CQF Lecture One

The Random Behaviour of Assets

The log-normal random walk.

$$dS = \mu S \, dt + \sigma S \, dX$$

In class, we have looked at the Stochastic Differential Equation as a popular model for stock price S.

On the set of prices from Lecture Excel file (SP500.xlsx) or any equity/index of your own choice, test the robustness of the assumption that

$$R_i = \left[\mu \delta t + \sigma \sqrt{\delta t} \, \phi_i\right]$$

or in the absence or negligible drift, $\mu \delta t \approx 0$

$$R_i = \sigma \sqrt{\delta t} \, \phi_i .$$

R_i represents the returns over timestep δt, the $\phi_i \sim N(0,1)$ is standard Normal variable, and μ, σ are drift and diffusion assumed to be constant.

1. Scaling of σ estimate to the size of δt: compute in own column 1D returns, 2D returns and 5D returns, the latter will be $R_i = \frac{S_{t+5} - S_t}{S_t}$. For each of these columns, compute sample std dev. Adjust 2D returns std dev by $/\sqrt{2}$, 5D std dev by $/\sqrt{5}$ – are these comparable to 1D returns std dev?

2. Re-shuffle the dataset into two halves (eg, to even / odd observations) and compute μ, σ separately for each half (1D returns only). Compare.

3. Construct Quantile-Quantile plots for 1D and 5D returns. The Q-Q plot assumes Normal distribution on horizontal axis – the better the fit between two distributions (empirical and theoretical Normal), the more observations will be on the diagonal straight line.

4. Construct a histogram over historical returns scaled to be Normal (z-scores) and compare to Normal distribution (eg, histogram obtained over simulated $\phi_i \sim N(0,1)$).

$$R_i = \frac{\delta S}{S} = \frac{S_{t+1} - S_t}{S_t} = \frac{S_{t+1}}{S_t} - 1$$

In class, we computed the *simple returns* for clarity and to operate in discrete time. However, *log-returns* are supposed to be Normal under the SDE, a continuous-time model.

The exercises have been edited by CQF Faculty, Richard.Diamond@fitchlearning.com

The Q-Q plot.

In explicit steps is straightforward in Excel to build, because you can organise the data in matching columns. 'Historic' stands for the actual S&P 500 return, matched by 'Scaled' return (z-score), indexes and the Standard Normal Percentile.

The percentile corresponds to the cumulative probability given by i/N, this will be raising until $N/N = 1$. Illustration below has one past negative return that was in excess of 21 standard deviations!

Historic	Scaled	i	i/N	Standard
-0.22900	-21.20462	1	0.00009	-3.74534
-0.09470	-8.78146	2	0.00018	-3.56758
-0.09354	-8.67429	3	0.00027	-3.45987
-0.09219	-8.54970	4	0.00036	-3.38162
-0.08642	-8.01584	5	0.00045	-3.31983
-0.07922	-7.35036	6	0.00054	-3.26858
...	...	...	...	...

Table 1: Inputs for a Q-Q plot.

1. Scale historic log-returns R_t to be the Normal variables $Z_t = \frac{r_t - \mu}{\sigma}$. For a market index the average daily return $\mu \approx 0$, that is increasingly valid for a large sample.

2. Sort the scaled returns in the ascending order and create an index column $i = 1..N$.

3. The cumulative density – percentage of observations below this – will be i/N. Each observation 'adds' density (probability mass) of $1/N$.

4. The standardised percentile is obtained with the inverse Normal CDF $\Phi^{-1}(i/N)$.

Plot the scaled returns (Z-scores) from Step 1 against the Normal percentiles from Step 4. For the perfectly Normal log-returns the Q-Q plot would be a straight line.

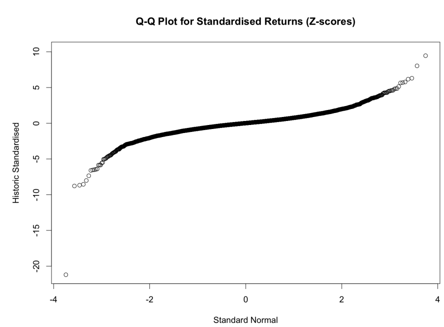

Figure 1: Q-Q Plot built step by step.