Advanced Engineering Mathematics

Vectors

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1 Vectors in $\mathbb{R}^n$

1.1 Definition and Representation of Vectors

Key Concept

A **vector** is a quantity that has both **magnitude** and **direction**. It is commonly represented as an ordered list of real numbers:

$$\mathbf{v} = [v_1, v_2, v_3, \ldots, v_n]$$

where $\mathbf{v} \in \mathbb{R}^n$.

A **scalar** is a quantity described only by magnitude (a single number), such as temperature, mass, or time.

Vectors can be interpreted geometrically as directed line segments (arrows), and algebraically as ordered tuples.

1.1.1 Position Vector

Key Concept

The **position vector** of a point $P(x_1, x_2, \ldots, x_n)$ is the vector drawn from the origin to the point:

$$\mathbf{r} = [x_1, x_2, \ldots, x_n]$$

Example

A point in 3D space is given by

$$P(2, -1, 5).$$

The position vector of P is the vector from the origin to P:

$$\mathbf{r} = [2, -1, 5].$$

This means the vector moves:

- 2 units in the x-direction,
- 1 unit in the negative y-direction,
- 5 units in the z-direction.

1.2 Basic Vector Operations

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and let c be a scalar.

1.2.1 Vector Addition

Key Concept

If

$$\mathbf{u} = [u_1, u_2, \ldots, u_n], \qquad \mathbf{v} = [v_1, v_2, \ldots, v_n],$$

then

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n].$$

1.2.2 Vector Subtraction

$$\mathbf{u} - \mathbf{v} = [u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n].$$

1.2.3 Scalar Multiplication

Key Concept

If c is a scalar and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, then

$$c\mathbf{v} = [cv_1, cv_2, \ldots, cv_n].$$

1.2.4 Basic Properties of Vector Operations

For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalar c, the following hold:

- Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- Distributive Law: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- Distributive Law: $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$

Example

Let

$$\mathbf{u} = [4, -2, 1], \qquad \mathbf{v} = [-1, 5, 3].$$

(a) Vector Addition

$$\mathbf{u} + \mathbf{v} = [4 + (-1), -2 + 5, 1 + 3] = [3, 3, 4].$$

(b) Vector Subtraction

$$\mathbf{u} - \mathbf{v} = [4 - (-1), -2 - 5, 1 - 3] = [5, -7, -2].$$

(c) Scalar Multiplication

$$3\mathbf{u} = 3[4, -2, 1] = [12, -6, 3].$$

(d) Linear Combination

$$2\mathbf{u} - \mathbf{v} = 2[4, -2, 1] - [-1, 5, 3] = [8, -4, 2] - [-1, 5, 3] = [9, -9, -1].$$

1.3 Norm, Magnitude, and Unit Vectors

1.3.1 Magnitude (Norm)

Key Concept

The **magnitude** (or **Euclidean norm**) of a vector

$$\mathbf{v} = [v_1, v_2, \ldots, v_n]$$

is defined as

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

This measures the length of the vector in $\mathbb{R}^n$.

1.3.2 Unit Vector

Key Concept

A **unit vector** is a vector of magnitude 1. If $\mathbf{v} \neq \mathbf{0}$, then the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Example

Let $\mathbf{v} = [3, 4]$.

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Thus the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{5}[3, 4] = \left[\frac{3}{5}, \frac{4}{5}\right].$$

Example

Let

$$\mathbf{v} = [6, -2, 3].$$

Step 1: Find the magnitude

$$\|\mathbf{v}\| = \sqrt{6^2 + (-2)^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7.$$

Step 2: Find the unit vector in the direction of v

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{7}[6, -2, 3] = \left[\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right].$$

Thus, $\mathbf{u}$ has the same direction as $\mathbf{v}$ but has length 1.

1.4 Standard Basis and Component Form

1.4.1 Standard Basis Vectors

Key Concept

The **standard basis vectors** in $\mathbb{R}^n$ are the vectors:

$$\mathbf{e}_1 = [1, 0, 0, \dots, 0], \quad \mathbf{e}_2 = [0, 1, 0, \dots, 0], \quad \dots, \quad \mathbf{e}_n = [0, 0, 0, \dots, 1].$$

Any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as a linear combination of these basis vectors:

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n.$$

1.4.2 Component Form in $\mathbb{R}^3$

In $\mathbb{R}^3$, the standard basis is often written as:

$$\mathbf{i} = [1, 0, 0], \quad \mathbf{j} = [0, 1, 0], \quad \mathbf{k} = [0, 0, 1].$$

Thus any vector can be written as:

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Example

Let the vector be

$$\mathbf{a} = [5, -3, 2].$$

Using the standard basis vectors

$$\mathbf{i} = [1, 0, 0], \quad \mathbf{j} = [0, 1, 0], \quad \mathbf{k} = [0, 0, 1],$$

we can rewrite **a** in component form as:

$$\mathbf{a} = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.$$

This representation is useful in physics and engineering because each term represents a vector component along an axis direction.

1.5 Vector Applications

Vectors appear naturally in engineering and physics because many real-world quantities require both magnitude and direction. In practice, vectors are used to model motion, forces, fields, and system behaviors in 2D and 3D space.

1.5.1 Displacement

Key Concept

Displacement is the vector that describes the change in position of an object. It points from the initial position to the final position.

A displacement from point $P(x_1, y_1, z_1)$ to point $Q(x_2, y_2, z_2)$ is represented as:

$$\mathbf{d} = \overrightarrow{PQ} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1].$$

The magnitude of the displacement vector represents the straight-line distance traveled:

$$\|\mathbf{d}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example

A robot moves from point

$$P(2, -1, 4)$$

to point

$$Q(7, 3, 1).$$

The displacement vector is

$$\mathbf{d} = [7 - 2,\ 3 - (-1),\ 1 - 4] = [5, 4, -3].$$

The distance traveled in a straight line is

$$\|\mathbf{d}\| = \sqrt{5^2 + 4^2 + (-3)^2} = \sqrt{25 + 16 + 9} = \sqrt{50}.$$

Thus, the robot displaced by $\mathbf{d} = [5, 4, -3]$ and traveled $\sqrt{50}$ units.

Engineering Note

In engineering applications, displacement is important in robotics, navigation systems, CNC motion, and kinematics.

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1.5.2 Velocity

Key Concept

Velocity is a vector that describes how fast an object is moving **and** in what direction it is moving.

If an object moves with displacement $\mathbf{d}$ over time interval Δt, then its average velocity is

$$\mathbf{v}_{avg} = \frac{\mathbf{d}}{\Delta t}.$$

The magnitude of velocity is the **speed**:

$$\text{speed} = \|\mathbf{v}\|.$$

Example

A drone travels with displacement vector

$$\mathbf{d} = [12, -4, 6] \text{ meters}$$

in a time interval of $\Delta t = 3$ seconds.
The average velocity is

$$\mathbf{v}_{avg} = \frac{\mathbf{d}}{3} = \left[\frac{12}{3}, \frac{-4}{3}, \frac{6}{3}\right] = [4, -\tfrac{4}{3}, 2] \text{ m/s}.$$

The speed of the drone is

$$\|\mathbf{v}_{avg}\| = \sqrt{4^2 + \left(-\tfrac{4}{3}\right)^2 + 2^2} = \sqrt{16 + \frac{16}{9} + 4} = \sqrt{\frac{180 + 16}{9}} = \sqrt{\frac{196}{9}} = \frac{14}{3} \text{ m/s}.$$

Thus, the drone moves with speed $\frac{14}{3}$ m/s.

Engineering Note

Velocity vectors are heavily used in mechanical engineering, robotics, control systems, fluid flow, and navigation.

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1.5.3 Force and Resultant Force

Key Concept

Force is a vector quantity because it has both magnitude and direction. Multiple forces acting on an object can be combined using vector addition.

Engineering Note

In statics and dynamics, forces are vectors. The **resultant force** is obtained by adding all forces acting on a body:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n.$$

If $\mathbf{R} = \mathbf{0}$, then the system is in equilibrium.

1.5.4 Resultant of Two Forces

Example

Suppose two forces act on an object:

$$\mathbf{F}_1 = [3, 2], \qquad \mathbf{F}_2 = [1, -4].$$

Then the resultant force is:

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = [3 + 1,\ 2 - 4] = [4, -2].$$

The magnitude of the resultant is:

$$\|\mathbf{R}\| = \sqrt{4^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20}.$$

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1.5.5 Resultant Force with Three Forces in 3D

Example

Suppose a mechanical structure is subjected to three forces:

$$\mathbf{F}_1 = [10, -2, 5]\ \text{N}, \qquad \mathbf{F}_2 = [-6, 4, 1]\ \text{N}, \qquad \mathbf{F}_3 = [2, 0, -3]\ \text{N}.$$

The resultant force is

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3.$$

Add the components:

$$\mathbf{R} = [10 - 6 + 2,\ -2 + 4 + 0,\ 5 + 1 - 3] = [6, 2, 3]\ \text{N}.$$

The magnitude of the resultant force is

$$\|\mathbf{R}\| = \sqrt{6^2 + 2^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7\ \text{N}.$$

Thus, the net force acting on the structure is $\mathbf{R} = [6, 2, 3]$ N with magnitude 7 N.

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1.5.6 Equilibrium Condition

Key Concept

A body is said to be in **equilibrium** if the net force acting on it is zero:

$$\mathbf{R} = \mathbf{0}.$$

This means the object has no acceleration.

Example

A box is pulled by two forces:

$$\mathbf{F}_1 = [8, 3] \text{ N}, \qquad \mathbf{F}_2 = [-5, -7] \text{ N}.$$

Find the force $\mathbf{F}_3$ that must be applied so the system is in equilibrium.
The equilibrium condition is:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}.$$

Thus,

$$\mathbf{F}_3 = -(\mathbf{F}_1 + \mathbf{F}_2).$$

Compute the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$:

$$\mathbf{F}_1 + \mathbf{F}_2 = [8 - 5,\ 3 - 7] = [3, -4].$$

So the balancing force is:

$$\mathbf{F}_3 = -[3, -4] = [-3, 4] \text{ N}.$$

Therefore, a force of $[-3, 4]$ N must be applied for equilibrium.

Engineering Note

This concept is extremely important in statics problems such as trusses, beams, bridges, and structural design.

2 Dot Product

2.1 Definition and Formula

Key Concept

The **dot product** (or **inner product**) of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos \gamma,$$

where γ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (with $0 \leq \gamma \leq \pi$).

If $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_n]$, then the dot product can be computed by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Example

Let $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$.

$$\mathbf{a} \cdot \mathbf{b} = 1(3) + 2(-2) + 0(1) = 3 - 4 + 0 = -1.$$

Key Concept

The dot product of a vector with itself gives the square of its magnitude:

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2.$$

2.2 Properties of Dot Product

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and let k be a scalar.

Key Concept

The dot product satisfies the following key properties:

(a) **Commutativity:** $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(b) **Distributivity:** $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(c) **Scalar Linearity:** $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b})$

(d) **Positive Definiteness:** $\mathbf{a} \cdot \mathbf{a} \geq 0$, and $\mathbf{a} \cdot \mathbf{a} = 0$ iff $\mathbf{a} = \mathbf{0}$

Example

Let $\mathbf{a} = [2, 1]$, $\mathbf{b} = [-1, 3]$, and $\mathbf{c} = [4, 0]$.
Compute $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$:

$$\mathbf{b} + \mathbf{c} = [-1 + 4, 3 + 0] = [3, 3]$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = [2, 1] \cdot [3, 3] = 2(3) + 1(3) = 9.$$

Now compute $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$:

$$\mathbf{a} \cdot \mathbf{b} = 2(-1) + 1(3) = 1, \qquad \mathbf{a} \cdot \mathbf{c} = 2(4) + 1(0) = 8$$

$$\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 1 + 8 = 9.$$

So distributivity is verified.

2.3 Angle and Orthogonality

2.3.1 Angle Between Two Vectors

If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then

$$\cos\gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}, \qquad \gamma = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right).$$

Example

Let $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$.
From earlier, $\mathbf{a} \cdot \mathbf{b} = -1$. Also,

$$\|\mathbf{a}\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}, \qquad \|\mathbf{b}\| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}.$$

Thus,

$$\cos\gamma = \frac{-1}{\sqrt{5}\sqrt{14}} = \frac{-1}{\sqrt{70}}.$$

2.3.2 Orthogonality

Key Concept

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are **orthogonal** (perpendicular) if and only if

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

Example

Check if $\mathbf{u} = [2, -1]$ and $\mathbf{v} = [1, 2]$ are orthogonal:

$$\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(2) = 2 - 2 = 0.$$

Hence, $\mathbf{u} \perp \mathbf{v}$.

2.4 Projection and Components

2.4.1 Scalar Component of a in the Direction of b

Key Concept

If $\mathbf{b} \neq \mathbf{0}$, the **scalar projection** (component) of $\mathbf{a}$ onto $\mathbf{b}$ is

$$\text{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}.$$

If $\mathbf{b}$ is a unit vector, then $\text{comp}_{\mathbf{b}}(\mathbf{a}) = \mathbf{a} \cdot \mathbf{b}$.

2.4.2 Vector Projection of a onto b

Key Concept

If $\mathbf{b} \neq \mathbf{0}$, the **vector projection** of $\mathbf{a}$ onto $\mathbf{b}$ is

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}.$$

Example

Let $\mathbf{a} = [3, 4]$ and $\mathbf{b} = [1, 2]$.
Compute:

$$\mathbf{a} \cdot \mathbf{b} = 3(1) + 4(2) = 11, \qquad \|\mathbf{b}\|^2 = 1^2 + 2^2 = 5.$$

So,

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{11}{5} \right) [1, 2] = \left[\frac{11}{5}, \frac{22}{5} \right].$$

2.5 Applications

2.5.1 Work Done by a Constant Force

Key Concept

If a constant force $\mathbf{F}$ displaces an object by $\mathbf{d}$, then the **work** done is

$$W = \mathbf{F} \cdot \mathbf{d} = \|\mathbf{F}\| \, \|\mathbf{d}\| \cos \gamma.$$

Example

A force $\mathbf{F} = [6, -3, 0]$ moves an object by $\mathbf{d} = [2, 5, 0]$.

$$W = \mathbf{F} \cdot \mathbf{d} = 6(2) + (-3)(5) + 0(0) = 12 - 15 = -3.$$

Since $W < 0$, the force opposes the displacement (work is done *against* the force).

2.5.2 Decomposing a Vector into Parallel and Perpendicular Parts

Engineering Note

In engineering (statics/dynamics), it is common to decompose a force into:

- a component **parallel** to a surface or direction
- a component **perpendicular** (normal) to the surface

This is done using projection.

If $\mathbf{b} \neq \mathbf{0}$, the component of $\mathbf{a}$ parallel to $\mathbf{b}$ is

$$\mathbf{a}_{\parallel} = \mathrm{proj}_{\mathbf{b}}(\mathbf{a}),$$

and the perpendicular component is

$$\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel}.$$

3 Cross Product

3.1 Definition and Geometric Meaning

Key Concept

The **cross product** (or **vector product**) of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is a vector

$$\mathbf{a} \times \mathbf{b}$$

that is **perpendicular** to both $\mathbf{a}$ and $\mathbf{b}$.

3.1.1 Magnitude and Area Interpretation

Let γ be the angle between $\mathbf{a}$ and $\mathbf{b}$ (with $0 \leq \gamma \leq \pi$). If $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin \gamma.$$

Engineering Note

Geometric meaning: $\|\mathbf{a} \times \mathbf{b}\|$ equals the **area of the parallelogram** formed by $\mathbf{a}$ and $\mathbf{b}$ as adjacent sides.

3.1.2 When is the Cross Product Zero?

If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Also, if $\mathbf{a}$ and $\mathbf{b}$ are parallel (same or opposite direction), then $\gamma = 0$ or $\gamma = \pi$, so $\sin \gamma = 0$, hence:

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Example

Let $\mathbf{a} = [3, 0, 0]$ and $\mathbf{b} = [5, 0, 0]$.
These vectors are parallel (both point along the x-axis), so the area of the parallelogram is zero. Therefore,

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Example

Suppose $\|\mathbf{a}\| = 4$, $\|\mathbf{b}\| = 6$, and the angle between them is $\gamma = 30°$.

Then

$$\|\mathbf{a} \times \mathbf{b}\| = (4)(6)\sin 30^\circ = 24\left(\frac{1}{2}\right) = 12.$$

So the parallelogram area formed by **a** and **b** is 12 square units.

3.2 Right-Hand Rule and Orientation

3.2.1 Direction of the Cross Product

Key Concept

The direction of $\mathbf{a} \times \mathbf{b}$ is determined by the **Right-Hand Rule**:

- Point your right-hand fingers in the direction of **a**,
- curl them toward **b** through the smaller angle,
- your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

This means the order matters:

$$\mathbf{a} \times \mathbf{b} \text{ points opposite to } \mathbf{b} \times \mathbf{a}.$$

3.2.2 Right-Handed Coordinate Systems

In a standard right-handed Cartesian coordinate system, the unit basis vectors satisfy:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Reversing the order flips the sign:

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

Example

Compute $\mathbf{j} \times \mathbf{i}$.
Using the right-handed relationships:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \Rightarrow \quad \mathbf{j} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k}.$$

Engineering Note

Engineering meaning: In torque and rotational motion, changing the order of vectors changes the rotation direction (clockwise vs counterclockwise), so orientation is physically important.

3.3 Cross Product Computation

Let

$$\mathbf{a} = [a_1, a_2, a_3], \qquad \mathbf{b} = [b_1, b_2, b_3].$$

3.3.1 Component Formula

Key Concept

The cross product in component form is:

$$\mathbf{a} \times \mathbf{b} = \Big[a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\Big].$$

3.3.2 Determinant Method

A convenient memory tool is the determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Expanding along the first row gives:

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1).$$

Example

Compute $\mathbf{a} \times \mathbf{b}$ for

$$\mathbf{a} = [1, 2, 3], \qquad \mathbf{b} = [4, 0, -1].$$

Using the component formula:

$$\mathbf{a} \times \mathbf{b} = \big[2(-1) - 3(0),\ 3(4) - 1(-1),\ 1(0) - 2(4)\big] = [-2,\ 13,\ -8].$$

So,

$$\mathbf{a} \times \mathbf{b} = [-2, 13, -8].$$

Example

Verify perpendicularity for the result above by dotting:
Let $\mathbf{v} = \mathbf{a} \times \mathbf{b} = [-2, 13, -8]$.

$$\mathbf{a} \cdot \mathbf{v} = [1, 2, 3] \cdot [-2, 13, -8] = 1(-2) + 2(13) + 3(-8) = -2 + 26 - 24 = 0.$$

$$\mathbf{b} \cdot \mathbf{v} = [4, 0, -1] \cdot [-2, 13, -8] = 4(-2) + 0(13) + (-1)(-8) = -8 + 8 = 0.$$

Thus $\mathbf{v}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, as expected.

3.4 Properties of the Cross Product

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and k be a scalar.

3.4.1 Linearity and Distributive Laws

Key Concept

Cross product satisfies:

(a) **Scalar Linearity:**

$$(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (k\mathbf{b})$$

(b) **Distributive Property:**

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

3.4.2 Anti-Commutativity

Key Concept

The cross product is **anti-commutative**:

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}).$$

3.4.3 Not Associative

Engineering Note

Cross product is **not associative**. In general:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}.$$

So parentheses matter.

Example

Verify anti-commutativity using an easy pair:

$$\mathbf{a} = \mathbf{i} = [1, 0, 0], \qquad \mathbf{b} = \mathbf{j} = [0, 1, 0].$$

We know:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

Switching the order:

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}.$$

So,

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}).$$

Example

Demonstrate non-associativity using basis vectors:

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = \mathbf{i} \times (\mathbf{i}) = \mathbf{0}.$$

But

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{k} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

This example gives equality, so it doesn't disprove associativity.
To show non-associativity more clearly, use:

$$\mathbf{a} = [1, 1, 0], \quad \mathbf{b} = [1, 0, 1], \quad \mathbf{c} = [0, 1, 1].$$

Compute $\mathbf{b} \times \mathbf{c}$:

$$\mathbf{b} \times \mathbf{c} = \big[0 \cdot 1 - 1 \cdot 1,\ 1 \cdot 0 - 1 \cdot 1,\ 1 \cdot 1 - 0 \cdot 0\big] = [-1, -1, 1].$$

Then

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = [1, 1, 0] \times [-1, -1, 1] = [1(-?) - 0(-?), \ \ldots] = [1, -1, 0].$$

Now compute $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = [1, 1, 0] \times [1, 0, 1] = [1 \cdot 1 - 0 \cdot 0, \ 0 \cdot 1 - 1 \cdot 1, \ 1 \cdot 0 - 1 \cdot 1] = [1, -1, -1].$$

Then

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = [1, -1, -1] \times [0, 1, 1] = [0, \ -1, \ 1].$$

Since

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = [1, -1, 0] \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = [0, -1, 1],$$

cross product is not associative.

3.5 Scalar Triple Product and Volume Interpretation

3.5.1 Definition

Key Concept

The **scalar triple product** of $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ is defined as

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

It produces a **scalar** (a real number).

3.5.2 Determinant Form

If

$$\mathbf{a} = [a_1, a_2, a_3], \quad \mathbf{b} = [b_1, b_2, b_3], \quad \mathbf{c} = [c_1, c_2, c_3],$$

then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

3.5.3 Geometric Meaning: Volume

Engineering Note

The absolute value of the scalar triple product gives the **volume of the parallelepiped** formed by $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$V = \left|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\right|.$$

If the volume is zero, the vectors lie in the same plane (they are coplanar), meaning they are linearly dependent in $\mathbb{R}^3$.

Example

Let

$$\mathbf{a} = [1, 2, 3], \quad \mathbf{b} = [2, 0, 1], \quad \mathbf{c} = [1, 1, 0].$$

Step 1: Compute $\mathbf{b} \times \mathbf{c}$:

$$\mathbf{b} \times \mathbf{c} = [\, 0 \cdot 0 - 1 \cdot 1,\; 1 \cdot 1 - 2 \cdot 0,\; 2 \cdot 1 - 0 \cdot 1\,] = [-1, 1, 2].$$

Step 2: Dot with $\mathbf{a}$:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = [1, 2, 3] \cdot [-1, 1, 2] = 1(-1) + 2(1) + 3(2) = -1 + 2 + 6 = 7.$$

Thus, the volume of the parallelepiped is

$$V = |7| = 7 \text{ cubic units.}$$

Example

If $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [2, 4, 6]$, and $\mathbf{c} = [0, 1, 0]$, notice that $\mathbf{b} = 2\mathbf{a}$, so $\mathbf{a}$ and $\mathbf{b}$ are parallel.
Then $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar, and the parallelepiped collapses into a flat shape. Hence,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0 \quad \Rightarrow \quad V = 0.$$