Advanced Engineering Mathematics

Vectors

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1 Vectors in $\mathbb{R}^n$

1.1 Definition and Representation of Vectors

Key Concept

A **vector** is a quantity that has both **magnitude** and **direction**. It is commonly represented as an ordered list of real numbers:

$$\mathbf{v} = [v_1, v_2, v_3, \dots, v_n]$$

where $\mathbf{v} \in \mathbb{R}^n$.

A **scalar** is a quantity described only by magnitude (a single number), such as temperature, mass, or time.

Vectors can be interpreted geometrically as directed line segments (arrows), and algebraically as ordered tuples.

1.1.1 Position Vector

Key Concept

The **position vector** of a point $P(x_1, x_2, \dots, x_n)$ is the vector drawn from the origin to the point:

$$\mathbf{r} = [x_1, x_2, \dots, x_n]$$

1.2 Basic Vector Operations

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and let c be a scalar.

1.2.1 Vector Addition

Key Concept

If

$$\mathbf{u} = [u_1, u_2, \dots, u_n], \qquad \mathbf{v} = [v_1, v_2, \dots, v_n],$$

then

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \dots, u_n + v_n].$$

1.2.2 Vector Subtraction

$$\mathbf{u} - \mathbf{v} = [u_1 - v_1, u_2 - v_2, \dots, u_n - v_n].$$

1.2.3 Scalar Multiplication

Key Concept

If c is a scalar and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, then

$$c\mathbf{v} = [cv_1, cv_2, \ldots, cv_n].$$

1.2.4 Basic Properties of Vector Operations

For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalar c, the following hold:

- Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- Distributive Law: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- Distributive Law: $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$

1.3 Norm, Magnitude, and Unit Vectors

1.3.1 Magnitude (Norm)

Key Concept

The **magnitude** (or **Euclidean norm**) of a vector

$$\mathbf{v} = [v_1, v_2, \ldots, v_n]$$

is defined as

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

This measures the length of the vector in $\mathbb{R}^n$.

1.3.2 Unit Vector

Key Concept

A **unit vector** is a vector of magnitude 1. If $\mathbf{v} \neq \mathbf{0}$, then the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Example

Let $\mathbf{v} = [3, 4]$.

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Thus the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{5}[3, 4] = \left[\frac{3}{5}, \frac{4}{5}\right].$$

1.4 Standard Basis and Component Form

1.4.1 Standard Basis Vectors

Key Concept

The **standard basis vectors** in $\mathbb{R}^n$ are the vectors:

$$\mathbf{e}_1 = [1, 0, 0, \ldots, 0], \quad \mathbf{e}_2 = [0, 1, 0, \ldots, 0], \quad \ldots, \quad \mathbf{e}_n = [0, 0, 0, \ldots, 1].$$

Any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as a linear combination of these basis vectors:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

1.4.2 Component Form in $\mathbb{R}^3$

In $\mathbb{R}^3$, the standard basis is often written as:

$$\mathbf{i} = [1, 0, 0], \quad \mathbf{j} = [0, 1, 0], \quad \mathbf{k} = [0, 0, 1].$$

Thus any vector can be written as:

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

1.5 Vector Applications

Vectors appear naturally in engineering and physics because many real-world quantities require both magnitude and direction.

1.5.1 Displacement

A displacement from point $P(x_1, y_1, z_1)$ to point $Q(x_2, y_2, z_2)$ is represented as:

$$\mathbf{d} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1].$$

1.5.2 Velocity

Velocity is a vector that describes both speed and direction of motion.

1.5.3 Force and Resultant Force

Engineering Note

In statics and dynamics, forces are vectors. The **resultant force** is obtained by adding all forces acting on a body:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n.$$

If $\mathbf{R} = \mathbf{0}$, then the system is in equilibrium.

Example

Suppose two forces act on an object:

$$\mathbf{F}_1 = [3, 2], \qquad \mathbf{F}_2 = [1, -4].$$

Then the resultant force is:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = [3 + 1,\ 2 - 4] = [4, -2].$$

The magnitude of the resultant is:

$$\|\mathbf{R}\| = \sqrt{4^2 + (-2)^2} = \sqrt{20}.$$