Lecture 2 – Sensitivity Analysis

Module 6 – Duality in Linear Programming
CS/ISyE/ECE 524

Sensitivity Analysis – back to Top Brass *again*

Primal problem

$$\max_{x_s,x_f}\ 12x_f + 9x_s$$
$$\text{s.t. } 4x_f + 2x_s \le 4800$$
$$x_f + x_s \le 1750$$
$$x_f \le 1000$$
$$x_s \le 1500$$
$$x_f, x_s \ge 0$$

Dual problem

$$\min_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}\ 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$
$$\text{s.t. } 4\lambda_1 + \lambda_2 + \lambda_3 \ge 12$$
$$2\lambda_1 + \lambda_2 + \lambda_4 \ge 9$$
$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$$

Sensitivity analysis is a tool to understand how we can (or can't) affect our solution by changing specific parameters.

For example: Suppose we had the option to buy more wood at a price of $1 per board foot. Should we buy more wood? If so, how much?

Sensitivity Analysis – changing primal constraints

Primal problem

$$\max_{x_s, x_f} \; 12x_f + 9x_s$$

$$\text{s.t. } 4x_f + 2x_s \leq 4800$$
$$x_f + x_s \leq 1750$$
$$x_f \leq 1000$$
$$x_s \leq 1500$$
$$x_f, x_s \geq 0$$

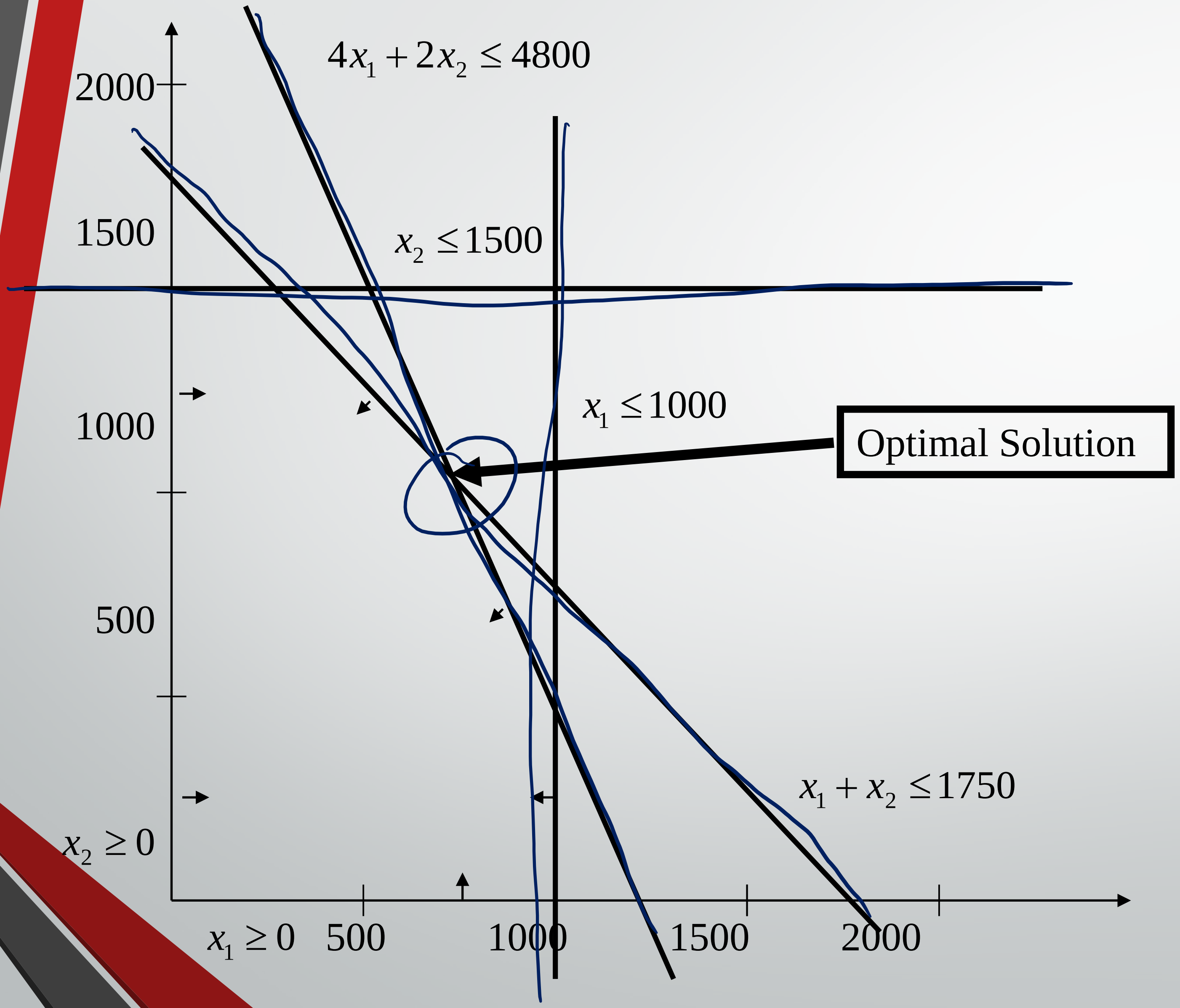
Dual problem

$$\min_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \; 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

$$\text{s.t. } 4\lambda_1 + \lambda_2 + \lambda_3 \geq 12$$
$$2\lambda_1 + \lambda_2 + \lambda_4 \geq 9$$
$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

- When we change a primal *constraint*, we change the dual *cost*
- As long as we don't change the constraint "too much," the dual variable values don't change (but the optimal cost does!)
- Because p^* = d^* (by Strong Duality), changing 4800 to 4801 increases *both* p^* and d^* by λ_1

Graphical solution of Top Brass



How do we interpret this?

- We are prevented from making more trophies by the constraints

$4x_1 + 2x_2 \le 4800$ and $x_1 + x_2 \le 1750$

- We are *not* prevented from making more trophies by the number of brass soccer balls or the number of brass footballs
- We would be willing to pay up to \$(increase in profit)/(increase in rhs) per unit for more wood or more plaques
  - This value is called the **shadow price** of the constraint
- We certainly wouldn't want to pay anything for more brass balls

What are shadow prices?

Question: How do we find shadow prices?

Let's think about what the units are on our variables:

- In Top Brass, x_f and x_s (the primal variables) have units of "number of football trophies" and "number of soccer trophies." The total profit, then, is

  (\$) = (12 \$/football trophy)*(x_f football trophies) + (9 \$/soccer trophy)*(x_s soccer trophies)

- Dual variables have units too. We can figure out what they should be from the cost function:

  (\$) = (4800 ft of wood)*(λ_1?/?) + (1750 plaques)*(λ_2?/?) + …

  - The units on λ_1 must be \$/ft of wood. λ_2 must be \$/plaque

- How does this help?

Previously, we figured out if we increase the available wood by 1 board foot, we increase the profit by $\$1 * \lambda_1$, so the shadow price is $(1 * \lambda_1)/1 \; = \; \lambda_1$. ***The dual variable values are the shadow prices!***

Original question: Suppose we had the option to buy more wood at a price of $1 per board foot. Should we buy more wood? If so, how much?

Bringing it all together in Top Brass

- Recall that at an optimal dual solution we have:

$$\lambda_1 = 1.5; \quad \lambda_2 = 6; \quad \lambda_3 = 0; \quad \lambda_4 = 0$$

- Now that we know about shadow prices, we can see we'd be willing to pay up to $1.50/ft for more wood
  - So $1/ft is a good deal for us!
- We'd be willing to pay up to $6/plaque for more plaques
- We'd be willing to pay $0 for more brass footballs or brass soccer balls
  - This makes sense from the graphical solution: these constraints weren't binding, so having more won't do anything for us

But how much can we buy before this is no longer a good deal?

Conceptually: We should increase the right-hand side of the wood constraint – which corresponds to increasing the "wood" dual objective coefficient – until $\lambda_1 = 1.5$ is no longer optimal

Sensitivity Analysis – changing primal costs

Primal problem

$$\max_{x_s, x_f} \; 12x_f + 9x_s$$

$$\text{s.t. } 4x_f + 2x_s \le 4800$$
$$x_f + x_s \le 1750$$
$$x_f \le 1000$$
$$x_s \le 1500$$
$$x_f, x_s \ge 0$$

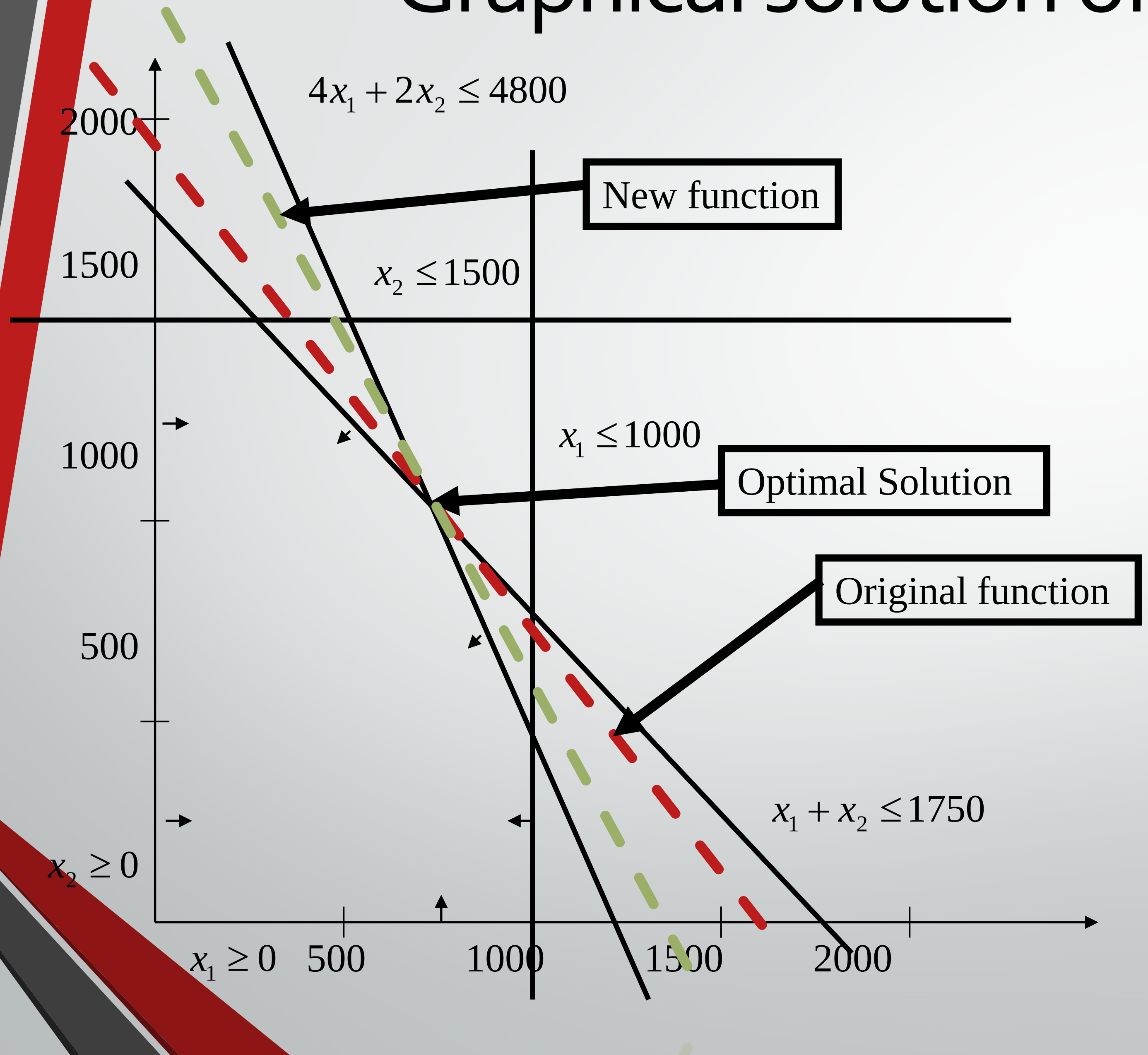
Dual problem

$$\min_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \; 4800\lambda_1 + 1750\lambda_2 + 1000\lambda_3 + 1500\lambda_4$$

$$\text{s.t. } 4\lambda_1 + \lambda_2 + \lambda_3 \ge 12$$
$$2\lambda_1 + \lambda_2 + \lambda_4 \ge 9$$
$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$$

- When we change a primal *cost*, we change the dual *constraint*
- The feasible region doesn't change when we change the primal objective
  - Increase in coefficients means value will not decrease
  - Decrease in coefficients means value will not increase
- Because $p^* = d^*$, changing 12 to 13 increases *both* p^* and d^* by x_f

Graphical solution of Top Brass



What happens if we change the cost function?

- Feasible region stays the same
- *Slope* of objective function changes
- As long as we don't change the slope "too much," the same point stays optimal
  - Just need to plug in variable values and calculate new cost
  - As before, we won't look at what "too much" means (just solve new model if you're worried)

Sensitivity in general: Primal Constraints

Primal problem

$$\max_x c^T x$$
$$\text{s.t. } Ax \leq b + \epsilon$$
$$x \geq 0$$

Dual problem

$$\min_\lambda (b+\epsilon)^T \lambda$$
$$\text{s.t. } A^T\lambda \geq c$$
$$\lambda \geq 0$$

What happens if we add a vector ϵ to the right-hand side vector b?

- The optimal x^* (and therefore p^*) may change since we are changing the feasible set of (P). Let's say the new values are x'^* and p'^*
- As long as ϵ is "small enough", the optimal λ won't change (since the feasible set of (D) isn't changing)
- Before: $p* = b^T\lambda^*$. After: $p'^* = d'^* = b^T\lambda^* + \epsilon^T\lambda^*$
- Therefore: $(p'^* - p^*) = \epsilon^T\lambda^*$

Sensitivity in general: Primal Objective Coefficients

Primal problem

$$\max_{x}\ (c+\epsilon)^T x$$
$$\text{s.t. } Ax \le b$$
$$x \ge 0$$

Dual problem

$$\min_{\lambda}\ b^T\lambda$$
$$\text{s.t. } A^T\lambda \ge c+\epsilon$$
$$\lambda \ge 0$$

What happens if we add a vector ϵ to the cost vector c?

- As long as ϵ is "small enough", the optimal x^* *will not change* since the feasible set of (P) isn't changing.
- The new $p'^* = c^T x^* + \epsilon^T x^*$
- Before: $d^* = p^* \ = \ b^T\lambda^*$. After: $p'^* \ = \ d'^* \ = c^T x^* + \epsilon^T x^*$
- Therefore: $(p'^* \ - \ p^*) \ = \epsilon^T x^*$
- The dual variable values change, because the dual feasible region changes!