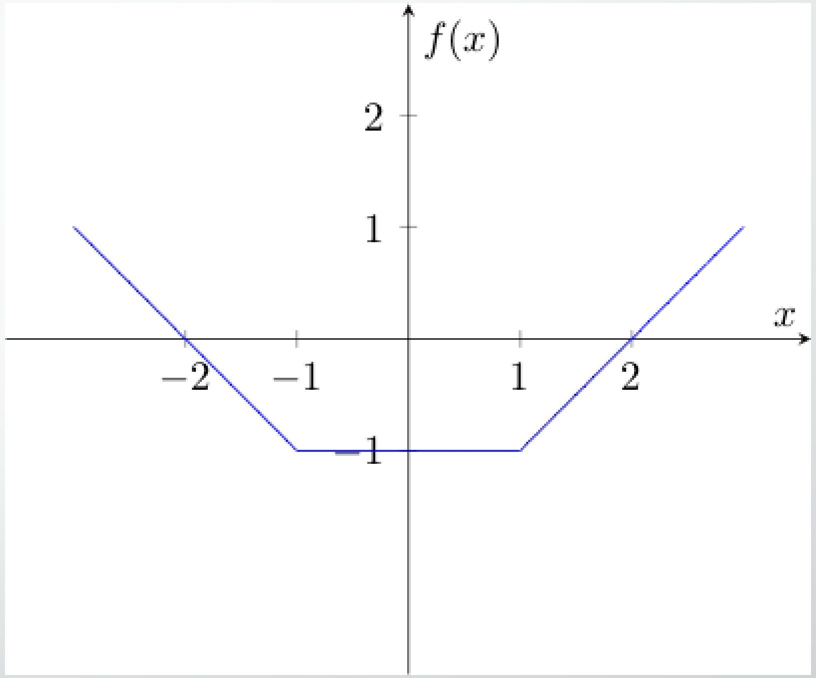
Lecture 2 – Modeling Piecewise Linear Functions

Module 5 – Special Cases of Linear Programs
CS/ISyE/ECE 524

Piecewise Linear Functions

- Some problems don't look like LPs, but can easily be transformed into an LP
- An important example is **minimizing convex piecewise linear functions**
  - If we are maximizing, we want the function to be *concave*



Example

$$\min_{x\in\mathbb{R}} \; f(x)$$
$$\text{s.t.} \quad Ax \geq b$$
$$x \geq 0$$

Where

$$f(x) = \begin{cases} -x - 2 & \text{if } -\infty < x \leq -1 \\ -1 & \text{if } -1 < x \leq 1 \\ x - 2 & \text{if } 1 < x < \infty \end{cases}$$

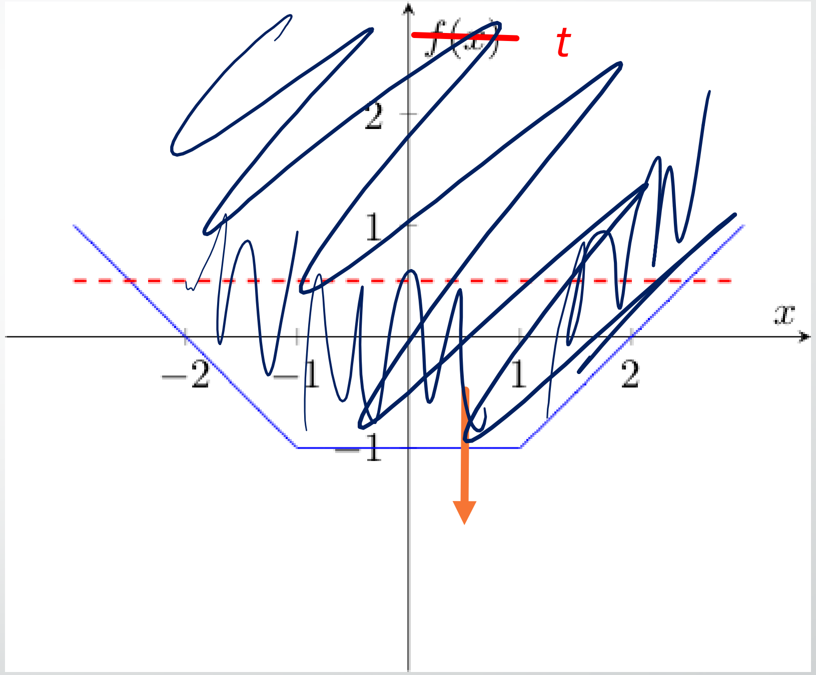
Converting to Epigraph Form

- Convert the problem to **epigraph form** by adding a new variable t and turning the cost function into a constraint

$$\min_{x \in \mathbb{R}} \ f(x) \quad \text{s.t.} \quad Ax \geq b, \quad x \geq 0$$

$$\Rightarrow$$

$$\min_{x,t} \ t \quad \text{s.t.} \quad Ax \geq b, \quad t \geq f(x), \quad x \geq 0$$

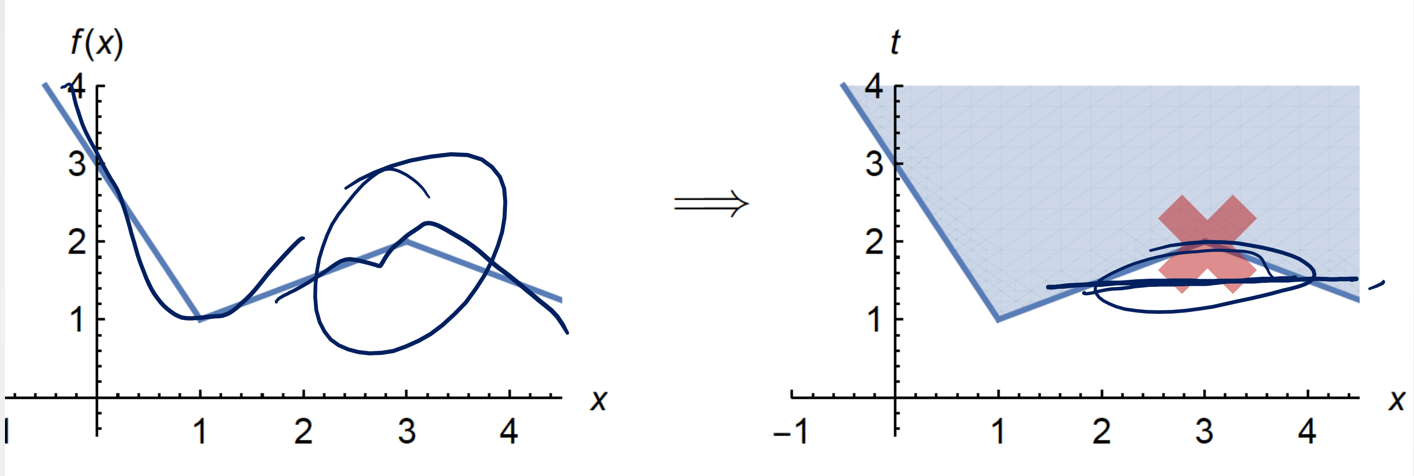


- The new feasible set is still a polyhedron!

$$\min_{x,t} \ \{t \mid Ax \geq b; \quad t \geq -x - 2; t \geq -1; t \geq x - 2; \quad x \geq 0\}$$

The **epigraph** of a function $f(x)$ is the set of points that lie on or above its graph:

$$epi\ f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t \geq f(x)\}$$



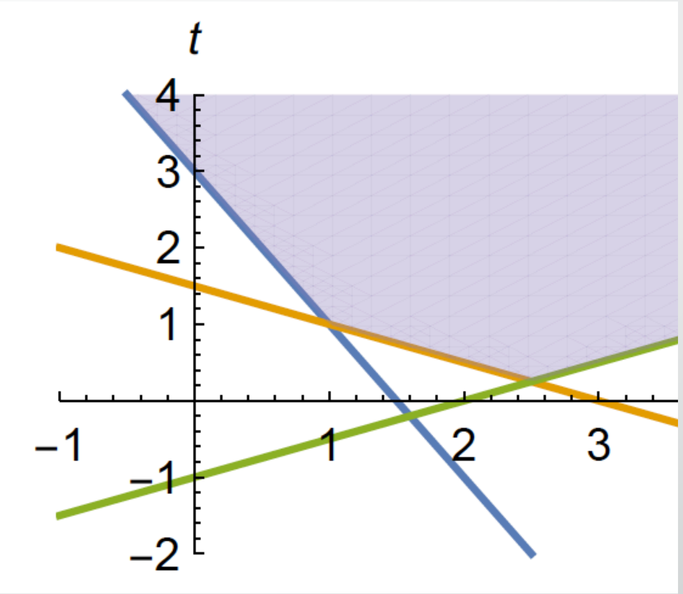
IMPORTANT:

The epigraph trick only works if the epigraph is a **polyhedron**

- The epigraph in this image is **not a polyhedron** (which means it cannot be a feasible region of any LP)

A special case: minimax problems

$$f(x) = \max_{i=1,\dots,k} \{a_i^T x + b_i\}$$



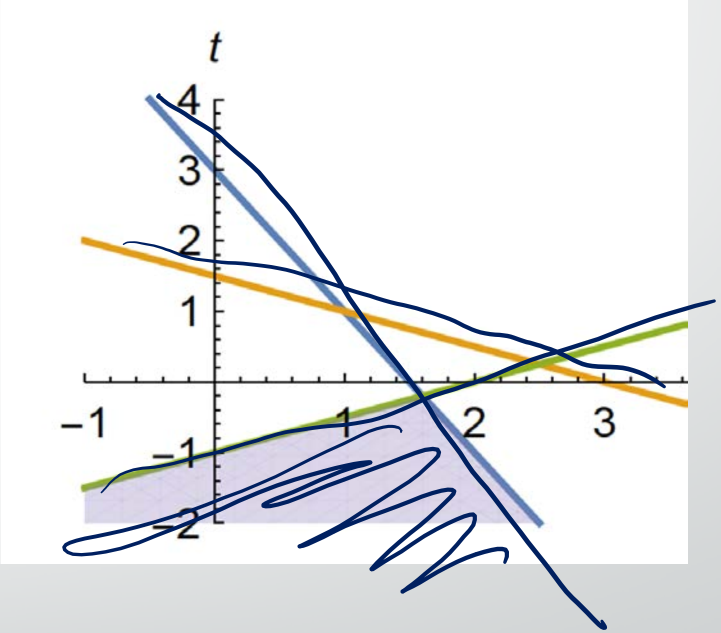
- Fact: The maximum of a set of linear functions is always convex!
  - Implies that the max of a set of linear functions is a *convex piecewise linear function*
  - If we are minimizing the function, we can use the epigraph trick!

$$\min_x \max_{i=1,\dots,k} \{a_i^T x \quad b_i\}$$

$$\begin{aligned} &\min_{x,t} \ t \\ &\text{s.t.} \quad t \geq a_i^T x + b_i \ \ \forall i = 1, \dots, k \end{aligned}$$

Another special case: maximin problems

$$f(x) = \min_{i=1,\ldots,k} \{a_i^T x + b_i\}$$



- Fact: The minimum of a set of linear functions is always concave!
  - Implies that the min of a set of linear functions is a *concave piecewise linear function*
  - If we are maximizing the function, we can use the epigraph trick!

$$\max_{x} \min_{i=1,\ldots,k} \{a_i^T x \;\; b_i\}$$

$$\begin{aligned} &\max_{x,t} \; t \\ &\text{s.t.} \quad t \le a_i^T x + b_i \;\; \forall i = 1, \ldots, k \end{aligned}$$

One more special case: absolute value problems

- The function $f(x) = |x|$ can be rewritten as:

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- This is a convex piecewise linear function!
- If we minimize $f(x)$ we can use the epigraph trick:

$$\min_{x \in \mathbb{R}} \ |x| \quad \text{s.t.} \quad Ax \geq b$$

$$\Rightarrow$$

$$\begin{aligned} \min_{x,t} \ & t \\ \text{s.t.} \ & Ax \geq b \\ & t \geq x \\ & t \geq -x \end{aligned}$$

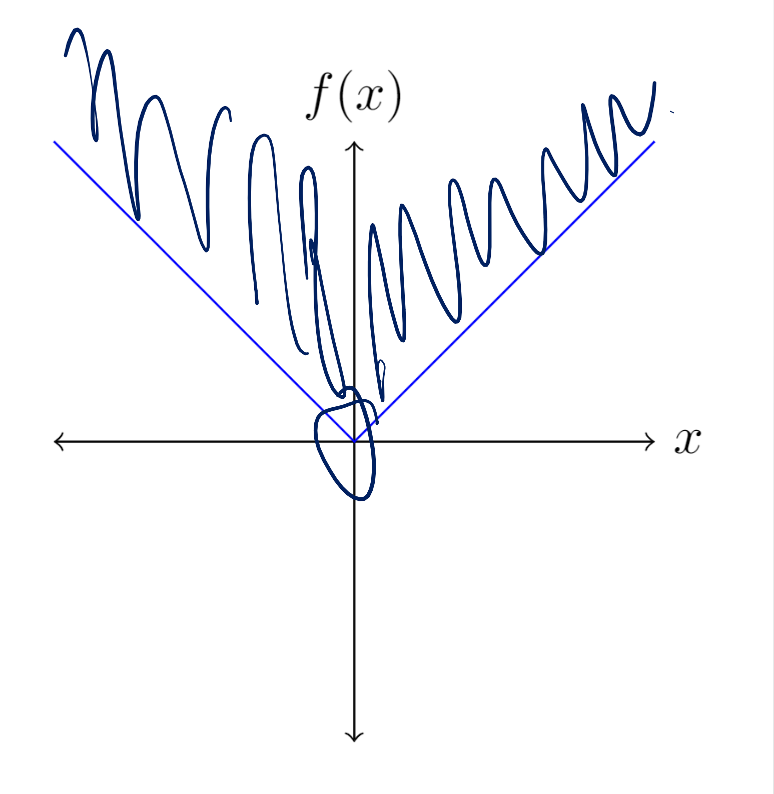
- This even works for sums of absolute values:

$$\min_{x,y \in \mathbb{R}} \ |x| + |y| \quad \text{s.t.} \quad Ax \geq b$$

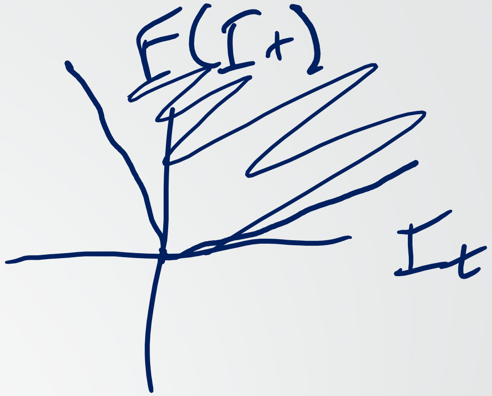
$$\Rightarrow$$

$$\begin{aligned} \min_{x,t,s} \ & t + s \\ \text{s.t.} \ & Ax \geq b \\ & t \geq x; t \geq -x \\ & s \geq y; s \geq -y \end{aligned}$$

$f(x) = |x|$



Back to ShoeCo….



- Recall the function for inventory costs with backlogging is:

$$F(I_t) = \begin{cases} 3I_t & \text{if } I_t \geq 0 \\ -20I_t & \text{if } I_t < 0 \end{cases}$$

- This is a piecewise linear function of inventory and we are minimizing costs
  - Question: do we want the function to be convex or concave? (And which is it?)
- Since this is a convex piecewise linear function and we are minimizing, we can use the epigraph trick!

Reformulate as:

$$\begin{aligned} \min_{I,\tau,\ldots} \ & \textstyle\sum_{t\in \mathrm{T}} \tau_t + \text{other costs} \\ \text{s.t.} \ & \tau_t \geq 3I_t \ \ \forall t = 1, 2, 3, 4 \\ & \tau_t \geq -20I_t \ \ \forall t = 1, 2, 3, 4 \\ & \text{other constraints} \end{aligned}$$

An equivalent formulation

In Julia:
ShoeCo with backlogging

Another way to reformulate $F(I_t)$:

1. Introduce new variables L_t and S_t (think of these as "leftover" and "shortage" variables)
2. Both variables nonnegative: $L_t \geq 0, S_t \geq 0$
3. Define inventory level as (Leftover – Shortage): $I_t = L_t - S_t$
4. The cost function can be written as: $\sum_{t \in T}(3L_t + 20S_t)$

Why does this work?

- If we're *minimizing* costs, only one of L_t and S_t will ever be positive in an optimal solution

Some notes:

- This trick works for any convex piecewise linear function (of x) we are minimizing
- We can even use this trick for constraints of the form $|x| \leq b$
  - I.e., model $|x| \leq b$ as $x = w - v; w, v \geq 0; w + v \leq b$

A minimax example

Han Solo is flying the *Millennium Falcon* through an asteroid field. His passengers' stress levels are functions of the speed Han flies (we'll call the speed x):

- Leia: $-2x + 20$
- Chewbacca: $-\frac{1}{10}x + 10$
- C-3Po: $x + 1.75$

To make his life easier, Han wants to minimize the maximum stress level of his passengers. The *Falcon* has a maximum speed of 12 parsecs/min. He must go at least 6 parsecs/min to outrun the Empire. How fast should Han fly?

In Julia: Han's Stressful Flight