Advanced Engineering Mathematics

Vectors

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1 Vectors in $\mathbb{R}^n$

1.1 Definition and Representation of Vectors

Key Concept

A **vector** is a quantity that has both **magnitude** and **direction**. It is commonly represented as an ordered list of real numbers:

$$\mathbf{v} = [v_1, v_2, v_3, \ldots, v_n]$$

where $\mathbf{v} \in \mathbb{R}^n$.

A **scalar** is a quantity described only by magnitude (a single number), such as temperature, mass, or time.

Vectors can be interpreted geometrically as directed line segments (arrows), and algebraically as ordered tuples.

1.1.1 Position Vector

Key Concept

The **position vector** of a point $P(x_1, x_2, \ldots, x_n)$ is the vector drawn from the origin to the point:

$$\mathbf{r} = [x_1, x_2, \ldots, x_n]$$

Example

A point in 3D space is given by

$$P(2, -1, 5).$$

The position vector of P is the vector from the origin to P:

$$\mathbf{r} = [2, -1, 5].$$

This means the vector moves:

- 2 units in the x-direction,
- 1 unit in the negative y-direction,
- 5 units in the z-direction.

1.2 Basic Vector Operations

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and let c be a scalar.

1.2.1 Vector Addition

Key Concept

If

$$\mathbf{u} = [u_1, u_2, \ldots, u_n], \qquad \mathbf{v} = [v_1, v_2, \ldots, v_n],$$

then

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n].$$

1.2.2 Vector Subtraction

$$\mathbf{u} - \mathbf{v} = [u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n].$$

1.2.3 Scalar Multiplication

Key Concept

If c is a scalar and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, then

$$c\mathbf{v} = [cv_1, cv_2, \ldots, cv_n].$$

1.2.4 Basic Properties of Vector Operations

For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalar c, the following hold:

- Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- Distributive Law: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- Distributive Law: $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$

Example

Let

$$\mathbf{u} = [4, -2, 1], \qquad \mathbf{v} = [-1, 5, 3].$$

(a) Vector Addition

$$\mathbf{u} + \mathbf{v} = [4 + (-1), -2 + 5, 1 + 3] = [3, 3, 4].$$

(b) Vector Subtraction

$$\mathbf{u} - \mathbf{v} = [4 - (-1), -2 - 5, 1 - 3] = [5, -7, -2].$$

(c) Scalar Multiplication

$$3\mathbf{u} = 3[4, -2, 1] = [12, -6, 3].$$

(d) Linear Combination

$$2\mathbf{u} - \mathbf{v} = 2[4, -2, 1] - [-1, 5, 3] = [8, -4, 2] - [-1, 5, 3] = [9, -9, -1].$$

1.3 Norm, Magnitude, and Unit Vectors

1.3.1 Magnitude (Norm)

Key Concept

The **magnitude** (or **Euclidean norm**) of a vector

$$\mathbf{v} = [v_1, v_2, \dots, v_n]$$

is defined as

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

This measures the length of the vector in $\mathbb{R}^n$.

1.3.2 Unit Vector

Key Concept

A **unit vector** is a vector of magnitude 1. If $\mathbf{v} \neq \mathbf{0}$, then the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Example

Let $\mathbf{v} = [3, 4]$.

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Thus the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{5}[3, 4] = \left[\frac{3}{5}, \frac{4}{5}\right].$$

Example

Let

$$\mathbf{v} = [6, -2, 3].$$

Step 1: Find the magnitude

$$\|\mathbf{v}\| = \sqrt{6^2 + (-2)^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7.$$

Step 2: Find the unit vector in the direction of v

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{7}[6, -2, 3] = \left[\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right].$$

Thus, $\mathbf{u}$ has the same direction as $\mathbf{v}$ but has length 1.

1.4 Standard Basis and Component Form

1.4.1 Standard Basis Vectors

Key Concept

The **standard basis vectors** in $\mathbb{R}^n$ are the vectors:

$$\mathbf{e}_1 = [1, 0, 0, \ldots, 0], \quad \mathbf{e}_2 = [0, 1, 0, \ldots, 0], \quad \ldots, \quad \mathbf{e}_n = [0, 0, 0, \ldots, 1].$$

Any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as a linear combination of these basis vectors:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

1.4.2 Component Form in $\mathbb{R}^3$

In $\mathbb{R}^3$, the standard basis is often written as:

$$\mathbf{i} = [1, 0, 0], \quad \mathbf{j} = [0, 1, 0], \quad \mathbf{k} = [0, 0, 1].$$

Thus any vector can be written as:

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Example

Let the vector be

$$\mathbf{a} = [5, -3, 2].$$

Using the standard basis vectors

$$\mathbf{i} = [1, 0, 0], \quad \mathbf{j} = [0, 1, 0], \quad \mathbf{k} = [0, 0, 1],$$

we can rewrite **a** in component form as:

$$\mathbf{a} = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.$$

This representation is useful in physics and engineering because each term represents a vector component along an axis direction.

1.5 Vector Applications

Vectors appear naturally in engineering and physics because many real-world quantities require both magnitude and direction. In practice, vectors are used to model motion, forces, fields, and system behaviors in 2D and 3D space.

1.5.1 Displacement

Key Concept

Displacement is the vector that describes the change in position of an object. It points from the initial position to the final position.

A displacement from point $P(x_1, y_1, z_1)$ to point $Q(x_2, y_2, z_2)$ is represented as:

$$\mathbf{d} = \overrightarrow{PQ} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1].$$

The magnitude of the displacement vector represents the straight-line distance traveled:

$$\|\mathbf{d}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example

A robot moves from point

$$P(2, -1, 4)$$

to point

$$Q(7, 3, 1).$$

The displacement vector is

$$\mathbf{d} = [7 - 2,\ 3 - (-1),\ 1 - 4] = [5, 4, -3].$$

The distance traveled in a straight line is

$$\|\mathbf{d}\| = \sqrt{5^2 + 4^2 + (-3)^2} = \sqrt{25 + 16 + 9} = \sqrt{50}.$$

Thus, the robot displaced by $\mathbf{d} = [5, 4, -3]$ and traveled $\sqrt{50}$ units.

Engineering Note

In engineering applications, displacement is important in robotics, navigation systems, CNC motion, and kinematics.

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1.5.2 Velocity

Key Concept

Velocity is a vector that describes how fast an object is moving **and** in what direction it is moving.

If an object moves with displacement $\mathbf{d}$ over time interval Δt, then its average velocity is

$$\mathbf{v}_{avg} = \frac{\mathbf{d}}{\Delta t}.$$

The magnitude of velocity is the **speed**:

$$\text{speed} = \|\mathbf{v}\|.$$

Example

A drone travels with displacement vector

$$\mathbf{d} = [12, -4, 6] \text{ meters}$$

in a time interval of $\Delta t = 3$ seconds.
The average velocity is

$$\mathbf{v}_{avg} = \frac{\mathbf{d}}{3} = \left[\frac{12}{3}, \frac{-4}{3}, \frac{6}{3}\right] = [4, -\tfrac{4}{3}, 2] \text{ m/s}.$$

The speed of the drone is

$$\|\mathbf{v}_{avg}\| = \sqrt{4^2 + \left(-\tfrac{4}{3}\right)^2 + 2^2} = \sqrt{16 + \frac{16}{9} + 4} = \sqrt{\frac{180 + 16}{9}} = \sqrt{\frac{196}{9}} = \frac{14}{3} \text{ m/s}.$$

Thus, the drone moves with speed $\frac{14}{3}$ m/s.

Engineering Note

Velocity vectors are heavily used in mechanical engineering, robotics, control systems, fluid flow, and navigation.

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1.5.3 Force and Resultant Force

Key Concept

Force is a vector quantity because it has both magnitude and direction. Multiple forces acting on an object can be combined using vector addition.

Engineering Note

In statics and dynamics, forces are vectors. The **resultant force** is obtained by adding all forces acting on a body:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n.$$

If $\mathbf{R} = \mathbf{0}$, then the system is in equilibrium.

1.5.4 Resultant of Two Forces

Example

Suppose two forces act on an object:

$$\mathbf{F}_1 = [3, 2], \qquad \mathbf{F}_2 = [1, -4].$$

Then the resultant force is:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = [3 + 1,\ 2 - 4] = [4, -2].$$

The magnitude of the resultant is:

$$\|\mathbf{R}\| = \sqrt{4^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20}.$$

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1.5.5 Resultant Force with Three Forces in 3D

Example

Suppose a mechanical structure is subjected to three forces:

$$\mathbf{F}_1 = [10, -2, 5]\ \text{N}, \qquad \mathbf{F}_2 = [-6, 4, 1]\ \text{N}, \qquad \mathbf{F}_3 = [2, 0, -3]\ \text{N}.$$

The resultant force is

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3.$$

Add the components:

$$\mathbf{R} = [10 - 6 + 2,\ -2 + 4 + 0,\ 5 + 1 - 3] = [6, 2, 3]\ \text{N}.$$

The magnitude of the resultant force is

$$\|\mathbf{R}\| = \sqrt{6^2 + 2^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7\ \text{N}.$$

Thus, the net force acting on the structure is $\mathbf{R} = [6, 2, 3]$ N with magnitude 7 N.

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1.5.6 Equilibrium Condition

Key Concept

A body is said to be in **equilibrium** if the net force acting on it is zero:

$$\mathbf{R} = \mathbf{0}.$$

This means the object has no acceleration.

Example

A box is pulled by two forces:

$$\mathbf{F}_1 = [8, 3] \text{ N}, \qquad \mathbf{F}_2 = [-5, -7] \text{ N}.$$

Find the force $\mathbf{F}_3$ that must be applied so the system is in equilibrium.
The equilibrium condition is:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}.$$

Thus,

$$\mathbf{F}_3 = -(\mathbf{F}_1 + \mathbf{F}_2).$$

Compute the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$:

$$\mathbf{F}_1 + \mathbf{F}_2 = [8 - 5,\ 3 - 7] = [3, -4].$$

So the balancing force is:

$$\mathbf{F}_3 = -[3, -4] = [-3, 4] \text{ N}.$$

Therefore, a force of $[-3, 4]$ N must be applied for equilibrium.

Engineering Note

This concept is extremely important in statics problems such as trusses, beams, bridges, and structural design.

2 Dot Product

2.1 Definition and Formula

Key Concept

The **dot product** (or **inner product**) of two vectors **a** and **b** is defined by

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos \gamma,$$

where γ is the angle between **a** and **b** (with $0 \le \gamma \le \pi$).

If $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_n]$, then the dot product can be computed by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Example

Let $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$.

$$\mathbf{a} \cdot \mathbf{b} = 1(3) + 2(-2) + 0(1) = 3 - 4 + 0 = -1.$$

Key Concept

The dot product of a vector with itself gives the square of its magnitude:

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2.$$

2.2 Properties of Dot Product

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and let k be a scalar.

Key Concept

The dot product satisfies the following key properties:

(a) **Commutativity:** $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(b) **Distributivity:** $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(c) **Scalar Linearity:** $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b})$

(d) **Positive Definiteness:** $\mathbf{a} \cdot \mathbf{a} \ge 0$, and $\mathbf{a} \cdot \mathbf{a} = 0$ iff $\mathbf{a} = \mathbf{0}$

Example

Let $\mathbf{a} = [2, 1]$, $\mathbf{b} = [-1, 3]$, and $\mathbf{c} = [4, 0]$.
Compute $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$:

$$\mathbf{b} + \mathbf{c} = [-1 + 4, 3 + 0] = [3, 3]$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = [2, 1] \cdot [3, 3] = 2(3) + 1(3) = 9.$$

Now compute $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$:

$$\mathbf{a} \cdot \mathbf{b} = 2(-1) + 1(3) = 1, \qquad \mathbf{a} \cdot \mathbf{c} = 2(4) + 1(0) = 8$$

$$\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 1 + 8 = 9.$$

So distributivity is verified.

2.3 Angle and Orthogonality

2.3.1 Angle Between Two Vectors

If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then

$$\cos\gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}, \qquad \gamma = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right).$$

Example

Let $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$.
From earlier, $\mathbf{a} \cdot \mathbf{b} = -1$. Also,

$$\|\mathbf{a}\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}, \qquad \|\mathbf{b}\| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}.$$

Thus,

$$\cos\gamma = \frac{-1}{\sqrt{5}\sqrt{14}} = \frac{-1}{\sqrt{70}}.$$

2.3.2 Orthogonality

Key Concept

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are **orthogonal** (perpendicular) if and only if

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

Example

Check if $\mathbf{u} = [2, -1]$ and $\mathbf{v} = [1, 2]$ are orthogonal:

$$\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(2) = 2 - 2 = 0.$$

Hence, $\mathbf{u} \perp \mathbf{v}$.

2.4 Projection and Components

2.4.1 Scalar Component of a in the Direction of b

Key Concept

If $\mathbf{b} \neq \mathbf{0}$, the **scalar projection** (component) of $\mathbf{a}$ onto $\mathbf{b}$ is

$$\text{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}.$$

If $\mathbf{b}$ is a unit vector, then $\text{comp}_{\mathbf{b}}(\mathbf{a}) = \mathbf{a} \cdot \mathbf{b}$.

2.4.2 Vector Projection of a onto b

Key Concept

If $\mathbf{b} \neq \mathbf{0}$, the **vector projection** of $\mathbf{a}$ onto $\mathbf{b}$ is

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}.$$

Example

Let $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [1, 2]$.
Compute:

$$\mathbf{a} \cdot \mathbf{b} = 3(1) + 4(2) = 11, \qquad \|\mathbf{b}\|^2 = 1^2 + 2^2 = 5.$$

So,

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{11}{5} \right) [1, 2] = \left[\frac{11}{5}, \frac{22}{5} \right].$$

2.5 Applications

2.5.1 Work Done by a Constant Force

Key Concept

If a constant force $\mathbf{F}$ displaces an object by $\mathbf{d}$, then the **work** done is

$$W = \mathbf{F} \cdot \mathbf{d} = \|\mathbf{F}\| \, \|\mathbf{d}\| \cos \gamma.$$

Example

A force $\mathbf{F} = [6, -3, 0]$ moves an object by $\mathbf{d} = [2, 5, 0]$.

$$W = \mathbf{F} \cdot \mathbf{d} = 6(2) + (-3)(5) + 0(0) = 12 - 15 = -3.$$

Since $W < 0$, the force opposes the displacement (work is done *against* the force).

2.5.2 Decomposing a Vector into Parallel and Perpendicular Parts

Engineering Note

In engineering (statics/dynamics), it is common to decompose a force into:

- a component **parallel** to a surface or direction
- a component **perpendicular** (normal) to the surface

This is done using projection.

If $\mathbf{b} \neq \mathbf{0}$, the component of $\mathbf{a}$ parallel to $\mathbf{b}$ is

$$\mathbf{a}_{\parallel} = \mathrm{proj}_{\mathbf{b}}(\mathbf{a}),$$

and the perpendicular component is

$$\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel}.$$

3 Cross Product

3.1 Definition and Geometric Meaning

Key Concept

The **cross product** (or **vector product**) of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is a vector

$$\mathbf{a} \times \mathbf{b}$$

that is **perpendicular** to both $\mathbf{a}$ and $\mathbf{b}$.

3.1.1 Magnitude and Area Interpretation

Let γ be the angle between $\mathbf{a}$ and $\mathbf{b}$ (with $0 \le \gamma \le \pi$). If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin \gamma.$$

Engineering Note

Geometric meaning: $\|\mathbf{a} \times \mathbf{b}\|$ equals the **area of the parallelogram** formed by $\mathbf{a}$ and $\mathbf{b}$ as adjacent sides.

3.1.2 When is the Cross Product Zero?

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Also, if $\mathbf{a}$ and $\mathbf{b}$ are parallel (same or opposite direction), then $\gamma = 0$ or $\gamma = \pi$, so $\sin \gamma = 0$, hence:

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Example

Let $\mathbf{a} = [3, 0, 0]$ and $\mathbf{b} = [5, 0, 0]$.
These vectors are parallel (both point along the x-axis), so the area of the parallelogram is zero. Therefore,

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Example

Suppose $\|\mathbf{a}\| = 4$, $\|\mathbf{b}\| = 6$, and the angle between them is $\gamma = 30°$.

Then

$$\|\mathbf{a} \times \mathbf{b}\| = (4)(6)\sin 30^\circ = 24\left(\frac{1}{2}\right) = 12.$$

So the parallelogram area formed by **a** and **b** is 12 square units.

3.2 Right-Hand Rule and Orientation

3.2.1 Direction of the Cross Product

Key Concept

The direction of $\mathbf{a} \times \mathbf{b}$ is determined by the **Right-Hand Rule**:

- Point your right-hand fingers in the direction of **a**,
- curl them toward **b** through the smaller angle,
- your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

This means the order matters:

$$\mathbf{a} \times \mathbf{b} \text{ points opposite to } \mathbf{b} \times \mathbf{a}.$$

3.2.2 Right-Handed Coordinate Systems

In a standard right-handed Cartesian coordinate system, the unit basis vectors satisfy:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Reversing the order flips the sign:

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

Example

Compute $\mathbf{j} \times \mathbf{i}$.
Using the right-handed relationships:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \Rightarrow \quad \mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

Engineering Note

Engineering meaning: In torque and rotational motion, changing the order of vectors changes the rotation direction (clockwise vs counterclockwise), so orientation is physically important.

3.3 Cross Product Computation

Let

$$\mathbf{a} = [a_1, a_2, a_3], \qquad \mathbf{b} = [b_1, b_2, b_3].$$

3.3.1 Component Formula

Key Concept

The cross product in component form is:

$$\mathbf{a} \times \mathbf{b} = \Big[a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\Big].$$

3.3.2 Determinant Method

A convenient memory tool is the determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Expanding along the first row gives:

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1).$$

Example

Compute $\mathbf{a} \times \mathbf{b}$ for

$$\mathbf{a} = [1, 2, 3], \qquad \mathbf{b} = [4, 0, -1].$$

Using the component formula:

$$\mathbf{a} \times \mathbf{b} = \big[2(-1) - 3(0),\ 3(4) - 1(-1),\ 1(0) - 2(4)\big] = [-2,\ 13,\ -8].$$

So,

$$\mathbf{a} \times \mathbf{b} = [-2, 13, -8].$$

Example

Verify perpendicularity for the result above by dotting:
Let $\mathbf{v} = \mathbf{a} \times \mathbf{b} = [-2, 13, -8]$.

$$\mathbf{a} \cdot \mathbf{v} = [1, 2, 3] \cdot [-2, 13, -8] = 1(-2) + 2(13) + 3(-8) = -2 + 26 - 24 = 0.$$

$$\mathbf{b} \cdot \mathbf{v} = [4, 0, -1] \cdot [-2, 13, -8] = 4(-2) + 0(13) + (-1)(-8) = -8 + 8 = 0.$$

Thus $\mathbf{v}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, as expected.

3.4 Properties of the Cross Product

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and k be a scalar.

3.4.1 Linearity and Distributive Laws

Key Concept

Cross product satisfies:

(a) **Scalar Linearity:**
$$(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (k\mathbf{b})$$

(b) **Distributive Property:**

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

3.4.2 Anti-Commutativity

Key Concept

The cross product is **anti-commutative**:

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}).$$

3.4.3 Not Associative

Engineering Note

Cross product is **not associative**. In general:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

So parentheses matter.

Example

Verify anti-commutativity using an easy pair:

$$\mathbf{a} = \mathbf{i} = [1, 0, 0], \qquad \mathbf{b} = \mathbf{j} = [0, 1, 0].$$

We know:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

Switching the order:

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}.$$

So,

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}).$$

Example

Demonstrate non-associativity using basis vectors:

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = \mathbf{i} \times (\mathbf{i}) = \mathbf{0}.$$

But

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{k} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

This example gives equality, so it doesn't disprove associativity.
To show non-associativity more clearly, use:

$$\mathbf{a} = [1, 1, 0], \quad \mathbf{b} = [1, 0, 1], \quad \mathbf{c} = [0, 1, 1].$$

Compute $\mathbf{b} \times \mathbf{c}$:

$$\mathbf{b} \times \mathbf{c} = \left[0 \cdot 1 - 1 \cdot 1,\ 1 \cdot 0 - 1 \cdot 1,\ 1 \cdot 1 - 0 \cdot 0\right] = [-1, -1, 1].$$

Then

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = [1, 1, 0] \times [-1, -1, 1] = [1(-?) - 0(-?),\ \dots] = [1, -1, 0].$$

Now compute $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = [1, 1, 0] \times [1, 0, 1] = [1 \cdot 1 - 0 \cdot 0,\ 0 \cdot 1 - 1 \cdot 1,\ 1 \cdot 0 - 1 \cdot 1] = [1, -1, -1].$$

Then

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = [1, -1, -1] \times [0, 1, 1] = [0,\ -1,\ 1].$$

Since

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = [1, -1, 0] \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = [0, -1, 1],$$

cross product is not associative.

3.5 Scalar Triple Product and Volume Interpretation

3.5.1 Definition

Key Concept

The **scalar triple product** of $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ is defined as

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

It produces a **scalar** (a real number).

3.5.2 Determinant Form

If

$$\mathbf{a} = [a_1, a_2, a_3], \quad \mathbf{b} = [b_1, b_2, b_3], \quad \mathbf{c} = [c_1, c_2, c_3],$$

then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

3.5.3 Geometric Meaning: Volume

Engineering Note

The absolute value of the scalar triple product gives the **volume of the parallelepiped** formed by $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| \,.$$

If the volume is zero, the vectors lie in the same plane (they are coplanar), meaning they are linearly dependent in $\mathbb{R}^3$.

Example

Let

$$\mathbf{a} = [1, 2, 3], \quad \mathbf{b} = [2, 0, 1], \quad \mathbf{c} = [1, 1, 0].$$

Step 1: Compute $\mathbf{b} \times \mathbf{c}$:

$$\mathbf{b} \times \mathbf{c} = [\, 0 \cdot 0 - 1 \cdot 1,\ 1 \cdot 1 - 2 \cdot 0,\ 2 \cdot 1 - 0 \cdot 1 \,] = [-1, 1, 2].$$

Step 2: Dot with $\mathbf{a}$:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = [1, 2, 3] \cdot [-1, 1, 2] = 1(-1) + 2(1) + 3(2) = -1 + 2 + 6 = 7.$$

Thus, the volume of the parallelepiped is

$$V = |7| = 7 \text{ cubic units}.$$

Example

If $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [2, 4, 6]$, and $\mathbf{c} = [0, 1, 0]$, notice that $\mathbf{b} = 2\mathbf{a}$, so $\mathbf{a}$ and $\mathbf{b}$ are parallel.
Then $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar, and the parallelepiped collapses into a flat shape. Hence,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0 \quad \Rightarrow \quad V = 0.$$

4 Lines and Planes in $\mathbb{R}^2$ and $\mathbb{R}^3$

4.1 Equations of Lines

A line is one of the most fundamental geometric objects in mathematics and engineering. In vector calculus, lines are often represented using vectors because this form is very convenient for describing direction and motion.

4.1.1 Vector Equation of a Line

Key Concept

A line passing through a point $P(x_0, y_0, z_0)$ with direction vector $\mathbf{v} = [a, b, c]$ is given by:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v},$$

where

$$\mathbf{r}_0 = [x_0, y_0, z_0], \qquad t \in \mathbb{R}.$$

Here:

- $\mathbf{r}$ is the position vector of any point on the line,
- $\mathbf{r}_0$ is the position vector of a known point on the line,
- $\mathbf{v}$ gives the direction of the line,
- t is a parameter that moves along the line.

4.1.2 Parametric Equations of a Line

From the vector equation:

$$\mathbf{r} = [x, y, z] = [x_0, y_0, z_0] + t[a, b, c],$$

we obtain the parametric equations:

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct.$$

4.1.3 Symmetric Form of a Line

If $a, b, c \neq 0$, then we can eliminate t and write:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

This is called the **symmetric equation** of the line.

Example

Find the equation of the line passing through $P(2, -1, 3)$ and having direction vector

$$\mathbf{v} = [4, 2, -1].$$

Vector form:

$$\mathbf{r} = [2, -1, 3] + t[4, 2, -1].$$

Parametric form:

$$x = 2 + 4t, \qquad y = -1 + 2t, \qquad z = 3 - t.$$

Symmetric form:

$$\frac{x-2}{4} = \frac{y+1}{2} = \frac{z-3}{-1}.$$

Engineering Note

In engineering, parametric line equations are useful in robotics path planning, motion control, computer graphics, and modeling the trajectory of moving objects.

4.2 Equations of Planes

A plane is a flat two-dimensional surface that extends infinitely in three dimensions. In 3D geometry, planes are commonly described using a point and a normal vector.

4.2.1 Plane Equation Using a Normal Vector

Key Concept

A plane passing through point $P(x_0, y_0, z_0)$ with normal vector

$$\mathbf{n} = [A, B, C]$$

is given by the equation:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0,$$

where $\mathbf{r} = [x, y, z]$ and $\mathbf{r}_0 = [x_0, y_0, z_0]$.

Expanding:

$$[A, B, C] \cdot [x - x_0,\ y - y_0,\ z - z_0] = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

This simplifies to the standard plane equation:

$$Ax + By + Cz = D,$$

where $D = Ax_0 + By_0 + Cz_0$.

4.2.2 Plane Equation in Standard Form

Key Concept

The standard equation of a plane is:

$$Ax + By + Cz = D,$$

where $\mathbf{n} = [A, B, C]$ is the normal vector of the plane.

4.2.3 Parametric Form of a Plane

A plane can also be expressed using two non-parallel direction vectors $\mathbf{u}$ and $\mathbf{v}$:

Key Concept

If a plane passes through point P and contains direction vectors $\mathbf{u}$ and $\mathbf{v}$, then the plane can be written as:

$$\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v},$$

where $s, t \in \mathbb{R}$.

Example

Find the plane passing through $P(1, 2, 3)$ with normal vector $\mathbf{n} = [2, -1, 4]$.
Using:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0,$$

we have:

$$[2, -1, 4] \cdot [x - 1, y - 2, z - 3] = 0.$$

Expanding:

$$2(x - 1) - (y - 2) + 4(z - 3) = 0$$

$$2x - 2 - y + 2 + 4z - 12 = 0$$

$$2x - y + 4z - 12 = 0.$$

Thus, the equation of the plane is:

$$2x - y + 4z = 12.$$

Engineering Note

In engineering, planes are used in structural mechanics, CAD modeling, robotics orientation, and describing surfaces in 3D space.

4.3 Parallelism and Perpendicularity

Understanding whether lines and planes are parallel or perpendicular is essential in engineering applications such as design, architecture, robotics, and machine alignment.

4.3.1 Parallel Lines

Key Concept

Two lines are parallel if their direction vectors are scalar multiples of each other.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are direction vectors:

$$\mathbf{v}_1 = k\mathbf{v}_2$$

for some scalar k, then the lines are parallel.

Example

Check if lines with direction vectors

$$\mathbf{v}_1 = [2, 4, -1], \qquad \mathbf{v}_2 = [1, 2, -0.5]$$

are parallel.
Observe:

$$\mathbf{v}_2 = \left[\frac{2}{2}, \frac{4}{2}, \frac{-1}{2}\right] = [1, 2, -0.5].$$

Thus,

$$\mathbf{v}_1 = 2\mathbf{v}_2.$$

So the lines are parallel.

4.3.2 Perpendicular Lines

Key Concept

Two lines are perpendicular if their direction vectors satisfy:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0.$$

Example

Check if the vectors

$$\mathbf{v}_1 = [3, -2, 1], \qquad \mathbf{v}_2 = [2, 3, 0]$$

are perpendicular.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 3(2) + (-2)(3) + 1(0) = 6 - 6 + 0 = 0.$$

Thus, the lines are perpendicular.

4.3.3 Parallel Planes

Key Concept

Two planes are parallel if their normal vectors are scalar multiples.

For planes:

$$A_1x + B_1y + C_1z = D_1$$

$$A_2x + B_2y + C_2z = D_2$$

they are parallel if:

$$[A_1, B_1, C_1] = k[A_2, B_2, C_2].$$

4.3.4 Perpendicular Planes

Key Concept

Two planes are perpendicular if their normal vectors are orthogonal:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 0.$$

Example

Determine whether the planes

$$2x - y + 3z = 5$$

and

$$x + 2y - z = 4$$

are perpendicular.
Normal vectors:

$$\mathbf{n}_1 = [2, -1, 3], \qquad \mathbf{n}_2 = [1, 2, -1].$$

Dot product:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 2(1) + (-1)(2) + 3(-1) = 2 - 2 - 3 = -3.$$

Since the dot product is not zero, the planes are not perpendicular.

4.4 Distance and Projections

Distance formulas are important in geometry, physics, and engineering analysis. Vectors allow us to compute distances systematically in 2D and 3D.

4.4.1 Distance Between Two Points

Key Concept

The distance between two points

$$P(x_1, y_1, z_1), \qquad Q(x_2, y_2, z_2)$$

is:

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example

Find the distance between $P(1, -2, 4)$ and $Q(5, 1, -2)$.

$$d = \sqrt{(5 - 1)^2 + (1 - (-2))^2 + (-2 - 4)^2}$$

$$d = \sqrt{4^2 + 3^2 + (-6)^2} = \sqrt{16 + 9 + 36} = \sqrt{61}.$$

4.4.2 Distance from a Point to a Plane

Key Concept

The distance from a point $P(x_0, y_0, z_0)$ to the plane

$$Ax + By + Cz = D$$

is:

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Example

Find the distance from the point $P(2, -1, 3)$ to the plane

$$2x - y + 2z = 6.$$

Here, $A = 2$, $B = -1$, $C = 2$, $D = 6$.

$$d = \frac{|2(2) + (-1)(-1) + 2(3) - 6|}{\sqrt{2^2 + (-1)^2 + 2^2}}$$

$$d = \frac{|4 + 1 + 6 - 6|}{\sqrt{4 + 1 + 4}} = \frac{|5|}{\sqrt{9}} = \frac{5}{3}.$$

Thus, the point is $\frac{5}{3}$ units away from the plane.

Engineering Note

Distance to a plane is useful in robotics (distance to a surface), computer graphics (collision detection), and mechanical design (clearance checking).

4.5 Intersections

In engineering and geometry, intersections represent where objects meet, such as beams meeting at joints, or planes cutting through structures.

4.5.1 Intersection of a Line and a Plane

To find where a line intersects a plane:

1. Write the line in parametric form.
2. Substitute into the plane equation.
3. Solve for the parameter t.
4. Substitute back to find the intersection point.

Example

Find the intersection of the line

$$x = 1 + 2t, \qquad y = 3 - t, \qquad z = 2 + 4t$$

with the plane

$$x + y + z = 10.$$

Substitute into the plane equation:

$$(1 + 2t) + (3 - t) + (2 + 4t) = 10.$$

Simplify:

$$1 + 2t + 3 - t + 2 + 4t = 10$$

$$6 + 5t = 10$$

$$5t = 4$$

$$t = \frac{4}{5}.$$

Substitute back into the parametric equations:

$$x = 1 + 2\left(\frac{4}{5}\right) = 1 + \frac{8}{5} = \frac{13}{5}$$

$$y = 3 - \frac{4}{5} = \frac{15}{5} - \frac{4}{5} = \frac{11}{5}$$

$$z = 2 + 4\left(\frac{4}{5}\right) = 2 + \frac{16}{5} = \frac{26}{5}$$

Thus, the intersection point is:

$$\left(\frac{13}{5}, \frac{11}{5}, \frac{26}{5}\right).$$

4.5.2 Intersection of Two Planes

Two planes in $\mathbb{R}^3$ generally intersect in a line (unless they are parallel or identical).

Engineering Note

If two planes are not parallel, then their intersection is a straight line. This is important in CAD modeling and structural intersection analysis.

Example

Consider the planes:

$$x + y + z = 6$$

$$2x - y + z = 3.$$

Subtract the second from the first:

$$(x + y + z) - (2x - y + z) = 6 - 3$$

$$-x + 2y = 3$$

$$x = 2y - 3.$$

Let $y = t$, then:

$$x = 2t - 3.$$

Substitute into the first plane:

$$(2t - 3) + t + z = 6$$

$$3t - 3 + z = 6$$

$$z = 9 - 3t.$$

Thus the intersection line is:

$$x = 2t - 3, \qquad y = t, \qquad z = 9 - 3t.$$

Vector form:

$$\mathbf{r} = [-3, 0, 9] + t[2, 1, -3].$$

5 Vector Spaces

5.1 Definition and Examples

Vector spaces generalize the idea of vectors beyond geometry. In engineering, vector spaces appear in signal processing, control systems, machine learning, and circuit analysis.

Key Concept

A **vector space** is a set V of objects called vectors, together with two operations:

- vector addition
- scalar multiplication

such that the following axioms are satisfied:

(a) $\mathbf{u} + \mathbf{v} \in V$

(b) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(c) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(d) There exists a zero vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$

(e) For every $\mathbf{v}$, there exists $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

(f) $c\mathbf{v} \in V$ for scalar c

(g) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(h) $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$

(i) $c(d\mathbf{v}) = (cd)\mathbf{v}$

(j) $1\mathbf{v} = \mathbf{v}$

5.1.1 Common Examples of Vector Spaces

- $\mathbb{R}^n$ (all real-valued vectors with n components)

- The set of all polynomials of degree $\leq n$
- The set of all $m \times n$ matrices
- The set of all continuous functions on an interval

Example

Show that $\mathbb{R}^2$ is a vector space.
Let $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$.
Addition is defined by:

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$$

which is still in $\mathbb{R}^2$.
Scalar multiplication is defined by:

$$c\mathbf{u} = [cu_1, cu_2]$$

which is also still in $\mathbb{R}^2$.
All axioms are satisfied because addition and multiplication follow the usual arithmetic rules of real numbers. Hence, $\mathbb{R}^2$ is a vector space.

Engineering Note

Vector spaces are the mathematical foundation behind eigenvalues, Fourier transforms, signals, machine learning models, and robotics kinematics.

5.2 Subspaces

A subspace is a smaller vector space inside a larger one. It is very common in linear algebra and engineering analysis.

Key Concept

A subset W of a vector space V is called a **subspace** of V if:

(a) $\mathbf{0} \in W$

(b) If $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$

(c) If $\mathbf{u} \in W$ and c is a scalar, then $c\mathbf{u} \in W$

These conditions are often called the **subspace test**.

Example

Determine if the set

$$W = \{[x, y] \in \mathbb{R}^2 \mid y = 2x\}$$

is a subspace of $\mathbb{R}^2$.

Step 1: Check if the zero vector is included.

If $x = 0$, then $y = 2(0) = 0$, so $[0, 0] \in W$.

Step 2: Closure under addition.

Let $\mathbf{u} = [x_1, 2x_1]$ and $\mathbf{v} = [x_2, 2x_2]$ be in W.

Then

$$\mathbf{u} + \mathbf{v} = [x_1 + x_2, 2x_1 + 2x_2] = [x_1 + x_2, 2(x_1 + x_2)].$$

This satisfies the condition $y = 2x$, so $\mathbf{u} + \mathbf{v} \in W$.

Step 3: Closure under scalar multiplication.

Let c be a scalar. Then:

$$c\mathbf{u} = c[x_1, 2x_1] = [cx_1, 2(cx_1)].$$

This still satisfies $y = 2x$. Thus $c\mathbf{u} \in W$.

Therefore, W is a subspace of $\mathbb{R}^2$.

Example

Determine if the set

$$S = \{[x, y] \in \mathbb{R}^2 \mid x + y = 1\}$$

is a subspace.

The zero vector $[0, 0]$ does not satisfy $0 + 0 = 1$. So $[0, 0] \notin S$.

Thus, S is not a subspace.

5.3 Span and Linear Combinations

5.3.1 Linear Combination

Key Concept

A **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is any vector of the form:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

where $c_1, c_2, \ldots, c_k$ are scalars.

5.3.2 Span

Key Concept

The **span** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is the set of all linear combinations:

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

The span represents all possible vectors that can be formed using the given vectors.

Example

Let

$$\mathbf{v}_1 = [1, 0], \qquad \mathbf{v}_2 = [0, 1].$$

Any vector in $\mathbb{R}^2$ can be written as:

$$[x, y] = x[1, 0] + y[0, 1].$$

Thus,

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbb{R}^2.$$

Example

Let

$$\mathbf{v}_1 = [2, 4], \qquad \mathbf{v}_2 = [1, 2].$$

Notice that:

$$\mathbf{v}_1 = 2\mathbf{v}_2.$$

So the span of these vectors is just the set of all multiples of $\mathbf{v}_2$. Hence,

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_2\},$$

which forms a line through the origin.

Engineering Note

In engineering, span represents the set of all signals or solutions that can be formed using a given set of basis functions or vectors.

5.4 Linear Independence

Linear independence is a key concept because it tells us whether vectors provide new information or if one is redundant.

Key Concept

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are **linearly independent** if the only solution to:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

is

$$c_1 = c_2 = \cdots = c_k = 0.$$

If there exists a nonzero solution, then the vectors are **linearly dependent**.

Example

Determine if the vectors

$$\mathbf{v}_1 = [1, 2], \qquad \mathbf{v}_2 = [2, 4]$$

are linearly independent.
We check if:

$$c_1[1, 2] + c_2[2, 4] = [0, 0].$$

This gives the system:

$$c_1 + 2c_2 = 0$$

$$2c_1 + 4c_2 = 0.$$

The second equation is just twice the first, so there are infinitely many solutions. For example, if $c_2 = 1$, then $c_1 = -2$.
Thus, there is a nontrivial solution, so the vectors are linearly dependent.

Example

Determine if the vectors

$$\mathbf{u} = [1, 0, 2], \qquad \mathbf{v} = [0, 1, 3], \qquad \mathbf{w} = [1, 1, 0]$$

are linearly independent.
We solve:

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}.$$

That is:

$$c_1[1,0,2] + c_2[0,1,3] + c_3[1,1,0] = [0,0,0].$$

Component-wise:

$$c_1 + c_3 = 0$$

$$c_2 + c_3 = 0$$

$$2c_1 + 3c_2 = 0.$$

From the first two equations:

$$c_1 = -c_3, \qquad c_2 = -c_3.$$

Substitute into the third:

$$2(-c_3) + 3(-c_3) = 0 \Rightarrow -5c_3 = 0 \Rightarrow c_3 = 0.$$

Then $c_1 = 0$ and $c_2 = 0$.
Thus the only solution is the trivial solution, so the vectors are linearly independent.

5.5 Basis and Dimension

5.5.1 Basis

Key Concept

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a **basis** for a vector space V if:

(a) The vectors span V

(b) The vectors are linearly independent

This means every vector in V can be written uniquely as a linear combination of the basis vectors.

5.5.2 Dimension

Key Concept

The **dimension** of a vector space is the number of vectors in any basis of the space.

For example:

$$\dim(\mathbb{R}^2) = 2, \qquad \dim(\mathbb{R}^3) = 3.$$

Example

Show that the vectors

$$\mathbf{v}_1 = [1, 0], \qquad \mathbf{v}_2 = [0, 1]$$

form a basis for $\mathbb{R}^2$.

Step 1: Check spanning.

Any vector $[x, y] \in \mathbb{R}^2$ can be written as:

$$[x, y] = x[1, 0] + y[0, 1].$$

So they span $\mathbb{R}^2$.

Step 2: Check independence.

Solve:

$$c_1[1, 0] + c_2[0, 1] = [0, 0].$$

This implies:

$$c_1 = 0, \qquad c_2 = 0.$$

So they are linearly independent.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$ and $\dim(\mathbb{R}^2) = 2$.

Engineering Note

A basis is like a coordinate system. In engineering, it is used in signal decomposition, Fourier series, state-space models, and machine learning feature spaces.

5.6 Coordinates Relative to a Basis

5.6.1 Coordinate Representation

Key Concept

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Any vector $\mathbf{x} \in V$ can be written uniquely as:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates of $\mathbf{x}$ relative to the basis** B.

We often write the coordinate vector as:

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

5.6.2 Why Coordinates Change With Basis

In standard coordinates, vectors are written using $\mathbf{e}_1, \mathbf{e}_2, \ldots$. But if we use a different basis, the coordinates of the same vector may change.

Example

Let the basis in $\mathbb{R}^2$ be:

$$B = \{\mathbf{v}_1 = [1,1],\ \mathbf{v}_2 = [1,-1]\}.$$

Write the vector

$$\mathbf{x} = [4,2]$$

as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
We want:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2.$$

Substitute:

$$[4,2] = c_1[1,1] + c_2[1,-1].$$

Component-wise:

$$4 = c_1 + c_2$$

$$2 = c_1 - c_2.$$

Add the equations:

$$4 + 2 = 2c_1 \Rightarrow 6 = 2c_1 \Rightarrow c_1 = 3.$$

Then:

$$4 = 3 + c_2 \Rightarrow c_2 = 1.$$

Thus:

$$\mathbf{x} = 3\mathbf{v}_1 + 1\mathbf{v}_2.$$

So the coordinate vector of $\mathbf{x}$ relative to basis B is:

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Engineering Note

In engineering, using different bases is extremely useful. For example, Fourier analysis changes a signal from the time domain basis into a frequency domain basis.

6 Gram-Schmidt Orthogonalization

6.1 Motivation and Concept

In many engineering and mathematical applications, it is useful to represent vectors using **orthogonal** or **orthonormal** bases instead of arbitrary bases.

This is because orthogonal bases make computations easier, especially in:

- projections
- least squares approximation
- Fourier analysis
- signal processing
- numerical computations

Key Concept

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is called **orthogonal** if

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \quad \text{for } i \neq j.$$

Key Concept

A set of vectors is called **orthonormal** if:

(a) the vectors are orthogonal, and

(b) each vector has magnitude 1.

That is,

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Engineering Note

Orthonormal bases are extremely useful because they simplify dot products:

$$\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n \quad \Rightarrow \quad c_i = \mathbf{x} \cdot \mathbf{u}_i$$

when $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is orthonormal.

The **Gram-Schmidt process** is an algorithm that converts a set of linearly independent vectors into an orthogonal (or orthonormal) set spanning the same subspace.

6.2 Projection Review

Before applying Gram-Schmidt, we must recall the concept of projection.

6.2.1 Vector Projection Formula

Key Concept

The projection of $\mathbf{v}$ onto $\mathbf{u}$ is given by:

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}.$$

This formula gives the component of $\mathbf{v}$ that lies in the direction of $\mathbf{u}$.

6.2.2 Geometric Interpretation

Engineering Note

Projection is essentially "shadowing" a vector onto another direction. In engineering, projection is used in force decomposition, signal approximation, and coordinate transformations.

Example

Let

$$\mathbf{v} = [3, 4], \qquad \mathbf{u} = [1, 2].$$

Compute the projection of $\mathbf{v}$ onto $\mathbf{u}$.
First compute the dot products:

$$\mathbf{v} \cdot \mathbf{u} = 3(1) + 4(2) = 11$$

$$\mathbf{u} \cdot \mathbf{u} = 1^2 + 2^2 = 5.$$

Thus,

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{11}{5}\right)[1, 2] = \left[\frac{11}{5}, \frac{22}{5}\right].$$

So the part of $\mathbf{v}$ in the direction of $\mathbf{u}$ is:

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left[\frac{11}{5}, \frac{22}{5}\right].$$

6.3 Gram-Schmidt Algorithm

Suppose we are given a set of linearly independent vectors:

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}.$$

The goal is to generate an orthogonal set:

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$$

that spans the same subspace.

6.3.1 Step-by-Step Process

Key Concept

The Gram-Schmidt process is defined as:

$$\mathbf{u}_1 = \mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2)$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3)$$

and so on.

In general:

$$\mathbf{u}_k = \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k).$$

6.3.2 Why It Works

Each new vector $\mathbf{u}_k$ is obtained by subtracting from $\mathbf{v}_k$ its components in the directions of all previous $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}$.

Thus, the remaining part is orthogonal to all previous vectors.

Engineering Note

Gram-Schmidt is like "removing overlap" from vectors. It forces the new vector to be perpendicular to all previously formed vectors.

6.4 Orthonormalization

The Gram-Schmidt process produces an **orthogonal set**. To convert this into an **orthonormal set**, we simply normalize each vector.

6.4.1 Normalization

Key Concept

Given a nonzero vector $\mathbf{u}$, its corresponding unit vector is:

$$\mathbf{e} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Thus, if Gram-Schmidt produces:

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n,$$

then the orthonormal vectors are:

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}, \quad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}, \quad \dots, \quad \mathbf{e}_n = \frac{\mathbf{u}_n}{\|\mathbf{u}_n\|}.$$

Engineering Note

Orthonormal vectors simplify calculations dramatically because:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Example

Let $\mathbf{u} = [6, -2, 3]$.
Compute its unit vector.

$$\|\mathbf{u}\| = \sqrt{6^2 + (-2)^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7.$$

Thus the normalized vector is:

$$\mathbf{e} = \frac{1}{7}[6, -2, 3] = \left[\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right].$$

6.5 Worked Example in $\mathbb{R}^2$

Example

Use Gram-Schmidt to convert the vectors

$$\mathbf{v}_1 = [1, 1], \qquad \mathbf{v}_2 = [2, 0]$$

into an orthogonal and orthonormal set.
Step 1: Set the first orthogonal vector

$$\mathbf{u}_1 = \mathbf{v}_1 = [1, 1].$$

Step 2: Compute the projection of $\mathbf{v}_2$ onto $\mathbf{u}_1$
First compute dot products:

$$\mathbf{v}_2 \cdot \mathbf{u}_1 = [2, 0] \cdot [1, 1] = 2(1) + 0(1) = 2$$

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = [1,1] \cdot [1,1] = 1 + 1 = 2.$$

Thus:

$$\mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = \left(\frac{2}{2}\right)\mathbf{u}_1 = 1[1,1] = [1,1].$$

Step 3: Compute $\mathbf{u}_2$

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = [2,0] - [1,1] = [1,-1].$$

So the orthogonal set is:

$$\{\mathbf{u}_1, \mathbf{u}_2\} = \{[1,1],[1,-1]\}.$$

Step 4: Normalize to get an orthonormal set

Compute magnitudes:

$$\|\mathbf{u}_1\| = \sqrt{1^2 + 1^2} = \sqrt{2}, \qquad \|\mathbf{u}_2\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

Thus:

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{2}}[1,1] = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{2}}[1,-1] = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right].$$

Therefore, the orthonormal set is:

$$\left\{\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]\right\}.$$

6.6 Worked Example in $\mathbb{R}^3$

Example

Apply Gram-Schmidt to the vectors

$$\mathbf{v}_1 = [1,1,0], \qquad \mathbf{v}_2 = [1,0,1], \qquad \mathbf{v}_3 = [0,1,1].$$

Step 1: Compute $\mathbf{u}_1$

$$\mathbf{u}_1 = \mathbf{v}_1 = [1,1,0].$$

Step 2: Compute $\mathbf{u}_2$

First compute:

$$\text{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1.$$

Dot products:

$$\mathbf{v}_2 \cdot \mathbf{u}_1 = [1,0,1] \cdot [1,1,0] = 1(1) + 0(1) + 1(0) = 1$$

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = [1,1,0] \cdot [1,1,0] = 1 + 1 + 0 = 2.$$

Thus:

$$\text{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = \left(\frac{1}{2}\right)[1,1,0] = \left[\frac{1}{2}, \frac{1}{2}, 0\right].$$

Now subtract:

$$\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = [1,0,1] - \left[\frac{1}{2}, \frac{1}{2}, 0\right] = \left[\frac{1}{2}, -\frac{1}{2}, 1\right].$$

Step 3: Compute $\mathbf{u}_3$

We compute:

$$\mathbf{u}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \text{proj}_{\mathbf{u}_2}(\mathbf{v}_3).$$

First projection:

$$\mathbf{v}_3 \cdot \mathbf{u}_1 = [0,1,1] \cdot [1,1,0] = 0(1) + 1(1) + 1(0) = 1$$

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = 2$$

$$\text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) = \left(\frac{1}{2}\right)[1,1,0] = \left[\frac{1}{2}, \frac{1}{2}, 0\right].$$

Second projection requires:

$$\mathbf{v}_3 \cdot \mathbf{u}_2 = [0,1,1] \cdot \left[\frac{1}{2}, -\frac{1}{2}, 1\right] = 0\left(\frac{1}{2}\right) + 1\left(-\frac{1}{2}\right) + 1(1) = \frac{1}{2}.$$

Now compute $\mathbf{u}_2 \cdot \mathbf{u}_2$:

$$\mathbf{u}_2 \cdot \mathbf{u}_2 = \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + (1)^2 = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}.$$

Thus:

$$\text{proj}_{\mathbf{u}_2}(\mathbf{v}_3) = \left(\frac{\frac{1}{2}}{\frac{3}{2}}\right)\mathbf{u}_2 = \left(\frac{1}{3}\right)\left[\frac{1}{2}, -\frac{1}{2}, 1\right] = \left[\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}\right].$$

Now compute $\mathbf{u}_3$:

$$\mathbf{u}_3 = [0,1,1] - \left[\frac{1}{2}, \frac{1}{2}, 0\right] - \left[\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}\right].$$

Combine the subtractions:

$$\mathbf{u}_3 = \left[0 - \frac{1}{2} - \frac{1}{6},\ 1 - \frac{1}{2} + \frac{1}{6},\ 1 - 0 - \frac{1}{3}\right].$$

$$\mathbf{u}_3 = \left[-\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right].$$

Thus, the orthogonal set is:

$$\mathbf{u}_1 = [1,1,0], \quad \mathbf{u}_2 = \left[\frac{1}{2}, -\frac{1}{2}, 1\right], \quad \mathbf{u}_3 = \left[-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right].$$

(Optional) Check orthogonality:

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = [1,1,0] \cdot \left[\frac{1}{2}, -\frac{1}{2}, 1\right] = \frac{1}{2} - \frac{1}{2} + 0 = 0.$$

$$\mathbf{u}_1 \cdot \mathbf{u}_3 = [1,1,0] \cdot \left[-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right] = -\frac{2}{3} + \frac{2}{3} + 0 = 0.$$

So $\mathbf{u}_1$ is orthogonal to both $\mathbf{u}_2$ and $\mathbf{u}_3$.

Engineering Note

The orthonormal version can be obtained by dividing each $\mathbf{u}_i$ by its magnitude. This is often done in numerical methods and QR factorization.