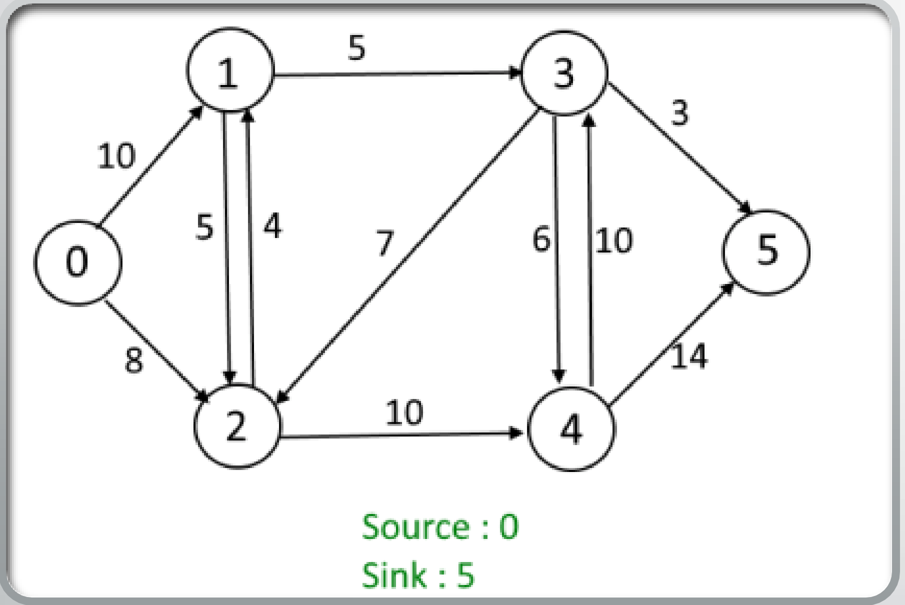
Lecture 7 – Network Flow Problems: Max-Flow Problems

Module 5 – Special Cases of Linear Programs
CS/ISyE/ECE 524

Max-flow problems

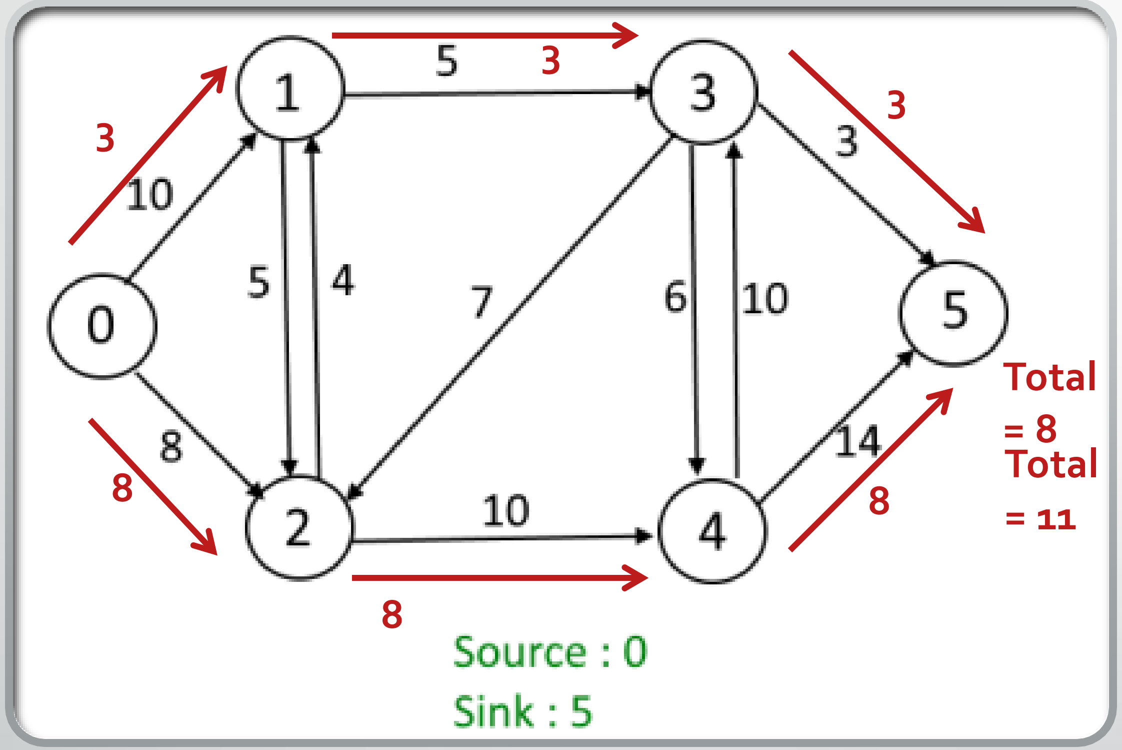


The **max-flow problem** is a type of MCNF problem that has *edge capacities.*

The goal of the max-flow problem is to find the maximum amount of material (flow) that can be pushed from the source to the sink.

- Flow begins at source
- Flow enters sink at the end
- Flow can split
- Edges have capacity limits
- Notions of supply and demand don't really apply (just want to know how much you'd be able to get through the network)

Max-flow problems



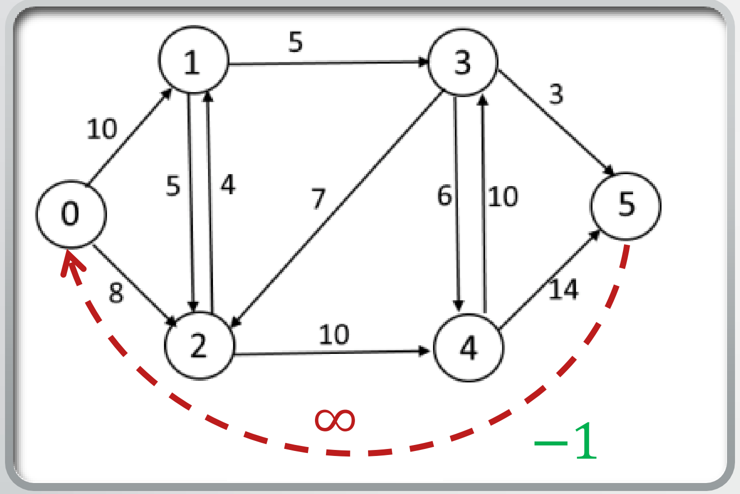
Suppose we have a network of pipes with the given capacities (in gallons/second)

- Want to pump as much water through the network as possible
- One feasible flow is to send 8 gallons of water from $0 - 2 - 4 - 5$
- We can add another 3 gallons along $0 - 1 - 3 - 5$
- Total flow from source (0) to sink (5) is 11 gallons/second

Questions

- Can we improve this flow? (> 11 gallons)
- Can you give me a guaranteed upper bound on the optimal (max) flow?

Modeling max-flow problems as MCNF problems



How do we model a max-flow problem as a MCNF?

1. Recall that supply/demand are meaningless, so all nodes should have $b_i = 0$
   - In other words, all nodes should be intermediate nodes (not sources or sinks)
2. Add a “dummy” arc that goes from the **sink** to the **source** that has **infinite capacity**
   - Now think of this as flow *repeatedly* moving through the network
   - If flow is balanced (flow in=flow out), max flow from source to sink == max flow on dummy arc!
3. Last step is to determine costs
   - All “real” arcs have 0 cost
   - Dummy arc: **cost = -1**
   - To minimize cost, want as much as possible flowing on dummy arc (maximize flow!)

The general max-flow model

Data

- N – the set of nodes
- $E \cup \{t, s\}$ – the set of given edges plus a dummy arc from sink to source
- c_{ij} – cost of arc $(i, j) \in E$ (-1 for edge $\{t, s\}$, 0 else)
- u_{ij} – capacity of arc $(i, j) \in E$ (∞ for edge $\{t, s\}$)

Variables

x_{ij} = total flow on edge $(i, j) \in E$ (nonnegative)

In Julia: Max Flow Example

Objective

- Minimize cost:

$$\min_{\mathbf{x}} \; -x_{ts}$$

Constraints

- Balance flow:

$$\sum_{j \in N} x_{kj} = \sum_{i \in N} x_{ik} \quad \forall k \in N$$

  - Since there is no supply/demand, all nodes will have flow in = flow out (no b_is)
- Not exceed capacity:

$$x_{ij} \le u_{ij} \quad \forall (i, j) \in E$$