Advanced Engineering Mathematics

SERIES SOLUTIONS

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1 Solutions About Ordinary Points

In this section, we introduce the **power series method** for solving linear differential equations near an *ordinary point*. This method is foundational in engineering mathematics and serves as a bridge between differential equations and special functions encountered in applied sciences.

1.1 Definition

Key Concept

An **ordinary point** of a differential equation is a point where all coefficient functions are analytic (i.e., can be expressed as power series).

1.1.1 General Form of the Differential Equation

We consider second-order linear differential equations of the form

$$p(x)y'' + q(x)y' + r(x)y = 0, \tag{1}$$

where $p(x)$, $q(x)$, and $r(x)$ are functions of x. At an ordinary point, the differential equation may be divided by $p(x)$, yielding

$$y'' + \frac{q(x)}{p(x)}y' + \frac{r(x)}{p(x)}y = 0. \tag{2}$$

Since $p(x_0) \neq 0$, the functions

$$\frac{q(x)}{p(x)}, \qquad \frac{r(x)}{p(x)}$$

or

$$y'' + P(x)y' + Q(x)y = 0, \tag{3}$$

where

$$P(x) = \frac{q(x)}{p(x)}, \qquad Q(x) = \frac{r(x)}{p(x)}.$$

Key Concept

A point $x = x_0$ is an **ordinary point** if both $P(x)$ and $Q(x)$ are analytic at x_0.

Engineering Note

In practice, most equations with polynomial coefficients have ordinary points everywhere except where division by zero occurs.

1.1.2 Test for Analyticity

A function $f(x)$ is said to be **analytic at a point** x_0 if it can be represented by a convergent power series in some open interval containing x_0. That is,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for all x sufficiently close to x_0.

Key Concept

A practical test for analyticity is that the function and all of its derivatives exist and are finite in a neighborhood of the point.

In the context of the differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

the point $x = x_0$ is an ordinary point if:

- $P(x)$ is analytic at x_0, and
- $Q(x)$ is analytic at x_0.

Engineering Note

If $P(x)$ and $Q(x)$ are rational functions, then analyticity fails only where their denominators vanish. These locations mark the boundary between ordinary and singular points.

This test allows us to classify points *before* attempting to construct a solution.

Example

Determine whether $x = 0$ is an ordinary point of the differential equation

$$x^2y'' + (x + 1)y' - y = 0.$$

Step 1: Write the equation in standard form
Divide the equation by x^2:

$$y'' + \frac{x + 1}{x^2}y' - \frac{1}{x^2}y = 0.$$

Thus,

$$P(x) = \frac{x+1}{x^2}, \qquad Q(x) = -\frac{1}{x^2}.$$

Step 2: Test analyticity at $x = 0$

Both $P(x)$ and $Q(x)$ contain the term $\frac{1}{x^2}$, which is undefined at $x = 0$. Therefore, neither function is analytic at $x = 0$.

Conclusion:

$$x = 0 \text{ is } \textit{not} \text{ an ordinary point.}$$

Engineering Note

The failure of analyticity is caused by division by zero. This point must be treated as a singular point.

1.2 Why Ordinary Points Matter

Classifying a point as ordinary is not merely a formal step—it determines whether standard solution techniques apply.

- At an ordinary point, all coefficient functions behave well locally.
- The differential equation can be reduced to a standard form.
- Solutions exist, are unique, and vary smoothly with initial conditions.

Key Concept

At an ordinary point, local solutions behave "nicely" and do not exhibit blow-up, discontinuities, or undefined behavior.

This favorable behavior is what ultimately permits solutions to be constructed systematically using series expansions, which will be developed in the next section.

Engineering Note

If a point fails the analyticity test, standard solution assumptions break down and more advanced techniques are required.

1.3 Why Power Series Solutions Work

1.3.1 Power Series Assumption

We know that:

- Analytic functions can be represented as power series.
- Power series can be differentiated and integrated term-by-term within their radius of convergence.

Hence, near an ordinary point x_0, we may *assume* a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{4}$$

This assumption is not a guess—it is justified by the theory of power series.

1.3.2 Derivatives of the Assumed Solution

Differentiating term-by-term,

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \tag{5}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}. \tag{6}$$

These expressions are substituted directly into the differential equation.

1.3.3 Method of Solution

The power series method about an ordinary point follows a systematic process:

1. Assume a power series solution centered at x_0.
2. Compute y' and y''.
3. Substitute into the differential equation.
4. Rewrite all terms using the same power of $(x - x_0)$.
5. Equate coefficients of like powers.
6. Obtain a **recurrence relation** for a_n.
7. Use initial conditions (if given) to find constants.

Example

Solve the differential equation

$$y'' - y = 0$$

about the ordinary point $x_0 = 0$ using a power series.

Step 1: Assume a power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Step 2: Compute derivatives

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Step 3: Substitute into the equation

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Step 4: Align powers of x

Re-index the first sum by letting $n \to n+2$:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Step 5: Equate coefficients

$$(n+2)(n+1) a_{n+2} - a_n = 0.$$

Recurrence Relation:

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}.$$

Structure of the Solution

The recurrence relation shows that:

- Even-indexed coefficients depend only on a_0.
- Odd-indexed coefficients depend only on a_1.

Thus, the general solution is a linear combination of two power series:

$$y(x) = a_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + a_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right).$$

Engineering Note

These series are recognized as the Maclaurin series for $\cosh x$ and $\sinh x$, respectively.

Another Example:

Example

Solve the differential equation

$$(1+x)y'' - xy' - y = 0$$

about $x_0 = 0$.
Since

$$p(x) = 1 + x \quad \text{and} \quad p(0) = 1 \neq 0,$$

the point $x_0 = 0$ is an **ordinary point**. Hence, a power series solution about $x = 0$ exists.

Step 1: Power Series Assumption
Assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \tag{7}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \tag{8}$$

Step 2: Substitution into the Differential Equation
Substituting into

$$(1+x)y'' - xy' - y = 0,$$

we obtain

$$(1+x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Distributing terms,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Step 3: Reindexing the Series
From Step 2 we had

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

We rewrite *each* sum in powers of x^n.

- For the first sum, let $n \to n+2$:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

- For the second sum, let $m = n-1$ (so $m \geq 1$), then rename $m \to n$:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1)n\, a_{n+1} x^n.$$

- The remaining sums already involve x^n:

$$-\sum_{n=1}^{\infty} n a_n x^n, \qquad -\sum_{n=0}^{\infty} a_n x^n.$$

Thus the equation becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n(n+1)a_{n+1} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Step 4: Coefficient Comparison
For $n = 0$:

$$2a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{a_0}{2}.$$

For $n \geq 1$:

$$(n+2)(n+1)a_{n+2} + \big[n(n-1) - n - 1\big]a_n = 0.$$

Simplifying,

$$n(n-1) - n - 1 = n^2 - 2n - 1.$$

Thus, the recurrence relation is

$$\boxed{a_{n+2} = \frac{2n+1-n^2}{(n+2)(n+1)}\, a_n, \qquad n \geq 0.}$$

Step 5: Meaning of a_0 and a_1

From the assumed series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

we have

$$\boxed{a_0 = y(0)}.$$

Also,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \Rightarrow \quad \boxed{a_1 = y'(0)}.$$

So a_0 and a_1 are the two arbitrary constants determined by initial conditions.

Step 6: Write the two linearly independent series solutions

Set $\{a_0 = 1,\ a_1 = 0\}$ to define $y_1(x)$:

$$y_1(x) = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{5}{24}x^4 - \frac{19}{120}x^5 + \frac{101}{720}x^6 + \cdots$$

Set $\{a_0 = 0,\ a_1 = 1\}$ to define $y_2(x)$:

$$y_2(x) = x + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{6}x^5 - \frac{5}{36}x^6 + \frac{31}{252}x^7 + \cdots$$

Step 5: Structure of the Solution

- Even-indexed coefficients depend only on a_0.
- Odd-indexed coefficients depend only on a_1.

Hence, the general solution about $x = 0$ is

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where y_1 and y_2 are linearly independent power series solutions.

Engineering Note

Variable-coefficient equations generally lead to non-terminating recurrence relations. Even when closed-form solutions do not exist, power series solutions remain valid within their radius of convergence.

Example

Find the first four terms in each portion of the series solution about $x_0 = -2$ for

$$y'' - xy = 0.$$

Step 1: Shift the expansion point. Let

$$t = x + 2 \quad \Longrightarrow \quad x = t - 2,$$

and write y as a power series in t:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Since $\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}t}$, we have

$$y''(t) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n.$$

Substitute into $y'' - xy = 0$:

$$y'' - (t-2)y = 0 \quad \Longrightarrow \quad y'' + 2y - ty = 0.$$

Step 2: Substitute the series and align powers.

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n + 2\sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1} = 0.$$

Rewrite the last sum as an t^n-series:

$$\sum_{n=0}^{\infty} a_n t^{n+1} = \sum_{n=1}^{\infty} a_{n-1} t^n.$$

So we get

$$\sum_{n=0}^{\infty}\Big((n+2)(n+1)a_{n+2}+2a_n\Big)t^n-\sum_{n=1}^{\infty}a_{n-1}t^n=0.$$

Step 3: Coefficient equations (recurrence).
For $n=0$:

$$2a_2+2a_0=0 \quad\Longrightarrow\quad a_2=-a_0.$$

For $n\geq 1$:

$$(n+2)(n+1)a_{n+2}+2a_n-a_{n-1}=0 \quad\Longrightarrow\quad \boxed{a_{n+2}=\frac{a_{n-1}-2a_n}{(n+2)(n+1)},\quad n\geq 1.}$$

Step 4: Build the two portions (two linearly independent series).

Portion 1 (take $a_0=1,\ a_1=0$). Using the recurrence:

$$a_2=-1,\quad a_3=\frac{1}{6},\quad a_4=\frac{1}{6},\ \dots$$

So the first four terms are

$$\boxed{y_1(t)=1-t^2+\frac{1}{6}t^3+\frac{1}{6}t^4+\cdots}$$

Portion 2 (take $a_0=0,\ a_1=1$). Using the recurrence:

$$a_2=0,\quad a_3=-\frac{1}{3},\quad a_4=\frac{1}{12},\quad a_5=\frac{1}{30},\ \dots$$

So the first four terms are

$$\boxed{y_2(t)=t-\frac{1}{3}t^3+\frac{1}{12}t^4+\frac{1}{30}t^5+\cdots}$$

Final series solution about $x_0=-2$. Since $t=x+2$,

$$\boxed{\begin{aligned}y(x)=a_0&\Big\{1-(x+2)^2+\frac{1}{6}(x+2)^3+\frac{1}{6}(x+2)^4+\cdots\Big\}\\&+a_1\Big\{(x+2)-\frac{1}{3}(x+2)^3+\frac{1}{12}(x+2)^4+\frac{1}{30}(x+2)^5+\cdots\Big\}\end{aligned}}$$

1.3.4 Radius of Convergence

We know that:

- Power series solutions converge within a radius determined by the nearest singularity.
- The solution is guaranteed to be valid at least up to the closest point where $P(x)$ or $Q(x)$ becomes non-analytic.

Key Concept

For an ordinary point, the power series solution always exists and converges in some neighborhood of the point.

2 Solutions About Singular Points

In the previous section, we developed power series solutions under the assumption that the expansion point is an ordinary point. However, many important differential equations arising in engineering and physics fail to satisfy this assumption.

Key Concept

A **singular point** of a differential equation is a point where the leading coefficient $p(x)$ vanishes or where the coefficient functions become non-analytic.

At such points, the standard power series method generally fails, and modified techniques are required.

2.1 Classification of Singular Points

Consider the second-order linear differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

A point $x = x_0$ is a **singular point** if $p(x_0) = 0$.

Not all singular points behave in the same way. We distinguish between two important types.

2.1.1 Regular Singular Points

Key Concept

A point $x = x_0$ is called a **regular singular point** if

$$(x - x_0)\frac{q(x)}{p(x)} \quad \text{and} \quad (x - x_0)^2\frac{r(x)}{p(x)}$$

are analytic at x_0.

At a regular singular point, the failure of analyticity is mild and controlled. Although standard power series solutions fail, solutions of a modified form may still exist.

Example

Determine the nature of the point $x = 0$ for the differential equation

$$x^2y'' + xy' - y = 0.$$

Step 1: Identify the coefficient functions
Here,

$$p(x) = x^2, \quad q(x) = x, \quad r(x) = -1.$$

Since $p(0) = 0$, the point $x = 0$ is a singular point.
Step 2: Test for regular singularity
Compute

$$(x - 0)\frac{q(x)}{p(x)} = x \cdot \frac{x}{x^2} = 1,$$

$$(x - 0)^2\frac{r(x)}{p(x)} = x^2 \cdot \frac{-1}{x^2} = -1.$$

Both expressions are constants and therefore analytic at $x = 0$.
Conclusion:

$$\boxed{x = 0 \text{ is a regular singular point.}}$$

2.1.2 Irregular Singular Points

Key Concept

A point $x = x_0$ is an **irregular singular point** if it is singular but not regular singular.

At irregular singular points, the behavior of solutions can be highly unstable, and no general power series method is guaranteed to work.

Engineering Note

Most physically meaningful differential equations encountered in engineering have at most regular singular points.

Example

Determine the nature of the point $x = 0$ for the differential equation

$$x^3 y'' + y' + xy = 0.$$

Step 1: Identify the coefficient functions

$$p(x) = x^3, \quad q(x) = 1, \quad r(x) = x.$$

Since $p(0) = 0$, the point $x = 0$ is a singular point.
Step 2: Test for regular singularity
Compute

$$(x-0)\frac{q(x)}{p(x)} = x \cdot \frac{1}{x^3} = \frac{1}{x^2},$$

$$(x-0)^2\frac{r(x)}{p(x)} = x^2 \cdot \frac{x}{x^3} = 1.$$

The expression $\frac{1}{x^2}$ is not analytic at $x = 0$.
Conclusion:

$$\boxed{x = 0 \text{ is an irregular singular point.}}$$

2.2 Why Ordinary Power Series Fail

At a singular point, dividing the equation by $p(x)$ introduces non-analytic terms into the coefficients. As a result:

- Standard power series substitutions lead to undefined expressions, or
- Recurrence relations fail to determine coefficients uniquely.

Key Concept

The breakdown of the ordinary power series method is caused by the behavior of the coefficients, not by the solution itself.

This observation motivates the search for a more flexible form of series solution.

Example

Proof via Example: Ordinary power series may fail at a singular point.
Consider the differential equation

$$x^2y'' + xy' - y = 0$$

about the point $x_0 = 0$.
Step 1: Show that $x = 0$ is singular.
Here $p(x) = x^2$, so $p(0) = 0$. Hence $x = 0$ is a singular point.

Step 2: Attempt an ordinary power series solution.
Assume (ordinary power series)

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute into the ODE:

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Simplify each term:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Now combine into one series (noting the $n = 0$ term explicitly):

$$(-a_0) + \sum_{n=1}^{\infty} \big(n(n-1) + n - 1\big) a_n x^n = 0.$$

Since the power series is identically zero, every coefficient must be zero.

Step 3: Coefficient comparison.
For x^0:

$$-a_0 = 0 \quad \Rightarrow \quad a_0 = 0.$$

For $n \geq 1$:

$$\big(n(n-1) + n - 1\big) a_n = 0 \quad \Rightarrow \quad (n^2 - 1) a_n = 0.$$

Thus:

$$(n^2 - 1)a_n = 0 \Rightarrow \begin{cases} a_n = 0, & n \neq 1, \\ a_1 \text{ is free}, & n = 1. \end{cases}$$

So the ordinary power series solution becomes

$$y = a_1 x.$$

Step 4: Why this shows the ordinary power series method fails.
A second-order linear differential equation should admit two linearly independent solutions. However, the ordinary power series assumption around $x = 0$ produced only *one* solution,

$$y = a_1 x,$$

and forced all other coefficients to be zero.

The loss of the second independent solution is not because the differential equation has only one solution, but because the assumed form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

is too restrictive at a singular point.

Conclusion: At singular points, dividing by $p(x)$ introduces non-analytic coefficients, and ordinary power series may fail to capture the full solution space. This motivates the need for a more flexible series form (Frobenius method), which allows non-integer exponents.

2.3 Frobenius Method

At regular singular points, ordinary power series solutions are often insufficient to capture the full behavior of solutions. To address this limitation, a generalized series expansion is introduced.

2.3.1 Motivation for the Frobenius Method

To accommodate singular behavior at the expansion point, we extend the standard power series assumption.

Key Concept

Instead of assuming a solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

we allow solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r},$$

where r is a constant to be determined.

This generalized assumption forms the basis of the **Frobenius Method**.

Engineering Note

The exponent r allows the solution to absorb singular behavior that ordinary power series cannot capture.

2.3.2 Assumed Form of the Frobenius Solution

Let $x = x_0$ be a regular singular point of the second-order linear differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

At such a point, dividing the equation by $p(x)$ introduces coefficients that may contain singular terms. As a result, ordinary power series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

are often too restrictive.

To overcome this limitation, we assume a more general series form.

Key Concept

A **Frobenius solution** about $x = x_0$ is assumed in the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}, \qquad a_0 \neq 0,$$

where r is a constant to be determined.

The exponent r allows the solution to include non-integer or negative powers of $(x - x_0)$, which are essential for capturing the behavior of solutions near singular points.

- If $r = 0$, the Frobenius series reduces to an ordinary power series.
- If $r > 0$, the solution vanishes at the singular point.
- If $r < 0$, the solution becomes unbounded as $x \to x_0$.

The condition $a_0 \neq 0$ ensures that $(x - x_0)^r$ is the dominant term near the singular point and prevents redundancy in the choice of r.

Engineering Note

Choosing $a_0 = 0$ would simply shift the index of the series and fail to capture the leading behavior of the solution.

Example 1: Why the Exponent r Is Necessary

Example

Consider the differential equation

$$x^2 y'' + y = 0$$

about the point $x_0 = 0$.
Here, $p(x) = x^2$, so $x = 0$ is a regular singular point.
If we attempt an ordinary power series solution,

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

substitution leads to recurrence relations that force all coefficients to vanish except the trivial solution.
However, if we instead assume a Frobenius form,

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

the lowest-power term captures the singular behavior introduced by the x^2 coefficient, allowing nontrivial solutions to emerge.

Engineering Note

The failure of the ordinary series is not due to the equation lacking solutions, but due to the inability of integer powers alone to represent them.

Example 2: Interpretation of the Leading Term

Example

Assume a Frobenius solution of the form

$$y(x) = a_0(x - x_0)^r + a_1(x - x_0)^{r+1} + \cdots .$$

Near $x = x_0$, the dominant behavior of the solution is governed by

$$y(x) \approx a_0(x - x_0)^r.$$

Thus:

- If $r > 0$, the solution approaches zero at the singular point.
- If $r = 0$, the solution approaches a finite nonzero constant.
- If $r < 0$, the solution diverges as $x \to x_0$.

Key Concept

The exponent r determines the qualitative behavior of the solution near the singular point.

2.3.3 Derivatives of the Frobenius Series

Differentiating term-by-term,

$$y'(x) = \sum_{n=0}^{\infty}(n + r)a_n(x - x_0)^{n+r-1}, \tag{9}$$

$$y''(x) = \sum_{n=0}^{\infty}(n + r)(n + r - 1)a_n(x - x_0)^{n+r-2}. \tag{10}$$

These expressions are substituted into the differential equation.

Engineering Note

The presence of r shifts the lowest power of $(x - x_0)$ and allows cancellation of singular terms introduced by dividing by $p(x)$.

Example

Consider the differential equation

$$x^2y'' + xy' - y = 0$$

about the point $x_0 = 0$.

Step 1: Assume a Frobenius series solution

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad a_0 \neq 0.$$

Then the derivatives are

$$y'(x) = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1},$$
$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}.$$

Step 2: Substitute into the differential equation

Substitute into

$$x^2y'' + xy' - y = 0:$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Simplify each term so all powers are x^{n+r}:

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Combine the sums:

$$\sum_{n=0}^{\infty} \Big[(n+r)(n+r-1) + (n+r) - 1\Big] a_n \, x^{n+r} = 0.$$

Inside the bracket,

$$(n+r)(n+r-1)+(n+r)-1=(n+r)\big((n+r-1)+1\big)-1=(n+r)^2-1.$$

Thus,

$$\sum_{n=0}^{\infty}\big((n+r)^2-1\big)a_n x^{n+r}=0.$$

Step 3: Indicial equation (lowest power term)
The lowest power occurs at $n=0$, giving the coefficient of x^r:

$$(r^2-1)a_0=0.$$

Since $a_0\neq 0$, we must have

$$r^2-1=0 \quad\Longrightarrow\quad \boxed{r=1 \text{ or } r=-1.}$$

Step 4: Determine coefficients (recurrence/constraints)
From

$$\big((n+r)^2-1\big)a_n=0 \qquad \text{for all } n\geq 0,$$

we see that for each chosen r, the coefficient a_n can be nonzero only when

$$(n+r)^2-1=0.$$

Case 1: $r=1$
Then

$$(n+1)^2-1=0 \quad\Longrightarrow\quad n^2+2n=0 \quad\Longrightarrow\quad n(n+2)=0.$$

For $n\geq 0$, the only solution is $n=0$. Hence, a_0 may be nonzero and all $a_n=0$ for $n\geq 1$.
Therefore,

$$\boxed{y_1(x)=a_0x^1=a_0x.}$$

Case 2: $r=-1$
Then

$$(n-1)^2-1=0 \quad\Longrightarrow\quad n^2-2n=0 \quad\Longrightarrow\quad n(n-2)=0.$$

For $n\geq 0$, this allows $n=0$ and $n=2$.
Thus, a_0 and a_2 may be nonzero, while all other $a_n=0$.

So,

$$y_2(x) = a_0 x^{-1} + a_2 x^{2-1} = a_0 x^{-1} + a_2 x.$$

But the x term is already represented by y_1, so an independent solution is

$$\boxed{y_2(x) = b_0 x^{-1}.}$$

Step 5: General solution

Two linearly independent solutions are

$$y_1(x) = x, \qquad y_2(x) = x^{-1}.$$

Hence the general solution is

$$\boxed{y(x) = C_1 x + C_2 \frac{1}{x}.}$$

Example

Suppose a Frobenius solution near $x_0 = 0$ has the form

$$y(x) = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \cdots.$$

Step 1: Identify the dominant term as $x \to 0$.

The Frobenius series is

$$y(x) = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \cdots.$$

As $x \to 0$, higher powers of x (such as $x^{r+1}, x^{r+2}, \dots$) become smaller in magnitude relative to x^r. Therefore, the leading (dominant) term is

$$\boxed{a_0 x^r}.$$

Step 2: The sign of r affects the behavior near $x = 0$.

Since the dominant behavior is controlled by x^r:

- If $r > 0$, then $x^r \to 0$ as $x \to 0$, so $y(x) \to 0$ (the solution vanishes at the singular point).
- If $r = 0$, then $x^r = 1$, so $y(x) \to a_0$ (the solution approaches a finite nonzero constant, assuming $a_0 \neq 0$).

- If $r < 0$, then $x^r \to \infty$ as $x \to 0$, so the solution becomes unbounded (blows up) near the singular point.

Step 3: The condition $a_0 \neq 0$ is necessary.
The Frobenius method chooses r so that the first nonzero term of the series is the $a_0 x^r$ term. If $a_0 = 0$, then the series effectively starts at a higher power:

$$y(x) = a_1 x^{r+1} + a_2 x^{r+2} + \cdots ,$$

which means the exponent r no longer represents the true leading behavior. In that case, we could re-index the series and replace r with a larger value.
Thus, requiring

$$\boxed{a_0 \neq 0}$$

ensures that x^r is genuinely the dominant term and that the exponent r is meaningful and uniquely determined by the differential equation.

2.3.4 The Indicial Equation

After substitution, all terms are rewritten so that they involve powers of $(x - x_0)^{n+r}$.

The coefficient of the *lowest power* of $(x - x_0)$ must vanish. This requirement leads to an algebraic equation in r, called the **indicial equation**.

Key Concept

The **indicial equation** determines the allowable values of r for which a Frobenius solution exists.

Solving the indicial equation yields one or more possible values of r, each corresponding to a potential solution.

Example

Determine the indicial equation for the differential equation

$$x^2 y'' + x y' - y = 0$$

about the point $x_0 = 0$.
Step 1: Verify that $x = 0$ is a regular singular point.

Here,

$$p(x) = x^2, \quad q(x) = x, \quad r(x) = -1,$$

and since

$$x\frac{q(x)}{p(x)} = 1, \qquad x^2\frac{r(x)}{p(x)} = -1,$$

both expressions are analytic at $x = 0$, the point is a regular singular point.

Step 2: Assume a Frobenius solution.
Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad a_0 \neq 0.$$

Then

$$y'(x) = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1},$$
$$y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}.$$

Step 3: Substitute into the differential equation.
Substituting into

$$x^2y'' + xy' - y = 0,$$

gives

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

Simplifying each term,

$$\sum_{n=0}^{\infty}\Big[(n+r)(n+r-1) + (n+r) - 1\Big]a_n x^{n+r} = 0.$$

Step 4: Identify the lowest power of x.
The lowest power of x occurs when $n = 0$, giving the term proportional to x^r:

$$\Big[r(r-1) + r - 1\Big]a_0 x^r.$$

Since $a_0 \neq 0$, the coefficient of x^r must vanish.

Step 5: Form the indicial equation.

$$r(r-1) + r - 1 = 0 \quad \Longrightarrow \quad \boxed{r^2 - 1 = 0.}$$

Indicial roots:

$$\boxed{r = 1, \quad r = -1.}$$

Engineering Note

Each root of the indicial equation corresponds to a possible Frobenius solution near the regular singular point.

2.3.5 Cases for the Roots of the Indicial Equation

Let r_1 and r_2 be the roots of the indicial equation, with $r_1 \geq r_2$.

- If $r_1 - r_2$ is not an integer, two linearly independent Frobenius solutions exist.
- If $r_1 = r_2$, only one Frobenius solution is guaranteed; a second solution may involve logarithmic terms.
- If $r_1 - r_2$ is a positive integer, the second solution may or may not exist in Frobenius form and often requires special treatment.

Key Concept

The nature of the roots of the indicial equation determines the structure of the solution space.

2.3.6 Recurrence Relation

Once a value of r is chosen, equating coefficients of like powers of $(x - x_0)^{n+r}$ produces a recurrence relation of the form

$$a_{n+k} = F(n, r)\, a_n,$$

which determines all coefficients in terms of a_0.

Example

Derive the recurrence relation for the differential equation

$$x^2 y'' + xy' - y = 0$$

about the regular singular point $x_0 = 0$.
From the indicial equation, the roots are

$$r = 1 \quad \text{and} \quad r = -1.$$

We now derive the recurrence relation for each case.

Step 1: General coefficient equation
From the Frobenius substitution and simplification, we obtained

$$\sum_{n=0}^{\infty} \big((n+r)^2 - 1\big) a_n x^{n+r} = 0.$$

Since the series is identically zero, the coefficient of each power of x^{n+r} must vanish:

$$\big((n+r)^2 - 1\big) a_n = 0, \qquad n \geq 0.$$

Step 2: Case $r = 1$
Substitute $r = 1$:

$$\big((n+1)^2 - 1\big) a_n = 0 \quad \Longrightarrow \quad (n^2 + 2n) a_n = 0.$$

For $n \geq 1$, this forces

$$a_n = 0.$$

Thus, only a_0 remains arbitrary, and the Frobenius solution reduces to

$$y_1(x) = a_0 x.$$

Step 3: Case $r = -1$
Substitute $r = -1$:

$$\big((n-1)^2 - 1\big) a_n = 0 \quad \Longrightarrow \quad (n^2 - 2n) a_n = 0.$$

This allows nonzero coefficients when

$$n = 0 \quad \text{or} \quad n = 2.$$

Hence, the Frobenius solution takes the form

$$y_2(x) = a_0 x^{-1} + a_2 x.$$

Since the x term is already represented by the $r = 1$ solution, the second independent solution is

$$y_2(x) = b_0 x^{-1}.$$

Engineering Note

In this example, the recurrence relation does not generate an infinite series. Instead, it restricts which coefficients may be nonzero. In many problems, the recurrence relation generates infinitely many coefficients, producing a full power series solution.

2.3.7 Interpretation and Scope

The Frobenius Method provides a systematic way to construct solutions near regular singular points.

Engineering Note

Frobenius solutions are local solutions. Their validity is restricted to a neighborhood of the singular point and depends on the nature of the coefficient functions.

2.3.8 Practice Problems

The following problems are intended to reinforce the application of the Frobenius Method at regular singular points. For each problem:

- Identify the type of singular point.
- Assume a Frobenius series solution.
- Derive the indicial equation.
- Obtain the recurrence relation.

Problem 1: Cauchy–Euler Type Equation

Solve the differential equation

$$x^2 y'' - 2xy' + 2y = 0$$

about the point $x_0 = 0$ using the Frobenius Method.

Example

Step 1: Frobenius form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Step 2: Substitute

$$\begin{aligned} x^2 y'' - 2xy' + 2y &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - 2 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + 2 \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{n=0}^{\infty} \Big[(n+r)(n+r-1) - 2(n+r) + 2 \Big] a_n x^{n+r}. \end{aligned}$$

Thus, for all $n \geq 0$,

$$\Big[(n+r)(n+r-1) - 2(n+r) + 2 \Big] a_n = 0.$$

Step 3: Indicial equation (lowest power $n = 0$)

$$r(r-1) - 2r + 2 = 0 \quad \Longrightarrow \quad r^2 - 3r + 2 = 0 \quad \Longrightarrow \quad \boxed{r = 1,\ 2.}$$

Step 4: Solutions This is an Cauchy–Euler equation, so the Frobenius solutions reduce to power functions:

$$\boxed{y(x) = C_1 x + C_2 x^2.}$$

Problem 2: Regular Singular Point with Infinite Series

Solve

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

about $x_0 = 0$.

Example

Step 1: Frobenius form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}.$$

Step 2: Substitute and simplify

$$\begin{aligned} x^2 y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r}, \\ x y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r}, \\ (x^2-1) y &= \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r}. \end{aligned}$$

So,

$$\sum_{n=0}^{\infty} \Big((n+r)^2 - 1\Big) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0.$$

Reindex the last sum ($n \to n-2$):

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+r}.$$

Hence,

$$\Big((r)^2 - 1\Big) a_0 x^r + \Big((r+1)^2 - 1\Big) a_1 x^{r+1} + \sum_{n=2}^{\infty} \Big(\big((n+r)^2 - 1\big) a_n + a_{n-2}\Big) x^{n+r} = 0.$$

Step 3: Indicial equation From the lowest power term x^r:

$$(r^2 - 1) a_0 = 0 \quad \Longrightarrow \quad \boxed{r = \pm 1.}$$

Also, the x^{r+1} term gives

$$\big((r+1)^2 - 1\big) a_1 = 0.$$

For both $r = 1$ and $r = -1$, this coefficient is nonzero, so

$$\boxed{a_1 = 0.}$$

Step 4: Recurrence relation ($n \geq 2$)

$$\big((n+r)^2 - 1\big)a_n + a_{n-2} = 0 \quad \Longrightarrow \quad \boxed{a_n = -\frac{a_{n-2}}{(n+r)^2 - 1}, \qquad n \geq 2.}$$

This generates two series (even indices only) for each choice of r.

Case 1: $r = 1$
For $n \geq 2$,

$$a_n = -\frac{a_{n-2}}{(n+1)^2 - 1} = -\frac{a_{n-2}}{n(n+2)}.$$

Starting with a_0:

$$a_2 = -\frac{a_0}{2 \cdot 4} = -\frac{a_0}{8}, \qquad a_4 = -\frac{a_2}{4 \cdot 6} = \frac{a_0}{192}, \ \dots$$

So one solution is

$$\boxed{y_1(x) = a_0 \left(x - \frac{x^3}{8} + \frac{x^5}{192} - \cdots \right).}$$

Case 2: $r = -1$
For $n \geq 2$,

$$a_n = -\frac{a_{n-2}}{(n-1)^2 - 1} = -\frac{a_{n-2}}{n(n-2)}.$$

Here, at $n = 2$ the denominator is $2 \cdot 0$, so the recurrence breaks. This signals that the second independent solution requires special treatment (often involving logarithmic terms).

Engineering Note

This is the integer-difference complication: the indicial roots differ by an integer $(1 - (-1) = 2)$. In such cases, only one Frobenius series is guaranteed; the second solution may involve a logarithm.

Thus, the Frobenius method guarantees the series solution for $r = 1$ above, and the second independent solution is not obtained directly from the same recurrence.

Problem 3: Integer Difference Between Indicial Roots

Solve
$$x^2y'' + 3xy' + y = 0$$
about $x_0 = 0$.

Example

Step 1: Frobenius form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}.$$

Step 2: Substitute

$$\begin{aligned} x^2y'' + 3xy' + y &= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + 3\sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{n=0}^{\infty} \Big[(n+r)(n+r-1) + 3(n+r) + 1\Big] a_n x^{n+r}. \end{aligned}$$

Thus, for all $n \geq 0$,

$$\Big[(n+r)(n+r-1) + 3(n+r) + 1\Big] a_n = 0.$$

Step 3: Indicial equation (lowest power $n = 0$)

$$\begin{gathered} r(r-1) + 3r + 1 = 0 \\ r^2 + 2r + 1 = 0 \\ \boxed{(r+1)^2 = 0 \;\Rightarrow\; r = -1 \text{ (double root)}} \end{gathered}$$

Step 4: Solutions This is an Euler–Cauchy equation. The double root indicates the second solution involves a logarithm:

$$\boxed{y(x) = C_1 x^{-1} + C_2 x^{-1} \ln x.}$$

3 Special Functions

In the study of differential equations, many important engineering and physical systems lead to solutions that cannot be expressed using elementary functions such as polynomials, exponentials, logarithms, or trigonometric functions. Instead, these systems produce solutions known as **special functions**.

Special functions commonly arise from:

- Power series solutions about ordinary points,
- Frobenius series solutions about regular singular points,
- Sturm–Liouville boundary value problems,
- Separation of variables in partial differential equations.

Key Concept

Special functions are solutions of important differential equations that occur frequently in applied mathematics, physics, and engineering.

Many special functions are defined directly by their power series expansions and recurrence relations. These functions form the foundation of mathematical modeling in areas such as heat transfer, wave propagation, quantum mechanics, vibrations, and electromagnetics.

3.1 Bessel Functions

Bessel functions arise when solving differential equations in cylindrical coordinates. They appear naturally in problems involving circular membranes, heat conduction in cylinders, and electromagnetic waveguides.

3.1.1 Bessel's Differential Equation

The standard form of Bessel's equation of order ν is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

Key Concept

The solutions of Bessel's equation are called **Bessel functions of the first and second kind**, denoted by $J_\nu(x)$ and $Y_\nu(x)$.

3.1.2 Bessel Function of the First Kind

A Frobenius solution about $x = 0$ yields the Bessel function of the first kind:

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}.$$

For integer $\nu = n$, the series becomes

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n}.$$

3.1.3 Bessel Function of the Second Kind

The second linearly independent solution is called the Bessel function of the second kind:

$$Y_\nu(x).$$

This solution cannot always be obtained from a simple Frobenius expansion and often involves logarithmic terms when ν is an integer.

Engineering Note

In engineering applications, $J_\nu(x)$ is typically finite at $x = 0$, while $Y_\nu(x)$ is singular at $x = 0$.

3.1.4 Applications of Bessel Functions

Bessel functions appear in many physical systems such as:

- Vibrations of a circular drum membrane,
- Heat conduction in cylindrical objects,
- Electromagnetic wave propagation in circular waveguides,
- Fluid flow in pipes.

3.2 Legendre Functions

Legendre functions arise naturally in problems with spherical symmetry, such as gravitational fields, electrostatic potentials, and wave propagation in spherical coordinates.

3.2.1 Legendre's Differential Equation

The Legendre equation is

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0.$$

Key Concept

The solutions of Legendre's equation are called **Legendre functions**. When n is a nonnegative integer, the polynomial solutions are called **Legendre polynomials**, denoted by $P_n(x)$.

3.2.2 Legendre Polynomials

For integer n, the solution $P_n(x)$ is a polynomial of degree n. The first few Legendre polynomials are:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

3.2.3 Orthogonality Property

Legendre polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} P_m(x)P_n(x)\, dx = 0 \qquad \text{for } m \neq n.$$

Engineering Note

Orthogonality is crucial in Fourier-type expansions and in solving boundary value problems using series methods.

3.2.4 Applications of Legendre Functions

Legendre functions appear in:

- Electrostatic potential problems (Laplace's equation in spherical coordinates),
- Gravitational field modeling,
- Spherical harmonics,
- Quantum mechanics (angular momentum).

3.3 Hermite Polynomials

Hermite polynomials arise in systems involving Gaussian-type solutions and are essential in quantum mechanics and probability theory.

3.3.1 Hermite's Differential Equation

Hermite's equation is

$$y'' - 2xy' + 2ny = 0,$$

where n is a nonnegative integer.

Key Concept

Polynomial solutions of Hermite's equation are called **Hermite polynomials**, denoted by $H_n(x)$.

3.3.2 First Few Hermite Polynomials

The first few Hermite polynomials are:

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 4x^2 - 2, \qquad H_3(x) = 8x^3 - 12x.$$

3.3.3 Applications of Hermite Polynomials

Hermite polynomials appear in:

- Quantum harmonic oscillator,
- Probability and statistics (Gaussian distributions),
- Signal processing and approximation theory.

3.4 Laguerre Polynomials

Laguerre polynomials arise in problems involving radial solutions in quantum mechanics and certain boundary value problems.

3.4.1 Laguerre's Differential Equation

The Laguerre equation is

$$xy'' + (1 - x)y' + ny = 0,$$

where n is a nonnegative integer.

Key Concept

Polynomial solutions of Laguerre's equation are called **Laguerre polynomials**, denoted by $L_n(x)$.

3.4.2 First Few Laguerre Polynomials

The first few Laguerre polynomials are:

$$L_0(x) = 1, \qquad L_1(x) = 1 - x, \qquad L_2(x) = 1 - 2x + \frac{x^2}{2}.$$

3.4.3 Applications of Laguerre Polynomials

Laguerre polynomials are commonly used in:

- Quantum mechanics (hydrogen atom radial solutions),
- Electrical engineering models involving exponential decay,
- Approximation methods in numerical analysis.

3.5 Airy Functions

Airy functions arise from differential equations with turning points and appear in wave propagation and quantum mechanics.

3.5.1 Airy's Differential Equation

The Airy equation is

$$y'' - xy = 0.$$

Key Concept

The solutions of Airy's equation are the **Airy functions** $Ai(x)$ and $Bi(x)$.

3.5.2 Series Form of Airy Functions

The Airy functions can be expressed as power series solutions. Their series expansions lead to two linearly independent solutions that behave differently as x becomes large.

Engineering Note

Airy functions are important in approximating solutions near boundary transition regions, where ordinary approximations fail.

3.5.3 Applications of Airy Functions

Airy functions appear in:

- Diffraction and optics,
- Quantum tunneling problems,
- Beam deflection and elastic bending models.

3.6 Chebyshev Polynomials

Chebyshev polynomials are important in approximation theory and numerical methods. They are closely related to trigonometric functions.

3.6.1 Chebyshev Differential Equation

The Chebyshev equation is

$$(1 - x^2)y'' - xy' + n^2y = 0.$$

Key Concept

The polynomial solutions of this equation are called **Chebyshev polynomials**, denoted by $T_n(x)$.

3.6.2 Trigonometric Definition

Chebyshev polynomials can be defined as

$$T_n(x) = \cos(n \arccos x).$$

3.6.3 Applications of Chebyshev Polynomials

Chebyshev polynomials are widely used in:

- Polynomial approximation (minimax approximation),
- Numerical integration and interpolation,
- Digital signal processing filter design.

3.7 Hypergeometric Functions

Hypergeometric functions are among the most general special functions. Many other special functions (such as Bessel, Legendre, and Hermite functions) can be written as special cases of hypergeometric functions.

3.7.1 Hypergeometric Differential Equation

The hypergeometric equation is

$$x(1-x)y'' + \big[c - (a+b+1)x\big]y' - ab\,y = 0,$$

where a, b, and c are constants.

Key Concept

The solution of the hypergeometric equation is called the **Gaussian hypergeometric function**, written as

$${}_2F_1(a, b; c; x).$$

3.7.2 Series Definition

The hypergeometric function is defined by the power series

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

where $(a)_n$ is the **Pochhammer symbol** defined as

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad (a)_0 = 1.$$

3.7.3 Importance of Hypergeometric Functions

Engineering Note

The hypergeometric function is a "master function" because many special functions can be expressed as hypergeometric series.

3.7.4 Applications of Hypergeometric Functions

Hypergeometric functions appear in:

- Advanced mathematical physics models,
- Fluid dynamics and relativity,
- Engineering problems involving complex boundary conditions,
- Reduction of many differential equations into a single general form.

Summary of Special Functions

- Bessel functions arise in cylindrical coordinate systems.
- Legendre functions arise in spherical coordinate systems.
- Hermite and Laguerre polynomials arise in quantum mechanical systems.
- Airy functions model turning-point behavior in wave propagation.
- Chebyshev polynomials are essential in approximation and numerical analysis.
- Hypergeometric functions provide a general framework that includes many other special functions.

Key Concept

Special functions form the bridge between differential equation theory and real-world engineering applications.