CQF Lecture on Understanding Volatility

Exercises

1. Explain what actual and implied volatilities are, and what is their relationship? Name three assumptions made in estimation of actual volatility from the market option prices.

2. The market price for a European put with strike 100 is quoted at \$5.57 for the asset value at \$100. Option expiry is one year, and interest rate is 5% p.a. How do you find its implied volatility?

3. Assume **a time-dependent** volatility function $\sigma(t)$. Consistent with Black-Scholes framework, the implied volatility $\sigma_i(t, T)$ measured at time t of an European option expiring at time T must satisfy

$$\sigma_i(t, T) = \sqrt{\frac{1}{T - t} \int_t^T \sigma^2(s)\ ds}$$

   Solve the inverse problem (an integral equation) to show that, at calibration time t^*, the volatility function $\sigma(t)$ must be consistent the implied volatility σ_i as follows:

$$\sigma^2(t) = 2(t - t^*)\,\sigma_i(t^*, t)\,\frac{\partial \sigma_i(t^*, t)}{\partial t} + \sigma_i^2(t^*, t)$$

4. Suppose implied volatilities are observable at $T_i, i = 0, 1, 2, \cdots, n$, with $T_0 = t^*$ is the date of calibration (fitting). Assuming that the actual volatility function is **piecewise constant**, show that for $T_{i-1} < t < T_i$ the total variance is (this is discretised Q3)

$$\sigma^2(t) = \frac{(T_i - t^*)\,\sigma_i^2(t^*, T_i) - (T_{i-1} - t^*)\,\sigma_i^2\,(t^*, T_{i-1})}{T_i - T_{i-1}}$$

5. Suppose that we know the actual volatility σ_a to realise (it can be a good forecast of average volatility) and can trade options at the implied volatility σ_i. We have a choice of calculating a hedge $\Delta = N(d_1)$ using implied or actual. Assume the asset follows the GBM with continuous dividend rate D, and an option denoted by $V_i(S, t; \sigma)$.

   Within the Black-Scholes framework, what is **the P&L from a replicated option** (Mark-to-Market value over dt) if one calculates Δ_a with actual volatility. What can we say about the total P&L?

   What about replicating with the implied volatility?

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