15.2 The Merton model

In 1974, R. Merton introduced in his seminal paper [Merton, 1974] an approach to default modeling based on option pricing theory. The basic idea of this approach is the key to understanding many credit portfolio models used in practice.

Our goal is to model the default of a company that holds a loan with maturity T and nominal K. We denote by A_t the value of this company (asset value) at time t. Of course this quantity is usually not explicitly observable in reality, but implicitly information on the value of a company can be gained from its balance sheet. At time T the lender gets his loan completely repaid if and only if $A_T \geq K$. More precisely at time T the firm value is distributed according to

firm value	equity	loan	
$A_T \geq K$	$A_T - K$	K	no default
$A_T < K$	0	A_T	default

In this model the default event D is just $D = \{A_T < K\}$ and the default probability for the time horizon T is

$$p = \mathbf{P}(D) = \mathbf{P}(A_T < K).$$

At time T the value S_T of the equity (stock) is

$$S_T = \max(A_T - K, 0),$$

which is just a call option on the firm value with strike K. The value B_T of the loan at time T is correspondingly

$$B_T = \min(K, A_T) = K - \max(K - A_T, 0),$$

i.e., the lender has implicitly given a risk free loan and additionally shorted a put option on the firm value.

If we now assume a Black & Scholes type model for the value of the firm (!!), i.e., the value of the firm follows a lognormal distribution,

$$A_T = A_0 \exp(\sigma_A W_T + rT - \sigma_A^2 T/2), \tag{24}$$

with a Wiener process W, the asset volatility σ_A and the risk free interest rate r. Then everything can be calculated explicitly:

$$\begin{aligned}
S_0 &= A_0 \mathrm{N}(d_1) - K \exp(-rT) \mathrm{N}(d_2) \\
p &= 1 - \mathrm{N}(d_2) \\
B_0 &= K \exp(-rT) - (K \exp(-rT) \mathrm{N}(-d_2) - A_0 \mathrm{N}(-d_1)) \\
&= A_0 \mathrm{N}(-d1) + K \exp(-rT) \mathrm{N}(d_2)
\end{aligned}$$

with

$$\begin{aligned}
d_1 &= \frac{\ln(A_0/K) + rT + \sigma_A^2 T/2}{\sigma_A \sqrt{T}} \\
d_2 &= \frac{\ln(A_0/K) + rT - \sigma_A^2 T/2}{\sigma_A \sqrt{T}}.
\end{aligned}$$

In the example of $A_0 = 100, K = 70, r = 5\%, \sigma_A = 20\%, T = 1$ we obtain the following values

p	S_0	$B^{\text{risky}}(0,T)$	$B^{\text{riskless}}(0,T)$
2,66%	33,54	94,94%	95,12%

For the details of the calculation see the EXCEL spreadsheet `MertonModel.xls`.

One of the striking ingredients of the formulas above is the ratio between the present value of the loan amount K and today's value of the firm A_0

$$\frac{K \exp(-rT)}{A_0}$$

which is called **Leverage**.

Exercise 15.1. *How does the Merton model implicitly also define the loss given default* LGD *or, equivalently, the recovery rate* R*?*

Computer Exercise 15.2. *For the example above and for different levels of K determine in EXCEL the implied yield spread $s(T)$ of the risky bonds relative to a risk free bond. For varying T plot the corresponding spread curves.*

Critical input to the model is today's value of the firm A_0 and its volatility σ_A which are both hard to observe directly. In case the firm is also financed by publicly traded equity one can use equity information such as the stock price S_0 and its volatility σ_S to back out the corresponding quantities for the firm value process. The Ito-formula yields the following (approximative) relationship between asset and equity volatility

$$\sigma_S S_0 \approx \sigma_A A_0 \mathrm{N}(d_1). \tag{25}$$

Together with the already mentioned formula

$$S_0 = A_0 \mathrm{N}(d_1) - K \exp(-rT) \mathrm{N}(d_2), \tag{26}$$

we have two equations involving the known quantities S_0, σ_S that can be solved for the unknowns A_0, σ_A.

Computer Exercise* 15.3. *Develop an EXCEL spreadsheet to calculate the firm value A_0 and its volatility σ_A from a given equity price S_0 and equity volatility σ_S! Use the EXCEL solver.*

15.3 The KMV model

The San Francisco based company KMV (Kealhofer, McQuown, Vasicek), which is now part of Moody's, is one of the world's leading providers of quantitative credit analysis tools to lenders, investors, and banks. The model developed by KMV was one of the first portfolio credit risk models and it is based on the Merton idea. The model by KMV delivers for example default probabilities and correlations out of observable quantities like leverage, equity price, volatility and macro economic factors (see [Crosbie and Bohn, 2002]). However, in this section we restrict ourselves to the problem of default dependence which is critical for all portfolio investigations.

15.3.1 Dependent defaults in the Merton model

We start by reformulating the Merton model (24). Consider the **asset returns**

$$\begin{aligned}\ln(A_T/A_0) &= \sigma_A W_T + rT - \sigma_A^2 T/2 \sim \mathrm{N}(rT - \sigma_A^2 T/2, \sigma_A^2 T) \\ &= \sigma_A \sqrt{T}\; Y + (r - \sigma_A^2/2)T,\end{aligned}$$

where the random variable Y is the **standardized asset return**, i.e., it possesses a standard normal distribution, $Y \sim \mathrm{N}(0,1)$. The default event D can then be written as

$$D = \{A_T < K\} = \left\{ Y < \frac{\ln(K/A_0) - rT + \sigma_A^2 T/2}{\sigma_A \sqrt{T}} \right\} = \{Y < c\}$$

with $c = -d_2$ as the **default boundary** for the standardized asset return Y and with a default probability of $p = \mathbf{P}(D) = \mathrm{N}(c)$. Given the default probability p, for example as a result of a rating analysis or logit analysis, the corresponding default boundary c is

$$c = \mathrm{N}^{(-1)}(p). \tag{27}$$

For two dependent firms (customers, counter parties) with asset values A_T^1, A_T^2 we set up a Merton model for each of them,

$$D_i = \{Y_i < c_i\}, \ \ i = 1, 2,$$

with default boundaries c_1, c_2. To model the dependencies between the defaults we assume now that Y_1, Y_2 are **correlated variables**. Their correlation

$$\rho^A = \mathrm{Corr}(Y_1, Y_2) = \mathrm{Corr}(\ln(A_T^1), \ln(A_T^2))$$

is the so-called **asset correlation**. The probability of a joint default until time T is then

$$\mathbf{P}(D_1 \cap D_2) = \mathbf{P}(Y_1 < c_1, Y_2 < c_2) = \mathrm{N}_2(c_1, c_2, \rho^A),$$

with N_2 denoting the cumulative distribution function of the 2-dimensional standard normal distribution.

If we model dependent defaults based upon the idea of two correlated Merton models, the problem reduces to the specification of the asset correlations in practice. One could try to estimate the asset correlation from a time series of historical stock prices[10] if available.

However, KMV and most other applications of the Merton approach model correlations by a **factor model** for the asset returns Y_i:

$$Y_i = \sum_{j=1}^{N} \omega_{ij} X_j + \gamma_i E_i. \tag{28}$$

The variables $X_1, \ldots, X_N$, E_i are independent standard normal distributed. The quantities X_j, $j = 1, \ldots, N$ describe systematic factors influencing the asset return, such as region, industry, economic environment etc. Finally the variable E_i is the firm-specific residual factor. The weight ω_{ij} measures the impact of the systematic factor X_j on the value of firm i. Since Y_i should be standard normal distributed we have $\gamma_i^2 = 1 - \sum_{j=1}^{N} \omega_{ij}^2$.

The asset correlation of two firms is then determined from the weights of the returns Y_1, Y_2

[10] Either using the stock price as proxy for the asset value or backing out the implied asset value considering the stock as call option on the value of the firm.

with respect to joint systematic factors:

$$\rho^A = \sum_{j=1}^{N} \omega_{1j}\omega_{2j}.$$

Designing a factor model like (28) requires an extensive and profound analysis detecting the driving systematic factors and their weights. However, the statistical techniques to be used are standard. Even in case of the KMV model the details of their factor model are not published and remain proprietary.

15.3.2 1-Factor dependence and the Basel II formula

Consider a portfolio of n loans with default probabilities $p_i = \mathbf{P}(D_i), i = 1, \ldots, n$ and one and the same asset correlation ρ between any two asset returns Y_i, Y_j. In terms of the factor models this yields

$$Y_i = \sqrt{\rho}X + \sqrt{1-\rho}E_i, i = 1, \ldots, n,$$

with a single joint systematic economic factor X. For credit i the conditional probability of default $p_i(x)$ given that the systematic factors takes the value $X = x$ is then

$$\begin{aligned} p_i(x) = \mathbf{P}(D_i | X = x) &= \mathbf{P}(Y_i < \mathrm{N}^{(-1)}(p_i) | X = x) \\ &= \mathbf{P}(\sqrt{\rho}x + \sqrt{1-\rho}E_i < \mathrm{N}^{(-1)}(p_i)) \\ &= \mathbf{P}\left(E_i < \frac{\mathrm{N}^{(-1)}(p_i) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \\ &= \mathrm{N}\left(\frac{\mathrm{N}^{(-1)}(p_i) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right). \end{aligned} \tag{29}$$

The factor variable X can be interpreted as state of the economy. Given the unconditional default probability $p_i = \mathbf{P}(D_i)$, depending on the state of the economy the actual default probability varies.

Computer Exercise 15.4. *In EXCEL plot the conditional default probabilities $p_i(x)$ as a function of the states x of the economy. Which values of x are "good" resp. "bad" states of the economy? Do the analysis for various fixed values of the unconditional probability p_i.*

Given the state of the economy $X = x$, the defaults are **conditionally independent**:

$$\mathbf{P}(D_1 \cap D_2 | X = x) = \mathbf{P}(D_1 | X = x) \cdot \mathbf{P}(D_2 | X = x).$$

In case of a homogeneous portfolio, i.e., all loans possess the same default probability $p_i = p$, the number of defaults in the portfolio given that the economy is in state $X = x$ follows a Binomial distribution:

$$\mathbf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{D_i} = k \middle| X = x\right) = \binom{n}{k} p(x)^k (1 - p(x))^{n-k}, \ k = 0, \ldots, n \tag{30}$$

with

$$p(x) = \mathrm{N}\left(\frac{\mathrm{N}^{(-1)}(p) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right).$$

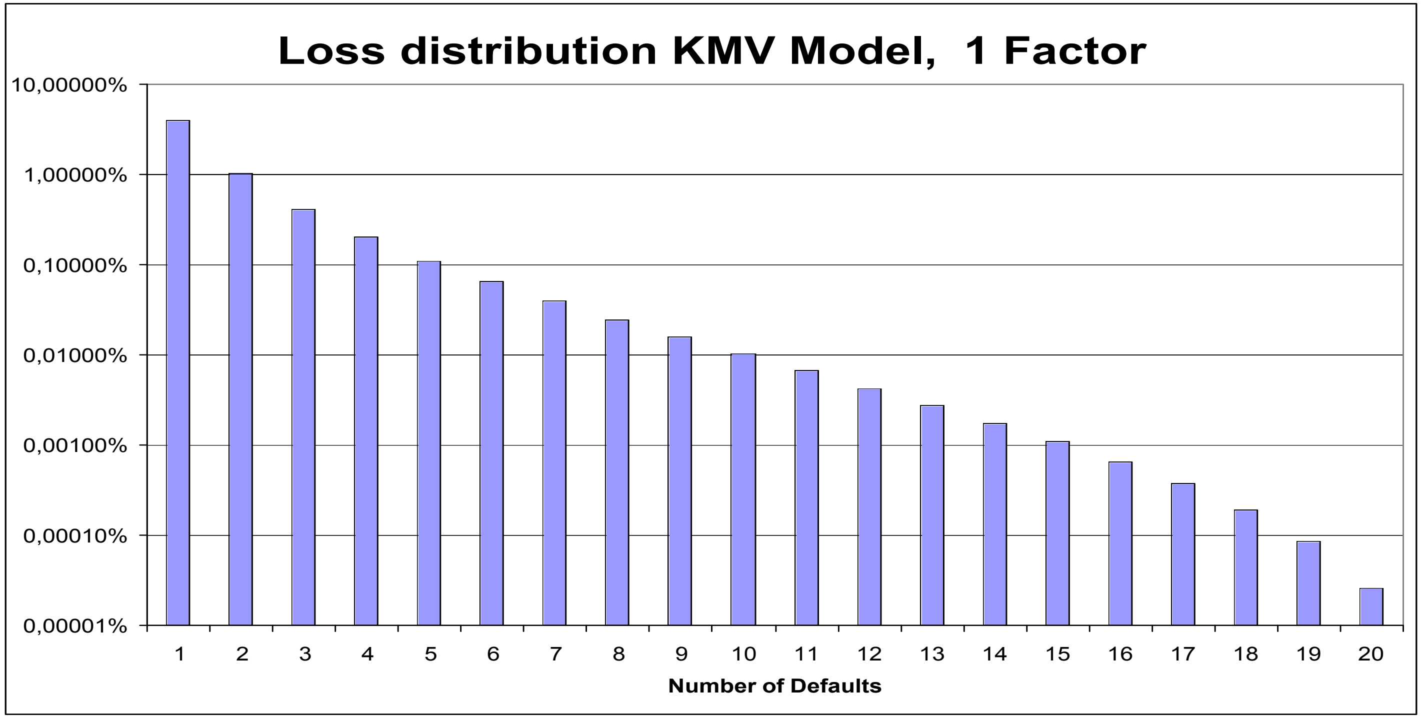
The unconditional distribution of the number of defaults is obtained by integrating over the state x of the economy,

$$\mathbf{P}\left(\sum_{i=1}^{n} \mathbf{1}_{D_i} = k\right) = \binom{n}{k} \int_{-\infty}^{\infty} p(x)^k (1-p(x))^{n-k} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \mathrm{d}x, \;\; k = 0, \ldots, n.$$

Usually this has to be calculated numerically on a computer.

Here is an example of the loss distribution[11] for $n = 20$ loans, a default probability of $p = 0.5\%$ and an asset correlation of $\rho^A = 50\%$. The probability of no default in the whole portfolio is 94.07%.

[11] Probabilities in logarithmic scale.


Loss distribution KMV Model, 1 Factor
10,00000%
1,00000%
0,10000%
0,01000%
0,00100%
0,00010%
0,00001%
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
Number of Defaults

For details of the calculation see the EXCEL spreadsheet `KMV1Factor.xls`.

Computer Exercise 15.5. *In the above example assume that each loan is of nominal amount 100. Calculate the economic capital,* **CVaR**, *for this example and a confidence of* $\alpha = 99\%$*! Assume a recovery rate of* $R = 20\%$ *for each loan.*

We are now in the position to understand the rationale behind the Basel II formula (22). The expected loss $\mathbb{E}(L|X = x)$ of the portfolio given a state $X = x$ of the economy is given by

$$\mathbb{E}(L|X = x) = \mathbb{E}\left(\sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot \mathbf{1}_{D_i} \Big| X = x\right) = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot p_i(x).$$

Now imagine an “infinitely large” portfolio ($n \to \infty$) with “infinitely small” loans ($\mathrm{EAD}_i \to 0$) possessing a 1 factor dependence structure. Such a portfolio is called an **infinitely fine-grained** portfolio.

According to the law of large numbers the sum of independent small variables converges with probability 1 to its expected value[12], therefore

$$\mathbf{P}\left(L = \mathbb{E}(L|X=x) = \sum_i \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot p_i(x) \Big| X = x\right) = 1.$$

To calculate the economic capital for the portfolio we need the quantile Q_α of the distribution of the loss L. Q_α is by definition the biggest value such that $\mathbf{P}(L \leq Q_\alpha) = \alpha$. In our situation

$$\begin{aligned}
\alpha &= \mathbf{P}(L \leq Q_\alpha) \\
&= \int_{-\infty}^{\infty} \mathbf{P}(L \leq Q_\alpha | X = x) \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \mathbf{1}_{\{\sum_i \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot p_i(x) \leq Q_\alpha\}} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \mathrm{d}x.
\end{aligned}$$

The quantity $\sum_i \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot p_i(x)$ is strictly decreasing in x and we can easily solve for the value of the quantile Q_α. Indeed, the set of values x with $x \geq N^{(-1)}(1-\alpha)$ has a probability of α and exactly for those

[12] For more detailed mathematical arguments see, e.g. [Blum et al., 2003], Section 2.5.1.

values x it holds that

$$\sum_i \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot p_i(x) \leq \sum_i \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot p_i(N^{(-1)}(1-\alpha)).$$

Consequently,

$$Q_\alpha = \sum_i \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot p_i(N^{(-1)}(1-\alpha)).$$

This quantity is, up to the adjustment for the expectation of L, just the economic capital of the portfolio. The summand

$$\mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot p_i(N^{(-1)}(1-\alpha))$$

can be interpreted as risk contribution due to the ith sub-position.

For $\alpha = 99.9\%$ taking into account that $N^{(-1)}(1-0.999) = -N^{(-1)}(0.999)$ and replacing this in formula (29) we end up with

$$\mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot p_i(N^{(-1)}(1-\alpha)) = \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot \mathrm{N}\left(\frac{\mathrm{N}^{(-1)}(p_i) + \sqrt{\rho}N^{(-1)}(0.999)}{\sqrt{1-\rho}}\right),$$

which is exactly the corresponding expression in the Basel II formula (22).