Lecture 1 – Linear Programs and Common Notation

Module 4 – Linear Programming Basics
CS/ISyE/ECE 524

Linear Programming Basics Module Learning Outcomes

By the end of this module, you should be able to…

- Give a formal definition of a linear program (LP)
- Transform any LP into standard form
- Graphically deduce whether an LP
  - has exactly one optimal solution
  - has infinitely many optimal solutions,
  - is infeasible, or
  - is unbounded.
- Begin to gain an intuition for the properties of LPs

Expressing Linear Programs in Math Notation

What *is* a linear program, anyway?

Variables

- Could be unrestricted:

  x_i free for some $i \in \{1, \dots, n\}$

- Could have **box constraints**:

  $\ell_i \leq x_i \leq u_i$ for some $i \in \{1, \dots, n\}$

Objective function

Either:

$$\max_x c^T x + d$$

Or:

$$\min_x c^T x + d$$

Constraints

- Inequalities:

  $$Ax \leq b$$

- Equalities:

  $$Ax = b$$

- A combination of both

A **linear program** is an optimization model with:

- Real-valued variables ($x \in \mathbb{R}^n$)
- A linear (or affine) objective function that we either maximize or minimize
- Affine constraints

Expressing a linear program in **standard form**

One of the most common ways to write down the variables, objective function, and constraints of a linear program is using **standard form:**

$$
\begin{aligned}
\max_{x \in \mathbb{R}^n} \quad & c^T x + d \\
\text{subject to} \quad & Ax \leq b \\
& x \geq 0
\end{aligned}
$$

A linear program in **standard form** has the following properties:

- The *objective* is maximized
- All *constraints* are "less than or equal to"
- All *variables* are bounded below by 0 and are unbounded above

Note: There are many equivalent ways of writing down a linear program. We will only focus on a couple possibilities.

Standard Form Example

Recall the "Top Brass" model from the first module:

$$\max_{x_s, x_f} \; 12x_f + 9x_s$$
$$\text{s.t. } 4x_f + 2x_s \le 4800$$
$$x_f + x_s \le 1750$$
$$0 \le x_f \le 1000$$
$$0 \le x_s \le 1500$$

Rewrite as operations on matrices

$$\max_{x_s, x_f} \; \begin{bmatrix} 12 \\ 9 \end{bmatrix}^T \begin{bmatrix} x_f \\ x_s \end{bmatrix}$$
$$\text{s.t. } \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_f \\ x_s \end{bmatrix} \le \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}$$
$$\begin{bmatrix} x_f \\ x_s \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Top Brass is already in standard form!

$$x = \begin{bmatrix} x_f \\ x_s \end{bmatrix}, A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}, c = \begin{bmatrix} 12 \\ 9 \end{bmatrix}$$

What to do if your LP isn't in standard form?

(7 helpful tricks)

- Convert between max and min by taking the negative of the objective:

$$\min_x \; c^T x = -\max_x \; (-c^T x)$$

- Reverse direction of inequalities by flipping the sign:

$$Ax \leq b \Leftrightarrow (-A)x \geq (-b)$$

- Turn equality into inequality by splitting it into two new inequalities:

$$Ax = b \;\; \Leftrightarrow Ax \geq b \text{ and } Ax \leq b$$

- Turn inequalities into equalities by adding a **slack variable**:

$$Ax \leq b \Leftrightarrow Ax + s = b \text{ and } s \geq 0$$

What to do if your LP isn't in standard form?

(7 helpful tricks continued)

- Replace free (unrestricted) variables with two new (non-negative) variables:

$$x \in \mathbb{R} \Leftrightarrow \mathrm{u} \geq 0, v \geq 0, \text{and } x = u - v$$

- Put variable bounds in constraints to create "free" variables:

$$p \leq x \leq q \Leftrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathrm{x} \leq \begin{bmatrix} q \\ -p \end{bmatrix}, x \text{ free}$$

- Shift bounds on bounded variables to make 0 the lower bound:

$$p \leq x \leq q \Leftrightarrow 0 \leq v, v \leq q \ - p, \text{and } v = x \ - p$$

Practice with Standard Form

Turn the following LP into standard form:

$$\begin{aligned} \min_{x_1, x_2} \quad & x_1 + x_2 \\ \text{s.t. } & 5x_1 - 3x_2 = 7 \\ & 2x_1 + x_2 \geq 2 \\ & 1 \leq x_2 \leq 4 \\ & x_1 \text{ free} \end{aligned}$$

Question: What are the issues with this LP?

In Julia: Standard Form

$$\begin{aligned} \max_{u,v,w} \quad & -u + v - w - 1 \\ \text{s.t. } & 5u - 5v - 3w \leq 10 \\ & -5u + 5v + 3w \leq -10 \\ & -2u + 2v - w \leq -1 \\ & w \leq 3 \\ & u, v, w \geq 0 \end{aligned}$$

Note: Often some additional work is required after solving standard-form problems. In this example we have:

- original cost= -(new cost)
- $x_1 = u - v$
- $x_2 = w + 1$