Black-Scholes Model - Solutions

Throughout this exercise you may use assume (where appropriate) the following results without proof

$$
\begin{aligned}
d_1 &= \frac{\log(S/E) + \left(r - D + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\log(S/E) + \left(r - D - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\
N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\phi^2/2} d\phi
\end{aligned}
$$

where $S \geq 0$ is the spot price, $t \leq T$ is the time, $E > 0$ is the strike, $T > 0$ the expiry date, $r \geq 0$ the interest rate, D is the dividend yield and σ is the volatility of S.

1. The Black–Scholes formula for a European call option $C(S,t)$ is given by

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2).$$

a) By differentiating with respect to S and σ show that the delta and vega are given by

$$\Delta = e^{(-D(T-t))} N(d_1), \text{ and } \upsilon = \sqrt{\frac{T-t}{2\pi}} S e^{(-D(T-t))} e^{\left(\frac{-d_1{}^2}{2}\right)}.$$

Note:
$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} \quad \text{and} \quad \frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T-t}$$

So

$$
\begin{aligned}
\Delta &= \frac{\partial C}{\partial S} \\
&= e^{(-D(T-t))} N(d_1) + S e^{(-D(T-t))} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_1{}^2}{2}\right)} \frac{\partial d_1}{\partial S} - E\exp(-r(T-t)) \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{d_2{}^2}{2}\right)} \frac{\partial d_2}{\partial S} \\
&= e^{(-D(T-t))} N(d_1) + \frac{1}{\sqrt{2\pi}} \frac{\partial d_1}{\partial S} \underbrace{\left(S e^{(-D(T-t))} e^{\left(-\frac{d_1{}^2}{2}\right)} - E e^{(-r(T-t))} e^{\left(-\frac{d_2{}^2}{2}\right)} \right)}_{=0} \\
&= e^{(-D(T-t))} N(d_1) \quad \text{because the term in the bracket above is zero.}
\end{aligned}
$$

$$
\begin{aligned}
\upsilon &= \frac{\partial C}{\partial \sigma} \\
&= Se^{(-D(T-t))}\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{d_1{}^2}{2}\right)}\frac{\partial d_1}{\partial \sigma} - Ee^{(-r(T-t))}\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{d_2{}^2}{2}\right)}\frac{\partial d_2}{\partial \sigma} \\
&= Se^{(-D(T-t))}\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{d_1{}^2}{2}\right)}\left(\frac{\partial d_2}{\partial \sigma} + \sqrt{T-t}\right) - \frac{1}{\sqrt{2\pi}}Ee^{(-r(T-t))}e^{\left(-\frac{d_2{}^2}{2}\right)}\frac{\partial d_2}{\partial \sigma} \\
&= \sqrt{\frac{T-t}{2\pi}}Se^{(-D(T-t))}e^{\left(-\frac{d_1{}^2}{2}\right)} + \frac{\partial d_2}{\partial \sigma}\frac{1}{\sqrt{2\pi}}\left[\underbrace{Se^{(-D(T-t))}e^{\left(-\frac{d_1{}^2}{2}\right)} - Ee^{(-r(T-t))}e^{\left(-\frac{d_2{}^2}{2}\right)}}_{=0}\right] \\
&= \sqrt{\frac{T-t}{2\pi}}Se^{(-D(T-t))}e^{\left(-\frac{d_1{}^2}{2}\right)} \quad \left(= \sqrt{\frac{T-t}{2\pi}}Ee^{(-r(T-t))}e^{\left(-\frac{d_2{}^2}{2}\right)}\right)
\end{aligned}
$$

2. Given that S is defined by the SDE

$$dS = a\,(S,t)\,dt + b\,(S,t)\,dW \tag{2.1}$$

where a and b are given functions of S and t, show **using** Itô's lemma that any function $V\,(S,t)$ satisfies the SDE

$$dV = \left(\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial S} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2}\right)dt + b\frac{\partial V}{\partial S}dW$$

where we have assumed that all partial derivatives exist.
Hence derive the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right) \tag{2.2}$$

for the fair price of an option based on a security S which satisfies (2.1) with r the risk-free interest rate.
Show (by substitution) that $V\,(S,t) = e^{-\alpha t}S^2$ is a solution of (2.2) provided

$$b^2 = (\alpha - r)\,S^2$$

and α is a constant.
The first part of this problem is trivial. Follow the derivation of the BSE as done in the notes. The only difference here is that $a\,(S,t)$ and $b\,(S,t)$ replace μS and σS. For the second part simply substitute $V\,(S,t) = e^{-\alpha t}S^2$ in (2.2); the following terms are needed

$$\frac{\partial V}{\partial t} = -\alpha e^{-\alpha t}S^2; \ \frac{\partial V}{\partial S} = 2e^{-\alpha t}S; \ \frac{\partial^2 V}{\partial S^2} = 2e^{-\alpha t}$$

$$
\begin{aligned}
-\alpha e^{-\alpha t}S^2 + \frac{1}{2}b^2 \times 2e^{-\alpha t} &= r\left(e^{-\alpha t}S^2 - 2e^{-\alpha t}S^2\right) \\
-\alpha S^2 + b^2 &= -rS^2 \rightarrow b^2 = (\alpha - r)\,S^2.
\end{aligned}
$$

3. The Black–Scholes formula for a European call option $C(S,t)$ is

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2)$$

From this expression, find the Black–Scholes value of the call option in the following limits:

a. (time tends to expiry) $t \to T^-$, $\sigma > 0$ (*this depends on* S/E); $\exp(-r(T-t))$, $\exp(-D(T-t)) \to 1$

We know

$$\begin{aligned}
d_{12} &= \frac{\log(S/E) + \left(r - D \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \frac{\left(r - D \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \left(r - D \pm \frac{1}{2}\sigma^2\right)\frac{(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \frac{\left(r - D \pm \frac{1}{2}\sigma^2\right)}{\sigma}\sqrt{T-t} = \frac{\log(S/E)}{\sigma\sqrt{T-t}} + O\left(\sqrt{T-t}\right)
\end{aligned}$$

$$d_{12} \to \frac{\log(S/E)}{\sigma\sqrt{T-t}} + O\left(\sqrt{T-t}\right) \to \begin{cases} \infty & S > E \\ 0 & S = E \\ -\infty & S < E \end{cases} \quad \text{so} \quad C \to \begin{cases} S - E & S > E \\ 0 & S = E \\ 0 & S < E \end{cases}$$

b. (volatility tends to zero) $\sigma \to 0^+$, $t < T$; (*this depends on* $S\exp(-D(T-t))/E\exp(-r(T-t))$)
Again start with what we know, i.e.

$$\begin{aligned}
d_{12} &= \frac{\log(S/E) + \left(r - D \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \frac{(r-D)(T-t)}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\frac{\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \frac{(r-D)(T-t)}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\sigma\sqrt{T-t}
\end{aligned}$$

$$d_{12} \to \frac{\log(S/E) + (r-D)(T-t)}{\sigma\sqrt{T-t}} + O(\sigma)$$

Now we employ a small trick in the first quotient

$$\begin{aligned}
&= \frac{\log(S\exp(-D(T-t))/E\exp(-r(T-t)))}{\sigma\sqrt{T-t}} + O(\sigma) \\
&\to \begin{cases} \infty & Se^{(-D(T-t))} > Ee^{(-r(T-t))} \\ 0 & Se^{(-D(T-t))} = Ee^{(-r(T-t))} \\ -\infty & Se^{(-D(T-t))} < Ee^{(-r(T-t))} \end{cases} \\
\text{so } C &\to \max\left[Se^{(-D(T-t))} - Ee^{(-r(T-t))}, 0\right]
\end{aligned}$$

4. The value of an option $V\,(S,t)$ satisfies the Black–Scholes equation. Write the option value in the form

$$V(S,t) = \exp(-r(T-t))q(S,t). \tag{4.1}$$

Show that the function $q(S,t)$ satisfies the equation

$$\frac{\partial q}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 q}{\partial S^2} + (r-D)S\frac{\partial q}{\partial S} = 0.$$

This is the backward Kolmogorov equation, used for calculating the expected value of stochastic quantities.
Substitute

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \exp(-r(T-t))\frac{\partial}{\partial t}q(S,t) + rV(S,t), \\
\frac{\partial V}{\partial S} &= \exp(-r(T-t))\frac{\partial q}{\partial S} \quad \& \\
\frac{\partial^2 V}{\partial S^2} &= \exp(-r(T-t))\frac{\partial^2 q}{\partial S^2}
\end{aligned}$$

from (4.1) into the BSE, all the exponentials cancel out and the above equation is left.
Thus the value of an option can be expressed in the form

$$V\,(S,t) = \exp\left(-r(T-t)\right)\mathbb{E}\left[\text{Payoff}\,(S)\right]$$

This is not a real expectation, but taken under the risk-neutral random walk (so r replaces μ) and forms the basis of Monte Carlo methods applied to finance. More on this later.