Numerical Methods Problems

1. Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad t > 0, \quad 0 < x < L \tag{1.1}$$

where the unknown function $u = u(x,t)$; c^2 is a constant. To discretize the equation, take N and M steps for x and t respectively, so

$$\begin{aligned} x &= n\delta x \qquad 0 \leq n \leq N \\ t &= m\delta t \qquad 0 \leq m \leq M, \end{aligned}$$

where $\delta x = \frac{L}{N}$; $\delta t = \frac{T}{M}$. By using the following approximations

$$\begin{aligned} \frac{\partial u}{\partial t}(n\delta x, m\delta t) &\sim \frac{u_n^{m+1} - u_n^m}{\delta t}, \\ \frac{\partial^2 u}{\partial x^2}(n\delta x, m\delta t) &\sim \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{\delta x^2} \end{aligned}$$

and writing $r = c^2 \dfrac{\delta t}{\delta x^2}$, derive the following **forward marching scheme** for (1.1)

$$u_n^{m+1} = Au_{n-1}^m + Bu_n^m + Cu_{n+1}^m, \tag{1.2}$$

where A, B, C should be stated.
Assume an initial disturbance E_n^m given by

$$E_n^m = \overline{a}^m e^{in\omega}, \tag{1.3}$$

which is oscillatory of amplitude $\overline{a}$ and frequency ω; $i = \sqrt{-1}$. By substituting (1.3) into (1.2), obtain a stability condition for this scheme.

2. Consider the pricing equation for the value of a derivative security $V(S,t)$,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0, \tag{2.1}$$

where $S \geq 0$ is the spot price of the underlying equity, $0 < t \leq T$ is the time, $r \geq 0$ the constant rate of interest, and σ is the constant volatility of S. The variables (t, S) can be written as

$$t = m\delta t \quad 0 \leq m \leq M; \quad S = n\delta S \qquad 0 \leq n \leq N,$$

where $(\delta t, \delta S)$ are fixed step sizes in turn. $V(S,t)$ is written discretely as V_n^m. An Explicit Finite Difference Method is to be developed to solve (2.1) using a *backward marching scheme*. Derive a difference equation in the form

$$V_n^{m-1} = a_n V_{n-1}^m + b_n V_n^m + c_n V_{n+1}^m$$

where a_n, b_n, c_n should be defined; you may use the following as a starting point,

3. A binary call option is to be priced. Discuss a Monte Carlo method to do this.