Stochastic Calculus and Itô's lemma

Throughout this problem sheet, you may assume that W_t or X_t Brownian Motion (Wiener Process); $W_0 = 0$.

1. Let ϕ be a random variable which follows a standardised normal distribution, i.e. $\phi \sim N(0,1)$. Calculate the expected value and variance given by $\mathbb{E}[\psi]$ and $\mathbb{V}[\psi]$, in turn, where $\psi = \sqrt{dt}\phi$. dt is a small time-step. **Note: No integration is required.**
   Firstly we know $\mathbb{V}[\phi] = \mathbb{E}[\phi^2] = 1$ from the definition of $N(0,1)$.
   $\mathbb{E}[\psi] = \mathbb{E}\left[\sqrt{dt}\phi\right] = \sqrt{dt}\mathbb{E}[\phi]$, because dt is not a RV and we also know that $\mathbb{E}[\phi] = 0, \therefore \mathbb{E}[\psi] = 0$. $\mathbb{V}[\psi] = \mathbb{E}[\psi^2] - \mathbb{E}[\psi]^2 \rightarrow \mathbb{E}[dt\phi^2] \Rightarrow \mathbb{V}[\psi] = dt\mathbb{E}[\phi^2] = dt$.

2. Consider the following examples of SDEs for a diffusion process G. Write these in standard form, i.e.

$$dG = A(G,t)dt + B(G,t)dW_t.$$

   Give the drift and diffusion for each case.

   a. $df + dW_t - dt + 2\mu t f dt + 2\sqrt{f}dW_t = 0$

$$df = (1 - 2\mu t f)\,dt + \left(-1 - 2\sqrt{f}\right)dW_t$$

   b. $\dfrac{dy}{y} = (A + By)\,dt + (Cy)\,dW_t$

$$dy = \left(Ay + By^2\right)dt + \left(Cy^2\right)dW_t$$

   c. $dS = (\nu - \mu S)dt + \sigma dW_t + 4dS$

$$\begin{aligned} dS - 4dS &= (\nu - \mu S)dt + \sigma dW_t \\ dS &= -\tfrac{1}{3}(\nu - \mu S)dt - \tfrac{1}{3}\sigma dW_t \end{aligned}$$

3. Use Itô's lemma to obtain a SDE for each of the following functions:

a. $f(W_t) = (W_t)^n$

$$df = nW_t^{n-1}dW_t + \frac{1}{2}n(n-1)W_t^{n-2}dt$$

b. $y(W_t) = \exp(W_t)$

$$\begin{aligned} dy &= \exp(W_t)\,dW_t + \frac{1}{2}\exp(W_t)\,dt \text{ or} \\ \frac{dy}{y} &= \frac{1}{2}dt + dW_t \end{aligned}$$

c. $g(W_t) = \ln W_t$

$$dg = -\frac{1}{2W_t^2}dt + \frac{1}{W_t}dW_t$$

d. $h(W_t) = \sin W_t + \cos W_t$

$$dh = (\cos W_t - \sin W_t)\,dW_t - \frac{1}{2}(\sin W_t + \cos W_t)\,dt$$

e. $f(W_t) = a^{W_t}$, where the constant $a > 1$

$$\begin{aligned} f(W_t) &= a^{W_t} \Rightarrow \log f = W_t \log a \Rightarrow \frac{1}{f}f'(W) = \log a \Rightarrow f'(W_t) = (\log a)\,f \\ \text{therefore } f'(W_t) &= (\log a)\,a^{W_t} \quad \text{and hence} \quad f^{\prime/}(W_t) = (\log a)^2 a^{W_t} \\ df &= (\log a)\,a^{W_t}dW_t + \frac{1}{2}(\log a)^2 a^{W_t}dt \\ \text{or } \frac{df}{f} &= \frac{1}{2}(\log a)^2 dt + (\log a)\,dW_t \end{aligned}$$

4. Using the formula below for stochastic integrals, for a function $F(W_t, t)$,

$$\int_0^t \frac{\partial F}{\partial W_\tau}dW_\tau = F(W_t, t) - F(W_0, 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial^2 F}{\partial W_\tau^2}\right)d\tau$$

show that we can write

a. $\int_0^t W_\tau^3 dW_\tau = \frac{1}{4}W_t^4 - \frac{3}{2}\int_0^t W_\tau^2 d\tau$. Here we have ordinary derivatives and no $\frac{\partial F}{\partial t}$

$$\frac{dF}{dW_t} = W_t^3 \longrightarrow F(W_t) = \frac{1}{4}W_t^4(t) \longrightarrow \frac{d^2F}{dW_t^2} = 3W_t^2(t)$$

which substituted into the formula gives the result

b. $\int_0^t \tau dW_\tau = tW_t - \int_0^t W_\tau d\tau$

$$\frac{\partial F}{\partial W_t} = t \longrightarrow F(W_t, t) = tW_t \Rightarrow \frac{\partial^2 F}{\partial W_t^2} = 0 \text{ and } \frac{\partial F}{\partial t} = W_t$$

substituting all of these terms in to the formula

$$\int_0^t \tau dW_\tau = tW_t - 0 - \int_0^t \left(W_\tau + \frac{1}{2}\times 0\right)d\tau = tW_t - \int_0^t W_\tau d\tau$$

c. $\int_0^t (W_\tau + \tau)\, dW_\tau = \frac{1}{2}W_t^2 + tW_t - \int_0^t \left(W_t + \frac{1}{2}\right) d\tau$

$$\frac{\partial F}{\partial W_t} = W_t + t \longrightarrow F\left(W_t\right) = \frac{1}{2}W_t^2 + tW_t \longrightarrow \frac{\partial F}{\partial t} = W_t$$

and $\frac{\partial^2 F}{\partial W^2} = 1$, therefore leading to the required result.

5. This PDE has the same structure as the Black-Scholes Equation and the working here is used in part to reduce it to a one dimensional heat equation - hence very useful problem (much more on this later). Start by differentiating

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(\beta v + \frac{\partial v}{\partial t}\right) e^{\alpha x + \beta t} \\ \frac{\partial u}{\partial x} &= \left(\alpha v + \frac{\partial v}{\partial x}\right) e^{\alpha x + \beta t} \\ \frac{\partial^2 u}{\partial x^2} &= \left(\alpha^2 v + 2\alpha \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) e^{\alpha x + \beta t} \end{aligned}$$

Substituting into the PDE we have

$$\left(\beta v + \frac{\partial v}{\partial t}\right) = \left(\alpha^2 v + 2\alpha \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) + a\left(\alpha v + \frac{\partial v}{\partial x}\right) + bv,$$

rearrange to give

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (2\alpha + a)\frac{\partial v}{\partial x} + \left(\alpha^2 + a\alpha + b - \beta\right) v.$$

To eliminate $\frac{\partial v}{\partial x}$ and v requires setting, in turn,

$$\begin{aligned} 2\alpha + a &= 0 \\ \alpha^2 + a\alpha + b - \beta &= 0. \end{aligned}$$

Hence the choice is

$$\alpha = -\frac{1}{2}a \text{ and } \beta = b - \frac{1}{4}a^2$$