Exotics - exercises

1. Consider an option which pays a continuous cash-flow to the holder at a rate proportional to the square of the underlying asset's price, so that during a time interval dt the holder receives $S^2 dt$. Suppose that at expiry the value of the option is
$$V(S,T) = S^2.$$
The underlying evolution follows geometric Brownian motion
$$dS = \mu S dt + \sigma S dX.$$
Derive the Black-Scholes partial differential equation for this "power" option and show that it is
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = -S^2.$$
By assuming a solution of the form
$$V(S,t) = \phi(t) S^2$$
show that
$$\phi(t) = \frac{1}{\sigma^2 + r}\left(\left(\sigma^2 + r + 1\right) e^{\left(\sigma^2+r\right)(T-t)} - 1\right).$$

2. Consider separable solutions of the Black–Scholes equation
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S\frac{\partial V}{\partial S} - rV = 0, \tag{2.1}$$
of the form
$$V(S,t) = f(S)g(t),$$
Show that (2.1) can be expressed as the following first order differential equation (2.2a) and Cauchy-Euler equation (2.2b)
$$\frac{dg}{dt} - \lambda g = 0 \tag{2.2a}$$
$$\tfrac{1}{2}\sigma^2 S^2 f'' + (r - D) S f' + (\lambda - r) f = 0, \tag{2.2b}$$
for some (universal) constant λ, where the following notation is used
$$f' = \frac{df}{dS},\ f'' = \frac{d^2 f}{dS^2}.$$

You may assume that (2.2b) has a solution of the form $f(S) = S^\alpha$. Solve these to obtain the following solutions for (2.1) :

   i for distinct roots of the A.E (2.2b) (A, B - constants)
$$V(S,t) = e^{\lambda t} S^{\frac{1}{2} - \frac{r-D}{\sigma^2}} \left(AS^{\alpha_+} + BS^{\alpha_-}\right)$$

   ii for a repeated root of the A.E (2.2b) (ε, ζ - constants)
$$V(S,t) = e^{\left(\left(r + \frac{\sigma^2}{2}\left(\frac{r-D}{\sigma^2} + \frac{1}{2}\right)^2\right)t\right)} S^{\left(\frac{1}{2} - \frac{r-D}{\sigma^2}\right)} (\varepsilon + \zeta \log S)$$
   where
$$\bar{d}_\pm = \pm\sqrt{\left(\frac{r-D}{\sigma^2} - \frac{1}{2}\right)^2 - \frac{2(\lambda - r)}{\sigma^2}}.$$

3. Assume that an asset price S evolves according to the SDE

$$\frac{dS}{S} = (\mu - D)\, dt + \sigma dX,$$

where μ and σ are constants. In particular S pays out a continuous dividend stream equal to $DS\ dt$ during the infinitesimal time interval dt, where D the dividend yield is constant.
Now suppose a European style derivative security is written on this asset with the properties that at expiry the holder receives the asset and prior to expiry the derivative pays a continuous cash flow $C(S,t)\, dt$ during each time interval of length dt.
Show that the option price satisfies the following partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)\, S \frac{\partial V}{\partial S} - rV = -C(S,t)\,.$$

Suppose that the cash flow $C(S,t)$ has the form $C(S,t) = f(t)\, S$. By writing $V = \phi(t)\, S$ and assuming a final condition at time T given by

$$V(S,T) = S,$$

show that the delta of the derivative security is

$$\Delta(S,t) = e^{-D(T-t)} + \int_t^T e^{-D(\tau-t)} f(\tau)\, d\tau.$$

4. An asset S follows a Geometric Brownian Motion $dS = \mu S dt + \sigma S dW$, where μ and σ are constants. We wish to value an option that pays off at expiry T an amount which is a function of the path taken by the asset between time zero and expiry. Assuming that an option value V depends on S, t and a quantity

$$I(t) = \int_0^t f(S,\tau)\, d\tau,$$

where f is a specified function and r the risk free interest rate, the option pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t)\, \frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0,$$

for the function $V(S,I,t)$.
For an arithmetic strike Asian call option the payoff at time T is

$$\max\left(S - \frac{1}{T}\int_0^T S(t)\, dt, 0\right).$$

By writing the value of this option as

$$V(S,I,t) = SW(R,t)\,,$$

where $R = I/S$, show that the partial differential equation for $W(R,t)$ is given by

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + (1 - rR)\, \frac{\partial W}{\partial R} = 0.$$