# Finite Difference Methods

## Model Problem

Consider the following Black-Scholes pricing problem for the value of a European Call Option $V = V(S,t)$ :

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

$$\begin{aligned} V(0,t) &= 0 \\ \lim_{S\to\infty} V(S,t) &\to S \\ V(S,T) &= \max(S - E, 0), \\ S &\in [0,\infty), \quad t \in (0,T) \end{aligned} \tag{2}$$

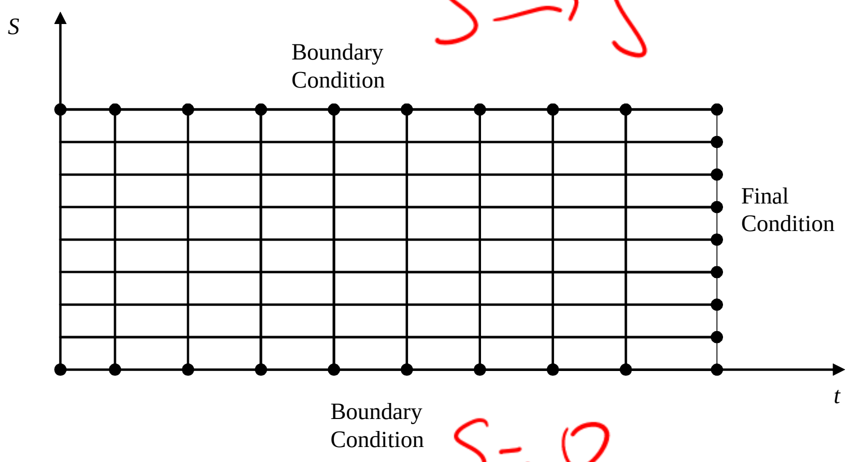
where $S$ is the spot price of the underlying financial asset, $t$ is the time, $E > 0$ is the strike price, $T$ the expiry date, $r \geq 0$ the interest rate, $D$ is the dividend yield and $\sigma$ is the volatility of $S$.

Final

This is an ~~Initial~~ Value Problem (IVP). It consists of a parabolic pde (similar to a diffusion equation) which is first order in time $t$ and second order in the asset price $S$. It is solved using two boundary conditions at $S = 0$ and the limit $S \to \infty$. The initial condition which is in fact a *final condition*, in this case, is defined at the end of the time horizon $t = T$. So knowing the value of $V$ for the final time $T$ allows us to time-step backwards to the current (starting) time $t = 0$ and hence calculate the option value for all values of $t$.

There are two steps to solving a PDE numerically using the FDM. Firstly the partial derivative terms in the equation are replaced by approximations. Secondly the region over which the equation is to be solved is decomposed into an area consisting of vertical and horizontal lines to produce a grid. The equation is then solved at each of the grid points.

The problem is to be solved over the following region:




$S \to S^*$

$S^* \sim 3E$

Payoff

$S = 0$

This is to be solved by a Finite Difference (FD) Scheme. By expressing $S = n\delta S$ and $t = T - m\delta t$, we will obtain a difference equation for the Black-Scholes equation. $V(S,t) = V(n\delta S, T - m\delta t) = V_n^m$. $\delta S = \frac{S^*}{N}$ where $S^* \gg E$ is a suitably large value of $S$ ; $\delta t = \frac{T}{M}$. Taking $N$ and $M$ steps for $S$ and $t$ respectively, so

$$
\begin{aligned}
S &= n\delta S \qquad 0 \le n \le N \\
t &= T - m\delta t \qquad 0 \le m \le M.
\end{aligned}
$$

$$\frac{\partial V}{\partial t} \sim \frac{V_n^m - V_n^{m+1}}{\delta t}, \qquad \Theta$$

$$\Delta \qquad \frac{\partial V}{\partial S} \sim \frac{V_{n+1}^m - V_{n-1}^m}{2\delta S}, \quad \frac{\partial^2 V}{\partial S^2} \sim \frac{V_{n-1}^m - 2V_n^m + V_{n+1}^m}{\delta S^2}. \qquad \Gamma$$

Substituting (3) in (1) gives

$$\frac{V_n^m - V_n^{m+1}}{\delta t} + \frac{1}{2} n^2 \sigma^2 \left( V_{n-1}^m - 2V_n^m + V_{n+1}^m \right) +$$

$$\frac{1}{2} (r - D)\, n \left( V_{n+1}^m - V_{n-1}^m \right) - r V_n^m = 0.$$

and rearrange to obtain a *forward marching* scheme in time.

Explicit

$$V_n^{m+1} =$$

$$\begin{aligned} & V_n^m + \delta t \left( \frac{1}{2} n^2 \sigma^2 \left( V_{n-1}^m - 2V_n^m + V_{n+1}^m \right) \right) \\ & + \delta t \left( \frac{1}{2} (r - D)\, n \left( V_{n+1}^m - V_{n-1}^m \right) - r V_n^m \right) \\ \equiv \; & F\left( V_{n-1}^m, V_n^m, V_{n+1}^m \right) \end{aligned}$$

Now for the RHS collect coefficients of each variable term $V$, to get

$$V_n^{m+1} = \alpha_n V_{n-1}^m + \beta_n V_n^m + \gamma_n V_{n+1}^m \tag{4}$$

where

$$\begin{aligned} \alpha_n &= \frac{1}{2}\left(n^2\sigma^2 - n\left(r - D\right)\right)\delta t, \\ \beta_n &= 1 - \left(r + n^2\sigma^2\right)\delta t, \\ \gamma_n &= \frac{1}{2}\left(n^2\sigma^2 + n\left(r - D\right)\right)\delta t \end{aligned} \tag{5}$$

(4) is a linear difference equation. We will use this to march forwards in time, i.e. given a solution at time step $m$ we can use (4) to approximate a solution at the next time step $(m+1)$ .The difference equation (4) is used inside the grid.

Boundary conditions:

At $S = 0$, i.e. $n = 0$, the BSE becomes

$$\begin{aligned} \frac{\partial V}{\partial t} &= rV \Rightarrow \\ V_0^{m+1} &= (1 - r\delta t)\, V_0^m \end{aligned}$$

This also follows from

$$\alpha_0 = 0 = \gamma_0,\ \beta_0 = 1 - r\delta t.$$

As $S$ becomes very large, i.e. $S \to \infty$ $(S^*)$ the probability of it becoming lower than the Exercise becomes negligible, therefore $\Delta = \Delta(t)$ only, hence $\Gamma \to 0$.

The problem arises at $n = N$. We cannot use our difference equation at the boundary, as we end up with a term $V_{N+1}^m$, which is not defined. So we use the gamma condition mentioned above.

We know $\Gamma \sim \dfrac{V_{n-1}^m - 2V_n^m + V_{n+1}^m}{\delta S^2} = 0$, which upon rearranging gives at $n = N$

$$V_{N+1}^m = 2V_N^m - V_{N-1}^m$$

and substituting in the difference equation gives

$$V_N^{m+1} = (\alpha_N - \gamma_N)\, V_{N-1}^m + (\beta_N + 2\gamma_N)\, V_N^m.$$

Subst in (4) for $n = N$

In summary, the scheme is

$$\left.\begin{array}{c} V_n^{m+1} = \alpha_n V_{n-1}^m + \beta_n V_n^m + \gamma_n V_{n+1}^m \\ M > m \geq 1; \quad 1 \leq n \leq N-1 \end{array}\right\} \quad \text{D.E}$$

$$\left.\begin{array}{c} V_n^M = \max\left(n\delta S - E, 0\right) \\ 0 \leq n \leq N; \end{array}\right\} \quad \text{Final Payoff Condition}$$

$$\left.\begin{array}{c} V_0^{m+1} = \beta_0 V_0^m \\ M \geq m \geq 1 \end{array}\right\} \text{BC at} \quad (S = 0)$$

$$\left.\begin{array}{c} V_N^{m+1} = \left(\alpha_N - \gamma_N\right) V_{N-1}^m + \left(\beta_N + 2\gamma_N\right) V_N^m \\ M \geq m \geq 1; \; S = N\delta S \end{array}\right\} \text{BC at} \quad S^*$$

## Fourier Stability (Von Neumann's) Method

A method is called step-wise unstable if for a fixed grid (i.e. $\delta t, \delta S$ constant) there exists an initial perturbation which "blows up" as $t \longrightarrow \infty$, i.e. as we march in time. Here in a backward marching scheme we have $t \longrightarrow 0$ $(m \longrightarrow 0)$. ignore
The question we wish to answer is "do small errors propagate along the grid and grow exponentially?". We hope not!

Assume an initial disturbance which is proportional to $\exp(in\omega)$. We therefore study the propagation of perturbations created at any given point in time.

IF $\widehat{V}_n^m$ is an approximation to the exact solution $V_n{}^m$ then

$$\widehat{V}_n^m = V_n^m + E_n^m$$

where $E_n{}^m$ is the associated error. Then $E_n{}^m$ also satisfies the difference equation (4) to give

$$E_n^{m+1} = \alpha_n E_{n-1}^m + \beta_n E_n^m + \gamma_n E_{n+1}^m.$$

Fourier modes

$i = \sqrt{-1}$

Put

$$E_n^m = \overline{a}^m \exp\left(in\omega\right) \tag{6}$$

which is oscillatory of amplitude $\overline{a}$ and frequency $\omega$. Substituting (6) into (4) gives

$$\overline{a}^{\cancel{m}+1} e^{\cancel{(in\omega)}} = \alpha_n \cancel{\overline{a}^m} e^{i(\cancel{n}-1)\omega} + \beta_n \cancel{\overline{a}^m} \cancel{e^{in\omega}} + \gamma_n \cancel{\overline{a}^m} e^{i(\cancel{n}+1)\omega}$$

which becomes

$$\overline{a} = \alpha_n e^{-i\omega} + \beta_n + \gamma_n e^{i\omega}.$$

Now stability criteria arises from the balancing of the time dependency and diffusion terms, so that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0$$

From (5) we take the following contributions

$$\alpha_n = \frac{1}{2} n^2 \sigma^2 \delta t, \;\; \beta_n = 1 - n^2 \sigma^2 \delta t, \;\; \gamma_n = \frac{1}{2} n^2 \sigma^2 \delta t$$

$\cos 2x = 1 - 2\sin^2 x$

$\cos 2x - 1 = -2\sin^2 x$

$x = \frac{\omega}{2}$

$$
\begin{aligned}
\bar{a} &= \frac{1}{2} n^2 \sigma^2 \delta t \left( e^{i\omega} + e^{-i\omega} \right) + 1 - n^2 \sigma^2 \delta t \\
\bar{a} &= n^2 \sigma^2 \delta t \left( \cos \omega - 1 \right) + 1.
\end{aligned}
$$

we have

$$\bar{a} = 1 - 2n^2\sigma^2 \sin^2 \frac{\omega}{2} \delta t$$

$\left(\sin(\omega/2)\right)^2$

$n = N$

For stability $\bar{a}$ must be bounded, i.e. $|\bar{a}| < 1$, i.e.

$$-1 < 1 - 2n^2\sigma^2 \sin^2 \frac{\omega}{2} \delta t < 1$$

which upon simplifying we find is

$-2 < \ldots < 0$

$$\delta t < \frac{1}{\sigma^2 N^2} \tag{7}$$

$1 > 2N^2\sigma^2\delta t$

so $\delta t \sim O\left(N^{-2}\right)$.

The beauty of the explicit method lies in its simplicity, both in numerical and computational terms.

However, the main disadvantage is associated with the stability criteria, given by (7).

This condition puts severe constraints on the viability of the method. If it is not satisfied, we will observe exponentially growing oscillations in our numerical solution as we iterate backwards in time.

Given that $\delta t = O\left(\frac{1}{N^2}\right)$, we see that the accuracy can be improved by increasing the number of asset steps $N$. However doubling $N$ requires the use of four times as many time-steps, to satisfy the stability condition.

# Early Exercise Feature - American Options

If we can exercise an option during some time interval, before its expiry date, then the *no-arbitrage* argument tells us that the value of the option $V$ can not be less than the payoff $P(S,t)$ during that time period, so

$$V \geq P(S,t).$$

In the explicit scheme, the early exercise constraint can be implemented in a most trivial manner. Consider the time interval $T$ in which the option may be exercised. As we step backwards in time, the option value is computed. If this price is less than the payoff during $T$, it is set equal to the payoff. So at each time step, we solve the explicit scheme to obtain the option price $\overline{V}$. Then check the condition $\overline{V} < P(S,t)$? If this is true then the option price $V = P(S,t)$, else $V = U$.

This strategy of checking the early exercise constraint is called the *cutoff* method.

Thus the explicit FDM can be expressed in compact form as

$$\begin{aligned} U_n^{m-1} &= \alpha_n V_{n-1}^m + \beta_n V_n^m + \gamma_n V_{n+1}^m \\ V_n^{m-1} &= \begin{cases} U_n^{m-1} & \text{if } U_n^{m-1} \geq P_n^{m-1} \\ P_n^{m-1} & \text{if } U_n^{m-1} < P_n^{m-1} \end{cases} \end{aligned}$$

where $P_n^m$ is the FDA for the payoff at $(S,\ t)$, i.e. $P_n^m =$Payoff$(n\delta S,\ m\delta t)$ as opposed to simply $P_n^M$ at time $t = T$. So we have a time dependent payoff function, defined (at each time step) for the life of the option at the time the contract is written.

# Variable Parameters

There are a number of parameters in the BSE, which need not be constant. FDM can easily handle problems involving non-constant parameters. Suppose the volatility, dividend yield and interest rates are functions of asset price and time, such that the pricing equation becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma\left(S,t\right)^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left(r\left(S,t\right) - D\left(S,t\right)\right) S \frac{\partial V}{\partial S} = r\left(S,t\right) V.$$

The explicit FD scheme now becomes

$$V_n^{m+1} = \alpha_n^m V_{n-1}^m + \beta_n^m V_n^m + \gamma_n^m V_{n+1}^m$$

where

$$\begin{aligned} \alpha_n &= \frac{1}{2}\Big(\left(\sigma_n^m\right)^2 n^2 - \left(r_n^m - D_n^m\right) n\Big)\,\delta t, \\ \beta_n &= 1 - \Big(r_n^m + \left(\sigma_n^m\right)^2 n^2\Big)\,\delta t, \\ \gamma_n &= \frac{1}{2}\Big(\left(\sigma_n^m\right)^2 n^2 + \left(r_n^m - D_n^m\right) n\Big)\,\delta t \end{aligned}$$

So the problem of variable drift/diffusion has been trivially implemented.

## The Greeks

Having discussed finite difference techniques for approximating derivatives, obtaining the greeks becomes a simple task of defining *theta*, *delta* and *gamma*:

$$\begin{aligned}\theta_n^m &\sim \frac{V_n^m - V_n^{m-1}}{\delta t}, \quad \Delta_n^m \sim \frac{V_{n+1}^m - V_{n-1}^m}{2\delta S}, \\ \Gamma_n^m &\sim \frac{V_{n-1}^m - 2V_n^m + V_{n+1}^m}{\delta S^2}.\end{aligned}$$

These are the simple greeks which involve derivatives with respect to a variable. For more advanced greeks (based upon parameters), such as *vega* and *rho*, these can be calculated using FDM by expressing modifications of the BSE. To illustrate this idea let us introduce a convenient form of shorthand by defining the options *vega* $\upsilon$ as

$$\upsilon\,(S,t) = \frac{\partial V}{\partial \sigma}.$$

Now by differentiating the (earlier) Black-Scholes problem (equation and payoff)

$$\frac{\partial}{\partial\sigma}\left\{\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-D)\,S\frac{\partial V}{\partial S}-rV=0\right\}$$

and

$$\frac{\partial}{\partial\sigma}\left\{\max\left(S-E,\ 0\right)\right\}$$

the equation becomes

$$\left\{\frac{\partial}{\partial t}\frac{\partial V}{\partial\sigma}+\tfrac{1}{2}S^2\frac{\partial}{\partial\sigma}\left(\sigma^2\frac{\partial^2 V}{\partial S^2}\right)+(r-D)\,S\frac{\partial}{\partial S}\frac{\partial V}{\partial\sigma}-r\frac{\partial V}{\partial\sigma}=0\right\}$$

to give the final form

$$\frac{\partial\upsilon}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2\upsilon}{\partial S^2}+(r-D)\,S\frac{\partial\upsilon}{\partial S}-r\upsilon=-\sigma S^2\frac{\partial^2 V}{\partial S^2}$$

together with the payoff condition which becomes

$$\frac{\partial V\left(S,T\right)}{\partial\sigma}=\upsilon\left(S,T\right)=0.$$

The resulting PDE is the BSE with a forcing term (on the right hand side) due to the diffusion. Note - in order to obtain this we have assumed the existence of continuous second order partial derivatives so for example

$$\frac{\partial}{\partial \sigma}\frac{\partial V}{\partial t} \equiv \frac{\partial}{\partial t}\frac{\partial V}{\partial \sigma} \rightarrow \frac{\partial v}{\partial t}.$$

We also note that in arriving at $v(S,T) = 0$ we have assumed that the payoff is independent of the volatility.

# Implicit Finite Difference Approximations

We now introduce the Implicit Finite Difference (FD) Scheme.

By using the same notation as in the explicit case we will obtain an implicit difference method for the Black-Scholes equation.

We begin with the Taylor-series approximations for the 1st and 2nd order derivatives, where $V(S,t) = V(n\delta S,\ T - m\delta t) = V_n^m$ :

$$
\begin{aligned}
\frac{\partial V}{\partial t} &\sim \frac{V_n^m - V_n^{m+1}}{\delta t} + O(\delta t)\,, \\
\frac{\partial V}{\partial S} &\sim \frac{V_{n+1}^{m-1} - V_{n-1}^{m-1}}{2\delta S} + O\left(\delta S^2\right), \\
\frac{\partial^2 V}{\partial S^2} &\sim \frac{V_{n-1}^{m-1} - 2V_n^{m-1} + V_{n+1}^{m-1}}{\delta S^2} + O\left(\delta S^2\right)
\end{aligned}
$$

Substituting in the BSE gives and rearranging to obtain another *forward marching* scheme in time gives the linear system

$$a_n V_{n-1}^{m+1} + b_n V_n^{m+1} + c_n V_{n+1}^{m+1} = V_n^m$$

where

$$\begin{aligned} a_n &= -\frac{1}{2}\left(\sigma^2 n^2 - n\,(r-D)\right)\delta t,\ \ b_n = 1 + \left(\sigma^2 n^2 + r\right)\delta t, \\ c_n &= -\frac{1}{2}\left(\sigma^2 n^2 + n\,(r-D)\right)\delta t \end{aligned} \tag{7a}$$

This expression is accurate to $O\left(\delta S^2, \delta t\right)$.

The chief attraction of this method lies in its instability - it is stable for all values of $\delta t$ and we call this scheme *unconditionally stable.*

As we know $V_n^m$ before $V_n^{m-1}$, the system is implicit for the $V_n^{m-1}$ term.

$$\left.\begin{array}{c} a_n V_{n-1}^{m-1} + b_n V_n^{m-1} + c_n V_{n+1}^{m-1} = V_n^m \\ M \geq m \geq 1; \quad 1 \leq n \leq N-1 \end{array}\right\} \text{ D.E}$$

$$\left.\begin{array}{c} V_n^M = \max\left(n\delta S - E, 0\right) \\ 0 \leq n \leq N; \end{array}\right\} \text{ Final Payoff Condition}$$

$$\left.\begin{array}{c} \left(1 + r\delta t\right) V_0^{m-1} = V_0^m \\ M \geq m \geq 1 \end{array}\right\} \text{Boundary condition at } \quad (S = 0)$$

$$\left.\begin{array}{c} \widehat{a}_N V_{N-1}^{m-1} + \widehat{b}_N V_N^{m-1} = V_N^m \\ M \geq m \geq 1; \quad S = N\delta S \end{array}\right\} \text{Boundary condition at } \quad S^*$$

We can write the problem as a system of linear equations, called a *linear system*,

$$\begin{array}{rcl} a_1 V_0^{m-1} + b_1 V_1^{m-1} + c_1 V_2^{m-1} & = & V_1^m \\ a_2 V_1^{m-1} + b_2 V_2^{m-1} + c_2 V_3^{m-1} & = & V_2^m \\ & & \vdots \\ & & \vdots \\ a_{N-1} V_{N-2}^{m-1} + b_{N-1} V_{N-1}^{m-1} + c_{N-1} V_N^{m-1} & = & V_{N-1}^m \end{array}$$

$$
\begin{pmatrix}
b_0 & c_0 & \cdots & \cdots & \cdots & 0 \\
a_1 & b_1 & c_1 & 0 & & \vdots \\
0 & a_2 & b_2 & c_2 & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
\vdots & & & a_{N-1} & b_{N-1} & c_{N-1} \\
0 & \cdots & \cdots & 0 & \widehat{a}_N & \widehat{b}_N
\end{pmatrix}
\begin{pmatrix}
V_0^{m-1} \\ V_1^{m-1} \\ V_2^{m-1} \\ \vdots \\ V_{N-1}^{m-1} \\ V_N^{m-1}
\end{pmatrix}
=
\begin{pmatrix}
V_0^m \\ V_1^m \\ V_2^m \\ \vdots \\ V_{N-1}^m \\ V_N^m
\end{pmatrix}
$$

So we are solving

$$\mathbf{A}\underline{V}^{m-1} = \underline{V}^m.$$

at each time step for the unknown vector $\underline{V}^{m-1}$. We note that the matrix $A$ is extremely sparse. A matrix consisting of a main diagonal together with one above (*super-diagonal*) and one below (*sub-diagonal*) is called a *tri-diagonal matrix*.

This linear system can now be solved directly or iteratively using e.g. the Gauss-Seidel Method.

## Stability of the Implicit Scheme

Suppose that the payoff contains small errors. These will propagate as we march backwards in time. As with the explicit scheme write

$$\widehat{V}_n^m = {V_n}^m + {E_n}^m$$

where ${E_n}^m$ is the associated error. Then ${E_n}^m$ also satisfies the difference equation (8) to give

$$a_n E_{n-1}^{m+1} + b_n E_n^{m+1} + c_n E_{n+1}^{m+1} = E_n^m. \tag{10}$$

By general Fourier analysis we consider harmonic perturbations of the form

$$E_n^m = \overline{a}^m \, \mathsf{exp}\left(in\omega\right). \tag{11}$$

So oscillatory of amplitude $\overline{a}$ and frequency $\omega$. Substituting (11) into (10) gives

$$a_n \overline{a}^{m+1} e^{i(n-1)\omega} + b_n \overline{a}^{m+1} e^{in\omega} + c_n \overline{a}^{m+1} e^{i(n+1)\omega} = \overline{a}^m e^{(in\omega)}$$

which on simplification becomes

$$a_n e^{-i\omega} + b_n + c_n e^{i\omega} = \frac{1}{\overline{a}}.$$

Now stability criteria arises from the balancing of the time dependency and diffusion terms, so that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0$$

From (9) we take the following contributions

$$a_n = -\frac{1}{2}n^2\sigma^2\delta t,\ b_n = 1 + n^2\sigma^2\delta t,\ c_n = -\frac{1}{2}n^2\sigma^2\delta t$$

$$\begin{aligned} \frac{1}{\overline{a}} &= 1 + n^2\sigma^2\delta t - \frac{1}{2}n^2\sigma^2\delta t \underbrace{\left(e^{i\omega} + e^{-i\omega}\right)}_{=2\cos\omega} \\ &= 1 + n^2\sigma^2\left(1 - \cos\omega\right)\delta t. \end{aligned}$$

Again using the trigonometric identity $\cos 2x = 1 - 2\sin^2 x$, we have

$|\bar{a}| < 1$

$$\bar{a} = \frac{1}{1 + 2n^2\sigma^2 \sin^2 \frac{\omega}{2}\delta t}.$$

In order that errors do not grow as we step backwards in time, we require $|\bar{a}| < 1$, this means we want

$$\left|1 + 2n^2\sigma^2 \sin^2 \frac{\omega}{2}\delta t\right| > 1$$

which upon consideration we find is valid for

$$\delta t > 0,$$

so stability is guaranteed for all values of $\delta t$ and the disturbances will die out as we time-step back from expiry.

# The Crank-Nicolson Scheme

The fully implicit method has the same order of accuracy (in time and asset price) as the explicit scheme but is unconditionally stable , i.e. $\forall\ \delta t > 0$.

The Crank-Nicolson method whilst being unconditionally stable has the additional advantage that it is also second order accurate in time.

Consider a PDE being satisfied at the midpoint $\left(n\delta S, \left(m - \frac{1}{2}\right)\delta t\right)$ and replace $V_S$ and $V_{SS}$ by means of a FD approximation at the $m^{\text{th}}$ and $(m+1)^{\text{th}}$ time steps, i.e.

$$\frac{1}{2}\left(m + (m-1)\right) \rightarrow \left(m - \frac{1}{2}\right).$$

The method is regarded as an **equally** weighted average of the explicit and implicit schemes with advantage over both individual cases due to accuracy being $O\left(\delta t^2\right)$.

We can then to $O\left(\delta t^2\right)$ write the FD approximations as

$$
\begin{aligned}
\frac{\partial V}{\partial t}\left(n\delta S,\left(m-\tfrac{1}{2}\right)\delta t\right) &\sim \frac{V_n^m - V_n^{m-1}}{\delta t} \\
\frac{\partial V}{\partial S}\left(n\delta S,\left(m-\tfrac{1}{2}\right)\delta t\right) &\sim \frac{1}{2}\frac{\partial V}{\partial S}(n\delta S, m\delta t) + \\
&\quad\ \frac{1}{2}\frac{\partial V}{\partial S}(n\delta S, (m-1)\,\delta t) \\
\frac{\partial^2 V}{\partial S^2}\left(n\delta S,\left(m-\tfrac{1}{2}\right)\delta t\right) &\sim \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(n\delta S, m\delta t) + \\
&\quad\ \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(n\delta S, (m-1)\,\delta t)
\end{aligned}
$$

and substitute into the BSE as earlier - keeping note of the fact that we are stepping backwards in time. The resulting difference equation is

$$
a_n V_{n-1}^{m-1} + b_n V_n^{m-1} + c_n V_{n+1}^{m-1} = A_n V_{n-1}^m + B_n V_n^m + C_n V_{n+1}^m
$$

where

$$
\begin{aligned}
a_n &= -\frac{1}{4}\left(\sigma^2 n^2 - n\,(r-D)\right)\delta t \\
b_n &= 1 + \frac{1}{2}\left(\sigma^2 n^2 + r\right)\delta t \\
c_n &= -\frac{1}{4}\left(\sigma^2 n^2 + n\,(r-D)\right)\delta t. \\
A_n &= \frac{1}{4}\left(\sigma^2 n^2 - n\,(r-D)\right)\delta t \\
B_n &= 1 - \frac{1}{2}\left(\sigma^2 n^2 + r\right)\delta t \\
C_n &= \frac{1}{4}\left(\sigma^2 n^2 + n\,(r-D)\right)\delta t.
\end{aligned}
$$

and the matrix inversion problem we solve is

$$\mathbf{A}\underline{V}^{m-1} = \mathbf{B}\underline{V}^{m}.$$

# The $\theta-$Method

While the attraction of the fully implicit scheme lies in its unconditionally stability, it is only first order accurate in time. The Crank–Nicolson scheme is an equally weighted average of both implicit and explicit methods and enjoys second order accuracy in time. Now we construct a generalisation of the Crank-Nicolson to obtain a weighted average of the two schemes with a weighted parameter $\theta$ such that $\theta \in [0, 1]$. On a simple heat equation it would be

$$\begin{aligned} \frac{V_n^{m+1} - V_n^m}{\delta t} &= \theta \left( \frac{V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}}{\delta S^2} \right) \\ &\quad + (1-\theta) \left( \frac{V_{n+1}^{m} - 2V_n^{m} + V_{n-1}^{m}}{\delta S^2} \right), \end{aligned}$$

i.e. $\theta \times$Implicit $+ (1-\theta) \times$Explicit. For the BSE we have

$$a_n V_{n-1}^{m-1} + b_n V_n^{m-1} + c_n V_{n+1}^{m-1} = A_n V_{n-1}^m + B_n V_n^m + C_n V_{n+1}^m$$

where

$$
\begin{aligned}
a_n &= -\frac{1}{2}\theta\left(\sigma^2 n^2 - n\,(r-D)\right)\delta t \\
b_n &= 1+\theta\left(\sigma^2 n^2 + r\right)\delta t \\
c_n &= -\frac{1}{2}\theta\left(\sigma^2 n^2 + n\,(r-D)\right)\delta t. \\
A_n &= \frac{1}{2}(1-\theta)\left(\sigma^2 n^2 - n\,(r-D)\right)\delta t \\
B_n &= 1-(1-\theta)\left(\sigma^2 n^2 + r\right)\delta t \\
C_n &= \frac{1}{2}(1-\theta)\left(\sigma^2 n^2 + n\,(r-D)\right)\delta t.
\end{aligned}
$$

When $\theta = 0, \frac{1}{2}$ and 1 the $\theta$ method becomes the Explicit, Crank Nicolson and Fully Implicit scheme, in turn.

The $\theta$ method is a generalisation of the CN method. For a general value of $\theta$

the local truncation error is

$$O\left(\frac{1}{2}\delta t + \frac{1}{12}\delta S^2 - \theta\delta t, \delta S^4, \delta t^2\right).$$

When $\theta = 0$, $1/2$ or $1$ we get the results we have seen so far. But if

$$\theta = \frac{1}{2} - \frac{\delta S^2}{12\delta t}$$

then the local truncation error is improved. The implementation of the method is no harder than the Crank–Nicolson scheme.

# Three time-level methods

Numerical schemes are not restricted to the use of just two time levels. So far we have considered $\left(V^{m+1}, V^{m}\right)$ or $\left(V^{m}, V^{m-1}\right)$. We can construct many algorithms using three or more time levels, i.e.

$$V^{m+1} = g\left(V^{m-1}, V^{m}\right)$$

Again, we would do this if it gave us a better local truncation error or had better convergence properties. We already know that the centred time difference

$$\frac{V_n^{m+1} - V_n^{m-1}}{2\delta t}$$

is unstable for all values of $\delta t$. However the scheme

$$\frac{\left(V_n^{m+1} - V_n^{m-1}\right)}{2\delta t} = \frac{1}{\delta S^2}\left(V_{n+1}^{m} \underbrace{- V_n^{m+1} - V_n^{m-1}}_{=-2V_n^m} + V_{n-1}^{m}\right)$$

# Basic Monte Carlo Techniques

## Motivation

Earlier we derived the Black-Scholes problem to price a European option $V(S,t)$, where the underlying asset follows GBM

$$dS = \mu S dt + \sigma S dW.$$

The resulting PDE and payoff $P(S)$ at expiry $T$ which is satisfied by $V(S,t)$ is

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= 0, \\ V(S,T) &= P(S). \end{aligned}$$

The BSE is a linear parabolic PDE and as such the solution can be expressed as an integral of the form

$$V(S,t) = e^{-r(T-t)} \int_0^\infty \widetilde{p}\left(S,t;S',T\right) V\left(S',T\right) dS'$$

where $\widetilde{p}\,(S, t; S', T)$ represents the transition density and is the solution of the backward Kolmogorov problem

$$\begin{aligned} \frac{\partial \widetilde{p}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widetilde{p}}{\partial S^2} + rS\frac{\partial \widetilde{p}}{\partial S} &= 0, \\ \widetilde{p}\left(S, t; S', T\right) &= \delta\left(S' - S\right). \end{aligned}$$

We have discussed that the function $\widetilde{p}\,(S, t; S', T)$ can be considered as one of two entities.

Firstly in the PDE framework it can be thought of as a Green's function for the general backward problem

$$\begin{aligned} \frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} &= 0, \\ U\,(S, T) &= f\,(S)\,. \end{aligned}$$

Secondly, and more importantly for this section in probabilistic terms it is the probability density function for the risk-neutral random walk mentioned earlier. This is also called the risk-neutral measure.

We can write the value of the option in the form

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[P\left(S\right)\right]$$

which is the present value of the expected payoff wrt the risk-neutral probability density $\mathbb{Q}$ and recall

$$\mathbb{E}^{\mathbb{Q}}\left[P\left(S\right)\right] = \int_0^{\infty} \widetilde{p}\left(S,t;S',T\right)P\left(S'\right)dS'.$$

The precise form of the integral obtained in the BSE work is

$$\int_0^{\infty} e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,\text{Payoff}(S')\frac{dS'}{S'}. \qquad \text{(A)}$$

This expression works because the equation is linear - so we just need to specify the payoff condition. It can be applied to any European option on a single lognormal underlying asset.

Equation (A) gives us the risk-neutral valuation. $e^{-r(T-t)}$ present values to today time $t$. The integral is the expected value of the payoff with respect to the lognormal transition pdf. The future state is $(S', T)$ and today is $(S, t)$. So it represents the probability of going from $(S, t) \longrightarrow (S', T)$.

Also note the presence of the risk-free IR $r$ in the pdf. So the expected payoff is as if the underlying evolves according to the *risk-neutral* random walk

$$\frac{dS}{S} = rdt + \sigma dW.$$

The real world drift $\mu$ is now replaced by the risk-free return $r$. The delta hedging has eliminated all the associated risk. This means that if two investors agree on the volatility they will also agree on the price of the derivatives even if they disagree on the drift. This brings us on to the idea of *risk-neutrality* and risk-neutral pricing.

So we can think of the option as discounted expectation of the payoff under the assumption that $S$ follows the risk neutral random walk

$$V(S, t) = e^{-r(T-t)} \int_0^\infty \widetilde{p}\left(S, t; S', T\right) V\left(S', T\right) dS'$$

where $p(S,t;S',T)$ represents the transition density and gives the probability of going from $(S,t)$ to $(S',T)$ under $\frac{dS}{S}=rdt+\sigma dW$, i.e. the risk-neutral random walk.

So clearly we have a definition for $\widetilde{p}$, i.e. the lognormal density given by

$$\widetilde{p}\left(S,t;S',T\right)=\frac{1}{\sigma S'\sqrt{2\pi(T-t)}}e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}.$$

Two important points

- $\widetilde{p}(S,t;S',T)$ is a Green's for the BSE. As the PDE is linear we can write the solution down as the integrand consisting of this function and the final condition.

- The BSE is essentially the backward Kolmogorov equation whose solution is the transition density $\widetilde{p}(S,t;S',T)$ with $(S',T)$ fixed and varying $(S,t)$; but with the discounting factor.

We know from earlier that the SDE

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

with constant $r$ and $\sigma$ has the solution

$$S_T = S_0 \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right)T + \sigma\phi\sqrt{T}\right\},$$

for some time horizon $T$; with $\phi \sim N(0,1)$; $W_t \sim N(0,t)$ and can be written $\phi\sqrt{T}$.

It is often more convenient to express in time stepping form

$$S_{t+\delta t} = S_t \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t}\right\}.$$

In general a closed form solution of an arbitrary SDE is difficult if e.g.

1. $r = r(t)$ and $\sigma = \sigma(S,t)$, i.e. the parameters are no longer constant

2. the SDE is complicated.

The need for Monte Carlo requires numerical integration of stochastic differential equations. Previously we considered the Forward **Euler-Maruyama** method. Why did this work?

Consider

$$dX_t = a\left(X_t, t\right) dt + b\left(X_t, t\right) dW_t \tag{1}$$

The simplest scheme for solving (1) is using the E-M method. That is

$$\int_{t_n}^{t_{n+1}} dX_s = \int_{t_n}^{t_{n+1}} a\left(X_s, s\right) ds + \int_{t_n}^{t_{n+1}} b\left(X_s, s\right) dW_s$$

$$X_{n+1} = X_n + \int_{t_n}^{t_{n+1}} a\left(X_s, s\right) ds + \int_{t_n}^{t_{n+1}} b\left(X_s, s\right) dW_s$$

Using the left hand integration rule:

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} a\left(s, X_s\right) ds &\approx a\left(t_n, X_n\right) \int_{t_n}^{t_{n+1}} ds = a\left(t_n, X_n\right) \delta t \\
\int_{t_n}^{t_{n+1}} b\left(s, X_s\right) ds &\approx b\left(t_n, X_n\right) \int_{t_n}^{t_{n+1}} dW_s = b\left(t_n, X_n\right) \Delta W_n
\end{aligned}$$

$$X_{n+1} = X_n + a\left(t_n, X_n\right)\delta t + b\left(t_n, X_n\right)\Delta W_n$$

where $\Delta W_n = \left(W_{n+1} - W_n\right).$

Monte-Carlo methods are centred on evaluating definite integrals as expectations (or averages). Before studying this in greater detail, we consider the simple problem of estimating expectations of functions of uniformly distributed random numbers.

Motivating Example: Estimate $\theta = \mathbb{E}\left[e^{U^2}\right]$, where $U \sim U(0,1)$.

We note that $\mathbb{E}\left[e^{U^2}\right]$ can be expressed in integral form, i.e.

$$\mathbb{E}\left[e^{U^2}\right] = \int_0^1 e^{x^2} p(x)\, dx$$

where $p(x)$ is the density function of a $U(0,1)$

$$p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

hence

$$\mathbb{E}\left[e^{U^2}\right] = \int_0^1 e^{x^2} dx.$$

This integral does not have an analytical solution. The theme of this section is to consider solving numerically, using simulations. We use the Monte Carlo simulation procedures:

1. Generate a sequence $U_1, U_2, ..., U_n \sim U(0,1)$ where $U_i$ are i.i.d (independent and identically distributed)

2. Compute $Y_i = e^{U_i^2}$ $(i = 1, ..., n)$

3. Estimate $\theta$ by

$$\widehat{\theta}_n \equiv \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} e^{U_i^2}$$

i.e. use the sample mean of the $e^{U_i^2}$ terms.

## Monte Carlo Integration

When a closed form solution for evaluating an integral is not available, numerical techniques are used. The purpose of Monte Carlo schemes is to use simulation methods to approximate integrals in the form of expectations.

Suppose $f(\cdot)$ is some function such that $f : [0, 1] \to \mathbb{R}$. The basic problem is to evaluate the integral

$$I = \int_0^1 f(x)\, dx$$

i.e. diagram

Consider e.g. the earlier problem $f(x) = e^{x^2}$, for which an analytical solution cannotbe obtained.

Note that if $U \sim U(0, 1)$ then

$$\mathbb{E}[f(U)] = \int_0^1 f(u)\, p(u)\, du$$

where the density $p(u)$ of a uniformly distributed random variable $U(0,1)$ is given earlier. Hence

$$\begin{aligned}\mathbb{E}[f(U)] &= \int_0^1 f(u)\,p(u)\,du \\ &= I.\end{aligned}$$

So the problem of estimating $I$ becomes equivalent to the exercise of estimating $\mathbb{E}[f(U)]$ where $U \sim U(0,1)$.

Very often we will be concerned with an arbitrary domain, other than $[0,1]$. This simply means that the initial part of the problem will involve seeking a transformation that converts $[a,b]$ to the domain $[0,1]$. We consider two fundamental cases.

1. Let $f(\cdot)$ be a function s.t. $f:[a,b] \longrightarrow \mathbb{R}$ where $-\infty < a < b < \infty$. The problem is to evaluate the integral

$$I = \int_a^b f(x)\,dx.$$

In this case consider the following substitution

$$y = \frac{x-a}{b-a}$$

which gives $dy = dx/(b-a)$. This gives

$$\begin{aligned} I &= (b-a)\int_0^1 f(y \times (b-a)+a)\,dy \\ &= (b-a)\,\mathbb{E}[f(U \times (b-a)+a)] \end{aligned}$$

where $U \sim U(0,1)$. Hence $I$ has been expressed as the product of a constant and expected value of a function of a $U(0,1)$ random number; the latter can be estimated by simulation.

2. Let $g(\cdot)$ be some function s.t. $g : [0,\infty) \to \mathbb{R}$ where $-\infty < a < b < \infty$. The problem is to evaluate the integral

$$I = \int_0^\infty g(x)\,dx,$$

provided $I < \infty$. So this is the area under the curve $g(x)$ between 0 and $\infty$. In this case use the following substitution

$$y = \frac{1}{1+x}$$

which is equivalent to $x = -1 + \frac{1}{y}$. This gives

$$\begin{aligned} dy &= -dx/(1+x)^2 \\ &= -y^2 dx. \end{aligned}$$

The resulting problem is

$$\begin{aligned} I &= \int_0^1 \frac{g\left(\frac{1}{y} - 1\right)}{y^2} dy \\ &= \mathbb{E}\left[\frac{g\left(-1 + \frac{1}{U}\right)}{U^2}\right] \end{aligned}$$

where $U \sim U(0,1)$. Hence $I$ has again been expressed as the expected value of a function of a $U(0,1)$ random number; to be estimated by simulation.