Monte Carlo Techniques

Earlier we derived the Black-Scholes problem to price a European option $V(S,t)$, where the underlying asset follows GBM

$$dS = \mu S dt + \sigma S dW.$$

The resulting PDE and payoff $P(S)$ at expiry T which is satisfied by $V(S,t)$ is

PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

Terminal condition

$$V(S,T) = P(S).$$

The BSE is a linear parabolic PDE and as such the solution can be expressed as an integral of the form

at T stock $= S'$

$$V(S,t) = e^{-r(T-t)} \underbrace{\int_0^\infty \underbrace{\tilde{p}\left(S,t;S',T\right)} V\left(S',T\right) dS'}_{E[V(S',T)]}.$$

L3.1

PV

Payoff

$\widetilde{p}\,(S,t;S',T)$ represents the transition density and is the solution of the backward Kolmogorov problem

$$\begin{aligned} \frac{\partial \widetilde{p}}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 \widetilde{p}}{\partial S^2}+rS\frac{\partial \widetilde{p}}{\partial S} &= 0, \\ \widetilde{p}\left(S,t;S',T\right) &= \delta\left(S'-S\right). \end{aligned}$$

We have discussed that the function $\widetilde{p}\,(S,t;S',T)$ can be considered as one of two entities.

Firstly in the PDE framework it can be thought of as a Green's function for the general backward problem

$$\begin{aligned} \frac{\partial U}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2}+rS\frac{\partial U}{\partial S} &= 0, \\ U\,(S,T) &= f\,(S)\,. \end{aligned}$$

Secondly, and more importantly for this section in probabilistic terms it is the probability density function for the risk-neutral random walk mentioned earlier. This is also called the risk-neutral measure.

$V(S,t) = e^{-r(T-t)}\, U(S,t)$

We can write the value of the option in the form

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[P\left(S\right)\right]$$

which is the present value of the expected payoff wrt the risk-neutral probability density $\mathbb{Q}$ and recall

$$\mathbb{E}^{\mathbb{Q}}\left[P\left(S\right)\right] = \int_0^\infty \widetilde{p}\left(S,t;S',T\right) P\left(S'\right) dS'.$$

The precise form of the integral obtained in the BSE work is

$$\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,\text{Payoff}(S')\frac{dS'}{S'}. \qquad \text{(A)}$$

This expression works because the equation is linear - so we just need to specify the payoff condition. It can be applied to any European option on a single lognormal underlying asset.

Equation (A) gives us the risk-neutral valuation. $e^{-r(T-t)}$ present values to today time t.

The integral is the expected value of the payoff with respect to the lognormal transition pdf. The future state is (S', T) and today is (S, t). So it represents the probability of going from $\underbrace{(S, t) \longrightarrow (S', T)}$.

Also note the presence of the risk-free IR r in the pdf. So the expected payoff is as if the underlying evolves according to the *risk-neutral* random walk

$\mathbb{Q}$

$$\frac{dS}{S} = r dt + \sigma dW.$$

The real world drift μ is now replaced by the risk-free return r. The delta hedging has eliminated all the associated risk. This means that if two investors agree on the volatility they will also agree on the price of the derivatives even if they disagree on the drift. This brings us on to the idea of *risk-neutrality* and $\boxed{\text{risk-neutral pricing.}}$ Since ML2 path indep.

So we can think of the option as discounted expectation of the payoff under the assumption that S follows the risk neutral random walk

$$V(S,t) = e^{-r(T-t)} \int_0^\infty \widetilde{p}\left(S, t; S', T\right) V\left(S', T\right) dS'$$

where $p\left(S, t; S', T\right)$ represents the transition density and gives the probability of going from (S, t) to $\left(S', T\right)$ under $\frac{dS}{S} = r dt + \sigma dW$, i.e. the risk-neutral random walk.

So clearly we have a definition for $\widetilde{p}$, i.e. the lognormal density given by

$$\widetilde{p}\left(S, t; S', T\right) = \frac{1}{\sigma S' \sqrt{2\pi(T-t)}} e^{-\left(\log(S/S') + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2 / 2\sigma^2(T-t)}.$$

MIL5

Two important points

Not of concern

- $\widetilde{p}\left(S, t; S', T\right)$ is a Green's for the BSE. As the PDE is linear we can write the solution down as the integrand consisting of this function and the final condition.

- The BSE is essentially the backward Kolmogorov equation whose solution is the transition density $\widetilde{p}\,(S,t;S',T)$ with (S',T) fixed and varying (S,t); but with the discounting factor.

BSE : BKE + discounting term $(-rV)$

- Large availability of software
- Complex path dependency can be easily incorporated

Monte Carlo weaknesses

- The method can be slow for very low dimensions (1-3)
- Accuracy comes at the expense of computational cost due to the large number of simulations required
- The method does not cope well with embedded decisions - early exercise features.

Monte Carlo – Integration

We know from earlier that the SDE $\boxed{\frac{dS_t}{S_t} = rdt + \sigma dW_t}$ with constant r and σ has the solution

$$S_T = S_0 \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right) \boxed{T} + \sigma\phi\sqrt{T}\right\},$$

Ito on log S

for some time horizon T; with $\phi \sim N(0,1)$; $W_t \sim N(0,t)$ and can be written $\phi\sqrt{T}$.

It is often more convenient to express in time stepping form

$\sigma_{imp.}$

$t \to t + \delta t$

$$S_{t+\delta t} = S_t \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t}\right\}.$$

In general a closed form solution of an arbitrary SDE is difficult if e.g. $r = r(t)$ and $\sigma = \sigma(S,t)$, i.e. the parameters are no longer constant; or the SDE is complicated.

The need for Monte Carlo requires numerical integration of stochastic differential equations. Previously we considered the Forward **Euler-Maruyama** method. Why did this work?

Martingales

Numerical Analysis

Consider a stochastic process X_t

$$dX_t = a\left(X_t, t\right) dt + b\left(X_t, t\right) dW_t \quad \text{Standard B.M.} \tag{1}$$

The simplest scheme for solving (1) is using the E-M method. That is

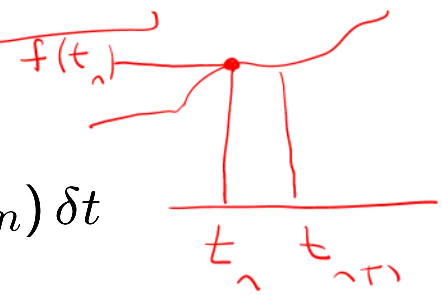
$t_n \quad t_{n+1}$

$$\int_{t_n}^{t_{n+1}} dX_s = \int_{t_n}^{t_{n+1}} a\left(X_s, s\right) ds + \int_{t_n}^{t_{n+1}} b\left(X_s, s\right) dW_s \qquad \boxed{*}$$

$X_{t_{n+1}} - X_{t_n}$

$$X_{n+1} = X_n + \int_{t_n}^{t_{n+1}} a\left(X_s, s\right) ds + \int_{t_n}^{t_{n+1}} b\left(X_s, s\right) dW_s$$

Using the left hand integration rule:



$$\int_{t_n}^{t_{n+1}} a\left(s, X_s\right) ds \;\approx\; a\left(t_n, X_n\right) \int_{t_n}^{t_{n+1}} ds = a\left(t_n, X_n\right) \delta t$$

$$\int_{t_n}^{t_{n+1}} b\left(s, X_s\right) ds \;\approx\; b\left(t_n, X_n\right) \int_{t_n}^{t_{n+1}} dW_s = b\left(t_n, X_n\right) \Delta W_n$$

$$X_{n+1} = X_n + a\left(t_n, X_n\right) \delta t + b\left(t_n, X_n\right) \Delta W_n$$

where $\boxed{\Delta W_n = \left(W_{n+1} - W_n\right).}$ $\quad W_{t_{n+1}} - W_{t_n}$

$\delta t = t_{n+1} - t_n$

$\Delta W_n \sim \phi \quad \Delta t$

δW ??

The Forward **Euler-Maruyama** method for GBM gives

$$\frac{\delta S_t}{S_t} = \frac{S_{t+\delta t} - S_t}{S_t} \sim r\delta t + \sigma\phi\sqrt{\delta t}$$

i.e

$$S_{t+\delta t} \sim S_t \left(1 + r\delta t + \sigma\phi\sqrt{\delta t}\right).$$

Now do a Taylor series expansion of the exact solution, i.e.

$$e^{\left(r-\frac{1}{2}\sigma^2\right)\delta t+\sigma\phi\sqrt{\delta t}} \sim 1 + \left(r - \frac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t} + \frac{1}{2}\sigma^2\phi^2\delta t$$

so we have

E-M

$$S_{t+\delta t} \sim S_t \left(\boxed{1 + r\delta t + \sigma\phi\sqrt{\delta t}} + \boxed{\frac{1}{2}\sigma^2\left(\phi^2 - 1\right)\delta t} + ..\right)$$

Milstein correction

which differs from the Euler method at $O\left(\delta t\right)$ by the term $\frac{1}{2}\sigma^2\left(\phi^2 - 1\right)\delta t$. The term

$$\frac{1}{2}\left(\phi^2 - 1\right)\delta t,$$

$\delta t < \sqrt{\delta t}$ when δt small

is called the *Milstein correction*.

The same ϕ has to be used in both calculations

Milstein Integration

We approximate the solution of the SDE

$$dG_t = A\left(G_t, t\right) dt + B\left(G_t, t\right) dW_t$$

which is compact form for

$$G_{t+\delta t} = G_t + \int_t^{t+\delta t} A\left(G_s, s\right) ds + \int_t^{t+\delta t} B\left(G_s, s\right) dW_s,$$

by

$$G_{t+\delta t} \sim G_t + A\left(G_t, t\right) \delta t + B\left(G_t, t\right) \sqrt{\delta t}\phi + B\left(G_t, t\right) \frac{\partial}{\partial G_t} B\left(G_t, t\right) \cdot \frac{1}{2}\left(\phi^2 - 1\right) \delta t$$

Note: We use the same value of the random number $\phi \sim N\left(0, 1\right)$ in both of the expressions

$$B\left(G_t, t\right) \sqrt{\delta t}\phi$$

and

$$B\left(G_t, t\right) \frac{\partial}{\partial G_t} B\left(G_t, t\right) \cdot \frac{1}{2}\left(\phi^2 - 1\right) \delta t.$$

The error of the Milstein scheme is $O(\delta t)$ which makes it better than the Euler-Maruyama method which is $O\left(\delta t^{1/2}\right)$. The Milstein makes use of Itô's lemma to increase the accuracy of the approximation by adding the second order term.

Some texts express the scheme in difference form. So a SDE written

$$dY_t = A(Y_t, t)\,dt + B(Y_t, t)\,dW_t$$

can be discretized as

$$Y_{i+1} = \underbrace{Y_i + A\Delta t + B\Delta W_t}_{\text{EM}} + \overbrace{\tfrac{1}{2}B\frac{\partial B}{\partial Y_i}\left((\Delta W_t)^2 - \Delta t\right)}^{\text{Milstein}}$$

Applying Milstein to the earlier example of GBM

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where

$$\begin{aligned} A(S_t, t) &= rS_t \\ B(S_t, t) &= \sigma S_t \end{aligned}$$

gives

$$
\begin{aligned}
S_{t+\delta t} &\sim S_t + rS_t\delta t + \sigma S_t\sqrt{\delta t}\phi + \sigma S_t\frac{\partial}{\partial S_t}\sigma S_t \cdot \tfrac{1}{2}\sigma^2\left(\phi^2 - 1\right)\delta t \\
&= S_t\left(1 + r\delta t + \sigma\phi\sqrt{\delta t} + \tfrac{1}{2}\sigma^2\left(\phi^2 - 1\right)\delta t\right)
\end{aligned}
$$

As another example, the CIR model for the spot rate is

$$
dr_t = \left(\eta - \gamma r_t\right)dt + \sqrt{\alpha r_t}\,dW_t.
$$

So identifying

$$
\begin{aligned}
A\left(r_t, t\right) &= \eta - \gamma r_t \\
B\left(r_t, t\right) &= \sqrt{\alpha r_t}
\end{aligned}
$$

and substituting into the Milstein scheme gives

$$
\begin{aligned}
r_{t+\delta t} &\sim r_t + \left(\eta - \gamma r_t\right)\delta t + \sqrt{\alpha r_t\delta t}\phi + \sqrt{\alpha r_t}\frac{\partial}{\partial r_t}\sqrt{\alpha / r_t} \cdot \tfrac{1}{2}\left(\phi^2 - 1\right)\delta t \\
&= r_t + \left(\eta - \gamma r_t\right)\delta t + \sqrt{\alpha r_t\delta t}\phi + \tfrac{1}{4}\alpha\left(\phi^2 - 1\right)\delta t.
\end{aligned}
$$

$X \quad p(x) \quad \mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)p(x)\,dx$

If $X \sim U(0,1)$ $\quad \int_{-\infty}^{0} 0 + \int_{0}^{1} 1 + \int_{1}^{\infty} 0$

Monte-Carlo methods are centred on evaluating definite integrals as expectations (or averages). Before studying this in greater detail, we consider the simple problem of estimating expectations of functions of uniformly distributed random numbers. $f(U) = e^{U^2}$

Motivating Example Estimate $\theta = \mathbb{E}\left[e^{U^2}\right]$, where $U \sim U(0,1)$.

We note that $\mathbb{E}\left[e^{U^2}\right]$ can be expressed in integral form, i.e.

$$\mathbb{E}\left[e^{U^2}\right] = \int_0^1 e^{x^2} \underbrace{p(x)}_{1}\, dx$$

where $p(x)$ is the density function of a $U(0,1)$

$$p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

hence

solve numerically

$$\mathbb{E}\left[e^{U^2}\right] = \int_0^1 e^{x^2}\, dx.$$

integral problem $\nexists$ closed form soln

This integral does not have an analytical solution. The theme of this section is to consider solving numerically, using simulations. We use the Monte Carlo simulation procedures:

$e^{U_1^2}$ $e^{U_2^2}$ $e^{U_n^2}$

1. Generate a sequence $U_1, U_2, ..., U_n \sim U(0,1)$ where U_i are i.i.d (independent and identically distributed)

$U_i \sim U(0,1)$

2. Compute $Y_i = e^{U_i^2}$ $(i = 1, ..., n)$

$\mathbb{E}[e^{U^2}]$ $\int_0^1 e^{x^2} dx$

$p = 1$

3. Estimate θ by

$$
\begin{aligned}
\widehat{\theta}_n &\equiv \frac{1}{n}\sum_{i=1}^{n} Y_i \\
&= \frac{1}{n}\sum_{i=1}^{n} e^{U_i^2} = \mathbb{E}[e^{U^2}]
\end{aligned}
$$

i.e. use the sample mean of the $e^{U_i^2}$ terms.

$\int_0^1 f(x)\,dx = \mathbb{E}[f(U)]$

Simpson Trapezoidal quadrature

Monte Carlo Integration

When a closed form solution for evaluating an integral is not available, numerical techniques are used. The purpose of Monte Carlo schemes is to use simulation methods to approximate integrals in the form of expectations.

$$\mathbb{E}\left[|f(x)|\right] < \infty$$

Suppose $f(\cdot)$ is some function such that $f : [0,1] \to \mathbb{R}$. The basic problem is to evaluate the integral

$$I = \int_0^1 f(x)\, dx$$

previous motivating example

Consider e.g. the earlier problem $\underbrace{f(x) = e^{x^2}}$, for which an analytical solution cannotbe obtained.

$$\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2}$$

Note that if $U \sim U(0,1)$ then

$$\mathbb{E}[f(U)] = \underbrace{\int_0^1 f(u)\, \overbrace{p(u)}^{=1}\, du}$$

where the density $p(u)$ of a uniformly distributed random variable $U(0,1)$ is given earlier. Hence

$$\begin{aligned} \mathbb{E}[f(U)] &= \int_0^1 f(u)\, p(u)\, du \\ &= I. \end{aligned}$$

So the problem of estimating I becomes equivalent to the exercise of estimating $\mathbb{E}[f(U)]$ where $U \sim U(0,1)$.

Very often we will be concerned with an arbitrary domain, other than $[0,1]$. This simply means that the initial part of the problem will involve seeking a transformation that converts $[a,b]$ to the domain $[0,1]$. We consider two fundamental cases.

1. Let $f(\cdot)$ be a function s.t. $f : [a,b] \to \mathbb{R}$ where $-\infty < a < b < \infty$. The problem is to evaluate the integral

General problem

$$I = \int_{a \to 0}^{b \to 1} f(x)\, dx.$$

Can we convert I to $\int_0^1 f(u)\, du$, $u \sim U(0,1)$?

$x = (b-a)y + a$ $\quad f(x) \longrightarrow f((b-a)y+a)$

$dx = (b-a)dy$

In this case consider the following substitution

Special subst^{n} $\longrightarrow$
$$y = \frac{x-a}{b-a}$$
$x = a \quad y = 0$

$x = b \quad y = 1$

which gives $dy = dx/(b-a)$. This gives

$$\begin{aligned} I &= (b-a)\int_0^1 f(\overbrace{y \times (b-a) + a}^{x}) \overset{\times 1}{dy} \\ &= (b-a)\underbrace{\mathbb{E}[f(U \times (b-a) + a)]}_{} \end{aligned}$$

$y \to U \sim U(0,1)$

simulations

where $U \sim U(0,1)$. Hence I has been expressed as the product of a constant and expected value of a function of a $U(0,1)$ random number; the latter can be estimated by simulation.

Improper integral

2. Let $g(\cdot)$ be some function s.t. $g : [0,\infty) \to \mathbb{R}$ where $-\infty < a < b < \infty$. The problem is to evaluate the integral

$$I = \int_0^\infty g(x)\,dx, \longrightarrow \int_0^1$$

$$\int_0^\infty g(x) \longrightarrow \int_0^1 f(y)\,dy$$

provided $I < \infty$. So this is the area under the curve $g(x)$ between 0 and ∞. In this case use the following substitution

$\iiint f(x,y,t)\,dx\,dy\,dt$ transf^n

$x = 0 \quad y = 1$

$x = \infty \quad y = 0$

$$y = \frac{1}{1+x}$$

which is equivalent to $x = -1 + \frac{1}{y}$. This gives

$U_i \sim U(0,1)$

$Y_i = U_i^3$

Take average.

$$\begin{aligned} dy &= -dx/(1+x)^2 \\ &= -y^2 dx. \end{aligned}$$

$-\frac{dy}{y^2} = dx$

The resulting problem is

$\int_0^1 x^3\,dx = 0.25$

$$\begin{aligned} I &= \int_0^1 \frac{g\left(\overbrace{\frac{1}{y}-1}^{x}\right)}{y^2} dy \\ &= \mathbb{E}\left[\frac{g\left(-1+\frac{1}{U}\right)}{U^2}\right] \end{aligned}$$

expectation

where $U \sim U(0,1)$. Hence I has again been expressed as the expected value of a function of a $U(0,1)$ random number; to be estimated by simulation.

draw $X_i \sim N(0,1)$

$Y_i = X_i^4$ Take average

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-\frac{1}{2}x^2}\,dx = \mathbb{E}[X^4]$$

$= 3$