \cap \sigma_2| = n - 3$. Now set $\psi := (\sigma_1 \cap \sigma_2)^* \in C^{n-4}(\Delta^{[n]})$, then we have $\|\delta^{n-4}(\psi)\| = 3$ and $\left|supp(\delta^{n-4}(\psi)) \cap supp(\varphi)\right| \geq 2$. Thus, $\|\delta^{n-4}(\psi) + \varphi\| \leq \|\varphi\| - 1$ and φ can not be a cosystole, so we have $C_{max}(\Delta^{[n]}, n - 3) \leq \lfloor \frac{n}{2} \rfloor$ and we are done. □

3.4 Multi-suspensions

Definition 6. *Let $d \geq 1$, then we call*

$$\begin{aligned} sus^d_{n,k} : C^k(\Delta^{[n]}) &\longrightarrow C^{k+1}(\Delta^{[n+d]}) \\ \varphi &\longmapsto \left(\sum_{m=n+1}^{d} \left(\sum_{\sigma \in supp(\varphi)} (\sigma, m) \right) \right)^* \end{aligned}$$

*the **suspension map of degree d**. Note, that $sus^d_{n,k}$ can also be easily defined on chain complexes by:*

$$\begin{aligned} sus^d_{n,k} : C_k(\Delta^{[n]}) &\longrightarrow C_{k+1}(\Delta^{[n+d]}) \\ c &\longmapsto \sum_{\sigma \in supp(sus^d_{n,k}(c^*))} \sigma \end{aligned}$$

Lemma 4. *Let $\varphi \in C^k(\Delta^{[n]})$ and $\mathcal{F} = \{\alpha_1, \dots, \alpha_t\} \subset C_k(\Delta^{[n]})$ be a family of cycles, such that $\langle \varphi, \alpha_i \rangle = 1$ for all $i = 1, \dots, t$. Then there exists a family of cycles $\mathcal{F}' = \left\{\alpha'_{1,1}, \dots, \alpha'_{t,d}\right\} \subset C_{k+1}(\Delta^{[n+d]})$, such that $\left\langle sus^d_{n,k}(\varphi), \alpha'_{i,j} \right\rangle = 1$ for all $i = 1, \dots, t$ and $j = 1, \dots, d$.*

Proof. For each $i = 1, \dots, t$ let $c_i \in C_{k+1}(\Delta^{[n]})$, such that $\partial_k(c_i) = \alpha_i$. For a simplex $\sigma \in \binom{[n]}{k+1}$ and some $n+1 \leq j \leq n+d$ we have

$$\partial_k\left((\sigma, j)\right) = \left(\sum_{\sigma' \in supp(\partial_{k-1}(\sigma))} (\sigma', j)\right) + \sigma$$

So, we have $\partial_k \left(\sum_{\sigma \in supp(\alpha_i)} (\sigma, j)\right) = \alpha_i$, since $\partial_{k-1}(\alpha_i) = 0$ for all $i = 1, \dots, t$ and $j = n+1, \dots, n+d$. Thus, for all $i = 1, \dots, t$ and $j = n+1, \dots, n+d$,

$$\alpha_{i,j} := \sum_{\sigma \in supp(\alpha_i)} (\sigma, j) + c_i$$

defines a cycle (since we have $\partial_k \left(\sum_{\sigma \in supp(\alpha_i)} (\sigma, j) + c_i\right) = \alpha_i + \alpha_i = 0$) which can be naturally embedded into $C_{k+1}(\Delta^{[n+d]})$. Furthermore, we have:

$$\begin{aligned}
\left\langle sus^d_{n,k}(\varphi), \alpha_{i,j} \right\rangle &= \left\langle \left(\sum_{m=n+1}^{d} \left(\sum_{\sigma \in supp(\varphi)} (\sigma, m)\right)\right)^*, \sum_{\sigma \in supp(\alpha_i)} (\sigma, j) + c_i \right\rangle \\
&= \left\langle \left(\sum_{\sigma \in supp(\varphi)} (\sigma, j)\right)^*, \sum_{\sigma \in supp(\alpha_i)} (\sigma, j) + c_i \right\rangle \\
&= 1,
\end{aligned}$$

since we have $\langle \varphi, \alpha_i \rangle = \left\langle \left(\sum_{\sigma \in supp(\varphi)} \sigma\right)^*, \sum_{\sigma \in supp(\alpha_i)} \sigma \right\rangle = 1$. □

Definition 7. *Let S be some set, $\mathcal{F} \subseteq 2^S$ a family of subsets of S and $P = (p_1, \dots, p_m)$ a finite ordered piercing set of $\mathcal{F}$. Furthermore, set $P_0 := \emptyset$ and for each $i = 1, \dots, m$ set $P_i := \{F \in \mathcal{F} : F \cap \{p_i\} \neq \emptyset\} \setminus P_{i-1}$. Then the tuple $\lambda_P := (|P_1|, \dots, |P_m|)$ is called the* ***piercing sequence*** *of P.*

Proposition 4. *Let $\varphi \in C^k(\Delta^{[n]})$ and $\mathcal{F} = \{\alpha_1, \dots, \alpha_t\} \subset C_k(\Delta^{[n]})$ be a family of cycles, such that $\langle \varphi, \alpha_i \rangle = 1$, for all $i = 1, \dots, t$ and $P = (p_1, \dots, p_m)$ an ordered piercing set of $\mathcal{F}$, such that $\tau(\mathcal{F}) = m$. Then for any $d \geq 1$ we have:*

$$\|sus^d_{n,k}(\varphi)\|_{csy} \geq d \cdot |\{\beta \in \lambda_P : \beta \geq d\}| + \sum_{\beta \in \lambda_P,\, \beta < d} \beta$$

Proof. □

3.5 A combinatorial perspective

3.5.1 Cosystolic sets and boundary isomorphisms

Lemma 5. *Let $S \subset \Delta^{[n]}(k-1)$, then we have:*

1. $|\delta(S^C)| \leq |S|(n-k)$, *for k odd*
2. $|\delta(S^C)| \geq \binom{n}{k+1} - |S|(n-k)$, *for k even*

Proof. In general we obviously have $|\delta(S)| \leq |S|(n-k)$ and the result follows directly from the preceding remark. □

Definition 8. *Let $1 \leq k \leq n-1$, then the* ***cosystolic complex*** *$\mathcal{C}^k(n)$ is defined as follows:*

- *The elements of $\Delta^{[n]}(k)$ determine the vertices of $\mathcal{C}^k(n)$.*
- *A set of vertices forms a simplex of $\mathcal{C}^k(n)$, if it corresponds to a cosystolic set.*

$\mathcal{C}M(n)$ then conincides with the complex $\mathcal{C}^1(n)$.

Definition 9. *Let $t \geq 0$, then $C \subset \Delta^{[n]}(k)$ and $C' \subset \Delta^{[n+t]}(k+t)$ are called* ***boundary isomorphic****, if there exists a bijection $\varphi : C \to C'$, such that for all $D \subset C$, satisfying $|D| \geq 2$ we have:*

$$\left| \bigcap_{\sigma \in D} \partial(\sigma) \right| = \left| \bigcap_{\sigma \in D} \partial(\varphi(\sigma)) \right|$$

Then we call φ a ***boundary isomorphism****.*

Proposition 5. *If C and C' are boundary isomorphic, then $\delta(C)$ and $\delta(C')$ are boundary isomorphic.*

Proof. Let $C \subset \Delta^{[n]}(k)$ and $S \in \Delta^{[n]}(k+1)$, such that S has an odd number of faces in C and let F be the set consisting of those faces.
We will first consider the case, when we have $|F| > 2$. Let $\sigma, \sigma' \in F$, such that $\sigma \neq \sigma'$. Then we have $|\partial(\sigma) \cap \partial(\sigma')| = 1$, since σ and σ' belong to the same simplex S. By boundary isomorphy we get $|\partial(\varphi(\sigma)) \cap \partial(\varphi(\sigma'))| = 1$ (where φ denotes the boundary isomorphism) and so $\varphi(\sigma)$ and $\varphi(\sigma')$ are faces of the same simplex S', uniquely determined by S. We still have to show, that there can not exist more faces of S' in C' than those from $\varphi(F)$. Suppose, there exists a face $\tau \in C'$ of S', such that $\tau \neq \varphi(\sigma)$ for all $\sigma \in F$. Then we have $|\partial(\tau) \cap \partial(\varphi(\sigma))| = 1$ for all $\sigma \in F$. It follows by boundary isomophy, that $|\partial(\varphi^{-1}(\tau)) \cap \partial(\sigma)| = 1$ for all $\sigma \in F$, so $\varphi^{-1}(\sigma) \in C$ is another face of S, distinct from the others, but this is a contradiction to $\varphi^{-1}(\tau) \notin F$.
Thus, the number of faces of S in C equals the number of faces of S' in C' and furthermore there is a one-to-one correspondence between the simplices having an odd (> 2) number of faces in C and the simplices having an odd (> 2) number of faces in C' by the bijectivity of φ.
Now consider the case, when we have $|F| = 1$. By the same arguments as in the first case, there is also a one-to-one correspondence between the simplices having more than one face in C and the simplices having more than one face in C'. Let $F := \{\sigma\}$ and m be the number of simplices having σ and at least one more element from C as a face. By the preceding correspondence m equals the number of simplices having $\varphi(\sigma)$ and at least one more element from C' as a face. Hence, the number of simplices having only σ as a face in C equals $n - (k+1) - m$ and the number of simplices having only $\varphi(\sigma)$ as a face in C' equals $n + t - (k + t + 1) - m$ for some $t \in \mathbb{Z}$, but these

two numbers equal for all t and we get $|\delta(C)| = |\delta(C')|$ in general.
Furthermore, the preceding construction induces a bijection $\phi : \delta(C) \to \delta(C')$, which turnes out to be a boundary isomorphism. Let $D \subset C$, such that $|D| \geq 2$ and all $\sigma \in D$ share a common face. Then by the construction of ϕ all simplices $\phi(\sigma)$ must share a common face again. On the other hand, if the boundaries of the simplices in D are distinct, the boundaries of their images under ϕ are as well. □

We conjecture, that boundary isomorphy even preserves cosystolicity but until now it seems pretty difficult to say anything more about boundary isomorphic sets in general. Let us instead study a certain boundary isomorphism in particular.

3.5.2 Coning of cochains

Consider the following so called **coning** map:

$$\begin{aligned} \varepsilon : \Delta^{[n]}(k) &\longrightarrow \Delta^{[n+1]}(k+1) \\ \sigma &\longmapsto (\sigma, n+1), \end{aligned}$$

Lemma 6. *Let $C \subset \Delta^{[n]}(k)$, then C is boundary isomorphic to $\varepsilon(C)$.*

Proof. Obviously, ε is bijective. Now consider some set $D \subset C$, satisfying $|D| \geq 2$. Then we have:

$$\left| \bigcap_{\sigma \in D} \partial(\varepsilon(\sigma)) \right| = \left| \bigcap_{\sigma \in D} \partial((\sigma, n+1)) \right| = \left| \bigcap_{\sigma \in D} \partial(\sigma) \right|,$$

where the second equation is valid, since if two simplices $\sigma, \sigma' \in D$ contain the same face, then adding a vertex preserves this property in one dimension higher, just as deleting a vertex preserves it in one dimension lower. □

Lemma 7. *Let $C \subset \Delta^{[n]}(k)$ and $C' \subset \Delta^{[n+1]}(k+1)$ be boundary isomorphic, then C' is boundary isomorphic to $C'' := \varepsilon(C)$.*

Proof. Obviously, boundary isomorphy is an equivalence relation, so by Lemma 6 we are done. □

Let us now switch to the algebraic situation to study ε more intensively. Consider the following diagram of cochain complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{k-1}(\Delta^{[n]}, \mathbb{Z}_2) \arrow[r, "\delta"] \arrow[d, "i"] \arrow[dr, "\varepsilon"] & C^{k}(\Delta^{[n]}, \mathbb{Z}_2) \arrow[r, "\delta"] \arrow[d, "i"] \arrow[dr, "\varepsilon"] & C^{k+1}(\Delta^{[n]}, \mathbb{Z}_2) \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & C^{k-1}(\Delta^{[n+1]}, \mathbb{Z}_2) \arrow[r, "\delta"] & C^{k}(\Delta^{[n+1]}, \mathbb{Z}_2) \arrow[r, "\delta"] & C^{k+1}(\Delta^{[n+1]}, \mathbb{Z}_2) \arrow[r] & \cdots
\end{tikzcd}$$

FIGURE 3.1: Cochain complexes with coning maps

Here i is induced by the natural inclusion $\Delta^{[n]} \to \Delta^{[n+1]}$ and the coning map can be again translated from the combinatorial version by $\varepsilon(c) = supp^{-1}(\varepsilon(supp(c)))$. Another equivalent way to define the coning map algebraically is by $\varepsilon = \delta i + i\delta$. Note, that the maps i and ε are both norm preserving.

Lemma 8. *According to the maps defined above, we have the following relations:*

1. $supp(i\delta(c)) \subseteq supp(\delta i(c))$, *for any cochain c*
2. $\delta\varepsilon = \varepsilon\delta$
3. $\delta\varepsilon = \delta i\delta$
4. $\|\varepsilon(c)\| = \|\delta i(c)\| - \|i\delta(c)\|$, *for any cochain c*

Proof. (1) The support of $i\delta(c)$ consists of all simplices from $\Delta^{[n]}$ (in the appropriate dimension), which contain an odd number of simplices from $supp(c)$ as a face. These are clearly contained in the set of simplices from $\Delta^{[n+1]}$, which contain an odd number of simplices from $supp(c)$ as a face.
(2) $\delta\varepsilon = \delta(\delta i + i\delta) = \delta\delta i + \delta i\delta = \delta i\delta = i\delta\delta + \delta i\delta = (i\delta + \delta i)\delta = \varepsilon\delta$.
(3) Is already contained in the proof of (2).
(4) Follows immediately from (1). □

Proposition 6. *Let $c \in C^k(\Delta^{[n]}, \mathbb{Z}_2)$ not be a cosystole, then $\varepsilon(c)$ is not a cosystole.*

Proof. Let $c \in C^k(\Delta^{[n]}, \mathbb{Z}_2)$ not be a cosystole. Then there exists a cochain $d' \in C^{k-1}(\Delta^{[n]}, \mathbb{Z}_2)$, such that $\|\delta(d') + c\| < \|c\|$ and so, defining $d := \varepsilon(d')$ and using Lemma 8 (2) we get:

$$\begin{aligned}
\|\delta(d) + \varepsilon(c)\| &= \|\delta(\varepsilon(d')) + \varepsilon(c)\| \\
&= \|\varepsilon(\delta(d') + c)\| \\
&= \|\delta(d') + c\| \\
&< \|c\| = \|\varepsilon(c)\|
\end{aligned}$$

□

We conjecture that the reverse is also true and we can already prove a weaker statement. Before, consider the following small but important observation.
Note, that any cochain $d \in C^k(\Delta^{[n+1]}, \mathbb{Z}_2)$ can uniquely be represented as $d = \varepsilon(d_1) + i(d_2)$, with $d_1 \in C^{k-1}(\Delta^{[n]}, \mathbb{Z}_2)$ and $d_2 \in C^k(\Delta^{[n]}, \mathbb{Z}_2)$. Regarding the proof of Proposition 6, we see that if we write $d = \varepsilon(d')$ as $\varepsilon(d_1) + i(d_2)$, the second part $i(d_2)$ vanishes, espacially we have $\delta(d_2) = 0$. The following statement shows, that if we could always construct a $d = \varepsilon(d_1) + i(d_2)$ satisfying $\|\delta(d) + \varepsilon(c)\| < \|\varepsilon(c)\|$ and $\delta(d_2) = 0$, then the reverse of the preceding Proposition would be true as well.

Proposition 7. *Let $\varepsilon(c) \in C^{k+1}(\Delta^{[n+1]}, \mathbb{Z}_2)$ not be a cosystole. If there exists a cochain $d = \varepsilon(d_1) + i(d_2) \in C^k(\Delta^{[n+1]}, \mathbb{Z}_2)$, such that $\|\delta(d) + \varepsilon(c)\| < \|\varepsilon(c)\|$ and $\delta(d_2) = 0$, then c is not a cosystole.*

Proof. Let $d = \varepsilon(d_1) + i(d_2) \in C^k(\Delta^{[n+1]}, \mathbb{Z}_2)$ satisfy the assumptions. Then there exists a $d_2' \in C^{k-1}(\Delta^{[n]}, \mathbb{Z}_2)$, such that $d_2 = \delta(d_2')$. This is because $\Delta^{[n]}$ is contractible, so cohomology vanishes and kernel and image of δ coinside. Now we have:

$$\begin{aligned}\|\delta(d_1 + d_2') + c\| &= \|\delta(d_1) + d_2 + c\| \\ &= \|\varepsilon(\delta(d_1) + d_2 + c)\| \\ &= \|\delta\varepsilon(d_1) + \varepsilon(d_2) + \varepsilon(c)\| \\ &= \|\delta\varepsilon(d_1) + \delta i(d_2) + \varepsilon(c)\| \\ &= \|\delta(\varepsilon(d_1) + i(d_2)) + \varepsilon(c)\| \\ &< \|\varepsilon(c)\| = \|c\|\end{aligned}$$

Hence, c is not a cosystole. □

Now we have to deal with the following challenge. If we have some cochain $d = \varepsilon(d_1) + i(d_2)$, satisfying $\|\delta(d) + \varepsilon(c)\| < \|\varepsilon(c)\|$, but unfortunately not satisfying $\delta(d_2) = 0$, we have to find a better "substitude" for d, which does the same job, meaning we have to find a $d' = \varepsilon(d_1') + i(d_2')$, satisfying $\|\delta(d') + \varepsilon(c)\| < \|\varepsilon(c)\|$ and $\delta(d_2') = 0$, but until now this seems pretty difficult to find.
The central problem which makes it difficult to deal with cosystoles in general is, that compared with cut-minimal graphs we have almost no handy tools beside the original definition yet to show cosystolicity of a given cochain.

Chapter 4

Cut-minimal graphs and Cheeger graphs

In this chapter we start from the work of Kozlov (see [1]) in which a graph theoretical approach to the first Cheeger constant of a simplex was developed. In the course of this approach the so called cut-minimal graphs appeared, which exactly describe first dimensional cosystoles in a very intuitive way. Beside the benefit for the research on Cheeger constants, we think that exploring cut-minimal graphs could provide interesting structures which might be enlightening and helpful in other braches of combinatorics and topology as well. As a consequence of the first main result of this chapter (Theorem 8) we will determinine the dimension and partly the homology of a simplicial complex, which contains all information about the cut-minimal graphs on a certain number of vertices. In the second part of this chapter, we face the research on the first Cheeger constant of a simplex by investigating combinatorial properties of the Cheeger graphs which are exactly those cut-minimal graphs that determine this constant.

4.1 Cut-minimal graphs

4.1.1 Basic definitions and properties

The following definition and some words about motivation and intuition for it can be found in [1].

Definition 10. *Consider a graph $G = ([n], E)$. For any subsets $A, B \subset [n]$, define:*

$$E_G(A, B) := \{(v, w) \in E \,:\, v \in A, w \in B\}$$

and

$$NE_G(A, B) := \{(v, w) \notin E \,:\, v \in A, w \in B\}$$

*A graph $G = ([n], E)$ is called **cut-minimal**, if for every $S \subset [n]$ we have*

$$|E_G(S, [n] \setminus S)| \leq |NE_G(S, [n] \setminus S)|,$$

which is equivalent to

$$|E_G(S, [n] \setminus S)| \leq \frac{|S|(n - |S|)}{2}.$$

Note, that there is a one-to-one correspondence between the graphs on n vertices and the 1-chains (more precisely the elements of $C_1(\Delta^{[n]})$) as follows:
For a graph $G = ([n], E)$ set $c_G := \sum\limits_{e \in E} e \in C_1(\Delta^{[n]})$ and for a chain $c \in C_1(\Delta^{[n]})$ set $G_c := ([n], E)$, with $E := supp(c)$. Considering characteristic cochains we also

get a one-to-one corresponsing between graphs on n vertices and 1-cochains and we observe the following relation:

Lemma 9. *A graph $G = ([n], E)$ is cut-minimal if and only if the corresponding cochain c_G^* is a cosystole.*

Proof. □

Remark 1. *In fact for a graph G to be cut-minimal we only need to require the preceding condition holding for all $S \subset [n]$, such that $1 \leq |S| \leq \frac{n}{2}$, since for all $S \subset [n]$ we have $E_G(S, [n] \setminus S) = E_G([n] \setminus S, S)$ and $NE_G(S, [n] \setminus S) = NE_G([n] \setminus S, S)$.*

Example 3. *A graph $G = ([n], E)$ forming a circle by the edge set $E := \{(i, i+1) \ : \ 1 \leq i \leq n-1\} \cup \{(n, 1)\}$ is cut-minimal for all $n \geq 7$ as follows. One can easily see that for all $S \subset [n]$, such that $|S| \leq \frac{n}{2}$, we have $|E_G(S, [n] \setminus S)| \leq 2|S|$ and the inequality $2|S| \leq \frac{|S|(n-|S|)}{2}$ holds for all $n \geq |S| + 4$, so by $|S| \leq \frac{n}{2}$ the statement is true for all $n \geq 7$.*

Definition 11. *For any $n, k \in \mathbb{N}$ we define the set of all cut-minimal graphs on n vertices:*

$$CMG(n) := \{G = ([n], E) \ : \ G \text{ is cut-minimal}\},$$

and its subsets of all cut-minimal graphs on n vertices and k edges:

$$CMG_k(n) := \{G = ([n], E) \ : \ G \text{ is cut-minimal and } |E| = k\}.$$

4.1.2 Maximal cut-minimal graphs

In [1] there was the simplicial complex $\mathcal{C}^1(n)$ introduced, constructed as follows:

- The unordered pairs (i, j) (where $i, j \in [n], i \neq j$) form the vertices of $\mathcal{C}^1(n)$.
- A set of vertices forms a simplex of $\mathcal{C}^1(n)$, if and only if the corresponding graph is cut-minimal.

Later we will introduce the more general complex $\mathcal{C}^k(n)$ (see Definition 8).
The complex $\mathcal{C}^1(n)$ contains a lot of information about the cut-minimal graphs on a certain number of vertices and it might be useful to study its topological and simplicial structure.
The first thing to note is that the dimension of $\mathcal{C}^1(n)$ is obviously one more than the maximal number of edges a cut-minimal graphs can have which obviously coincides with the number $C_{max}(\Delta^{[n]}, 1)$ by Lemma 9.
Let us now determine the number $C_{max}(\Delta^{[n]}, 1)$ explicitly.

Proposition 8. $C_{max}(\Delta^{[n]}, 1) \geq \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$

Proof. Let us construct a graph G on n vertices as follows. Choose a set $V' \subset [n]$, such that $|V'| = \lceil \frac{n}{2} \rceil$ and connect each pair of vertices from V' by an edge. Then connect each pair of the remaining $\lfloor \frac{n}{2} \rfloor$ vertices by an edge. In other words our graph consists of two complete graphs. If n is even, they are identical, otherwise they differ by one vertex. In total we get $\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$ edges. We will show that this

graph is cut-minimal.
Let $S \subset [n]$ and define $A := S \cap V'$ and $B := S \setminus A$. If n is even, we have:

$$\begin{aligned}|E_G(S,[n]\setminus S)| &= |A|(\frac{n}{2}-|A|)+|B|(\frac{n}{2}-|B|)\\ &= \frac{n|A|}{2}-|A|^2+\frac{n|B|}{2}-|B|^2\\ &= \frac{n|S|}{2}-(|A|^2+|B|^2)\\ &\leq \frac{n|S|}{2}-\frac{|S|^2}{2}\\ &= \frac{|S|(n-|S|)}{2},\end{aligned}$$

where the inequality comes from:

$$\begin{aligned} & (|A|-|B|)^2 \geq 0\\ \iff\quad & |A|^2-2|A||B|+|B|^2 \geq 0\\ \iff\quad & \frac{|A|^2}{2}-|A||B|+\frac{|B|^2}{2}\geq 0\\ \iff\quad & |A|^2+|B|^2 \geq \frac{|A|^2}{2}+|A||B|+\frac{|B|^2}{2}\\ \iff\quad & |A|^2+|B|^2 \geq \frac{|S|^2}{2}.\end{aligned}$$

If n is odd, we have:

$$\begin{aligned}|E_G(S,[n]\setminus S)| &= |A|(\frac{n+1}{2}-|A|)+|B|(\frac{n-1}{2}-|B|)\\ &= \frac{n|A|}{2}+\frac{|A|}{2}-|A|^2+\frac{n|B|}{2}-\frac{|B|}{2}-|B|^2\\ &= \frac{n|S|}{2}-(|A|^2+|B|^2-\frac{|A|-|B|}{2})\\ &\leq \frac{n|S|}{2}-\frac{|S|^2}{2}\\ &= \frac{|S|(n-|S|)}{2},\end{aligned}$$

where the inequality comes from:

$$\begin{aligned} & (|A|-|B|)^2-(|A|-|B|)\geq 0\\ \iff\quad & (|A|-|B|)^2-|A|+|B|\geq 0\\ \iff\quad & |A|^2+|B|^2-|A|+|B|\geq 2|A||B|\\ \iff\quad & \frac{|A|^2}{2}+\frac{|B|^2}{2}-\frac{|A|}{2}+\frac{|B|}{2}\geq |A||B|\\ \iff\quad & |A|^2+|B|^2-\frac{|A|}{2}+\frac{|B|}{2}\geq \frac{|A|^2}{2}+\frac{2|A||B|}{2}+\frac{|B|^2}{2}\\ \iff\quad & |A|^2+|B|^2-\frac{|A|-|B|}{2}\geq \frac{(|A|+|B|)^2}{2}\\ \iff\quad & |A|^2+|B|^2-\frac{|A|-|B|}{2}\geq \frac{|S|^2}{2}.\end{aligned}$$

Hence, the constructed graph is cut-minimal. $\square$

Remark 2. *For further proofs and calculations it might be helpful to keep mind that we have:*

$$\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} = \frac{n^2 - 2n + 1}{4}, \qquad \textit{for } n \textit{ odd, and}$$
$$\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} = \frac{n^2 - 2n}{4}, \qquad \textit{for } n \textit{ even.}$$

Theorem 8. $C_{max}(\Delta^{[n]}, 1) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$

Proof. In a cut-minimal graph the maximum degree of each vertex is $\lfloor \frac{n-1}{2} \rfloor$, so if n is even, we have:

$$C_{max}(\Delta^{[n]}, 1) \leq \frac{n \lfloor \frac{n-1}{2} \rfloor}{2} = \frac{n^2 - 2n}{4} = 2\binom{\frac{n}{2}}{2} = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$$

If n is odd, the situation becomes more complicated. We only know that

$$C_{max}(\Delta^{[n]}, 1) \leq \frac{n \lfloor \frac{n-1}{2} \rfloor}{2} = \frac{n \frac{n-1}{2}}{2} = \frac{n^2 - n}{4},$$

but in this case unfortunately the right hand side is bigger than $\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$, so we have to find a smaller upper bound for $C_{max}(\Delta^{[n]}, 1)$. The following investigation shows, that a graph with $\frac{n^2-n}{4}$ edges can not be cut-minimal anymore, which will lead to the requested bound.
Consider a graph $G = ([n], E)$, and choose a vertex $v \in [n]$, such that $\deg_G(v) = \frac{n-1}{2}$. If such a vertex does not exist, we have

$$|E| \leq \frac{n(\frac{n-1}{2} - 1)}{2} = \frac{n^2 - 3n}{4} < \frac{n^2 - 2n + 1}{4} = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$$

and we are done. Now there exist exactly $\frac{n-1}{2}$ vertices $v_1, \ldots, v_{\frac{n-1}{2}} \in [n]$, such that $(v, v_i) \notin E$, for all $i = 1, \ldots, \frac{n-1}{2}$. If we had $\deg_G(v_i) = \frac{n-1}{2}$ for one of these vertices, we would get

$$|E_G(\{v, v_i\}, [n] \setminus \{v, v_i\})| = 2\frac{n-1}{2} = n - 1 > n - 2 = \frac{2(n-2)}{2},$$

so G would not be cut-minimal anymore. It follows that the degree of these $\frac{n-1}{2}$ vertices has to be at least one lower than assumed, so the number of edges has to be at least $\frac{n-1}{4}$ lower than assumed, which provides the new inequality:

$$\begin{aligned} C_{max}(\Delta^{[n]},1) &\leq \frac{n^2-n}{4} - \frac{n-1}{4} \\ &= \frac{n^2-2n+1}{4} \\ &= \frac{(n-1)^2}{4} \\ &= \binom{\frac{n+1}{2}}{2} + \binom{\frac{n-1}{2}}{2} \\ &= \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} \end{aligned}$$

Hence, we have $C_{max}(\Delta^{[n]},1) \leq \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$ in general and by Proposition 8 we are done. □

Note, that the proof of Proposition 8 even contains a description of the shape of a cut-minimal graph with the maximum number of edges, which furthermore provides information about how a top dimensional simplex is embedded in $\mathcal{C}^1(n)$. Now the following theorem does not only represent a first statement about the homology of $\mathcal{C}^1(n)$, but even shows that the construction in the proof of Proposition 8 is the only possible shape of such graphs (top dimensional simplices respectively). Let us first define some basic terminology, which we will mainly need in the next section, but it might already be helpful at this point to know exactly what we mean by deleting or adding edges in a graph.

Definition 12. *For a graph $G = ([n], E)$ and a family of edges $e_1, \dots, e_k \in E$ we define the **deletion** of $e_1, \dots, e_k$ in G as:*

$$G_{e_1,\dots,e_k} := ([n], E \setminus \{e_1, \dots, e_k\}).$$

*For a graph $G = ([n], E)$ and a family of edges $e_1, \dots, e_k \in NE_G([n],[n])$ we define the **addition** of $e_1, \dots, e_k$ to G as:*

$$G^{e_1,\dots,e_k} := ([n], E \cup \{e_1, \dots, e_k\}),$$

Theorem 9. $H_{C_{max}(\Delta^{[n]},1)-1}(\mathcal{C}^1(n)) \cong 0$

Proof. We will first show, that a top dimensional simplex in $\mathcal{C}^1(n)$ can only be represented as a graph of the type we constructed in the proof of Proposition 8.
Let n be even. If we set $n = 2t+2$, then by the number $C_{max}(\Delta^{[n]},1)$ and cut-minimality, the graph G corresponding to a top dimensional simplex in $\mathcal{C}^1(n)$ must be t-regular. Furthermore for any three vertices $v, w, u \in [n]$ by cut-minimality we have:

$$E_G(\{v,w,u\},[n] \setminus \{v,w,u\}) \leq \frac{3(2t-1)}{2} = 3t - \frac{3}{2},$$

so by t-regularity among any three vertices at least two of them must be adjacent. Now choose a vertex $v \in [n]$ and set A to be the set consisting of all vertices, which are not adjacent to v. Then we have $|A| = t+1$ by t-regularity and by the preceding result any two vertices in A have to be adjacent. Thus, A provides a complete graph on $t+1$ vertices, so by t-regularity $[n] \setminus A$ must also provide a complete graph on $t+1$ vertices. Hence, every top dimensional simplex in $\mathcal{C}^1(n)$ (for n even) corresponds

to a graph of that shape.
Let now n be odd. If we set $n = 2t + 3$, then by the number $C_{max}(\Delta^{[n]}, 1)$ and cut-minimality there exist at least $t + 2$ vertices having degree $t + 1$. Let A denote the set of these vertices. For any $v, w \in A$ we have:

$$E_G(\{v, w\}, [n] \setminus \{v, w\}) \leq \frac{2(2t + 3 - 2)}{2} = 2t + 1 < 2t + 2 = \deg_G(v) + \deg_G(w),$$

so all vertices from A are adjacent. By cut-minimality again we have $|A| = t + 2$, so A provides a complete graph on $t + 2$ vertices. The number of remaining edges $\binom{t+1}{2}$ shows, that the remaining $t + 1$ vertices must also provide a complete graph, which is disjoint from the first one, because any other constellation would destroy cut-minimality. So, again every top dimensional simplex in $\mathcal{C}^1(n)$ corresponds to a graph of the requested type.
Now we see that deleting an edge from such a graph (which is the same as deleting a vertex from a top dimensional simplex) produces a graph corresponding to a simplex which appears as a face of one top dimensional simplex, but can not be a face of another top dimensional simplex, since we can not construct a graph of the requested type by adding an edge at any other place than the place where we just deleted it. So, all top dimensional simplices are at most connected via simplices of codimension 2, which implies that $\mathcal{C}^1(n)$ can be continiously retracted to a complex of codimension 1 and so homology in top dimension vanishes. □

Definition 13. *A cut-minimal graph $G = ([n], E)$ is called **maximal**, if for all $e \in NE_G([n], [n])$ the addition G^e is not cut-minimal. We denote the set of all maximal cut-minimal graphs on n vertices by $MAX(n)$.*

Since a deletion of a cut-minimal graph is cut-minimal itself, it turnes out that we only have to determine all maximal cut-minimal graphs to find all cut-minimal graphs.

Definition 14. *Two graphs $G = ([n], E)$ and $G' = ([n], E')$ are called **isomorphic**, if there exists a map $f : [n] \to [n]$, such that $(i, j) \in E$ if and only if $(f(i), f(j)) \in E'$.*

Remark 3. *Note, that an isomorphism of graphs preserves all properties, which are studied in this chapter, especially cut-minimality and the constant $h(G)$, studied in the next section.*

Figure 1 illustrates, how the isomorphism classes of cut-minimal graphs on 6 vertices are arranged, where the connecting lines represent the relations between the classes refering to the deletion or addition of edges. Note, that the choosen representatives in the figure do not always satisfy the property of being a deletion or an addition of the shown representative of a class below or above, but in this case there is always another representative in the class which does. We see that beside the class of maximal cut-minimal graphs constructed in the proof of Proposition 8, we have three more classes of maximal cut-minimal graphs here.

Except for the maximal cut-minimal graphs with maximum number of edges we do not know anything about the remaining classes of cut-minimal graphs until now. The following statements now approach this challenge by providing new classes of cut-minimal graphs in general which do not appear as deletions of those largest maximal cut-minimal graphs.

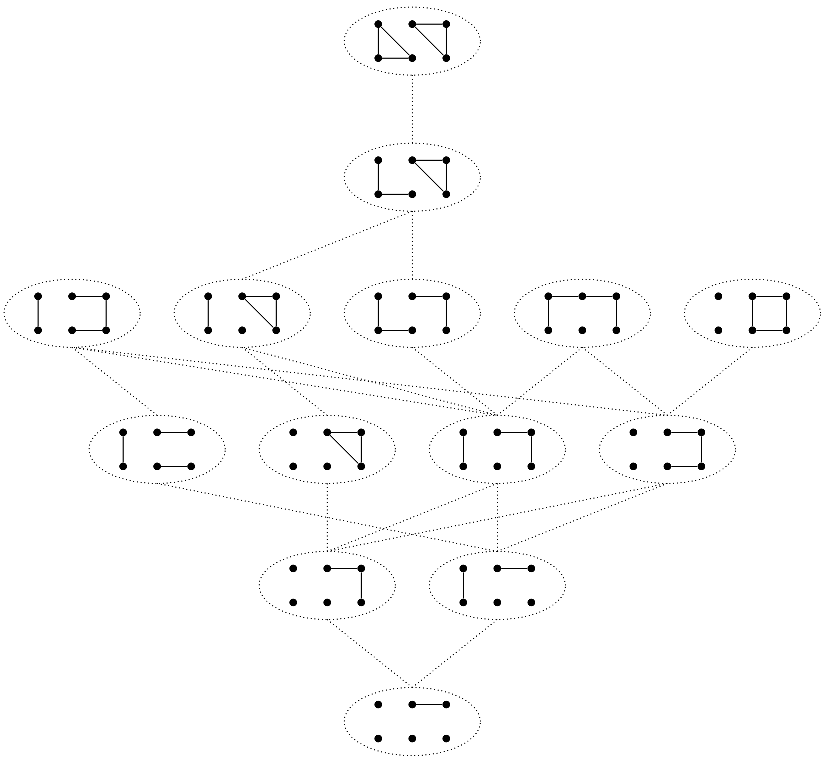

FIGURE 4.1: All cut-minimal graphs on 6 vertices (up to isomorphism)

Lemma 10. *Let $u_1, \dots, u_t, s_1, \dots, s_t \in \mathbb{N} \cup \{0\}$, such that for all $i = 1, \dots, t$ we have:*

$$u_i + s_i \leq \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j + s_j,$$

then we have:

$$\sum_{i=1}^{t} s_i (u_i - \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j) \leq 0$$

Proof. If we have

$$u_i \leq \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j$$

for all $i = 1, \dots, t$, then we are done, since all u_i's and s_i's are positive. So, let there exist a $k \in [t]$, such that

$$u_k > \sum_{\substack{j=1 \\ j \neq k}}^{t} u_j.$$

Obviously, there is at most one unique u_k satisfying this property. Now for all $i \neq k$ we have:

$$u_i - \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j \leq -(u_k - \sum_{\substack{j=1 \\ j \neq k}}^{t} u_j),$$

and furthermore we have

$$s_k < \sum_{\substack{j=1 \\ j \neq k}}^{t} s_j$$

by assumption and so we get:

$$\sum_{\substack{i=1 \\ i \neq k}}^{t} s_i(u_i - \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j) \leq -s_k(u_k - \sum_{\substack{j=1 \\ j \neq k}}^{t} u_j).$$

Hence, we have:

$$\sum_{i=1}^{t} s_i(u_i - \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j) \leq 0.$$

$\square$

Proposition 9. *Let $G = ([n], E)$ be a simple graph and let the largest connected component of G not contain more than $\frac{n}{2}$ vertices. Then G is cut-minimal.*

Proof. Let $C_1, \ldots, C_t \subset [n]$ be the connected components of G and let $S \subset [n]$, $S_i := S \cap C_i$ and $U_i := C_i \setminus S_i$. Then we have:

$$|E_G(S, [n] \setminus S)| \leq \sum_{i=1}^{t} |S_i||U_i|$$

and

$$|NE_G(S, [n] \setminus S)| \geq \sum_{i=1}^{t} (|S_i| \sum_{\substack{j=1 \\ j \neq i}}^{t} |U_j|).$$

So, we have to show that

$$\sum_{i=1}^{t} |S_i||U_i| \leq \sum_{i=1}^{t} (|S_i| \sum_{\substack{j=1 \\ j \neq i}}^{t} |U_j|),$$

which is equivalent to

$$\sum_{i=1}^{t} |S_i|(|U_i| - \sum_{\substack{j=1 \\ j \neq i}}^{t} |U_j|) \leq 0,$$

but this follows directly from Lemma 10, since by the assumption $|C_i| \leq \frac{n}{2}$ for all i, we have:

$$|S_i| + |U_i| = |C_i| \leq \sum_{\substack{j=1 \\ j \neq i}}^{t} |C_j| = \sum_{\substack{j=1 \\ j \neq i}}^{t} |S_j| + |U_j|.$$

$\square$

Let us now define and determine the counterpart of the number $C_{max}(\Delta^{[n]}, 1)$, namely the minimal number of edges, a non-cut-minimal graph can have.

Definition 15. *For any $n \geq 2$ we define the number:*

$$C_{min}(n) := \min_{\substack{G=([n],E),\\ G \text{ is not cut-minimal}}} |E|$$

Theorem 10. $C_{min}(n) = \lceil \frac{n}{2} \rceil$

Proof. We can always find a graph $G = ([n], E)$ with $\lceil \frac{n}{2} \rceil$ edges, such that for a vertex $v \in [n]$ we have

$$E_G(\{v\}, [n] \setminus \{v\}) = \left\lceil \frac{n}{2} \right\rceil > \left\lfloor \frac{n-1}{2} \right\rfloor,$$

so it is not cut-minimal and we have $C_{min}(n) \leq \lceil \frac{n}{2} \rceil$.
On the other hand if we have a graph $G = ([n], E)$ with $|E| = \lceil \frac{n}{2} \rceil - 1$, then it must be cut-minimal, since $\lceil \frac{n}{2} \rceil - 1 = \lfloor \frac{n-1}{2} \rfloor$, so we get $C_{min}(n) \geq \lceil \frac{n}{2} \rceil$ and we are done. □

Obviously, $\mathcal{CM}(n)$ contains all possible simplices of dimension lower than $C_{min}(n)$, which leads to the following observation.

Corollary 2. $H_k(\mathcal{CM}(n)) \cong 0$ *for all* $1 \leq k \leq \lceil \frac{n}{2} \rceil - 3$

Proof. By Theorem 10 $\mathcal{CM}(n)$ has a full k-skeleton for all $k \leq \lceil \frac{n}{2} \rceil - 2$ and we are done. □

Since adding a vertex to a cut-minimal graph will never destroy its cut-minimality, we can define the following natural inclusion.

Definition 16. *For every $n \in \mathbb{N}$ define:*

$$\begin{aligned} i_n \quad : \quad CMG(n) &\longrightarrow CMG(n+1) \\ ([n], E) &\longmapsto ([n+1], E) \end{aligned}$$

Furthermore, we see that maximality of cut-minimal graphs always becomes destroyed by adding a vertex to them.

Proposition 10. *Let $G \in MAX(n)$, then we have $i_n(G) \notin MAX(n+1)$.*

Proof. Let $G = ([n], E) \in MAX(n)$. If n is odd, Theorem 8 gives:

$$|E| \leq \frac{(n-1)^2}{4} = \frac{n^2 - 2n + 1}{4} < \frac{n^2 - n}{4} = \frac{n\frac{n-1}{2}}{2} \quad \text{for } n \geq 3,$$

so there exists a $v \in [n]$, such that $\deg_G(v) < \frac{n-1}{2}$ (*). Now define
$G' := ([n+1], E \cup (v, n+1))$ and let $S \subset [n+1]$, such that $1 \leq |S| \leq \frac{n+1}{2}$, then we have:

$$\begin{aligned} |E_{G'}(S, [n+1] \setminus S)| &\leq |E_G(S \setminus \{n+1\}, [n] \setminus S)| + 1 \\ &\leq \frac{|S|(n - |S|)}{2} + 1 \\ &= \frac{|S|(n + \frac{2}{|S|} - |S|)}{2} \\ &\leq \frac{|S|(n + 1 - |S|)}{2}, \end{aligned}$$

for all $S \subset [n+1]$, such that $|S| \geq 2$. For $|S| = 1$, the upper condition is also satisfied by (*).
Hence, G' is cut-minimal and so we have $i_n(G) = ([n+1], E) \notin MAX(n+1)$.
If n is even, define $G' := ([n+1], E \cup (v, n+1))$ for some arbitrary $v \in [n]$. Then by the same calculations as in the first part, we have

$$|E_{G'}(S, [n+1] \setminus S)| \leq \frac{|S|(n+1-|S|)}{2},$$

for all $S \subset [n+1]$, such that $|S| \geq 2$, and for $|S| = 1$ we have:

$$\begin{aligned} |E_{G'}(S, [n+1] \setminus S)| &\leq \max\{\deg_{G'}(v) : v \in [n+1]\} \\ &= \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \\ &= \frac{n-2}{2} + 1 = \frac{n}{2} = \frac{(n+1)-1}{2}. \end{aligned}$$

Hence, G' is cut-minimal and $i_n(G) \notin MAX(n+1)$. □

4.2 Cheeger graphs

In this chapter we want to recall the notion of Cheeger graphs and the first Cheeger constant of a simplex as introduced in [1] but then not try to determine the Cheeger constant for certain simplices more precisely but investigate the combinatorial structure of Cheeger graphs and develop some methods which might help to check if a given cut-minimal graph is a Cheeger graph or not.

4.2.1 Basic definitions and properties

The following definition is completely adopted from [1].

Definition 17. *Consider a graph $G = ([n], E)$, then we set:*

$$T(G) := \{(v, e) \ : \ v \in [n], e = (w, u) \in E, v \notin e, |\{(v, w), (v, u), (w, u)\} \cap E\| \text{ is odd}\}.$$

We have:

$$|T(G)| = \sum_{e \in E} t(e),$$

where for an edge $e = (v, w)$, we set

$$t(e) := \sum_{u \in [n] \,:\, u \neq v, w} \tau_e(u),$$

with

$$\tau_e(u) := \begin{cases} 1, & \text{if } (v, u), (w, u) \notin E \\ \frac{1}{3}, & \text{if } (v, u), (w, u) \in E \\ 0, & \text{otherwise} \end{cases}$$

Furthermore, we adopt the number:

$$h(G) := \frac{|T(G)|}{|E|}$$

and call a cut-minimal graph $G = ([n], E)$ a ***Cheeger graph****, if*

$$h(G) = \min_{G' \in CMG(n)} h(G').$$

The ***first Cheeger constant of a simplex*** $h_1(\Delta^{[n]})$ *is then defined by:*

$$h_1(\Delta^{[n]}) := h(G)$$

where G is some Cheeger graph on n vertices.

We already know by [1] that $\frac{n}{3} \leq h_1(\Delta[n]) \leq \lceil \frac{n}{3} \rceil$ and the lower bound is archieved if n is not a power of 2. If two graphs G and G' belong to the same isomorphism class, we obviously have $|T(G)| = |T(G')|$ and $h(G) = h(G')$, so taking up the example from the preceding section, Figure 2 shows the numbers $h(G)$ for all cut-minimal graphs on 6 vertices with the same partially ordering as in Figure 1 and we can see that there is one Cheeger graph attaining the Cheeger constant $\frac{8}{4}$.

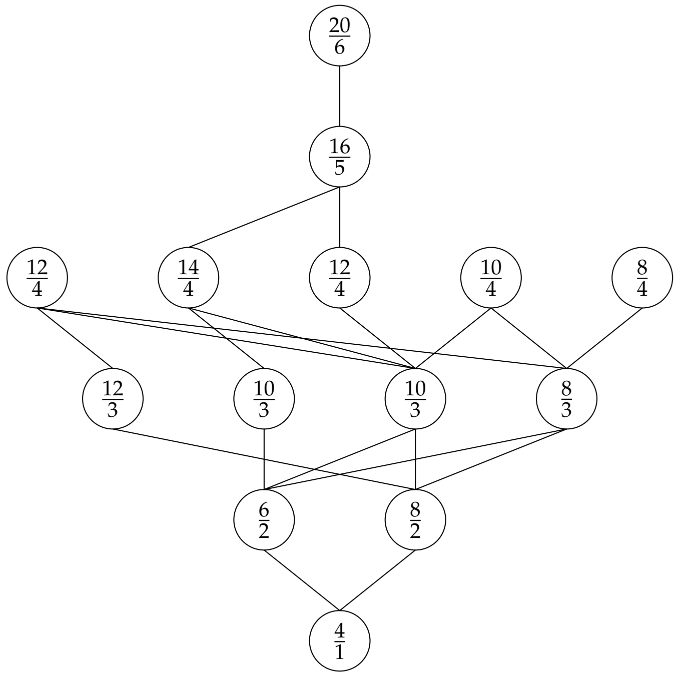

FIGURE 4.2: The numbers $h(G)$ for all cut-minimal graphs on 6 vertices

4.2.2 Deletions and additions of edges

We will now investigate how the set $T(G)$ (and so the constant $h(G)$ as well) changes when we delete or add one or more edges in a given graph G. This will give us some methods to determine that a given graph satisfying certain conditions is not a Cheeger graph.

Definition 18. *Let $G = ([n], E)$ be a simple graph. For an edge $e \in E$ we define the numbers:*

$$\psi_e^G := |\{e' \in E \,:\, e' \text{ and } e \text{ share a vertex}\}|,$$

where two edges $e = (v_1, v_2)$ and $e' = (v_3, v_4)$ are said to be sharing a vertex, if $|(v_1, v_2) \cap (v_3, v_4)| = 1$ and

$$\phi_e^G := |\{A \subseteq E \,:\, A \text{ is a triangle in } G \text{ and } e \in A\}|,$$

where a triangle is a set of the type $A = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$. If G contains no triangle, which is equivalent to $\sum_{e \in E} \phi_e^G = 0$, then we call G ***triangle-free***.

Lemma 11. *Let $G = ([n], E)$ be a cut-minimal graph and $e \in E$, then we have:*

$$\psi_e^G \leq \begin{cases} n-3, & \text{for } n \text{ odd} \\ n-4, & \text{for } n \text{ even} \end{cases}$$

and

$$\phi_e^G \leq \begin{cases} \frac{n-3}{2}, & \text{for } n \text{ odd} \\ \frac{n-4}{2}, & \text{for } n \text{ even} \end{cases}$$

Proof. Let $e := (v_1, v_2)$, then we have $\deg_G(v_1), \deg(v_2) \leq \lfloor \frac{n-1}{2} \rfloor$, so we get:

$$\psi_e^G = \deg_G(v_1) + \deg_G(v_2) - 2 \leq 2 \left\lfloor \frac{n-1}{2} \right\rfloor - 2$$

Since $2 \lfloor \frac{n-1}{2} \rfloor = n - 1$ for n odd and $n - 2$ for n even we get the claimed result. The second part follows directly from the first one, since for each triangle containing e, we have exactly two edges sharing a vertex with e. □

Since a Cheeger graph G can be characterized by the property that neither deleting edges from it nor adding edges to it (provided that the corresponding graph is still cut-minimal) nor a combination of both actions can decrease the number $h(G)$, it might be helpful to know more about how those actions change $h(G)$. The next Lemma gives exactly this information and the following statements then develop conditions which being satisfied by a certain graph guarantee that it is not a Cheeger graph.

Lemma 12. *Let $G = ([n], E)$ be a simple graph, $e \in E$ and $e' \in NE_G([n], [n])$. Then we have:*

1. $|T(G)| \leq |E|(n-2) - 2\psi_e^G + 2\phi_e^G$
2. $|T(G_e)| = |T(G)| - (n-2) + 2(\psi_e^G - 2\phi_e^G)$
3. $|T(G^{e'})| = |T(G)| + (n-2) - 2(\psi_{e'}^{G^{e'}} - 2\phi_{e'}^{G^{e'}})$

Proof. (1) The maximum number $|T(G)|$ can become is obviously $|E|(n-2)$, namely in the case when no two edges share a vertex. Now for an edge
$e = (v_1, v_2) \in E$ contained in a triangle $\{(v_1, v_2), (v_1, v_3), (v_2, v_3)\}$, we have $\tau_e(v_3) = \frac{1}{3}$ instead of 1, so $t(e)$ decreases at least by $\frac{2}{3}$. By 3 edges participating at each triangle, we then overall lose at least $2\phi_e^G$.
If we have two edges $(v_1, v_2), (v_2, v_3) \in E$, such that $(v_1, v_3) \notin E$, meaning two edges which share a vertex but which are not contained in the same triangle, then we get $\tau_{(v_1,v_2)}(v_3) = \tau_{(v_2,v_3)}(v_1) = 0$ instead of 1, so overall we lose at least $2(\psi_e^G - 2\phi_e^G)$, since $\psi_e^G - 2\phi_e^G$ is the number of edges which share a vertex with e but which are not

contained in the same triangle with e.
All in all we get:

$$|T(G)| \leq |E|(n-2) - 2\phi_e^G - 2(\psi_e^G - 2\phi_e^G) = |E|(n-2) - 2\psi_e^G + 2\phi_e^G$$

(2) The maximum number by which $|T(G)|$ can decrease when deleting e from G is $n-2$. This is the case when e is not connected to the rest of the graph. We do not have to consider the case when e is contained in a triangle because even though $t(e)$ is smaller than $n-2-\frac{2}{3}$ then, the numbers $t(e')$ and $t(e'')$ refering to the other two edges e' and e'' contained in the triangle also decrease by $\frac{1}{3}$ when deleting e. So, we still have to consider the case, when we have an edge e' sharing a vertex with e but not being contained in the same triangle with e. Then $|T(G)|$ decreases by one less than $n-2$ and even increases by 1 when deleting e, since $t(e')$ increases by 1.
Hence, we get:

$$|T(G_e)| = |T(G)| - (n-2) + 2(\psi_e^G - 2\phi_e^G)$$

(3) By part (2) of this Lemma we have:

$$|T(G)| = |T(G_{e'}^{e'})| = |T(G^{e'})| - (n-2) + 2(\psi_{e'}^{G^{e'}} - 2\phi_{e'}^{G^{e'}})$$

which is equivalent to:

$$|T(G^{e'})| = |T(G)| + (n-2) - 2(\psi_{e'}^{G^{e'}} - 2\phi_{e'}^{G^{e'}}).$$

□

Proposition 11. *Let $G = ([n], E)$ be a simple graph. If there exists an edge $e \in E$, such that $2(|E|-1)\psi_e^G - 2(2|E|-1)\phi_e^G < 0$, then we have $h(G_e) < h(G)$.*

Proof. By Lemma 12 (1) we get:

$$\begin{aligned}
& |E|(n-2) - (2\psi_e^G - 2\phi_e^G) \geq |T(G)| \\
\iff\quad & |E||T(G)| - |E|(n-2) + (2\psi_e^G - 2\phi_e^G) \leq |T(G)|(|E|-1) \\
\iff\quad & \frac{|E||T(G)| - |E|(n-2) + (2\psi_e^G - 2\phi_e^G)}{|E|(|E|-1)} \leq \frac{|T(G)|(|E|-1)}{|E|(|E|-1)} \\
\iff\quad & \frac{|T(G)| - (n-2)}{|E|-1} + \frac{2\psi_e^G - 2\phi_e^G}{|E|(|E|-1)} \leq \frac{|T(G)|}{|E|} = h(G) \qquad (4.1)
\end{aligned}$$

and by Lemma 12 (2) we get:

$$h(G_e) = \frac{|T(G)| - (n-2)}{|E|-1} + \frac{2\psi_e^G - 4\phi_e^G}{|E|-1} \qquad (4.2)$$

Now consider:

$$
\begin{aligned}
& 2(|E|-1)\psi_e^G - 2(2|E|-1)\phi_e^G < 0 \\
\iff\quad & 2(|E|-1)\psi_e^G - (4|E|-2)\phi_e^G < 0 \\
\iff\quad & 2(|E|-1)\psi_e^G - 4|E|\phi_e^G + 2\phi_e^G < 0 \\
\iff\quad & 2|E|\psi_e^G - 4|E|\phi_e^G < 2\psi_e^G - 2\phi_e^G \\
\iff\quad & \frac{2\psi_e^G - 4\phi_e^G}{|E|-1} < \frac{2\psi_e^G - 2\phi_e^G}{|E|(|E|-1)}
\end{aligned}
$$

So, by (1) and (2) we have:

$$h(G_e) < h(G)$$

□

Example 4. *Let $G = ([n], E)$ be a simple graph containing a triangle, which is only connected to the rest of the graph via one vertex or isolated, meaning a set $\{(v_1, v_2), (v_2, v_3), (v_1, v_3)\} \subseteq E$, such that $deg_G(v_1) = deg_G(v_2) = 2$, then G is not a Cheeger graph as follows. We set $e := (v_1, v_2)$, then we have $\phi_e^G = 1$ and $\psi_e^G = 2$. This gives us:*

$$2(|E|-1)\psi_e^G - 2(2|E|-1)\phi_e^G = 4(|E|-1) - 2(2|E|-1) = -2 < 0$$

and by Proposition 11 we get:

$$h(G_e) < h(G),$$

so G can not be a Cheeger graph.

Now we want to generalize the second and third part of Lemma 12 to a situation where we consider more than only one edge being deleted or added.

Lemma 13. *Let $G = ([n], E)$ be a simple graph and $e_1, \ldots, e_k \in E$, then we have:*

$$|T(G_{e_1,\ldots,e_k})| = |T(G)| - k(n-2) + 2\left(\sum_{i=1}^{k} \psi_{e_i}^{G_{e_1,\ldots,e_{i-1}}} - 2\sum_{i=1}^{k} \phi_{e_i}^{G_{e_1,\ldots,e_{i-1}}}\right)$$

In this formula we set $G_{e_0} = G$ to avoid writing it in a more complicated way.
For $e_1, \ldots, e_k \in NE_G([n], [n])$ we have:

$$|T(G^{e_1,\ldots,e_k})| = |T(G)| + k(n-2) - 2\left(\sum_{i=1}^{k} \psi_{e_i}^{G^{e_1,\ldots,e_i}} - 2\sum_{i=1}^{k} \phi_{e_i}^{G^{e_1,\ldots,e_i}}\right)$$

Proof. By Lemma 12 (2) we get:

$$
\begin{aligned}
|T(G_{e_1,\ldots,e_k})| &= |T(G_{e_1,\ldots,e_{k-1}})| - (n-2) + 2\left(\psi_{e_k}^{G_{e_1,\ldots,e_{k-1}}} - 2\phi_{e_k}^{G_{e_1,\ldots,e_{k-1}}}\right) \\
&= |T(G_{e_1,\ldots,e_{k-2}})| - (n-2) + 2\left(\psi_{e_{k-1}}^{G_{e_1,\ldots,e_{k-2}}} - 2\phi_{e_{k-1}}^{G_{e_1,\ldots,e_{k-2}}}\right) \\
&\quad - (n-2) + 2\left(\psi_{e_k}^{G_{e_1,\ldots,e_{k-1}}} - 2\phi_{e_k}^{G_{e_1,\ldots,e_{k-1}}}\right) \\
&= |T(G_{e_1,\ldots,e_{k-2}})| - 2(n-2) \\
&\quad + 2\left(\psi_{e_{k-1}}^{G_{e_1,\ldots,e_{k-2}}} + \psi_{e_k}^{G_{e_1,\ldots,e_{k-1}}} - 2\left(\phi_{e_{k-1}}^{G_{e_1,\ldots,e_{k-2}}} + \phi_{e_k}^{G_{e_1,\ldots,e_{k-1}}}\right)\right)
\end{aligned}
$$

so inductively we have:

$$|T(G_{e_1,\dots,e_k})| = |T(G)| - k(n-2) + 2(\sum_{i=1}^{k} \psi_{e_i}^{G_{e_1,\dots,e_{i-1}}} - 2\sum_{i=1}^{k} \phi_{e_i}^{G_{e_1,\dots,e_{i-1}}})$$

The second part works analogously using Lemma 12 (3). □

Proposition 12. *Let $G = ([n], E)$ be a simple graph. If there exist edges $e_1, \dots, e_k \in E$ and $e \in E$ such that:*

$$\frac{2(\sum_{i=1}^{k} \psi_{e_i}^{G_{e_1,\dots,e_{i-1}}} - 2\sum_{i=1}^{k} \phi_{e_i}^{G_{e_1,\dots,e_{i-1}}})}{|E| - k} < \frac{k(2\psi_e^G - 2\phi_e^G)}{|E|(|E| - k)},$$

then we have:

$$h(G_{e_1,\dots,e_k}) < h(G)$$

Proof. By Lemma 12 (1) we get:

$$\begin{aligned}
& |E|(n-2) - (2\psi_e^G - 2\phi_e^G) \geq |T(G)| \\
\iff\quad & |E||T(G)| - k|E|(n-2) + k(2\psi_e^G - 2\phi_e^G) \leq |T(G)|(|E| - k) \\
\iff\quad & \frac{|E||T(G)| - k|E|(n-2) + k(2\psi_e^G - 2\phi_e^G)}{|E|(|E| - k)} \leq \frac{|T(G)|(|E| - k)}{|E|(|E| - k)} \\
\iff\quad & \frac{|T(G)| - k(n-2)}{|E| - k} + \frac{k(2\psi_e^G - 2\phi_e^G)}{|E|(|E| - k)} \leq \frac{|T(G)|}{|E|} = h(G)
\end{aligned} \tag{4.3}$$

and Lemma 13 gives us:

$$h(G_{e_1,\dots,e_k}) = \frac{|T(G)| - k(n-2)}{|E| - k} + \frac{2(\sum_{i=1}^{k} \psi_{e_i}^{G_{e_1,\dots,e_{i-1}}} - 2\sum_{i=1}^{k} \phi_{e_i}^{G_{e_1,\dots,e_{i-1}}})}{|E| - k}$$

So, by (3) and the assumption we have:

$$h(G_{e_1,\dots,e_k}) < h(G)$$

□

4.2.3 Adding vertices to Cheeger graphs

After we studied how adding or deleting edges in a cut-minimal graph G affects its constant $h(G)$ we are now interested in the consequences of adding a vertex to G. We will find out that in the most cases the property of being a Cheeger graph will be destroyed by this action. Using this knowledge we can construct a lower bound on the number of edges in Cheeger graphs holding in the most cases.

Lemma 14. *Let $G = ([n], E)$ be a cut-minimal graph, then we have:*

$$h(i_n(G)) = h(G) + 1$$

Proof. Adding an isolated vertex to a graph $G = ([n], E)$ obviously increases $|T(G)|$ by $|E|$, so we get:

$$h(i_n(G)) = \frac{|T(G)| + |E|}{|E|} = \frac{|T(G)|}{|E|} + \frac{|E|}{|E|} = h(G) + 1$$

□

A direct consequence of this relation is that if n is not divisible by 3, Cheeger graphs in $CMG(n)$ can not appear as an embedding of graphs from $CMG(n-1)$.

Proposition 13. *For all $G \in CMG(n)$, such that $3 \nmid n$ the graph $i_n(G) \in CMG(n+1)$ is not a Cheeger graph.*

Proof. We only need to consider Cheeger graphs in $CMG(n)$, since for any $G \in CMG(n)$ which is not a Cheeger graph, there exists a graph $G' \in CMG(n)$, such that $h(G') < h(G)$, so by Lemma 14 we get:

$$h(i_n(G')) = h(G') + 1 < h(G) + 1 = h(i_n(G))$$

Let now $G \in CMG(n)$ be a Cheeger graph. Then by a result from [1] we get:

$$\frac{n}{3} \leq h(G) \leq \left\lceil \frac{n}{3} \right\rceil$$

which implies that:

$$h(i_n(G)) = h(G) + 1 \geq \frac{n}{3} + 1 \geq \left\lceil \frac{n+1}{3} \right\rceil,$$

where the last inequality is sharp if and only if $3 \nmid n$.
So, if $3 \nmid n$ then $i_n(G)$ is not a Cheeger graph, since by [1] we have:

$$h_1(\Delta^{[n+1]}) \leq \left\lceil \frac{n+1}{3} \right\rceil < h(i_n(G))$$

□

We even have a similar version of the previous statement if n is divisible by 3 with the restriction that $n+1$ must not be a power of 2.

Proposition 14. *Let $G \in CMG(n)$, such that $3 \mid n$ and $n+1 \neq 2^t$ for some $t \in \mathbb{N}$. Then $i_n(G)$ is not a Cheeger graph.*

Proof. Since n is divisible by 3 we have $h(G) = \frac{n}{3}$ by [1] and so we get $h(i_n(G)) = \frac{n}{3} + 1$ by Lemma 14. Now, since $n+1$ is not a power of 2 we also know that $h_1(\Delta^{[n+1]}) = \frac{n+1}{3}$ by [1]. Hence, we have:

$$h(i_n(G)) > h_1(\Delta^{[n+1]})$$

□

Combining the last two statements, we can calculate a lower bound for the number of edges in Cheeger graphs, except for the case when the number of vertices n is a power of 2 and $n-1$ is divisible by 3.

Proposition 15. *Let $G = ([n], E)$ be a Cheeger graph, such that n is not a power of 2 or $n-1$ is not divisible by 3. Then we have $|E| \geq \lceil \frac{n-1}{2} \rceil$.*

Proof. Assume we have $|E| < \lceil \frac{n-1}{2} \rceil$. Then by Theorem 10 and since G must contain an isolated vertex, there exists a graph $G' \in CMG(n-1)$, such that $i_{n-1}(G') = G$. Now if we have $n \neq 2^t$ for some $t \in \mathbb{N}$, then G is not a Cheeger graph, since we either get $3 \nmid n-1$ and we are done by Proposition 13 or we get $3 \mid n-1$ and we are done by Proposition 14. On the other hand, if we have $n = 2^t$ but $3 \nmid n-1$, then we immediately see that G is not a Cheeger graph by Proposition 13. $\square$

Chapter 5

Perspectives

Appendix A

A theorem about partitioning consecutive numbers

A.1 Preliminaries

As an introducing example to the main theorem we want to prove later, consider the following "staircase shaped" Young tableau:

1				
2	2			
3	3	3		
4	4	4	4	
5	5	5	5	5

FIGURE A.1: "Staircase shaped" Young tableau consisting of 15 boxes

We have one box in the first line and one more in every following line. Now, we can "rebuild" this tableau by reordering the lines, such that we have not necessarily one box in the first line anymore, but always have still one more box in every following line and such that we do not have to split the single lines, as the numbers show:

4	4	4	4	2	2	1	
5	5	5	5	5	3	3	3

FIGURE A.2: Rebuilt Young tableau

The interesting question which will be answered by our main theorem is, if there is always a possibility to reorder the lines of a Young tableau of the first type without having to split them to build a Young tableau of the second type, provided that the numbers of boxes are the same. The question, how many different Young tableaux of this type exist, meaning Young tableaux consisting of a fixed number of boxes such that based on any line the following line has one box more, has already been answered in [5]. Namely, for every odd divisor of the number of boxes, there exists exactly one Young tableau of this type.
Note, that in this paper we set $\mathbb{N}$ to be the set of natural numbers without 0 and a sequence of consecutive numbers always denotes a sequence
$n_1, \ldots, n_t \in \mathbb{N}$ (or $\mathbb{Z}$), such that $n_{i+1} = n_i + 1$. Furthermore, we use the notation $[n] := 1, 2, 3, \ldots, n$.

Lemma 15. *Let $n, a, b \in \mathbb{N}$, such that $n < a \leq b$ and $\sum_{i=1}^{n} i = \sum_{i=a}^{b} i$. Then we have $n \geq 2(b - a + 1)$.*

Proof. Note, that $b-a+1$ is the number of summands in $\sum_{i=a}^{b} i$. Then we have:

$$\begin{aligned} & \sum_{i=1}^{n} i = \sum_{i=a}^{b} i \\ \iff & \frac{n(n+1)}{2} = (a+b)\frac{b-a+1}{2} \\ \iff & \frac{n}{2}(n+1) = (b-a+1)\frac{a+b}{2} \end{aligned}$$

Now, by assumption we have $\frac{a+b}{2} \geq n+1$, so we get $\frac{n}{2} \geq b-a+1$, which is equivalent to $n \geq 2(b-a+1)$. □

Lemma 16. *For every $m \in \mathbb{N}$ and every $l \in \mathbb{Z}$ there exist pairs of numbers $(x_1, x'_1), \ldots, (x_m, x'_m) \in \mathbb{N}^2$, such that $x_1, x'_1, \ldots, x_m, x'_m$ are all distinct, $x'_i - x_i = i$ for every $i = 1\ldots, m$, $l = \min\{x_1, x'_1, \ldots, x_m, x'_m\}$ and $\max\{x_1, x'_1, \ldots, x_m, x'_m\} \leq 2m + l$.*

Proof. Consider the following pairs (y_i, y'_i) for i odd:

$$\left(y_{2\lceil \frac{m}{2}\rceil-1}, y'_{2\lceil \frac{m}{2}\rceil-1}\right) = \left(1, 2\left\lceil \frac{m}{2}\right\rceil\right), \ldots, (y_1, y'_1) = \left(\left\lceil \frac{m}{2}\right\rceil, \left\lceil \frac{m}{2}\right\rceil + 1\right)$$

and the following for i even:

$$\left(y_{2\lfloor \frac{m}{2}\rfloor}, y'_{2\lfloor \frac{m}{2}\rfloor}\right) = \left(2\left\lceil \frac{m}{2}\right\rceil + 1, 2m+1\right), \ldots, (y_2, y'_2) = \left(\left\lceil \frac{m}{2}\right\rceil + m, 2m+2-\left\lfloor \frac{m}{2}\right\rfloor\right),$$

where $(y_{i+2}, y'_{i+2}) = (y_i - 1, y'_i + 1)$ for all $i = 1, \ldots, m-2$. Obviously, these pairs satisfy the assumption $y'_i - y_i = i$ and we have $\min\{y_1, y'_1, \ldots, y_m, y'_m\} = 1$ and $\max\{y_1, y'_1, \ldots, y_m, y'_m\} = 2m+1$ for every $m \in \mathbb{N}$. (In fact, we even have $\max\{y_1, y'_1\} = 2m < 2m+1$ for the case $m = 1$). Now, set $x_i := y_i + l - 1$ and $x'_i = y'_i + l - 1$ for all $i = 1, \ldots, m$ and we are done. □

A.2 The main theorem

Theorem 11. *Let $n, a, b \in \mathbb{N}$ ($b \geq a$), such that $\sum_{i=1}^{n} i = \sum_{i=a}^{b} i$, then for every $a \leq t \leq b$ there exists a subset $U_t \subseteq [n]$, such that $U_i \cap U_j = \emptyset$ for all $i \neq j$, $[n] = \bigcup_{a\leq t\leq b} U_t$ and $\sum_{i\in U_t} i = t$.*

Proof. Without loss of generality let $a > n$. Otherwise, we have $U_t = \{t\}$ for all $a \leq t \leq n$ and $\sum_{i=1}^{a-1} i = \sum_{i=n+1}^{b} i$, so the remaining problem is reduced to the requested case, since we have $n+1 > a-1$.

Now, denote the number of summands in the sum $\sum_{i=a}^{b} i$ by $s := b-a+1$ and set:

$$\begin{aligned} P &= \{p_1, \ldots, p_s\} := \{n-2s+1, n-2s+2, \ldots, n-s-1, n-s\}, \\ Q &= \{q_1, \ldots, q_s\} := \{n-s+1, n-s+2, \ldots, n-1, n\}, \\ R &= \{r_1, \ldots, r_s\} := \{a, a+1, \ldots, b-1, b\} \end{aligned}$$

We have $P, Q, R \subset \mathbb{N}$, since $n - 2s + 1 > 0$ holds by Lemma 15. Furthermore, we see that we have $p_i + q_j = 2n - 2s + 1$ for all i, j, such that $i + j = s + 1$. Now, set $c := 2n - 2s + 1$ and for every t such that $r_t - c < 0$ consider pairs $\left\{p_{i_t}, q_{j_t}\right\}$ and $\left\{p_{i'_t}, q_{j'_t}\right\}$, such that $i_t + j_t = i'_t + j'_t = s + 1$, $q_{j'_t} - q_{j_t} = c - r_t$ and all appearing numbers in all pairs are distinct (we actually only have to require that all $q_{j_t}, q_{j'_t}$ are distinct, then all the rest is distinct by the condition $i_t + j_t = i'_t + j'_t = s + 1$). We can find those kinds of tuples of pairs for every t satisfying $r_t - c < 0$ by Lemma 16 as follows. For every t, such that $r_t - c < 0$ there exists a t', such that $r_{t'} - c = -(r_t - c)$, since otherwise we had $\sum\limits_{i=1}^{s} p_i + q_i > \sum\limits_{i=1}^{s} r_i$, because the numbers $r_1 - c, \ldots, r_s - c$ are obviously consecutive, but this is a contradiction to the assumption $\sum\limits_{i=1}^{n} i = \sum\limits_{i=a}^{b} i$. By this observation, there must still exist a t, such that $r_t - c = 0$, since $r_1, \ldots, r_s$ are consecutive numbers, so letting m denote the maximum number, such that there exists a t satisfying $r_t - c = -m$, we have at least $2m + 1$ consecutive numbers in Q. Now, choose such a set of $2m + 1$ consecutive numbers from Q and let l denote the minimum number in this set, then applying Lemma 16 we get the requested tuples of pairs satisfying $q_{j'_t} - q_{j_t} = c - r_t$. Finally, set $U_{r_t} := \left\{p_{i'_t}, q_{j_t}\right\}$ and $U_{r_t + 2(c - r_t)} := \left\{p_{i_t}, q_{j'_t}\right\}$ for every t satisfying $r_t - c < 0$ and we have $\sum\limits_{i \in U_{r_t}} i = r_t$ and $\sum\limits_{i \in U_{r_t + 2(c - r_t)}} i = r_t + 2(c - r_t)$. For the remaining t's satisfying $r_t - c \geq 0$ we set $U_{r_t} := \{p_{i_t}, q_{j_t}\}$, where $\{p_{i_t}, q_{j_t}\}$ can be any of the remaining pairs, satisfying $p_{i_t} + q_{j_t} = c$. For those t's we have $\sum\limits_{i \in U_{r_t}} i = c$. So, the problem is reduced to a smaller one of the type $\sum\limits_{i=1}^{n-2s} i = \sum\limits_{i=r_k - c}^{r_s - c} i$, where k is the minimal number such that $r_k - c > 0$. By induction we are done as follows. The sets U_{r_t} and $U_{r_t + 2(c - r_t)}$, where $r_t - c < 0$ stay the same till the end, whereas the other sets U_{r_t}, where we still have $\sum\limits_{i \in U_{r_t}} i = c < r_t$ will be "filled up" during the next steps of induction. $\square$

The reader may develop a better intuition for the preceding proof by considering the following example.

Example 5. *Consider $n = 14$, $a = 15$ and $b = 20$, then we obviously have $\sum\limits_{i=1}^{n} i = \sum\limits_{i=a}^{b} i$. Now, according to the proof of Theorem 11 we have $s = 6$ and we consider the following pairs:*

$$\begin{aligned} &\{p_1, q_6\} = \{3, 14\}, \quad \{p_2, q_5\} = \{4, 13\}, \quad \{p_3, q_4\} = \{5, 12\}, \\ &\{p_4, q_3\} = \{6, 11\}, \quad \{p_5, q_2\} = \{7, 10\}, \quad \{p_6, q_1\} = \{8, 9\} \end{aligned}$$

We see that we have $c = 17$, so by subtracting c from the numbers $15, \ldots, 20$ only the last 3 numbers stay strictly positive. Now we "swap" the second components of two pairs just as we did in the preceding proof and construct the following sets:

$$\begin{aligned} &U_{15} = \{3, 12\}, \quad U_{16} = \{6, 10\}, \quad U_{17} = \{8, 9\}, \\ &U_{18} = \{7, 11\}, \quad U_{19} = \{5, 14\}, \quad U_{20} = \{4, 13\} \end{aligned}$$

We can see that we already have $\sum_{i \in U_t} i = t$ *for all* $t = 15, \ldots, 19$ *and* $\sum_{i \in U_{20}} i = c = 17 < 20$*, so the problem is reduced to* $\sum_{i=1}^{2} i = 3$ *and inductively we will be done by the next step. Formally this means, that* $U_{15}, \ldots, U_{19}$ *stay the same and we "fill up" the remaining set* U_{20} *with the remaining numbers* $1, 2$*, such that we get* $U_{20} = \{13, 4, 1, 2\}$.

There are obviously many more ways of partitioning numbers in our sense than the one way the "algorithm" in the proof of Theorem 11 gives us, but until now this is the only working way we know in general and we even still do not know, how many partitions leading to the requested result exist.

Bibliography

[1] Dmitry N. Kozlov, *The first Cheeger constant of a simplex*, arXiv 1610:07136, 2016

[2] N. Linial, R. Meshulam, *Homological connectivity of random 2-complexes*, Combinatorica 26, 2006, no. 4, 475-487

[3] M. Gromov, *Singularities, expanders and topology of maps. Part 2. From combinatorics to topology via algebraic isoperimetry*, Geom. Funct. Anal. 20, (2010), no. 2, 416-526.

[4] M. Wallach and R. Meshulam, *Homological connectivity of random k-dimensional complexes*, Random Structures Algorithms 34, 2009, no. 3, 408–417

[5] J. J. Sylvester and F. Franklin, *A Constructive Theory of Partitions, Arranged in Three Acts, an Interact and an Exodion*, American Journal of Mathematics Vol. 5, No. 1 (1882), pp. 251-330

[6] Dmitry N. Kozlov and Roy Meshulam; *Quantitative aspects of acyclicity*, arXiv:1802.03210 [math.CO], 2018

[7] Peter Keevash, *Hypergraph Turán problems*, http://people.maths.ox.ac.uk/keevash/papers/turan-survey.pdf

[8] Anders Björner and Martin Tancer, *Combinatorial Alexander Duality - A Short and Elementary Proof*, Discrete Comput Geom (2009) 42: 586–593 DOI 10.1007/s00454-008-9102-x, Springer Science+Business Media, LLC 2008

[9] Tim Lindemann and Kai Michael Renken, *Personal communication*, 2018