Lecture 1 – Multiperiod Planning Problems

Module 5 – Special Cases of Linear Programs

CS/ISyE/ECE 524

Special Cases of Linear Programs Module Learning Outcomes

By the end of this module, you should be able to…

- Recognize the multi-period planning problem structure
  - Build a linear programming model of a multi-period planning problem
  - Give an example of a multi-period planning problem
- Reformulate a model with piecewise linear objective into a standard form linear programming model
- Recognize the "minimax" (maximin) problem structure
  - Build a linear programming model of a minimax problem
  - Give an example of a minimax problem

Special Cases of Linear Programs Module Learning Outcomes

By the end of this module, you should be able to…

- Give basic definitions of combinatorial optimization problems and be able to describe how they relate to network flow problems
  - Graph
  - Node (vertex)
  - Arc (edge)
  - Flow
- Recognize when a problem falls into the network flow structure and build a linear programming model of the problem
- Give examples of different types of network flow problem (e.g., shortest path, assignment, max flow, matching)
- Explain the concept of total unimodularity
  - Understand the relationship between TU matrices and MCNF problems

Multi-Period Planning Problems

What are those?!?!
(Or: A shoe-based motivating example)

ShoeCo needs to plan their production of shoes for the next 4 months after introducing some new kicks. Specifically, ShoeCo needs to:

- Meet the expected monthly shoe demand on time
- Hire and/or lay off workers at the beginning of each month
- Make overtime decisions
- Minimize total cost of operations

(Yes, this is a Dead Meme.
No, I will not stop using it.)

ShoeCo is a ***multi-period planning problem***

- These are optimization problems that have a **temporal** component
- Decisions must be made over a **sequence of discrete time periods**
  - Most multi-period planning problems have a "planning horizon":
  $$T = \{1, 2, \ldots, T^{\max}\}$$
- Usually distinguished by **inventory** or **carry-over** variables
  - Decisions in each time period are **coupled** and must be jointly optimized
  - Decisions that look really good now might negatively affect the future
- These problems can be tricky! Decision variables not always obvious
  - **Decision variables are not always things you decide directly!**

Let's get specific…

Your contact at ShoeCo has given you the following information*:

- We have a 4-month planning horizon:
$$T = \{1, 2, 3, 4\}; T^{\max} = 4$$
- We must meet demand for shoes each month:
$$d_1 = 3000; d_2 = 5000; d_3 = 2000; d_4 = 1000$$
- We currently have 500 shoes in inventory: $I_0 = 500$
- We currently have 100 workers employed: $W_0 = 100$
- Workers are paid $1500/month for working 160 hours/month
- Workers can work overtime (max of 20 hours/worker/month) at a rate of $13/hour

*To avoid confusion, assume “shoes” here refers to “pairs of shoes” (shoes would never be sold individually)

Let's get specific...

- To make a pair of shoes, we need 4 hours of labor and $15 worth of raw material
- It costs $1600 to hire a new worker and $2000 to fire a worker
- It costs $3 to hold a pair of shoes in inventory at the end of a month

Your mission, should you choose to accept it:

Minimize all costs

- Costs include labor, production, hiring, firing, inventory

What decision variables do we need?

- **Hint:** Try writing the objective function if you are having trouble thinking of the variables

We need (at least) the following decision variables

- x_t: # of shoes to produce in month $t = 1, \ldots, 4$
- I_t: Ending inventory of shoes in month $t = 0,1, \ldots, 4$
- w_t: # workers available in month $t = 0,1, \ldots, 4$
- o_t: # overtime hours used in month $t = 1, \ldots, 4$
- h_t: # workers hired at the beginning of month $t = 1, \ldots, 4$
- f_t: # workers fired at the beginning of month $t = 1, \ldots, 4$

Objective: minimize total costs

- Raw material costs: $\sum_{t \in T} \$15 x_t$
- Regular labor costs: $\sum_{t \in T} \$1500 w_t$
- Overtime labor costs: $\sum_{t \in T} \$13 o_t$
- Hiring costs: $\sum_{t \in T} \$1600 h_t$
- Firing costs: $\sum_{t \in T} \$2000 f_t$
- Inventory costs: $\sum_{t \in T} \$3 I_t$

Constraints on shoe production

Limit on total monthly production

- No explicit upper bound given
- Implicitly determined by workers available:

$$4x_t \leq 160w_t + o_t \quad \forall t = 1, 2, 3, 4$$

- Or, in general:

$$\alpha x_t \leq Hw_t + o_t \quad \forall t \in T$$

Where α is a parameter for how many labor hours are needed to make a pair of shoes and H is a parameter for the total hours/worker available each month.

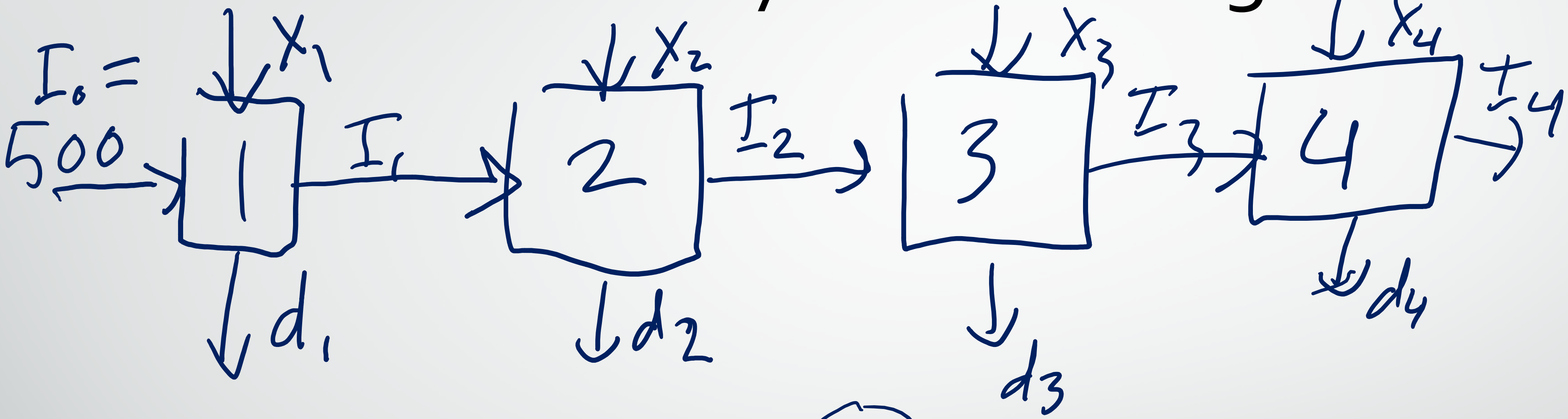
Demand must be met (on time)

- Equivalent to non-negative ending inventory every month (i.e., no backlogging): $I_t \geq 0 \quad \forall t \in T$
- This means our inventory must be a function of the production and demand each month:

$$I_{t-1} + x_t = d_t + I_t \quad \forall t \in T$$

- We also have a starting inventory: $I_0 = 500$

The inventory balance diagram



$$500 + x_1 = I_1 + d_1$$
$$I_1 + x_2 = I_2 + d_2$$
$$I_2 + x_3 = I_3 + d_3$$
$$I_3 + x_4 = I_4 + d_4$$

Flow balance

$$I_{t-1} + x_t = d_t + I_t$$

Constraints on shoe production

Upper bound on overtime hours/month

- Depends on the number of workers
- We are planning in *aggregate* (not tracking individual workers), so we just need:

$$o_t \leq 20w_t \quad \forall t = 1, 2, 3, 4$$

- Or, in general:

$$o_t \leq Ow_t \quad \forall t \in T$$

Where O is a parameter for the max overtime hours/worker/month.

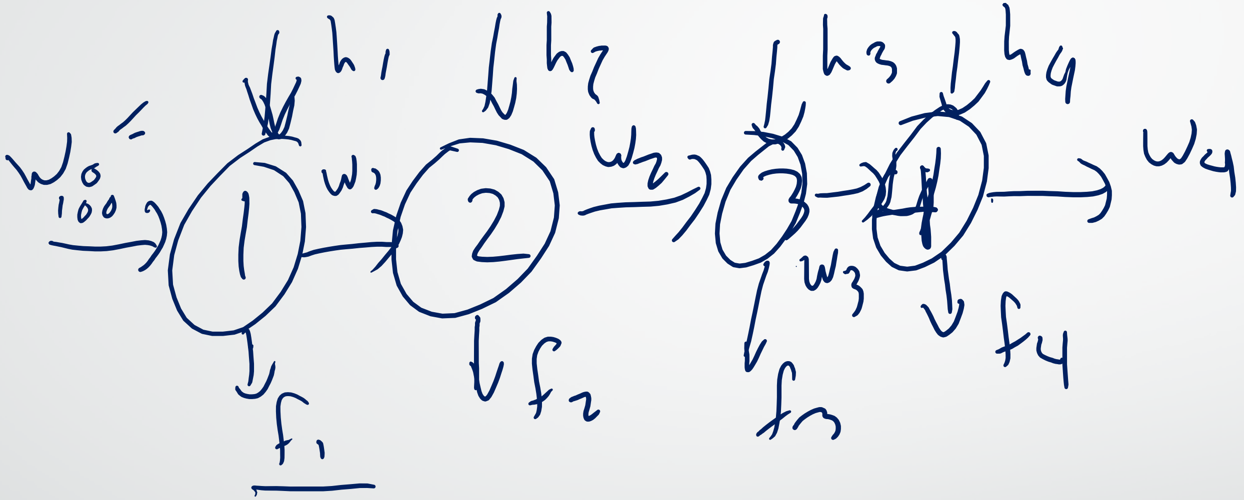
We have to "balance" the number of workers

- Works the same way as inventory balance:

$$w_t = w_{t-1} + h_t - f_t \quad \forall t \in T$$

- We have an initial number of workers: $w_0 = 100$

The worker balance diagram



$$w_1 + h_2 = w_2 + f_2$$

$$w_2 = \underline{w_1} + \underbrace{h_2 - f_2}$$

Full Math Model of ShoeCo Problem

$$\min \sum_{t \in T} (15x_t + 1500w_t + 13o_t + 1600h_t + 2000f_t + 3I_t)$$

$$\text{s.t.} \quad \alpha x_t \le Hw_t + o_t \qquad \forall t \in T$$

$$o_t \le Ow_t \qquad \forall t \in T$$

$$I_{t-1} + x_t = d_t + I_t \qquad \forall t \in T$$

$$w_t = w_{t-1} + h_t - f_t \qquad \forall t \in T$$

$$I_0 = 500$$

$$w_0 = 100$$

$$x_t, I_t, w_t, h_t, f_t \ge 0 \qquad \forall t \in T$$

ShoeCo.ipynb

Backlogging

Now suppose we don't have to meet the forecast demands in every period

Often too stringent a requirement in the real world!

But we must meet it *eventually*

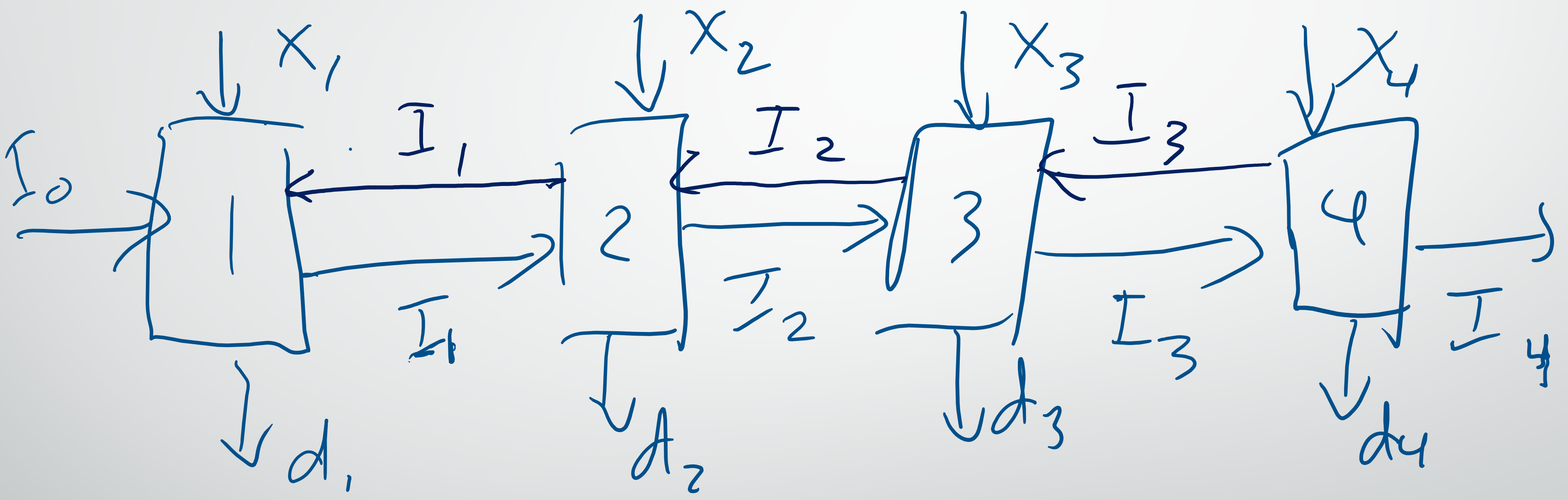
Suppose we have a *shortage cost* of $20/unit per month

Think of this as allowing inventory to go negative

This is called **backlogging**

Question: How should the minimum cost compare to the original ShoeCo minimum cost?

Backlogging inventory balance diagram



Modeling Backlogging

- The new function for inventory costs (incorporating shortage costs) is:

$$F(I_t) = \begin{cases} 3I_t \ \text{ if } I_t \geq 0 \\ -20I_t \ \text{ if } I_t < 0 \end{cases}$$

- To enforce the requirement that all demand is met *eventually*, we simply add the constraint:

$$I_4 \geq 0 \ \ (I_{T^{max}} \geq 0 \text{ in general})$$

Question: Is this a linear function of inventory?

No. But it is a convex piecewise linear function of inventory that we are minimizing! (So??)