Exotic Options

In this lecture. . .

- the names and contract details for many basic types of exotic options
- how to classify exotic options according to important features
- how to think about derivatives in a way that makes it easy to compare and contrast different contracts
- pricing exotics using Monte Carlo simulation
- pricing exotics via partial differential equations

By the end of this lecture you will be able to

- characterize most exotic contracts according to a list of important features

- price exotics using Monte Carlo simulations

- interpret the pricing of many exotics in terms of partial differential equations

Introduction

Exotic contracts are traded **over the counter** (**OTC**), meaning that they are designed by the relevant counterparties and are not available as exchange-traded contracts.

Exotic options include contracts with features making them more complex to price and to hedge than vanillas.

Often one takes volatilities ‘implied’ by the market prices of vanillas and put them into the pricing model for exotics.

Important features to look out for

when classify exotic options

- Time dependence
- Cashflows
- Path dependence
- Dimensionality
- Order
- Embedded decisions

As for as examples go "The sky is the limit"

Bermudan options

Euro America

Euro ∩ American.

It is common for contracts that allow early exercise to permit the exercise only at certain specified times, and not at *all* times before expiry.

Theoretical vs. Market

For example, exercise may only be allowed on Thursdays between certain times. An option with such intermittent exercise opportunities is called a **Bermudan option**.

All that this means mathematically is that the constraint (**??**) is only 'switched on' at these early exercise dates.

$V = V(S, t)$

1. Time dependence

$\exists$ special dates, on which something special may happen

Here we are concerned with time dependence in the option contract.

For example, discrete cashflows necessarily involve time dependence.

Another example, early exercise might only be permitted on certain dates or during certain periods. This intermittent early exercise is a characteristic of **Bermudan options**.

Similarly, the position of the barrier in a knock-out option may change with time. Every month it may be reset at a higher level than the month before.

- These contracts are referred to as **time inhomogeneous**.

$\Downarrow$

non-exotics are time indep.

When there is time dependence in a contract we might expect

- jumps in option values and/or the greeks

- to have to worry about time discretization in numerical schemes

t_d

$t + dt$

2. Cashflows

Imagine a contract that pays the holder an amount q at time t_i. The contract could be a bond and the payment a coupon.

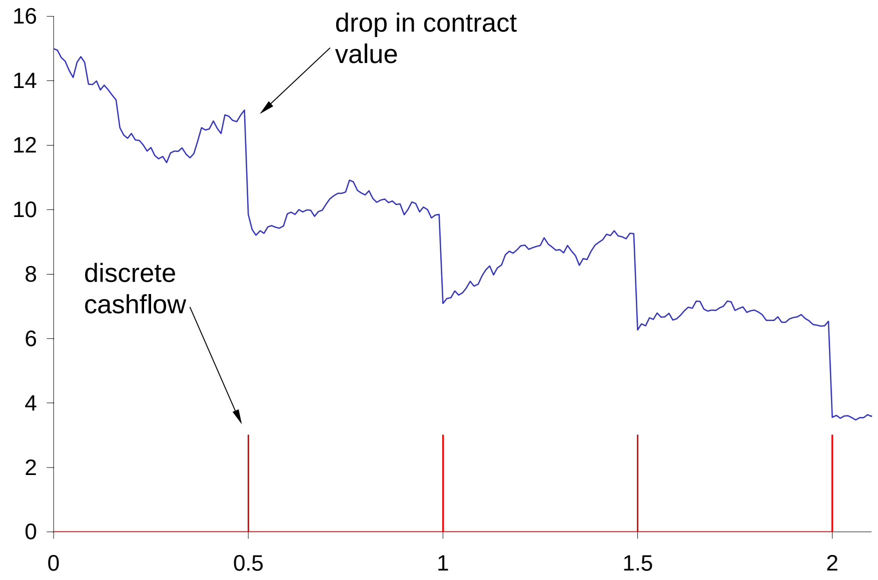
If we use $V(t)$ to denote the contract value and t_i^- and t_i^+ to denote just before and just after the cashflow date then simple arbitrage considerations lead to

cashflow on date t_i

- $$V(t_i^-) = V(t_i^+) + q.$$

This is a **jump condition**. mathematical term

The value of the contract jumps by the amount of the cashflow. The behavior of the contract value across the payment date is shown in the figure.



$q = q(S) \leftarrow$ q_i

A discrete cashflow and its effect on a contract value.

If the contract is contingent on an underlying variable so that we have $V(S,t)$ then we can accommodate cashflows that depend on the level of the asset S i.e. we could have $q(S)$. See related exercise sheet question

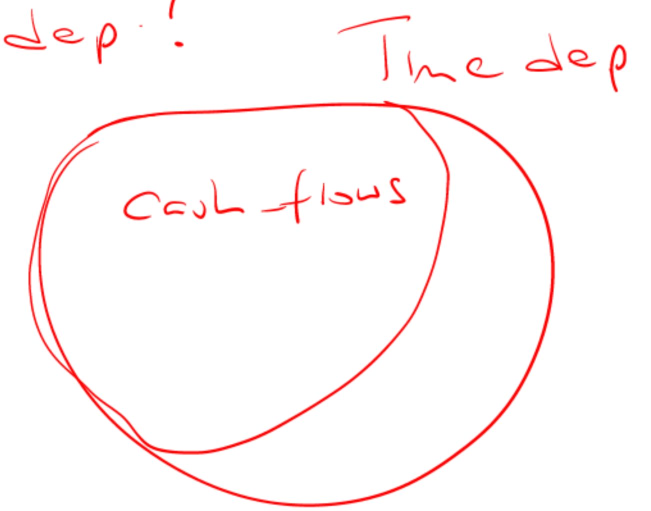
That's an example of a **discrete cashflow**.

Some contracts specify **continuous cashflows**. There may be a payment every day.

When there are cashflows we expect

- option values to jump
- the greeks to jump

cash flows $\subseteq$ Time dep.?



3. Path dependence

Path-dependent contracts have payoffs, and therefore values, that depend on the history of the asset price path.

An asset starts at A and ends at Z at expiration. If the contract is path dependent the route taken from A to Z matters. If it is not path dependent then the route does not matter.

Path dependence comes in two main forms:

- Strong path dependence $\longrightarrow$ extra variable(s)
- Weak path dependence $\longrightarrow$ does not introduce new dimensionality

Strong path dependence

Of particular interest, mathematical and practical, are the **strongly path-dependent contracts**.

These have payoffs that depend on some property of the asset price path in addition to the value of the underlying at the present moment in time; in the equity option language, we cannot write the value as $V(S, t)$.

- The contract value is a function of at least one more independent variable.

$V(S, I, t)$

$V(S_1, S_2, t)$ $V(S, r, t)$

One **Example:**

The Asian option has a payoff that depends on the average value of the underlying asset from inception to expiry. We must keep track of more information about the asset price path than simply its present position.

I — some sampling

The extra information that we need is contained in the 'running average.' This is the average of the asset price from inception until the present, when we are valuing the option.

Asian — I will handle some form of averaging.


30
25
20
15
10
5
0
Asset
Continuous arithmetic average

252 prices Average → uses all 252 prices cts
→ first price of each month discrete

Path dependency also comes in **discrete** and **continuous** varieties depending on whether the path-dependent quantity is **sampled** discretely or continuously.

Discrete arithmetic average $A = \frac{1}{n}\sum_{i=1}^{n} S(t_i)$

Discrete running arith average $A_i = \frac{1}{i}\sum_{k=1}^{i} S(t_k)$

$\to I(t) = \int_0^t f(S)\, d\tau$

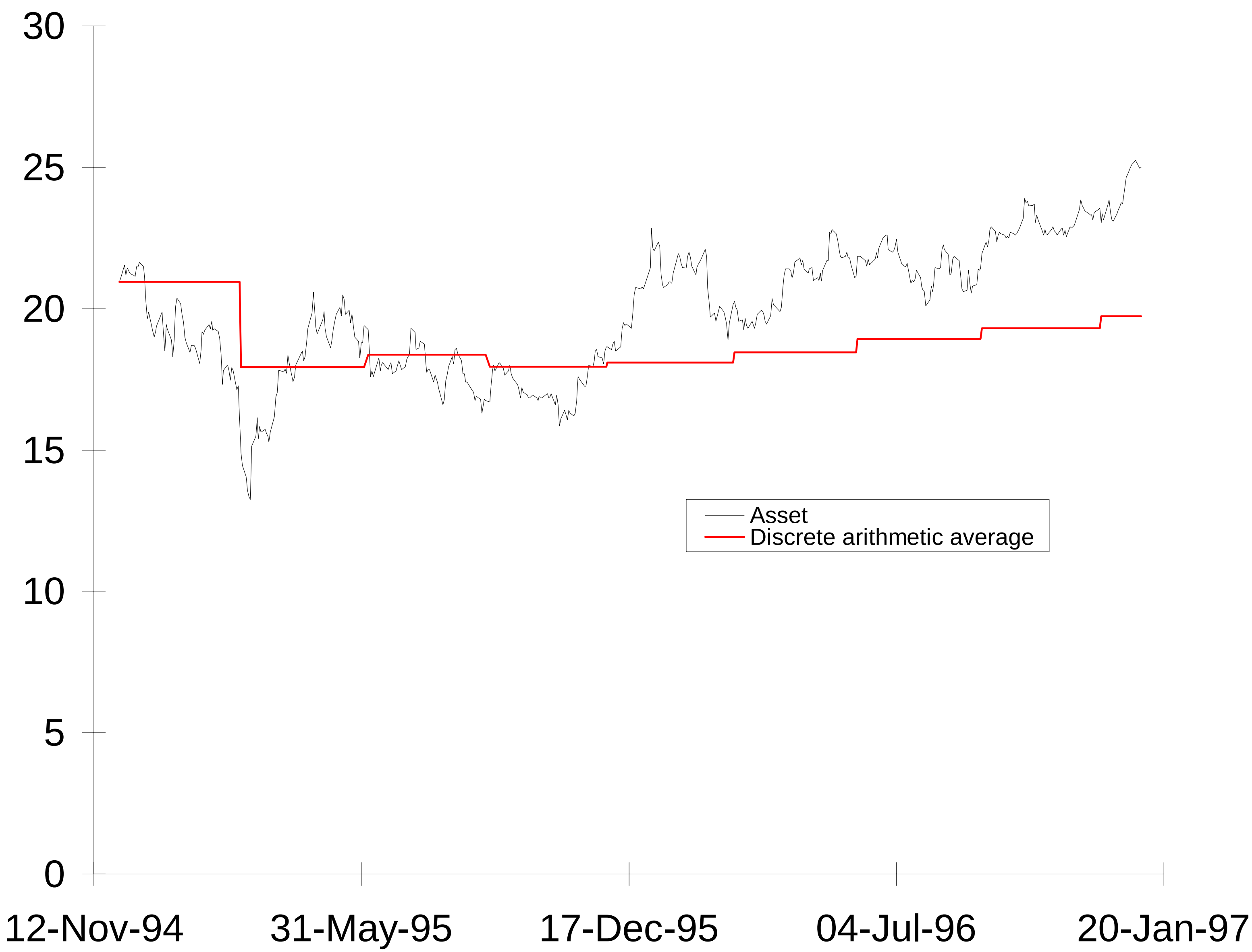
Continuous time equivalents $A(T) = \frac{1}{T}\int_0^T S(t)\, dt$; $A(t) = \frac{1}{t}\int_0^t S(\tau)\, d\tau$

Geometric Averaging: $A_g = \left[\prod_{i=1}^{n} S(t_i)\right]^{1/n}$ let's rearrange

$\log A_g = \frac{1}{n}\log\prod_{i=1}^{n} S(t_i) = \frac{1}{n}\sum_{i=1}^{n}\log S(t_i)$ Now take exp

$A_g = \exp\left[\frac{1}{n}\sum_{i=1}^{n}\log S(t_i)\right]$ cts time: $A_g = \exp\left[\frac{1}{T}\int_0^T \log S(t)\, dt\right]$

discrete



30
25
20
15
10
5
0
12-Nov-94
31-May-95
17-Dec-95
04-Jul-96
20-Jan-97
Asset
Discrete arithmetic average

When there is strong path dependence in a contract we might expect

- to have to solve in higher dimensions

(Sampling)

(We have to keep track of a new **state variable** such as the average to date.)

Asian tails.

Weak path dependence

- Options whose value depends on the asset history, but can still be written as $V(S, t)$ are said to be **weakly path dependent**.

New B.Cs level of underlying $S = S_B$

One of the most common reasons for weak path dependence in a contract is a **barrier**. Barrier (or knock-in, or knock-out) options are triggered by the action of the underlying hitting a prescribed value at some time before expiry.

For example, as long as the asset remains below 150, the contract will have a call payoff at expiry. However, should the asset reach this level before expiry then the option becomes worthless; the option has 'knocked out.' Knock-in zero to begin. Must activate barrier

Knock-out : starts off as plain vanilla. Must not hit

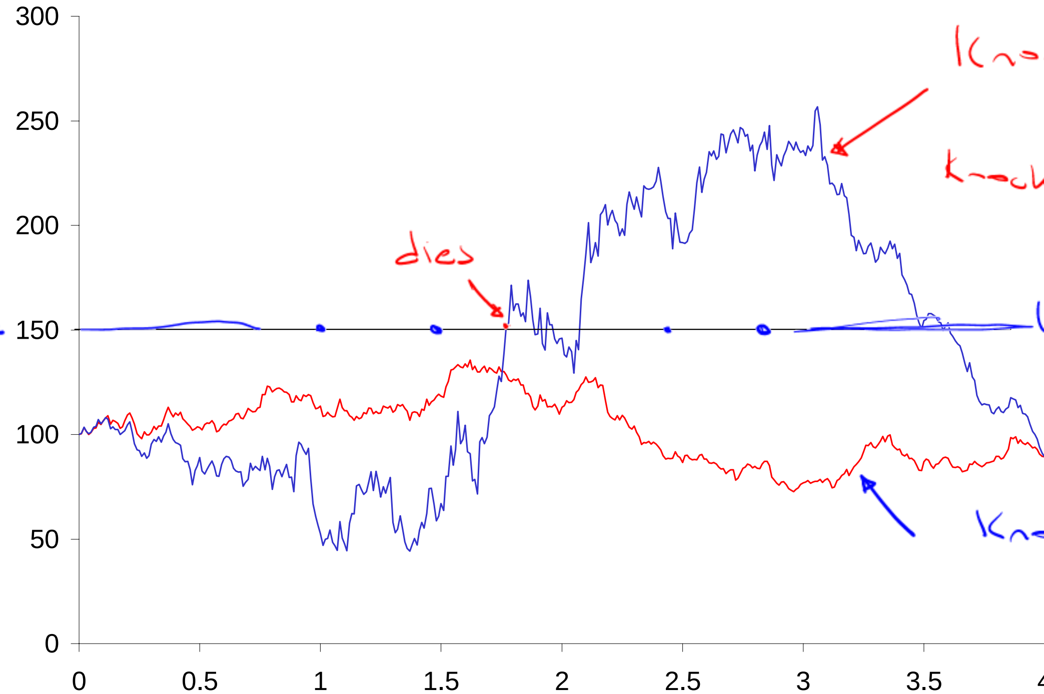
prescribed barrier S_B else "dies"

Lifelong learning: PhD and colleague

$150 > S_0 = 100$

$S_B = 150$

$S_0 = 100$



Knock-in → holder gets payoff.

knock out – worthless

Up barrier $S = B_u$

Knock-in: doesn't come alive. Stays worthless

Double barrier

Multiple barriers

Resetting barrier (time dep.)

Knock-out: holder gets payoff at expiry.

Two paths having the same value at expiry but with completely different payoffs.

partial barriers

discrete barriers

- Weak path dependency does not add any extra dimensions.

(So a barrier option still only has two dimensions, S and t.)

4. Dimensionality

Dimensionality refers to the number of underlying independent variables.

$V(S_1, \ldots, S_n, t)$ $\qquad \dfrac{dS_i}{S_i} = \mu_i\, dt + \sigma_i\, dX^{(i)}$

multi-factor problem

- The vanilla option has two independent variables, S and t, and is thus two dimensional.

$$\mathbb{E}\left[dX^{(p)} dX^{(q)}\right] = \rho_{pq}\, dt$$

- The weakly path-dependent contracts have the same number of dimensions as their non-path-dependent cousins, i.e. a barrier call option has the same two dimensions as a vanilla call.

But some contracts require us to go in to extra dimensions!

There are two distinct reasons why we need more dimensions. . .

- More sources of randomness Multi-factor models

  $dX^{(i)}$ $1 \leq i \leq n$

- Strong path dependence $V = V(S, I, t)$

More dimensions caused by more sources of randomness

We will get higher dimensions if we have more sources of randomness

- If we have an option on **10** underlyings ('best of' for example) we will have **11** dimensions (S_1, S_2,. . . ,S_{10} and t)

But we will also get more dimensions if we have other types of randomness, such as volatility.

- If we have an option on **10** underlyings and we use a stochastic volatility model for each asset we will have **21** dimensions (S_1, S_2,. . . ,S_{10} and t, and also σ_1, σ_2,. . . ,σ_{10})

Each new dimensions introduces extra 'diffusion' terms. (What does this mean for the governing PDE?)

More dimensions caused by strong path dependency

We will get higher dimensions if we an option that is strongly path dependent.

- If we have an option that pays off the maximum of the average stock price we will have **4** dimensions (S and t, but also a state variable for the average and another for the maximum of the average!)

S; I (averaging var.) ; t ; V

We'll see the theory of this later, but the effect on the governing PDE is to sometimes add new terms that are not diffusive! (But no new parameters!)

When the problem is of high dimensions we might expect

- to have restrictions on the kind of numerical solution we employ. The higher the number of dimensions, the more likely we are to want to use Monte Carlo simulations.

5. The order of an option

Thus far we have considered 1st order options

The basic, vanilla options are of first order. Their payoffs depend only on the underlying asset, the quantity that we are *directly* modeling. Other, path-dependent, contracts can still be of first order if the payoff only depends only on properties of the asset price path.

- **Higher order** refers to options whose payoff, and hence value, is contingent on the value of *another* option.

The obvious second-order options are compound options, for example, a call option giving the holder the right to buy a put option. The compound option expires at some date T_1 and the option on which it is contingent, expires at a later time T_2. Technically speaking, such an option is weakly path dependent.

From a practical point of view, the compound option raises some important modeling issues.

- The payoff for the compound option depends on the *market* value of the underlying option, and not on the theoretical price.

If you hold a compound option, and want to exercise the first option then you must take possession of the underlying option. High order option values are very sensitive to the basic pricing model and should be handled with care.

$C \circ C$

$C \circ P$

$P \circ C$

$P \circ P$

Schematic diagram of exotic option classification system:

- **Cashflows**
  - **Discrete** q_i
  - **Continuous** $q(S)$
- **Time dependence**
- **Path dependence**
  - **Weak**
    - **Discrete**
    - **Continuous**
  - **Strong**
    - **Discrete**
    - **Continuous**
- **Dimensionality**
  - **Multi factor**
  - **Strong path dependency**
- **Order**
- **Embedded decisions** (MC struggles here)

Pricing methodologies

Now let's look at the (numerical) pricing of exotic options.

Two main methods:

- Pricing via simulations, Monte Carlo

- Formulating the pricing problem in terms of partial differential equations, for solving by finite-difference methods

Pricing via expectations, Monte Carlo simulation

We can value options in the Black–Scholes world by taking the present value of the expected payoff under a risk-neutral random walk.

Simply simulate the random walk

$$dS = rS\,dt + \sigma S\,dX$$

for many paths, calculate the payoff for each path—and this means calculating the value of the path-dependent quantity which is usually very simple to do—take the average payoff over all the paths and then take the present value of that average.

That is the option's fair value.

When and when not to use MC

This is a very general and powerful technique.

When to use MC:

- Good when there are a large number of dimensions
- Useful for path-dependent contracts (even if low dimensions!) for which a partial differential equation approach is tedious to set up
- Some models $\overbrace{\text{(e.g. HJM)}}^{\text{Dr. Diamond}}$ are built for MC, not easy (or impossible) to write as PDE

When not to use MC:

- The main disadvantage is that it is hard to value options with embedded decisions using MC simulation

Upwind differencing C.F.D

Partial differential equations and finite differences

To be able to turn the valuation of a derivatives contract into the solution of a partial differential equation is a big step forward.

- The partial differential equation approach is one of the best ways to price a contract because of its flexibility and because of the large body of knowledge that has grown up around the fast and accurate numerical solution of these problems

- But there is effort involved in setting up the PDE for numerical solution. (In contrast, Monte Carlo can be used 'straight out of the bag')

Let's look at setting up the PDE approach for two examples, a **barrier** option and an **Asian** option.

Both of these can be priced via Monte Carlo but finite-difference solution of the PDEs will be faster.

- Is it worth the effort? Sometimes you might do initial pricing via MC (just to get a 'number') and then you'll spend a bit of time coding up finite differences before it goes into the bank's 'system.'

After we've looked at these two problems we'll do a general theory of path-dependent options.

Barrier options

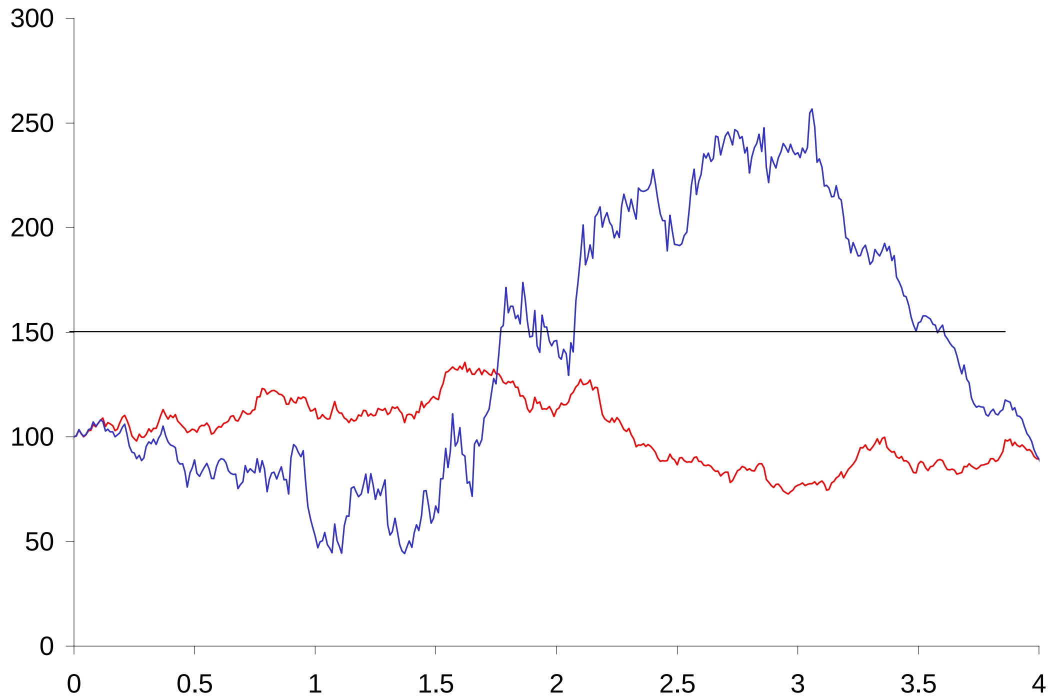
- **Barrier options** have a payoff that is contingent on the underlying asset reaching some specified level before expiry.

The critical level is called the barrier, there may be more than one.

Barrier options come in two main varieties, the ‘in’ barrier option (or **knock-in**) and the ‘out’ barrier option (or **knock-out**). The former only have a payoff if the barrier level is reached before expiry and the latter only have a payoff if the barrier is *not* reached before expiry.

These contracts are weakly path dependent.

Example: An up-and-out call option. This has a call payoff at expiration unless the barrier has been triggered some time before expiration.

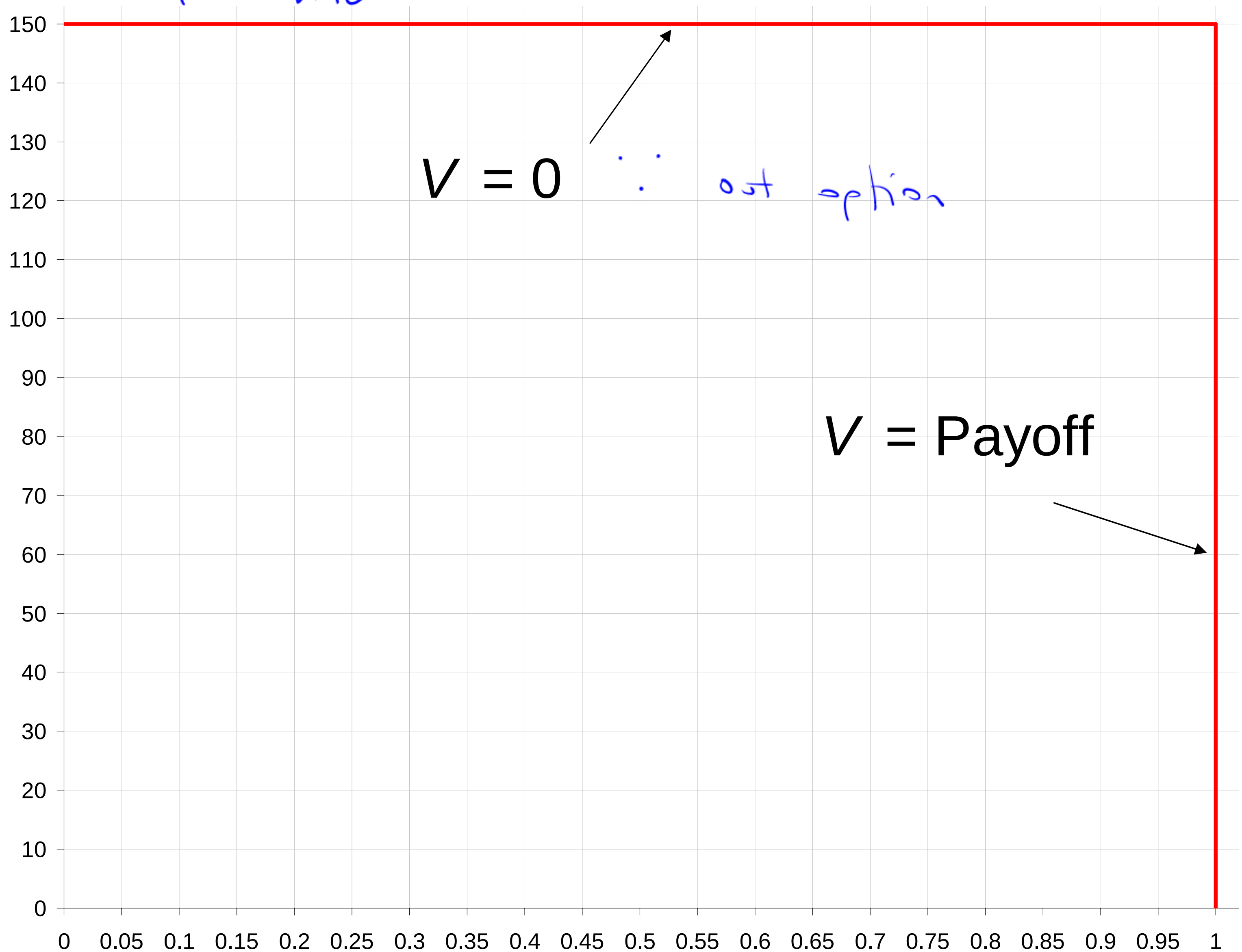


This is easily solved by Monte Carlo simulation or by finite-difference methods.

The latter is much preferable. And because the barrier option is weakly path dependent the relevant PDE is exactly the classical Black–Scholes equation! We just have to figure out **initial** and **boundary conditions**.

We still have 2 B.C.s

Up barrier



Asian options

- **Asian options** have a payoff that depends on the average value of the underlying asset over some period before expiry.

They are strongly path dependent. Their value prior to expiry depends on the path taken.

Fixed strike vs. Floating strike

A is some average

Call option $\max(A - E, 0)$ fixed strike Asian

In this example A replaces $S(T)$ from a plain vanilla.

Call option $\max(S(T) - A, 0)$ floating strike $\therefore$ A is now no longer fixed.

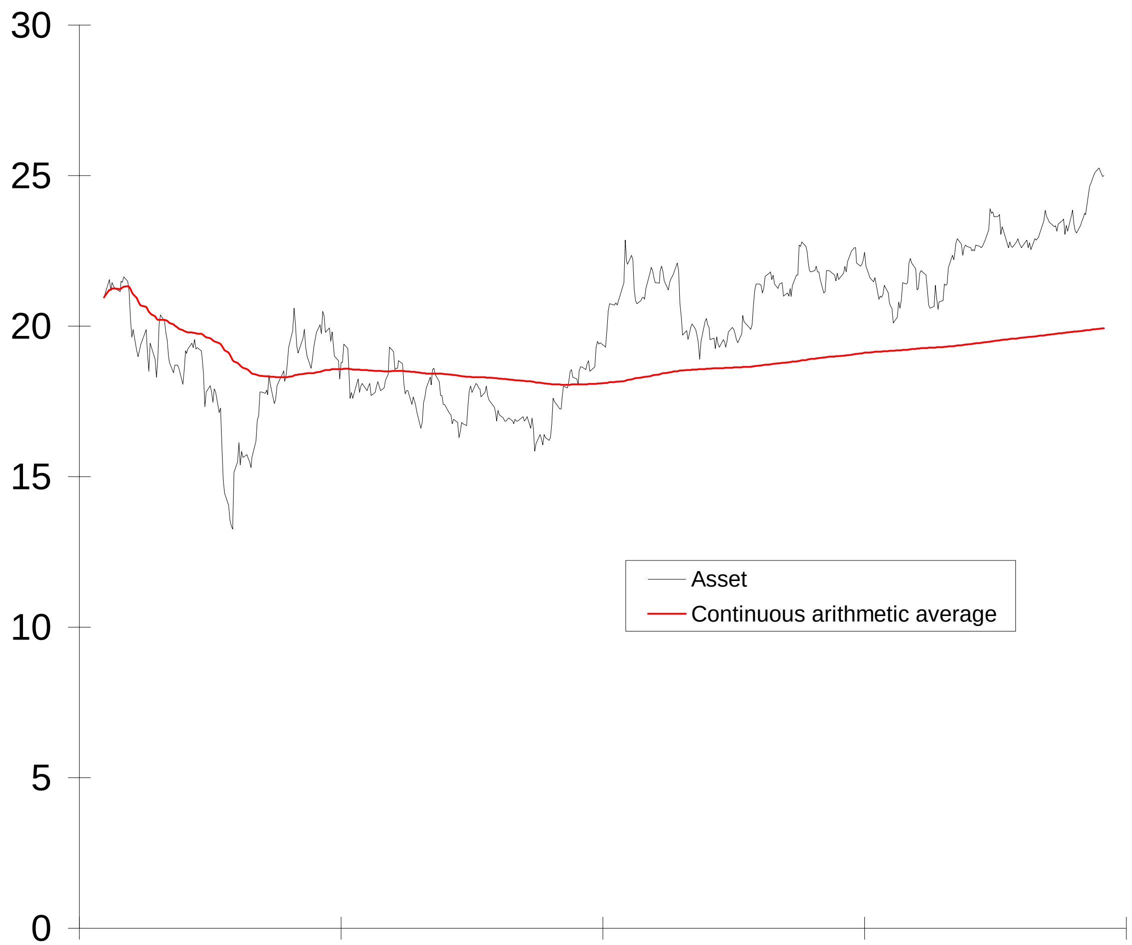
The average used in the calculation of the option's payoff can be defined in many different ways.

It can be an **arithmetic average** or a **geometric average**, for example.

The data could be **continuously sampled**, so that every realized asset price over the given period is used. More commonly, for practical and legal reasons, the data is usually **sampled discretely**.

$$\max\left[e^{\frac{1}{T}\int_0^T \log S\,dt} - S(T),\ 0\right]$$

How is the continuously sampled arithmetic average, for example, defined mathematically?



The final payoff is a function of

$$A = \frac{1}{T}\int_0^T S(\tau)\, d\tau.$$

(The averaging started at time $t = 0$.)

But the running average, and hence our new **state variable** is

$$A = \frac{1}{t}\int_0^t S(\tau)\, d\tau.$$

We need a theory for options with payoff depending on integrals.

A theory for strong path dependence

Continuous

We will now look at

- pricing many strongly path-dependent contracts in the Black–Scholes partial differential equation framework
- how to handle both continuously sampled and discretely sampled paths
- jump conditions for differential equations

We will now see how to generalize the Black–Scholes analysis, delta hedging and no arbitrage, to the pricing of many more derivative contracts, specifically contracts that are strongly path dependent.

Path-dependent quantities represented by an integral

We start by assuming that the underlying asset follows the log-normal random walk

$$dS = \mu S \, dt + \sigma S \, dX.$$

Imagine a contract that pays off at expiry, T, an amount that is a function of the path taken by the asset between time zero and expiry.

average will incorporate the path

- Let us suppose that this path-dependent quantity can be represented by an integral of some function of the asset over the period zero to T:

$$I(T) = \int_0^T \underbrace{f(S, \tau)} d\tau.$$

This is not such a strong assumption, many of the path-dependent quantities in exotic derivative contracts, such as averages, can be written in this form with a suitable choice of $f(S, t)$.

$$I(t) = \int_0^t f(S, \tau)\, d\tau$$

cts arith average $\quad f \equiv S$

cts geom average $\quad f \equiv \log S$

You might think that we need to model and remember S at every single moment between now and expiration.

This may look like a problem with an infinite number of variables.

It is much easier than this.

For the basic Asian option it turns out, as we shall see, that not all of the past of the asset matters, only one ‘functional’ of it.

Prior to expiry we have information about the possible final value of S (at time T) in the present value of S (at time t).

For example, the higher S is today, the higher it will probably end up at expiry.

Similarly, we have information about the possible final value of I in the value of the integral to date:

average Asian

realised max / min.

$$m = \min_{1 \le i \le n} S(t_i)$$

$$I(t) = \int_0^t f(S, \tau)\, d\tau. \qquad (1)$$

As we get closer to expiry, so we become more confident about the final value of I.

$$M = \max_{1 \le i \le n} S(t_i)$$

- One can imagine that the value of the option is therefore not only a function of S and t, but also a function of I; I will be our new independent variable, called a **state variable**.

We see in the next section how this observation leads to a pricing equation.

In anticipation of an argument that will use Itô's lemma, we need to know the stochastic differential equation satisfied by I.

This could not be simpler. $I_t + \underbrace{dI}$

- Incrementing t by dt in (1) we find that

$\frac{dI}{dt} = f(S,t)$

$$dI = f(S,t)\,dt + 0 \times dX \tag{2}$$

Observe that I is a smooth function (except at discontinuities of f) and from (2) we can see that its stochastic differential equation does not contain any stochastic terms.

Continuous sampling: The pricing equation

We will derive the pricing partial differential equation for a contract that pays some function of our new variable I.

- The value of the contract is now a function of the three variables, $V(S, I, t)$.

Itô on V ?

$$V(t+dt, S+dS, I+dI) = V(t, S, I) +$$

$$dV = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial I} \underbrace{dI}_{f\,dt} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \underbrace{dS^2}_{\sigma^2 S^2 dt}$$

Set up a portfolio containing one of the path-dependent option and short a number Δ of the underlying asset:

fix across dt

hedged portfolio.
$$\Pi = V(S, I, t) - \Delta S.$$

$t \to t + dt$ $\quad d\Pi = dV - \Delta dS$

The change in the value of this portfolio is given by

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial I} dI + \left(\frac{\partial V}{\partial S} - \Delta \right) dS.$$

how can we eliminate risk i.e. $\Delta = \frac{\partial V}{\partial S}$

$\Rightarrow$ No-arb. $d\Pi = r\,\Pi\, dt$

- Choosing

$$\Delta = \frac{\partial V}{\partial S}$$

to hedge the risk, and using (2), we find that

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t)\frac{\partial V}{\partial I} \right) dt.$$

This change is risk free, and thus earns the risk-free rate of interest r, leading to the pricing equation...

$\boxed{\frac{\partial}{\partial I}(I) = 1}$ $\Rightarrow$ $\cancel{\frac{dI}{dt}}$ S $\frac{\partial}{\partial t} S = 0$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \underbrace{f(S,t)\frac{\partial V}{\partial I}}_{\text{new term}} + rS\frac{\partial V}{\partial S} - rV = 0$$

This is to be solved subject to $\frac{d}{dx}(x) = 1$

$$V(S, I, T) = \text{Payoff}(S, I).$$

This completes the formulation of the valuation problem.

$\therefore$ there is no diffusion in dI $\Rightarrow$ No $\frac{\partial^2 V}{\partial I^2}$ term.

Example:

Continuing with the arithmetic Asian example, we have

$$I = \int_0^t S \, d\tau,$$

$f \equiv S$

so that the equation to be solved is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0.$$

1. combine existing variables

Similarity reductions

2. reduce dim$^{\underline{n}}$ of the problem

As long as the stochastic differential equation for the path-dependent quantity only contains references to S, t and the path-dependent quantity itself then the value of the option depends on three variables.

Unless we are very lucky, the value of the option must be calculated numerically.

- Some options have a particular structure that permits a reduction in the dimensionality of the problem by use of a similarity variable.

The dimensionality of the continuously sampled arithmetic average strike option can be reduced from three to two.

floating

The payoff for the continuously sampled arithmetic average strike call option is

$$\max\left(S - \frac{1}{T}\underbrace{\int_0^T S(\tau)\,d\tau}_{I},\ 0\right).$$

This can be written as $I \max\left(\frac{S}{I} - \frac{1}{T}, 0\right)$

$$I \max\left(\boxed{R} - \frac{1}{T},\ 0\right)$$

$R = \frac{S}{I}$

where

$$\underbrace{I = \int_0^t S(\tau)\,d\tau}$$

and

$$R = \frac{S}{\int_0^t S(\tau)\,d\tau}. \quad = \frac{S}{I}$$

new idea

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0$$

In view of the form of the payoff function, it seems plausible that the option value takes the form

$$\frac{\partial R}{\partial I} = -\frac{S}{I^2} \qquad \frac{\partial R}{\partial S} = \frac{1}{I}$$

$$\frac{\partial V}{\partial t} = I\frac{\partial W}{\partial t}$$

proposition

$$\frac{\partial V}{\partial S} = I\frac{\partial W}{\partial S} = I\frac{\partial R}{\partial S}\frac{\partial W}{\partial R} = I \times \frac{1}{I}\frac{\partial W}{\partial R} = \frac{\partial W}{\partial R}$$

$$V(S, I, t) = IW(R, t), \quad \text{with} \quad R = \frac{S}{I}.$$

new var.

We find that W satisfies

$$\frac{\partial V}{\partial I} = W + I\frac{\partial W}{\partial I} = W + I\left(\frac{\partial R}{\partial I}\frac{\partial W}{\partial R}\right)$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial W}{\partial R}\right) = \frac{\partial R}{\partial S}\frac{\partial}{\partial R}\frac{\partial W}{\partial R} = \frac{1}{I}\frac{\partial^2 W}{\partial R^2}$$

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + R(r - R)\frac{\partial W}{\partial R} - (r - R)W = 0$$

Laplace Transform.

$$= W - I\left(\frac{S}{I^2}\right)\frac{\partial W}{\partial R} = W - R\frac{\partial W}{\partial R}$$

with final condition

Subst in PDE & ÷ thro by I.

$$W(R, T) = \max\left(R - \frac{1}{T}, 0\right)$$

Path-dependent quantities represented by an updating rule

For practical and legal reasons path-dependent quantities are never measured continuously.

There is minimum time step between sampling of the path-dependent quantity.

- From a practical viewpoint it is difficult to incorporate every single traded price into an average, for example. Data can be unreliable and the exact time of a trade may not be known accurately.

If the time between samples is small we can confidently use a continuous-sampling model, the error will be small.

If the time between samples is long, or the time to expiry itself is short we must build this into our model. This is the goal of this section.

When path-dependent quantities are sampled discretely we have summations instead of integrals.

Example: The discretely sampled Asian option

We saw how to use the continuous running integral in the valuation of Asian options.

But what if that integral is replaced by a discrete sum?

In practice, the payoff for an Asian option depends on

$$A_M = \frac{I_M}{M} = \frac{1}{M} \sum_{k=1}^{M} S(t_k), \tag{3}$$

where M is the total number of sampling dates.

This is the discretely sampled average.

This is simply a discrete version of our earlier continuous integral.

Example: The payoff for a discretely sampled arithmetic average strike put is then

$$\max\left(A_M - S, 0\right).$$

Must we remember every single $S(t_k)$ to price the option? That would be an $M+2$-dimensional problem!

Recall that when valuing the continuously sampled Asian option we only had to remember the value of a single new quantity, the average to date.

Is the same true with the discretely sampled Asian?

Can we write the expression for the running discretely sampled average in a form that does not require us to remember every single $S(t_k)$?

Yes.

Running arith average – discrete time av

$$A_i = \frac{1}{i}\sum_{k=1}^{i} S(t_k)$$

$\exists$ many sampling dates $k=1, 2, 3, \ldots, i$

So that

$\frac{1}{2} S_1 + \frac{1}{2} S_2$

A_1

previous stock

today's stock price

$t=1$

$t=2$

$t=3$

$$A_1 = S(t_1),$$

$$A_2 = \frac{S(t_1) + S(t_2)}{2} = \frac{1}{2}A_1 + \frac{1}{2}S(t_2),$$

$$A_3 = \frac{S(t_1) + S(t_2) + S(t_3)}{3} = \frac{2}{3}A_2 + \frac{1}{3}S(t_3), \cdots.$$

$= \frac{S_1 + S_2}{3} + \frac{1}{3} S_3$

• This can be expressed as an **updating rule**

$$A_i = \frac{1}{i}\underbrace{S(t_i)}_{\text{today's stock}} + \frac{i-1}{i}\underbrace{A_{i-1}}_{\text{last average}}.$$

No

Generalization

An **updating rule** is an algorithm for defining the path-dependent quantity in terms of the current 'state of the world.'

- The path-dependent quantity is measured on the **sampling dates** t_i, and takes the value I_i for $t_i \leq t < t_{i+1}$.

- At the sampling date t_i the quantity I_{i-1} is updated according to a rule such as

  some general rule   time

  $$I_i = F(\underbrace{S(t_i)}_{\text{today's stock}}, I_{i-1}, i).$$

  updated rule   last sample

  Note how, in this simplest example, the new value of I is determined by only the old value of I and the value of the underlying on the sampling date, and the sampling date.

Another example: the Lookback option

We will see how to use this for pricing in the next section. But first, another example.

The lookback option has a payoff that depends on the maximum or minimum of the realized asset price.

$$\text{Payoff}(M) \qquad \max_{1 \le i \le n} S(t_i)$$

$$\text{Payoff}(m) \qquad \min_{1 \le i \le n} S(t_i)$$

$M_1 = \max S(t_1)$

$t_2: \max(S(t_2), M_1)$

If the payoff depends on the maximum sampled at times t_i then we have

$$I_1 = S(t_1), \quad I_2 = \max(S(t_2), I_1), \quad I_3 = \max(S(t_3), I_2) \cdots .$$

- The updating rule is therefore simply

last update.

$$\underbrace{I_i}_{\text{today's Update}} = \underbrace{\max}_{\text{rule}}(S(t_i), I_{i-1}).$$

today's stock price

How do we use these updating rules in the pricing of derivatives?

Discrete sampling: The pricing equation

- We anticipate that the option value will be a function of three variables, $V(S, I, t)$.

The first step in the derivation is the observation that the stochastic differential equation for I is degenerate:

$$dI = 0.$$

This is because the variable I can only change at the discrete set of dates t_i. This is true if $t \neq t_i$ for any i.

- So provided we are not *on* a sampling date the quantity I is constant, the stochastic differential equation for I reflects this, and the pricing equation is simply the basic Black–Scholes equation:

B.S.E

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

+ updating rule F.D.M

How does the equation know about the path dependency?

- Across a sampling date the option value is continuous.

As we get closer and closer to the sampling date we become more and more sure about the value that I will take according to the updating rule.

Since the outcome on the sampling date is known and since *no money changes hands* there cannot be any jump in the value of the option.

This is a simple application of the no arbitrage principle.

We introduce the notation t_i^- to mean the time infinitesimally before the sampling date t_i and t_i^+ to mean infinitesimally just after the sampling date.

Continuity of the option value is represented mathematically by

becomes

t_i

$$V(S, I_{i-1}, t_i^-) := V(S, I_i, t_i^+).$$

before just after

- In terms of the updating rule, we have

updating rule

$$\boxed{V(S, I, t_i^-) = V(S, F(S(t_i), I, i), t_i^+)}$$

This is called a **jump condition**.

Test

$$\boxed{V \geq \text{payoff}}$$

Check

How to apply discrete sampling condition,

Examples: Summary.

- To price an arithmetic Asian option with the average sampled at times t_i solve the Black–Scholes equation for $V(S, A, t)$ with

Asian option

$$V(S, A, t_i^-) = V\left(S, \frac{i-1}{i}A + \frac{1}{i}S, t_i^+\right).$$

See earlier updating rule

- To price a lookback depending on the maximum sampled at times t_i solve the Black–Scholes equation for $V(S, M, t)$ with

$$V(S, M, t_i^-) = V\left(S, \max(S, M), t_i^+\right).$$

Let's see how this is applied.

① classification of exotics

② PDE + general discussion

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③ Updating rules