Expansion detected by cycles

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Abstract

1 On the hitting number of sets

Let S be a set and $\mathcal{F} \subseteq 2^S$ a family of subsets. Then

$$\tau(\mathcal{F}) := \min\{|P| : P \subseteq S, P \cap F \neq \emptyset, \text{for all } F \in \mathcal{F}\}$$

is called the **hitting number** of $\mathcal{F}$ and

$$\Delta_{\mathcal{F}} := \left\{ S \subseteq \mathcal{F} : \bigcap_{F \in S} F \neq \emptyset \right\}$$

is called the **cut complex** of $\mathcal{F}$.

Let now X be a simplicial complex on the vertex set V and $S \subseteq X$ a set of simplices of X, then we call S a **covering** of X if for every vertex $v \in V$ there exists a simplex $\sigma \in S$, such that $v \in \sigma$.

The first obvious observation is that for every set S and every family of subsets $\mathcal{F} \subseteq 2^S$ we have:

$$\tau(\mathcal{F}) = \min\{|S| : S \text{ is a covering of } \Delta_{\mathcal{F}}\}$$

2 The cut complex for families of cycles

Denote the complete simplex on n vertices by $\Delta^{[n]}$ and always work with coefficients in $\mathbb{Z}_2$. Let now $\varphi \in C^k(\Delta^{[n]})$ be a k-cochain. Then we can define the family

$$T_\varphi := \{\text{supp}(\partial\sigma) : \sigma \in \text{supp}(\delta\varphi)\}$$

where the sets are the supports of the boundaries of the simplices in the support of the coboundary of φ.

Let us now study the structure of the cut complex of T_φ. First, we see that

$$H_i(\Delta_{T_\varphi}) = 0$$

holds for all $i \geq 2$, as follows:
If we have two 2-simplices $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$ in Δ_{T_φ} which share an edge $\{v_2, v_3\}$, then the 3-simplex $\{v_1, v_2, v_3, v_4\}$ has to be contained in Δ_{T_φ} as well.

References