Universität Bremen

DISSERTATION

Cosystoles and Cheeger constants of the simplex

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Abstract

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Cosystoles and Cheeger constants of the simplex

by Kai Michael RENKEN

The central interest of this thesis is to develop tools to determine the cosystolicity of a cochain, a property which is important to determine the Cheeger constants of a simplicial complex. We develop a general theory about the cosystolic norm of a cochain, in which we establish an interesting connection between that norm and the piercing number of a certain set system (see Chapter 2). In Chapter 3 we restrict our research to 1-dimensional cosystoles of a simplex which are slightly easier to understand, so we can provide more explicit results for that case, including the explicit determination of the largest 1-dimensional cosystoles of a simplex and a rough insight, how all cosystoles of a simplex in a certain dimension can be arranged in a certain simplicial complex. In the appendix we solve a beautiful combinatorial ordering problem, which is not directly related to the main subject of this thesis but arose during considerations about that and is worth to be stated as well.

Acknowledgements

The acknowledgments and the people to thank go here, don't forget to include your project advisor…

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Dedicated to my mother

Chapter 0

Preliminaries

Let us shortly recall some basic algebraic and combinatorial concepts, which we will use within this thesis.

0.1 Simplicial complexes

Let S be some set (whose elements are called **vertices**) and $X \subseteq 2^S$ a family of subsets of S (we will use the notation 2^S for the power set of S within the whole thesis), such that for all $\sigma \in X$ and all $\sigma' \subseteq \sigma$ we have $\sigma' \in X$. Then we call X an **(abstract) simplicial complex**. We just use the notation "simplicial complex" in this thesis, because we will only consider abstract simplicial complexes and are not interested in their geometric realization.

An element of a simplicial complex X is called **simplex** and a **face** of a simplex $\sigma \in X$ is a simplex $\sigma' \in X$, such that $\sigma' \subset \sigma$ and $|\sigma'| = |\sigma| - 1$. Furthermore, we denote the k-**skeleton** of a simplicial complex X by

$$X(k) := \{\sigma \in X : \ |\sigma| \leq k+1\},$$

and the **uniform** k-**skeleton** of X by

$$X^{(k)} := \{\sigma \in X : \ |\sigma| = k+1\}$$

Let $\sigma \subset S$ be a simplex and $s \in S \setminus \sigma$, then the simplex constructed by adding s to σ is denoted by $(\sigma, s) := \sigma \cup \{s\}$.
For a simplicial complex X we call $\dim(X) := \max\{|\sigma| - 1 : \sigma \in X\}$ the **dimension** of X (if it exists).
A simplicial complex is called **finite** if its vertex set is finite and **finite dimensional** if its dimension is finite.
The most frequently considered simplicial complex in this thesis will be the complex induced by the standard simplex on n-vertices. It can be considered as the complete power set of $[n] := |\{i \in \mathbb{N} : 1 \leq i \leq n\}|$ and we will denote it by $\Delta^{[n]} := 2^{[n]}$.

0.2 Chain- / Cochain complexes & Homology / Cohomology

Let X be a simplicial complex and $0 \leq k \leq \dim(X)$, then

$$C_k(X, \mathbb{Z}_2) := \left\{ \sum_{i \in I} c_i \sigma_i : \sigma_i \in X, c_i \in \mathbb{Z}_2 \right\}$$

is called the k-th **chain group** of X, where I is some index set. (The elements of $C_k(X, \mathbb{Z}_2)$ are called k-**chains**)
Note, that in general we have more possible coefficient systems than $\mathbb{Z}_2$ and X can be any topological space, but we will restrict ourselves to simplicial complexes in this thesis. Furthermore, since we only consider chain groups with $\mathbb{Z}_2$-coefficients in this thesis, we will use the notation $C_k(X) := C_k(X, \mathbb{Z}_2)$.
The linear map $\partial_k : C_{k+1}(X) \to C_k(X)$ defined on a simplex $\sigma = (v_0, \dots, v_{k+1}) \in X$ as

$$\partial_k(\sigma) = \sum_{i=0}^{k+1} (-1)^i (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{k+1})$$

is called the k-th **boundary map**. (Recall, that the boundary maps have the property $\partial_k \circ \partial_{k+1} = 0$)

The k-th **homology group** of X is then defined as:

$$H_k(X) := \frac{\ker(\partial_{k-1})}{\operatorname{Im}(\partial_k)},$$

where the elements from $\ker(\partial_{k-1})$ are called k-**cycles** and the elements from $\operatorname{Im}(\partial_k)$ are called k-**boundaries**.
Dualizing this concept, we get the k-th **cochain group** of X by

$$C^k(X) := C^k(X, \mathbb{Z}_2) := \{\varphi : C_k(X) \to \mathbb{Z}_2 : \varphi \text{ is a linear map}\},$$

whose elements are called k-**cochains**, the k-th **coboundary map**
$\delta^k : C^k(X) \to C^{k+1}(X)$ by $\delta^k(\varphi) := \varphi \circ \partial_k$, and the k-th cohomology group of X by:

$$H^k(X) := \frac{\ker(\delta^k)}{\operatorname{Im}(\delta^{k-1})}$$

Furthermore, the sequence

$$\cdots \xrightarrow{\partial_{k+1}} C_{k+1}(X) \xrightarrow{\partial_k} C_k(X) \xrightarrow{\partial_{k-1}} C_{k-1}(X) \xrightarrow{\partial_{k-2}} \cdots$$

is called a **chain complex** and

$$\cdots \xrightarrow{\delta^{k-2}} C^{k-1}(X) \xrightarrow{\delta^{k-1}} C^k(X) \xrightarrow{\delta^k} C^{k+1}(X) \xrightarrow{\delta^{k+1}} \cdots$$

is called a **cochain complex**.

Since, we are working with $\mathbb{Z}_2$-coeficients only, there is a very intuitive way to talk about chains (cochains, respectively). A k-chain is a linear combination of simplices with coefficients in $\mathbb{Z}_2$, so it can just be considered as a subset of the uniform k-skeleton of the underlying simplicial complex X. Furthermore, there is a one-to-one correspondence between chains and cochains, so to every chain $c \in C_k(X)$ we can associate its characteristic cochain we denote by $c^* \in C^k(X)$ and for every cochain $\varphi \in C^k(X)$ there exists a unique chain $c \in C_k(X)$, such that we have $c^* = \varphi$.
Let $c \in C_k(X)$ be some chain and $\varphi \in C^k(X)$ some cochain, then we denote the **evaluation** of φ on c as

$$\langle \varphi, c \rangle := \varphi(c) \in \mathbb{Z}_2,$$

and the **support** of φ as

$$\operatorname{supp}(\varphi) := \{\sigma \in X : \langle \varphi, \sigma \rangle = 1\}$$

Furthermore, we define the support of a chain $c \in C_k(X)$ as $\operatorname{supp}(c) := \operatorname{supp}(c^*)$.
Note, that for simplicity we will omit natural inclusion maps of the type
$i : \Delta^{[n]} \longrightarrow \Delta^{[n+d]}$ for some $n, d \in \mathbb{N}$ when calculating with chains / cochains, so that for $\varphi \in C^k(\Delta^{[n]})$, $\psi \in C^k(\Delta^{[n+d]})$ we will write $\varphi + \psi \in C^k(\Delta^{[n+d]})$ instead of $i(\varphi) + \psi \in C^k(\Delta^{[n+d]})$. It should always be clear from the context what we mean.

0.3 Graphs & Hypergraphs

Let V be some set and $E \subseteq \binom{V}{2}$ (we will always use the notation
$\binom{V}{k} := \{S \in 2^V : |S| = k\}$ to denote the set of all subsets of cardinality k of a set V). Then the pair $G = (V, E)$ is called a **(simple) graph**, where the elements of V are called **vertices** and the elements of E are called **edges**. Since we only consider simple graphs (undirected graphs with no loops or double edges) in this thesis, we will just call them graphs. Even though we only consider undirected graphs, we want to stick to the common notation and denote an edge by $e = (v, w)$ instead of using set brackets $e = \{v, w\}$.
Note, that a simple graph can be considered as a 1-dimensional simplicial complex, where V is the 0-skeleton and E is the uniform 1-skeleton.
According to the terminology of simplicial complexes we call a graph **finite**, if the number of vertices is finite.
A graph $G = (V, E)$ is called **complete**, if $E = \binom{V}{2}$ and a graph $G' = (V', E')$ is called a **subgraph** of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$. Furthermore, we call a subgraph $G' = (V', E')$ of $G = (V, E)$ a **spanning subgraph** if $E' \subseteq E$ and $V' = V$.
Let $G = (V, E)$ be a graph, then $\deg_G(v) := |\{w \in V : (v, w) \in E\}|$ is called the **degree** of the vertex $v \in V$.
A **hypergraph** is a pair $H = (V, E)$, where the edge set $E \subseteq 2^V$ can be any set of

subsets of V. Note, that every simplicial complex is a hypergraph, but not every hypergraph is a simplicial complex, since subsets of an edge do not have to be an edge in a hypergraph. If all edges of a hypergraph have the same cardinality k, then we call it a k-**uniform hypergraph**. Analogously to the terminology of graphs, a hypergraph $H' = (V', E')$ is called a **subhypergraph** of the hypergraph $H = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

Chapter 1

Introduction

The classical Cheeger constant of a graph is a well-studied object and can be intuitively be imagined as a measure of connectivity of a graph, as follows: From a connected graph, we can delete edges to make it become disconnected, and so there exists a smallest (in terms of numbers of vertices) connected component. Now, the Cheeger constant is the smallest quotient that can appear by deviding the number of removed edges by the size of the smallest of the resulting connected components. So, formally the Cheeger constant of a graph $G = (V, E)$ is then defined by:

$$h(G) := \min \left\{ \frac{|\delta(A)|}{|A|} : A \subset V, 1 \leq |A| \leq \frac{|V|}{2} \right\},$$

with $\delta(A) := \{e = (v, w) \in E : v \in A, w \in V \setminus A\}$.

There is a lot of literature that studies this Cheeger constant for arbitrary graphs, whereas it is pretty easy to determine for the complete graph on n-vertices K_n, where we have:

$$h(K_n) = \left\lceil \frac{n}{2} \right\rceil$$

In this thesis we want to investigate a higher-dimensional generalization of this Cheeger constant that was first introduced by Lineal and Meshulam (see [2]) and later independently by Gromov (see [3]) and is defined by the following construction:

Let X be a simplicial complex and $\varphi \in C^k(X)$, such that $\|\delta^{k-1}(\phi) + \varphi\| \geq \|\varphi\|$ for every $\phi \in C^{k-1}(X)$, where δ^{k-1} denotes the coboundary map $C^{k-1}(X) \to C^k(X)$, then we call φ a k-**cosystole**.
For general cochains $\varphi \in C^k(X)$ we define the **cosystolic norm** of φ by:

$$\|\varphi\|_{csy} := \min \left\{ \|\delta^{k-1}(\phi) + \varphi\| : \phi \in C^{k-1}(X) \right\}$$

Furthermore, any $c \in C^k(X)$, satisfying $c = \delta^{k-1}(\phi) + \varphi$ and $\|c\| = \|\varphi\|_{csy}$ is called a **cosystolic form** of φ.

The quotient

$$\|\varphi\|_{exp} := \frac{\|\delta^k(\varphi)\|}{\|\varphi\|_{csy}}$$

is called the **coboundary expansion** of φ and

$$h_k(X) := \min_{\substack{\varphi \in C^k(X) \\ \varphi \notin \mathrm{Im}(\delta^{k-1})}} \|\varphi\|_{exp}$$

is called the k-th **Cheeger constant** of X.

A cosystole $\varphi \in C^k(X)$ is called a **Cheeger cosystole** if $h_k(X) = \|\varphi\|_{exp}$.

We could even define the Cheeger constants more generally for polyhedral complexes (see [6]), but in this thesis we will only focus simplicial complexes.
Note, that the classical Cheeger constant of a graph coincides with the 0-th Cheeger constant $h_0(X)$. For larger k's the value of $h_k(X)$ is not even known for all standard simplices $X = \Delta^{[n]}$. By now we only have the estimate

$$\frac{n}{k+2} \leq h_k(\Delta^{[n]}) \leq \left\lceil \frac{n}{k+2} \right\rceil \tag{1.1}$$

that was proven by Wallach and Meshulam (see [4]), so we have the exact value $h_k(\Delta^{[n]}) = \frac{n}{k+2}$ when $k+2$ devides n. In [1] Kozlov showed that the upper bound is archieved when $k = n-3$, so we have $h_{n-3}(\Delta^{[n]}) = 2$ and furthermore he showed that $h_1(\Delta^{[n]}) = \frac{n}{3}$ even holds for every n that is not a power of 2.

The classical 0-th Cheeger constant of a graph is still pretty easy to understand intuitively, whereas the higher-dimensional generalizations raise the question what a measure of connectivity could mean in those cases. The following observation might help the reader to develop this intuition. For a graph $G = (V, E)$ the classical Cheeger constant $h(G)$ equals 0 if and only if G is disconnected, since for any non-empty proper subset of vertices $A \subset V$, the set $\delta(A)$ is empty, if and only if there is no edge between A and $V \setminus A$. More generally, for a simplicial complex X, the k-th Cheeger constant $h_k(X)$ equals zero, if and only if the k-th homology group $H_k(X)$ of X is not trivial, as follows:
For any cochain $\varphi \in C^k(X)$ we have $\|\varphi\|_{csy} > 0$ if and only if $\varphi \notin \mathrm{Im}(\delta^{k-1})$ and $\delta^k(\varphi) = 0$ if and only if $\varphi \in \ker(\delta^k)$, but the existence of a cochain satisfying these two properties is equivalent to $\mathrm{Im}(\delta^{k-1}) \subsetneq \ker(\delta^k)$, which just means that $H^k(X) \neq 0$.
So, we have to study the Cheeger constants of those simplicial complexes whose homology vanishes and the most obvious example of those complexes is the standard simplex $\Delta^{[n]}$.

In the first part of this thesis we will develop some theory about the cosystolic norm of cochains, since a better understanding of cosystoles seems to be the key knowledge to determine the Cheeger constants. In the second part we will focus on the special case of 1-cosystoles and the first Cheeger constant, where we have an interesting graph theoretical approach introduced by Kozlov in [1], that seems to be suited well to investigate those cosystoles in a combinatorial way. The last chapter adresses an interesting purely combinatorial observation about partitioning consecutive numbers, which is not directly related to the topic of Cheeger constants, but arose during our research and might be helpful in other branches of combinatorics.

Chapter 2

High-dimensional cosystoles and Cheeger constants

When we want to determine whether a cochain is a cosystole or not (or in other words, to determine the cosystolic norm of a cochain) until now we only have the original definition of cosystolicity, which does not seem to be very useful. In this chapter we want to develop tools to get hands on this problem, especially by estimating the cosystolic norm in various situations and investigating the structure how cosystoles in certain simplicial complexes are arranged. Furthermore, at the end of this chaper we will develop some interesting statements about the shape of Cheeger cosystoles.

2.1 Piercing sets and the cycle detection theorem

The following definition is adopted from [6].

Definition 1. *Let V be some set and $\mathcal{F} \subseteq 2^V$ a family of finite subsets of V. A subset $P \subseteq V$ is called a* ***piercing set*** *of $\mathcal{F}$ if we have $P \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The minimal cardinality of a piercing set of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is called the* ***piercing number*** *of $\mathcal{F}$. For a $v \in V$ and an $F \in \mathcal{F}$ we say that F* ***is pierced by*** *v, if $v \in F$.*

Example 1. *Let $V := \{1, 2, 3, 4, 5\}$ and $\mathcal{F} := \{\{1, 2\}\{2, 3, 4\}, \{1, 5\}, \{2, 4, 5\}\}$, then we have $\tau(\mathcal{F}) = 2$, since for example $P := \{2, 5\}$ is a minimal piercing set of $\mathcal{F}$.*

Later we will talk about piercing numbers and piercing sets of families of k-chains, which we define as follows:

Definition 2. *Let X be a simplicial complex and $\mathcal{F} \subseteq C_k(X)$ a family of k-chains. The* ***piercing sets*** *and the* ***piercing number*** *of $\mathcal{F}$ are defined as the piercing sets and the piercing number of the family $\{\mathrm{supp}(F) : F \in \mathcal{F}\}$.*

In [6] Kozlov stated the following useful method to bound the cosystolic norm of a cochain.

Theorem 1 (The cycle detection theorem)**.** *Let X be a simplicial complex and $\varphi \in C^k(X)$. Let now $\mathcal{F} = \{\alpha_1, \ldots, \alpha_t\}$ be a family of k-cycles in $C_k(X)$, such that $\langle \varphi, \alpha_i \rangle = 1$ for all*

$1 \le i \le t$, *then we have:*

$$\|\varphi\|_{csy} \ge \tau(\mathcal{F})$$

Proof. Let $\psi \in C^{k-1}(X)$, then for any $1 \le i \le t$ we have:

$$\begin{aligned}\langle \varphi + \delta^{k-1}(\psi), \alpha_i\rangle &= \langle \varphi, \alpha_i\rangle + \langle \delta^{k-1}(\psi), \alpha_i\rangle \\ &= \langle \varphi, \alpha_i\rangle + \langle \psi, \partial_{k-1}(\alpha_i)\rangle \\ &= \langle \varphi, \alpha_i\rangle + \langle \psi, 0\rangle \\ &= \langle \varphi, \alpha_i\rangle = 1\end{aligned}$$

This means that we have $\mathrm{supp}(\varphi + \delta^{k-1}(\psi)) \cap \mathrm{supp}(\alpha_i) \neq \emptyset$ for all $1 \le i \le t$, so $\mathrm{supp}(\varphi + \delta^{k-1}(\psi))$ is a piercing set of $\mathcal{F}$ and we get:

$$\|\varphi + \delta^{k-1}(\psi)\| = |\mathrm{supp}(\varphi + \delta^{k-1}(\psi))| \ge \tau(\mathcal{F})$$

Since ψ was chosen arbitrarily we get the requested result. □

The following corollary (a special case of the preceding theorem) was also stated by Kozlov in [6].

Corollary 1. *Let X be a simplicial complex and $\varphi \in C^k(X)$.*
Let now $\mathcal{F} = \{\alpha_1, \dots, \alpha_{\|\varphi\|}\}$ be a family of k-cycles in $C_k(X)$, such that $\langle \varphi, \alpha_i\rangle = 1$ for all $1 \le i \le \|\varphi\|$ and $\mathrm{supp}(\alpha_i) \cap \mathrm{supp}(\alpha_j) = \emptyset$ for all $i \neq j$, then φ is a cosystole.

Proof. Since the supports of the cycles $\alpha_1, \dots, \alpha_{\|\varphi\|}$ are pairwise disjoint, we obviously have $\tau(\mathcal{F}) = \|\varphi\|$ and using the cycle detection theorem we are done. □

Example 2. *Consider the cochain*

$$\varphi = (\{1,2,4\} + \{2,5,6\} + \{3,4,6\} + \{2,4,6\})^* \in C^2(\Delta^{[6]})$$

(Figure 2.1 illustrates how the support of this cochain can be imagined) and the family of cycles

$$\begin{aligned}\mathcal{F} = \{&\{1,2,3\} + \{1,2,4\} + \{1,4,3\} + \{2,4,3\},\\ &\{1,2,5\} + \{1,2,6\} + \{1,5,6\} + \{2,5,6\},\\ &\{3,4,5\} + \{3,4,6\} + \{3,5,6\} + \{4,5,6\},\\ &\{1,3,5\} + \{1,3,6\} + \{1,4,5\} + \{1,4,6\}+\\ &\{2,3,5\} + \{2,3,6\} + \{2,4,5\} + \{2,4,6\}\} \subset C_2(\Delta^{[6]}).\end{aligned}$$

It is easy to check that we have $\langle \varphi, \alpha\rangle = 1$ for all $\alpha \in \mathcal{F}$ and $\tau(\mathcal{F}) = 4$, so we get $\|\varphi\|_{csy} \ge 4$ by the cycle detection theorem. We can immediately see, that φ is a cosystole, either by the fact, that we always have $\|\varphi\|_{csy} \le \|\varphi\|$, so in our case we have $\|\varphi\| = 4 = \|\varphi\|_{csy}$, or by using the preceding corollary, since the supports of the cycles in $\mathcal{F}$ are pairwise disjoint. Note, that φ is even a Cheeger cosystole, since we have $\|\varphi\|_{exp} = \frac{6}{4}$, so we know that for the *case $n = 6$ and $k = 2$ the lower bound of the estimate 1.1 is archieved, even if $k + 2$ does not devide n.*

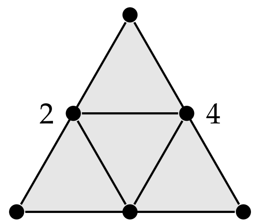

FIGURE 2.1: The support of a 2-Cheeger cosystole (The gray triangles represent the four 2-simplices)

To get the best possible results using the cycle detection theorem, the challenge is now for a certain cochain φ to find families of cycles such that they have a large piercing number and φ evaluates to 1 on every cycle. The following construction seems to be suited well to get hands on this problem.
For a cochain $\varphi \in C^k(X)$ we define the following set of cycles:

$$\mathcal{T}_\varphi := \left\{ \partial_k(\sigma) : \sigma \in \text{supp}(\delta^k(\varphi)) \right\}$$

Proposition 1. *Let $\varphi \in C^k(X)$, then we have:*

$$\|\varphi\|_{csy} \geq \tau(\mathcal{T}_\varphi)$$

Proof. By the definition of the coboundary map, we obviously have $\langle \varphi, \partial_k(\sigma) \rangle = 1$ for all $\partial_k(\sigma) \in \mathcal{T}_\varphi$ and so by the cycle detection theorem we are done. □

Note that in many cases, even the equality $\|\varphi\|_{csy} = \tau(\mathcal{T}_\varphi)$ can be reached, but unfortunately this is not always true as the following example shows.

Example 3. *Let $\varphi \in C^2(\Delta^{[6]})$ be the cosystole from Example 2. We have:*

$$\begin{aligned}
\mathcal{T}_\varphi = \{ &\{1,2,3\} + \{1,2,4\} + \{1,3,4\} + \{2,3,4\}, \\
&\{1,2,4\} + \{1,2,5\} + \{1,4,5\} + \{2,4,5\}, \\
&\{2,3,5\} + \{2,3,6\} + \{2,5,6\} + \{3,5,6\}, \\
&\{1,2,5\} + \{1,2,6\} + \{1,5,6\} + \{2,5,6\}, \\
&\{3,4,5\} + \{3,4,6\} + \{3,5,6\} + \{4,5,6\}, \\
&\{1,3,4\} + \{1,3,6\} + \{1,4,6\} + \{3,4,6\} \}.
\end{aligned}$$

It is easy to check that we get $\tau(\mathcal{T}_\varphi) = 3$, but as shown in Example 1 we have $\|\varphi\|_{csy} = 4$.

It seems to be very difficult to determine the piercing number of $\mathcal{T}_\varphi$ explicitely, but the concept of piercing complexes might be useful on the way to solve this problem.

2.2 Piercing complexes

Let V be some set, $\mathcal{F} \subset 2^V$ a family of subsets and P a piercing set of $\mathcal{F}$. Then for any $v \in V$ obviously $P \cup \{v\}$ is also a piercing set of $\mathcal{F}$. We can use this fact to construct a simplicial complex, which contains all information about the piercing sets for a given family of sets as follows:

Definition 3. *Let V be a set and $\mathcal{F} \subset 2^V$ a family of subsets. Then the* ***piercing complex*** *of $\mathcal{F}$ is defined as:*

$$\Delta_{\mathcal{F}} := \left\{ V' \subseteq V, : (V \setminus V') \cap F \neq \emptyset \textit{ for all } F \in \mathcal{F} \right\}$$

So, $\Delta_{\mathcal{F}}$ consists of all subsets of V, such that their complements in V are piercing sets of $\mathcal{F}$ and indeed, $\Delta_{\mathcal{F}}$ defines a simplicial complex, since deleting an element from the complement of a piercing set is equivalent to adding an element to a piercing set, which preserves the condition of being a piercing set.

Example 4. *Let V be an arbitrary set and $\mathcal{F} := 2^V$ its power set. Then the piercing complex $\Delta_{\mathcal{F}}$ is empty, since even the complement of a single vertex $v \in V$ is not a piercing set of $\mathcal{F}$. More general for an arbitrary set V we have that $\Delta_{\mathcal{F}}$ is empty if and only if $\{v\} \in \mathcal{F}$, for all $v \in V$.*
On the other hand $\Delta_{\mathcal{F}}$ is a complete simplex on $|V|$ vertices if and only if $\mathcal{F}$ is empty, since only in this case even the empty set is a piercing set of $\mathcal{F}$.

We can now reformulate the question of determining the piercing number $\tau(\mathcal{F})$ by asking for the dimension of $\Delta_{\mathcal{F}}$, since we have the equality:

$$\tau(\mathcal{F}) = |V| - \dim(\Delta_{\mathcal{F}}) - 1$$

Since our main interest in this section will be to investigate the piercing complex of $\mathcal{T}_\varphi$ for a given cochain $\varphi \in C^k(X)$ (where X is some simplicial complex) we will use a shorter notation for this piercing complex and set $\Delta_\varphi := \Delta_{\mathcal{T}_\varphi}$. Then the preceding formula turns to:

$$\tau(\mathcal{T}_\varphi) = |X^{(k)}| - \dim(\Delta_\varphi) - 1$$

Theorem 2. *Let X be a simplicial complex and $\varphi \in C^k(X)$, then we have:*

$$\tilde{H}_i(\Delta_\varphi) \cong 0 \quad \textit{for all } i \leq k-1$$

Proof. Let $\varphi \in C^k(X)$ be chosen arbitrarily. Then for all $\sigma \in \operatorname{supp}(\delta^k(\varphi))$ we have $|\operatorname{supp}(\partial_k(\sigma))| = k+2$. Therefore, for all $S \subset X^{(k)}$ such that $|S| \leq k+1$ we have that

$X^{(k)} \setminus S$ is a piercing set of $\mathcal{T}_\varphi$. This just means, that Δ_φ has a full k-skeleton, so we immediately get $\tilde{H}_i(\Delta_\varphi) \cong 0$ for all $i \leq k-1$. $\square$

Definition 4. *Let X be a simplicial complex on the vertex set V. Then the simplicial complex*

$$X^\vee := \{\sigma \subseteq V : V \setminus \sigma \notin X\}$$

is called the ***Alexander dual*** *of X.*

The following theorem can be found in [8].

Theorem 3 (The Alexander duality theorem)**.** *Let X be a simplicial complex on n vertices and $X^\vee$ its Alexander dual. Then we have:*

$$\tilde{H}_i(X) \cong \tilde{H}^{n-i-3}(X)$$

Definition 5. *Let V be some set and $\mathcal{F} \subseteq 2^V$ a family of subsets of V. Then the simplicial complex*

$$\Delta[\mathcal{F}] := \{\sigma \subseteq V : \textit{there exists an } F \in \mathcal{F} \textit{ such that } \sigma \subseteq F\}$$

is called the ***induced complex*** *of $\mathcal{F}$.*

The following statement was developed in [9].

Proposition 2. *Let V be some set and $\mathcal{F} \subseteq 2^V$ a family of subsets of V. Then we have:*

$$\Delta\left[\bar{\mathcal{F}}\right]^\vee = \Delta_{\mathcal{F}}$$

where we set $\bar{\mathcal{F}} := \{V \setminus F : F \in \mathcal{F}\}$.

Proof. We have:

$$\begin{aligned}
& \sigma \in \Delta\left[\bar{\mathcal{F}}\right]^\vee \\
\iff & V \setminus \sigma \notin \Delta\left[\bar{\mathcal{F}}\right] \\
\iff & \nexists F \in \bar{\mathcal{F}} : V \setminus \sigma \subseteq F \\
\iff & \nexists F \in \bar{\mathcal{F}} : (V \setminus \sigma) \cap (V \setminus F) = \emptyset \\
\iff & \nexists F' \in \mathcal{F} : (V \setminus \sigma) \cap F' = \emptyset \\
\iff & V \setminus \sigma \text{ is a piercing set of } \mathcal{F} \\
\iff & \sigma \in \Delta_{\mathcal{F}}
\end{aligned}$$

$\square$

Theorem 4. *Let X be a finite simplicial complex and $\varphi \in C^k(X)$, then we have:*

$$\tilde{H}_k(\Delta_\varphi) \cong 0$$

Proof. For all $\partial_k(\sigma) \in \mathcal{T}_\varphi$ we have $|\text{supp}(\partial_k(\sigma))| = k+2$, so we get:

$$\dim(\Delta[\bar{\mathcal{T}}_\varphi]) = |X^{(k)}| - (k+2) - 1 = |X^{(k)}| - k - 3$$

Now, there exist no two simplices of dimension $|X^{(k)}| - k - 3$ in $\Delta[\bar{\mathcal{T}}_\varphi]$, such that they have a face in common, so we have:

$$\tilde{H}_{|X^{(k)}|-k-3}(\Delta[\bar{\mathcal{T}}_\varphi]) \cong 0$$

By the Alexander duality theorem and Proposition 2 we get:

$$\tilde{H}_k(\Delta_\varphi) \cong \tilde{H}^k(\Delta_\varphi) = \tilde{H}^{|X^{(k)}|-(|X^{(k)}|-k-3)-3}(\Delta_\varphi) \cong 0,$$

where the first isomorphy is true, because we consider homology / cohomology over a field and Δ_φ is finite. □

2.3 Large cosystoles of a simplex

In this section we focus our investigations onto the question, what is the largest norm, a cosystole can attain when the underlying simplicial complex is a complete simplex, where the first statement we make is still valid for general simplicial complexes.

Let us first introduce a simplicial complex which represents all the information about cosystoles we need.

Definition 6. *Let X be a simplicial complex and $1 \leq k \leq \dim(X)$. Then the simplicial complex*

$$\mathcal{C}_k(X) := \left\{ S \subseteq X^{(k)} : \left(\sum_{\sigma \in S} \sigma \right)^* \text{ is a cosystole} \right\}$$

*is called the **k-cosystolic complex** of X.*

Obviously, $\mathcal{C}_k(X)$ defines a simplicial complex, since removing a simplex from the support of a cosystole and considering the corresponding cochain again preserves cosystolicity.

Now, for any simplicial complex X and any $1 \leq k \leq \dim(X)$, we define the following number (the largest norm, a cosystole can attain):

$$C_{max}(X,k) := \max \left\{ \|\varphi\|_{csy} : \varphi \in C^k(X) \right\}$$

Note, that we have the relation $\dim(\mathcal{C}_k(X)) = C_{max}(X,k) - 1$.

For the sake of completeness we still define the following number of the smallest norm a non-cosystolic cochain can attain:

$$C_{min}(X,k) := \min \left\{ \|\varphi\| : \varphi \in C^k(X),\, \|\varphi\| > \|\varphi\|_{csy} \right\}$$

Lemma 1. *If k is odd, then we have:*

$$C_{max}(X,k) \geq \tau(\mathcal{T}_\varphi),$$

with $\varphi = \left(\sum_{\sigma \in X^{(k)}} \sigma\right)^*$.

Proof. Since k is odd we have $\langle \varphi, c \rangle = 1$ for all $c \in \mathcal{T}_\varphi$, so by the cycle detection theorem we get $\|\varphi\|_{csy} \geq \tau(\mathcal{T}_\varphi)$ and we are done. □

A direct consequence of the preceding lemma is the following estimate. For simplicity we want to introduce the notation $\mathcal{T}_n^k := \mathcal{T}_\varphi$ for $\varphi := \left(\sum_{\sigma \in \binom{[n]}{k+1}} \sigma\right)^*$.

Proposition 3. *Let k be odd, then we have:*

$$C_{max}(\Delta^{[n]},k) \geq \left\lceil \frac{\binom{n}{k+2}}{n-k-1} \right\rceil$$

Proof. Obviously, we have $|\mathcal{T}_n^k| = \binom{n}{k+2}$, so since any simplex $\sigma \in \binom{[n]}{k+1}$ intersects the support of exactly $n-k-1$ cycles from $\mathcal{T}_n^k$, any piercing set of $\mathcal{T}_n^k$ must contain at least $\left\lceil \frac{\binom{n}{k+2}}{n-k-1} \right\rceil$ elements and by Lemma 1 we are done. □

Using the preceding lemma we can determine a lower bound of the maximal size of the 1-dimensional cosystoles in a simplex. A more elementary proof of this estimate is given as Proposition 8) in the following chapter, where we will even see, that equality can be reached, shown in Theorem 7.

Proposition 4. $C_{max}(\Delta^{[n]},1) \geq \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor$

Proof. Asking for the smallest piercing set of $\mathcal{T}_n^1$ is equivalent to asking for the largest triangle-free graph (i.e. a graph on n vertices, containing as many edges as posssible, but no complete graph on 3 vertices as a subgraph) and taking the complement. Mantel's theorem (see [7]) says, that a triangle-free graph on n vertices has at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, so we immediately get:

$$\tau(\mathcal{T}_n^1) = \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor$$

and by Lemma 1 we are done. □

Unfortunately, determining the piercing number of $\mathcal{T}_n^k$ for $k \geq 2$, or equivalently, determining the largest k-uniform hypergraph on n-vertices, containing no complete k-uniform hypergraph on $k+2$ vertices as a subhypergraph, seems to be very difficult (see [7]), so we can not use the preceding procedure to say something about $C_{max}(\Delta^{[n]},k)$ for larger k's in general.

Eventhough, we can exactly determine this number for the ultimate and the penultimate proper dimension as follows.

Theorem 5. *$C_{max}(\Delta^{[n]}, n-2) = 1$, for all $n \geq 3$*

Proof. Let $\sigma \in \binom{[n]}{n-1}$ be chosen arbitrarily, $\varphi := \sigma^* \in C^{n-2}(\Delta^{[n]})$ and
$\mathcal{F} := \{\alpha\}$, where α is the boundary of the single $(n-1)$-dimensional simplex in $\Delta^{[n]}$. Obviously, we have $\langle \varphi, \alpha \rangle = 1$, since $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\alpha) = \mathrm{supp}(\varphi)$ and $\tau(\mathcal{F}) = 1$, so by the cycle detection theorem we have $\|\varphi\|_{csy} \geq 1$.
Now, let $\sigma_1, \sigma_2 \in \binom{[n]}{n-1}$ be chosen arbitrarily again ($\sigma_1 \neq \sigma_2$) and $c := \sigma_1 \cap \sigma_2$. Then we have $\delta^{n-3}(c^*) + \sigma_1^* + \sigma_2^* = 0$, so there exists no $(n-2)$-cosystole attaining norm 2 and we are done. □

Lemma 2. *For $n \geq 4$ we have:*

$$\tau(\mathcal{T}_n^{n-3}) = \left\lceil \frac{n}{2} \right\rceil$$

Proof. For each $\sigma \in \binom{[n]}{n-2}$ there exist exactly two cycles $\alpha_1, \alpha_2 \in \mathcal{T}_n^{n-3}$
($\alpha_1 \neq \alpha_2$), such that $\sigma \in \mathrm{supp}(\alpha_1) \cap \mathrm{supp}(\alpha_2)$, so the largest possible number of cycles from $\mathcal{T}_n^{n-3}$ that can be pierced by one simplex is two. Furthermore, we have $\left|\mathcal{T}_n^{n-3}\right| = \binom{n}{n-1} = n$, so we get $\tau(\mathcal{T}_n^{n-3}) \geq \left\lceil \frac{n}{2} \right\rceil$.
On the other hand, for all $\alpha_1, \alpha_2 \in \mathcal{T}_n^{n-3}$ there exists a $\sigma \in \binom{[n]}{n-2}$, such that
$\sigma \in \mathrm{supp}(\alpha_1) \cap \mathrm{supp}(\alpha_2)$, so we get $\tau(\mathcal{T}_n^{n-3}) \leq \left\lceil \frac{n}{2} \right\rceil$. □

Lemma 3. *Let $S \subset \binom{[n]}{n-2}$, such that $|S| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$, then there exist $\sigma, \sigma' \in S$ ($\sigma \neq \sigma'$), such that $|\sigma \cap \sigma'| = n-3$.*

Proof. For $\sigma, \sigma' \in \binom{[n]}{n-2}$ the condition $|\sigma \cap \sigma'| < n-3$ is equivalent to the condition $([n] \setminus \sigma) \cap ([n] \setminus \sigma') = \emptyset$. Since we obviously have $|[n] \setminus \sigma| = 2$ for all $\sigma \in \binom{[n]}{n-2}$ we can find at most $\left\lfloor \frac{n}{2} \right\rfloor$ simplices $\sigma_1, \ldots, \sigma_{\left\lfloor \frac{n}{2} \right\rfloor} \in \binom{[n]}{n-2}$, such that the sets $[n] \setminus \sigma_1, \ldots, [n] \setminus \sigma_{\left\lfloor \frac{n}{2} \right\rfloor}$ are pairwise disjoint and we are done. □

Theorem 6. *$C_{max}(\Delta^{[n]}, n-3) = \left\lfloor \frac{n}{2} \right\rfloor$, for all $n \geq 4$*

Proof. Let $S \subset \binom{[n]}{n-2}$ be a minimal piercing set of $\mathcal{T}_n^{n-3}$ as constructed in the proof of Lemma 2 and $\varphi := S^* \in C^{n-3}(\Delta^{[n]})$.
If n is even, then $n-3$ is odd, so using Lemma 1 and Lemma 2 we have:

$$C_{max}(\Delta^{[n]}, n-3) \geq \tau(\mathcal{T}_n^{n-3}) = \frac{n}{2}$$

If n is odd then by the construction of φ there exists exactly one $\alpha' \in \mathcal{T}_n^{n-3}$, such that $\langle \varphi, \alpha' \rangle = 0$, since $|\mathrm{supp}(\alpha') \cap \mathrm{supp}(\varphi)| = 2$. Set $\mathcal{F} := \mathcal{T}_n^{n-3} \setminus \alpha'$, then we have $\langle \varphi, \alpha \rangle = 1$ for all $\alpha \in \mathcal{F}$ and $\tau(\mathcal{F}) \geq \tau(\mathcal{T}_n^{n-3}) - 1 = \left\lceil \frac{n}{2} \right\rceil - 1 = \left\lfloor \frac{n}{2} \right\rfloor$ by Lemma 2, so by the cycle detection theorem we get $C_{max}(\Delta^{[n]}, n-3) \geq \left\lfloor \frac{n}{2} \right\rfloor$. One the other hand let $\varphi \in C^{n-3}(\Delta^{[n]})$, such that $\|\varphi\| = \left\lfloor \frac{n}{2} \right\rfloor + 1$, then by Lemma 3 there exist $\sigma_1, \sigma_2 \in \mathrm{supp}(\varphi)$, such that $|\sigma_1 \cap \sigma_2| = n-3$. Now set $\psi := (\sigma_1 \cap \sigma_2)^* \in C^{n-4}(\Delta^{[n]})$, then we have $\|\delta^{n-4}(\psi)\| = 3$ and $\left|\mathrm{supp}(\delta^{n-4}(\psi)) \cap \mathrm{supp}(\varphi)\right| \geq 2$. Thus, we have

$\|\delta^{n-4}(\psi) + \varphi\| \leq \|\varphi\| - 1$ and φ can not be a cosystole, so we get $C_{max}(\Delta^{[n]}, n-3) \leq \lfloor \frac{n}{2} \rfloor$ and we are done. □

2.4 Embeddings of cosystoles

When searching for cosystoles another approach is to consider cochains, which are already proven to be cosystolic and try to, roughly speaking, change or extend them to create new cosystoles. The most obvious way to do this is just to add a vertex to the underlying simplex by the embedding:

$$\begin{aligned} i_n^k : C^k(\Delta^{[n]}) &\longrightarrow C^k(\Delta^{[n+1]}) \\ \varphi &\longmapsto \left(\sum_{\sigma \in \operatorname{supp}(\varphi)} \sigma \right)^* \end{aligned}$$

The first observation is, that cosystolicity is preserved by this embedding.

Lemma 4. *Let $\varphi \in C^k(\Delta^{[n]})$ be a cosystole, then $i_n^k(\varphi) \in C^k(\Delta^{[n+1]})$ is a cosystole.*

Proof. Let $\phi \in C^{k-1}(\Delta^{[n+1]})$ and consider the partition $\operatorname{supp}(\phi) = C_1 \cup C_2$ with $C_1 := \{\sigma \in \operatorname{supp}(\phi) : n+1 \in \sigma\}$ and $C_2 = \{\sigma \in \operatorname{supp}(\phi) : n+1 \notin \sigma\}$. Now set $\phi_1 := C_1^*$ and $\phi_2 := C_2^*$, then we have

$$\delta^{k-1}(\phi) + i_n^k(\varphi) = \delta^{k-1}(\phi_1) + \delta^{k-1}(\phi_2) + i_n^k(\varphi),$$

but $\|\delta^{k-1}(\phi_2) + i_n^k(\varphi)\| \geq \|i_n^k(\varphi)\|$ holds by assumption and $\operatorname{supp}(\delta^{k-1}(\phi_1)) \cap \operatorname{supp}(i_n^k(\varphi)) = \emptyset$ holds by the construction of ψ_1, so we get

$$\|\delta^{k-1}(\phi) + i_n^k(\varphi)\| \geq \|i_n^k(\varphi)\|$$

and we are done. □

Furthermore, if φ is a cosystole, the coboundary expansions of φ and $i_n^k(\varphi)$ are related as follows.

Proposition 5. *Let $\varphi \in C^k(\Delta^{[n]})$ be a cosystole, then we have:*

$$\|i_n^k(\varphi)\|_{exp} = \|\varphi\|_{exp} + 1$$

Proof. By Lemma 4 we know that $i_n^k(\varphi)$ is a cosystole and we have $\|\delta^k(i_n^k(\varphi))\| = \|\delta^k(\varphi)\| + \|\varphi\|$, since for every simplex σ in the support of φ, we get the additional simplex $(\sigma, n+1)$ in the support of $\delta^k(i_n^k(\varphi))$. Hence, we have:

$$\|i_n^k(\varphi)\|_{exp} = \frac{\|\delta^k(\varphi)\| + \|\varphi\|}{\|\varphi\|} = \|\varphi\|_{exp} + 1$$

□

2.5 Multi-suspensions

Another approach to create new cosystoles is to not only include a cochain into a higher dimensional simplex, but even increase the dimension of the cochain's support, which can be realized by the so called multi-suspensions.

Definition 7. *Let $0 \leq k \leq n-1$ and $d \geq 1$, then we call*

$$\begin{aligned} \operatorname{sus}_{n,k}^d : C^k(\Delta^{[n]}) &\longrightarrow C^{k+1}(\Delta^{[n+d]}) \\ \varphi &\longmapsto \left(\sum_{m=n+1}^{d} \left(\sum_{\sigma \in \operatorname{supp}(\varphi)} (\sigma, m) \right) \right)^* \end{aligned}$$

*the **suspension map of degree d**. Note, that $\operatorname{sus}_{n,k}^d$ can also be easily defined on chain complexes by:*

$$\begin{aligned} \operatorname{sus}_{n,k}^d : C_k(\Delta^{[n]}) &\longrightarrow C_{k+1}(\Delta^{[n+d]}) \\ c &\longmapsto \sum_{\sigma \in \operatorname{supp}(\operatorname{sus}_{n,k}^d(c^*))} \sigma \end{aligned}$$

Lemma 5. *Let $\varphi \in C^k(\Delta^{[n]})$ and $\mathcal{F} = \{\alpha_1, \ldots, \alpha_t\} \subset C_k(\Delta^{[n]})$ be a family of cycles, such that $\langle \varphi, \alpha_i \rangle = 1$ for all $i = 1, \ldots, t$. Then for each $d \geq 1$ there exists a family of cycles $\mathcal{F}' = \left\{\alpha'_{1,1}, \ldots, \alpha'_{t,d}\right\} \subset C_{k+1}(\Delta^{[n+d]})$, such that $\left\langle \operatorname{sus}_{n,k}^d(\varphi), \alpha'_{i,j} \right\rangle = 1$ for all $i = 1, \ldots, t$ and $j = 1, \ldots, d$.*

Proof. For each $i = 1, \ldots, t$ let $c_i \in C_{k+1}(\Delta^{[n]})$, such that $\partial_k(c_i) = \alpha_i$. For a simplex $\sigma \in \binom{[n]}{k+1}$ and some $n+1 \leq j \leq n+d$ we have

$$\partial_k\left((\sigma, j)\right) = \left(\sum_{\sigma' \in \operatorname{supp}(\partial_{k-1}(\sigma))} (\sigma', j) \right) + \sigma$$

So, we get

$$
\begin{aligned}
\partial_k\left(\sum_{\sigma\in\operatorname{supp}(\alpha_i)}(\sigma,j)\right) &= \sum_{\sigma\in\operatorname{supp}(\alpha_i)}\partial_k\left((\sigma,j)\right)\\
&= \sum_{\sigma\in\operatorname{supp}(\alpha_i)}\left(\left(\sum_{\sigma'\in\operatorname{supp}(\partial_{k-1}(\sigma))}(\sigma',j)\right)+\sigma\right)\\
&= \left(\sum_{\sigma\in\operatorname{supp}(\alpha_i)}\ \sum_{\sigma'\in\operatorname{supp}(\partial_{k-1}(\sigma))}(\sigma',j)\right)+\alpha_i\\
&= 0+\alpha_i\\
&= \alpha_i,
\end{aligned}
$$

since we have

$$
\sum_{\sigma\in\operatorname{supp}(\alpha_i)}\ \sum_{\sigma'\in\operatorname{supp}(\partial_{k-1}(\sigma))}\sigma' = \partial_{k-1}(\alpha_i) = 0
$$

for all $i = 1,\ldots,t$.
Thus, for all $i = 1,\ldots,t$ and $j = n+1,\ldots,n+d$,

$$
\alpha_{i,j} := \sum_{\sigma\in\operatorname{supp}(\alpha_i)}(\sigma,j)+c_i
$$

defines a cycle, since we have

$$
\partial_k\left(\sum_{\sigma\in\operatorname{supp}(\alpha_i)}(\sigma,j)+c_i\right) = \alpha_i+\alpha_i = 0.
$$

Furthermore, we get

$$
\begin{aligned}
\left\langle \operatorname{sus}^d_{n,k}(\varphi),\alpha_{i,j}\right\rangle &= \left\langle \left(\sum_{m=n+1}^{d}\left(\sum_{\sigma\in\operatorname{supp}(\varphi)}(\sigma,m)\right)\right)^*, \sum_{\sigma\in\operatorname{supp}(\alpha_i)}(\sigma,j)+c_i\right\rangle\\
&= \left\langle \left(\sum_{\sigma\in\operatorname{supp}(\varphi)}(\sigma,j)\right)^*, \sum_{\sigma\in\operatorname{supp}(\alpha_i)}(\sigma,j)+c_i\right\rangle\\
&= 1,
\end{aligned}
$$

since we have

$$
\langle\varphi,\alpha_i\rangle = \left\langle\left(\sum_{\sigma\in\operatorname{supp}(\varphi)}\sigma\right)^*, \sum_{\sigma\in\operatorname{supp}(\alpha_i)}\sigma\right\rangle = 1.
$$

□

Definition 8. *Let V be some set, $\mathcal{F}\subseteq 2^V$ a family of subsets of V and $P=(p_1,\ldots,p_m)$ a finite ordered piercing set of $\mathcal{F}$. Furthermore, set $P_0 := \emptyset$ and for each $i = 1,\ldots,m$ set*

$P_i := \{F \in \mathcal{F} : F \cap \{p_i\} \neq \emptyset\} \setminus P_{i-1}$. Then the tuple $\lambda_P := (|P_1|, \ldots, |P_m|)$ is called the ***piercing sequence*** *of P.*

Proposition 6. *Let $\varphi \in C^k(\Delta^{[n]})$ and $\mathcal{F} = \{\alpha_1, \ldots, \alpha_t\} \subset C_k(\Delta^{[n]})$ be a family of cycles, such that $\langle \varphi, \alpha_i \rangle = 1$, for all $i = 1, \ldots, t$ and $P = (p_1, \ldots, p_m)$ an ordered piercing set of $\mathcal{F}$, such that $\tau(\mathcal{F}) = m$. Then for any $d \geq 1$ we have:*

$$\|\mathrm{sus}^d_{n,k}(\varphi)\|_{csy} \geq d \cdot |\{\beta \in \lambda_P : \beta \geq d\}| + \sum_{\beta \in \lambda_P,\, \beta < d} \beta$$

Proof. □

2.6 On Cheeger cosystoles when k+2 does not devide n

In [4] and [6] Meshulam and Kozlov gave examples of Cheeger cosystoles in $C^k(\Delta^{[n]})$ when $k+2$ devides n. Recall, that is this case the Cheeger constant is explicitly determined by $h_k(\Delta^{[n]}) = \frac{n}{k+2}$. For the case when $k+2$ does not devide n, we can still explore some interesting properties about Cheeger cosystoles in terms of their norm.

Proposition 7. *Let $\varphi \in C^k(\Delta^{[n]})$ be a Cheeger cosystole, such that $\|\varphi\|_{exp} = \frac{n}{k+2}$, then $k+2$ divides n or $k+2$ devides $\|\varphi\|$.*

Proof. We have $\|\delta^k(\varphi)\| = \frac{n\|\varphi\|}{k+2}$, since

$$\frac{\|\delta^k(\varphi)\|}{\|\varphi\|} = \|\varphi\|_{exp} = \frac{n}{k+2} = \frac{n\|\varphi\|}{(k+2)\|\varphi\|} = \frac{\frac{n\|\varphi\|}{k+2}}{\|\varphi\|},$$

but $\|\delta^k(\varphi)\|$ is a natural number, so $k+2$ must devide n or $\|\varphi\|$. □

Chapter 3

Cut-minimal graphs and Cheeger graphs of a simplex

In this chapter we start from the work of Kozlov (see [1]) in which a graph theoretical approach to the first Cheeger constant of a simplex was developed. In the course of this approach the so called cut-minimal graphs appeared, which exactly describe first dimensional cosystoles in a very intuitive way. As a consequence of the first main result of this chapter (Theorem 7) we will determinine the dimension and partly the homology of the simplicial complex $\mathcal{C}_1(\Delta^{[n]})$. In the second part of this chapter, we face the research on the first Cheeger constant of a simplex by investigating combinatorial properties of the Cheeger graphs which are exactly those cut-minimal graphs that determine this constant.

3.1 Cut-minimal graphs

3.1.1 Basic definitions and properties

The following definition and some words about motivation and intuition for it can be found in [1].

Definition 9. *Consider a graph $G = ([n], E)$. For any subsets $A, B \subset [n]$, define:*

$$E_G(A,B) := \{(v,w) \in E \,:\, v \in A, w \in B\}$$

and

$$NE_G(A,B) := \{(v,w) \notin E \,:\, v \in A, w \in B\}$$

*A graph $G = ([n], E)$ is called **cut-minimal**, if for every $S \subset [n]$ we have*

$$|E_G(S, [n] \setminus S)| \leq |NE_G(S, [n] \setminus S)|,$$

which is equivalent to

$$|E_G(S, [n] \setminus S)| \leq \frac{|S|(n - |S|)}{2}.$$

Note, that there is a one-to-one correspondence between the graphs on n vertices and the 1-chains (more precisely the elements of $C_1(\Delta^{[n]})$) as follows:

For a graph $G = ([n], E)$ set $c_G := \sum_{e \in E} e \in C_1(\Delta^{[n]})$ and for a chain $c \in C_1(\Delta^{[n]})$ set $G_c := ([n], E)$, with $E := \mathrm{supp}(c)$. Considering characteristic cochains we also get a one-to-one corresponsing between graphs on n vertices and 1-cochains and it is easy to see that a graph $G = ([n], E)$ is cut-minimal if and only if the corresponding cochain c_G^* is a cosystole.

Remark 1. *In fact for a graph G to be cut-minimal we only need to require the preceding condition holding for all $S \subset [n]$, such that $1 \leq |S| \leq \frac{n}{2}$, since for all $S \subset [n]$ we have $E_G(S, [n] \setminus S) = E_G([n] \setminus S, S)$ and $NE_G(S, [n] \setminus S) = NE_G([n] \setminus S, S)$.*

Example 5. *A simple cycle respresented by the graph $G = ([n], E)$ with $E := \{(i, i+1) : 1 \leq i \leq n-1\} \cup \{(n, 1)\}$ is cut-minimal for all $n \geq 7$ as follows. One can easily see that for all $S \subset [n]$, such that $|S| \leq \frac{n}{2}$, we have $|E_G(S, [n] \setminus S)| \leq 2|S|$ and the inequality $2|S| \leq \frac{|S|(n-|S|)}{2}$ holds for all $n \geq |S| + 4$, so by $|S| \leq \frac{n}{2}$ the statement is true for all $n \geq 7$.*

Definition 10. *For any $n \geq 1$ we define the set of all cut-minimal graphs on n vertices:*

$$CMG(n) := \{G = ([n], E) : G \text{ is cut-minimal}\}$$

3.1.2 Maximal cut-minimal graphs

Let us study the the topological and simplicial structure of the simplicial complex $\mathcal{C}_1(\Delta^{[n]})$ which was, more generally, already introduced in the preceding chapter. Understanding this complex will help us to explore combinatorial properties of the cut-minimal graphs on a certain number of vertices.

The first thing we will do is to determine the maximum number of edges, a cut-minimal graph on a certain number of vertices can have, which immediately gives us the dimension of $\mathcal{C}_1(\Delta^{[n]})$.

Proposition 8. $C_{max}(\Delta^{[n]}, 1) \geq \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$

Proof. Let us construct a graph G on n vertices as follows. Choose a set $V' \subset [n]$, such that $|V'| = \lceil \frac{n}{2} \rceil$ and connect each pair of vertices from V' by an edge. Then connect each pair of the remaining $\lfloor \frac{n}{2} \rfloor$ vertices by an edge. In other words our graph consists of two complete graphs. If n is even, they are identical, otherwise they differ by one vertex. In total we get $\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$ edges. We will show that this graph is cut-minimal.

Let $S \subset [n]$ and define $A := S \cap V'$ and $B := S \setminus A$. If n is even, we have:

$$\begin{aligned}
|E_G(S,[n] \setminus S)| &= |A|(\frac{n}{2} - |A|) + |B|(\frac{n}{2} - |B|) \\
&= \frac{n|A|}{2} - |A|^2 + \frac{n|B|}{2} - |B|^2 \\
&= \frac{n|S|}{2} - (|A|^2 + |B|^2) \\
&\leq \frac{n|S|}{2} - \frac{|S|^2}{2} \\
&= \frac{|S|(n - |S|)}{2},
\end{aligned}$$

where the inequality comes from:

$$\begin{aligned}
& (|A| - |B|)^2 \geq 0 \\
\iff \quad & |A|^2 - 2|A||B| + |B|^2 \geq 0 \\
\iff \quad & \frac{|A|^2}{2} - |A||B| + \frac{|B|^2}{2} \geq 0 \\
\iff \quad & |A|^2 + |B|^2 \geq \frac{|A|^2}{2} + |A||B| + \frac{|B|^2}{2} \\
\iff \quad & |A|^2 + |B|^2 \geq \frac{|S|^2}{2}.
\end{aligned}$$

If n is odd, we have:

$$\begin{aligned}
|E_G(S,[n] \setminus S)| &= |A|(\frac{n+1}{2} - |A|) + |B|(\frac{n-1}{2} - |B|) \\
&= \frac{n|A|}{2} + \frac{|A|}{2} - |A|^2 + \frac{n|B|}{2} - \frac{|B|}{2} - |B|^2 \\
&= \frac{n|S|}{2} - (|A|^2 + |B|^2 - \frac{|A| - |B|}{2}) \\
&\leq \frac{n|S|}{2} - \frac{|S|^2}{2} \\
&= \frac{|S|(n - |S|)}{2},
\end{aligned}$$

where the inequality comes from:

$$
\begin{aligned}
& (|A|-|B|)^2 - (|A|-|B|) \geq 0 \\
\iff\quad & (|A|-|B|)^2 - |A| + |B| \geq 0 \\
\iff\quad & |A|^2 + |B|^2 - |A| + |B| \geq 2|A||B| \\
\iff\quad & \frac{|A|^2}{2} + \frac{|B|^2}{2} - \frac{|A|}{2} + \frac{|B|}{2} \geq |A||B| \\
\iff\quad & |A|^2 + |B|^2 - \frac{|A|}{2} + \frac{|B|}{2} \geq \frac{|A|^2}{2} + \frac{2|A||B|}{2} + \frac{|B|^2}{2} \\
\iff\quad & |A|^2 + |B|^2 - \frac{|A|-|B|}{2} \geq \frac{(|A|+|B|)^2}{2} \\
\iff\quad & |A|^2 + |B|^2 - \frac{|A|-|B|}{2} \geq \frac{|S|^2}{2}.
\end{aligned}
$$

Hence, the constructed graph is cut-minimal. □

Note, that we already proved the equivalent statement as Proposition 4, since a simple calculation immediately shows that $\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} = \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor$. Eventhough, it seemed to be worth to state this alternative proof, as it gets along with pure combinatorics, without using any algebraic properties.

Remark 2. *For further proofs and calculations it might be helpful to keep mind that we have:*

$$
\begin{aligned}
\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} &= \frac{n^2-2n+1}{4}, && \textit{for } n \textit{ odd, and} \\
\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} &= \frac{n^2-2n}{4}, && \textit{for } n \textit{ even.}
\end{aligned}
$$

Now, we are able to state the main theorem of this section, saying that the preceding estimate is even an upper bound.

Theorem 7. $C_{max}(\Delta^{[n]}, 1) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$

Proof. In a cut-minimal graph the maximum degree of each vertex is $\left\lfloor \frac{n-1}{2} \right\rfloor$, so if n is even, we have:

$$
C_{max}(\Delta^{[n]}, 1) \leq \frac{n \left\lfloor \frac{n-1}{2} \right\rfloor}{2} = \frac{n^2-2n}{4} = 2\binom{\frac{n}{2}}{2} = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}
$$

If n is odd, the situation becomes more complicated. We only know that

$$
C_{max}(\Delta^{[n]}, 1) \leq \frac{n \left\lfloor \frac{n-1}{2} \right\rfloor}{2} = \frac{n \frac{n-1}{2}}{2} = \frac{n^2-n}{4},
$$

but in this case unfortunately the right hand side is bigger than $\binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$, so we have to find a smaller upper bound for $C_{max}(\Delta^{[n]}, 1)$. The following investigation shows, that a graph with $\frac{n^2-n}{4}$ edges can not be cut-minimal anymore, which will lead to the requested bound.

Consider a graph $G = ([n], E)$, and choose a vertex $v \in [n]$, such that $\deg_G(v) = \frac{n-1}{2}$. If such a vertex does not exist, we have

$$|E| \leq \frac{n(\frac{n-1}{2} - 1)}{2} = \frac{n^2 - 3n}{4} < \frac{n^2 - 2n + 1}{4} = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$$

and we are done. Now there exist exactly $\frac{n-1}{2}$ vertices $v_1, \ldots, v_{\frac{n-1}{2}} \in [n]$, such that $(v, v_i) \notin E$, for all $i = 1, \ldots, \frac{n-1}{2}$. If we had $\deg_G(v_i) = \frac{n-1}{2}$ for one of these vertices, we would get

$$|E_G(\{v, v_i\}, [n] \setminus \{v, v_i\})| = 2\frac{n-1}{2} = n - 1 > n - 2 = \frac{2(n-2)}{2},$$

so G would not be cut-minimal anymore. It follows that the degree of these $\frac{n-1}{2}$ vertices has to be at least one lower than assumed, so the number of edges has to be at least $\frac{n-1}{4}$ lower than assumed, which provides the new inequality:

$$\begin{aligned} C_{max}(\Delta^{[n]}, 1) &\leq \frac{n^2 - n}{4} - \frac{n-1}{4} \\ &= \frac{n^2 - 2n + 1}{4} \\ &= \frac{(n-1)^2}{4} \\ &= \binom{\frac{n+1}{2}}{2} + \binom{\frac{n-1}{2}}{2} \\ &= \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} \end{aligned}$$

Hence, we have $C_{max}(\Delta^{[n]}, 1) \leq \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$ in general and by Proposition 8 we are done. $\square$

A formula for the dimension of $\mathcal{C}_1(\Delta^{[n]})$ follows immediately from the preceding theorem.

Corollary 2. $\dim(\mathcal{C}_1(\Delta^{[n]}) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} - 1$

Note, that the proof of Proposition 8 even contains a description of the shape of a cut-minimal graph with the maximum number of edges, which furthermore provides information about how a top dimensional simplex is embedded in $\mathcal{C}_1(\Delta^{[n]})$. Now, the following theorem, which was developed in [10], does not only represent a first statement about the homology of $\mathcal{C}_1(\Delta^{[n]})$, but even shows that the construction in the proof of Proposition 8 is the only possible shape of such graphs (top dimensional simplices respectively). Let us first define some basic terminology, which we will mainly need in the next section, but it might already be helpful at this point to know exactly what we mean by deleting or adding edges in a graph.

Definition 11. *For a graph $G = ([n], E)$ and a family of edges $e_1, \ldots, e_k \in E$ we define the* ***deletion*** *of $e_1, \ldots, e_k$ in G as:*

$$G_{e_1,\ldots,e_k} := ([n], E \setminus \{e_1, \ldots, e_k\}).$$

For a graph $G = ([n], E)$ and a family of edges $e_1, \ldots, e_k \in NE_G([n], [n])$ we define the ***addition*** *of $e_1, \ldots, e_k$ to G as:*

$$G^{e_1,\ldots,e_k} := ([n], E \cup \{e_1, \ldots, e_k\}),$$

Theorem 8. $H_{\dim(\mathcal{C}_1(\Delta^{[n]})}(\mathcal{C}_1(\Delta^{[n]})) \cong 0$

Proof. We will first show, that a top dimensional simplex in $\mathcal{C}_1(\Delta^{[n]})$ can only be represented as a graph of the type we constructed in the proof of Proposition 8.
Let n be even. If we set $n = 2t + 2$, then by the number $C_{max}(\Delta^{[n]}, 1)$ and cut-minimality, the graph G corresponding to a top dimensional simplex in $\mathcal{C}_1(\Delta^{[n]})$ must be t-regular. Furthermore for any three vertices $v, w, u \in [n]$ by cut-minimality we have:

$$E_G(\{v, w, u\}, [n] \setminus \{v, w, u\}) \leq \frac{3(2t-1)}{2} = 3t - \frac{3}{2},$$

so by t-regularity among any three vertices at least two of them must be adjacent. Now choose a vertex $v \in [n]$ and set A to be the set consisting of all vertices, which are not adjacent to v. Then we have $|A| = t + 1$ by t-regularity and by the preceding result any two vertices in A have to be adjacent. Thus, A provides a complete graph on $t + 1$ vertices, so by t-regularity $[n] \setminus A$ must also provide a complete graph on $t + 1$ vertices. Hence, every top dimensional simplex in $\mathcal{C}_1(\Delta^{[n]})$ (for n even) corresponds to a graph of that shape.
Let now n be odd. If we set $n = 2t + 3$, then by the number $C_{max}(\Delta^{[n]}, 1)$ and cut-minimality there exist at least $t + 2$ vertices having degree $t + 1$. Let A denote the set of these vertices. For any $v, w \in A$ we have:

$$E_G(\{v, w\}, [n] \setminus \{v, w\}) \leq \frac{2(2t+3-2)}{2} = 2t + 1 < 2t + 2 = \deg_G(v) + \deg_G(w),$$

so all vertices from A are adjacent. By cut-minimality again we have $|A| = t + 2$, so A provides a complete graph on $t + 2$ vertices. The number of remaining edges $\binom{t+1}{2}$ shows, that the remaining $t + 1$ vertices must also provide a complete graph, which is disjoint from the first one, because any other constellation would destroy cut-minimality. So, again every top dimensional simplex in $\mathcal{C}_1(\Delta^{[n]})$ corresponds to a graph of the requested type.
Now we see that deleting an edge from such a graph (which is the same as deleting a vertex from a top dimensional simplex) produces a graph corresponding to a simplex which appears as a face of one top dimensional simplex, but can not be a face of another top dimensional simplex, since we can not construct a graph of the requested type by adding an edge at any other place than the place where we just

deleted it. So, all top dimensional simplices are at most connected via simplices of codimension 2, which implies that $\mathcal{C}_1(\Delta^{[n]})$ can be continiously retracted to a complex of codimension 1 and so homology in top dimension vanishes. $\square$

Definition 12. *A cut-minimal graph $G = ([n], E)$ is called **maximal**, if for all $e \in NE_G([n], [n])$ the addition G^e is not cut-minimal. We denote the set of all maximal cut-minimal graphs on n vertices by $MAX(n)$.*

Since deleting in edge in a cut-minimal graph produces a cut-minimal graph again, it turnes out that we only have to determine all maximal cut-minimal graphs to find all cut-minimal graphs.

Definition 13. *Two graphs $G = ([n], E)$ and $G' = ([n], E')$ are called **isomorphic**, if there exists a map $f : [n] \to [n]$, such that $(i, j) \in E$ if and only if $(f(i), f(j)) \in E'$.*
*For a graph $G = ([n], E)$ the **isomorphism class** of G is the set*

$$\left\{G' = ([n], E') : G' \text{ and } G \text{ are isomorphic}\right\}$$

Remark 3. *Note, that an isomorphism of graphs preserves most properties studied in this chapter, especially cut-minimality and the constant $h(G)$, studied in the next section.*

Figure 3.1 illustrates, how the isomorphism classes of cut-minimal graphs on 6 vertices are arranged, where the connecting lines represent the relations between the classes refering to the deletion or addition of edges. Note, that the choosen representatives in the figure do not always satisfy the property of being a deletion or an addition of the shown representative of a class below or above, but in this case there is always another representative in the class which does. We see that beside the class of maximal cut-minimal graphs constructed in the proof of Proposition 8, we have three more classes of maximal cut-minimal graphs here.

Except for the maximal cut-minimal graphs with maximum number of edges we do not know anything about the remaining classes of cut-minimal graphs until now. The following statements now approach this challenge by providing new classes of cut-minimal graphs in general which do not appear as deletions of those largest maximal cut-minimal graphs.

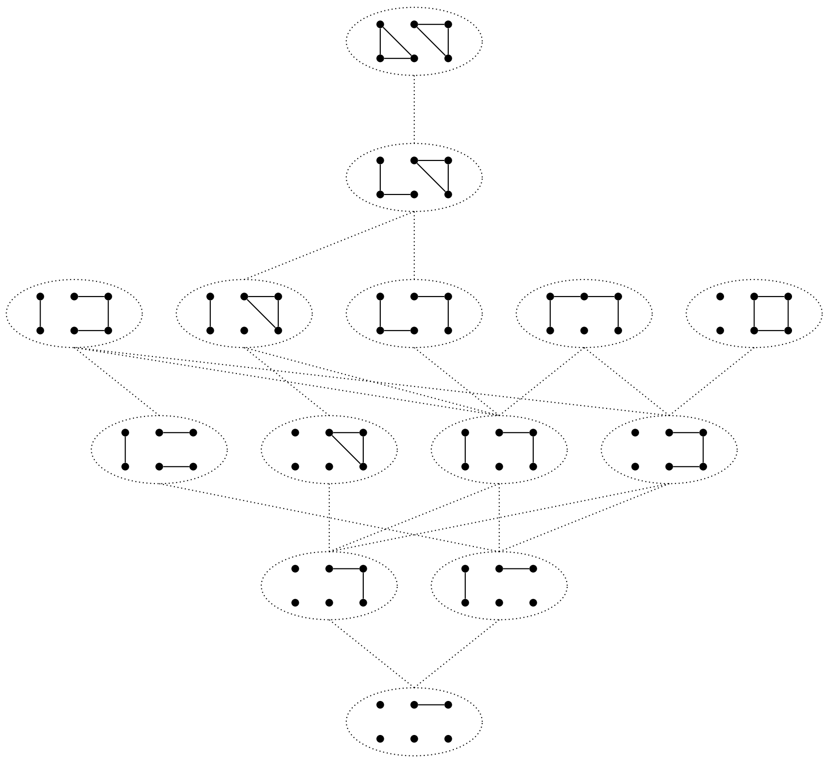

FIGURE 3.1: All cut-minimal graphs on 6 vertices (up to isomorphism)

Lemma 6. *Let $u_1, \dots, u_t, s_1, \dots, s_t \in \mathbb{N} \cup \{0\}$, such that for all $i = 1, \dots, t$ we have:*

$$u_i + s_i \le \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j + s_j,$$

then we have:

$$\sum_{i=1}^{t} s_i(u_i - \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j) \le 0$$

Proof. If we have

$$u_i \le \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j$$

for all $i = 1, \dots, t$, then we are done, since all u_i's and s_i's are positive. So, let there exist a $k \in [t]$, such that

$$u_k > \sum_{\substack{j=1 \\ j \neq k}}^{t} u_j.$$

Obviously, there is at most one unique u_k satisfying this property. Now for all $i \neq k$ we have:

$$u_i - \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j \le -(u_k - \sum_{\substack{j=1 \\ j \neq k}}^{t} u_j),$$

and furthermore we have

$$s_k < \sum_{\substack{j=1 \\ j \neq k}}^{t} s_j$$

by assumption and so we get:

$$\sum_{\substack{i=1 \\ i \neq k}}^{t} s_i(u_i - \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j) \leq -s_k(u_k - \sum_{\substack{j=1 \\ j \neq k}}^{t} u_j).$$

Hence, we have:

$$\sum_{i=1}^{t} s_i(u_i - \sum_{\substack{j=1 \\ j \neq i}}^{t} u_j) \leq 0.$$

□

Proposition 9. *Let the largest connected component of a graph $G = ([n], E)$ not contain more than $\frac{n}{2}$ vertices, then G is cut-minimal.*

Proof. Let $C_1, \ldots, C_t \subset [n]$ be the connected components of G and let $S \subset [n]$, $S_i := S \cap C_i$ and $U_i := C_i \setminus S_i$. Then we have:

$$|E_G(S, [n] \setminus S)| \leq \sum_{i=1}^{t} |S_i||U_i|$$

and

$$|NE_G(S, [n] \setminus S)| \geq \sum_{i=1}^{t}(|S_i| \sum_{\substack{j=1 \\ j \neq i}}^{t} |U_j|).$$

So, we have to show that

$$\sum_{i=1}^{t} |S_i||U_i| \leq \sum_{i=1}^{t}(|S_i| \sum_{\substack{j=1 \\ j \neq i}}^{t} |U_j|),$$

which is equivalent to

$$\sum_{i=1}^{t} |S_i|(|U_i| - \sum_{\substack{j=1 \\ j \neq i}}^{t} |U_j|) \leq 0,$$

but this follows directly from Lemma 6, since by the assumption $|C_i| \leq \frac{n}{2}$ for all i, we have:

$$|S_i| + |U_i| = |C_i| \leq \sum_{\substack{j=1 \\ j \neq i}}^{t} |C_j| = \sum_{\substack{j=1 \\ j \neq i}}^{t} |S_j| + |U_j|.$$

□

Let us now and determine the counterpart of the number $C_{max}(\Delta^{[n]}, 1)$, namely the minimal number of edges, a non-cut-minimal graph can have, as it was already (more generally) defined in the preceding chapter.

Theorem 9. $C_{min}(\Delta^{[n]}, 1) = \lceil \frac{n}{2} \rceil$

Proof. We can always find a graph $G = ([n], E)$ with $\lceil \frac{n}{2} \rceil$ edges, such that for a vertex $v \in [n]$ we have

$$E_G(\{v\}, [n] \setminus \{v\}) = \left\lceil \frac{n}{2} \right\rceil > \left\lfloor \frac{n-1}{2} \right\rfloor,$$

so it is not cut-minimal and we have $C_{min}(\Delta^{[n]}, 1) \leq \lceil \frac{n}{2} \rceil$.
On the other hand if we have a graph $G = ([n], E)$ with $|E| = \lceil \frac{n}{2} \rceil - 1$, then it must be cut-minimal, since $\lceil \frac{n}{2} \rceil - 1 = \lfloor \frac{n-1}{2} \rfloor$, so we get $C_{min}(\Delta^{[n]}, 1) \geq \lceil \frac{n}{2} \rceil$ and we are done. □

Obviously, $\mathcal{C}_1(\Delta^{[n]})$ contains all simplices of dimension lower than $C_{min}(\Delta^{[n]}, 1)$, which leads to the following observation.

Corollary 3. $H_k(\mathcal{C}_1(\Delta^{[n]})) \cong 0$ *for all* $1 \leq k \leq \lceil \frac{n}{2} \rceil - 3$

Proof. By Theorem 9 $\mathcal{C}_1(\Delta^{[n]})$ has a full k-skeleton for all $k \leq \lceil \frac{n}{2} \rceil - 2$ and we are done. □

Since adding a vertex to a cut-minimal graph will always preserve its property to be cut-minimal (see Lemma 4), we can define the following natural embedding:

$$\begin{aligned} i_n \quad : \quad CMG(n) &\longrightarrow CMG(n+1) \\ ([n], E) &\longmapsto ([n+1], E) \end{aligned}$$

Note, that the embedding i_n is just the graphic version of the more general embedding i_n^k for cochains from the preceding chapter, restricted to cut-minimal graphs. Now, the first thing we see is that maximality of cut-minimal graphs always becomes destroyed by embedding them.

Proposition 10. *Let* $G \in MAX(n)$*, then we have* $i_n(G) \notin MAX(n+1)$*.*

Proof. Let $G = ([n], E) \in MAX(n)$. If n is odd, Theorem 7 gives:

$$|E| \leq \frac{(n-1)^2}{4} = \frac{n^2 - 2n + 1}{4} < \frac{n^2 - n}{4} = \frac{n\frac{n-1}{2}}{2} \quad \text{for } n \geq 3,$$

so there exists a $v \in [n]$, such that $\deg_G(v) < \frac{n-1}{2}$ (*). Now define
$G' := ([n+1], E \cup (v, n+1))$ and let $S \subset [n+1]$, such that $1 \leq |S| \leq \frac{n+1}{2}$, then we

have:

$$\begin{aligned}|E_{G'}(S,[n+1]\setminus S)| &\leq |E_G(S\setminus\{n+1\},[n]\setminus S)|+1\\ &\leq \frac{|S|(n-|S|)}{2}+1\\ &= \frac{|S|(n+\frac{2}{|S|}-|S|)}{2}\\ &\leq \frac{|S|(n+1-|S|)}{2},\end{aligned}$$

for all $S \subset [n+1]$, such that $|S| \geq 2$. For $|S| = 1$, the upper condition is also satisfied by (*).
Hence, G' is cut-minimal and so we have $i_n(G) = ([n+1], E) \notin MAX(n+1)$.
If n is even, define $G' := ([n+1], E \cup (v, n+1))$ for some arbitrary $v \in [n]$. Then by the same calculations as in the first part, we have

$$|E_{G'}(S,[n+1]\setminus S)| \leq \frac{|S|(n+1-|S|)}{2},$$

for all $S \subset [n+1]$, such that $|S| \geq 2$, and for $|S| = 1$ we have:

$$\begin{aligned}|E_{G'}(S,[n+1]\setminus S)| &\leq \max\{\deg_{G'}(v) : v \in [n+1]\}\\ &= \left\lfloor \frac{n-1}{2} \right\rfloor + 1\\ &= \frac{n-2}{2}+1 = \frac{n}{2} = \frac{(n+1)-1}{2}.\end{aligned}$$

Hence, G' is cut-minimal and $i_n(G) \notin MAX(n+1)$. □

3.2 Cheeger graphs

In this section we want to recall the notion of Cheeger graphs as introduced in [1] and study some interesting properties concerning the combinatorial structure of Cheeger graphs, which might be helpful on our way to determine the first Cheeger constant of a simplex $\Delta^{[n]}$ when n is not a power of 2.

3.2.1 Basic definitions and properties

The following definition is completely adopted from [1].

Definition 14. *Consider a graph $G = ([n], E)$, then we set:*

$$T(G) := \{(v,e) \,:\, v \in [n], e = (w,u) \in E, v \notin e, |\{(v,w),(v,u),(w,u)\} \cap E\}| \text{ is odd}\}.$$

We have:

$$|T(G)| = \sum_{e \in E} t(e),$$

where for an edge $e = (v, w)$, we set

$$t(e) := \sum_{u \in [n] \,:\, u \neq v,w} \tau_e(u),$$

with

$$\tau_e(u) := \begin{cases} 1, & \text{if } (v,u),(w,u) \notin E \\ \frac{1}{3}, & \text{if } (v,u),(w,u) \in E \\ 0, & \text{otherwise} \end{cases}$$

Furthermore, we adopt the number:

$$h(G) := \frac{|T(G)|}{|E|}$$

*and call a cut-minimal graph $G = ([n], E)$ a **Cheeger graph**, if*

$$h(G) = \min_{G' \in CMG(n)} h(G').$$

*The **first Cheeger constant of a simplex** $h_1(\Delta^{[n]})$ is then defined by:*

$$h_1(\Delta^{[n]}) := h(G)$$

where G is some Cheeger graph on n vertices.

We already know by [1] that $\frac{n}{3} \leq h_1(\Delta[n]) \leq \lceil \frac{n}{3} \rceil$ and the lower bound is archieved if n is not a power of 2. If two graphs G and G' belong to the same isomorphism class, we obviously have $|T(G)| = |T(G')|$ and $h(G) = h(G')$, so taking up the example from the preceding section, Figure 3.2 shows the numbers $h(G)$ for all cut-minimal graphs on 6 vertices with the same partially ordering as in Figure 3.1 and we can see that there is one Cheeger graph attaining the Cheeger constant $\frac{8}{4}$.

3.2.2 A restriction on the size of connected components in Cheeger graphs

Proposition 11. *Let $G = ([n], E)$ be a Cheeger graph. Then G must have a connected component of size at least $n - \lceil \frac{n}{3} \rceil$.*

Proof. Let $C \subseteq [n]$ be the largest connected component of G, then we obviously have

$$\left\lceil \frac{n}{3} \right\rceil \geq h(G) \geq \frac{|E|(n - |C|)}{|E|} = n - |C|,$$

which is equivalent to

$$|C| \geq n - \left\lceil \frac{n}{3} \right\rceil.$$

□

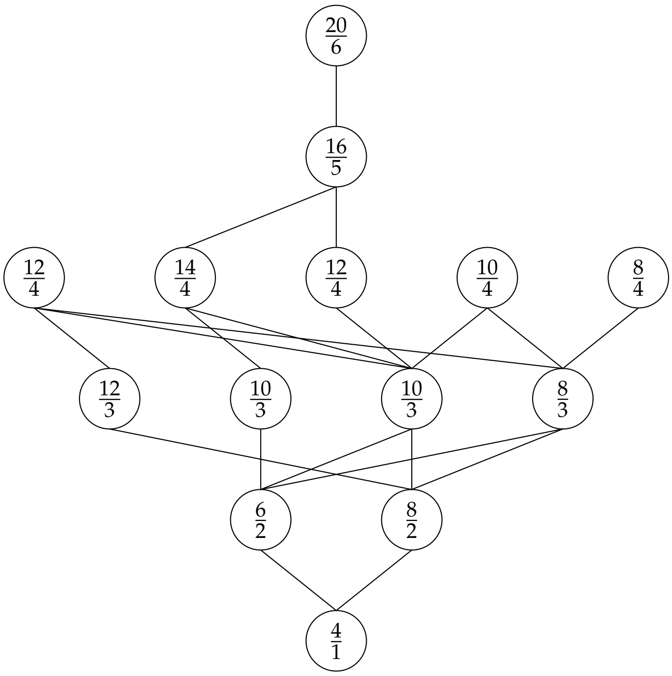

FIGURE 3.2: The numbers $h(G)$ for all cut-minimal graphs on 6 vertices

3.2.3 Embeddings of Cheeger graphs

Based on the fact that adding a vertex to a cut-minimal graph will always result in a cut-minimal graph again from the preceding section, we will now study the consequences of adding a vertex to a Cheeger graph. It will turn out that in most cases the resulting graph will not be a Cheeger graph anymore. Using this fact we will be able to find a lower bound on the number of edges in Cheeger graphs holding in the most cases.

The following lemma is just the graphic version of Lemma 4.

Lemma 7. *Let $G = ([n], E)$ be a cut-minimal graph, then we have:*

$$h(i_n(G)) = h(G) + 1$$

A direct consequence of this relation is that if n is not divisible by 3, Cheeger graphs in $CMG(n)$ can not appear as an embedding of graphs from $CMG(n-1)$.

Proposition 12. *For all $G \in CMG(n)$, such that $3 \nmid n$ the graph $i_n(G) \in CMG(n+1)$ is not a Cheeger graph.*

Proof. We only need to consider Cheeger graphs in $CMG(n)$, since for any $G \in CMG(n)$ which is not a Cheeger graph, there exists a graph $G' \in CMG(n)$, such that $h(G') < h(G)$, so by Lemma 7 we get:

$$h(i_n(G')) = h(G') + 1 < h(G) + 1 = h(i_n(G))$$

Let now $G \in CMG(n)$ be a Cheeger graph. Then by the estimate $\frac{n}{3} \leq h(G) \leq \lceil \frac{n}{3} \rceil$ we get

$$h(i_n(G)) = h(G) + 1 \geq \frac{n}{3} + 1 \geq \left\lceil \frac{n+1}{3} \right\rceil,$$

where the last inequality is sharp if and only if $3 \nmid n$.
So, if $3 \nmid n$ then $i_n(G)$ is not a Cheeger graph, since by [1] we have:

$$h_1(\Delta^{[n+1]}) \leq \left\lceil \frac{n+1}{3} \right\rceil < h(i_n(G))$$

□

We even have a similar version of the previous statement if n is divisible by 3 with the restriction that $n+1$ must not be a power of 2.

Proposition 13. *Let $G \in CMG(n)$, such that $3 \mid n$ and $n+1 \neq 2^t$ for some $t \in \mathbb{N}$. Then $i_n(G)$ is not a Cheeger graph.*

Proof. Since n is divisible by 3 we have $h(G) = \frac{n}{3}$ by [1] and so we get $h(i_n(G)) = \frac{n}{3} + 1$ by Lemma 7. Now, since $n+1$ is not a power of 2 we also know that $h_1(\Delta^{[n+1]}) = \frac{n+1}{3}$ by [1]. Hence, we have:

$$h(i_n(G)) > h_1(\Delta^{[n+1]})$$

□

Combining the last two statements, we can calculate a lower bound for the number of edges in Cheeger graphs, except for the case when the number of vertices n is a power of 2 and $n-1$ is divisible by 3.

Proposition 14. *Let $G = ([n], E)$ be a Cheeger graph, such that n is not a power of 2 or $n-1$ is not divisible by 3. Then we have $|E| \geq \lceil \frac{n-1}{2} \rceil$.*

Proof. Assume we have $|E| < \lceil \frac{n-1}{2} \rceil$. Then by Theorem 9 and since G must contain an isolated vertex, there exists a graph $G' \in CMG(n-1)$, such that $i_{n-1}(G') = G$. Now if we have $n \neq 2^t$ for some $t \in \mathbb{N}$, then G is not a Cheeger graph, since we either get $3 \nmid n-1$ and we are done by Proposition 12 or we get $3 \mid n-1$ and we are done by Proposition 13. On the other hand, if we have $n = 2^t$ but $3 \nmid n-1$, then we immediately see that G is not a Cheeger graph by Proposition 12. □

3.2.4 The case when n is a power of 2

From [1] we know that $h(n) > \frac{n}{3}$ can only be valid, if n is a power of 2. An interesting question is, if this inequality is always strict for such n. For a graph $G = ([n], E)$ and a vertex $v \in [n]$ let us introduce the notation $A_v := \{w \in [n] : (v, w) \in E\}$, so we have $|A_v| = \deg_G(v)$. The following useful fact was given by Kozlov (see [1]).

Lemma 8. *Let $G = ([n], E)$ be a cut-minimal graph, then we have $h(n) = \frac{n}{3}$ if and only if for each vertex $v \in [n]$ we have $|E(A_v, [n] \setminus A_v)| = |NE(A_v, [n] \setminus A_v)|$.*

We can now make a statement about the possible degrees a vertex in a Cheeger graph on $n = 2^t$ vertices can attain, assuming the Cheeger constant in this case is exactly $\frac{n}{3}$.

Proposition 15. *Let $n = 2^t$ and $G = ([n], E)$ be a cut-minimal graph, such that $h(G) = \frac{n}{3}$. Then for each vertex $v \in [n]$ we have $|A_v| \in \left\{0 \leq i \leq \frac{n-2}{2} : n \text{ is even } \right\} \setminus \{2\}$.*

Proof. The proof of the fact, that $|A_v|$ is even can be found in [1]. Since G is cut-minimal and n is even, we have:

$$|A_v| = |E(\{v\}, [n] \setminus \{v\})| \leq \left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n-2}{2}$$

Now, assume there exists a $v \in [n]$, such that $|A_v| = 2$, then by Lemma 8 we have

$$|E(A_v, [n] \setminus A_v)| = |NE(A_v, [n] \setminus A_v)| = \frac{2(n-2)}{2} = n-2,$$

so there must exist a $w \in A_v$, such that $|A_w| \geq \frac{n-2}{2} = \frac{2^t-2}{2} = 2^{t-1} - 1$, but $2^{t-1} - 1$ is odd and $|A_w|$ can not be strictly bigger than $\frac{n-2}{2}$, which leads to a contradiction. □

Example 6. *In [1] Kozlov already used the fact, that a vertex in such a graph can not attain an odd degree to prove the equality $h(8) > \frac{8}{3}$. Applying the preceding slightly stronger proposition we get this result immediately, since then every vertex in a Cheeger graph $G = ([8], E)$, such that $h(G) = \frac{8}{3}$ must attain degree 0, which is obviously not possible.*

Chapter 4

A theorem about partitioning consecutive numbers

4.1 Preliminaries

As an introducing example to the main theorem we want to prove later, consider the following "staircase shaped" Young tableau:

1				
2	2			
3	3	3		
4	4	4	4	
5	5	5	5	5

FIGURE 4.1: "Staircase shaped" Young tableau consisting of 15 boxes

We have one box in the first line and one more in every following line. Now, we can "rebuild" this tableau by reordering the lines, such that we have not necessarily one box in the first line anymore, but always have still one more box in every following line and such that we do not have to split the single lines, as the numbers show:

4	4	4	4	2	2	1	
5	5	5	5	5	3	3	3

FIGURE 4.2: Rebuilt Young tableau

The interesting question which will be answered by our main theorem is, if there is always a possibility to reorder the lines of a Young tableau of the first type without having to split them to build a Young tableau of the second type, provided that the numbers of boxes are the same. The question, how many different Young tableaux of this type exist, meaning Young tableaux consisting of a fixed number of boxes such that based on any line the following line has one box more, has already been answered in [5]. Namely, for every odd divisor of the number of boxes, there exists exactly one Young tableau of this type.

Note, that in this paper we set $\mathbb{N}$ to be the set of natural numbers without 0 and a sequence of consecutive numbers always denotes a sequence

$n_1, \ldots, n_t \in \mathbb{N}$ (or $\mathbb{Z}$), such that $n_{i+1} = n_i + 1$. Furthermore, we use the notation $[n] := 1, 2, 3, \ldots, n$.

Lemma 9. *Let $n, a, b \in \mathbb{N}$, such that $n < a \leq b$ and $\sum_{i=1}^{n} i = \sum_{i=a}^{b} i$. Then we have $n \geq 2(b - a + 1)$.*

Proof. Note, that $b - a + 1$ is the number of summands in $\sum_{i=a}^{b} i$. Then we have:

$$\begin{aligned} & \sum_{i=1}^{n} i = \sum_{i=a}^{b} i \\ \Longleftrightarrow \quad & \frac{n(n+1)}{2} = (a+b)\frac{b-a+1}{2} \\ \Longleftrightarrow \quad & \frac{n}{2}(n+1) = (b-a+1)\frac{a+b}{2} \end{aligned}$$

Now, by assumption we have $\frac{a+b}{2} \geq n + 1$, so we get $\frac{n}{2} \geq b - a + 1$, which is equivalent to $n \geq 2(b - a + 1)$. □

Lemma 10. *For every $m \in \mathbb{N}$ and every $l \in \mathbb{Z}$ there exist pairs of numbers $(x_1, x_1'), \ldots, (x_m, x_m') \in \mathbb{N}^2$, such that $x_1, x_1', \ldots, x_m, x_m'$ are all distinct, $x_i' - x_i = i$ for every $i = 1 \ldots, m$, $l = \min\{x_1, x_1', \ldots, x_m, x_m'\}$ and $\max\{x_1, x_1', \ldots, x_m, x_m'\} \leq 2m + l$.*

Proof. Consider the following pairs (y_i, y_i') for i odd:

$$\left(y_{2\lceil \frac{m}{2} \rceil - 1}, y'_{2\lceil \frac{m}{2} \rceil - 1}\right) = \left(1, 2\left\lceil \frac{m}{2} \right\rceil\right), \ldots, (y_1, y_1') = \left(\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{m}{2} \right\rceil + 1\right)$$

and the following for i even:

$$\left(y_{2\lfloor \frac{m}{2} \rfloor}, y'_{2\lfloor \frac{m}{2} \rfloor}\right) = \left(2\left\lceil \frac{m}{2} \right\rceil + 1, 2m + 1\right), \ldots, (y_2, y_2') = \left(\left\lceil \frac{m}{2} \right\rceil + m, 2m + 2 - \left\lfloor \frac{m}{2} \right\rfloor\right),$$

where $(y_{i+2}, y'_{i+2}) = (y_i - 1, y_i' + 1)$ for all $i = 1, \ldots, m - 2$. Obviously, these pairs satisfy the assumption $y_i' - y_i = i$ and we have $\min\{y_1, y_1', \ldots, y_m, y_m'\} = 1$ and $\max\{y_1, y_1', \ldots, y_m, y_m'\} = 2m + 1$ for every $m \in \mathbb{N}$. (In fact, we even have $\max\{y_1, y_1'\} = 2m < 2m + 1$ for the case $m = 1$). Now, set $x_i := y_i + l - 1$ and $x_i' = y_i' + l - 1$ for all $i = 1, \ldots, m$ and we are done. □

4.2 The main theorem

Theorem 10. *Let $n, a, b \in \mathbb{N}$ $(b \geq a)$, such that $\sum_{i=1}^{n} i = \sum_{i=a}^{b} i$, then for every $a \leq t \leq b$ there exists a subset $U_t \subseteq [n]$, such that $U_i \cap U_j = \emptyset$ for all $i \neq j$, $[n] = \bigcup_{a \leq t \leq b} U_t$ and $\sum_{i \in U_t} i = t$.*

Proof. Without loss of generality let $a > n$. Otherwise, we have $U_t = \{t\}$ for all $a \leq t \leq n$ and $\sum_{i=1}^{a-1} i = \sum_{i=n+1}^{b} i$, so the remaining problem is reduced to the requested case, since we have $n+1 > a-1$.
Now, denote the number of summands in the sum $\sum_{i=a}^{b} i$ by $s := b-a+1$ and set:

$$
\begin{aligned}
P &= \{p_1, \ldots, p_s\} := \{n-2s+1, n-2s+2, \ldots, n-s-1, n-s\}, \\
Q &= \{q_1, \ldots, q_s\} := \{n-s+1, n-s+2, \ldots, n-1, n\}, \\
R &= \{r_1, \ldots, r_s\} := \{a, a+1, \ldots, b-1, b\}
\end{aligned}
$$

We have $P, Q, R \subset \mathbb{N}$, since $n-2s+1 > 0$ holds by Lemma 9. Furthermore, we see that we have $p_i + q_j = 2n-2s+1$ for all i, j, such that $i+j = s+1$. Now, set $c := 2n-2s+1$ and for every t such that $r_t - c < 0$ consider pairs $\{p_{i_t}, q_{j_t}\}$ and $\left\{p_{i'_t}, q_{j'_t}\right\}$, such that $i_t + j_t = i'_t + j'_t = s+1$, $q_{j'_t} - q_{j_t} = c - r_t$ and all appearing numbers in all pairs are distinct (we actually only have to require that all $q_{j_t}, q_{j'_t}$ are distinct, then all the rest is distinct by the condition $i_t + j_t = i'_t + j'_t = s+1$). We can find those kinds of tuples of pairs for every t satisfying $r_t - c < 0$ by Lemma 10 as follows. For every t, such that $r_t - c < 0$ there exists a t', such that $r_{t'} - c = -(r_t - c)$, since otherwise we had $\sum_{i=1}^{s} p_i + q_i > \sum_{i=1}^{s} r_i$, because the numbers $r_1 - c, \ldots, r_s - c$ are obviously consecutive, but this is a contradiction to the assumption $\sum_{i=1}^{n} i = \sum_{i=a}^{b} i$. By this observation, there must still exist a t, such that $r_t - c = 0$, since $r_1, \ldots, r_s$ are consecutive numbers, so letting m denote the maximum number, such that there exists a t satisfying $r_t - c = -m$, we have at least $2m+1$ consecutive numbers in Q. Now, choose such a set of $2m+1$ consecutive numbers from Q and let l denote the minimum number in this set, then applying Lemma 10 we get the requested tuples of pairs satisfying $q_{j'_t} - q_{j_t} = c - r_t$. Finally, set $U_{r_t} := \left\{p_{i'_t}, q_{j_t}\right\}$ and $U_{r_t+2(c-r_t)} := \left\{p_{i_t}, q_{j'_t}\right\}$ for every t satisfying $r_t - c < 0$ and we have $\sum_{i \in U_{r_t}} i = r_t$ and $\sum_{i \in U_{r_t+2(c-r_t)}} i = r_t + 2(c - r_t)$. For the remaining t's satisfying $r_t - c \geq 0$ we set $U_{r_t} := \{p_{i_t}, q_{j_t}\}$, where $\{p_{i_t}, q_{j_t}\}$ can be any of the remaining pairs, satisfying $p_{i_t} + q_{j_t} = c$. For those t's we have $\sum_{i \in U_{r_t}} i = c$. So, the problem is reduced to a smaller one of the type $\sum_{i=1}^{n-2s} i = \sum_{i=r_k-c}^{r_s-c} i$, where k is the minimal number such that $r_k - c > 0$. By induction we are done as follows. The sets U_{r_t} and $U_{r_t+2(c-r_t)}$, where $r_t - c < 0$ stay the same till the end, whereas the other sets U_{r_t}, where we still have $\sum_{i \in U_{r_t}} i = c < r_t$ will be "filled up" during the next steps of induction. $\square$

The reader may develop a better intuition for the preceding proof by considering the following example.

Example 7. *Consider $n = 14$, $a = 15$ and $b = 20$, then we obviously have $\sum_{i=1}^{n} i = \sum_{i=a}^{b} i$.*

Now, according to the proof of Theorem 10 we have $s = 6$ and we consider the following pairs:

$$\{p_1, q_6\} = \{3, 14\}, \quad \{p_2, q_5\} = \{4, 13\}, \quad \{p_3, q_4\} = \{5, 12\},$$
$$\{p_4, q_3\} = \{6, 11\}, \quad \{p_5, q_2\} = \{7, 10\}, \quad \{p_6, q_1\} = \{8, 9\}$$

We see that we have $c = 17$, so by subtracting c from the numbers $15, \ldots, 20$ only the last 3 numbers stay strictly positive. Now we "swap" the second components of two pairs just as we did in the preceding proof and construct the following sets:

$$U_{15} = \{3, 12\}, \quad U_{16} = \{6, 10\}, \quad U_{17} = \{8, 9\},$$
$$U_{18} = \{7, 11\}, \quad U_{19} = \{5, 14\}, \quad U_{20} = \{4, 13\}$$

We can see that we already have $\sum_{i \in U_t} i = t$ for all $t = 15, \ldots, 19$ and

$\sum_{i \in U_{20}} i = c = 17 < 20$, so the problem is reduced to $\sum_{i=1}^{2} i = 3$ and inductively we will be done by the next step. Formally this means, that $U_{15}, \ldots, U_{19}$ stay the same and we "fill up" the remaining set U_{20} with the remaining numbers $1, 2$, such that we get $U_{20} = \{13, 4, 1, 2\}$.

There are obviously many more ways of partitioning numbers in our sense than the one way the "algorithm" in the proof of Theorem 10 gives us, but until now this is the only working way we know in general and we even still do not know, how many partitions leading to the requested result exist.

Chapter 5

Perspectives

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