Advanced Engineering Mathematics

Series

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Module 1: Infinite Series

This lecture introduces the fundamental theory of infinite series as used in Advanced Engineering Mathematics. Emphasis is placed on convergence concepts and power series representations, which are essential in engineering analysis, modeling, and later topics such as differential equations and signal processing.

Series

A **series** is the sum of the terms of a sequence. If $\{a_n\}$ is a sequence, then the corresponding series is written as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The value of a series is determined by the behavior of its **partial sums**

$$S_N = \sum_{n=1}^{N} a_n.$$

- The series **converges** if $\lim_{N\to\infty} S_N$ exists and is finite.
- The series **diverges** if this limit does not exist or is infinite.

Necessary Condition for Convergence:

$$\lim_{n\to\infty} a_n = 0.$$

If this condition is not satisfied, the series must diverge.

1.1 Power Series and Their Convergence

Definition of a Power Series

A **power series** is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where:

- c_n are constant coefficients,
- x_0 is a fixed real number called the **center** of the series,
- $(x - x_0)^n$ represents powers of the variable x.

Unlike ordinary numerical series, a power series depends on the value of x. As a result, a power series may converge for some values of x and diverge for others.

General Behavior of Power Series

For a given power series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

there exists a real number $R \geq 0$, called the **radius of convergence**, such that:

$$\begin{cases} \text{The series converges absolutely if } |x - x_0| < R, \\ \text{The series diverges if } |x - x_0| > R, \\ \text{The series may converge or diverge if } |x - x_0| = R. \end{cases}$$

The radius of convergence can be determined from the coefficients of the series through:

$$\boxed{(a) \quad R = \frac{1}{\lim\limits_{n \to \infty} \sqrt[n]{|c_n|}}} \quad \boxed{(b) \quad R = \frac{1}{\lim\limits_{n \to \infty} \left| \dfrac{c_{n+1}}{c_n} \right|}}$$

provided the limit exists.

The interval

$$(x_0 - R,\ x_0 + R)$$

together with any endpoints where the series converges is called the **interval of convergence**.

Convergence, Divergence, and Absolute Convergence

Let

$$\sum_{n=0}^{\infty} c_n(x - x_0)^n$$

be a power series.

- The series is said to **converge** at a value $x = b$ if the numerical series

  $$\sum_{n=0}^{\infty} c_n(b - x_0)^n$$

  converges.

- The series **diverges** at $x = b$ if the corresponding numerical series diverges.

- The series is **absolutely convergent** at $x = b$ if

  $$\sum_{n=0}^{\infty} |a_n(b - x_0)^n|$$

  converges.

Important Result: If a power series converges at a point $x = b$, then it converges absolutely for all values of x such that $|x - x_0| < |b - x_0|$.

Focus on Convergence of Power Series

The convergence of a power series depends primarily on the distance of x from the center x_0. The farther x is from x_0, the more likely the series is to diverge.

To determine where a power series converges, the following steps are followed:

1. Apply a convergence test to find the radius of convergence R.
2. Determine the interval $|x - x_0| < R$.
3. Test the endpoints $x = x_0 \pm R$ separately.

Convergence Tests for Power Series

1. Ratio Test Let the power series be

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

and define

$$a_n = c_n (x - x_0)^n.$$

Compute

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- If the limit is less than 1, the series converges absolutely.
- If the limit is greater than 1, the series diverges.
- If the limit equals 1, the test is inconclusive.

Examples: Radius and Interval of Convergence Using the Ratio Test

Example 1: Power Series of e^x Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Let

$$a_n = \frac{x^n}{n!}.$$

Applying the Ratio Test,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \frac{|x|}{n+1} = 0.$$

Since the limit is zero for all real values of x, the series converges for every x.

$$\boxed{R = \infty}$$

Thus, the interval of convergence is

$$\boxed{(-\infty, \infty)}.$$

Example 2: Geometric Series $\dfrac{1}{1-x}$ Consider the power series

$$\sum_{n=0}^{\infty} x^n.$$

Let

$$a_n = x^n.$$

Applying the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| = |x|.$$

For convergence,

$$|x| < 1.$$

Thus, the radius of convergence is

$$\boxed{R = 1}.$$

Endpoint Testing:

- At $x = -1$:

$$\sum_{n=0}^{\infty} (-1)^n \quad \text{diverges.}$$

- At $x = 1$:

$$\sum_{n=0}^{\infty} 1 \quad \text{diverges.}$$

Hence, the interval of convergence is

$$\boxed{(-1,\, 1)}.$$

Example 3: Series Involving Factorials Consider the power series

$$\sum_{n=0}^{\infty} n!\, x^n.$$

Let

$$a_n = n!\, x^n.$$

Applying the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x|.$$

For convergence,

$$(n+1)|x| < 1.$$

As $n \to \infty$, this inequality holds only when $x = 0$.

$$\boxed{R = 0}.$$

Thus, the series converges only at $x = 0$, and the interval of convergence is

$$\boxed{\{0\}}.$$

Summary of Results:

Series	**Radius of Convergence**	**Interval of Convergence**
$\sum \frac{x^n}{n!}$	$R = \infty$	$(-\infty, \infty)$
$\sum x^n$	$R = 1$	$(-1, 1)$
$\sum n!x^n$	$R = 0$	$\{0\}$

2. Root Test The **Root Test** may also be applied:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

The conclusions are the same as those of the Ratio Test.

3. Endpoint Testing Once the radius of convergence R is found, the series must be tested at

$$x = a - R \quad \text{and} \quad x = a + R.$$

At these endpoints, the power series reduces to a numerical series. Common tests used include:

- p-series test
- Alternating series test
- Comparison test

1.2 Divergence of Series

A series $\sum a_n$ is said to **diverge** if the sequence of partial sums does not approach a finite limit.

Key Divergence Tests

- nth-Term Test for Divergence
- Harmonic Series

Example 1: nth-Term Test

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

Since

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0,$$

the series diverges.

Example 2: Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

This series diverges even though $a_n \to 0$.

Example 3: Comparison

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This is a p-series with $p = \frac{1}{2} < 1$, hence it diverges.

1.3 Absolute Convergence

A series $\sum a_n$ is **absolutely convergent** if

$$\sum |a_n| \text{ converges.}$$

Absolute convergence guarantees convergence of the original series.

Example 1

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

The absolute series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges; hence the given series is absolutely convergent.

Example 2

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

The absolute series diverges, but the original series converges conditionally.

Example 3

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$$

The absolute series behaves like $\sum \frac{1}{n}$ and diverges. The original series converges conditionally.

1.4 Taylor Series

If a function $f(x)$ has derivatives of all orders at $x = a$, then its Taylor series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Example 1: Taylor Series of e^x about $a = 1$

$$e^x = e \left[1 + (x-1) + \frac{(x-1)^2}{2!} + \cdots \right]$$

Example 2: Taylor Series of $\ln x$ about $a = 1$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

Example 3: Polynomial Approximation Approximate $\sqrt{x}$ near $x = 4$:

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \cdots$$

1.5 Maclaurin Series

A Maclaurin series is a Taylor series centered at $a = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Example 1: e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Example 2: $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Example 3: $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Engineering Insight: Power series allow engineers to approximate complex functions, solve differential equations, and analyze system behavior near operating points.