Understanding Volatility

In this lecture. . .

- Volatility of Black-Scholes' world
- Implied volatility and the shape of Greeks. Vega, and why it is dangerous
- Volatility Arbitrage: should you hedge using implied or actual volatility?
- Calibration: fitting volatility term structure, an attempt
- Introduction to smile and local volatility

By the end of this lecture you will

- know about the practical uses of volatility
- be aware about historical volatility (estimated) vs a calibrated model
- understand risk decomposition with Greeks and effects of the implied volatility
- uncover how to utilise delta-replication for arbitrage
- have an example of a term structure fitting, and what drives the skew (how ‘fear’ and ‘greed’ transpire)

A caution

This lecture is not one-flow derivation, where one follows a single equation or works towards the known final mathematical result.

Instead, the lecture assumes and requires Black-Scholes PDE and its solutions to be known, in order to make progress with further practical and theoretical aspects.

Much of today's coverage comes from the need to work around the limitations of Black-Scholes.

Introduction

Black-Scholes is a convenient valuation model that matches the market price V to volatility σ_{BS}, *under the risk-neutral condition*.

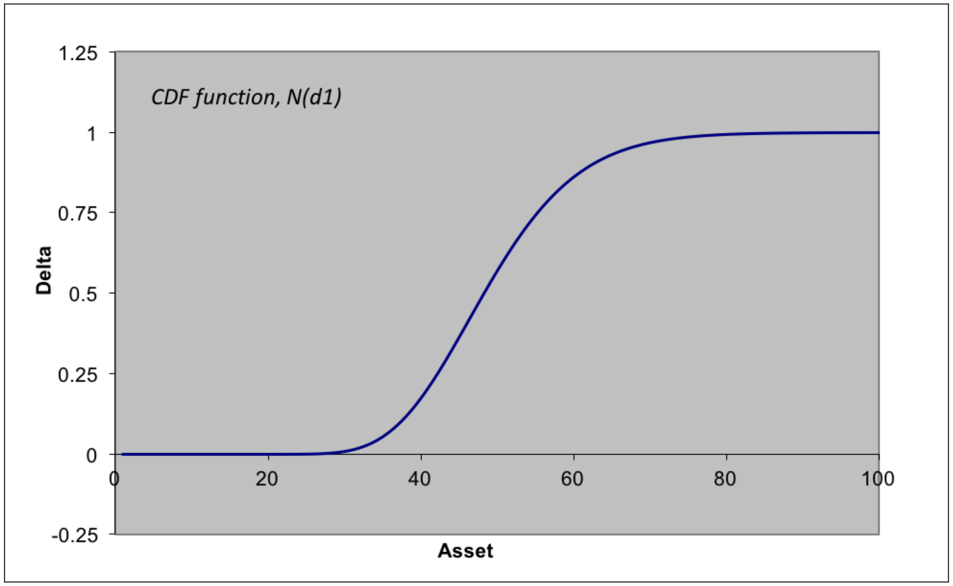
That assumes the exposure to asset price direction is delta-hedged. Exposure to the magnitude of move $(\Delta S)^2$ remains.

Option traders (sell-side) have always compensated for the limitations of the Black-Scholes valuation by adjusting the implied volatility at which they sell. The main limitation is imperfect delta-hedging.

Delta hedging

Hedging is governed by the shape of $\frac{\partial V}{\partial S} = N(d_1)$ Normal CDF.

The more price drops the less shares we need to hold. At 0% or 100% hedged we are not much sensitive to change in BS Delta .



Not selling fast enough means extra losses (right-to-left)

(left-to-right) Not buying fast enough means missed profits

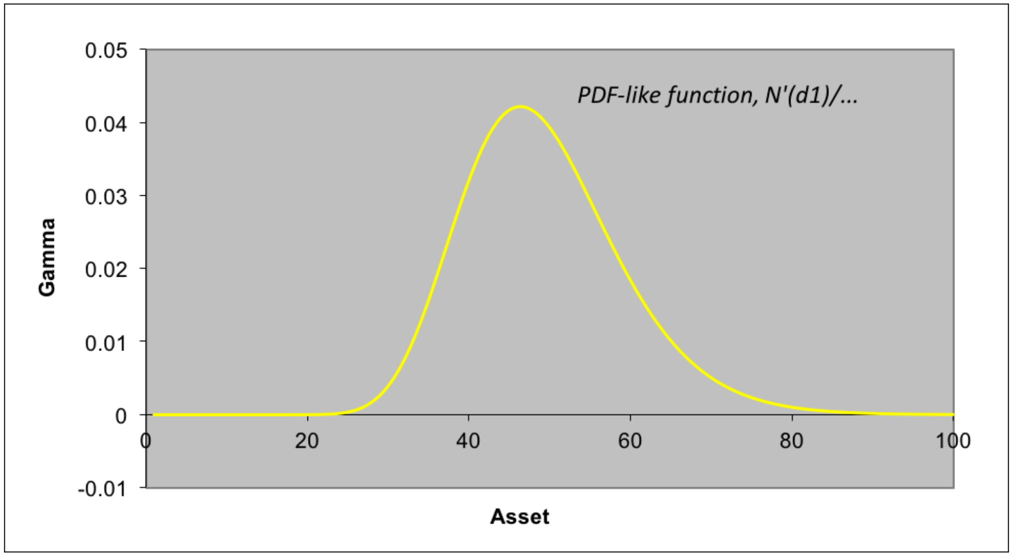
Consider a price peaked and dropping. Hedger problem is not being able to sell **fast enough**: the peak was at \$200, she supposed to reduce from $\Delta_{\times 100} = 99$ to 90 at \$190 but was able to sell at \$185 only. \$5 slippage loss incurred per share.

The risk-neutral condition of the Black-Sholes assumes the ability to delta-hedge continuously: no bid/ask, no transaction cost, in other words, full liquidity.

- Buying and selling large quantities Δ_{BS} *timely* (fast enough) is problematic.

Hedging put exposure in particular requires short-selling in volume – that can be fine for the futures on major index S&P500 but a pain for single stock.

Gamma



"Fast enough" equates to Gamma.

Traders professional language to say **short gamma**, as in "not enough gamma" or failure to anticipate that Gamma increases.

Inability to enter or exist market at right times also referred to as being short Gamma but that is a vernacular.

Rule of Thumb: when implied volatility is low, Gamma function peaks high. We saw that right on Excel demo.

The is also when hedgers (market-makers) likely to suffer a loss. Consider P&L from a hedged position in an option, where option price has already been decomposed into Greeks:

$$\begin{aligned}
\text{P\&L}_{\Delta t, \Delta S} &= \cancel{\Delta(\Delta S)} + \frac{1}{2}\Gamma(\Delta S)^2 + \underbrace{\Theta \Delta t} - \cancel{\Delta(\Delta S)} \\
& \qquad \text{the Black-Sholes } \Theta_i = -\frac{1}{2}\sigma_i^2 S^2 \Gamma_i \\
&= \frac{1}{2}\Gamma(\Delta S)^2 + -\frac{1}{2}\Gamma_i S^2 \sigma_i^2 (\Delta t) \\
&= \frac{1}{2}\Gamma S^2 \left[\left(\frac{\Delta S}{S}\right)^2 - \sigma_i^2 \Delta t \right].
\end{aligned}$$

Risk-neutral condition: given continuous delta-hedging at known σ_a input, assumed constant over the life of option, the pricing PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Closed-form solution for vanilla call and put, known as the Black–Scholes formulæ

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$P(S,t) = C(S,t) + Ee^{-r(T-t)} - S$$

"What volatility must I use to get the correct market price?"

Put into the Black-Scholes formulæ, it gives *a theoretical price*, which is equal to the observed market price.

$$V(S,t;K,T;r,\sigma) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Example: A trader sees on his screen that a put option with one year until expiry and a strike of 100 at the money is trading at \$5.57 and a short-term interest rate of 5%.

How do we know which volatility to put into the $d_{1,2}$? Well, approximately here $\sigma_i \approx 20\%$

$$V(S, t; K, T; r, \sigma) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

To obtain the implied volatility, we have to numerically solve **a root-finding problem**.

We will explore the ready working in Lecture Solutions.

Implied Volatility

Often described as **the market's view of volatility** to realise over the lifetime of the particular option.

Why this view is naïve?

Option prices are influenced by models, and supply & demand (fear and greed).

- There is expiration time T for each implied volatility: (0, 3M), (0, 6M), etc.

- 'Over the lifetime'

Actual/Local Volatility

Amount of randomness in asset return at any instant of time.

- There is no 'timescale' associated with actual volatility.

- It is an instantaneous quantity.

Example: The actual volatility is 20%. . . (asset drops) now it is 26%. . . (asset rises slightly) now it is 24%. . .

$$dS = \mu S dt + \sigma_a S dX$$

Delta hedging improved

Traders tinker with σ_i value they use to compute d_1 for

$$\Delta_{BS} = N(d_1)$$

Minimum Variance (MV) delta takes into account both, changes in S and expected changes in σ_{imp} as a result of changes in S,

$$\begin{aligned} \Delta_{MV} &= \Delta_{BS} + \text{Vega}_{BS} \frac{\partial E(\sigma_{imp})}{\partial S} \\ \frac{\partial E(\sigma_{imp})}{\partial S} &\approx \frac{a + b\Delta_{BS} + c\Delta^2_{BS}}{S\sqrt{T}} \end{aligned}$$

The expected rate of change of the implied volatility wrt changes in the stock price is moderated by Vega.

Vega

An option value can change even when the underlying doesn't move.

A market in panic will sell expensive Puts (and so will be Calls).

$$\text{Vega}_{BS} = \frac{\partial V}{\partial \sigma} = S\sqrt{T}\,N'(d_1)$$

Practitioners refer to Vega risk but it requires experience in specific markets and advanced models to discuss it.

- potential dramatic changes in the level of implied volatility make Vega important.

Example: An option has Vega of 37.5, that means that if implied volatility σ_i goes up by 1 point (from 20% to 21%), the cash option price ($\times$100 shares) will change by \$3,750.

Vega is a **bastard Greek**, taken with regard to parametrised σ.

Inherently, $\frac{\partial V}{\partial \sigma}$ depends on what kind of function we assume for σ. We will see that calibrated $\sigma(t)$ is just piecewise constant.

After selling exotics to clients, sell-side hedges Vega risk out (by purchasing vanilla calls and puts) – trading book management.

Hedging with Greeks:

Suppose the following contracts are available to us:

	Delta	**Gamma**	**Vega**
Stock	1	0	0
Option A	0.4	0.026	27
Option B	0.6	0.018	36

Consider two strategies:

1. Construct a portfolio that is Delta and Gamma-neutral but positive Vega. (Why need one?)

2. Option B is an OTC exotic, you are going to sell it for more than it is worth. Construct Delta and Vega-neutral portfolio.

Strategy 1: If we expect implied volatility to rise in short run.

A quantity A of option A, B of option B, and C of the stock.

Delta neutral: $C + 0.4A + 0.6B = 0$

Gamma neutral: $0.026A + 0.018B = 0$

Positive vega: $27A + 36B = 1$ (1 unit or percent)

How much Vega risk we can take would be constrained by cash limits. The solution suggests short stock, short contract A, and go long contract B:

$$C = -0.01867, \quad A = -0.04 \quad \text{and} \quad B = 0.05778.$$

Strategy 2: Sell OTC Option B to make a profit, but be delta and vega neutral.

A quantity A of option A, -1 of option B, and C of the stock.

Delta neutral: $C + 0.4A + 0.6 \times (-1) = 0$

Vega neutral: $27A + 36 \times (-1) = 0$

The solution is

$$C = 0.06667, \qquad \text{and} \qquad A = 1.3333.$$

Please remember you can add Greeks arithmetically in order to calculate the total exposure for a basket of options.

Greeks are 'risk factors' for all purposes including FRTB.

Delta says by how much the option price moves, if asset price moves (market risk on asset). **Gamma** estimates by how much Delta might change.

Vega says by how much the option moves if a perception of volatility changes (market risk of options market).

P&L from a derivative(s) position be decomposed with its Greeks:

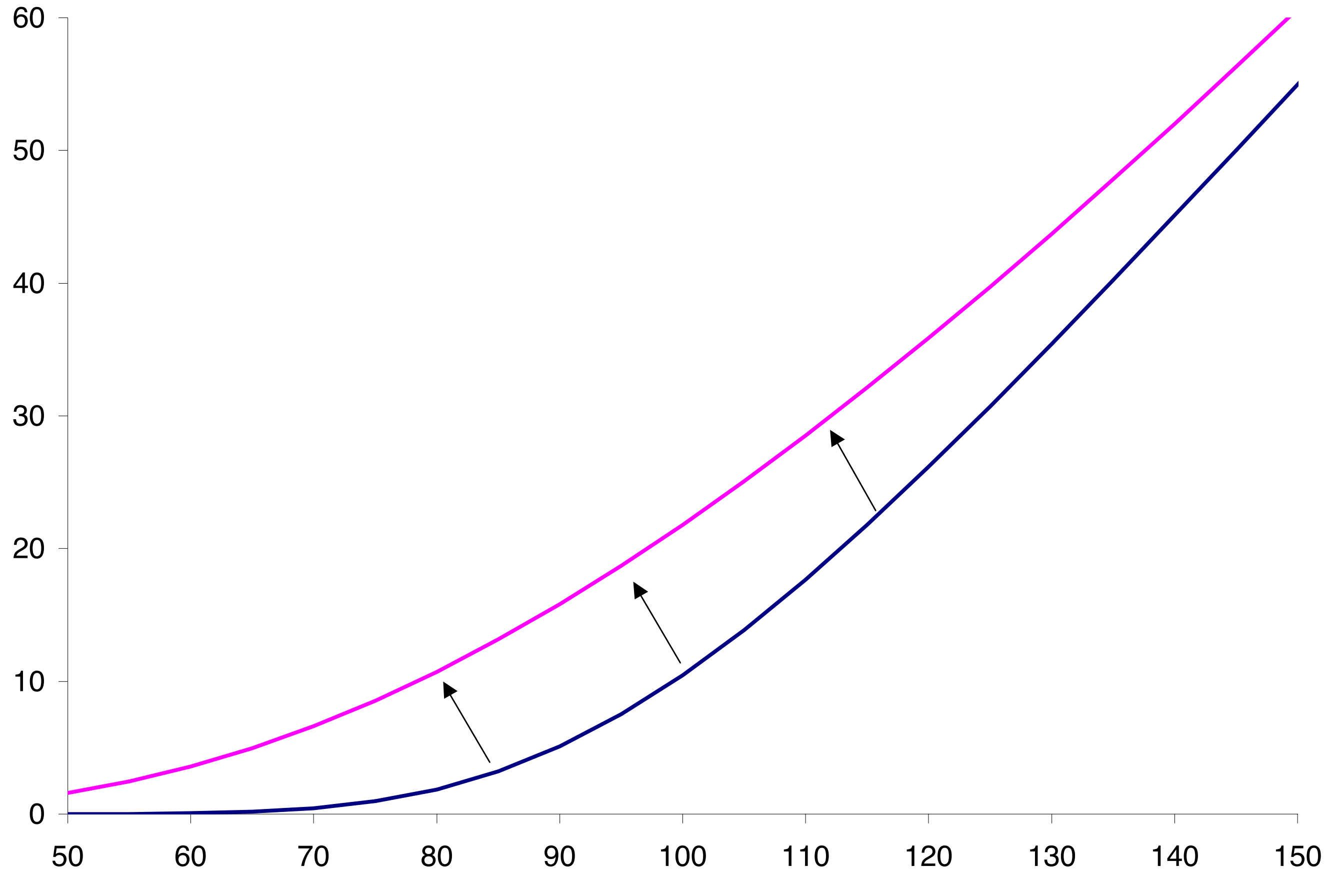
$$\mathbf{P\&L} = \Delta \times (\Delta S) + \frac{1}{2}\Gamma \times (\Delta S)^2 + \Theta \times (\Delta t) + \nu \times (\Delta \sigma) + \ldots$$

This is helpful when you need to compute the **VaR** (but Theta).

Basel III sensitivities-based method to compute capital requirements utilises Delta, Gamma (called Curvature risk) and Vega. This is Standardised approach.

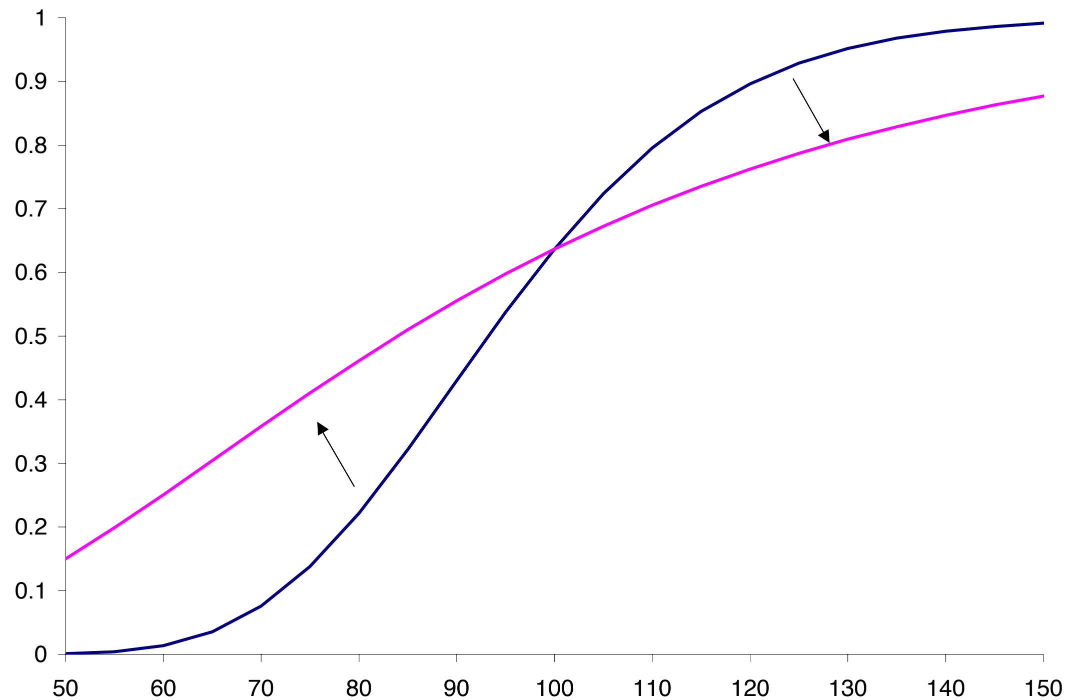
Internal models approach: backtesting and P&L attribution test.

Rule of Thumb 1: Increased IV will increase option value if Gamma is positive $\Gamma > 0$.



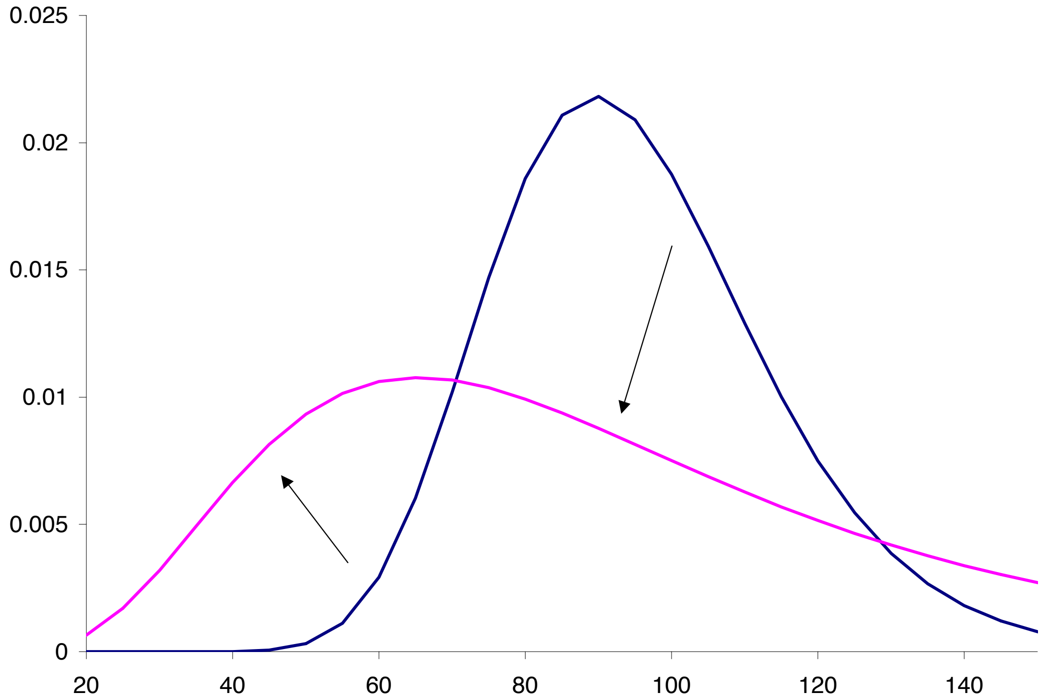
Rule of Thumb 2: Increasing volatility 'smoothes' out the curvature.

This is true for option price, also the Greeks. For example, Delta vs. the underlying asset.



Rule of Thumb 3: Increasing volatility makes Gamma less 'severe'

When volatility is low, Gamma can get very large.



Rule of Thumb 4: Delta-hedging Puzzle

A trading book has only a few at-the-money options. Each has several days before expiration. Volatility is low.

A trading desk that does delta-hedging lost a large sum amount of money.

How can that happen?

Two different approaches

We are going to map our thinking about information offered by implied volatility:

First, we will assume the market is wrong!
If we have a better idea of volatility than the market, the opportunity for **volatility arbitrage** opens. If we are right we will make money.

Second, we will assume the market is right.
The implied volatility offers a close-to-perfect information on actual volatility. We just need some maths to relate the two.

Why would the market be wrong in predicting?

Option prices are driven by models, default risk, and trading conditions, eg insurance buyers vs. earning carry vol sellers.

- **Out of the money puts** are a popular form of insurance. They are low-priced in dollar terms but not cheap in IV terms.

- When someone sells an OTM put, the premium has to cover **model risk** and uncertainty (discrete hedging, probability of jump) and **cost of funding** and exposure (CVA).

  Oh, and **Profit**!

Problems with Black–Scholes model

- Effects of discrete hedging. Discrete price effect (especially in the teenies)
- Transaction costs on the underlying. Illiquidity
- **Uncertainty in implied volatility**
- **Jumps**, 'flash crashes', discontinuous paths in asset price
- **Default risk** of the underlying asset
- Uncertainty about dividends
- Counterparty risk. CVA for OTC trades
- **Supply and demand. Feedback, market manipulation**

So what is Black–Scholes?

- Not a good model for the real prices or to predict asset volatility

- Not a good model to determine the ‘fair value’ of an option

- Not a good model to arbitrage options

- BUT a powerful toy model to interpret a market quote:
  - Understand risk factors (Greeks) and estimate an option’s manufacturing costs (delta replication)

Volatility Arbitrage

Should I hedge with implied or actual volatility?

Scenario: Implied volatility is $\sigma_i = 20\%$ but we believe that the asset will realise over a period the actual volatility of $\sigma_a = 35\%$.

- Should we use σ_a or σ_i to delta-hedge?

This is one of those questions that no one seems to know the answer to. Black–Scholes gives the formula:

$$\Delta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{s^2}{2}}\, ds \qquad \text{which is } N(d_1)$$

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Market participants can easily agree on S, r but not σ.

Case 1: Hedge with actual volatility, σ_a

So you believe an option at $\sigma_i = 20\%$ is mispriced... **how can you profit from this?**

Buy an option and delta-replicate: cash from *buying* V^i and *selling* Δ^a quantity of the stock:

$$-V^i + \Delta^a S$$

By continually selling stock we replicate a short position in a correctly priced option V^a.

Eventually, we shall earn a pile of money equal to option premium $\$V^a$... at the market's expense!

The profit to make is exactly the difference in the Black–Scholes theoretical prices

$$V(S,t;\sigma_a) - V(S,t;\sigma_i)$$

or simply

$$V^a - V^i$$

How do we know this guaranteed total profit?

Total profit is summation over all timesteps dt.

Let's do the maths **on the mark-to-market basis**, by which we mean to consider P&L over each time step.

'Today' at time t:

Option	V^i
Stock	$-\Delta^a S$
Cash	$-V^i + \Delta^a S$

'Tomorrow' at time $t + dt$:

Option	$V^i + dV^i$
Stock	$-\Delta^a S - \Delta^a dS$
Cash	$(-V^i + \Delta^a S)(1 + r\, dt)$

Δ^a calculated using the actual volatility in $N(d_1)$.

Therefore we have made marked to market,

$$dV^i - \Delta^a \, dS - (V^i - \Delta^a \, S) \, rdt \dagger \tag{1}$$

Because the option would be correctly valued at V^a then we have

$$dV^a - \Delta^a \, dS - (V^a - \Delta^a \, S) \, rdt = 0 \ddagger$$

This is profit from time t to $t + dt$ is

$$\begin{aligned}
\dagger - \ddagger &= dV^i - dV^a + r(V^a - \Delta^a \, S) \, dt - r(V^i - \Delta^a \, S) \, dt \\
&= dV^i - dV^a - r(V^i - V^a) \, dt \\
&= e^{rt} \, d\left(e^{-rt}(V^i - V^a)\right) \qquad \textit{by Integrating Factor } e^{-rt} \\
d(e^{-rt}V) &= e^{-rt}dV - re^{-rt}V dt = e^{-rt}(dV - rV dt)
\end{aligned}$$

PV-ing that increment of profit to t_0 gives

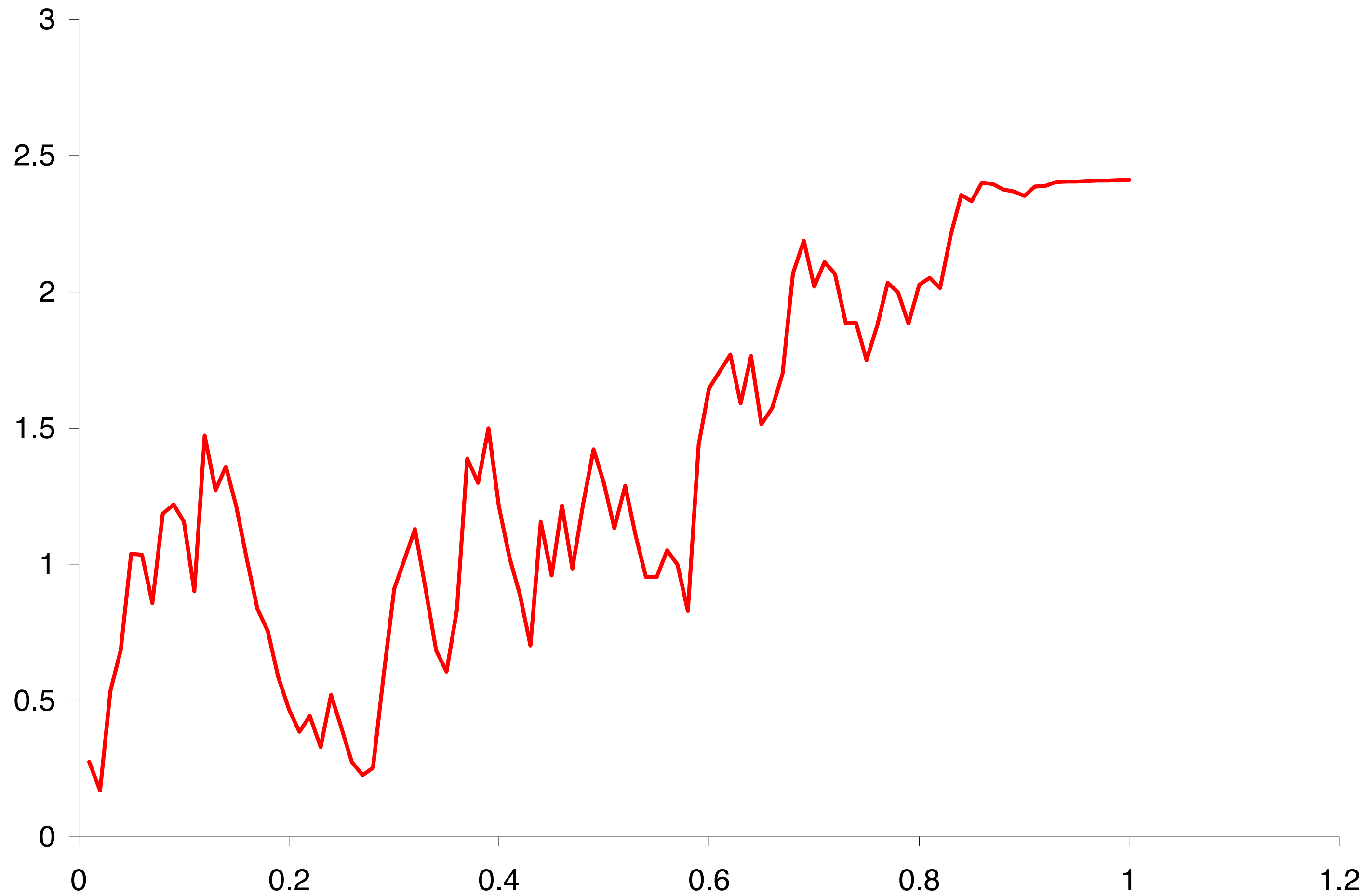
$$e^{-r(t-t_0)} \underbrace{e^{rt}\, d\left(e^{-rt}(V^i - V^a)\right)} = e^{rt_0}\, d\left(e^{-rt}(V^i - V^a)\right)$$

And **the total profit** from t_0 to expiration comes from summation (integration in continuous time)

$$e^{rt_0} \int_{t_0}^{T} d\left(e^{-rt}(V^i - V^a)\right) = V^a - V^i$$

The total profit is a known quantity.

As we said.



How the guaranteed profit is achieved is random. MtM P&L is affected by asset price and Gamma.

Case 2: Hedge with implied volatility, σ_i

We are balancing the random fluctuations in the value of option we bought dV^i **just** with the fluctuations in the stock price.

$$-V^i + \Delta^i S$$

- Such model is inconsistent: implied volatility changes and that risk measured by Vega. But the risk is not reflected in Δ_{BS}.

Portfolio cash does not depend on actual volatility! (See Excel)

Buy the option today for V^i, hedge using Δ^i of the stock.

The mark-to-market over dt is our MtM Equation † <u>with</u> $\Delta^a \to \Delta^i$

$$\underline{dV^i - \Delta^i \, dS} - r(V^i - \Delta^i S)\, dt$$

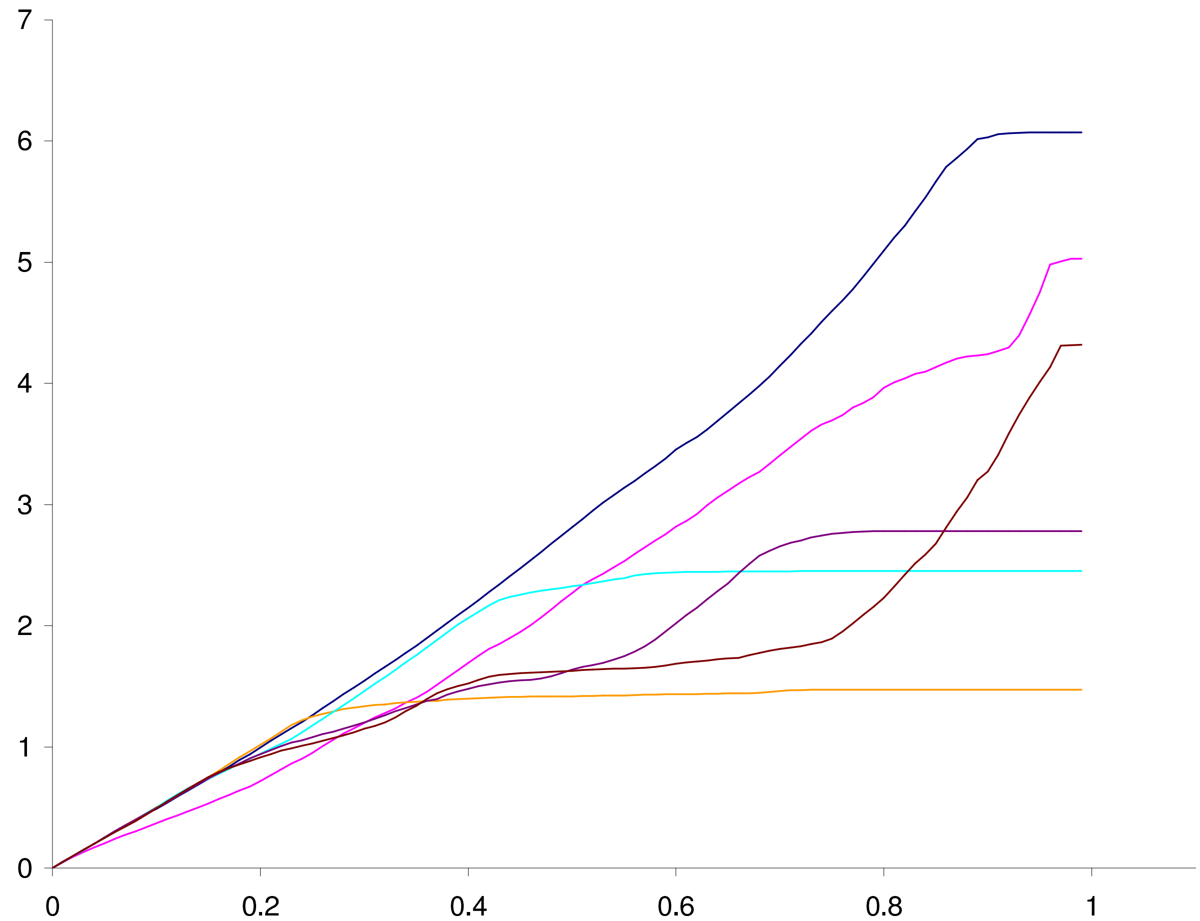
$$= \underline{\Theta^i \, dt + \tfrac{1}{2}\sigma_a^2 S^2 \Gamma^i \, dt} - r(V^i - \Delta^i S)\, dt$$

$$= \tfrac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2 \Gamma^i \, dt$$

To get **the total profit**, add up the present value of all moves over dt,

$$\frac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) \int_{t_0}^{T} e^{-r(t-t_0)} S_t^2 \, \Gamma^i \, dt$$

This is positive $\sigma_a > \sigma_i$ but not a guaranteed amount.



Using implied volatility σ_i as a prediction, while asset evolves according to its own actual volatility σ_a. **End result is uncertain.**

Calibration

Term Structure + Volatility Skew = Volatility Surface

Assuming the market 'knows' what actual volatility is going to be in the future – we should extract that knowledge of a function $\sigma(t)$!

That kind of fitting poses an inverse problem $\sigma(t) \quad \Leftarrow \quad \sigma_{BS}(T)$.

Term Dependence

Below is typical market pricing ofEuropean call options,all at the strike 110.

Expiry	**Value**	**IV**
3 months	4.74$	22.8%
6 months	6.72$	20.9%
9 months	8.22$	19.7%
12 months	9.63$	19.1%

Observation: clearly these prices cannot be correct if actual volatility is constant for the whole 12M.

σ(0, 3M), σ(3M, 6M), etc – gives piecewise linear approximation of $\sigma(t)$ which is form of σ_a.

If volatility varies with time $\sigma(t)$, the value of a vanilla option is a function of **an average volatility** between now and expiry,

$$\sigma_{BS} = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(\tau)d\tau}$$

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\frac{1}{T-t}\int_t^T \sigma^2(\tau)d\tau S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Black–Scholes formulæ still work.

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Black-Sholes solution breaks down (textbooks don't say it).

We would expect to know $\sigma(t)$ and use it for option pricing

$$\sigma(t) \quad \Rightarrow \quad V(\sigma_{BS}(T))$$

However, we do know **the answer**, the market option price V and have to find the matching fitted structure for $\sigma(t)$.

$$\sigma(t) \quad \Leftarrow \quad \sigma_{BS}(T)$$

This is an **inverse problem**, leading to an integral equation:

$$\sqrt{\frac{1}{T-t}\int_t^T \sigma^2(\tau)d\tau} = \text{ implied volatilities}$$

In practice, implied volatilities are only known at a finite number of expiration points with a fairly large increment.

Example:

Implied volatility σ_{imp}(0, 1M)= 30%

Implied volatility σ_{imp}(0, 2M)= 25%

Implied volatility σ_{imp}(0, 3M)= 26%

Fitting: fit a volatility function consistent with this term structure.

Assume $\sigma(t)$ is **piecewise constant**,

$$\frac{3}{12}\sigma_{imp}(0,3M)^2 = \frac{1}{12}\sigma_a^2[0,0.08] + \frac{1}{12}\sigma_a^2[0.08,0.17] + \frac{1}{12}\sigma_a^2[0.17,0.25]$$

Let's simplify $\sigma_1 = \sigma_a[0,0.08]$, $\sigma_2 = \sigma_a[0.08,0.17]$, and on

$$\begin{aligned}
\sigma_{imp}^2(0,3M) &= \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{3} \\
\sigma_{imp}^2(0,2M) &= \frac{\sigma_1^2+\sigma_2^2}{2} \\
\sigma_{imp}(0,1M) &= \sigma_1 = 0.3.
\end{aligned}$$

For the second period

Implied volatility σ_{imp}(0, 1M)= 30% $= \sigma_1$

Implied volatility σ_{imp}(0, 2M)= 25%

$$\sigma_2 = \sqrt{2\sigma^2_{imp}(0, 2M) - \sigma_1^2}$$

$$\sigma_2 = 0.187$$

consistent with the general result for piecewise volatility

$$\sigma(\mathbf{t}) = \sqrt{\frac{T_i - t^*}{T_i - T_{i-1}}\,\sigma^2_{imp}(T_i) - \frac{T_{i-1} - t^*}{T_i - T_{i-1}}\,\sigma^2_{imp}(T_{i-1})} \qquad (2)$$

So, the term structure is reconciled with

$$\frac{0.3^2 + 0.187^2}{2} = 0.25^2.$$

The second period [1M,2M] has much, much lower volatility than any over-the-life of option implied volatility $[0, T]$ suggested.

Please reivew Lecture Solutions for the treatment of inverse problem and general result Equation (2).

For the third period

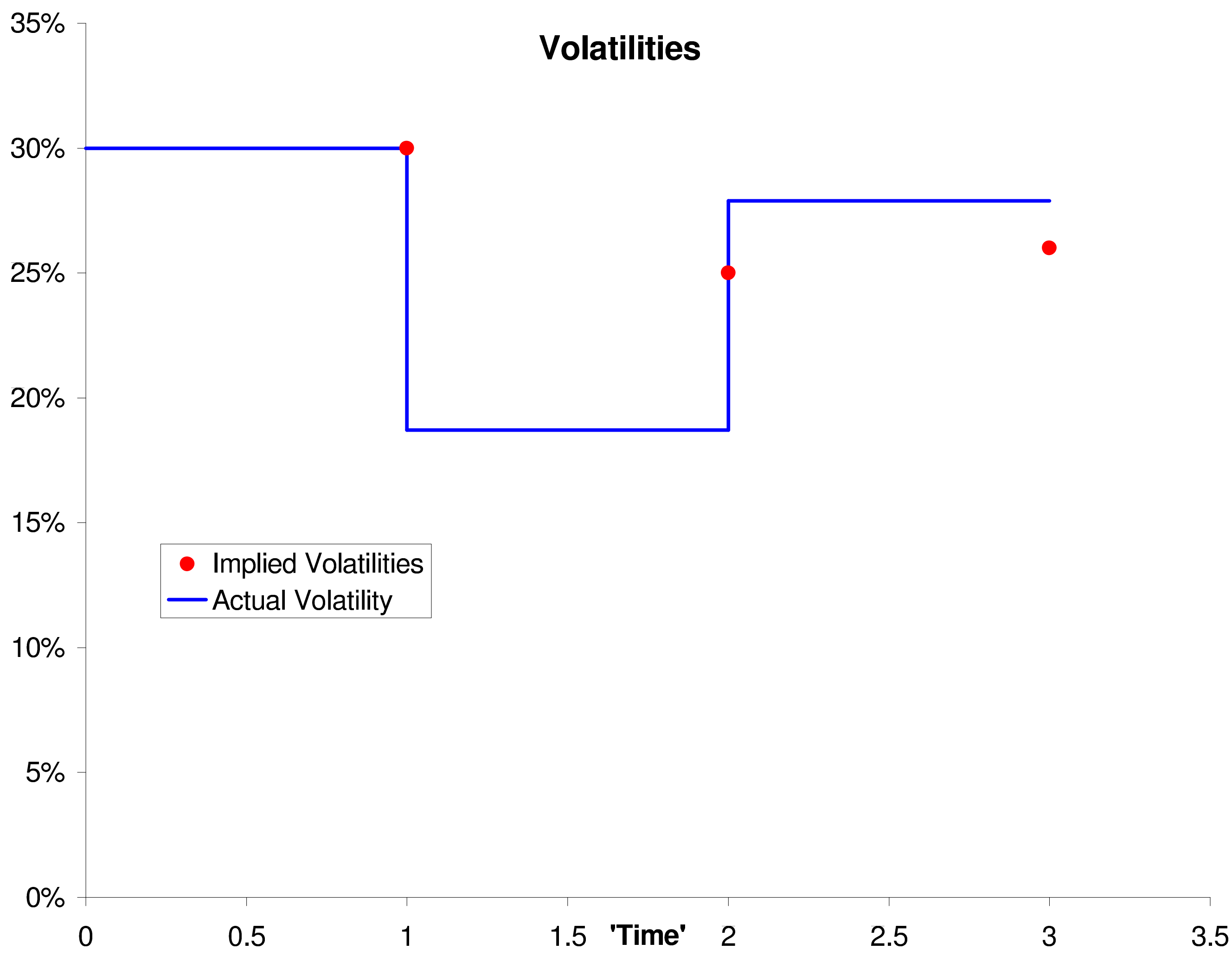
$$\sigma^2_{imp}(0,3M) = \frac{\sigma_1^2+\sigma_2^2+\sigma_3^2}{3}$$
$$\sigma_3 = \sqrt{3\sigma^2_{imp}(0,3M)-\sigma_1^2-\sigma_2^2}$$

$$\frac{1}{3}0.3^2+\frac{1}{3}0.187^2+\frac{1}{3}\sigma_3^2 = 0.26^2$$

The solution is $\sigma_3 = 0.279$.

For comparison, $\sigma_1 = 0.3$, $\sigma_2 = 0.187$, and $\sigma_3 = 0.279$

σ_{imp}(0, 1M)= 0.3, σ_{imp}(0, 2M)= 0.25, σ_{imp}(0, 3M)= 0.26.



Volatilities
35%
30%
25%
20%
15%
10%
5%
0%
0
0.5
1
1.5
'Time'
2
2.5
3
3.5
Implied Volatilities
Actual Volatility

Summary for our volatility fitting

- An iterative bootstrapping: assume initial σ_1, then use it to obtain σ_2, and then use both to obtain σ_3.

- The fitting is linear in time, or linear in variance $\sigma^2(T-t)$.

- This is piecewise constant $\sigma_1[t_0, t_1], \sigma_2[t_1, t_2], \sigma_3[t_2, t_3]$ fitting towards a continious function $\sigma(t)$.

- The solution is non-unique. Black-Scholes equation becomes 'parametrised' because it relies on our non-unique fitted input. Think of impact on Vega $\dfrac{\partial V}{\partial \sigma}$.

Volatility Skew

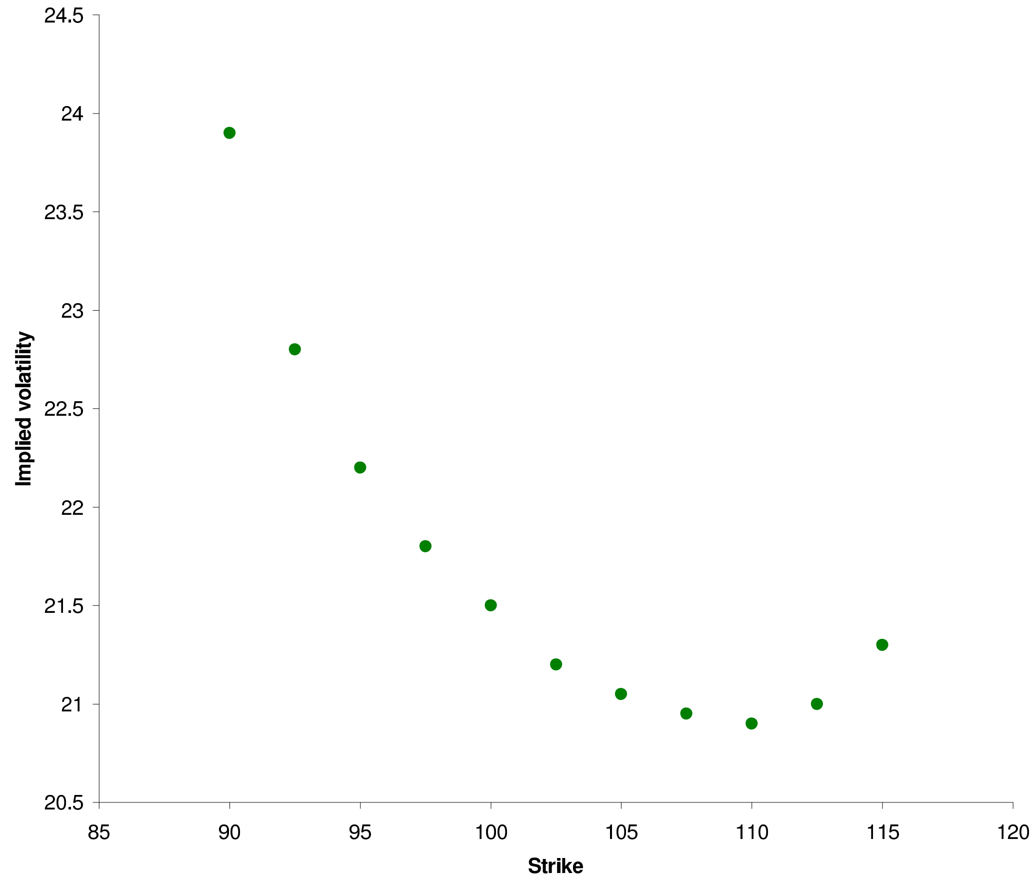
That's term structure dealt with... now 'strike structure' better known as the volatility skew or smile.

Now, we have two European call options with different strikes, both expiring in six months

Strike	Value	IV
100	12.84$	**21.5%**
110	6.72$	**20.9%**

These are two conflicting views on volatilities for the same expiry.

Concentrating on the same example, suppose call options are traded with strikes of 90, 92.5, 95, 97.5, 100, 102.5, etc.



The shape of this curve of implied volatility *vs.* strike K is called **the volatility skew** or smile.

Sometimes, the shape is upside down in a **frown**.

More sensibly, the skew is against Moneyness $\log(S/K)$.

- We related the over-pricing of OTM puts to the probability of default.

A few words and introduction to local volatility modeling, before we conclude.

Calibrating volatility that varies with expiration and strike

Just as we *assumed* actual volatility is a function of t to match implied volatility being a function of T, we now make actual volatility a function of S and t!

$$\sigma(S,t) \;\; \Rightarrow \;\; \sigma_i(K,T)$$

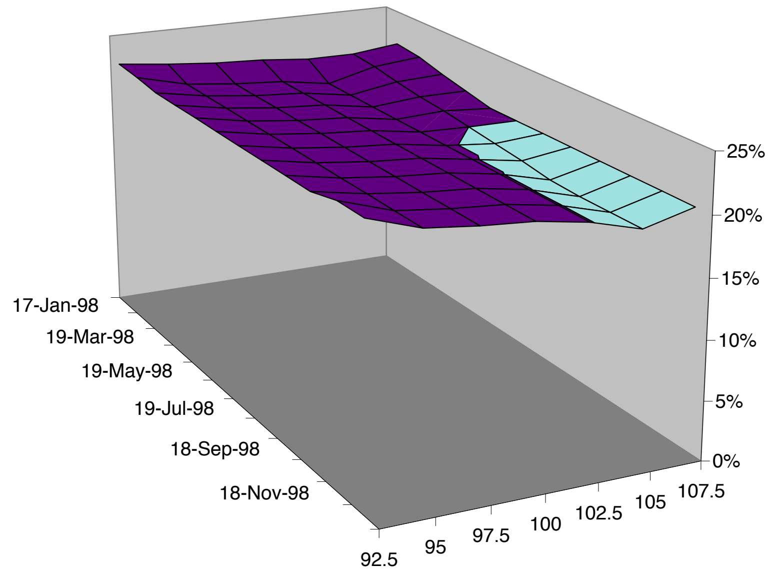
Again, in this problem we know the output (market prices). The calibration task becomes,

$$\sigma(S,t) \;\; \Leftarrow \;\; \sigma_i(K,T)$$

This is another **inverse problem**.

Volatility Surface

Implied volatility plotted against strike and maturity in 3D.

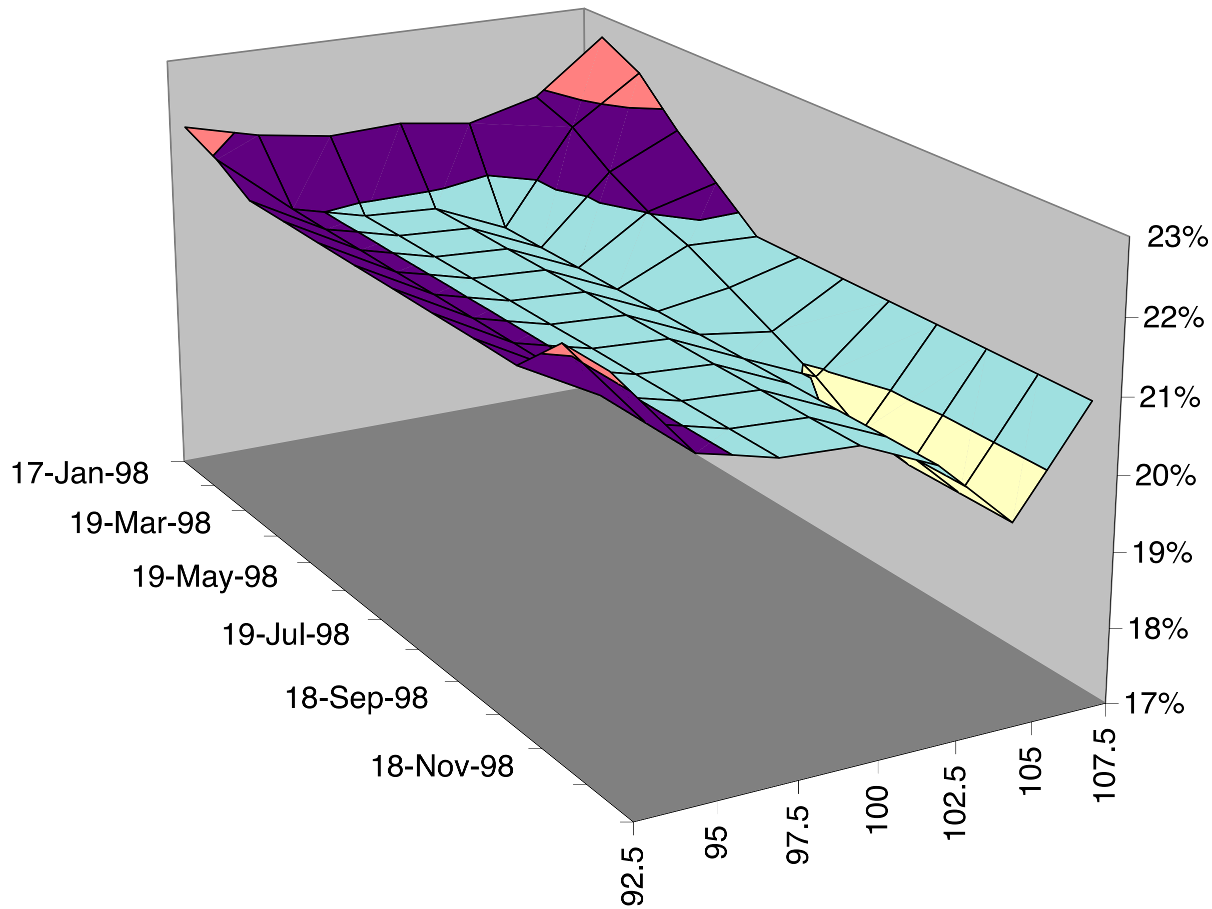


This **implied volatility surface** represents the constant BS volatility that gives each traded option value equal to the market price.

Now, think of $\sigma(S,t)$ as the market's view of the forward volatility **when** the asset price is S at time t.

- if these predictions come to pass – we can price more complex derivatives with the calibrated (fitted) local volatility.

A lot of maths omitted!



Local volatility surface calibrated from European call prices.

Rule of Thumb: Local Volatility

Local volatility $\sigma(S)$ changes approximately **twice as fast** with spot S as implied volatility σ_{imp} changes with strike K.

$$\begin{aligned} \sigma(S) &= \sigma_0 + \beta S \qquad \forall t \\ \Sigma_{imp}(S,K) &\approx \sigma(S) + \frac{\beta}{2}(K - S) \end{aligned}$$

This makes the local volatility surface to appear wilder or rougher.

How do I use this local volatility surface?

Prediction can work but also is a very naive.

- Not only do the predictions not come true but even if we come back a few days later to look at the 'prediction,' i.e. to refit the surface, we see that $\sigma(S, t)$ has changed throughout.

- BUT as long as we price our exotic contract consistently with the vanillas, *and simultaneously hedge with these vanillas*, then we are reducing our exposure to the model risk of $\sigma(S, t)$.

That's enough about calibration, *for now*!

For delta hedging

The Black-Scholes hedging strategy will perform **the worst** in fast sell-offs or slow rises. Negative skew markets are precisely the feature of the equities and equity indices.

The local-volatility hedging strategy will have problems in slow sell-offs and fast rise, both are low volatility environments.

There is more, Minimum Variance (MV) delta can be easily adopted for local volatility

$$\Delta_{MV} = \Delta_{BS} + \text{Vega}_{BS} \frac{\partial \sigma_{imp}}{\partial K}$$

Model Risk

Please... always know which volatility you, and others, are talking about.

There are different ways of modelling for actual volatility, each coming with some strong assumptions:

- uncertain volatility $[\sigma^{min}, \sigma^{max}]$
- deterministic volatility $\sigma(t)$
- local volatility $\sigma(S, t)$

Summary

Please take away the following important ideas:

- implied volatility is not the same as realised volatility

- we strive to know the actual volatility. Calibration of $\sigma(t)$ as a function is subject to assumptions

- Gamma- and Vega- hedging is difficult, we want to get as much milleage as possible from Delta hedging

- options can be used for making a profit from volatility models

Extra Slides

The different types of volatility

- Actual/Local

- Historical/Realized

- Implied

- Forward

Volatility is the most important parameter determining the value of an option $V(S,t;K,T;r,\sigma)$, yet it is also the hardest to measure.

Actual/Local Volatility

Amount of randomness in asset return at any instant of time.

- There is no 'timescale' associated with actual volatility.

- It is an instantaneous quantity.

Example: The actual volatility is 20%. . . (asset drops) now it is 26%. . . (asset rises slightly) now it is 24%. . .

$$dS = \mu S dt + \sigma_a S dX$$

Historical/Realized Volatility

A measure of the amount of randomness over some period in the past. Estimation method may differ from std. deviation.

- We should model historic volatility differently for the short timescale (day-to-day) and long-term horizon.

Computed from daily log-returns $r_t = \ln(S_t/S_{t-1})$,

$$\sigma_t = \sqrt{\frac{\sum^{N}(r_t - \mu)^2}{N-1}}$$

an average of squared daily differences from the mean $(r_t - \mu)^2$.

Timescale

Doesn't matter how many observations there: 21, 60, or 252.

σ_t calculated on daily differences $(r_t - \mu)^2$ so, timescale remains **daily**.

To scale to reflect a ten-day move, we use the additivity of variance:

$$
\begin{aligned}
\sigma_{t,10D} &= \sqrt{\sigma_t^2 + \sigma_t^2 + \sigma_t^2 + \ldots} \\
&= \sqrt{10\,\sigma_t^2} \\
&= \sigma_t \sqrt{10}
\end{aligned}
$$

EGARCH Example:

$$\ln \sigma_t^2 = \alpha_0 + \frac{\alpha_1 u_{t-1} + \gamma_1 |u_{t-1}|}{\sigma_{t-1}} + \beta_1 \ln \sigma_{t-1}^2$$

Thorough empirically-tested solution. Advantageous estimator.

Problem (for options): strong assumption of reversion to the long-term variance $\overline{\sigma} = \frac{\alpha_0}{1-\alpha}$.

Alternative: exponential smoothing (EWMA on variance) which simply puts higher weight on past return $\alpha_1 > \lambda$,

$$\sigma_t^2 = \alpha_1 u_{t-1}^2 + \lambda \sigma_{t-1}^2$$

We aim to forecast short-term volatility $< 3M$ (to use in hedging) and that requires creativity.

Implied Volatility

Often described as **the market's view of volatility** to realise over the lifetime of the particular option.

Why this view is naïve?

Option prices are influenced by models, and supply & demand (fear and greed).

- There is expiration time T for each implied volatility: (0, 3M), (0, 6M), etc.

- 'Over the lifetime'

Volatility is quoted in annualised terms.

Black-Scholes takes annualised volatility and that is why it is always $\sigma\sqrt{T-t}$.

Std. deviation, from daily returns (any sample size), requires $\sigma_t\sqrt{252}$.

A three-month option suggests

$$\sigma_t\sqrt{252/4} = \sigma_{BS}(0, 3M)\sqrt{1/4}$$

Forward Volatility and VIX

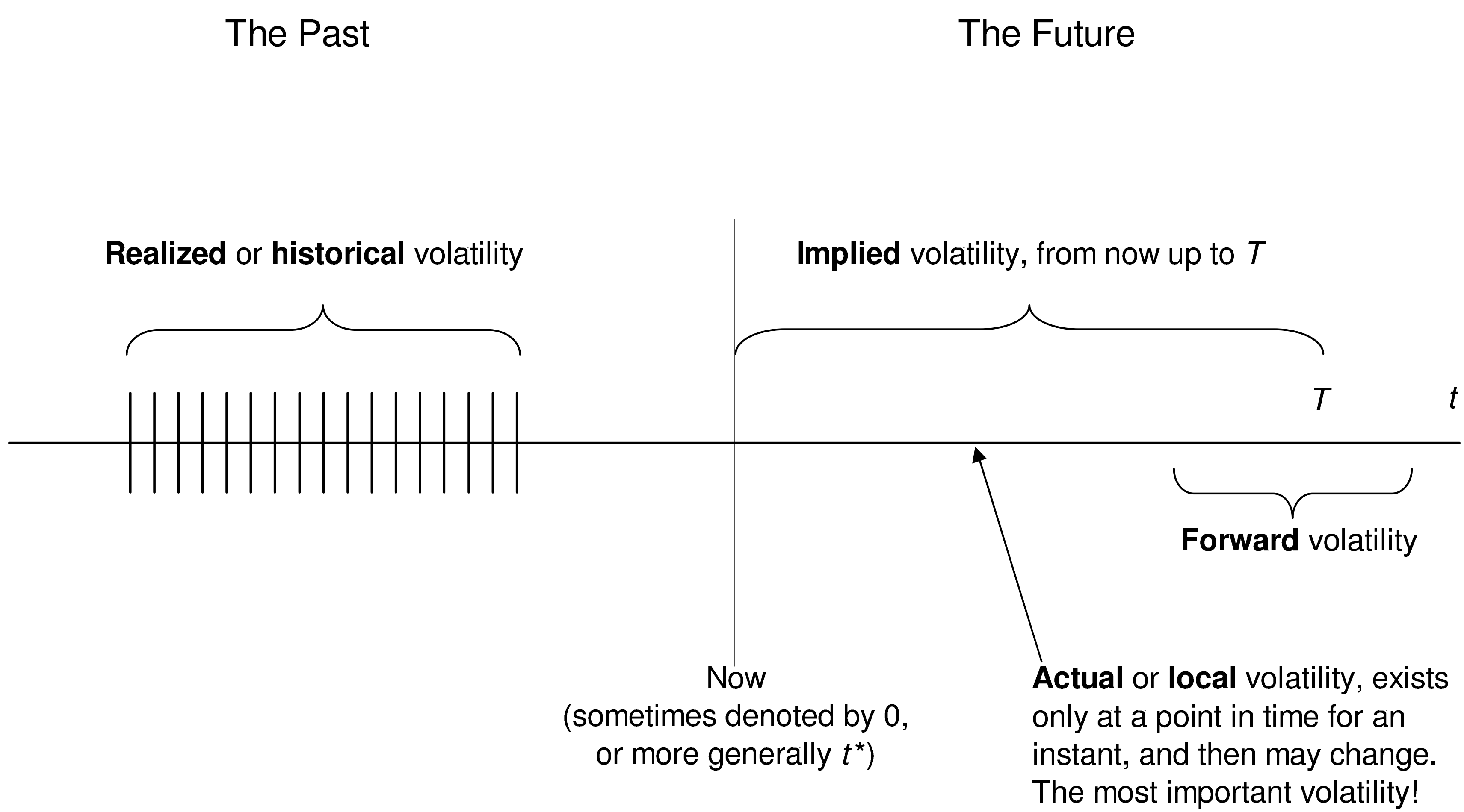
- Refers to variance over some period in the future:

$$\mathbf{E}\left[\int_0^T \sigma^2\, dt\right] = 2\left\{\int_{-\infty}^{0} \frac{p(k)}{k^2}\, dk + \int_0^{\infty} \frac{c(k)}{k^2}\, dk\right\}$$

where $p(k)$ and $c(k)$ are log-values of the strip of all traded put and call options (same expiry) – incl. far OTM and ITM strikes. Put-call parity $c(k) - p(k) = F - K$

Discretisation will lead to VIX Calculation (CBOE 2019),

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2}\, e^{RT}\, Q(K_i) \; - \; \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2.$$



VIX is forward price of the 30-day variance – of S&P500 returns.

σ_i is **a parameter** of the Black-Scholes which we treat as a variable.

The implied volatility is attributed to the underlying. However, the actual volatility exists even in the absence of options.

The proper measure of randomness in the asset is σ_a, which turns out to be an instantaneous quantity. We can't measure that quantity directly.

Vega

Q What happens to the value of an option when perceptions about volatility of the underlying change? **A** Implied volatility can change on its own.

An option value can change even when the underlying doesn't move.

A market that is panicking sells puts and calls more expensively.

We would like to know how sensitive an option value is *wrt* that,

$$\frac{\partial V}{\partial \sigma}$$

Example: An option has Vega of 37.5, that means that if implied volatility σ_i goes up by 1 unit (from 20% to 21%), the cash option price will change by \$37.5.

Usually, options are traded per 100 shares (US exchanges).

Vega is a **bastard greek**.

Bastard greeks are illegitimate because they involve differentiating with respect to a parameter, saliently *assumed constant* in the derivation of Black-Scholes.

Our model, and assumption made about the volatility, tells us the value of Vega.

Does the model see volatility as **a constant**, **random number** (stochastic variable), or **uncertain quantity** (parameter)?

Sensitivity *wrt* parameter is inherently inconsistent. This means that our market risk is subject to estimation error (fundamentally caused by model risk).

- Just because the Black-Scholes model Vega is small, it is simply dangerous to think there is no exposure.

Uncertain Volatility and Vega for an exotic

A Vega can be calculated as zero at precisely the point, at which there is a great deal of volatility risk.

Why is Vega important?

- **Hedging:** you can statically hedge one option with another.

  If option C_1 has a Vega of 37.5 and option C_2 has a Vega of 75 then, the spread $2C_1 - C_2$ will be *Vega-neutral*.

  The sell-side uses this to reduce risk when selling exotics and managing their option books.

- **Risk management:** if you buy an option to speculate on price movement, the level of implied volatility is a risk factor.

  The option value will be quoted as implied volatility and you want to know your exposure, e.g., $\partial V / \partial \sigma = 37.5$.

Let's step back a bit...

If σ is constant then given derivation assumptions we have the Black–Scholes **pricing equation**:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

And for vanilla calls and puts this has nice, **closed-form solutions**, known as the Black–Scholes formulæ.

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$P(S,t) = C(S,t) + Ke^{-r(T-t)} - S$$

Problems with Black–Scholes model

- Effects of *discrete hedging*
- Transaction costs on the underlying. Illiquidity
- **Uncertainty in volatility**
- Market jumps, crashes, discontinuous asset paths
- **Default risk** of the underlying
- Uncertainty in dividends
- Discrete price effect (especially in the teenies)
- Feedback, market manipulation
- **Supply and demand**

Out of the money puts are a popular form of insurance.

When someone sells a deep OTM put, the premium has to cover *replication costs*: discrete delta hedging, probability of a crash (aka fat tail), market frictions... business costs (CVA, funding, collateral).

Oh, and Profit!

Black–Scholes has only **one parameter** measuring replication costs: the volatility.

So we can't really back out just perfectly meaningful option value/asset volatility from option price. Why?

Counterparty issues

Models the sell-side uses in order to add their charges address

- Liquidity – bid/ask spreads of all instruments used affect an exotic structure or hedging solution

- Counterparty exposure / credit risk

- Collateralisation – mitigates credit risk but introduces valuation of collateral and usually FX considerations (‘quanto’)

- Cost of funding

Costs reflected in adjustments

Adjustments are taken against Mark-to-Market value (option price) to recognise credit risk, funding costs and operations with collateral – also discrete hedging and model-specific risk.

Risk-free Derivative = Risky Derivative + Adjustments

The major adjustment is CVA – the cost of buying protection on the counterparty that pays the portfolio value in case of default (Cesari, 2011; updated sources of market practice available).

The requirement to hedge against credit risk affects business costs of every regulated financial entity.

Back to the issue of information offered by the implied volatility...

One is usually taught to think of the implied volatility as the market's view of the future value of volatility.

People often say the nonsensical 'the market is always right.'

Yes and no.

If there is any useful information to have about the future behaviour of actual volatility, then one will engage in calibration.

Uses of implied volatility

Implied volatility has many uses, e.g., local volatility, market's view, market-making, hedging exotic options, etc.

Buy side/Hedge funds: make their own volatility forecasts. If it is different from implied, then arbitrage opportunities exist! They buy/sell options and make a profit if they are right.

Sell side/Investment banks: use implied volatility to tell them how to price exotics. They sell exotics with profit margin on top and hedge to lock in gains.

Useful sold exotic options with known analytical solutions are:

- Barrier Options (also partial)

- Autocalls (worst of)

- Passport options (on long/short P&L)

Sell-side business relies on information contained in implied volatility, subject to **calibration**: fitting of the volatility process (to have the term structure $\sigma(t)$ or even full local surface $\sigma(t, S)$.

Case 1: Hedge with actual volatility, σ_a - P&L Study

Set up a portfolio by *buying* the option for V^i and selling Δ^a quantity of the stock. Cash position is

$$-V^i + \Delta^a\, S$$

By selling Δ^a shares we are replicating a short position in a correctly priced option V^a.

Recall we derived the MtM over dt in result (1) as

$$dV^i - \Delta^a\, dS - (V^i - \Delta^a\, S)\, rdt \dagger$$

To study the sources of randomness in the P&L,

We can write **mark-to-market** from t to $t+dt$ by *invoking Itô lemma* to expand $\underline{dV^i}$ in expression (1)

$$\underline{\Theta^i\,dt+\tfrac{1}{2}\sigma_a^2S^2\Gamma^i\,dt+\Delta^i\,dS}-\Delta^a\,dS-r(V^i-\Delta^aS)\,dt$$

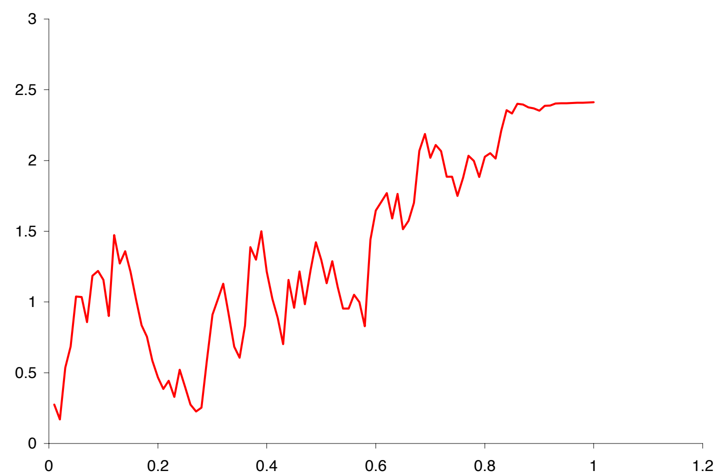
$$=\Theta^i\,dt+\mu S(\Delta^i-\Delta^a)\,dt+\tfrac{1}{2}\sigma_a^2S^2\Gamma^i\,dt-r(V^i-\Delta^aS)\,dt+(\Delta^i-\Delta^a)\sigma_aS\,dX$$

Using the fact that V^i satisfies Black-Scholes, can substitute $(\Theta^i-rV^i)dt$ with $(-r\Delta^iS-\frac{1}{2}\sigma_i^2S^2\Gamma^i)dt$.

$$=(\Delta^i-\Delta^a)\sigma_aS\,dX+(\mu-r)S(\Delta^i-\Delta^a)\,dt+\tfrac{1}{2}\left(\sigma_a^2-\sigma_i^2\right)S^2\Gamma^i\,dt$$

$$= \frac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2 \Gamma^i \, dt + \left(\Delta^i - \Delta^a\right) \left((\mu - r)S \, dt + \sigma_a S \, dX\right).$$

Sources of randomness: asset price S and dX, the factor from asset price SDE.



The fluctuation in the portfolio mark-to-market value is random and may even go negative. You could lose before you gain.