

Cosystoles and Cheeger Constants of the Simplex

Kai Renken

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An intuitive approach

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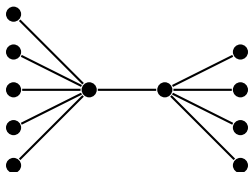


Figure: A "weakly" connected graph

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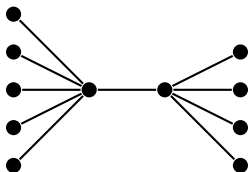


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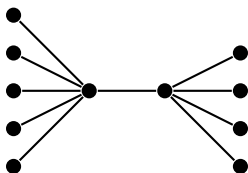


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Deleting the one edge in the middle will give a disconnected graph, consisting of two connected components, the smallest of them consisting of 5 vertices. The Cheeger constant of this graph is $\frac{1}{5}$.

The definition of the classical Cheeger constant

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Definition

Let $G = (V, E)$ be a (simple) graph. Then the **Cheeger constant** of G is defined by

$$h(G) = \min \left\{ \frac{|\delta(A)|}{|A|} : A \subset V, 1 \leq |A| \leq \frac{|V|}{2} \right\},$$

with $\delta(A) := \{e = (v, w) \in E : v \in A, w \in V \setminus A\}$.

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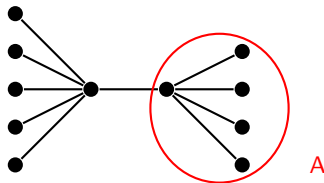


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Then we have:

$$h(K_n) = \left\lceil \frac{n}{2} \right\rceil$$

as follows:

The classical Cheeger constant of the simplex

For any subset $A \subset [n] := \{1, \dots, n\}$ we have:

$$\frac{|\delta(A)|}{|A|} = \frac{|A|(n - |A|)}{|A|} = n - |A|,$$

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$$\|\varphi\|_{\text{csy}} := \min \left\{ \|\delta^{k-1}(\phi) + \varphi\| : \phi \in C^{k-1}(X) \right\}$$

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A cochain $\varphi \in C^k(X)$ is called a **k -cosystole**, if it satisfies $\|\varphi\|_{\text{csy}} = \|\varphi\|$.

Coboundary expansion and the k -th Cheeger constant

Definition

For a cochain $\varphi \in C^k(X) \setminus \text{Im}(\delta^{k-1})$ the quotient

$$\|\varphi\|_{\text{exp}} := \frac{\|\delta^k(\varphi)\|}{\|\varphi\|_{\text{csy}}}$$

is called the **coboundary expansion** of φ and

$$h_k(X) := \min_{\substack{\varphi \in C^k(X) \\ \varphi \notin \text{Im}(\delta^{k-1})}} \|\varphi\|_{\text{exp}}$$

is called the k -th **Cheeger constant** of X .

A cosystole $\varphi \in C^k(X) \setminus \text{Im}(\delta^{k-1})$ satisfying $\|\varphi\|_{\text{exp}} = h_k(X)$ is called a **Cheeger cosystole**.

Relations to the classical Cheeger constant

The classical Cheeger constant of a graph can be considered as the 0-th Cheeger constant by defining the cosystolic norm of a 0-cochain $\varphi \in C^0(X)$ as

$$\|\varphi\|_{csy} := \min\{|\text{supp}(\varphi)|, |X^{(0)}| - |\text{supp}(\varphi)|\}$$

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The classical Cheeger constant of a graph equals 0 iff the graph is disconnected.

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which is equivalent to

$$H^k(X) \neq \{0\}$$

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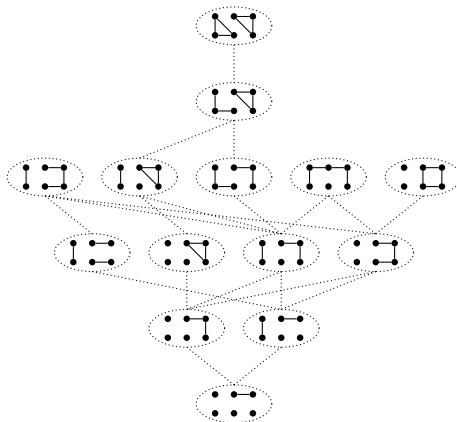


Figure: The supports of all 1-cosystoles of $\Delta^{[6]}$ (up to isomorphism)

Some values for higher dimensional Cheeger constants

Theorem (Wallach, Meshulam)

Let $\Delta^{[n]}$ be the standard simplex on n vertices and $1 \leq k \leq n-2$, then we have:

$$\frac{n}{k+2} \leq h_k(\Delta^{[n]}) \leq \left\lceil \frac{n}{k+2} \right\rceil$$

If n is divisible by $k+2$, then we have:

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Theorem (Kozlov)

Let $n > 2$ not be a power of 2, then we have:

$$h_1(\Delta^{[n]}) = \frac{n}{3}$$

Hitting sets and hitting numbers

Definition

Let V be some set and $\mathcal{F} \subseteq 2^V$ a family of subsets of V . A subset $P \subseteq V$ is called a **hitting set** of \mathcal{F} if we have $P \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The **hitting number** of \mathcal{F} is defined by

$$\tau(\mathcal{F}) := \min\{|P| : P \subseteq V \text{ is a hitting set of } \mathcal{F}\}$$

If \mathcal{F} is a family of chains / cochains, the hitting sets and the hitting number of \mathcal{F} are defined as the hitting sets and the hitting number of $\{\text{supp}(\varphi) : \varphi \in \mathcal{F}\}$.

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Example

Let $V := \{1, 2, 3, 4, 5\}$ and $\mathcal{F} := \{\{1, 2\}, \{2, 3, 4\}, \{1, 5\}, \{2, 4, 5\}\}$, then we have $\tau(\mathcal{F}) = 2$.

The cycle detection theorem

Theorem (Kozlov)

Let X be a simplicial complex, $k \geq 1$, and $\varphi \in C^k(X)$. Let now $\mathcal{F} = \{\alpha_1, \dots, \alpha_t\}$ be a family of k -cycles in $C_k(X)$, such that $\langle \varphi, \alpha_i \rangle = 1$ for all $1 \leq i \leq t$, then we have:

$$\|\varphi\|_{\text{csy}} \geq \tau(\mathcal{F})$$

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$$\begin{aligned}\langle \varphi + \delta^{k-1}(\psi), \alpha_i \rangle &= \langle \varphi, \alpha_i \rangle + \langle \delta^{k-1}(\psi), \alpha_i \rangle \\ &= \langle \varphi, \alpha_i \rangle + \langle \psi, \partial_{k-1}(\alpha_i) \rangle \\ &= \langle \varphi, \alpha_i \rangle + \langle \psi, 0 \rangle \\ &= \langle \varphi, \alpha_i \rangle = 1\end{aligned}$$

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This means that we have $\text{supp}(\varphi + \delta^{k-1}(\psi)) \cap \text{supp}(\alpha_i) \neq \emptyset$ for all $1 \leq i \leq t$, so $\text{supp}(\varphi + \delta^{k-1}(\psi))$ is a hitting set of \mathcal{F} and we get:

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Since ψ was chosen arbitrarily we are done. □

Maximal cosystoles

Definition

Let X be a simplicial complex and $1 \leq k \leq \dim(X)$, then

$$C_{\max}(X, k) := \max \left\{ \|\varphi\|_{\text{csy}} : \varphi \in C^k(X) \right\}$$

is the largest norm a k -cosystole in X can attain.

The largest cosystoles of the simplex

Theorem (Renken)

$$C_{\max}(\Delta^{[n]}, 1) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$$

The case when n is a power of 2

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Conjecture

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Theorem (Kozlov)

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Theorem (Renken)

$$h_1(\Delta^{[16]}) > \frac{16}{3}$$

Disjoint cycle expansion

Definition

Let X be a simplicial complex, $\mathcal{F} \subset C_k(X)$ a family of cycles, such that their supports are pairwise disjoint and

$$P(\mathcal{F}) := \{\varphi \in C^k(X) : |\text{supp}(\varphi) \cap \text{supp}(F)| = 1 \text{ for all } F \in \mathcal{F} \text{ and } \text{supp}(\varphi) \subset \bigcup_{F \in \mathcal{F}} \text{supp}(F)\}$$

Then we define

$$\gamma_{\mathcal{F}} := \frac{\min_{\varphi \in P(\mathcal{F})} \|\delta^k(\varphi)\|}{|\mathcal{F}|}$$

and we call

$$\gamma_k(X) := \min_{\mathcal{F} \in \mathfrak{C}} \gamma_{\mathcal{F}}$$

the k -th **disjoint cycle expansion** of X with

$$\mathfrak{C} := \{\mathcal{F} \subset C_k(X) : F \text{ is a cycle and } \text{supp}(F) \cap \text{supp}(F') = \emptyset \text{ for all } F, F' \in \mathcal{F} (F \neq F')\}$$

Hitting expansion

Definition

Let X be a simplicial complex, $\mathcal{F} \subset C_k(X)$ a family of cycles and

$$P'(\mathcal{F}) := \{\varphi \in C^k(X) : |\text{supp}(\varphi) \cap \text{supp}(F)| \text{ is odd for all } F \in \mathcal{F}\}$$

Then we define

$$\rho_{\mathcal{F}} := \frac{\min_{\varphi \in P'(\mathcal{F})} \|\delta^k(\varphi)\|}{\tau(\mathcal{F})}$$

and we call

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