Advanced Engineering Mathematics

SERIES

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Infinite Series

This lecture introduces the fundamental theory of infinite series as used in Advanced Engineering Mathematics. Emphasis is placed on convergence concepts and power series representations, which are essential in engineering analysis, modeling, and later topics such as differential equations and signal processing.

Series

A **series** is the sum of the terms of a sequence. If $\{a_n\}$ is a sequence, then the corresponding series is written as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The value of a series is determined by the behavior of its **partial sums**

$$S_n = \sum_{i=1}^{n} a_i.$$

- The series **converges** if $\lim_{n\to\infty} S_n$ exists and is finite.
- The series **diverges** if this limit does not exist or is infinite.

Necessary Condition for Convergence:

$$\lim_{n\to\infty} a_n = 0.$$

If this condition is not satisfied, the series must diverge.

Example

Example 1 Determine if the series

$$\sum_{n=1}^{\infty} n$$

converges or diverges.

Solution: The nth partial sum is

$$S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

As $n \to \infty$,

$$\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty,$$

so the sequence of partial sums diverges and thus the series diverges.

Example

Example 2 Determine if the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

converges or diverges, and if it converges, find its value.
Solution: A known formula for the partial sums of this series is

$$S_n = \sum_{i=2}^{n} \frac{1}{i^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}.$$

Taking the limit as $n \to \infty$,

$$\lim_{n\to\infty} S_n = \frac{3}{4}.$$

Thus the series converges and

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}.$$

Example

Example 3 Determine if the series

$$\sum_{n=0}^{\infty} (-1)^n$$

converges or diverges.
Solution: The partial sums oscillate:

$$s_0 = 1, \quad s_1 = 0, \quad s_2 = 1, \quad s_3 = 0, \quad \ldots$$

Since the sequence of partial sums does not approach a single value, this series diverges.

Example

Example 4 Determine if the series

$$\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$$

converges or diverges. If it converges, find the value.
Solution: A formula for the partial sums is

$$S_n = \sum_{i=1}^{n} \frac{1}{3^{i-1}} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right).$$

Taking the limit,

$$\lim_{n\to\infty} S_n = \frac{3}{2}.$$

Therefore, the series converges with

$$\sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = \frac{3}{2}.$$

Observation: In Examples 2 and 4, the terms of the series approach zero and the series converges; in Examples 1 and 3, they do not approach a limit that yields a convergent series. Thus, the limit of the series terms is an important preliminary check for convergence before applying tests.

1.1 Power Series and Their Convergence

Definition of a Power Series

A **power series** is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where:

- c_n are constant coefficients,
- x_0 is a fixed real number called the **center** of the series,
- $(x - x_0)^n$ represents powers of the variable x.

Unlike ordinary numerical series, a power series depends on the value of x. As a result, a power series may converge for some values of x and diverge for others.

General Behavior of Power Series

For a given power series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

there exists a real number $R \geq 0$, called the **radius of convergence**, such that:

$$\begin{cases} \text{The series converges absolutely if } |x - x_0| < R, \\ \text{The series diverges if } |x - x_0| > R, \\ \text{The series may converge or diverge if } |x - x_0| = R. \end{cases}$$

The radius of convergence can be determined from the coefficients of the series through:

$$\boxed{(a) \quad R = \frac{1}{\lim\limits_{n \to \infty} \sqrt[n]{|c_n|}}} \qquad \boxed{(b) \quad R = \frac{1}{\lim\limits_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|}}$$

provided the limit exists.

The interval

$$(x_0 - R,\, x_0 + R)$$

together with any endpoints where the series converges is called the **interval of convergence**.

Convergence, Divergence, and Absolute Convergence

Let

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

be a power series.

- The series is said to **converge** at a value $x = b$ if the numerical series

  $$\sum_{n=0}^{\infty} c_n (b - x_0)^n$$

  converges.

- The series **diverges** at $x = b$ if the corresponding numerical series diverges.
- The series is **absolutely convergent** at $x = b$ if

$$\sum_{n=0}^{\infty} |c_n(b - x_0)^n|$$

converges.

Important Result: If a power series converges at a point $x = b$, then it converges absolutely for all values of x such that $|x - x_0| < |b - x_0|$.

Focus on Convergence of Power Series

The convergence of a power series depends primarily on the distance of x from the center x_0. The farther x is from x_0, the more likely the series is to diverge.

To determine where a power series converges, the following steps are followed:

1. Apply a convergence test to find the radius of convergence R.
2. Determine the interval $|x - x_0| < R$.
3. Test the endpoints $x = x_0 \pm R$ separately.

Convergence Tests for Power Series

1. Ratio Test Let the power series be

$$\sum_{n=0}^{\infty} c_n(x - x_0)^n,$$

and define

$$a_n = c_n(x - x_0)^n.$$

Compute

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- If the limit is less than 1, the series converges absolutely.
- If the limit is greater than 1, the series diverges.
- If the limit equals 1, the test is inconclusive.

Examples: Radius and Interval of Convergence Using the Ratio Test

Example

Example 1: Power Series of e^x Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Let

$$a_n = \frac{x^n}{n!}.$$

Applying the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Since the limit is zero for all real values of x, the series converges for every x.

$$\boxed{R = \infty}$$

Thus, the interval of convergence is

$$\boxed{(-\infty, \infty)}.$$

Example

Example 2: Geometric Series $\frac{1}{1-x}$ Consider the power series

$$\sum_{n=0}^{\infty} x^n.$$

Let

$$a_n = x^n.$$

Applying the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = |x|.$$

For convergence,

$$|x| < 1.$$

Thus, the radius of convergence is

$$\boxed{R = 1}.$$

Endpoint Testing:

- At $x = -1$:

$$\sum_{n=0}^{\infty} (-1)^n \quad \text{diverges.}$$

- At $x = 1$:

$$\sum_{n=0}^{\infty} 1 \quad \text{diverges.}$$

Hence, the interval of convergence is

$$\boxed{(-1,\, 1)}.$$

Example

Example 3: Series Involving Factorials Consider the power series

$$\sum_{n=0}^{\infty} n!\, x^n.$$

Let

$$a_n = n!\, x^n.$$

Applying the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} (n+1)|x|.$$

For convergence,

$$(n+1)|x| < 1.$$

As $n \to \infty$, this inequality holds only when $x = 0$.

$$\boxed{R = 0}.$$

Thus, the series converges only at $x = 0$, and the interval of convergence is

$$\boxed{\{0\}}.$$

Summary of Results:

Series	Radius of Convergence	Interval of Convergence
$\sum \frac{x^n}{n!}$	$R = \infty$	$(-\infty, \infty)$
$\sum x^n$	$R = 1$	$(-1, 1)$
$\sum n! x^n$	$R = 0$	$\{0\}$

2. Root Test The **Root Test** may also be applied:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

The conclusions are the same as those of the Ratio Test.

Example

Example 4: Power Series with Exponential Coefficients Consider the power series

$$\sum_{n=0}^{\infty} \frac{(2x)^n}{n}.$$

Let

$$a_n = \frac{(2x)^n}{n}.$$

Apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{|2x|^n}{n}} = |2x| \cdot \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}}.$$

Since

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = 1,$$

we obtain

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = |2x|.$$

For convergence,

$$|2x| < 1 \quad \Rightarrow \quad |x| < \frac{1}{2}.$$

Thus, the radius of convergence is

$$\boxed{R = \frac{1}{2}}.$$

Endpoint Testing:

- At $x = \frac{1}{2}$:
$$\sum_{n=0}^{\infty} \frac{1}{n} \quad \text{diverges.}$$
- At $x = -\frac{1}{2}$:
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} \quad \text{converges (alternating series).}$$

Hence, the interval of convergence is

$$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right)}.$$

Example

Example 5: Power Series with Polynomial Growth Consider the power series

$$\sum_{n=0}^{\infty} n^2 x^n.$$

Let

$$a_n = n^2 x^n.$$

Apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n^2 |x|^n} = |x| \cdot \lim_{n\to\infty} \sqrt[n]{n^2}.$$

Since

$$\lim_{n\to\infty} \sqrt[n]{n^2} = 1,$$

we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = |x|.$$

For convergence,

$$|x| < 1.$$

Thus, the radius of convergence is

$$\boxed{R = 1}.$$

Endpoint Testing:

- At $x = 1$:

$$\sum_{n=0}^{\infty} n^2 \quad \text{diverges.}$$

- At $x = -1$:

$$\sum_{n=0}^{\infty} (-1)^n n^2 \quad \text{diverges.}$$

Hence, the interval of convergence is

$$\boxed{(-1,\, 1)}.$$

Example

Example 6: Power Series with Factorials in the Denominator Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}.$$

Let

$$a_n = \frac{x^n}{(n!)^2}.$$

Apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x|}{(n!)^{2/n}}.$$

Since $(n!)^{1/n} \to \infty$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$$

for all real x.

Therefore, the series converges for all x, and

$$\boxed{R = \infty}.$$

The interval of convergence is

$$\boxed{(-\infty, \infty)}.$$

Endpoint Testing: Common Tests and When to Use Them

After finding the radius of convergence R, the power series must be tested at the endpoints

$$x = x_0 - R \quad \text{and} \quad x = x_0 + R.$$

At each endpoint, the power series becomes a numerical series. Since the Ratio and Root Tests are inconclusive at endpoints, other convergence tests must be used.

1. p-Series Test A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a p-**series**.

- The series converges if $p > 1$.
- The series diverges if $p \leq 1$.

When to use: This test is used when the endpoint series simplifies to a rational expression involving powers of n, such as

$$\sum \frac{1}{n^2}, \quad \sum \frac{1}{\sqrt{n}}, \quad \sum \frac{1}{n}.$$

Example:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges} \qquad (p = 2 > 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges} \qquad (p = 1).$$

2. Alternating Series Test An alternating series has the form

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} b_n,$$

where $b_n > 0$ for all n.

The series converges if:

1. b_n is decreasing, and
2. $\lim_{n\to\infty} b_n = 0$.

When to use: This test is used when substituting an endpoint produces alternating signs, typically from terms such as $(-1)^n$.

Important Note: An alternating series may converge even if it does not converge absolutely.

Example:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} \quad \text{converges (conditionally).}$$

3. Comparison Test Let $\sum a_n$ and $\sum b_n$ be series with $0 \le a_n \le b_n$ for all sufficiently large n.

- If $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges.

When to use: This test is used when the endpoint series resembles a known series (such as a p-series) but does not match it exactly.

Example:

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} \quad \text{converges by comparison with } \sum \frac{1}{n^2}.$$

Choosing the Appropriate Test At an endpoint:

- If the series resembles $\frac{1}{n^p}$, use the ***p*-series test**.
- If the series alternates in sign, try the **Alternating Series Test** first.
- If the series resembles a known convergent or divergent series but is not exact, use the **Comparison Test**.

Each endpoint must be tested **independently**. A power series may converge at one endpoint and diverge at the other.

1.2 Divergence of Series

A series $\sum a_n$ is said to **diverge** if the sequence of partial sums

$$S_n = a_1 + a_2 + \cdots + a_n$$

does not approach a finite limit as $n \to \infty$.

Important Observation A necessary condition for convergence is

$$\lim_{n\to\infty} a_n = 0.$$

If this limit does not exist or is not zero, the series **must diverge**. However, this condition alone does not guarantee convergence.

Key Divergence Tests

- nth-Term Test for Divergence
- Harmonic Series
- Comparison with Known Divergent Series

1. nth-Term Test for Divergence

If

$$\lim_{n\to\infty} a_n \neq 0 \quad \text{or does not exist,}$$

then the series

$$\sum_{n=1}^{\infty} a_n$$

diverges.

Important Note: If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.

Example 1: nth-Term Test

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

Since

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0,$$

the series diverges by the nth-term test.

Example 2: Oscillating Terms

$$\sum_{n=1}^{\infty}(-1)^n$$

Here, the sequence $a_n = (-1)^n$ does not approach a limit. Therefore, the series diverges by the nth-term test.

2. Harmonic Series

The **harmonic series** is defined as

$$\sum_{n=1}^{\infty}\frac{1}{n}.$$

Although

$$\lim_{n\to\infty}\frac{1}{n} = 0,$$

the harmonic series **diverges**.

This example shows that the condition $\lim_{n\to\infty} a_n = 0$ is necessary but not sufficient for convergence.

Example 3: Constant Multiple of the Harmonic Series

$$\sum_{n=1}^{\infty}\frac{5}{n}$$

Since this is a constant multiple of the harmonic series, it also diverges.

3. Comparison with Divergent Series

If $0 \le a_n \le b_n$ for all sufficiently large n and

$$\sum b_n \quad \text{diverges},$$

then

$$\sum a_n \quad \text{also diverges}.$$

Example 4: Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This is a p-series with $p = \frac{1}{2} < 1$, which diverges. Therefore, any series comparable to it will also diverge.

Example 5: Rational Function

$$\sum_{n=1}^{\infty} \frac{n+1}{n}$$

Since

$$\frac{n+1}{n} = 1 + \frac{1}{n},$$

and the terms do not approach zero, the series diverges by the nth-term test.

Summary of Divergence Results

- If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.
- The harmonic series diverges even though its terms approach zero.
- Series comparable to divergent p-series with $p \leq 1$ diverge.

1.3 Absolute Convergence

A series $\sum a_n$ is **absolutely convergent** if

$$\sum |a_n| \text{ converges.}$$

Absolute convergence guarantees convergence of the original series.

Example 1

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

The absolute series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges; hence the given series is absolutely convergent.

Example 2

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

The absolute series diverges, but the original series converges conditionally.

Example 3

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$$

The absolute series behaves like $\sum \frac{1}{n}$ and diverges. The original series converges conditionally.

1.4 Taylor Series

Taylor series provide a way to represent a function as an infinite power series centered at a point $x = x_0$. This representation allows complicated functions to be approximated by polynomials near the center.

Definition of the Taylor Series

If a function $f(x)$ has derivatives of all orders at $x = x_0$, then the **Taylor series of** $f(x)$ **about** $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

This is a power series centered at x_0, with coefficients determined by the derivatives of $f(x)$ at $x = x_0$.

Interpretation of the Taylor Series

Each term of the Taylor series incorporates information about the function at the point $x = x_0$:

- The constant term gives the value $f(x_0)$.
- The linear term depends on $f'(x_0)$ and matches the slope at $x = x_0$.
- Higher-degree terms improve the accuracy of the approximation near x_0.

Thus, Taylor polynomials provide increasingly accurate local approximations of $f(x)$ as more terms are included.

Taylor Polynomials

The n**th Taylor polynomial** for $f(x)$ about $x = x_0$ is defined as

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

As n increases, $P_n(x)$ better approximates $f(x)$ near $x = x_0$, provided the Taylor series converges.

Example

Example 1: Taylor Series of e^x About $x_0 = 1$

Since all derivatives of e^x are equal to e^x, we have

$$f^{(n)}(1) = e.$$

Substituting into the Taylor series formula gives

$$e^x = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}.$$

Writing out the first few terms,

$$e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right].$$

This series converges for all real values of x.

Example

Example 2: Taylor Series of $\ln x$ About $x_0 = 1$

Let $f(x) = \ln x$. Its derivatives are

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f'''(x) = \frac{2}{x^3}, \quad \ldots$$

Evaluating at $x = 1$,

$$f^{(n)}(1) = (-1)^{n+1}(n-1)!.$$

Substituting into the Taylor series formula yields

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

This series converges for

$$0 < x \leq 2,$$

which corresponds to $|x-1| \leq 1$ with endpoint testing.

Example

Example 3: Polynomial Approximation of $\sqrt{x}$ Near $x = 4$

Let

$$f(x) = \sqrt{x}.$$

Compute derivatives:

$$f(4) = 2, \quad f'(4) = \frac{1}{4}, \quad f''(4) = -\frac{1}{32}.$$

Using these values, the second-degree Taylor polynomial about $x = 4$ is

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2.$$

This polynomial provides a good approximation to $\sqrt{x}$ for values of x close to 4.

Example

Example 4: Taylor Series of $\frac{1}{1+2x}$ About $x_0 = 1$

Consider the function

$$f(x) = \frac{1}{1+2x}.$$

We want the Taylor series centered at $x = 1$. First, rewrite the function in terms of $(x-1)$.

$$1 + 2x = 1 + 2(1 + (x-1)) = 3 + 2(x-1).$$

Thus,

$$f(x) = \frac{1}{3 + 2(x-1)}.$$

Factor out the constant term:

$$f(x) = \frac{1}{3} \cdot \frac{1}{1 + \frac{2}{3}(x-1)}.$$

Now use the geometric series formula

$$\frac{1}{1+u} = \sum_{n=0}^{\infty} (-1)^n u^n \quad \text{for } |u| < 1.$$

Let

$$u = \frac{2}{3}(x-1).$$

Then the Taylor series becomes

$$f(x) = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{3}(x-1) \right)^n.$$

Writing out the first few terms,

$$\frac{1}{1+2x} = \frac{1}{3} - \frac{2}{9}(x-1) + \frac{4}{27}(x-1)^2 - \frac{8}{81}(x-1)^3 + \cdots$$

Example

Interval of Convergence:
The geometric series converges when

$$\left|\frac{2}{3}(x-1)\right| < 1.$$

Solving,

$$|x-1| < \frac{3}{2}.$$

Thus, the radius of convergence is

$$\boxed{R = \frac{3}{2}},$$

and the interval of convergence is

$$\boxed{\left(-\frac{1}{2}, \frac{5}{2}\right)}.$$

Example

Example 5: Taylor Series of a Polynomial About a Nonzero Center

Find the Taylor series for

$$f(x) = x^3 - 10x^2 + 6$$

about $x_0 = 3$.

Step 1: Compute the derivatives

$$\begin{aligned}
f(x) &= x^3 - 10x^2 + 6, \\
f'(x) &= 3x^2 - 20x, \\
f''(x) &= 6x - 20, \\
f'''(x) &= 6, \\
f^{(n)}(x) &= 0 \quad \text{for } n \geq 4.
\end{aligned}$$

Step 2: Evaluate the derivatives at $x = 3$

$$\begin{aligned} f(3) &= -57, \\ f'(3) &= -33, \\ f''(3) &= -2, \\ f'''(3) &= 6. \end{aligned}$$

Step 3: Construct the Taylor series
The Taylor series about $x_0 = 3$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n.$$

Since all derivatives of order 4 and higher are zero, the series terminates after the cubic term.

$$\begin{aligned} x^3 - 10x^2 + 6 &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 \\ &= -57 - 33(x-3) - (x-3)^2 + (x-3)^3. \end{aligned}$$

Convergence and Validity of Taylor Series

A Taylor series represents the original function only on the interval where the series converges.

- Inside the interval of convergence, the Taylor series converges to $f(x)$.
- Outside this interval, the series may diverge or converge to a different value.

Therefore, convergence is essential in determining where a Taylor series can be used as a valid representation or approximation of a function.

1.5 Maclaurin Series

A **Maclaurin series** is a special case of the Taylor series obtained by choosing the center $x_0 = 0$. If a function $f(x)$ has derivatives of all orders at $x = 0$, then its Maclaurin series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Maclaurin series are especially useful because of their simple form and their connection to fundamental power series.

Interpretation of the Maclaurin Series

The Maclaurin series represents a function as an infinite polynomial whose coefficients are determined by the function and its derivatives at $x = 0$.

- The constant term gives the value $f(0)$.
- The linear term matches the slope at $x = 0$.
- Higher-degree terms improve the accuracy near $x = 0$.

Thus, Maclaurin polynomials provide increasingly accurate approximations of $f(x)$ for values of x close to zero.

Common Maclaurin Series

Example

Example 1: e^x Since all derivatives of e^x are equal to e^x, and $e^0 = 1$, we obtain

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This series converges for all real values of x.

Example

Example 2: $\sin x$ The derivatives of $\sin x$ alternate between $\sin x$ and $\cos x$, yielding

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

This series also converges for all real values of x.

Example

Example 3: $\cos x$ The derivatives of $\cos x$ alternate between $\cos x$ and $-\sin x$, giving

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Again, the radius of convergence is infinite.

Convergence of Maclaurin Series

Since Maclaurin series are power series centered at 0, their convergence is determined using the same tests discussed earlier.

- Many common Maclaurin series converge for all real x.
- Some converge only within a finite interval and must be tested at endpoints.

The interval of convergence determines where the series equals the original function.

Polynomial Approximation Using Maclaurin Series

Maclaurin series can be used to approximate functions near $x = 0$ by truncating the series after a finite number of terms.

Example 4: Approximation of $\sin x$ Using the first two nonzero terms,

$$\sin x \approx x - \frac{x^3}{3!}.$$

This approximation is accurate for small values of x.

1.5 Common Series and Power Series Formulas

This section summarizes the most frequently used series formulas encountered in Module 1. These formulas are used as building blocks for constructing Taylor and Maclaurin series and for analyzing convergence.

Key Concept

1. Geometric Series

$$\sum_{n=0}^{\infty} c_n \quad \text{where } c_n = ar^n$$

- Converges if $|r| < 1$
- Diverges if $|r| \geq 1$

When convergent,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Key Concept

2. Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

This series diverges, even though

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Key Concept

3. p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- Converges if $p > 1$
- Diverges if $p \leq 1$

Key Concept

4. Taylor Series Formula

If a function $f(x)$ has derivatives of all orders at $x = a$, then its Taylor series about $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Key Concept

5. Maclaurin Series Formula

The Maclaurin series is the Taylor series centered at $a = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Key Concept

6. Common Maclaurin Series

e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad R = \infty$$

$\sin x$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad R = \infty$$

$\cos x$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad R = \infty$$

$\dfrac{1}{1-x}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$\ln(1+x)$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \qquad -1 < x \le 1$$

7. Using Series Formulas

- Known series may be shifted using $(x-a)$.
- Constants may be factored out of a series.
- Algebraic substitutions can be used to generate new series.
- The interval of convergence must always be adjusted accordingly.