Probability Distributions and Stochastic Processes

At the heart of modern finance theory lies the uncertain movement of financial quantities. For modelling purposes we are concerned with the evolution of random events through time. *Stochastic Processes* are a family of random variables parametrized by time. Financial assets can be considered as stochastic processes.

A *diffusion process* is one that is continuous in space, while a *random walk* is a process that is discrete. The random path followed by the process is called a realization. Hence when referring to the path traced out by a financial variable will be termed as an asset price realization.

The mathematics can be achieved by the concept of a transition density function and is the connection between probability theory and differential equations.

Starting with a binomial random walk which is discrete we can obtain a continuous time process to obtain a partial differential equation for the transition probability density function (i.e. a time dependent PDF).

A simple symmetric random walk models the dynamics of a random variable, with value y at time t. The probability of an up/down move is $\alpha = 1/2$.

The Transition Probability Density Function

The transition pdf is denoted by

$$p\left(y, t; y', t'\right)$$

We can gain information such as the centre of the distribution, where the random variable might be in the long run, etc. by studying its probabilistic properties. So the density of particles diffusing from (y, t) to (y', t') .

Think of (y, t) as current (or backward) variables and (y', t') as futures ones.

The more basic assistance it gives is with

$$\mathbb{P}\left(a < y' < b \text{ at } t' \middle| \, y \text{ at } t\right) = \int_a^b p\left(y, t; y', t'\right) dy'$$

i.e. the probability that the random variable y' lies in the interval a and b,at a future time t', given it started out at time t with value y.

$p\left(y, t; y', t'\right)$ satisfies two equations:

Forward equation involving derivatives with respect to the future state (y', t') . Here (y, t) is a starting point and is 'fixed'.

Backward equation involving derivatives with respect to the current state (y, t) . Here (y', t') is a future point and is 'fixed'. The backward equation tells us the probability that we were at (y, t) given that we are now at (y', t') , which is fixed.

The mathematics: Start out at a point (y, t) . We want to answer the question, what is the probability density function of the position y' of the diffusion at a later time t'?

This is known as the **transition density function** written $p\left(y, t; y', t'\right)$ and represents the density of particles diffusing from (y, t) to (y', t')

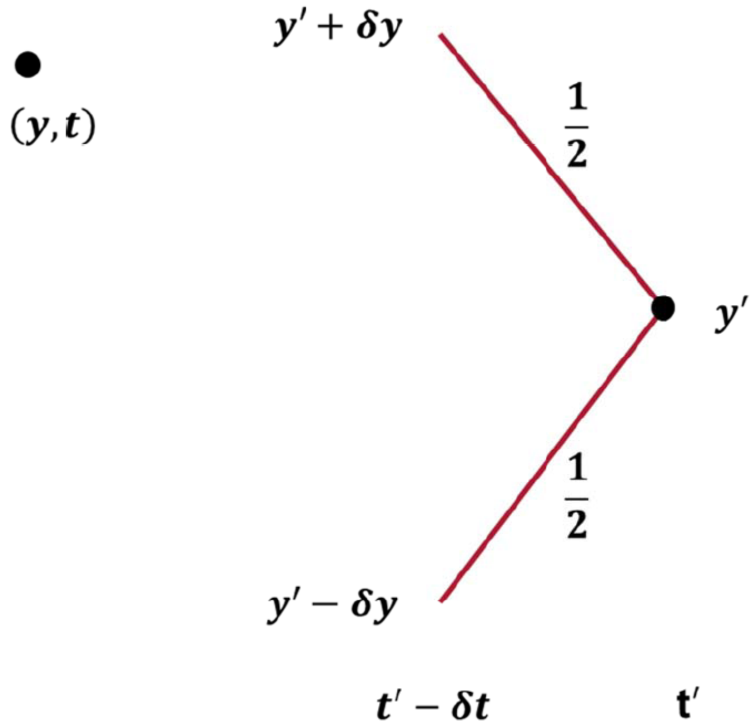
How can we find p?

Consider the following (two step) binomial random walk. So the random variable can either rise or fall with equal probability.

y is the random variable and δt is a time step. δy is the size of the move in y.

$\mathbb{P}[\delta y] = \mathbb{P}[-\delta y] = 1/2.$

Suppose we are at (y', t'), how did we get there? At the previous step time step we must have been at one of $(y' + \delta y, t' - \delta t)$ or $(y' - \delta y, t' - \delta t)$.



So

$$p(y', t') = \tfrac{1}{2} p(y' + \delta y, t' - \delta t) + \tfrac{1}{2} p(y' - \delta y, t' - \delta t)$$

Taylor series expansion gives

$$\begin{aligned}
p(y' + \delta y, t' - \delta t) &= p(y', t') - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + \ldots \\
p(y' - \delta y, t' - \delta t) &= p(y', t') - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + \ldots
\end{aligned}$$

Substituting into the above

$$\begin{aligned}
p(y', t') &= \tfrac{1}{2}\left(p(y', t') - \frac{\partial p}{\partial t'}\delta t + \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right) \\
&\quad + \tfrac{1}{2}\left(p(y', t') - \frac{\partial p}{\partial t'}\delta t - \frac{\partial p}{\partial y'}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2\right)
\end{aligned}$$

$$0 = -\frac{\partial p}{\partial t'}\delta t + \tfrac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2$$

$$\frac{\partial p}{\partial t'} = \tfrac{1}{2} \frac{\delta y^2}{\delta t} \frac{\partial^2 p}{\partial y'^2}$$

Now take limits. This only makes sense if $\frac{\delta y^2}{\delta t}$ is $O(1)$, i.e. $\delta y^2 \sim O(\delta t)$ and letting $\delta y,\ \delta t \longrightarrow 0$ gives the equation

$$\frac{\partial p}{\partial t'} = \tfrac{1}{2} \frac{\partial^2 p}{\partial y'^2}$$

This is called the **forward Kolmogorov equation.** Also called Fokker Planck equation.

It shows how the probability density of future states evolves, starting from (y, t).

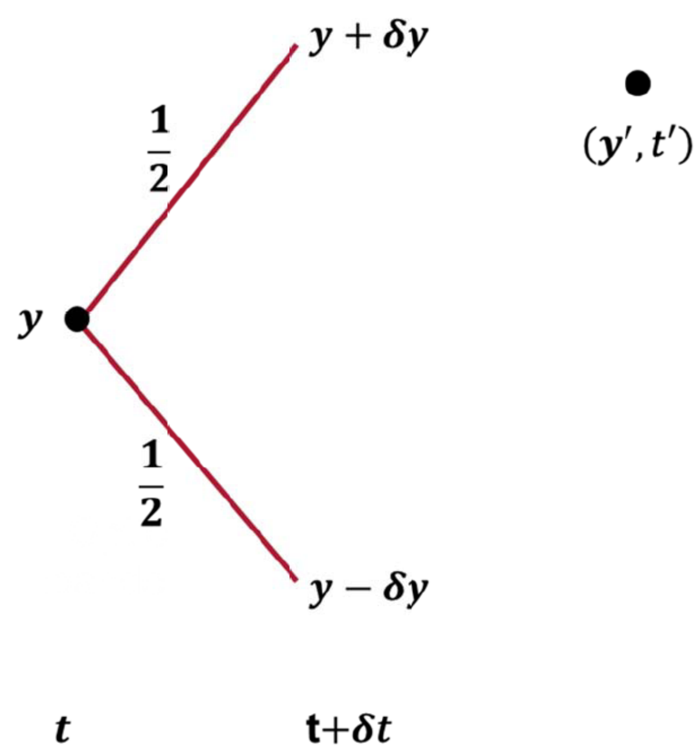
A particular solution of this is

$$p(y, t; y', t') = \frac{1}{\sqrt{2\pi (t' - t)}} \exp\left(-\frac{(y' - y)^2}{2(t' - t)} \right)$$

At $t' = t$ this is equal to $\delta(y' - y)$. The particle is known to start from (y, t) and its density is normal with mean y and variance $t' - t$.

Now we come to find the backward equation. This will be useful if we want to calculate probabilities of reaching a specified final state from various initial states. It will be a backward parabolic partial differential equation requiring conditions imposed in the future, and solved backwards in time. Whereas the forward equation had independent variable t' and y' the backward equation has variables t and y.

The **backward equation** tells us the probability that we are at (y', t') given that we are at (y, t) in the past. So (y', t') are now fixed and (y, t) are variables. So the probability of being at (y', t') given we are at y at t is linked to the probabilities of being at $(y + \delta y, t + \delta t)$ and $(y - \delta y, t + \delta t)$.



$$p(y,t;y',t') = \tfrac{1}{2}p(y+\delta y, t+\delta t; y', t') + \tfrac{1}{2}p(y-\delta y, t+\delta t; y', t')$$

Since (y', t') do not change, drop these for the time being and use a TSE on the right hand side

$p(y,t) =$

$$\begin{aligned} &\tfrac{1}{2}\left(p(y,t) + \frac{\partial p}{\partial t}\delta t + \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + \dots\right) + \\ &\tfrac{1}{2}\left(p(y,t) + \frac{\partial p}{\partial t}\delta t - \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + \dots\right) \end{aligned}$$

which simplifies to

$$0 = \frac{\partial p}{\partial t} + \tfrac{1}{2}\frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y^2}.$$

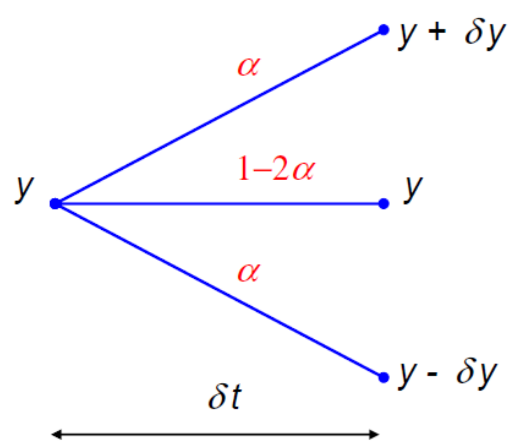
Putting $\frac{\delta y^2}{\delta t} = O(1)$ and taking limit gives the **backward equation**

$$-\frac{\partial p}{\partial t} = \tfrac{1}{2}c^2\frac{\partial^2 p}{\partial y^2}.$$

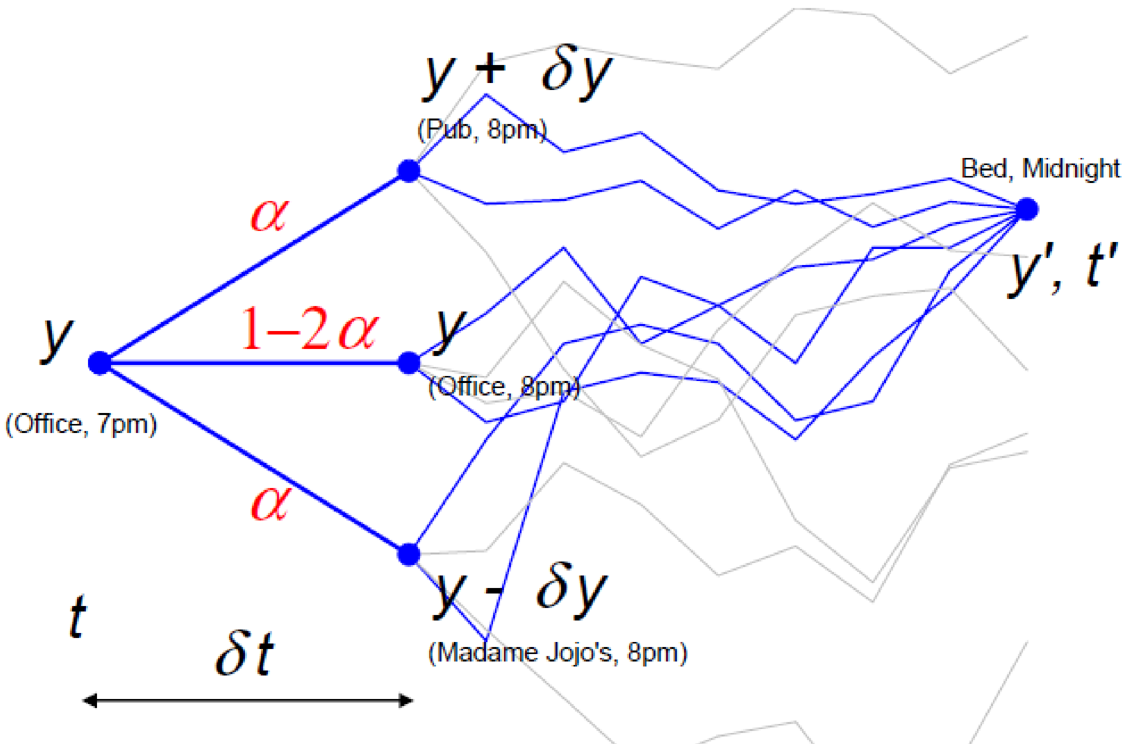
or commonly written as $\dfrac{\partial p}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 p}{\partial y^2} = 0$

The backward equation is particularly important in the context of finance, but also a source of much confusion. Illustrate with the 'real life' example that Wilmott uses.

Wilmott uses a *Trinomial* Random Walk



So 3 possible states at the next time step. Here $\alpha < 1/2$.



y + δy
(Pub, 8pm)
Bed, Midnight
α
y', t'
y
1–2α
y
(Office, 8pm)
(Office, 7pm)
α
y - δy
t
δt
(Madame Jojo's, 8pm)

- At 7pm you are at the office - this is the point (y, t)
- At 8pm you will be at one of three places:

  § The Pub - the point $(y + \delta y, t + \delta t)$;
  § Still at the office - the point $(y, t + \delta t)$;
  § Madame Jojo's - the point $(y - \delta y, t + \delta t)$

We are interested in the probability of being tucked up in bed at midnight (y', t') , given that we were at the office at 7pm.

Looking at the earlier figure, we can only get to bed at midnight via either

- the pub
- the office
- Madame Jojo's

at 8pm.

What happens after 8pm doesn't matter - we don't care, you may not even remember! We are only concerned with being in bed at midnight.

The earlier figure shows many different paths, only the ones ending up in 'our' bed are of interest to us.

In words: The probability of going from the office at 7pm to bed at midnight is

- the probability of going to the pub from the office and then to bed at midnight plus
- the probability of staying in the office and then going to bed at midnight plus
- the probability of going to Madame Jojo's from the office and then to bed at midnight

The above can be expressed mathematically as

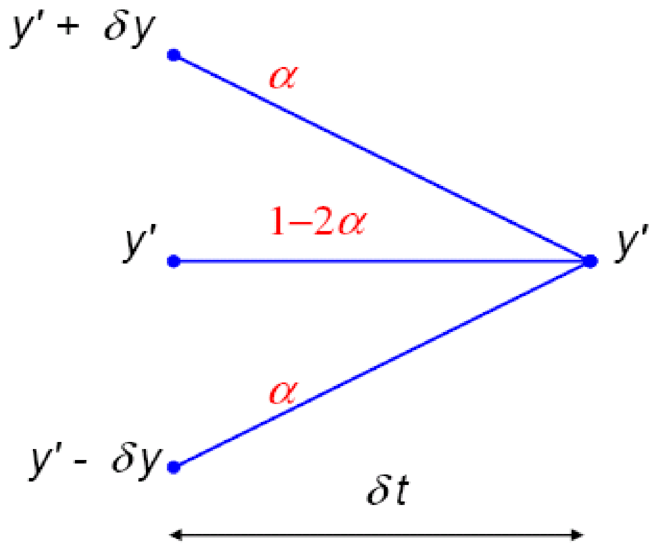
$$\begin{aligned} p\left(y, t; y', t'\right) &= \alpha p\left(y + \delta y, t + \delta t; y', t'\right) + (1 - 2\alpha)\, p\left(y, t + \delta t; y', t'\right) + \\ &\quad \alpha p\left(y - \delta y, t + \delta t; y', t'\right). \end{aligned}$$

Performing a Taylor expansion gives dropping y', t'

$$\begin{aligned} p\left(y, t\right) &= \alpha\left(p + \frac{\partial p}{\partial t}\delta t + \frac{\partial p}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + ..\right) \\ &\quad + (1 - 2\alpha)\left(p + \frac{\partial p}{\partial t}\delta t + ..\right) \\ &\quad \alpha\left(p + \frac{\partial p}{\partial t}\delta t - \frac{\partial p}{\partial y}\delta y + \frac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + ..\right). \end{aligned}$$

Most of the terms cancel and leave

$$0 = \delta t\frac{\partial p}{\partial t} + \alpha\delta y^2\frac{\partial^2 p}{\partial y^2} + ...$$



which becomes

$$0 = \frac{\partial p}{\partial t} + \alpha \frac{\delta y^2}{\delta t} \frac{\partial^2 p}{\partial y^2} + ...$$

and letting $\alpha \frac{\delta y^2}{\delta t} = c^2$ where c is non-zero and finite as $\delta t, \delta y \longrightarrow 0$, we have

$$\frac{\partial p}{\partial t} + c^2 \frac{\partial^2 p}{\partial y^2} = 0$$

Use of a trinomial random walk for the Forward Equation would have resulted in

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}.$$

This can be derived using the followingwhere the random walker is at y at an earlier time $t < t'$. The probability of being at y' at time t' is given by

$$p(y,t) = \alpha p(y + \delta y, t - \delta t) + (1 - 2\alpha) p(y, t - \delta t) + \alpha p(y - \delta y, t - \delta t)$$

Taylor series expansion gives

$$\begin{aligned} p(y + \delta y, t - \delta t) &= p(y,t) - \frac{\partial p}{\partial t}\delta t + \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + ... \\ p(y, t - \delta t) &= p(y,t) - \frac{\partial p}{\partial t}\delta t + ... \\ p(y - \delta y, t - \delta t) &= p(y,t) - \frac{\partial p}{\partial t}\delta t - \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + ... \end{aligned}$$

Substituting into (1.1)

$$\begin{aligned} p(y,t) &= \alpha \left(p(y,t) - \frac{\partial p}{\partial t}\delta t + \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + ... \right) \\ &\quad + (1 - 2\alpha) \left(p(y,t) - \frac{\partial p}{\partial t}\delta t + ... \right) \\ &\quad + \alpha \left(p(y,t) - \frac{\partial p}{\partial t}\delta t - \frac{\partial p}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 p}{\partial y^2}\delta y^2 + ... \right) \end{aligned}$$

$$\frac{\partial p}{\partial t}\delta t = \alpha \frac{\partial^2 p}{\partial y'^2}\delta y^2$$

$$\frac{\partial p}{\partial t'} = \alpha \frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y'^2}$$

Now take limits. This only makes sense if $\frac{\delta y^2}{\delta t}$ is $O(1)$, i.e. $\delta y^2 \sim O(\delta t)$ and letting the constant $c^2 = \alpha \frac{\delta y^2}{\delta t}$, gives the equation

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2},$$

This is the Forward Kolmogorov equation or Folker-Planck equation - it is a diffusion equation for the function p of two independent variables t' and y'. It is a forward parabolic partial differential equation, requiring initial conditions at time t and to be solved for $t' > t$. This equation is to be used if there is some special state now and we want to know what could happen later; i.e. knowing the current value of y and obtaining the distribution of values at some later date. y and t are rather like parameters in this problem (fixed), think of them as an initial condition (starting quantities) for the random walk. This is also an example of Brownian motion. When we get on to financial applications the quantity c will be related to volatility σ.

Solving the Forward Equation

The equation is

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}$$

for the unknown function $p = p(y', t')$. The idea is to obtain a solution in terms of Gaussian curves. Let's drop the primed notation (for convenience).

We assume a solution of the following form exists:

$$p(y,t) = t^a f\left(\frac{y}{t^b}\right)$$

where a, b are constants to be determined. In some textbooks start by specifying the value of a and b.

So create a new variable, from combining t and y

$$\xi = \frac{y}{t^b} = yt^{-b},$$

which is a dimensionless. We have the following derivatives

$$\frac{\partial \xi}{\partial y} = t^{-b}; \quad \frac{\partial \xi}{\partial t} = -byt^{-b-1}$$

we can now say

$$p(y,t) = t^a f(\xi)$$

therefore

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial \xi}\frac{\partial \xi}{\partial y} = t^a f'(\xi).t^{-b} = t^{a-b} f'(\xi)$$

$$\begin{aligned}
\frac{\partial^2 p}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial y}\left(t^{a-b} f'(\xi)\right) \\
&= \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi}\left(t^{a-b} f'(\xi)\right) \\
&= t^{a-b}\frac{1}{t^b}\frac{\partial}{\partial \xi} f'(\xi) = t^{a-2b} f''(\xi)
\end{aligned}$$

$$\frac{\partial p}{\partial t} = t^a \frac{\partial}{\partial t} f(\xi) + a t^{a-1} f(\xi)$$

we can use the chain rule to write

$$\frac{\partial}{\partial t} f(\xi) = \frac{\partial f}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} = -byt^{-b-1} f'(\xi)$$

so we have

$$\frac{\partial p}{\partial t} = a t^{a-1} f(\xi) - byt^{a-b-1} f'(\xi)$$

and then substituting these expressions in to the pde gives

$$a t^{a-1} f(\xi) - byt^{a-b-1} f'(\xi) = c^2 t^{a-2b} f''.$$

We know from ξ that

$$y = t^b \xi$$

hence the equation above becomes

$$a t^{a-1} f(\xi) - b\xi t^{a-1} f'(\xi) = c^2 t^{a-2b} f''.$$

For the similarity solution to exist we require the equation to be independent of t, i.e. $a - 1 = a - 2b \Longrightarrow b = 1/2$, therefore

$$af - \frac{1}{2}\xi f' = c^2 f''$$

thus we have so far

$$p = t^a f\left(\tfrac{y}{\sqrt{t}}\right)$$

which gives us a whole family of solutions dependent upon the choice of a.

We know that p represents a pdf, hence

$$\int_{\mathbb{R}} p(y,t)\, dy = 1 = \int_{\mathbb{R}} t^a f\left(\tfrac{y}{\sqrt{t}}\right) dy$$

change of variables $u = y/\sqrt{t} \longrightarrow du = dy/\sqrt{t}$ so the integral becomes

$$t^{a+1/2} \int_{-\infty}^{\infty} f(u)\, du = 1$$

which we need to normalize independent of time t. This is only possible if $a = -1/2$.

So the D.E becomes

$$-\frac{1}{2}(f + \xi f') = c^2 f''.$$

We have an exact derivative on the lhs, i.e. $\dfrac{d}{d\xi}(\xi f) = f + \xi f'$, hence

$$-\frac{1}{2}\frac{d}{d\xi}(\xi f) = c^2 f''$$

and we can integrate once to get

$$-\frac{1}{2}(\xi f) = c^2 f' + K.$$

We obtain K from the following information about a probability density, as $\xi \to \infty$

$$\begin{aligned} f(\xi) &\to 0 \\ f'(\xi) &\to 0 \end{aligned}$$

hence $K = 0$ in order to get the correct solution, i.e.

$$-\frac{1}{2}(\xi f) = c^2 f'$$

which can be solved as a simple first order variable separable equation:

$$\begin{aligned} -\tfrac{1}{2}(\xi f) &= c^2 \frac{df}{d\xi} \to \frac{df}{f} = -\tfrac{1}{2c^2}\xi d\xi \to \int \tfrac{df}{f} = -\tfrac{1}{2c^2}\xi d\xi \\ \log f &= -\tfrac{1}{4c^2}\xi^2 + C \end{aligned}$$

Taking exponentials of both sides

$$f(\xi) = e^{-\frac{1}{4c^2}\xi^2 + C} = e^C e^{-\frac{1}{4c^2}\xi^2}; \text{ now write } e^C = A$$

$$f(\xi) = A \exp\left(-\tfrac{1}{4c^2}\xi^2\right).$$

A is a normalizing constant, so write

$$A \int_{\mathbb{R}} \exp\left(-\tfrac{1}{4c^2}\xi^2\right) d\xi = 1.$$

Now substitute $x = \xi/2c$, so $2cdx = d\xi$

$$\underbrace{2cA \int_{\mathbb{R}} \exp\left(-x^2\right) dx}_{=\sqrt{\pi}} = 1,$$

which gives $A = 1/2c\sqrt{\pi}$. Returning to

$$p(y,t) = t^{-1/2} f(\xi)$$

becomes

$$p(y',t') = \frac{1}{2c\sqrt{\pi t'}} \exp\left(-\frac{y'^2}{4t'c^2}\right).$$

This is a pdf for a variable y that is normally distributed with mean zero and standard deviation $c\sqrt{2t}$, which we ascertained by the following comparison:

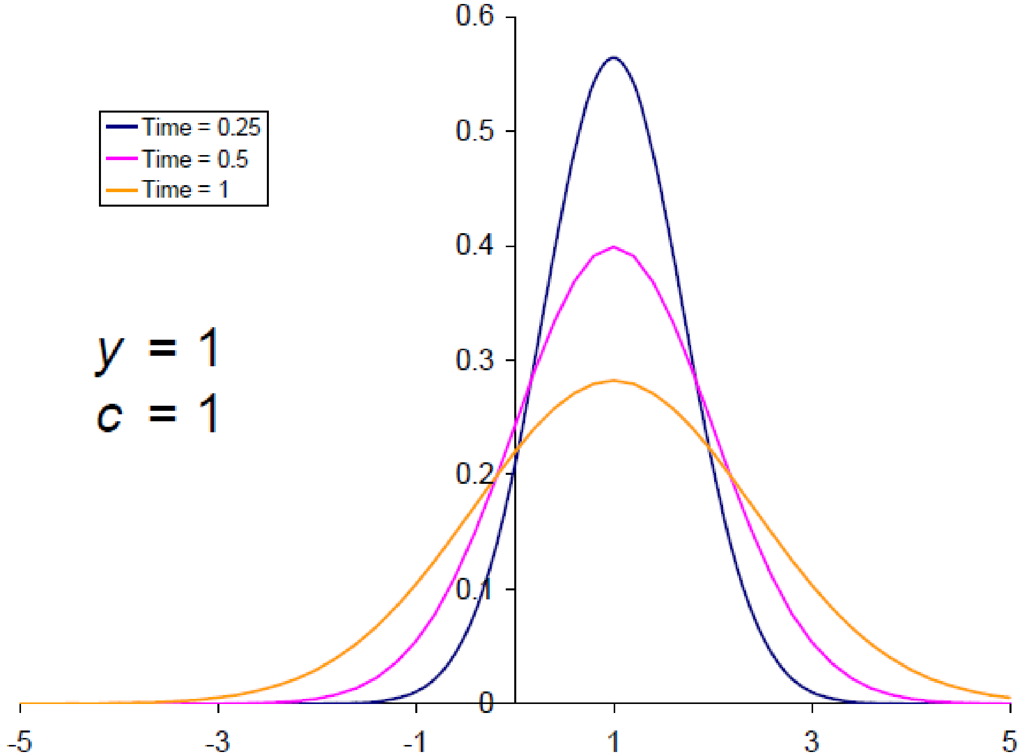
$$-\frac{1}{2}\frac{y'^2}{2t'c^2} : -\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}$$

i.e. $\mu \equiv 0$ and $\sigma^2 \equiv 2t'c^2$.

This solution is also called the **Source Solution** or **Fundamental Solution**.

If the random variable y' has value y at time t then we can generalize to

$$p(y,t;y',t') = \frac{1}{2c\sqrt{\pi(t'-t)}} \exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right)$$



At $t' = t$ this is now a Dirac delta function $\delta(y' - y)$. This particle is known to start from (y, t) and diffuses out to (y', t') with mean y and variance $(t' - t)$

Recall this behaviour of decay away from one point y, unbounded growth at that point and constant area means that $p(y, t; y', t')$ has turned in to a **Dirac delta function** $\delta(y' - y)$ as $t' \to t$.

Dirac delta function

This is written $\delta(x - x') = \lim_{\tau \longrightarrow 0} \delta(x - x')$, such that

$$\delta(x - x') = \begin{cases} \infty & x = x' \\ 0 & x \neq x' \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x - x')\, dx = 1$$

or $\int_0^{\infty} \delta(x - x')\, dx = 1$.

If $g(x)$ is a continuous function then

$$\int_{-\infty}^{\infty} g(x)\, \delta(x - x')\, dx = g(x')$$

So if we take a delta function and multiply it by any other function - and calculate the area under this product - this is simply the function $g(x)$ evaluated at the point $x = x'$. What is happening here?

The delta function picks out the value of the function at which it is singular (in this case x'). All other points are irrelevant because we are multiplying by zero.

Solutions of the heat equation

Earlier we derived the solution of the one dimensional heat/diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for the unknown function of the form $u(x,t) = t^a F\left(\frac{x}{t^b}\right)$. The corresponding solution derived using the similarity reduction technique is the *fundamental solution* or *source solution.*

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

Consider the following integral

$$\lim_{t \longrightarrow 0} \int_{-\infty}^{\infty} u(y,t) f(y)\, dy$$

which can be simplified by the substitution

$$s = \frac{y}{2\sqrt{t}} \Longrightarrow 2\sqrt{t}ds = dy$$

to give

$$\lim_{t \longrightarrow 0} \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left(-s^2\right) f\left(2\sqrt{t}s\right) 2\sqrt{t}ds.$$

In the limiting process we get

$$\begin{aligned} f(0) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-s^2\right) ds &= f(0) \frac{1}{\sqrt{\pi}} \sqrt{\pi} \\ &= f(0). \end{aligned}$$

Hence

$$\lim_{t \longrightarrow 0} \int_{-\infty}^{\infty} u(y,t) f(y)\, dy = f(0).$$

A slight extension of the above shows that

$$\lim_{t \longrightarrow 0} \int_{-\infty}^{\infty} u(x-y,t) f(y)\, dy = f(x),$$

where

$$u(x-y,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right).$$

Let's derive the result above. As earlier we begin by writing $s = \dfrac{x-y}{2\sqrt{t}} \Longrightarrow y = x - 2\sqrt{t}s$ and hence $dy = -2\sqrt{t}ds$. Under this transformation the limits are

$$\begin{aligned} y &= \infty \longrightarrow s = -\infty \\ y &= -\infty \longrightarrow s = \infty \end{aligned}$$

$$\tfrac{1}{2\sqrt{\pi t}} \int_{\infty}^{-\infty} \exp\left(-s^2\right) f\left(x - 2\sqrt{t}s\right) \left(-2\sqrt{t}ds\right) ds$$

$$
\begin{aligned}
& \lim_{t\longrightarrow 0}\tfrac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left(-s^2\right)f\left(x-2\sqrt{t}s\right)ds \\
= & \ f\left(x\right)\tfrac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left(-s^2\right)ds \\
= & \ f\left(x\right)\tfrac{1}{\sqrt{\pi}}\sqrt{\pi}
\end{aligned}
$$

and

$$\lim_{t\longrightarrow 0}\int_{-\infty}^{\infty}u\left(x-y,t\right)f\left(y\right)dy=f\left(x\right).$$

Since the heat equation is a constant coefficient PDE, if $u\left(x,t\right)$ satisfies it, then $u\left(x-y,t\right)$ is also a solution for any y.

Recall what it means for an equation to be linear:

Since the heat equation is linear,

1. if $u\left(x-y,t\right)$ is a solution, so is a multiple $f\left(y\right)u\left(x-y,t\right)$
2. we can add up solutions. Since $f\left(y\right)u\left(x-y,t\right)$ is a solution for any y, so too is the integral

$$\int_{-\infty}^{\infty}u\left(x-y,t\right)f\left(y\right)dy.$$

Recall, adding can be done in terms of an integral. So we can summarize by specifying the following initial value problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\
u\left(x,0\right) &= f\left(x\right)
\end{aligned}
$$

which has a solution

$$u\left(x,t\right)=\tfrac{1}{2\sqrt{\pi t}}\int_{\infty}^{-\infty}\exp\left(-\frac{\left(x-y\right)^2}{4t}\right)f\left(y\right)dy.$$

This satisfies the initial condition at $t=0$ because we have shown that at that point the value of this integral is $f\left(x\right)$. Putting $t<0$ gives a non-existent solution, i.e. the integrand will blow up.

This can also be solved by using the substitution

$$\widehat{s}=\frac{-\left(y-x\right)}{2\sqrt{t}}\longrightarrow -dy=2\sqrt{t}d\widehat{s}$$

$\int_{-\infty}^{0}$ becomes $\int_{\infty}^{\frac{x}{2\sqrt{t}}}$

$$
\begin{aligned}
& -\tfrac{1}{2\sqrt{\pi t}}\int_{\infty}^{\frac{x}{2\sqrt{t}}}\exp\left(-\widehat{s}^2\right)2\sqrt{t}d\widehat{s} \\
= & \ \tfrac{1}{2}\cdot\tfrac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\infty}\exp\left(-\widehat{s}^2\right)d\widehat{s} \\
= & \ \tfrac{1}{2}\operatorname{erf c}\left(\tfrac{x}{2\sqrt{t}}\right)
\end{aligned}
$$

so now we have a solution in terms of the complimentary error function.