Advanced Engineering Mathematics

SERIES SOLUTIONS

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1 Solutions About Ordinary Points

In this section, we introduce the **power series method** for solving linear differential equations near an *ordinary point*. This method is foundational in engineering mathematics and serves as a bridge between differential equations and special functions encountered in applied sciences.

1.1 Definition

Key Concept

An **ordinary point** of a differential equation is a point where all coefficient functions are analytic (i.e., can be expressed as power series).

1.1.1 General Form of the Differential Equation

We consider second-order linear differential equations of the form

$$p(x)y'' + q(x)y' + r(x)y = 0, \tag{1}$$

where $p(x)$, $q(x)$, and $r(x)$ are functions of x. At an ordinary point, the differential equation may be divided by $p(x)$, yielding

$$y'' + \frac{q(x)}{p(x)}y' + \frac{r(x)}{p(x)}y = 0. \tag{2}$$

Since $p(x_0) \neq 0$, the functions

$$\frac{q(x)}{p(x)}, \qquad \frac{r(x)}{p(x)}$$

or

$$y'' + P(x)y' + Q(x)y = 0, \tag{3}$$

where

$$P(x) = \frac{q(x)}{p(x)}, \qquad Q(x) = \frac{r(x)}{p(x)}.$$

Key Concept

A point $x = x_0$ is an **ordinary point** if both $P(x)$ and $Q(x)$ are analytic at x_0.

Engineering Note

In practice, most equations with polynomial coefficients have ordinary points everywhere except where division by zero occurs.

1.1.2 Test for Analyticity

A function $f(x)$ is said to be **analytic at a point** x_0 if it can be represented by a convergent power series in some open interval containing x_0. That is,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for all x sufficiently close to x_0.

Key Concept

A practical test for analyticity is that the function and all of its derivatives exist and are finite in a neighborhood of the point.

In the context of the differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

the point $x = x_0$ is an ordinary point if:

- $P(x)$ is analytic at x_0, and
- $Q(x)$ is analytic at x_0.

Engineering Note

If $P(x)$ and $Q(x)$ are rational functions, then analyticity fails only where their denominators vanish. These locations mark the boundary between ordinary and singular points.

This test allows us to classify points *before* attempting to construct a solution.

Example

Determine whether $x = 0$ is an ordinary point of the differential equation

$$x^2 y'' + (x + 1)y' - y = 0.$$

Step 1: Write the equation in standard form
Divide the equation by x^2:

$$y'' + \frac{x+1}{x^2} y' - \frac{1}{x^2} y = 0.$$

Thus,

$$P(x) = \frac{x+1}{x^2}, \qquad Q(x) = -\frac{1}{x^2}.$$

Step 2: Test analyticity at $x = 0$

Both $P(x)$ and $Q(x)$ contain the term $\frac{1}{x^2}$, which is undefined at $x = 0$. Therefore, neither function is analytic at $x = 0$.

Conclusion:

$$x = 0 \text{ is } \textit{not} \text{ an ordinary point.}$$

Engineering Note

The failure of analyticity is caused by division by zero. This point must be treated as a singular point.

1.2 Why Ordinary Points Matter

Classifying a point as ordinary is not merely a formal step—it determines whether standard solution techniques apply.

- At an ordinary point, all coefficient functions behave well locally.
- The differential equation can be reduced to a standard form.
- Solutions exist, are unique, and vary smoothly with initial conditions.

Key Concept

At an ordinary point, local solutions behave "nicely" and do not exhibit blow-up, discontinuities, or undefined behavior.

This favorable behavior is what ultimately permits solutions to be constructed systematically using series expansions, which will be developed in the next section.

Engineering Note

If a point fails the analyticity test, standard solution assumptions break down and more advanced techniques are required.

1.3 Why Power Series Solutions Work

1.3.1 Power Series Assumption

We know that:

- Analytic functions can be represented as power series.
- Power series can be differentiated and integrated term-by-term within their radius of convergence.

Hence, near an ordinary point x_0, we may *assume* a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{4}$$

This assumption is not a guess—it is justified by the theory of power series.

1.3.2 Derivatives of the Assumed Solution

Differentiating term-by-term,

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \tag{5}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}. \tag{6}$$

These expressions are substituted directly into the differential equation.

1.3.3 Method of Solution

The power series method about an ordinary point follows a systematic process:

1. Assume a power series solution centered at x_0.
2. Compute y' and y''.
3. Substitute into the differential equation.
4. Rewrite all terms using the same power of $(x - x_0)$.
5. Equate coefficients of like powers.
6. Obtain a **recurrence relation** for a_n.
7. Use initial conditions (if given) to find constants.

Example

Solve the differential equation

$$y'' - y = 0$$

about the ordinary point $x_0 = 0$ using a power series.

Step 1: Assume a power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Step 2: Compute derivatives

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Step 3: Substitute into the equation

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Step 4: Align powers of x

Re-index the first sum by letting $n \to n+2$:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Step 5: Equate coefficients

$$(n+2)(n+1) a_{n+2} - a_n = 0.$$

Recurrence Relation:

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}.$$

Structure of the Solution

The recurrence relation shows that:

- Even-indexed coefficients depend only on a_0.
- Odd-indexed coefficients depend only on a_1.

Thus, the general solution is a linear combination of two power series:

$$y(x) = a_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + a_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right).$$

Engineering Note

These series are recognized as the Maclaurin series for $\cosh x$ and $\sinh x$, respectively.

Another Example:

Example

Solve the differential equation

$$(1 + x)y'' - xy' - y = 0$$

about $x_0 = 0$.
Since

$$p(x) = 1 + x \quad \text{and} \quad p(0) = 1 \neq 0,$$

the point $x_0 = 0$ is an **ordinary point**. Hence, a power series solution about $x = 0$ exists.

Step 1: Power Series Assumption
Assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \tag{7}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \tag{8}$$

Step 2: Substitution into the Differential Equation
Substituting into

$$(1 + x)y'' - xy' - y = 0,$$

we obtain

$$(1+x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Distributing terms,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Step 3: Reindexing the Series

Rewrite each sum in terms of x^n:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Step 4: Coefficient Comparison

For $n = 0$:

$$2a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{a_0}{2}.$$

For $n \geq 1$:

$$(n+2)(n+1)a_{n+2} + \big[n(n-1) - n - 1\big]a_n = 0.$$

Simplifying,

$$n(n-1) - n - 1 = n^2 - 2n - 1.$$

Thus, the recurrence relation is

$$\boxed{a_{n+2} = \frac{2n+1-n^2}{(n+2)(n+1)}\, a_n, \qquad n \geq 0.}$$

Step 5: Structure of the Solution

- Even-indexed coefficients depend only on a_0.
- Odd-indexed coefficients depend only on a_1.

Hence, the general solution about $x = 0$ is

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where y_1 and y_2 are linearly independent power series solutions.

Engineering Note

Variable-coefficient equations generally lead to non-terminating recurrence relations. Even when closed-form solutions do not exist, power series solutions remain valid within their radius of convergence.

1.3.4 Radius of Convergence

We know that:

- Power series solutions converge within a radius determined by the nearest singularity.
- The solution is guaranteed to be valid at least up to the closest point where $P(x)$ or $Q(x)$ becomes non-analytic.

Key Concept

For an ordinary point, the power series solution always exists and converges in some neighborhood of the point.