

Exotics - Solutions

1. Consider an option which pays a continuous cash-flow to the holder at a rate proportional to the square of the underlying asset's price, so that during a time interval dt the holder receives $S^2 dt$. Suppose that at expiry the value of the option is

$$V(S, T) = S^2.$$

The underlying evolution follows geometric Brownian motion

$$dS = \mu S dt + \sigma S dX.$$

Derive the Black-Scholes partial differential equation for this "power" option and show that it is

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = -S^2.$$

By assuming a solution of the form

$$V(S, t) = \phi(t) S^2$$

show that

$$\phi(t) = \frac{1}{\sigma^2 + r} \left((\sigma^2 + r + 1) e^{(\sigma^2 + r)(T-t)} - 1 \right).$$

Solution: Start with delta hedged portfolio

$$\Pi = V(S, t) - \Delta S.$$

with

$$dS = \mu S dt + \sigma S dX$$

over one time-step dt , where Δ is fixed from t to $t + dt$

$$d\Pi = \underbrace{dV - \Delta dS}_{\text{changes in } V \text{ and } S} + \underbrace{S^2 dt}_{\text{cash flow from holding } V}$$

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \\ &\quad \frac{\partial V}{\partial S} dS - \Delta dS + S^2 dt \end{aligned}$$

Only source of risk is in dS , so choose $\Delta = \frac{\partial V}{\partial S}$ to eliminate it.

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S^2 \right) dt \\ &= r\Pi dt \\ &= r(V - \Delta S) dt = r \left(V - S \frac{\partial V}{\partial S} \right) dt \end{aligned}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S^2 = rV - rS \frac{\partial V}{\partial S}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = -S^2$$

At expiry we have $V(S, T) = S^2$. Look for a solution of the form $V(S, t) = \phi(t) S^2$, then

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \dot{\phi}(t) S^2, \quad \frac{\partial V}{\partial S} = 2S\phi(t), \quad \frac{\partial^2 V}{\partial S^2} = 2\phi(t) \\
V(S, T) &= \phi(T) S^2 = S^2 \implies \phi(T) = 1 \\
\dot{\phi}(t) S^2 + \sigma^2 S^2 \phi(t) + 2r S^2 \phi(t) - r S^2 \phi(t) &= -S^2 \implies \\
\dot{\phi}(t) + (\sigma^2 + r) \phi(t) &= -1, \quad \phi(T) = 1
\end{aligned}$$

use integrating factor

$$e^{(\sigma^2 + r)t}$$

$$\begin{aligned}
\frac{d}{dt} \left(\phi(t) e^{(\sigma^2 + r)t} \right) &= -e^{(\sigma^2 + r)t} \\
\int d \left(\phi(t) e^{(\sigma^2 + r)t} \right) &= -\int e^{(\sigma^2 + r)t} dt \\
\phi(t) e^{(\sigma^2 + r)t} &= -\frac{e^{(\sigma^2 + r)t}}{(\sigma^2 + r)} + A \\
\phi(t) &= -\frac{1}{(\sigma^2 + r)} + A e^{-(\sigma^2 + r)t}
\end{aligned}$$

we know $\phi(T) = 1$ so $A = \left(1 + \frac{1}{(\sigma^2 + r)}\right) e^{(\sigma^2 + r)t}$.

Hence

$$\phi(t) = \frac{1}{\sigma^2 + r} \left((\sigma^2 + r + 1) e^{(\sigma^2 + r)(T-t)} - 1 \right)$$

2. Consider separable solutions of the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV = 0, \quad (2.1)$$

of the form

$$V(S, t) = f(S)g(t),$$

Show that (2.1) can be expressed as the following first order differential equation (2.2a) and Cauchy-Euler equation (2.2b)

$$\frac{dg}{dt} - \lambda g = 0 \quad (2.2a)$$

$$\frac{1}{2} \sigma^2 S^2 f'' + (r - D) S f' + (\lambda - r) f = 0, \quad (2.2b)$$

for some (universal) constant λ , where the following notation is used

$$f' = \frac{df}{dS}, \quad f'' = \frac{d^2 f}{dS^2}.$$

You may assume that (2.2b) has a solution of the form $f(S) = S^\alpha$. Solve these to obtain the following solutions for (2.1) :

i for distinct roots of the A.E (2.2b) (A, B - constants)

$$V(S, t) = e^{\lambda t} S^{\frac{1}{2} - \frac{r-D}{\sigma^2}} (AS^{\alpha+} + BS^{\alpha-})$$

ii for a repeated root of the A.E (2.2b) (ε, ζ - constants)

$$V(S, t) = e^{\left(\left(r + \frac{\sigma^2}{2} \left(\frac{r-D}{\sigma^2} + \frac{1}{2} \right)^2 \right) t \right)} S^{\left(\frac{1}{2} - \frac{r-D}{\sigma^2} \right)} (\varepsilon + \zeta \log S)$$

where

$$\bar{d}_{\pm} = \pm \sqrt{\left(\frac{r-D}{\sigma^2} - \frac{1}{2} \right)^2 - \frac{2(\lambda - r)}{\sigma^2}}.$$

Start by differentiating

$$\begin{aligned}\frac{\partial V}{\partial t} &= f\dot{g} \\ \frac{\partial V}{\partial S} &= gf' \\ \frac{\partial^2 V}{\partial S^2} &= gf''\end{aligned}$$

and substitute in the BSE

$$f\dot{g} + \frac{1}{2}\sigma^2 S^2 gf'' + (r - D) Sgf' - rgf = 0.$$

Dividing through by gf

$$\frac{f\dot{g} + \frac{1}{2}\sigma^2 S^2 gf'' + (r - D) Sgf' - rgf}{gf} = 0$$

and arrange so that t dependency on lhs and S dependency on rhs,

$$\frac{\dot{g}}{g} = \frac{-\frac{1}{2}\sigma^2 S^2 f'' - (r - D) S f' + r f}{f} = \lambda$$

$$\dot{g} = \lambda g \rightarrow g(t) = ke^{\lambda t}.$$

Secondly a 2nd order Cauchy-Euler equation:

$$\frac{1}{2}\sigma^2 S^2 f'' + (r - D) S f' + (\lambda - r) f = 0$$

Putting

$$f(S) = S^\alpha$$

gives a quadratic in α

$$\alpha^2 + \left(\frac{2(r - D)}{\sigma^2} - 1 \right) \alpha + \frac{2}{\sigma^2} (\lambda - r) = 0$$

hence

$$\begin{aligned}\alpha_{\pm} &= \frac{1}{2} \left(1 - \frac{2(r - D)}{\sigma^2} \right) \pm \frac{1}{2} \sqrt{\left(\frac{2(r - D)}{\sigma^2} - 1 \right)^2 - \frac{8}{\sigma^2} (\lambda - r)} \\ \alpha_{\pm} &= \frac{1}{2} \left(1 - \frac{2(r - D)}{\sigma^2} \right) \pm \frac{1}{2} \sqrt{\frac{4(r - D)^2}{\sigma^4} + 1 - \frac{4(r - D)}{\sigma^2} - \frac{8(\lambda - r)}{\sigma^2}} \\ &= \frac{1}{2} \left(1 - \frac{2(r - D)}{\sigma^2} \right) \pm \sqrt{\frac{(r - D)^2}{\sigma^4} + \frac{1}{4} - \frac{(r - D)}{\sigma^2} - \frac{2(\lambda - r)}{\sigma^2}} \\ &= \frac{1}{2} \left(1 - \frac{2(r - D)}{\sigma^2} \right) \pm \sqrt{\left(\frac{r - D}{\sigma^2} - \frac{1}{2} \right)^2 - \frac{2(\lambda - r)}{\sigma^2}}\end{aligned}$$

2 cases to consider:

(1) Solution for distinct roots - $f(S) = aS^{\alpha_+} + bS^{\alpha_-}$

$$V(S, t) = e^{\lambda t} S^{\frac{1}{2} - \frac{r-D}{\sigma^2}} \left[AS^{\bar{d}_+} + BS^{\bar{d}_-} \right] \quad A, B - \text{constants}$$

where

$$\bar{d}_+ = \sqrt{\left(\frac{r - D}{\sigma^2} - \frac{1}{2} \right)^2 - \frac{2(\lambda - r)}{\sigma^2}}; \quad \bar{d}_- = -\sqrt{\left(\frac{r - D}{\sigma^2} - \frac{1}{2} \right)^2 - \frac{2(\lambda - r)}{\sigma^2}}$$

(2) Repeated Root - $f(S) = S^{\frac{1}{2} - \frac{r-D}{\sigma^2}} [a + b \log S]$. Now $\left(\frac{r-D}{\sigma^2} - \frac{1}{2}\right)^2 = \frac{2(\lambda-r)}{\sigma^2} \rightarrow \lambda = r + \frac{\sigma^2}{2} \left(\frac{r-D}{\sigma^2} - \frac{1}{2}\right)^2$ therefore

$$V(S, t) = \exp \left(\left(r + \frac{\sigma^2}{2} \left(\frac{r-D}{\sigma^2} - \frac{1}{2} \right)^2 \right) t \right) S^{(\frac{1}{2} - \frac{r-D}{\sigma^2})} [\varepsilon + \zeta \log S] \quad \varepsilon, \zeta - \text{constants}$$

3. Assume that an asset price S evolves according to the SDE

$$\frac{dS}{S} = (\mu - D) dt + \sigma dW,$$

where μ and σ are constants. In particular S pays out a continuous dividend stream equal to $DSdt$ during the infinitesimal time interval dt , where D the dividend yield is constant.

Now suppose a European style derivative security is written on this asset with the properties that at expiry the holder receives the asset and prior to expiry the derivative pays a continuous cash flow $C(S, t) dt$ during each time interval of length dt .

Show that the option price $V(S, t)$ satisfies the following partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV = -C(S, t).$$

Solution:

$$\begin{aligned} \Pi &= V - \Delta S \\ d\Pi &= dV - \Delta dS - \Delta DSdt + C(S, t) dt, \end{aligned}$$

because we are short the stock. Using the usual hedging argument gives

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \\ &\quad - \Delta (\mu Sdt + \sigma SdW) - \Delta DSdt + C(S, t) dt \end{aligned}$$

Put $\Delta = \frac{\partial V}{\partial S}$ to eliminate risk and use no-arbitrage

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - DS \frac{\partial V}{\partial S} + C(S, t) \right) dt \\ &= r \left(V - S \frac{\partial V}{\partial S} \right) dt. \end{aligned}$$

from which the BSE is obtained

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV = -C(S, t)$$

Given cash flow $C(S, t)$ has the form $C(S, t) = f(t) S$. Writing $V = \phi(t) S$ and assuming a final condition at time T given by

$$V(S, T) = S.$$

This means

$$V(S, T) = \phi(T) S = S \Rightarrow \phi(T) = 1$$

We show that the delta of the derivative security is

$$\Delta(S, t) = e^{-D(T-t)} + \int_t^T e^{-D(\tau-t)} f(\tau) d\tau.$$

First thing to note is that $\Delta(S, t) = \frac{\partial V}{\partial S} = \frac{\partial}{\partial S} \phi(t) S = \phi(t)$. So once $\phi(t)$ is calculated that is the delta.

Writing $C(S, t) = f(t) S$ gives

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV = -f(t) S$$

and we now use the transformation $V = \phi(t) S$ to convert to an ode which is a function of t alone.

$$\frac{\partial V}{\partial t} = \phi'(t) S; \quad \frac{\partial V}{\partial S} = \phi; \quad \frac{\partial^2 V}{\partial S^2} = 0$$

So the original problem reduces to

$$\begin{aligned} \frac{d\phi}{dt} + (r - D) \phi - r\phi &= -f(t) \\ \longrightarrow \frac{d\phi}{dt} - D\phi &= -f \end{aligned}$$

which is a first order linear equation i.e. integrating factor method. Integrating Factor is

$$\exp(-Dt)$$

so the ordinary differential equation becomes

$$\begin{aligned} e^{-Dt} \frac{d\phi}{dt} - D\phi e^{-Dt} &= -f e^{-Dt} \\ \frac{d}{dt} (e^{-Dt} \phi) &= -f e^{-Dt} \\ \int_t^T d(e^{-D\tau} \phi(\tau)) &= -\int_t^T f(\tau) e^{-D\tau} d\tau \\ (e^{-D\tau} \phi(\tau))|_t^T &= -\int_t^T f(\tau) e^{-D\tau} d\tau \\ e^{-DT} \phi(T) - e^{-Dt} \phi(t) &= -\int_t^T f(\tau) e^{-D\tau} d\tau \end{aligned}$$

and we know $\phi(T) = 1$, hence

$$\begin{aligned} e^{-DT} - e^{-Dt} \phi(t) &= -\int_t^T f(\tau) e^{-D\tau} d\tau \\ e^{-Dt} \phi(t) &= e^{-DT} + \int_t^T f(\tau) e^{-D\tau} d\tau \\ \phi(t) &= e^{-D(T-t)} + \int_t^T f(\tau) e^{-D(\tau-t)} d\tau \end{aligned}$$

So the option price $V(S, t) = \phi(t) S$ and $\Delta(S, t) = \frac{\partial V}{\partial S} = \phi(t) =$

$$e^{-D(T-t)} + \int_t^T f(\tau) e^{-D(\tau-t)} d\tau.$$

4. An asset S follows a Geometric Brownian Motion $dS = \mu S dt + \sigma S dW$, where μ and σ are constants. We wish to value an option that pays off at expiry T an amount which is a function of the path taken by the asset between time zero and expiry. Assuming that an option value V depends on S , t and a quantity

$$I(t) = \int_0^t f(S, \tau) d\tau,$$

where f is a specified function and r the risk free interest rate, the option pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S, t) \frac{\partial V}{\partial I} + rS \frac{\partial V}{\partial S} - rV = 0,$$

for the function $V(S, I, t)$.

For an arithmetic strike Asian call option the payoff at time T is

$$\max\left(S - \frac{1}{T} \int_0^T S(t) dt, 0\right).$$

By writing the value of this option as

$$V(S, I, t) = SW(R, t),$$

where $R = I/S$, show that the partial differential equation for $W(R, t)$ is given by

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + (1 - rR) \frac{\partial W}{\partial R} = 0.$$

Solution: If $V(S, I, t) = SW(R, t)$; $R = I/S$ then

$$\frac{\partial R}{\partial S} = -\frac{I}{S^2}; \quad \frac{\partial R}{\partial I} = \frac{1}{S}; \quad \frac{\partial}{\partial S} \equiv -\frac{I}{S^2} \frac{\partial}{\partial R}$$

$$\begin{aligned} \frac{\partial V}{\partial t} &= S \frac{\partial W}{\partial t}; \quad \frac{\partial V}{\partial I} = S \frac{\partial W}{\partial R} \frac{\partial R}{\partial I} = \frac{\partial W}{\partial R} \\ \frac{\partial V}{\partial S} &= W + S \frac{\partial W}{\partial R} \frac{\partial R}{\partial S} = W - R \frac{\partial W}{\partial R} \\ \frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S} \left(W - R \frac{\partial W}{\partial R} \right) = \frac{\partial R}{\partial S} \frac{\partial}{\partial R} \left(W - R \frac{\partial W}{\partial R} \right) \\ &= -\frac{I}{S^2} \frac{\partial}{\partial R} \left(W - R \frac{\partial W}{\partial R} \right) = -\frac{R}{S} \left(\frac{\partial W}{\partial R} - \frac{\partial W}{\partial R} - R \frac{\partial^2 W}{\partial R^2} \right) \\ &= \frac{R^2}{S} \frac{\partial^2 W}{\partial R^2} \end{aligned}$$

Substituting in the pricing PDE

$$\begin{aligned} S \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{R^2}{S} \frac{\partial^2 W}{\partial R^2} + S \frac{\partial W}{\partial R} + rS \left(W - R \frac{\partial W}{\partial R} \right) - rSW &= 0 \\ \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + \frac{\partial W}{\partial R} + r \left(W - R \frac{\partial W}{\partial R} \right) - rW &= 0 \\ \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + (1 - rR) \frac{\partial W}{\partial R} &= 0 \end{aligned}$$