# Martingales - Solutions

1. $X_t$ and $Y_t$ are two stochastic processes.

   **a.** Using the Itô rule for products $(X_tY_t)$ deduce the following *integration by parts formula*

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s.$$

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t dY_t.$$

Rearranging the above product rule

$$X_t dY_t = d(X_tY_t) - Y_t dX_t - dX_t dY_t$$

and integrating over $[0,t]$

$$\begin{aligned}\int_0^t X_s dY_s &= \int_0^t d(X_sY_s) - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s \\ &= X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s.\end{aligned}$$

   **b.** Derive the Itô rule for quotients $\left(\frac{X_t}{Y_t}\right)$

$$\begin{array}{lll} \frac{\partial f}{\partial X} = 1/Y & \frac{\partial f}{\partial Y} = -X/Y^2 & \frac{\partial^2 f}{\partial X^2} = 0 \\ \frac{\partial^2 f}{\partial Y^2} = 2X/Y^3 & \frac{\partial^2 f}{\partial X \partial Y} = -1/Y^2 = \frac{\partial^2 f}{\partial Y \partial X} & \end{array}$$

which gives

$$d\left(\frac{X}{Y}\right) = \frac{X}{Y}\left(\frac{dX}{X} - \frac{dY}{Y} - \frac{dXdY}{XY} + \left(\frac{dY}{Y}\right)^2\right)$$

There are a number of ways to express this result

$$d\left(\frac{X}{Y}\right) = \left(\frac{YdX - XdY - dXdY}{Y^2} + \frac{X}{Y^3}dY^2\right)$$

2. In this question $t \geq 0$.

   a. For which values of $k$ is the process

$$Y_t = W_t^4 - 6tW_t^2 + kt^2,$$

   a martingale? The problem is asking you to calculate the value of $k$ such that $Y_t$ has zero drift. Using Itô

$$dY_t = \left(\frac{\partial Y_t}{\partial t} + \frac{1}{2}\frac{\partial^2 Y_t}{\partial W^2}\right) dt + \frac{\partial Y_t}{\partial W} dW$$

$$\frac{\partial Y_t}{\partial t} = -6W_t^2 + 2kt; \quad \frac{\partial Y_t}{\partial W} = 4W_t^3 - 12tW_t; \quad \frac{\partial^2 Y_t}{\partial W^2} = 12W_t^2 - 12t$$

$$\begin{aligned} \frac{\partial Y_t}{\partial t} + \frac{1}{2}\frac{\partial^2 Y_t}{\partial W^2} &= 0 \rightarrow -6W_t^2 + 2kt + 6W_t^2 - 6t = 0 \\ k &= 3. \end{aligned}$$

   b. Is $X_t = \cosh(\theta W_t) e^{-\theta^2 t/4}$; $\theta \in \mathbb{R}$, a martingale?

$$F(W_t, t) = \cosh(\theta W_t) e^{-\theta^2 t/4}$$

   Using Itô

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}\right) dt + \frac{\partial F}{\partial W} dW$$

   So checking that $\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} = 0$, i.e. a driftless process.

$$\begin{aligned} \frac{\partial F}{\partial t} &= \cosh(\theta W_t) e^{-\theta^2 t/4} = -\frac{\theta^2}{4}\cosh(\theta W_t) e^{-\theta^2 t/4} \\ \frac{\partial F}{\partial W} &= \theta \sinh(\theta W_t) e^{-\theta^2 t/4}; \ \frac{\partial^2 F}{\partial W^2} = \theta^2 \cosh(\theta W_t) e^{-\theta^2 t/4} \end{aligned}$$

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} &= -\frac{\theta^2}{4}\cosh(\theta W_t) e^{-\theta^2 t/4} + \frac{1}{2}\left(\theta^2 \cosh(\theta W_t) e^{-\theta^2 t/4}\right) \\ &\neq 0 \end{aligned}$$

   Hence $X_t$ not a martingale.

3. Consider the Vasicek model

$$dr_t = \kappa(\theta - r_t) dt + \sigma dW_t,$$

where $\kappa, \ \theta, \sigma \in \mathbb{R}$. We are familar with the following solution for $s < t$

$$r_t = r_s e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma \int_s^t e^{-\kappa(t-u)} dW_u.$$

Show that as $t \rightarrow \infty$, the mean and variance become in turn

$$\begin{aligned} \mathbb{E}[r_t | r_s] &= \theta \\ \mathbb{V}[r_t | r_s] &= \frac{\sigma^2}{2\kappa} \end{aligned}$$

Hint: First calculate both mean and variance at time $t$. For the latter you can use the Itô isometry. For the mean we make use of the property that the Itô integral is a martngale

$$\begin{aligned} \mathbb{E}[r_t | r_s] &= e^{-\kappa(t-s)} r_s + \theta\left(1 - e^{-\kappa(t-s)}\right) \\ \mathbb{V}[r_t | r_s] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(t-s)}\right) \end{aligned}$$

For the variance note that $\mathbb{V}\left[r_s e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right)\right] = 0$ because it is a scalar. We note that as $t \to \infty$, the mean and variance become in turn

$$\begin{aligned}
\mathbb{E}\left[r_t \middle| r_s\right] &= \theta \\
\mathbb{V}\left[r_t \middle| r_s\right] &= \frac{\sigma^2}{2\kappa}
\end{aligned}$$