CQF FitchLearning

TUTORIAL

Statistical Essentials for VaR and ES

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Learning outcomes

- understand the first principles: **inverse percentile** for Normal Distribution, **conditional expectation** for ES (CVaR)
- be able to read probability notation maths for VaR and ES
- EXCEL walkthrough via VaR Backtesting
- NO PYTHON released today.

These slides might be released AFTER tutorial.

During the session, we refer to Market Risk Lecture SOLUTIONS – please download from CQF Portal.

How to use

- While we re-visit relevant first principles, expectations algebra, the tutorial is <u>not a substitute</u> for the core lectures, eg on Market Risk and VaR. If you have not worked with the material of the core lecture and not reviewed its <u>Exercises/Solutions</u> – you will experience difficulty to follow the material.
- The tutorial is delivered ‘from the desk’ and typically reviews a computational implementation (Excel, Python, R etc). The teaching is by presenting an example – which is usually limited to own scope.
- The tutorial is not a full lecture with a set program of content. Eg, video recording and slide annotations might not. Live URLs can be opened and advanced concepts might be introduced – eg EVT VaR.

Percentile VaR

3. Assume that P&L of an investment portfolio is a random variable that follows Normal distribution $X \sim N(\mu, \sigma^2)$. Use the definition of *VaR as a percentile* to derive analytical expression for VaR calculation.

$\boxed{\textbf{Solution:}}$

The probability of loss $x < 0$ being worse than VaR < 0 is

$$\begin{aligned} \Pr\left(x \leq \mathrm{VaR}(X)\right) &= 1 - c \\ \mathrm{VaR}_c(X) &= \inf\{x \,|\, \Pr(X > x) \leq 1 - c\} = \inf\{x \,|\, F_X(x) \geq c\} \end{aligned}$$

for or 99% confidence, the probability that X above loss x is less than $(1 - 0.99) = 0.01$.

If P&L X is a random variable then VaR(X) is also a random variable. In order to use the well-known Normal Distribution functions, we have to work with the Standard Normal variable

$$\begin{aligned} \Pr(\phi \leq \frac{\mathrm{VaR}(X) - \mu}{\sigma}) &= 1 - c \qquad \Longrightarrow \\ \mathrm{VaR}(X) &= \mu + \Phi^{-1}(1 - c) \times \sigma \end{aligned}$$

Inverse CDF for a probability distribution is known as 'percentile function'.

Analytical VaR / Parametric

$$\cancel{\Phi^{-1}}\left[\cancel{\Phi}\left(\frac{VaR(x)-\mu}{\sigma}\right)\right] = \Phi^{-1}[1-c]$$

$F_X(x) = \Phi(\ )$

inverse of CDF

$$VaR(x) - \mu = \Phi^{-1}[1-c] \times \sigma$$

$$VaR(x) = \mu + \underbrace{\Phi^{-1}[1-c] \times \sigma}$$

Factor

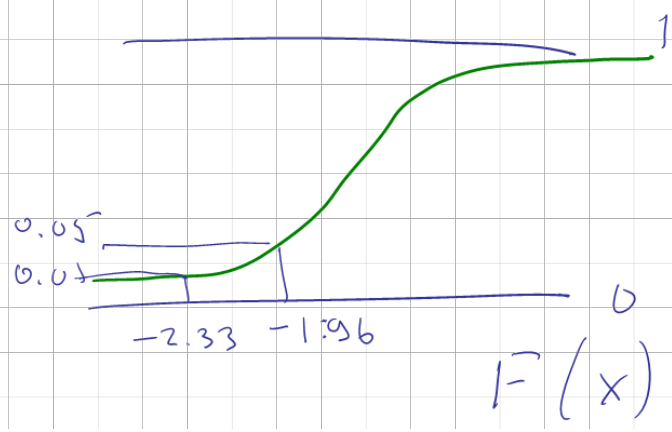
99% VaR

$$\Phi^{-1}[1-0.99] = \Phi^{-1}[0.01] = -2,32635$$

Standardised Percentile

NORMSINV(0.01)

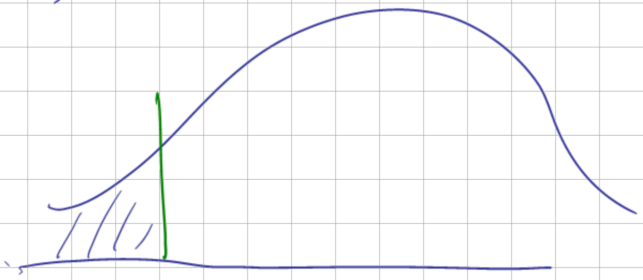
Percentile function



Empirical VaR / Historical Simulation

VaR

P&L returns $\sim N(\mu, \sigma^2)$



$VaR = \mu - 2.32635 \times \sigma$

↳ Analytical VaR

−2.33

↳ Analytical ES

−2.67

(−2.6652)

$ES = \mu - 2.6652 \times \sigma$

Standardised residuals, Z

Z
−2.7
−2.5
−2.33
−2.18
−1.98

$Z_t = \frac{X_t - \mu}{\sigma}$

Returns (sorted) X

X
−0.17
−0.155
−0.14
−0.12
−0.10
...

VaR

$\left.\begin{matrix} -0.17 \\ -0.155 \\ -0.14 \end{matrix}\right\}$ 1% of $N_{obs} = 300$

inf{ }

PERCENTILE(0,01

VaR = −0.14

$ES_{99\%} = -\frac{(0.17 + 0.155 + 0.14)}{3}$

↳ Historic Simulation $X \to Z$

↳ Monte Carlo Simulation

$Z_{sim} \to X$

Problem 1: Historical Simulation Compute the 1-day VAR at 90% confidence (both in percent and monetary terms) for a portfolio of £3 million whose recent daily returns have been: +1%, 0%, -1%, -2%, +1%, +3%, -1%, 0%, -3%, 0%

① VaR Historical Simulation

Π: 3m £

VaR(1d, 90%)

VaR(1d, X)

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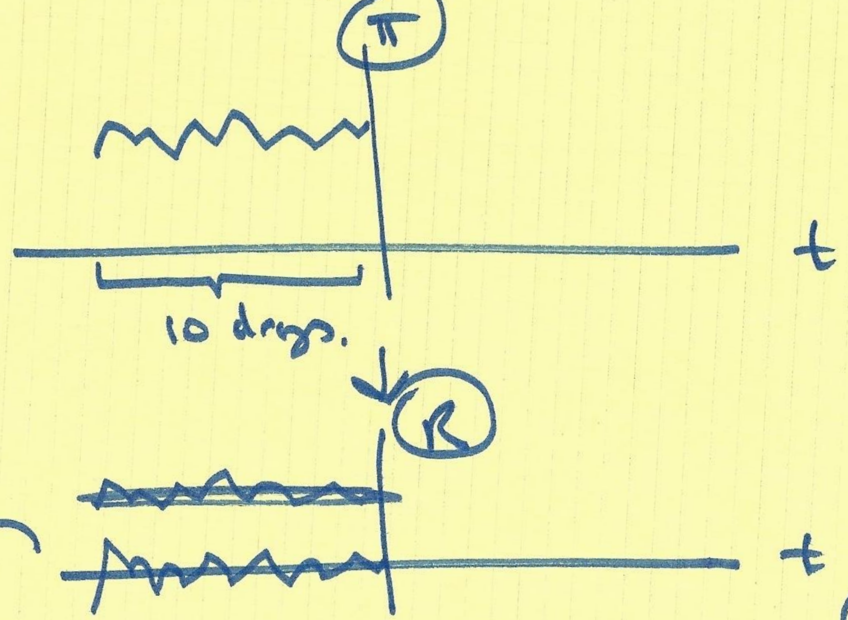
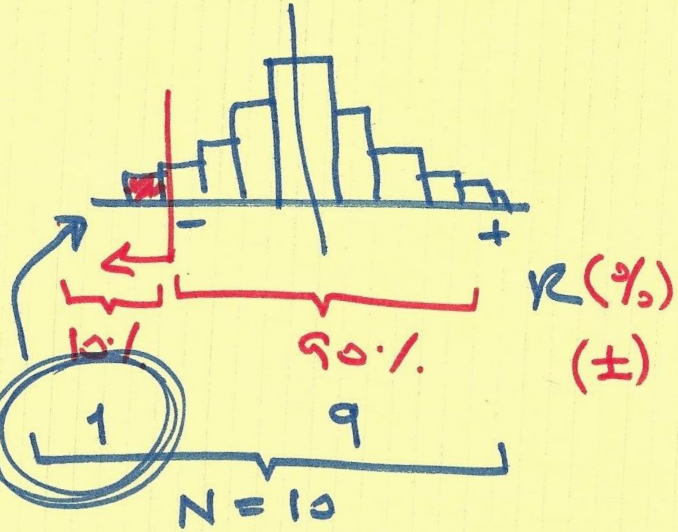




N=10

R	sorted R
+1%	−3%
0%	−2%
−1%	−1%
−2%	−1%
+1%	0%
+3%	0%
−1%	0%
0%	+1%
−3%	+1%
0%	+3%

→ $\text{VaR}(1d, 90\%) = \underline{+3\%}$ (%)

→ $\text{VaR}(1d, 90\%) = (-3\%)(3m£) = \underline{+90{,}000 £}$ (£)

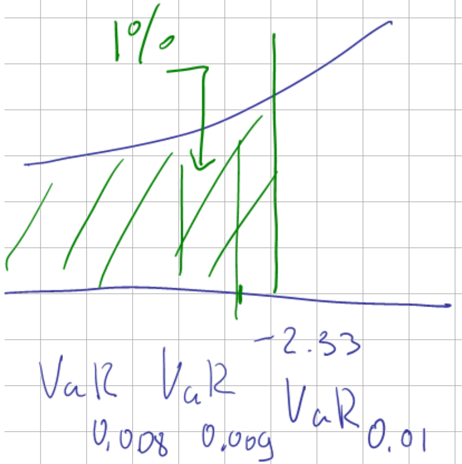
Expected Shortfall

ES

$$ES(X) = E[X \mid X \le VaR_c(X)]$$

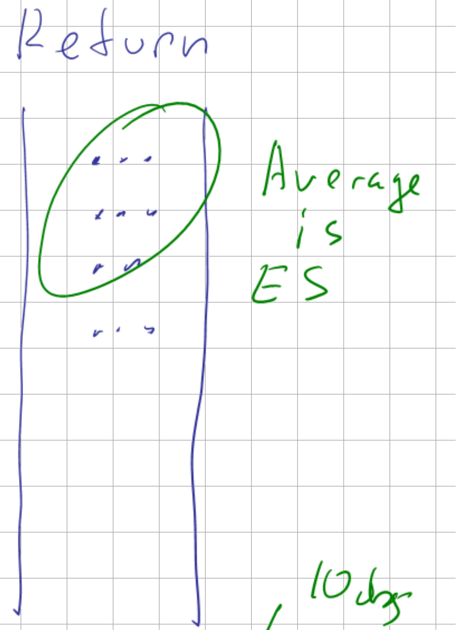
mean $E[X]$ but here

mean of the tail



$$ES(X) = \frac{1}{1-c}\int_0^{1-c} VaR_u \, du$$

$$\underbrace{\frac{\sum VaR_u}{0.01}}_{\text{averaging}}$$



$du = 0.001$ (10 obs)

or

0.0001 (100 obs)

$VaR_{99\%} \approx -2{,}33$

$VaR_{99.1\%} \approx -2.33$

$VaR_{99.2\%} \approx -2{,}34$

ES is

average worst outcome

mean of the tail values

Recap

- Analytical approach: distribution represented by a Factor
  - Give percentage (eg, 90%, 99%), Inverse Normal CDF returns factor value (eg, -2.33).
  - Multiply Factor x Volatility. Voilà!
  - **VaR is Parametric**

- Empirical approach: distribution is represented by a sorted column of returns
  - Manually search for value (return) that corresponds to 90% of observations
  - 1 year period is 252 days, translates into a tail of 2.5 observations for 90%
  - **VaR is Historical Simulation**

VaR Backtesting DEMO

Imagine that each morning you calculate 99%/10day VaR from available prior data only. Once ten days pass you compare that VaR number to the realised return and check if your prediction about the worst loss was breached. You are given a dataset of FTSE 100 index levels, continue in Excel.

- Calculate Value at Risk for each day t (starting on Day 21) as follows:

$$\text{VaR} = \mu_{10D} + \text{Factor} \times \sigma_{10D} \quad \dagger$$

where Factor is a percentile of the Standard Normal Distribution that 'cuts' 1% on the tail.

In Excel, you will have a final column with VaR_t as a percentage since calculation is done on returns.

Backtesting: what to improve?

C.1 Calculate the rolling 99%/10day Value at Risk for an investment in the market index using a sample standard deviation of log-returns, as follows:

- The rolling standard deviation for a sample of 21 is computed for days 1-21, 2-22, ..., there must be 21 observations in the sample. So, you have a time series of σ_t.
- Scale standard deviation to reflect a ten days move $\sigma_{10D} = \sqrt{10 \times \sigma^2}$ (we can add variances) and scale an average daily return as $\mu_{10D} = \mu \times 10$ where μ is a mean return of all data given.

- Did we drop the mean, mu?
- Volatility estimation – is it responsive enough? Why ARCH-filtered volatility might make the breach count **worse**?
- Is Normal Percentile an adequate predictor for the P&L in returns? What are the alternatives (t distribution percentile, Cornish-Fisher VaR)

RiskMetrics (EWMA)

$$\boldsymbol{\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)u_{t-1}^2}$$

<u>Low λ</u>

Low λ leads to more weight for the $(1-\lambda)u_{n-1}^2$ term, so the model is very responsive to the previous day's returns, i.e. news from the market.

<u>High λ</u>

High λ leads to a slow response to new information.

Example: The RiskMetrics database made available by by JP Morgan in 1994 uses the EWMA model with $\lambda = 0.94$ for updating daily estimation of variance across a range of markets.

Volatility Filtering (Risk Metrics, ARCH)

EWMA

$\cancel{0.94}$ 0.72

$$\sigma_{t+1}^2 = \alpha \sigma_t^2 + (1-\alpha) r_t^2$$

forecast

past vol^2 (all / nearly all obs avail.)

past ret^2

GARCH

$$\sigma_{t+1}^2 = \beta \sigma_t^2 + \alpha r_t^2 + \omega$$

"Monthly Avgs"

$r_t \equiv u_t$

or

$r_t \equiv \varepsilon_t$

$$\bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}$$

long term average variance (const)

Time Series Q&A

- Period of backtesting: regulators' approach is 200 days.
  - However, we are interested in prediction of *short-term volatility*, therefore, a 'rolling window' of 21-42 observations (returns).
  - Eg, rolling 42-day window over 200 days

- Since we use Normal Factor, the sample size should be close and above 25 (we don't want to go into small samples estimation).
  - Can we improve on Factor? Yes, it is possible to use **Cornish-Fisher** expansion for Value at Risk

REF Cornish-Fisher VaR

https://www.riskconcile.com/wp-content/uploads/2020/03/cornish_fisher_2.html

Basel II and legacy backtesting

- Independence of breaches in VaR: Christoffersen’s 1998 Exceedance Independence Test

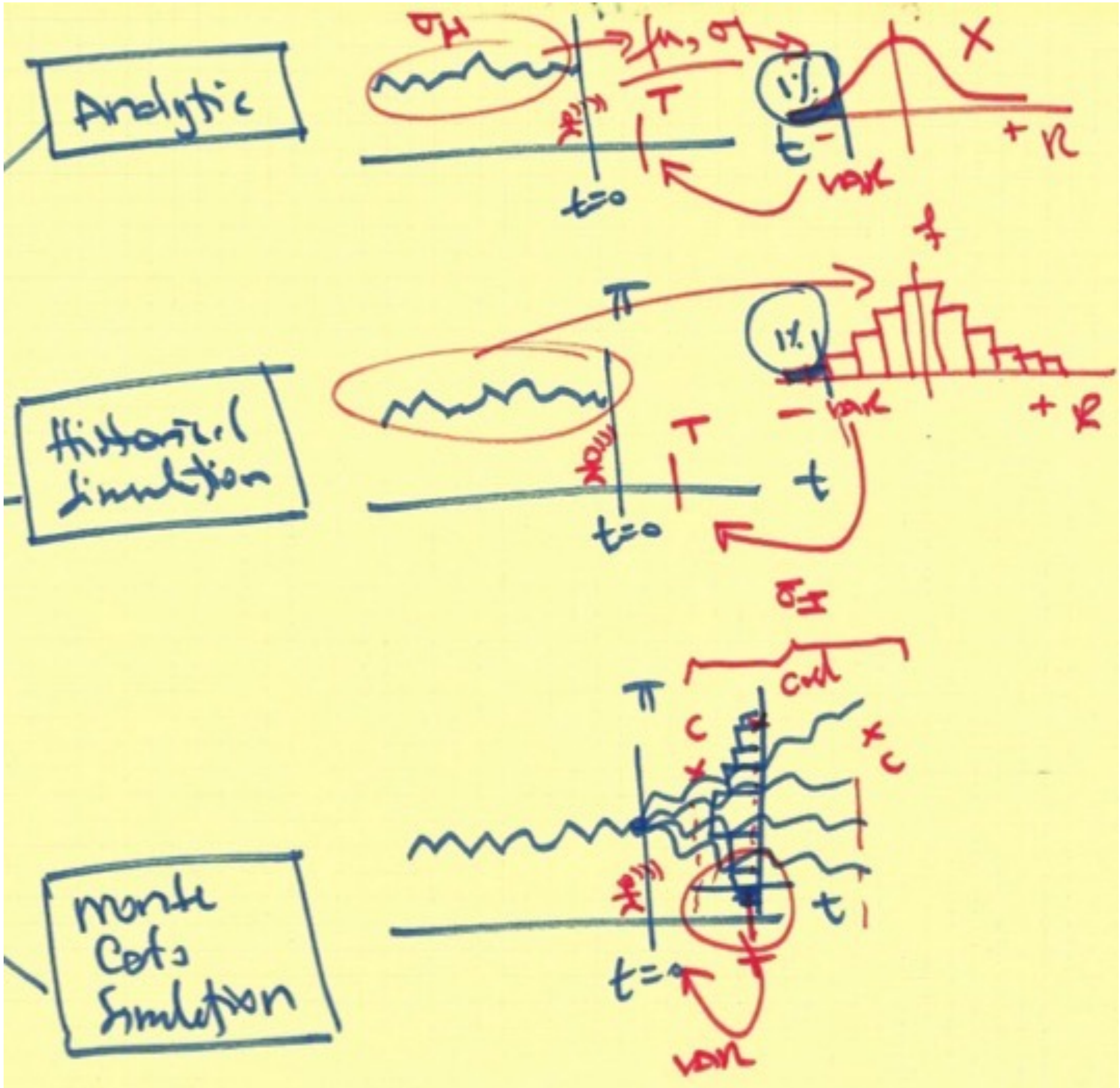
  Measures the dependence between consecutive days only

REF Christoffersen’s 1998 Exceedance Independence Test

https://www.value-at-risk.net/backtesting-independence-tests/

REF Statistical tests for VaR backtesting

https://www.mathworks.com/help/risk/overview-of-var-backtesting.html



Analytic
t=0
T
t
VaR
Historical Simulation
t=0
T
t
VaR
Monte Carlo Simulation
t=0
t
VaR