1 Partial Differential Equations

1.1 Introduction

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}$$

The formation (and solution) of PDE's forms the basis of a large number of mathematical models used to study physical situations arising in science, engineering and medicine. More recently their use has extended to the modelling of problems in finance and economics. We now look at the second type of DE, i.e. PDE's. These have partial derivatives instead of ordinary derivatives. One of the underlying equations in finance, the Black-Scholes equation for the price of an option $V(S,t)$ is an example of a linear PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \cancel{D})\, S \frac{\partial V}{\partial S} - \cancel{rV} = 0$$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + \mu S \frac{\partial P}{\partial S} = 0$$

providing σ, D, r are not functions of V or any of its derivatives.

L - lin. op

f, g sol^n's of PDE, $\lambda \in \mathbb{R}$

① $L(\lambda f) = \lambda L(f)$ scalar mult^n

② $L(f+g) = L(f) + L(g)$ vector add^n

1.2 Linear PDE

If we let $u = u(x,y)$, then the general form of a linear 2nd order PDE is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{1}$$

where the coefficients A, ..., G are functions of x & y.When

$$G(x,y) = \begin{cases} 0 & (1) \text{ is homogeneous} \\ \text{non-zero} & (1) \text{ is non-homogeneous} \end{cases}$$

Example:

a $\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0 \qquad A = 1 = C; \qquad G = 0 \Rightarrow$equation is homogeneous

b $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} = xy \qquad A = 1 : \ E = -1; \qquad G = xy \Rightarrow$equation is non-homogeneous

Equations such as (1) can be classified as one of three types. This depends only on the coefficients of the 2nd order derivatives, providing at least one of A, B and C is non-zero. Equation (1) is said to be

(i) hyperbolic $B^2 - 4AC > 0$ → $u_{tt} = c^2 u_{xx}$ wave

(ii) parabolic $B^2 - 4AC = 0$ → $u_t = c^2 u_{xx}$ heat

(iii) elliptic $B^2 - 4AC < 0$ → $u_{xx} + u_{yy} = f(x,y)$ Poisson

when $f = 0$ Poisson → Laplace

In the context of mathematical finance we are only interested in the 2nd type, i.e. parabolic. There are several methods for obtaining solutions of PDE's.We look at a simple (but useful) technique:

1.3 Method of Separation of Variables

Without loss of generality, we solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{*}$$

for the unknown function $u(x,t)$. In this method we assume existence of a solution which is a product of a function of x (only) and a function of y (only). So the form is

$$u(x,t) = X(x)\,T(t).$$

$\bullet \equiv \frac{d}{dt}$

$/ \equiv \frac{d}{ds}$

We substitute this in (*), so

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}(XT) = XT' \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(XT)\right) = \frac{\partial}{\partial x}(X'\,T) = X''T \end{aligned}$$

LHS

RHS

Therefore (*) becomes

$$XT' = c^2 X''T$$

dividing through by $c^2 X\,T$ gives

$$\frac{T'}{c^2 T} = \frac{X''}{X}. \quad = \text{CONSTANT}$$

The RHS is independent of t and LHS is independent of x. So each equation must be a constant. The convention is to write this constant as λ^2 or $-\lambda^2$

applied maths

Three possible cases:

1st $\dot{T} = c^2 T$ const.

2 ODE

Case 1: $\boxed{\lambda^2 > 0}$

2nd. $\boxed{X'' - \text{const.}\, X = 0}$

$\frac{T'}{c^2 T} = \frac{X''}{X} = \lambda^2$ leading to

$$\left.\begin{aligned} T' - \lambda^2 c^2 T = 0 \\ X'' - \lambda^2 X = 0 \end{aligned}\right\}$$

$\frac{dT}{dt} = \lambda^2 c^2 T$

$\frac{dT}{T} = \lambda^2 c^2 dt$

which have solutions, in turn

$$\left.\begin{aligned} T(t) = k \exp\left(c^2\lambda^2 t\right) \\ X(x) = A\cosh(\lambda x) + B\sinh(\lambda x) \end{aligned}\right\}$$

Exercise.

So solution is $u(x,t) = X\,T = k\exp\left(c^2\lambda^2 t\right)\{A\cosh(\lambda x) + B\sinh(\lambda x)\}$.

Therefore $\boxed{u = \exp\left(c^2\lambda^2 t\right)\{\alpha\cosh(\lambda x) + \beta\sinh(\lambda x)\}}$ $(\alpha = Ak;\ \ \beta = Bk)$

1st solve the eqn then rearrange

Case 2: $\boxed{-\lambda^2 < 0}$

$\frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda^2$ which gives

$$\left.\begin{aligned} T' + \lambda^2 c^2 T = 0 \\ X'' + \lambda^2 X = 0 \end{aligned}\right\}$$

$\cosh x = \frac{e^x + e^{-x}}{2}$

resulting in the solutions

$$\left.\begin{aligned} T = \overline{k}\exp\left(-c^2\lambda^2 t\right) \\ X = \overline{A}\cos(\lambda x) + \overline{B}\sin(\lambda x) \end{aligned}\right\}$$

$\sinh x = \frac{e^x - e^{-x}}{2}$

respectively. Hence

$T(t)$ $X(x)$

$$u(x,y) = \exp\left(-c^2\lambda^2 t\right)\{\gamma\cos(\lambda x) + \delta\sin(\lambda x)\}$$

where $\left(\gamma = \overline{kA};\quad \delta = \overline{kB}\right)$.

<u>**Case 3**</u>: $\lambda^2 = 0$

$$\left.\begin{array}{c} T' = 0 \\ X'' = 0 \end{array}\right\} \to \left\{\begin{array}{c} T(t) = \widetilde{A} \\ X = \widetilde{B}x + \widetilde{C} \end{array}\right\}$$

which gives the simple solution

$$\boxed{u(x,y) = \widehat{A}x + \widehat{C}}$$

where $\left(\widehat{A} = \widetilde{A}\widetilde{B};\quad \widehat{C} = \widetilde{B}\widetilde{C}\right)$.

1 Similarity Reduction ← Gaussian Curve

2 Sep. of vars ← Fourier modes (periodic behaviour)

$\boxed{\sigma = 0}$

$$\boxed{\frac{\partial V}{\partial t} + (r - D)S\frac{\partial V}{\partial S} = rV}$$

1.4 The Black-Scholes Equation (BSE)

Consider the BSE

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.$$

The option price is $V(S,t)-$ comparing with (1) we see $A=0=B$; $C=\frac{1}{2}\sigma^2S^2$; $D=1$; $E=rS$; $F=r$; $G=0$ therefore the BSE is a (linear) parabolic equation.

$\boxed{\text{Seek separable solution } V(S,t)=\Psi(t)\,\Phi(S)}$ So

$\cdot \equiv \frac{d}{dt}$

$$\frac{\partial V}{\partial t}=\Psi'\Phi,\ \frac{\partial V}{\partial S}=\Phi'\Psi\Rightarrow\frac{\partial^2 V}{\partial S^2}=\Phi''\Psi$$

$' \equiv \frac{d}{dS}$

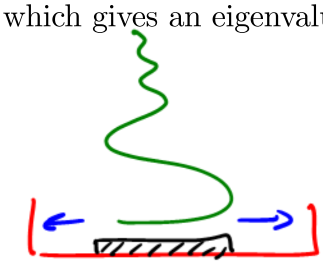
and we substitute in BSE and re-arrange so that all functions of S are on one side and all functions of t on the other, giving:

Indep. of S

Indep.

$$\boxed{\frac{\Psi'}{\Psi}}=\boxed{\frac{-\frac{1}{2}\sigma^2S^2\Phi''-rS\Phi'+r\Phi}{\Phi}}=\boxed{\text{constant}}\quad(\text{e.g. } c)\,.$$

[when using Fourier modes we usually ~~take the constant~~ to be $-\lambda^2$ which gives an eigenvalue problem.]



This now gives us two ODE's:

Firstly a 1st order ODE : $\Psi'=c\Psi\rightarrow\Psi=k\exp(ct)$

Secondly a 2nd order Cauchy-Euler equation:

$$\frac{1}{2}\sigma^2S^2\Phi''+rS\Phi'+(c-r)\,\Phi=0$$

So we look for the existence of solution of the form:

$$\Phi(S)=S^d$$

which upon substituting in the above gives a quadratic in d

A.E. $$d^{\,2}+\left(\frac{2r}{\sigma^2}-1\right)d-\frac{2}{\sigma^2}(r-c)=0$$

hence

$$d_\pm=\frac{1}{2}\left(1-\frac{2r}{\sigma^2}\right)\pm\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2c}{\sigma^2}}$$

<u>3 cases to consider</u>: $\in\mathbb{R}$

x^m $\quad$ $x^m\log x$

(1) Solution for distinct roots - $\Phi(S)=aS^{d_+}+bS^{d_-}$

$$V(S,t)=\exp(ct)\,S^{\frac{1}{2}-\frac{r}{\sigma^2}}\left[AS^{d_+}+BS^{d_-}\right]\qquad A,\ B\text{ - constants}$$

where (from now on)

$$d_+=\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2c}{\sigma^2}}\ ;\qquad d_-=-\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2c}{\sigma^2}}$$

(2) Repeated Root - $\Phi(S)=S^{\frac{1}{2}-\frac{r}{\sigma^2}}\left[a+b\log S\right]$

$(\quad)^2=\frac{2c}{\sigma^2}$

Now $\boxed{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2=\frac{2c}{\sigma^2}}\rightarrow c=\boxed{\frac{\sigma^2}{2}\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2}$ therefore

$$V(S,t)=\exp\left(\frac{\sigma^2}{2}\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2 t\right)S^{\frac{1}{2}-\frac{r}{\sigma^2}}\left[\varepsilon+\zeta\log S\right]\qquad \varepsilon,\,\zeta\text{ - constants}$$

(3) Complex Roots i.e. $\frac{2c}{\sigma^2}>\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2$ - $d_+=\alpha+i\beta$; $d_-=\alpha-i\beta$

$$\Phi(S)=S^{\alpha}\left[A\cos(\beta\ln S)+B\sin(\beta\ln S)\right]$$

where

$$\alpha=\left(\frac{1}{2}-\frac{r}{\sigma^2}\right);\quad \beta=\sqrt{\left|\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2-\frac{2c}{\sigma^2}\right|}$$

$p\pm iq$

$$V(S,t)=\exp(ct)\,S^{\frac{1}{2}-\frac{r}{\sigma^2}}\left[A\cos(\beta\ln S)+B\sin(\beta\ln S)\right]$$

Euler eq^{n}: $y=x^{p}\left[A\cos(q\ln x)+B\sin(q\ln x)\right]$

General sol^{n}

Function	F.T.
$f(x)$	$\hat{f}(\omega)$
$g(x)$	$\hat{g}(\omega)$
$h(x)$	$\hat{h}(\omega)$

$$f(x)\xrightarrow{\mathcal{F}(f(x))}\hat{f}(\omega)$$

$$\xleftarrow[\mathcal{F}^{-1}(\hat{f}(\omega))]{}$$

1.5 The Fourier Transform

Complex numbers.

Integral Transform

If $f = f(x)$ then consider

$x \longmapsto \omega$

$$\hat{f}(\omega) = \boxed{\frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx.$$

$< \infty$

If this special integral converges, it is called the *Fourier Transform* of $f(x)$. Similar to the case of Laplace Transforms, it is denoted as $\mathcal{F}(f)$, i.e.

$\mathcal{L}$ $\mathcal{F}$

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx = \hat{f}(\omega).$$

The *Inverse Fourier Transform* is then

$$\mathcal{F}^{-1}\left(\hat{f}(\omega)\right) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-ix\omega} dx = f(x).$$

The convergent property means that $\hat{f}(\omega)$ is bounded and we have

$$(*) \qquad \boxed{\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.}$$

Functions of this type $f(x) \in L_1(-\infty, \infty)$ and are called *square integrable.*

We know from integration that

$$\boxed{\left|\int_a^b f(x)\, dx\right| \le \int_a^b |f(x)|\, dx.}$$

Analysis

Hence

$$\begin{aligned} \left|\hat{f}(\omega)\right| &= \left|\int_{\mathbb{R}} f(x)\, e^{ix\omega} dx\right| \\ &\le \int_{\mathbb{R}} \left|f(x)\, e^{ix\omega}\right| dx \end{aligned}$$

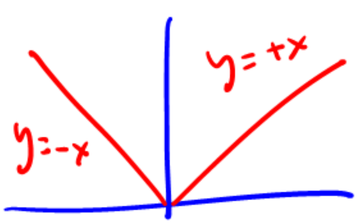
Using (**)

and Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ implies that $\left|e^{i\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, therefore

$$\boxed{\left|\hat{f}(\omega)\right| \le \int_{\mathbb{R}} |f(x)|\, dx < \infty.}$$

In addition to the boundedness of $\hat{f}(\omega)$, it is also continuous (requires a $\boxed{\delta - \epsilon \text{ proof}}$).

Example: Obtain the Fourier transform of $f(x) = e^{-|x|}$



$$\begin{aligned} \hat{f}(\omega) &= \mathcal{F}(f) = \int_{-\infty}^{\infty} f(x)\, e^{ix\omega} dx \\ &= \int_{-\infty}^{\infty} e^{-|x|} e^{ix\omega} dx \\ &= \int_{-\infty}^{0} e^{-|x|} e^{ix\omega} dx + \int_{0}^{\infty} e^{-|x|} e^{ix\omega} dx \\ &= \int_{-\infty}^{0} e^{x} e^{ix\omega} dx + \int_{0}^{\infty} e^{-x} e^{ix\omega} dx \end{aligned}$$

$-\infty \quad 0 \quad \infty$

$\mathcal{F}(e^x) = ?$

$\int_{-\infty}^{\infty}$

$$
\begin{aligned}
&= \int_{-\infty}^{0} \exp\left[(1+i\omega)x\right]dx + \int_{0}^{\infty} \exp\left[-(1-i\omega)x\right]dx \\
&= \frac{1}{(1+i\omega)} \exp\left[(1+i\omega)x\right]\Big|_{-\infty}^{0} + \\
&\quad \frac{1}{(1-i\omega)} \exp\left[-(1-i\omega)x\right]\Big|_{0}^{\infty} \\
&= \frac{1}{(1+i\omega)} + \frac{1}{(1-i\omega)} = \boxed{\frac{2}{(1+\omega^2)}}
\end{aligned}
$$

$\mathcal{F}\left(e^{-|x|}\right) = \frac{2}{1+\omega^2}.$

Our interest in differential equations continues, hence the reason for introducing this transform. We now look at obtaining Fourier transforms of derivative terms. We assume that $f(x)$ is continuous and $f(x) \to 0$ as $x \to \pm\infty$. Consider

$uv - \int uv'$

$$\mathcal{F}\{f'(x)\} = \int_{\mathbb{R}} f'(x)\, e^{ix\omega} dx$$

$v = e^{ix\omega}$ $\quad u' = f'(x)$

$v' = i\omega e^{ix\omega}$ $\quad u = f(x)$

which is simplified using integration by parts

$\hat{f}(\omega)$

Heston's stoch. vol.

$$f(x)\, e^{ix\omega}\Big|_{-\infty}^{\infty} - i\omega \int_{\mathbb{R}} f(x)\, e^{ix\omega} dx$$

$\to 0$

so

$$\mathcal{F}\{f'(x)\} = -i\omega \int_{\mathbb{R}} f(x)\, e^{ix\omega} dx = -i\omega \hat{f}(\omega).$$

$-i\omega \times -i\omega$

We can obtain the Fourier transform for the second derivative by performing integration by parts (twice) to give

$$\mathcal{F}\{f''(x)\} = (-i\omega)^2 \mathcal{F}\{f(x)\} = -\omega^2 \hat{f}(\omega).$$

Example: Solve the diffusion equation problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \\
\text{I.C.} \quad u(x,0) &= e^{-|x|}, \quad -\infty < x < \infty
\end{aligned}
$$

Here $u = u(x,t)$, so we begin by defining

$$\mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)\, e^{ix\omega} dx = \boxed{\hat{u}(\omega,t)}.$$

Now take Fourier transforms of our PDE, i.e.

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \boxed{\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\}}$$

$\frac{\partial}{\partial t} \underbrace{\mathcal{F}(u(x,t))}_{\hat{u}}$

to obtain

$$\boxed{\frac{d\hat{u}}{dt} = -\omega^2 \hat{u}(\omega,t).}$$

1^{st} order O.D.E.

We note that the second order PDE has been reduced to a first order equation of type variable separable. This has solution

$$\boxed{\hat{u}(\omega,t) = Ce^{-\omega^2 t}.}$$

$t=0$

We can find the constant of integration transforming the initial condition

$$
\begin{aligned}
\mathcal{F}\{u(x,0)\} &= \mathcal{F}\left\{e^{-|x|}\right\} \\
\boxed{\hat{u}(\omega,0)} &= \int_{-\infty}^{\infty} e^{-|x|} e^{ix\omega} dx = \frac{2}{(1+\omega^2)}.
\end{aligned}
$$

$\int \frac{d\hat{u}}{\hat{u}} = -\omega^2 \int dt$

$\log \hat{u} = -\omega^2 t + k$

Applying this to the solution $\widehat{u}(\omega, t)$ gives

At $t=0$

$$\boxed{\widehat{u}(\omega, 0) = C = \frac{2}{(1+\omega^2)},}$$

hence

$$\boxed{\widehat{u}(\omega, t) = \frac{2}{(1+\omega^2)} e^{-\omega^2 t}.}$$

We now use the inverse transform to get $u(x,t) = \mathcal{F}^{-1}(\widehat{u}(\omega, t))$

$$
\begin{aligned}
u(x,t) = \mathcal{F}^{-1}(\widehat{u}(\omega,t)) &= \int_{-\infty}^{\infty} \widehat{u}(\omega, t) e^{-ix\omega} d\omega \\
&= 2\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)} e^{-\omega^2 t} e^{-ix\omega} d\omega \\
&= 2\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)} e^{-\omega^2 t} (\cos x\omega - i \sin x)\, d\omega \\
&= 2\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)} e^{-\omega^2 t} \cos x\omega \ d\omega \\
&\quad -2i \boxed{\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)} e^{-\omega^2 t} \sin x\omega \ d\omega.} \to 0
\end{aligned}
$$

Euler's ID

$\int_{-a}^{a} f(x)\, dx = 0$

odd fn.

This now simplifies nicely because $\frac{1}{(1+\omega^2)} e^{-\omega^2 t} \sin x\omega$ is an odd function, hence

$$\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)} e^{-\omega^2 t} \sin x\omega \ d\omega = 0.$$

Therefore

Sol^n

$$\boxed{u(x,t) = 2\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)} e^{-\omega^2 t} \cos x\omega \ d\omega.}$$

Residues

$\int_{-\infty}^{\infty} f(x)\, dx \qquad \int_{0}^{2\pi} f(\theta)\, d\theta$

Complex analysis

$z = x + iy \qquad f(z) = u + iv$