Stochastic Calculus and Itô's lemma

Throughout this problem sheet, you may assume that W_t is a Brownian Motion (Wiener Process) and dW_t is its increment; and $W_0 = 0$. SDE is Stochastic Differential Equation.

1. Let ϕ be a random variable which follows a standardised normal distribution, i.e. $\phi \sim N(0,1)$.
   Calculate the expected value and variance given by $\mathbb{E}[\psi]$ and $\mathbb{V}[\psi]$, in turn, where $\psi = \sqrt{dt}\phi$. dt is a small time-step. **Note: No integration is required.**

2. Consider the following examples of SDEs for a diffusion process G. Write these in standard form, i.e.

$$dG = A(G,t)dt + B(G,t)dW_t.$$

   Give the drift and diffusion for each case.

   a. $df + dW_t - dt + 2\mu t f dt + 2\sqrt{f} dW_t = 0$
   b. $\dfrac{dy}{y} = (A + By)\,dt + (Cy)\,dW_t$
   c. $dS = (\nu - \mu S)dt + \sigma dW_t + 4dS$

3. Use Itô's lemma to obtain a SDE for each of the following functions:

   a. $f(W_t) = (W_t)^n$
   b. $y(W_t) = \exp(W_t)$
   c. $g(W_t) = \log W_t$
   d. $h(W_t) = \sin W_t + \cos W_t$
   e. $f(W_t) = a^{W_t}$, where the constant $a > 1$

4. Using the formula below for stochastic integrals, for a function $F(W_t, t)$,

$$\int_0^t \frac{\partial F}{\partial W_\tau} dW_\tau = F(W_t, t) - F(W_0, 0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2} \frac{\partial^2 F}{\partial W_\tau^2} \right) d\tau,$$

   show that we can write

a.

$$\int_0^t W_\tau^3 dW_\tau = \frac{1}{4} W_t^4 - \frac{3}{2} \int_0^t W_\tau^2 d\tau$$

b.

$$\int_0^t \tau dW_\tau = tW_t - \int_0^t W_\tau d\tau$$

c.

$$\int_0^t (W_\tau + \tau) \, dW_\tau = \frac{1}{2} W_t^2 + tW_t - \int_0^t \left(W_\tau + \frac{1}{2} \right) d\tau$$

5. Consider the linear parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x} + bu,$$

for the function $u(x,t)$; where a and b are constants. By using a substitution of the form

$$u(x,t) = e^{\alpha x + \beta t} v(x,t),$$

and suitable choice of α and β, show that the PDE can be reduced to the heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}.$$