Experiments with Pants Covers for the Modular Surface

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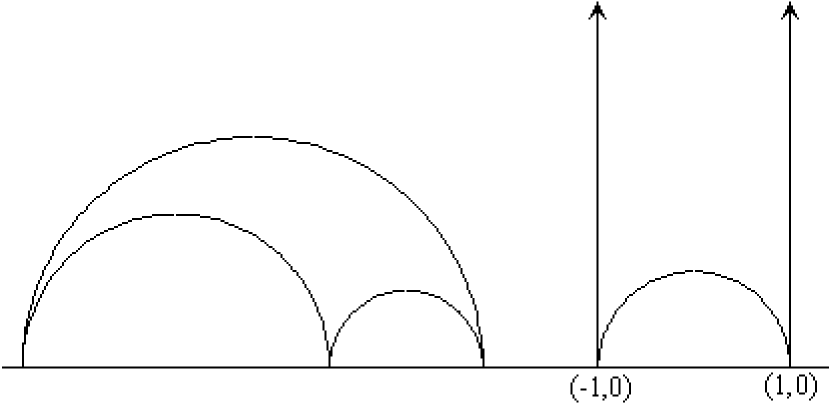
Motivation

- Let X, Y be two finite area hyperbolic Riemann surfaces, either both closed, or both punctured.
- Then the Ehrenpreis conjecture asserts that there exist finite covers $\tilde{X} \to X$, $\tilde{Y} \to Y$ with a homeomorphism $\tilde{X} \to \tilde{Y}$ with arbitrarily small distortion of angles.
- The closed case of this conjecture was settled by Kahn and Markovic in 2011.
- The cuspidal case remains open, and it is unclear what to believe.

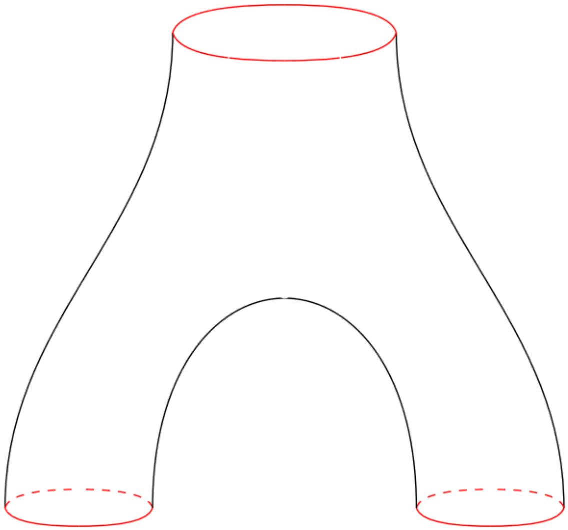
Immersing pants

- A pair of pants is topologically a sphere minus three disjoint disks.
- When a pants is immersed in a hyperbolic surface, its boundaries are loops on the surface, which we can take to be closed geodesics.
- We want to create a covering of a surface by stitching together immersed pants.
- Need boundaries to cancel out: If we use a geodesic, we must glue it to its inverse.
- We can only use finitely many pants (need a finite cover).

Geodesics in the half-plane



A pair of pants

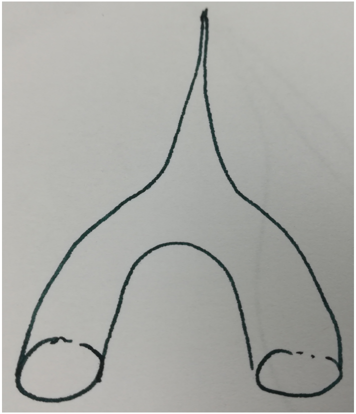


An immersion of a pants

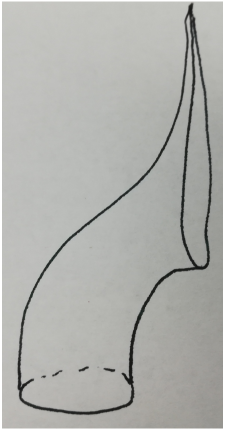
Handling cusps

- Since regular pants have no cusps, to cover a neighborhood of the puncture of the surface, we need to introduce ”degenerate pants”, where one cuff degenerates to a cusp.
- Convenient to “split” the degenerate pants into objects we call eyes.

A degenerate pants



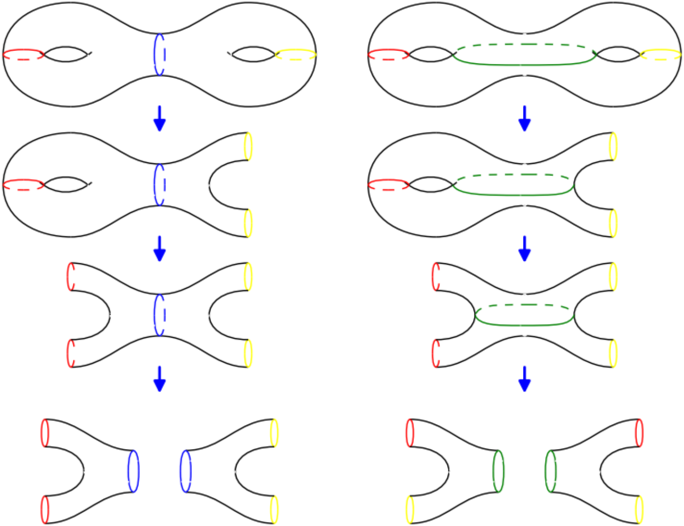
An eye



Gluing pants and eyes

- We can consider gluing two pants or a pants and an eye if they share a geodesic boundary, that is, if one has a cuff γ and the other has cuff γ^{-1}.
- The boundary operator takes a pants or an eye to a formal sum of the boundary components (closed and cuspidal geodesics).
- A collection of pants and eyes then can be glued to form a cover if it lies in the kernel of the boundary operator.

Example of boundary summing to zero

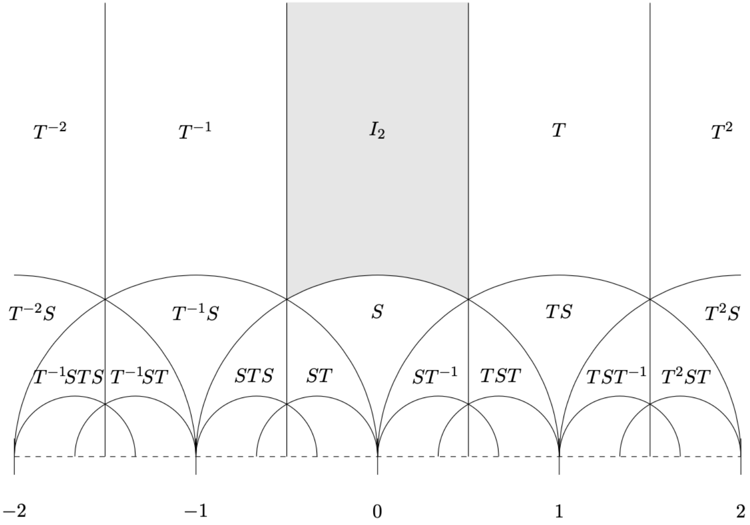


- The covers in the proof of the closed Ehrenpreis conjecture are constructed, as above, by gluing pants together.
- Moreover, the covers were constructed by constraining a geometric invariant called the shear, which measures the compatibility of two pants with a shared cuff.
- Intuitively, the shear property can be thought of as twisting along the cuffs.
- In particular, it is required that the shear be close to 1.

The modular surface

- $$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$$
  $$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$
  $$\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) / \{\pm I\}.$$
- The modular surface is
  $$\mathcal{M} = \mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H},$$
  where $\mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ via Möbius transformations:
  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \curvearrowright z = \frac{az + b}{cz + d}.$$
- We often look at the fundamental domain
  $$\mathcal{F} = \{z \in \mathbb{H} : |z| > 1 \text{ and } |\operatorname{Re}(z)| < 1/2\}$$
  as a convenient representation of $\mathcal{M}$.

Picture of the fundamental domain



Covering the modular surface

- We want to numerically construct pants and eyes with total boundary zero such that for each cuff γ of each pants or eye in the cover, we can find a pants containing γ^{-1} with shear close to 1 with respect to the original pants.
- To solve this problem numerically, we need to convert these geometric objects into objects accessible to the computer.

How to numerically deal with geodesics on the modular surface

- Each pair of pants can be represented by three compatible closed geodesics. How do we represent these geodesics?
- Continued fractions and cutting sequences!

From closed geodesics to continued fractions

- As the closed geodesic "loops" over $\mathcal{F}$, it periodically crosses the same sequence of boundaries; this corresponds in a nice way to the continued fraction of the positive ($t \to \infty$) limit point of the geodesic.
- As a geodesic traverses each $\mathrm{PSL}_2(\mathbb{Z})$-translate of $\mathcal{F}$ in a tessellation of $\mathbb{H}$, we can encode the border-crossings of the geodesic as a "cutting sequence" of L's and R's.
- Closedness of geodesic $\iff$ periodicity of cutting sequence.
- The periodic cutting sequences can be naturally interpreted as a continued fraction:

$$LRLRR = L^1R^1L^1R^2 \longleftrightarrow [\overline{1,1,1,2}]$$

$$\longleftrightarrow \alpha \in \mathbb{R}\setminus\mathbb{Q} \text{ s.t. } \alpha = 1 + \cfrac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{\alpha}}}}.$$

Generating geodesics

- $Q = [A, B, C]$ represents the (binary) quadratic form:

$$Q(x, y) = Ax^2 + Bxy + Cy^2.$$

- We are interested in forms such that A, B, C are integral, coprime, and such that the two roots of $Q(x, 1)$ lie in $(-1, 0)$ and $(1, \infty)$ respectively (this property is known as being *reduced*).
- Fact: a quadratic irrational α is reduced (meaning $\alpha > 1$, and its conjugate $-1 < \overline{\alpha} < 0$) *iff* it its continued fraction is exactly periodic.
- Under some action of $\mathrm{SL}_2(\mathbb{Z})$, we can form equivalence classes of quadratic forms.
- Theorem: There is a bijection between equivalence classes of reduced quadratic forms with $B^2 - 4AC > 0$ and primitive, oriented, closed geodesics.
- This theorem is proven by using roots of $Q(x, 1)$ to generate a continued fraction and therefore a geodesic.

Generating geodesics (cont.)

- We can get bounds on the possible coefficients of our quadratic forms and then iterate over the finitely many resulting forms to get one representative of each class.
- Thus, an exhaustive list of closed geodesics of a given length can be obtained by generating all quadratic forms satisfying these constraints.
- We store these geodesics as the continued fractions of their positive ($t \to \infty$) limit point.

Generating pants

- The condition that a triple of geodesics forms a pair of pants amounts to the combinatorial condition that the continued fraction of each geodesic is a ‘spliced’ version of the other two.
- We can then find all pants with cuffs of a given length by computing all geodesics of a given length and finding triples which satisfy the above combinatorial condition.

Generating Eyes

- An eye can be represented as a closed geodesic and a cuspidal geodesic generated by the closed geodesic.
- Unlike a closed geodesic, a cuspidal geodesic crosses only a finite number of boundaries, whence its associated continued fraction is finite and therefore rational.
- A cuspidal geodesic's trajectory corresponds to a single period of the trajectory of the associated closed geodesic.

Computing shears

- We begin by placing the two pants, each of which contains either γ or γ^{-1} in a canonical position in $\mathbb{H}$.
- Next we map the γ or γ^{-1} of the second pair of pants onto that of the first pair, and compute the shear between the two using elementary geometric considerations.
- We repeat the above process for computing shears between cuffs of pants and cuffs for eyes.
- tl;dr: we can compute shears between pants and eyes quickly by coordinate geometry!

Computationally finding a cover

- Recall that pants are represented by three closed geodesics and eyes are represented by one closed geodesic and one cuspidal geodesic.
- We can associate to each closed geodesic and each cuspidal geodesic an orientation.
- We can then take the list of positively oriented closed geodesics and positively oriented eyes as basis vectors.
- From this, each pants or eye can be written as a sum of three closed geodesics or a sum of one closed geodesic and a cuspidal geodesic.
- Thus, we can represent the boundary operator as a matrix, mapping formal sums of pants and eyes to sums of closed geodesics and cuspidal geodesics.
- Finding a cover is then equivalent to finding a non-zero element in the kernel of the matrix (with non-negative entries); this can be solved easily by linear programming.

Computationally finding a cover (cont.)

- The condition of having a cover such that the shears of pants and eyes that are glued lie in some small interval, $[1-\epsilon, 1+\epsilon]$ can be translated to a set of linear inequalities in the entries of the solution vector via a combinatorial theorem on matching (Hall's marriage theorem).
- We can then frame the question of finding numerical solutions to the modular domain entirely as a problem in linear programming.

Accomplishments and future directions

- Wrote code in Python for the above algorithms, available to all on Github: `https://github.com/KaiShaikh/REU`.
- Found covers by pants and eyes of the modular surface.
- Studied distributions of pants and eyes in solutions minimizing ℓ^1 and ℓ^2.
- Remaining to do: Find covers simultaneously for a pair of distinct surfaces and analyze distributions of pants and eyes.

Concluding remarks

- Hopefully, the algorithms, code and numerical examples we have developed will generate insight towards this angle of attack in a proof of the cusped Ehrenpreis conjecture.
- We thank our advisors, Profs. Alex Kontorovich and Jeremy Kahn, for their mentorship.
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