Lecture 1 – Introduction to Duality

Module 6 – Duality in Linear Programming
CS/ISyE/ECE 524

LP Duality Module
Learning Outcomes

By the end of this module, you should be able to…

- Find lower and upper bounds on any linear program's optimal objective
- Define duality in linear programs
  - Understand the relationship between primal and dual linear programs
  - State the strong duality theorem
- Perform sensitivity analysis on a linear program using duality
  - Understand relationship between duality and sensitivity analysis

LP Duality
Learning Outcomes

By the end of this module, you should be able to…

- Define complementary slackness
  - Use complementary slackness to find primal and/or dual solutions
- Write down the dual for the general form of the general min-cost network flow problem as well as specifically for
  - Longest path
  - Max flow
- Describe the real-world interpretation of the dual problem for some MCNF problems

Bounds on optimal objective values

Recall the "Top Brass" model from the first class:

$p^* \geq 12(1000) + 9(400) = 15600$

check feasibility:

$4(1000) + 2(400) = 4800 \leq 4800$ ✓

$(1000) + (400) = 1400 \leq 1750$ ✓

$0 \leq 1000 \leq 1000$ ✓

$0 \leq 400 \leq 1500$ ✓

Try (1000,400)

Let's call the optimal objective value (max profit) $\boldsymbol{p}^*$

Question: Is there a way to find **bounds** on p^*?

- Finding a *lower bound* is easy…we can use any feasible point!

Every feasible point gives a lower bound on the LP's optimal objective!

Finding the largest lower bound (best feasible point) amounts to solving the LP.

Bounds on optimal objective values

What about upper bounds?

$$\begin{aligned} \max_{x_s, x_f} \; & 12x_f + 9x_s \\ \text{s.t. } & 4x_f + 2x_s \leq 4800 \\ & x_f + x_s \leq 1750 \\ & 0 \leq x_f \leq 1000 \\ & 0 \leq x_s \leq 1500 \end{aligned}$$

Question: What is the best upper bound we can find by combining constraints in this manner?

- Finding an *upper bound* is not so easy….
- We can use constraints:
  1. We know $x_f \leq 1000$ and $x_s \leq 1500$
  2. Thus, $12x_f \leq 12 \times 1000 = 12000$ and $9x_s \leq 9 \times 1500 = 13500$
  3. So $p^* \leq 12{,}000 + 13500 = 25{,}500$
- 25,500 is very far away from the best lower bound 15,600
  - Can we do better?
  1. Rewrite: $12x_f + 9x_s = x_f + \left(4x_f + 2x_s\right) + 7\left(x_f + x_s\right)$
  2. We know
     $$x_f + \left(4x_f + 2x_s\right) + 7\left(x_f + x_s\right) \leq 1000 + 4800 + 7(1750)$$
  3. So $p^* \leq 18{,}050$
- Combining constraints in different ways gives us different upper bounds on p^*

Constraint multipliers

$$\max_{x_s, x_f} \; 12x_f + 9x_s$$
$$\text{s.t. } 4x_f + 2x_s \le 4800$$
$$x_f + x_s \le 1750$$
$$0 \le x_f \le 1000$$
$$0 \le x_s \le 1500$$

$\lambda_4 x_s \ge \lambda_4 1500$

Question: If we choose multipliers $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$, what values are we "allowed" to use to find an upper bound on p^*?

In general, what we've been doing is:

1. Combine all constraints with different multipliers (that sum to at least 12 for x_f and 9 for x_s):
   - $0(4x_f + 2x_s) + 0(x_f + x_s) + 12x_f + 9x_s$
   - $1(4x_f + 2x_s) + 7(x_f + x_s) + 1x_f + 0x_s$
2. Then plug in the maximum value the constraint can take:
   - $0(4x_f + 2x_s) + 0(x_f + x_s) + 12x_f + 9x_s \le 0 + 0 + 12(1000) + 9(1500) = 25{,}500$
   - $1(4x_f + 2x_s) + 7(x_f + x_s) + 1x_f + 0x_s \le 4800 + 7(1750) + 1000 + 0 = 18{,}050$

Rephrased question: For what values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$ is the following inequality *guaranteed* to hold for *any* feasible values of x_f, x_s?

≤ 0

$$12x_f + 9x_s \le \lambda_1(4x_f + 2x_s) + \lambda_2(x_f + x_s) + \lambda_3 x_f + \lambda_4 x_s \le \text{RHS}$$

Allowable constraint multipliers

$$\max_{x_s, x_f} \; 12x_f + 9x_s$$
$$\text{s.t. } 4x_f + 2x_s \leq 4800$$
$$x_f + x_s \leq 1750$$
$$0 \leq x_f \leq 1000$$
$$0 \leq x_s \leq 1500$$

- First, let's rearrange the inequality to get a constant on one side:

$$12x_f + 9x_s \leq \lambda_1(4x_f + 2x_s) + \lambda_2(x_f + x_s) + \lambda_3 x_f + \lambda_4 x_s \Rightarrow$$
$$0 \leq x_f(4\lambda_1 + \lambda_2 + \lambda_3 - 12) + x_s(2\lambda_1 + \lambda_2 + \lambda_4 - 9)$$

- Next, because $x_f, x_s \geq 0$, the only way to guarantee this holds for *any* feasible solution is if:
  - $4\lambda_1 + \lambda_2 + \lambda_3 - 12 \geq 0$ ***and*** $2\lambda_1 + \lambda_2 + \lambda_4 - 9 \geq 0$

Question: This gives us the allowable choices of λ, but if we want the ***best upper bound***, how do we find the ***best possible*** λ?

Rephrased question: Given that

$$p^* \leq \lambda_1(4x_f + 2x_s) + \lambda_2(x_f + x_s) + \lambda_3 x_f + \lambda_4 x_s$$
$$\leq 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4,$$

what choices of λ give us the smallest value for the right-hand side?

Best constraint multipliers

$$\max_{x_s, x_f} \; 12x_f + 9x_s$$
$$\text{s.t. } 4x_f + 2x_s \leq 4800$$
$$x_f + x_s \leq 1750$$
$$x_f \leq 1000$$
$$x_s \leq 1500$$
$$x_f, x_s \geq 0$$

Let's recap. We want to find values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that:

- $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ (otherwise things get funky with plugging in upper bounds on constraints)
- $4\lambda_1 + \lambda_2 + \lambda_3 \geq 12$, $2\lambda_1 + \lambda_2 + \lambda_4 \geq 9$ (so we can guarantee we have an upper bound)
- The value $4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$ is as small as possible.

We can rewrite this more compactly:

$$\min_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \; 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$
$$\text{s.t. } \quad 4\lambda_1 + \lambda_2 + \lambda_3 \geq 12$$
$$2\lambda_1 + \lambda_2 + \lambda_4 \geq 9$$
$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

This is just an LP!

Duality

(Or, the part where Prof. Smith always gets really excited)

Primal problem

$$\max_{x_s, x_f} \; 12x_f + 9x_s$$
$$\text{s.t. } 4x_f + 2x_s \le 4800$$
$$x_f + x_s \le 1750$$
$$x_f \le 1000$$
$$x_s \le 1500$$
$$x_f, x_s \ge 0$$

Dual problem

$$\min_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \; 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$
$$\text{s.t. } \; 4\lambda_1 + \lambda_2 + \lambda_3 \ge 12$$
$$2\lambda_1 + \lambda_2 + \lambda_4 \ge 9$$
$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$$

The problem of finding the lambda multipliers that give the best upper bound is called the **dual problem.** The original LP is called the **primal problem.**

Fact: *Every* primal LP has an associated dual!

Some observations:

- Primal is max, dual is min
- There is a dual variable for every primal constraint
- There is a dual constraint for every primal variable
- (any feasible primal point) $\le p^* \le d^* \le$ (any feasible dual point)

Duality in Top Brass with matrices

Primal problem

$$\max_{f,s} \begin{bmatrix} 12 \\ 9 \end{bmatrix}^T \begin{bmatrix} f \\ s \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} 4 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f \\ s \end{bmatrix} \leq \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}$$

$$f, s \geq 0$$

Dual problem

$$\min_{\lambda} \begin{bmatrix} 4800 \\ 1750 \\ 1000 \\ 1500 \end{bmatrix}^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} 4 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \geq \begin{bmatrix} 12 \\ 9 \end{bmatrix}$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

In Julia: Top Brass Dual

General LP Duality

Primal problem (P)

$$\max_x \; c^T x$$
$$\text{s.t.} \; Ax \le b$$
$$x \ge 0$$

Dual problem (D)

$$\min_\lambda \; b^T \lambda$$
$$\text{s.t.} \; A^T \lambda \ge c$$
$$\lambda \ge 0$$

Challenge: See if you can prove the Weak Duality Theorem for the general case.

Hint: This is exactly what we did for Top Brass.

Theorem (Weak Duality): If x and λ are feasible points of (P) and (D) (respectively), then:

$$c^T x \le p^* \le d^* \le b^T \lambda$$

Theorem (Strong Duality): If either p^* or d^* exist and are finite, then:

$$p^* = d^*$$

Possible outcomes for Primal/Dual pairs

Primal problem (P)

$$\max_{x}\ c^T x$$
$$\text{s.t.}\ Ax \leq b$$
$$x \geq 0$$

Dual problem (D)

$$\min_{\lambda}\ b^T \lambda$$
$$\text{s.t.}\ A^T \lambda \geq c$$
$$\lambda \geq 0$$

1. Optimal p^* is attained
2. (P) is unbounded: $p^* = \infty$
3. (P) is infeasible: $p^* = -\infty$

1. Optimal d^* is attained
2. (D) is unbounded: $d^* = -\infty$
3. (D) is infeasible: $d^* = \infty$

Question: Given Weak Duality, which primal/dual combinations are possible?

$p^* \leq d^*$, $p^* = d^*$

P	D
1	1
2	3
3	2
3	3

What if your LP is not in standard form?

If your LP is not in standard form and you need to take the dual:

1. Convert the LP to standard form
2. Write the dual
3. Make any simplifications

Exercise: What is the dual of the dual?

$$\begin{aligned} \min_{\lambda} \quad & b^{\mathsf{T}}\lambda \\ \text{s.t.} \quad & A^{\mathsf{T}}\lambda \geq c \\ & \lambda \geq 0 \end{aligned}$$

equiv

$$\begin{aligned} -\max_{\lambda} \quad & (-b)^{\mathsf{T}}\lambda \\ \text{s.t.} \quad & (-A^{\mathsf{T}})\lambda \leq (-c) \\ & \lambda \geq 0 \end{aligned}$$

dual

$$\begin{aligned} -\min_{z} \quad & (-c)^{\mathsf{T}}z \\ \text{s.t.} \quad & (-A)z \geq (-b) \\ & z \geq 0 \end{aligned}$$

equiv

$$\begin{aligned} \max_{z} \quad & c^{\mathsf{T}}z \\ \text{s.t.} \quad & Az \leq b \\ & z \geq 0 \end{aligned}$$

Common primal/dual pairs

Standard form:

$$\begin{aligned} \max_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_{\lambda} \quad & b^T \lambda \\ \text{s.t.} \quad & \lambda \geq 0 \\ & A^T \lambda \geq c \end{aligned}$$

Free form:

$$\begin{aligned} \max_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \text{ free} \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_{\lambda} \quad & b^T \lambda \\ \text{s.t.} \quad & \lambda \geq 0 \\ & A^T \lambda = c \end{aligned}$$

Mixed constraints:

$$\begin{aligned} \max_{x} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & Fx = g \\ & x \text{ free} \end{aligned} \quad \overset{\text{dual}}{\Longleftrightarrow} \quad \begin{aligned} \min_{\lambda,\mu} \quad & b^T \lambda + g^T \mu \\ \text{s.t.} \quad & \lambda \geq 0 \\ & \mu \text{ free} \\ & A^T \lambda + F^T \mu = c \end{aligned}$$

Dual Rules

Here are some useful rules for transitioning between primal and dual problems (either side can be primal or dual):

Minimization	**Maximization**
Nonnegative variable ≥	Inequality constraint ≤
Nonpositive variable ≤	Inequality constraint ≥
Free variable	Equality constraint =
Inequality constraint ≥	Nonnegative variable ≥
Inequality constraint ≤	Nonpositive variable ≤
Equality constraint =	Free variable

Key:

Dual variables ≡ Primal constraints

Dual constraints ≡ Primal variables

The same information but in math

$$\max_x \; c^T x \quad \text{(max)}$$
$$\text{s.t. } Ax \le b \quad \text{(constraint } \le)$$
$$x \ge 0 \quad \text{(variable } \ge)$$

$$\min_\lambda \; b^T \lambda \quad \text{(min)}$$
$$\text{s.t. } \lambda \ge 0 \quad \text{(variable } \ge)$$
$$A^T \lambda \ge c \quad \text{(constraint } \ge)$$

LP primal-dual pair with every possible situation:

$$\begin{aligned}
\max_{x,y,z} \;& c^T x + d^T y + f^T z \\
\text{s.t. } & Ax + By + Cz \le p \\
& Dx + Ey + Fz \ge q \\
& Gx + Hy + Jz = r \\
& x \ge 0 \\
& y \le 0 \\
& z \text{ free}
\end{aligned}$$

$$\begin{aligned}
\min_{\lambda,\eta,\mu} \;& p^T \lambda + q^T \eta + r^T \mu \\
\text{s.t. } & \lambda \ge 0 \\
& \eta \le 0 \\
& \mu \text{ free} \\
& A^T \lambda + D^T \eta + G^T \mu \ge c \\
& B^T \lambda + E^T \eta + H^T \mu \le d \\
& C^T \lambda + F^T \eta + J^T \mu = f
\end{aligned}$$

The real question: so what?

- Why should we care about the dual? (Other than the fact that it's really cool, of course!)
- Sometimes the dual is much easier to solve than the primal:
  - The dual is much easier than the primal in this case
  - Many solvers take advantage of duality to get solutions quickly
- Duality is very closely related to the idea of **sensitivity**: how *sensitive* is a solution to changes in the problem data? (e.g., if you change a constraint, how much does it affect the objective?)

$$\begin{aligned} \max_{x,y,z} \quad & 3x + y + 2z \\ \text{s.t.} \quad & x + 2y + z \leq 2 \\ & x, y, z \geq 0 \end{aligned} \quad \xleftrightarrow{\text{dual}} \quad \begin{aligned} \min_{\lambda} \quad & 2\lambda \\ \text{s.t.} \quad & \lambda \geq 3 \\ & 2\lambda \geq 1 \\ & \lambda \geq 2 \\ & \lambda \geq 0 \end{aligned}$$