solutions01

September 16, 2016

0.1 Linear regression

0.1.1 Part A

$f(\beta) = \sum_{i=1}^{N} \{ \frac{\omega_i}{2} (y_i - x_i\beta)^2 \}$
$= (y - X\beta)^T \omega (y - X\beta)$
$= y^T \omega y - 2y^T \omega X\beta + \beta^T X^T \omega X\beta$
Therefore,
$\frac{\partial f\beta}{\partial \beta} = 0 - 2y^T \omega X + 2\beta^T X^T \omega X = 0$
$X^T \omega y = (X^T \omega X)\beta$

0.1.2 Part B

pseudo code for my method (Cholesky Factorization) $X^T W X = L L^T $

```
A = X^T %*% W %*% X
b = X^T %*% W %*% y
L %*% L^T = X^T * W * X  # decomposation
beta = L^(-1) %*% (L^T)^(-1) %*% b
```

0.1.3 Part C

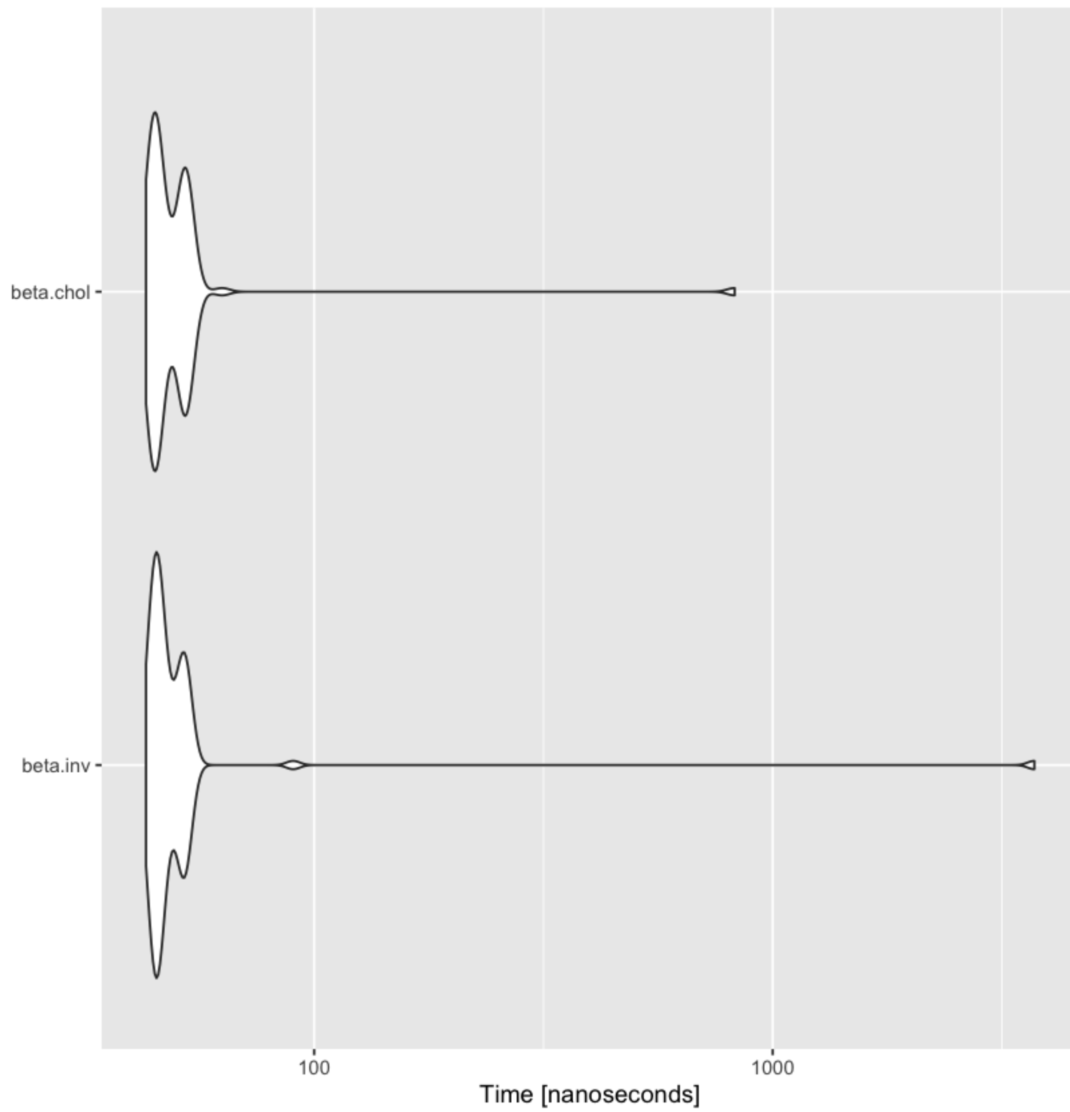
In [23]:
```
# generate simulated data
P = 1000  # variable size
N = 2000  # sample size
set.seed(1)
beta = rep(1,P)
X = matrix(rnorm(N*P), nrow=N)
y = X%*%beta + rnorm(N,sd=0.05)
W = diag(rep(1, N))
```

In [24]:
```
# import inversion and Cholesky methods
source("~/Box Sync/PhDCourses/SDS385Statistical_models_for_big_data/SDS385/solutions/exercises
library('microbenchmark')
library('ggplot2')
```

In [25]:
```
# experiment on inversion method
beta.inv = inversion.method(X, y, W)
```

In [26]:
```
# experiment on Cholesky method
beta.chol = Cholesky.method(X, y, W)
```

In [27]:
```
# timing
time = microbenchmark(beta.inv, beta.chol, times=100)
autoplot(time)
```



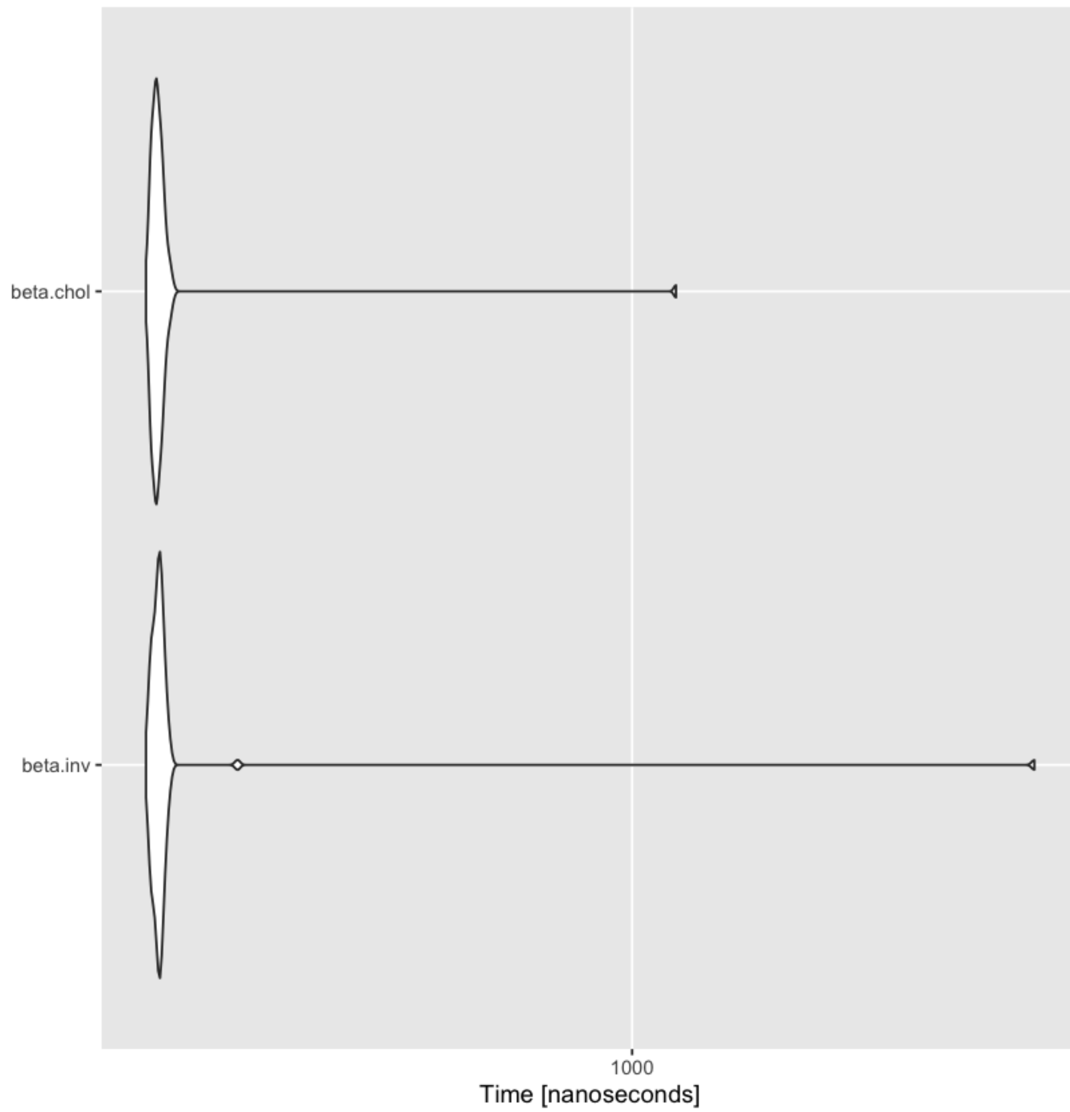
0.1.4 Part D

```
In [28]: # generate sparse data
         P = 1000  # variable size
         N = 2000  # sample size
         sparse = 0.1 # dense of sparse matrix
         set.seed(1)
         beta = rep(1,P)
         X = matrix(rnorm(N*P), nrow=N)
         mask = matrix(rbinom(N*P, 1, sparse), nrow=N)
         X = mask * X
         y = X%*%beta + rnorm(N,sd=0.05)
         W = diag(rep(1, N))
```

```
In [29]: # experiment on inversion method for sparse data
         beta.inv = inversion.method(X, y, W)
```

```
In [30]: # experiment on Cholesky method for sparse data
         beta.chol = Cholesky.method(X, y, W)
```

```
In [31]: # timing
         time = microbenchmark(beta.inv, beta.chol, times=100)
         autoplot(time)
```



0.2 Generalized linear regression

0.2.1 Part A

Negative log likelihood:

$$l(\beta) = -log\{\prod_{i=1}^{N} p(y_i|\beta)\}$$
$$= -log\{\prod_{i=1}^{N} C * \omega_i^{y_i} * (1-\omega_i)^{m_i - y_i}\}$$

$= -\sum_{i=1}^{N}\{logC + y_i * log\omega_i + (m_i - y_i) * log(1-\omega_i)\}$
$= -\sum_{i=1}^{N}\{y_i * log\frac{\omega_i}{1-\omega_i} + m_i * log(1-\omega_i) + C\}$
$= -\sum_{i=1}^{N}\{y_i * log\frac{\omega_i}{1-\omega_i} + m_i * log(1-\omega_i) + C\}$
$= -\sum_{i=1}^{N}\{y_i * loge^{x\beta} - m_i * log(1+e^{x\beta})\}$
Gradient of the expression:
$\nabla l(\beta) = \frac{\partial l(\beta)}{\partial \beta}$
$= -\sum_{i=1}^{N}\{\frac{y_i}{\omega_i*(1-\omega_i)} - \frac{m_i}{1-\omega_i}\}\frac{\partial \omega_i}{\partial \beta}$
$\frac{\partial \omega_i}{\partial \beta} = \frac{e^{x_i*\beta}*x_i}{(1+e^{x_i*\beta})^2}$
Therefore, $\nabla l(\beta) = -\sum_{i=1}^{N}(y_i - m_i * \omega_i) * x_i = -X^T * (y - m * w)$

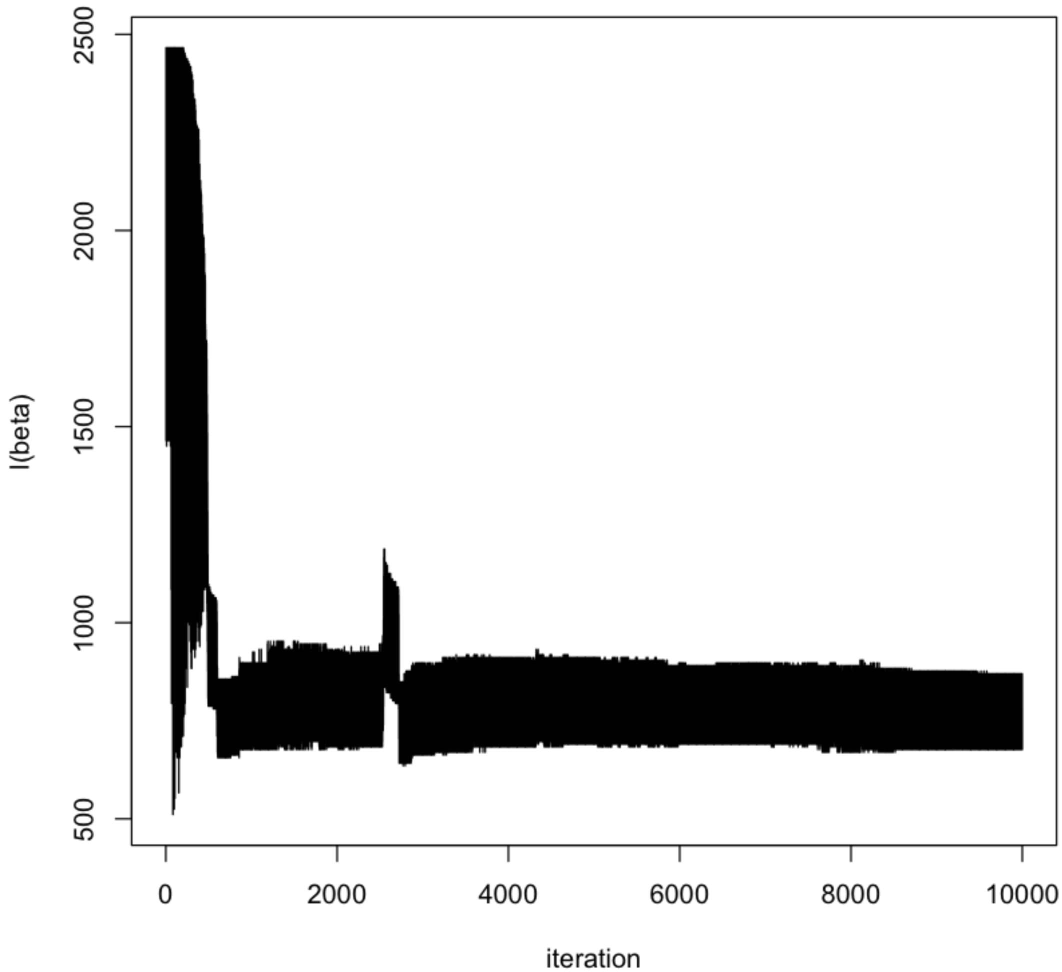
0.2.2 Part B

In [21]:
```
# import functions for gradient descent
source('~/Box Sync/PhDCourses/SDS385Statistical_models_for_big_data/SDS385/solutions/exercises(
```

In [22]:
```
setwd('~/Box Sync/PhDCourses/SDS385Statistical_models_for_big_data/SDS385/data')
# import wdbs.csv data
data <- read.csv('wdbc.csv', header=F)
y.BM <- data[ ,2]
y <- rep(0, length(y.BM))
y[y.BM == "B"] <- 1
y <- as.matrix(y)
x.variables <- as.matrix(data[ , 3:12])
x <- cbind(x.variables, rep(1,dim(x.variables)[1]))
colnames(x) <- NULL
```

In [24]:
```
# gradient descent
betas = matrix(1, 11, 1)
GD.results <- gradient.descent(x, y, betas, step.size=0.01, max.iter=10000)
```

In [26]:
```
# plot l(beta)
l.beta.GD = GD.results[[2]]
plot(1:10000, l.beta.GD, type='l', xlab='iteration', ylab='l(beta)')
```

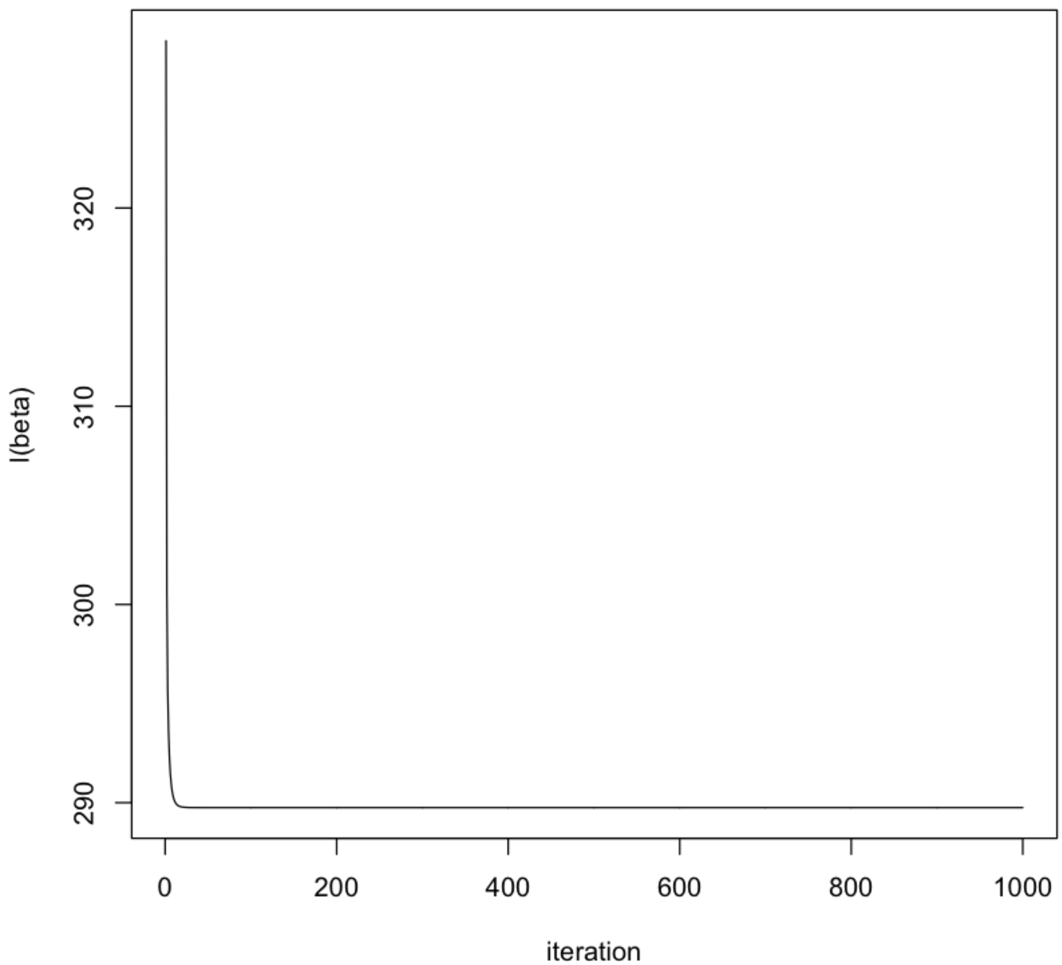


The log likelihood value never converage even with 10k iterations when using `wdbc.csv` data. So next I tried gradient descent algorithm on simulated data –

In [18]:
```
# generate simulated data
set.seed(666)
x1 = rnorm(1000)  # some continuous variables
x2 = rnorm(1000)
x = cbind(as.matrix(x1), as.matrix(x2))
x= cbind(x, rep(1,1000))  # add one column for intercept
z = 1 + 2*x[ ,1] + 3*x[ ,2]  # linear combination with a bias
prob = 1/(1+exp(-z))  # pass through an inv-logit function
y = as.matrix(rbinom(1000,1,prob))
```

In [20]:
```
# gradient descent on simulated data.
betas = matrix(1, 3, 1)
GD.results <- gradient.descent(x, y, betas, step.size=0.01, max.iter=1000)
```

```
In [17]: # plot l(beta)
         l.beta.GD = GD.results[[2]]
         plot(1:1000, l.beta.GD, type='l', xlab='iteration', ylab='l(beta)')
```



0.2.3 Part C

The second-order Taylot approximation:

$q(\beta;\beta_0) = l(\beta_0) + (\beta-\beta_0)l'(\beta) + \frac{1}{2}(\beta-\beta_0)^2 l''(\beta)$

Derive $l''(\beta)$ from $l'(\beta)$

$l'(\beta) = -\sum_{i=1}^{N}\{x_i^T(y_i - m_i * \omega_i)\}$

$l''(\beta) = -\sum_{i=1}^{N}\{x_i^T(0-m_i) * \frac{\partial \omega_i}{\partial \beta}\}$

$= \sum\{x_i^T m_i \frac{e^{x_i\beta}}{1+e^{x_i\beta}} x_i \frac{1}{1+e^{x_i\beta}}\}$

$= \sum\{x_i^T m_i \omega_i(1-\omega_i)x_i\}$

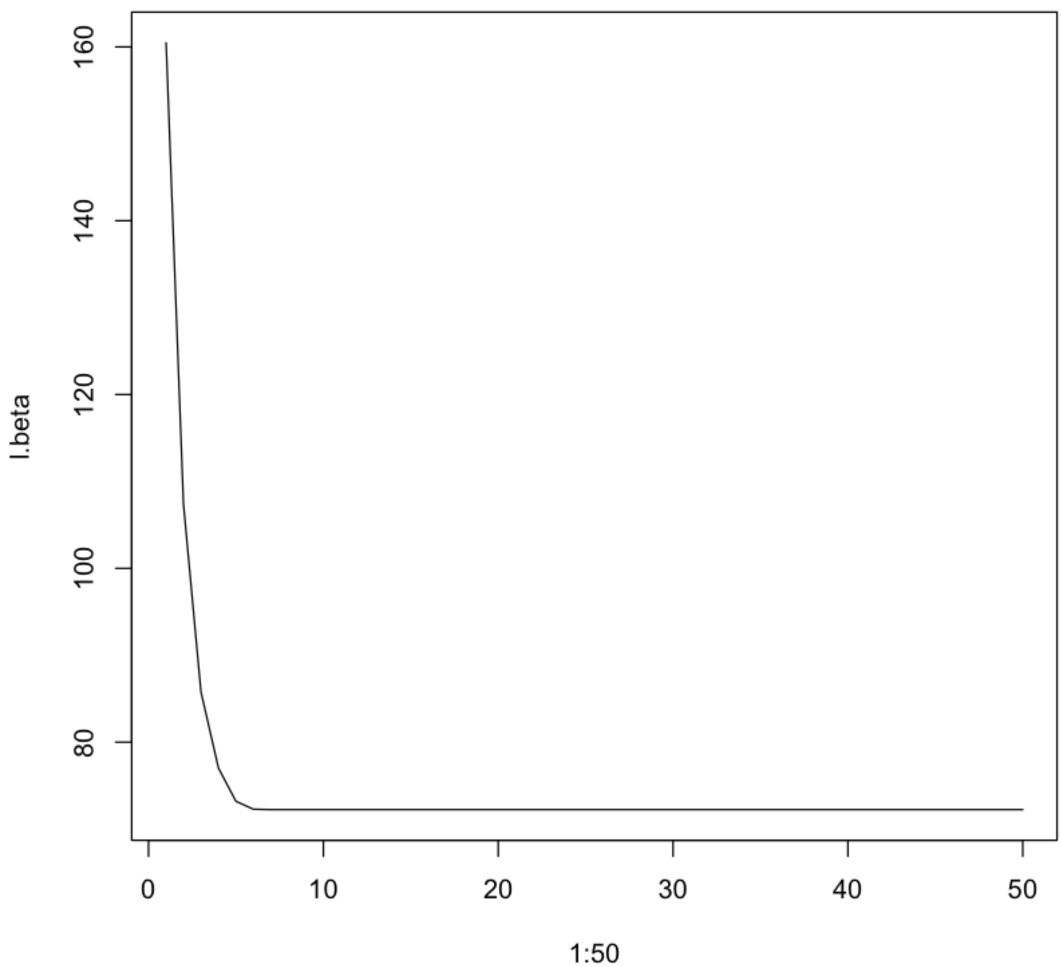
Therefore, $l''(\beta) = X^T SX$, where S is a diagonal matrix with elements $m_i\omega_i(1-\omega_i)$
Thus,
$q(\beta;\beta_0) = l(\beta_0) - (\beta-\beta_0)^T X^T(y - m_i\omega_i) + \frac{1}{2}(\beta-\beta_0)^T X^T SX(\beta-\beta_0)$
$q(\beta;\beta_0) = C - (\beta^T X^T - \beta_0^T X^T)(y - m_i\omega_i) + \frac{1}{2}(\beta^T X^T - \beta_0^T X^T)S(X\beta - X\beta_0)$
$= \frac{1}{2}(X\beta - X\beta_0 - S^{-1}(y-m\omega))^T S(X\beta - X\beta_0 - S^{-1}(y-m\omega)) + C$
Compare the above equation with the original format –
$z = X\beta + S^{-1}(y-m\omega)$
$W = S$

0.2.4 Part D

In [24]:
```
# import function for Newton's method.
source('~/Box Sync/PhDCourses/SDS385Statistical_models_for_big_data/SDS385/solutions/exercises(
```

In [25]:
```
# Newton's method
betas = matrix(0.0001, 11, 1)
newton.results <- my.newton(x, y, betas, max.iter=50)
```

In [26]:
```
# plot l(beta)
l.beta.newton = newton.results[[2]]
plot(1:50, l.beta.newton, type='l')
```



l.beta
160
140
120
100
80
0
10
20
30
40
50
1:50