Advanced Engineering Mathematics

Vectors

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1 Vectors in $\mathbb{R}^n$

1.1 Definition and Representation of Vectors

Key Concept

A **vector** is a quantity that has both **magnitude** and **direction**. It is commonly represented as an ordered list of real numbers:

$$\mathbf{v} = [v_1, v_2, v_3, \ldots, v_n]$$

where $\mathbf{v} \in \mathbb{R}^n$.

A **scalar** is a quantity described only by magnitude (a single number), such as temperature, mass, or time.

Vectors can be interpreted geometrically as directed line segments (arrows), and algebraically as ordered tuples.

1.1.1 Position Vector

Key Concept

The **position vector** of a point $P(x_1, x_2, \ldots, x_n)$ is the vector drawn from the origin to the point:

$$\mathbf{r} = [x_1, x_2, \ldots, x_n]$$

1.2 Basic Vector Operations

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and let c be a scalar.

1.2.1 Vector Addition

Key Concept

If

$$\mathbf{u} = [u_1, u_2, \ldots, u_n], \qquad \mathbf{v} = [v_1, v_2, \ldots, v_n],$$

then

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n].$$

1.2.2 Vector Subtraction

$$\mathbf{u} - \mathbf{v} = [u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n].$$

1.2.3 Scalar Multiplication

Key Concept

If c is a scalar and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, then

$$c\mathbf{v} = [cv_1, cv_2, \ldots, cv_n].$$

1.2.4 Basic Properties of Vector Operations

For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalar c, the following hold:

- Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- Distributive Law: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- Distributive Law: $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$

1.3 Norm, Magnitude, and Unit Vectors

1.3.1 Magnitude (Norm)

Key Concept

The **magnitude** (or **Euclidean norm**) of a vector

$$\mathbf{v} = [v_1, v_2, \ldots, v_n]$$

is defined as

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

This measures the length of the vector in $\mathbb{R}^n$.

1.3.2 Unit Vector

Key Concept

A **unit vector** is a vector of magnitude 1. If $\mathbf{v} \neq \mathbf{0}$, then the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Example

Let $\mathbf{v} = [3, 4]$.

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Thus the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{5}[3, 4] = \left[\frac{3}{5}, \frac{4}{5}\right].$$

1.4 Standard Basis and Component Form

1.4.1 Standard Basis Vectors

Key Concept

The **standard basis vectors** in $\mathbb{R}^n$ are the vectors:

$$\mathbf{e}_1 = [1, 0, 0, \dots, 0], \quad \mathbf{e}_2 = [0, 1, 0, \dots, 0], \quad \dots, \quad \mathbf{e}_n = [0, 0, 0, \dots, 1].$$

Any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as a linear combination of these basis vectors:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

1.4.2 Component Form in $\mathbb{R}^3$

In $\mathbb{R}^3$, the standard basis is often written as:

$$\mathbf{i} = [1, 0, 0], \quad \mathbf{j} = [0, 1, 0], \quad \mathbf{k} = [0, 0, 1].$$

Thus any vector can be written as:

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

1.5 Vector Applications

Vectors appear naturally in engineering and physics because many real-world quantities require both magnitude and direction.

1.5.1 Displacement

A displacement from point $P(x_1, y_1, z_1)$ to point $Q(x_2, y_2, z_2)$ is represented as:

$$\mathbf{d} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1].$$

1.5.2 Velocity

Velocity is a vector that describes both speed and direction of motion.

1.5.3 Force and Resultant Force

Engineering Note

In statics and dynamics, forces are vectors. The **resultant force** is obtained by adding all forces acting on a body:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n.$$

If $\mathbf{R} = \mathbf{0}$, then the system is in equilibrium.

Example

Suppose two forces act on an object:

$$\mathbf{F}_1 = [3, 2], \qquad \mathbf{F}_2 = [1, -4].$$

Then the resultant force is:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = [3 + 1,\ 2 - 4] = [4, -2].$$

The magnitude of the resultant is:

$$\|\mathbf{R}\| = \sqrt{4^2 + (-2)^2} = \sqrt{20}.$$

2 Dot Product

2.1 Definition and Formula

Key Concept

The **dot product** (or **inner product**) of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos \gamma,$$

where γ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (with $0 \leq \gamma \leq \pi$).

If $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_n]$, then the dot product can be computed by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Example

Let $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$.

$$\mathbf{a} \cdot \mathbf{b} = 1(3) + 2(-2) + 0(1) = 3 - 4 + 0 = -1.$$

Key Concept

The dot product of a vector with itself gives the square of its magnitude:

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2.$$

2.2 Properties of Dot Product

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and let k be a scalar.

Key Concept

The dot product satisfies the following key properties:

(a) **Commutativity:** $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(b) **Distributivity:** $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(c) **Scalar Linearity:** $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b})$

(d) **Positive Definiteness:** $\mathbf{a} \cdot \mathbf{a} \geq 0$, and $\mathbf{a} \cdot \mathbf{a} = 0$ iff $\mathbf{a} = \mathbf{0}$

Example

Let $\mathbf{a} = [2, 1]$, $\mathbf{b} = [-1, 3]$, and $\mathbf{c} = [4, 0]$.
Compute $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$:

$$\mathbf{b} + \mathbf{c} = [-1 + 4, 3 + 0] = [3, 3]$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = [2, 1] \cdot [3, 3] = 2(3) + 1(3) = 9.$$

Now compute $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$:

$$\mathbf{a} \cdot \mathbf{b} = 2(-1) + 1(3) = 1, \qquad \mathbf{a} \cdot \mathbf{c} = 2(4) + 1(0) = 8$$

$$\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 1 + 8 = 9.$$

So distributivity is verified.

2.3 Angle and Orthogonality

2.3.1 Angle Between Two Vectors

If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then

$$\cos \gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}, \qquad \gamma = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}\right).$$

Example

Let $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$.
From earlier, $\mathbf{a} \cdot \mathbf{b} = -1$. Also,

$$\|\mathbf{a}\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}, \qquad \|\mathbf{b}\| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}.$$

Thus,

$$\cos \gamma = \frac{-1}{\sqrt{5}\sqrt{14}} = \frac{-1}{\sqrt{70}}.$$

2.3.2 Orthogonality

Key Concept

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are **orthogonal** (perpendicular) if and only if

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

Example

Check if $\mathbf{u} = [2, -1]$ and $\mathbf{v} = [1, 2]$ are orthogonal:

$$\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(2) = 2 - 2 = 0.$$

Hence, $\mathbf{u} \perp \mathbf{v}$.

2.4 Projection and Components

2.4.1 Scalar Component of a in the Direction of b

Key Concept

If $\mathbf{b} \neq \mathbf{0}$, the **scalar projection** (component) of $\mathbf{a}$ onto $\mathbf{b}$ is

$$\text{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}.$$

If $\mathbf{b}$ is a unit vector, then $\text{comp}_{\mathbf{b}}(\mathbf{a}) = \mathbf{a} \cdot \mathbf{b}$.

2.4.2 Vector Projection of a onto b

Key Concept

If $\mathbf{b} \neq \mathbf{0}$, the **vector projection** of $\mathbf{a}$ onto $\mathbf{b}$ is

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}.$$

Example

Let $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [1, 2]$.
Compute:

$$\mathbf{a} \cdot \mathbf{b} = 3(1) + 4(2) = 11, \qquad \|\mathbf{b}\|^2 = 1^2 + 2^2 = 5.$$

So,

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{11}{5} \right) [1, 2] = \left[\frac{11}{5}, \frac{22}{5} \right].$$

2.5 Applications

2.5.1 Work Done by a Constant Force

Key Concept

If a constant force $\mathbf{F}$ displaces an object by $\mathbf{d}$, then the **work** done is

$$W = \mathbf{F} \cdot \mathbf{d} = \|\mathbf{F}\| \, \|\mathbf{d}\| \cos \gamma.$$

Example

A force $\mathbf{F} = [6, -3, 0]$ moves an object by $\mathbf{d} = [2, 5, 0]$.

$$W = \mathbf{F} \cdot \mathbf{d} = 6(2) + (-3)(5) + 0(0) = 12 - 15 = -3.$$

Since $W < 0$, the force opposes the displacement (work is done *against* the force).

2.5.2 Decomposing a Vector into Parallel and Perpendicular Parts

Engineering Note

In engineering (statics/dynamics), it is common to decompose a force into:

- a component **parallel** to a surface or direction
- a component **perpendicular** (normal) to the surface

This is done using projection.

If $\mathbf{b} \neq \mathbf{0}$, the component of $\mathbf{a}$ parallel to $\mathbf{b}$ is

$$\mathbf{a}_{\parallel} = \operatorname{proj}_{\mathbf{b}}(\mathbf{a}),$$

and the perpendicular component is

$$\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel}.$$