$$\frac{\partial V}{\partial t} + \underbrace{\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}}_{\text{diffusion}} + \underbrace{(r-D)S\frac{\partial V}{\partial S}}_{\text{drift}} \underbrace{- rV}_{\text{discounting}} = 0 \qquad (\dagger)$$

$\sigma, r, D \in \mathbb{R}$
σ - vol. of stock returns
r - risk free rate
D - continuous const. Dividend yield

We can look for separable solutions of this P.D.E

$dS = (\mu - D)S\,dt + \sigma S\,dX$

$$\boxed{V = V(S, t) = A(t)B(S)} \qquad (*)$$

Compare this to 1.] where we solved $\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}$ and look for a solution of the form

$$p(y', t') = \frac{1}{\sqrt{t'}} f\left(\frac{y'}{\sqrt{t'}}\right)$$

Subst $(*)$ in $(\dagger)$. $\dot{} \equiv \frac{d}{dt}$ $\quad {}' \equiv \frac{d}{dS}$

$$\frac{\partial V}{\partial t} = \dot{A}(t)B(S) \;; \quad \frac{\partial V}{\partial S} = A(t)B'(S) \;; \quad \frac{\partial^2 V}{\partial S^2} = A(t)B''(S)$$

$$\underbrace{\dot{A}(t)B(S)}_{\frac{\partial V}{\partial t}} + \frac{1}{2}\sigma^2 S^2 \underbrace{A(t)B''(S)}_{\frac{\partial^2 V}{\partial S^2}} + (r-D)S\underbrace{A(t)B'(S)}_{\frac{\partial V}{\partial S}} - r\underbrace{A(t)B(S)}_{V} = 0$$

$\div$ thro' by AB

$$\frac{\dot{A}(t)B(S)}{A(t)B(S)} + \frac{1}{2}\sigma^2 S^2 \frac{A(t)B''(S)}{A(t)B(S)} + (r-D)S\frac{A(t)B'(S)}{A(t)B(S)} - r\frac{A(t)B(S)}{A(t)B(S)} = 0$$

$$\frac{\dot{A}(t)}{A(t)} + \frac{1}{2}\sigma^2 S^2 \frac{B''(S)}{B(S)} + (r-D)S\frac{B'(S)}{B(S)} - r = 0$$

Rearrange $$\frac{\dot{A}(t)}{A(t)} = -\frac{1}{2}\sigma^2 S^2 \frac{B''(S)}{B(S)} - (r-D)S\frac{B'(S)}{B(S)} + r$$

Both sides must equal some function that is indep. of both t and S. That only possible choice of function is a constant

$$\frac{\dot{A}(t)}{A(t)} = -\frac{1}{2}\sigma^2 S^2 \frac{B''(S)}{B(S)} - (r-D)S\frac{B'(S)}{B(S)} + r = c$$

$\rightarrow$ a 1st order ODE $\quad \frac{\dot{A}}{A} = c \qquad \frac{dA}{dt} = cA$

$\longrightarrow$ a 2^{nd} order Cauchy – Euler eq$^{\underline{n}}$

$$-\frac{1}{2}\sigma^2 s^2 B''(s) - (r-D)s B'(s) + B(s)r - c B(s) = 0$$

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2B}{dS^2} + (r-D)s\frac{dB}{ds} - (c-r)B = 0$$

① $\frac{dA}{A} = c\, dt$ $\qquad \log A(t) = ct + K$

$$A(t) = \alpha e^{ct}$$

② $\exists$ a solution of the form $B(s) = S^{\lambda}$

Auxiliary Eq$^{\underline{n}}$: $\lambda^2 + \left(\frac{2}{\sigma^2}(r-D) - 1\right)\lambda - \frac{2}{\sigma^2}(c-r) = 0$

For Palani (p.224–225 primer Thankyou Carlo) :

$ax^2y'' + bxy' + cy = 0$ $\qquad$ Euler eq$^{\underline{n}}$

$\exists$ a sol$^{\underline{n}}$ of the form $y = x^m$

$\longrightarrow$ $am^2 + (b-a)m + c = 0$

There will be 3 solutions depending on roots of A.E

$m^+, m^- \in \mathbb{R}$: $\qquad y = \beta_1 x^{m^+} + \beta_2 x^{m^-}$ $\qquad \beta_i \in \mathbb{R}$

$m^+ = m^- = m \in \mathbb{R}$: $\qquad y = x^m(\beta_3 + \beta_4 \log x)$

$m = p \pm iq \in \mathbb{C}$ $\qquad y = x^p\left(\beta_5 \cos(q \log x) + \beta_6 \sin(q \log x)\right)$

After some messy working you will have 2 sol$^{\underline{n}}$s

① for A

② for B

$$V = A(t)\, B(s)$$

Perpetual American option

$V = V(S)$. Earlier PDE becomes:

$$\frac{\partial V}{\partial t} \longrightarrow 0 \qquad \partial \longrightarrow d$$

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 V}{dS^2} + rS\frac{dV}{dS} - rV = 0 \qquad (D=0)$$

Call: ① $S=0 \quad V=0$ B.C

Smooth pasting condition

$$\boxed{\begin{aligned} &② \quad V(S^*) = S^* - E \\ &③ \quad \left.\frac{dV}{dS}\right|_{S=S^*} = +1 \end{aligned}}$$

S^* – const. i.e a special value of S when we choose to strike

A.E: $V(S) = S^\lambda \qquad \frac{1}{2}\sigma^2\lambda^2 + (r - \frac{1}{2}\sigma^2)\lambda - r = 0$

$$\lambda^2 + \left(\frac{2r}{\sigma^2} - 1\right)\lambda - \frac{2}{\sigma^2}r = 0$$

$$(\lambda - 1)\left(\lambda + \frac{2r}{\sigma^2}\right) = 0 \qquad \lambda^+ = 1 \qquad \lambda^- = -\frac{2r}{\sigma^2}$$

$$\therefore V(S) = c_1 S + c_2 S^{-2r/\sigma^2} = c_1 S + c_2 \frac{1}{S^{2r/\sigma^2}}$$

Condition $\Rightarrow c_2 = 0 \qquad \therefore V(S) = c_1 S$

Exercise: To finish off use ②, ③

Extension problem:

put: ① $S \longrightarrow \infty \quad V = 0$

$$\boxed{\begin{aligned} &② \quad V(S^*) = -S^* + E \\ &③ \quad \left.\frac{dV}{dS}\right|_{S=S^*} = -1 \end{aligned}}$$

smooth pasting condition (High contact condition)