Lecture 3– Complementary Slackness

Module 6 – Duality in Linear Programming
CS/ISyE/ECE 524

Important property of duality: complementary slackness

- As we have already seen, at an optimal solution some inequality constraints are *tight* or *binding*. These are most commonly called **active constraints**. (E.g., the wood constraint.) These constraints have no slack and nonzero shadow prices.
- Some inequality constraints may remain loose or unbinding at optimality. E.g., the brass footballs constraint. These constraints have **slack** and their shadow prices are 0.

Theorem (Complementary Slackness): At an optimal solution to both the primal and dual problem (x^*, λ^*):
Either a primal constraint is active **or** its dual variable is 0.

AND

Either a dual constraint is active **or** its primal variable is 0.

We can even solve LPs with Complementary Slackness!

Primal problem

$$\min_x x_1 + x_2$$
$$\text{s.t. } 2x_1 + x_2 \geq 5$$
$$x_1 + 4x_2 \geq 6$$
$$x_1 \geq 1$$

Dual problem

$$\max_\lambda 5\lambda_1 + 6\lambda_2 + \lambda_3$$
$$\text{s.t. } 2\lambda_1 + \lambda_2 + \lambda_3 = 1$$
$$\lambda_1 + 4\lambda_2 = 1$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

Question:
Is the point $(x_1, x_2) = (1,3)$ optimal?

Steps:

1. Verify it is feasible (it is)
2. Second primal constraint has slack, so $\lambda_2 = 0$
3. $p^* = d^*$, so if it's optimal, we have $1 + 3 = 4 = 5\lambda_1 + \lambda_3$
4. Third dual constraint gives $\lambda_1 = 1$
5. For objective, we get $5 * 1 + \lambda_3 = 4$, so that $\lambda_3 = -1$
6. This is **not** a feasible point for the dual. Complementary slackness doesn't hold $\Rightarrow$ $(1,3)$ must ***not be optimal***!

Another one!

Primal problem

$$\min_x x_1 + x_2$$
$$\text{s.t. } 2x_1 + x_2 \geq 5$$
$$x_1 + 4x_2 \geq 6$$
$$x_1 \geq 1$$

Dual problem

$$\max_\lambda 5\lambda_1 + 6\lambda_2 + \lambda_3$$
$$\text{s.t. } 2\lambda_1 + \lambda_2 + \lambda_3 = 1$$
$$\lambda_1 + 4\lambda_2 = 1$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

Question:
Is the point $(x_1, x_2) =$ $(2,1)$ optimal?

Steps:

1. Verify it is feasible (it is)
2. Third primal constraint has slack, so $\lambda_3 = 0$
3. $p^* = d^*$, so if it's optimal we have: $2 + 1 = 3 = 5\lambda_1 + 6\lambda_2 = 3$
4. First and second dual constraints give $2\lambda_1 + \lambda_2 = 1,\ \lambda_1 + 4\lambda_2 = 1$
5. Solving these gives $(\lambda_1, \lambda_2, \lambda_3) = (\frac{3}{7}, \frac{1}{7}, 0)$ (a dual feasible solution)
6. Check $5\left(\frac{3}{7}\right) + 6\left(\frac{1}{7}\right) = \frac{21}{7} = 3.$
7. Complementary slackness holds $\Rightarrow$ (2,1) must be ***optimal***!