

# CQF Binomial Model Solutions

1. Binomial tree for share price  $S$  and for option price  $V$  is

$$\begin{array}{ccc} & 84 & 5 \text{ (= max(84 - 79, 0))} \\ 80 & & V \\ & 76 & 0 \text{ (= max(76 - 79, 0))} \end{array}$$

Now set up a Black-Scholes hedged portfolio,  $V - \Delta S$ , then binomial tree for its value is

$$\begin{array}{ccc} & 5 - 84\Delta & \\ V - 80\Delta & & \\ & -76\Delta & \end{array}$$

For risk-free portfolio choose  $\Delta$  such that  $5 - 84\Delta = -76\Delta \Rightarrow \Delta = \frac{5}{8}$ . So in absence of arbitrage,  $V - 80\Delta = -76\Delta$ , and  $V = 2.5$ .

2. Binomial tree for share price and option respectively is

$$\begin{array}{ccc} & 98 & 8 \text{ (= max(98 - 90, 0))} \\ 92 & & V \\ & 86 & 0 \text{ (= max(86 - 90, 0))} \end{array}$$

Now set up a Black-Scholes hedged portfolio,  $V - \Delta S$ , then binomial tree for its value is

$$\begin{array}{ccc} & 8 - 98\Delta & \\ V - 92\Delta & & \\ & -86\Delta & \end{array}$$

For risk-free portfolio choose  $\Delta$  such that  $8 - 98\Delta = -86\Delta \Rightarrow \Delta = \frac{2}{3}$ . So in absence of arbitrage, since portfolio is riskless, it must earn risk-free rate  $r = 2\%$  and  $V - 92\Delta = \exp(-0.02)(-86\Delta)$ , then  $V = \frac{2}{3}(92 - 86 \exp(-0.02)) = 5.14$ .

3. Binomial tree for share price and for option is

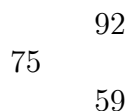
$$\begin{array}{ccc} & 17 & 130 \text{ (= max(17^2 - 159, 0))} \\ 15 & & V \\ & 13 & 10 \text{ (= max(13^2 - 159, 0))} \end{array}$$

Now set up a Black-Scholes hedged portfolio,  $V - \Delta S$ , then binomial tree for its value is

$$\begin{array}{ccc} & 130 - 17\Delta & \\ V - 15\Delta & & \\ & 10 - 13\Delta & \end{array}$$

For risk-free portfolio choose  $\Delta$  such that  $130 - 17\Delta = 10 - 13\Delta \Rightarrow \Delta = 30$ . So in absence of arbitrage,  $V - 15\Delta = 10 - 13\Delta$ , and  $V = 70$ .

4. The Binomial tree for share price is



Time is 3 months - i.e.  $\frac{1}{4}$  year, interest rate  $r = 0$ , so risk neutral probability that share price increases satisfies

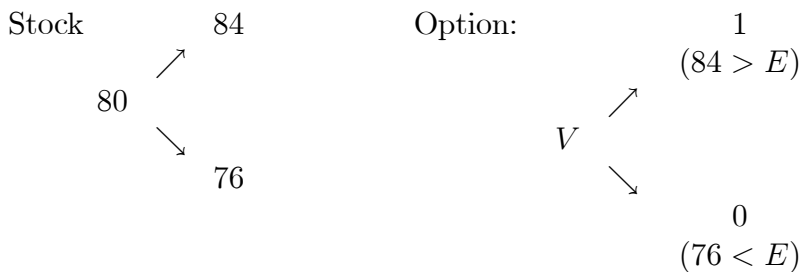
$$92q + 59(1 - q) = 75$$

Re-arranging gives

$$q = \frac{75 - 59}{33} = 0.485$$

So probability of a fall would be given by  $1 - q = 0.515$ .

5.



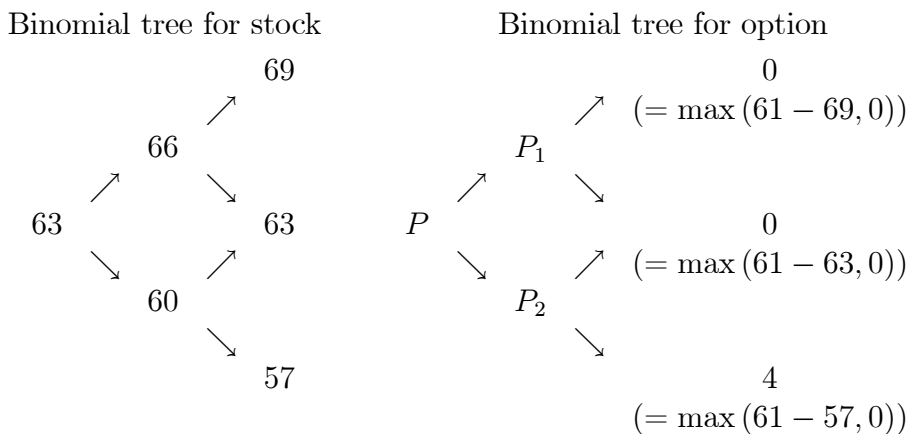
Now set up a Black-Scholes hedged portfolio,  $V - \Delta S$ , then binomial tree for its value is

$$\begin{array}{c} 1 - 84\Delta \\ V - 80\Delta \\ -76\Delta \end{array}$$

For risk-free portfolio choose  $\Delta$  such that  $1 - 84\Delta = -76\Delta \Rightarrow \Delta = 1/8$ . So in absence of arbitrage,  $V - 80\Delta = 1 - 84\Delta$ , and  $V = 0.5$ . Note how the option price is significantly cheaper than the European style security in

6. See spreadsheet

7.



We must find the values of  $P_1$  and  $P_2$  before we can solve for  $P$ .

If we set up a Black-Scholes delta hedged portfolio  $P_1 - \Delta_1 S$ , for  $P_1$ , then the binomial tree for its

value is

$$\begin{array}{c}
 \nearrow -69\Delta_1 \\
 P_1 - 66\Delta_1 \\
 \searrow -63\Delta_1
 \end{array}$$

For a risk-free portfolio, we choose  $\Delta_1$  s.t.  $-69\Delta_1 = -63\Delta_1$ , i.e.  $\Delta_1 = 0$ . Then in the absence of arbitrage it must earn at the risk-free interest-rate and

$$P_1 - 66\Delta_1 = e^{-0.04 \times (3/12)} (-69\Delta_1) \rightarrow P_1 = 0.$$

If we set up a Black-Scholes delta hedged portfolio  $P_2 - \Delta_2 S$ , for  $P_2$ , then the binomial tree for its value is

$$\begin{array}{c}
 \nearrow -63\Delta_2 \\
 P_2 - 60\Delta_2 \\
 \searrow 4 - 57\Delta_2
 \end{array}$$

For a risk-free portfolio, we choose  $\Delta_2$  s.t.  $-63\Delta_2 = 4 - 57\Delta_2$ , i.e.  $\Delta_2 = -2/3$ . Then in the absence of arbitrage it must earn at the risk-free interest-rate and

$$P_2 - 60\Delta_2 = e^{-0.04 \times (3/12)} (4 - 57\Delta_2) \rightarrow P_2 = 1.58.$$

We can now set up a Black-Scholes hedged portfolio,  $P - \Delta S$ , for  $P$ . The binomial tree for its value is

$$\begin{array}{c}
 \nearrow P_1 - 66\Delta \\
 P - 63\Delta \\
 \searrow P_2 - 60\Delta
 \end{array}$$

For a risk-free portfolio, we choose  $\Delta$  s.t.

$$P_1 - 66\Delta = P_2 - 60\Delta \rightarrow \Delta = \frac{P_1 - P_2}{6} = \frac{-P_2}{6} = -0.263.$$

Then in an arbitrage free market, the portfolio earns at the risk-free rate

$$\begin{aligned}
 P - 63\Delta &= e^{-0.04 \times (3/12)} (-66\Delta) \\
 P &= \Delta (63 - 65.34) = -0.263 \times -2.34 \\
 &= 0.6154
 \end{aligned}$$

Hence the value of the put option is £0.62.

8. To find  $V_1$  from portfolio:  $\Pi = V - \Delta S$ . Then from  $T_1$  to  $T$  we have

$$\Pi \longrightarrow \begin{cases} 15 - \Delta(\alpha + 20) \\ 0 - \Delta\alpha \end{cases}$$

so for risk free portfolio  $\Rightarrow$  choose  $15 - \Delta(\alpha + 20) = -\Delta\alpha \longrightarrow \Delta = 3/4$ . For no arbitrage we want

$$V_1 - \Delta(\alpha + 10) = -\Delta\alpha$$

since  $r = 0$ . Solving gives  $V_1 = 7.5$

For  $V_{-1}$  :

$$\Pi \longrightarrow \begin{cases} 0 - \Delta\alpha \\ 0 - \Delta(\alpha - 20) \end{cases} \Rightarrow \Delta = 0$$

Therefore  $V_{-1} = 0$ .

For  $V$  :

$$V - \Delta\alpha = \begin{cases} V_1 - \Delta(\alpha + 10) & \equiv \frac{15}{2} - \Delta(\alpha + 10) \\ V_{-1} - \Delta(\alpha - 10) & \equiv 0 - \Delta(\alpha - 10) \end{cases}$$

so  $\frac{15}{2} - 20\Delta = 0 \longrightarrow \Delta = 3/8$ .

Finally  $V - \Delta\alpha = -(\alpha - 10) \longrightarrow V = 10\Delta = 15/4$ . Hence

$$V = 15/4 \begin{matrix} & & 15 \\ & 15/2 & \\ & & 0 \\ & 0 & \\ & & 0 \end{matrix}$$