Expansion detected by cycles

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Abstract

1 Alternative Cheeger constant of the simplex

Theorem 1.1. *Let n not be a power of 2, then we have:*

$$\gamma_1(\Delta^{[n]}) = \frac{n}{3}$$

Proof. Since n is not a power of 2 we can write it as $n = c(2t+1)$. Now consider the staircase graph $G_n(\lambda)$ given by the partition $\lambda = c \cdot \mathrm{cor}(t)$. Since $G_n(\lambda)$ is bipartite we can partition the vertices of $G_n(\lambda)$ as $[n] = A \cup B \cup C$, with $A = \{v_1, \dots, v_{ct}\}$, $B = \{w_1, \dots, w_{ct}\}$ and $C = \{x_1, \dots, x_c\}$, such that C is the set of all isolated vertices and all edges of $G_n(\lambda)$ are contained in $E_{G_n(\lambda)}(A, B)$.
Construct a family of edge-disjoint cycles of the vertex set $[n]$ as follows:
For all edges (v_i, w_j) satisfying $i + j \leq ct$, such that (v_{ct-j+1}, w_{ct-i+1}) is not an edge in $G_n(\lambda)$ consider the cycle

$$C_{ij} := \{(v_i, w_j), (v_{ct-j+1}, w_{ct-i+1}), (v_i, v_{ct-j+1}), (w_j, w_{ct-i+1})\}$$

For all edges $e_{ij} = (v_i, w_j)$ satisfying $i + j \leq ct + 1$, such that $e'_{ij} = (v_{ct-j+1}, w_{ct-i+1})$ is also an edge in $G_n(\lambda)$ (for $i + j = ct + 1$ they are equal), the set

$$D := \{e_{ij}, e'_{ij} : i + j \leq ct + 1,\ e_{ij} \text{ and } e'_{ij} \text{ are edges in } G_n(\lambda)\}$$

can be partitioned into t sets $B_1, \dots, B_t$, each containing c^2 edges:

$$B_k := \{(v_i, w_j) : (k-1)c + 1 \leq i \leq kc,\ c(t-k) + 1 \leq j \leq c(t-k+1)\}$$

Now, each vertex from $A \cup B$ is only contained in edges from exactly one of the sets B_k. This means that for any $l = 1, \dots, c$ and any pair of edges $(v_i, w_j) \in B_{k_1}$ and $(v_{i'}, w_{j'}) \in B_{k_2}$ ($k_1 \neq k_2$) the cycles $\{(v_i, w_j), (v_i, x_l), (w_j, x_l)\}$ and $\{(v_{i'}, w_{j'}), (v_{i'}, x_l), (w_{j'}, x_l)\}$ are edge-disjoint. Furthermore, each set B_k itself is a complete balanced bipartite graph (i.e. a graph in which each of the

c vertices from A is adjecent to each of the c vertices from B) so we can partition it into c sets $B_k^1, \ldots, B_k^c$, such that all edges in B_k^l are disjoint, for every $l = 1, \ldots, c$. Thus, the cycles $\{(v_i, w_j), (v_i, x_l), (w_j, x_l)\}$ are edge-disjoint for all $(v_i, w_j) \in B_k^l$. The family of all these cycles united with the cycles C_{ij} we defined before gives a family of edge-disjoint cycles, such that every edge of $G_n(\lambda)$ is contained in exactly one cycle and every cycle containes exactly one of the edges from $G_n(\lambda)$. Hence, the hitting number of this □

Theorem 1.2. *Let $k+2$ devide n, then we have:*

$$\gamma_k(\Delta^{[n]}) = \frac{n}{k+2}$$

Proof. Follows immediately from [1] □

References

[1] Dmitry N. Kozlov, Roy Meshulam; Quantitative aspects of acyclicity