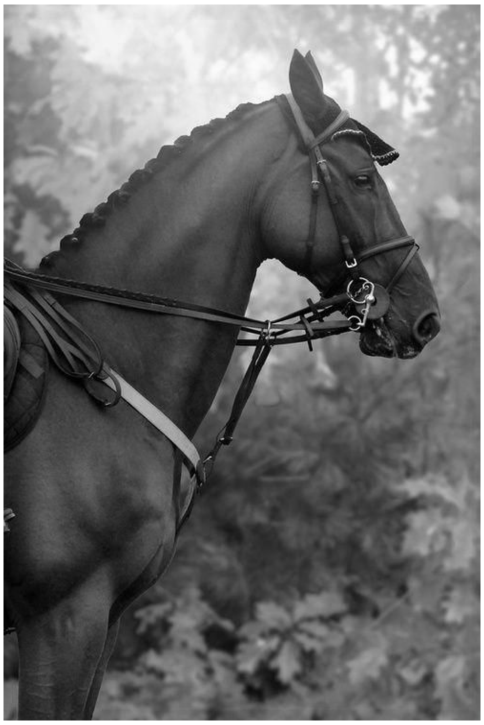
Martingales

Continuous time Martingales



In the mathematical finance literature, most articles written in the last two and half decades start with the words " Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, ...". Whether it is explicitly mentioned or not in the articles or texts, the probability space is always there somewhere in the background. The triple $(\Omega, \mathcal{F}, \mathbb{P})$, is called a *probability space.*and its inclusion in quantitative finance literature refects the increasing infuence of probability theory and probability theorists over the subject. It forms the foundation of the *probabilistic universe.*

3-tuple

Martingales are a key concept in probability and in mathematical finance. The term 'martingale' may refer to very different ideas e.g. a stochastic process that has no drift. Essentially, this is the idea of a fair (random) game.

This probability space comprises of three components

1. the sample space $\Omega = \{\omega_1, \ldots, \omega_n\}$
2. the filtration $\mathcal{F}_t =$ record of information $t \in [0, T]$

   $\mathcal{F}_n$ $n = 0, 1, 2, \ldots$
3. the probability measure $\mathbb{P}$. set theoretic extension of C.D.F

The filtration is an indication of how information about a probabilistic experiment builds up over time as more results become available. It can be thought of as an increasing family of events. Martingales are encountered through three distinct, but closely connected ideas. We focus on *Martingales* as a class of stochastic process.

condition

$$\mathbb{E}_n[S_6 \mid S_5] = S_5$$

discrete

$n = 5$

Conditional Expectations

What makes a conditional expectation different (from an unconditional one) is information (just as in the case of conditional probability). In our probability space, $(\Omega, \mathcal{F}, \mathbb{P})$ information is represented by the filtration $\mathcal{F}$; hence a conditional expectation with respect to the filtration seems a natural choice.

Y is a R.V.   Stoch. process   fil.

$$\mathbb{E}_m \left[Y_n \middle| \mathcal{F}_m \right]; \; n > m$$

$0 \quad \underbrace{\qquad\qquad}_{\mathcal{F}_m} \quad m \qquad n \qquad n > m$

is the expected value of the random variable conditional upon the filtration set $\mathcal{F}_m$.

$$\mathbb{E}_5 [S_6 | \mathcal{F}_5]$$

Adapted (Measurable) Process

cts. time

A stochastic process S_t is said to be adapted to the filtration $\mathcal{F}_t$ (or measurable with respect to $\mathcal{F}_t$, or $\mathcal{F}_t$ -adapted) if the value of S_t at time t is known given the information set $\mathcal{F}_t$.

measurable

t $\{\mathbb{N}\}$

Being adapted to an information set

10 a.m

$$\mathbb{E}_t \left[S_{t+1} \middle| \mathcal{F}_t \right]$$

this captures the information

know up to time t

Discrete Time Martingales

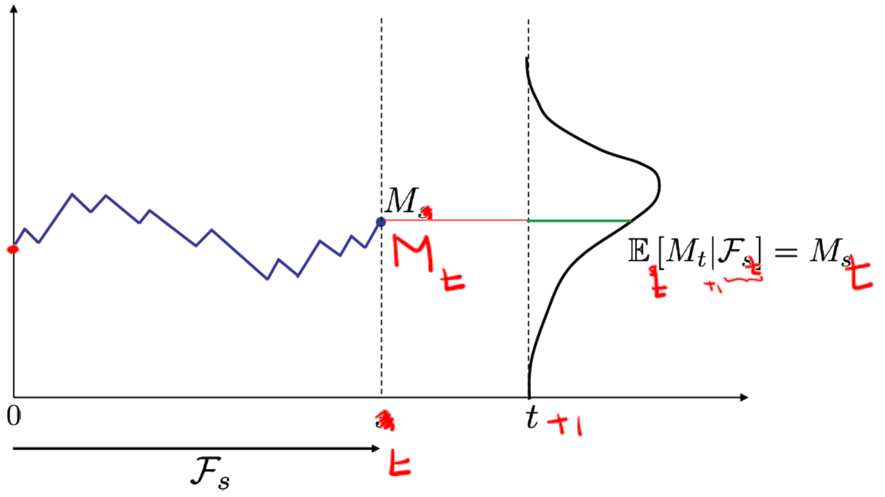
A discrete time stochastic process $\{M_t : t = 0, \ldots, T\}$ such that M_t is $\mathcal{F}_t$-measurable for $\mathbb{T} = \underbrace{\{0, \ldots, T\}}_{\text{Set of times}}$ is a **martingale** if

Integrability

$$\mathbb{E}\,|M_t| < \infty \qquad \mathbb{E}[\cdot] = \int_{-\infty}^{\infty}$$

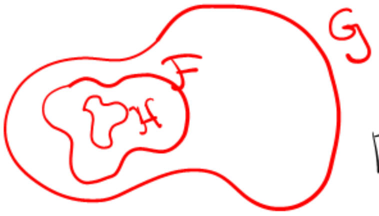
$$\mathbb{E}\,[M_{t+1}|\mathcal{F}_t] = M_t \tag{1}$$

s<t



The first equation represents a standard integrability condition.
The second equation tells you that the expected value of M at time $t+1$ conditional on all the information available up to time t is the value of M at time t. In short, a Martingale is a **driftless process**.

∃ only randomness   Constant mean process



$\mathcal{F} \subset \mathcal{G}$ $\quad$ $\mathcal{G}_t$: $\underline{\beta(t)\, dX_t}$

more

$\mathbb{E}\left[\,[\,[Y|\mathcal{G}]\,|\,\mathcal{F}]\,|\mathcal{H}\right] = \mathbb{E}[Y|\mathcal{F}]$

lowest level

If we take expectation on both sides of eqn. 1, then

Smallest filtration - is no information

$$\underbrace{\mathbb{E}\left[M_{t+1}\right]} = \underline{\mathbb{E}\left[M_t\right]} = \cdots = \mathbb{E}[M_0]$$

This is due to the **Tower Property** of conditional expectations.

Martingales are a very nice mathematical object. They "get rid of the drift" and enable us to focus on what probabilists consider is the most interesting part: the statistical properties of purely random processes.

Continuous Time Martingales

Next, we generalize our definitions to continuous time: A continuous time stochastic process

$$\left\{M_t : t \in \mathbb{R}^+\right\}$$

$t \in [0, \ldots, T]$

$n = 1, 2, 3, \ldots, n$

such that M_t is $\mathcal{F}_t$-measurable for $t \in \mathbb{R}^+$ is a **martingale** if

$$\mathbb{E}\left|M_t\right| < \infty \qquad \int_{-\infty}^{\infty}$$

and

$$\mathbb{E}_s[M_t|\mathcal{F}_s] = M_s, \quad 0 \leq s \leq t.$$

A Brownian motion is a *Martingale*.

Lévy's Martingale Characterisation: Let $W_t,\ t > 0$ be a stochastic process and let $\mathcal{F}_t$ be the filtration generated by it. W_t is a Brownian motion iff the following conditions are satisfied:

1. $W_0 = 0$ a.s.;

   $Q[B_t] = T$

2. the sample paths $t \mapsto W_t$ are continuous a.s.;

3. W_t is a martingale with respect to the filtration $\mathcal{F}_s$;

4. $\underbrace{|W_t|^2 - t}$ is a martingale with respect to the filtration $\mathcal{F}_s$.

   Quadratic Variation

The Lévy characterization can be contrasted with the classical definition of a Brownian motion as a stochastic process W_t satisfying:

$$dS = \underbrace{\mu t \, dt}_{} + \sigma S \, dX$$

1. $W_0 = 0$ a.s.;

2. the sample paths $t \mapsto W(t)$ are continuous a.s.;

3. **independent increments**: for $t_1 < t_2 < t_3 < t_4$ the increments $W_{t_4} - W_{t_3}$, $W_{t_2} - W_{t_1}$ are independent;

4. **normally distributed increments**: $W_t - W_s \sim N(0, |t-s|)$.

Lévy's characterization neither mentions independent increments nor normally distributed increments.

Instead, Lévy introduces two easily verifiable martingale conditions.

$$\mathbb{E}_s[W_t | W_s] = \mathbb{E}_s[W_t - W_s + W_s | W_s]$$

$$= \mathbb{E}_s[W_s | W_s] + \mathbb{E}_s[\underbrace{W_t - W_s}_{N(0,|t-s|)} | W_s]$$

$$= W_s + 0 = W_s$$

Itô Integrals and Martingales

Next, we explore the link between Itô integration and martingales. Consider the stochastic process $Y(t) = W^2(t)$. By Itô, we have

Itô on $Y(t)$

$d(W_t^2)$

$\int_0^T$

$$W^2(T) = T + \int_0^T 2W(t)dW(t)$$

Taking the expectation, we get

$$\mathbb{E}[W^2(T)] = T + \mathbb{E}\left[\underbrace{\int_0^T 2W(t)dW(t)}_{=0}\right]$$

Now, the quadratic variation property of Brownian motions implies that

$$\mathbb{E}[W^2(T)] = T$$

and hence

$$* \quad \mathbb{E}\left[\int_0^T 2W(t)dW(t)\right] = 0 \quad *$$

Martingale property of Integrals.

Therefore, the Itô integral

$$\int_0^T 2W(t)dW(t)$$

is a martingale.

$$\mathbb{E}\left[\int_0^T f(t, W_t)\right] = 0$$

In fact, this property is shared by all Itô integrals.

The Itô integral is a martingale

Let $g(t, W_t)$ be a function on $[0, T]$ and satisfying the technical condition. Then the Itô integral

$$\int_0^T \underbrace{\cancel{g(t, W_t)}}_{\text{Riemann Integral}} dW_t \qquad \int_0^T \sin t \, dt$$

is a martingale.

So, Itô integrals are martingales.

But does the converse hold? Can we represent any martingale as an Itô integral?

The answer is yes! (Martingale Representation Theorem)

Example We will show that

$$\mathbb{E}\left[W^2(T)\right] = T$$

using only Itô and the fact that Itô integrals are martingales.

Consider the function $F(t, W_t) = W_T^2$, then by Itô's lemma,

$$\begin{aligned} W_T^2 &= \overset{=0}{\cancel{W_0^2}} + \frac{1}{2}\int_0^T 2dt + \int_0^T 2W_t dW_t \\ &= \underbrace{\int_0^T dt}_{T} + 2\int_0^T W_t dW_t \end{aligned}$$

since $W_0 = 0$

Taking the expectation,

$$\mathbb{E}\left[W_T^2\right] = \underbrace{\mathbb{E}\left[\int_0^T dt\right]}_{T} + \underbrace{2\mathbb{E}\left[\int_0^T W_t dW_t\right]}_{=0}$$

Martingale property.

Now,

$$\int_0^T W_t dW_t$$

is an Itô integral and as a result $\mathbb{E}\left[\int_0^T W_t dW_t\right] = 0$
Moreover,

$$\mathbb{E}\left[\int_0^T dt\right] = \mathbb{E}\left[T\right] = T$$

We can conclude that

$$\mathbb{E}\left[W^2(T)\right] = T$$

As an aside, we can usually exchange the order of integration between the time integral and the expectation so that

$$\mathbb{E}\left[\int_0^T f(W_t)dt\right] = \int_0^T \mathbb{E}\left[f(W_t)\right] dt$$

This is due to an analysis result known as **Fubini's Theorem**.

Properties of Itô Integrals:

1. **Linearity**

$\sum S$

$S_i^2 = \left(\sum R_i\right)^2 = \sum R_i^2$

$$\left.\begin{aligned}&\int_0^T (\alpha f(W_t) + \beta g(W_t))\, dW_t \\ &= \alpha \int_0^T f(W_t) dW_t + \beta \int_0^T g(W_t) dW_t\end{aligned}\right\}$$

2. **Itô Isometry**

$dW^2 \to dt$

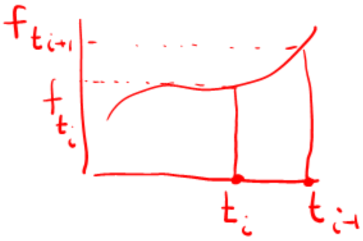
Scalar

$$\boxed{\mathbb{E}\left[\left(\int_0^T f_t dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T f_t^2 dt\right]}$$

Use to calculate the variance of a stochastic process

$V[Y_t] = \mathbb{E}[Y_t^2] - \mathbb{E}^2[Y_t]$

Indefinite Integral: $\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i)(t_{i+1}-t_i)$ Left hand rule



$$\int_0^T f(t)dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1})(t_{i+1}-t_i) \quad \text{right hand rule}$$

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} \tfrac{1}{2}(f(t_{i+1})+f(t_i))(t_{i+1}-t_i) \quad \text{trapezium rule}$$

All the above give the same value

Stochastic Integral

Itô Integral

① $\int_0^T f(t, W_t)\,dW_t = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, W_{t_i})(\underbrace{W_{t_{i+1}} - W_{t_i}})$ L H Rule

② $\int_0^T f(t, W_t)\,dW_t = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, W_{t_{i+1}})(W_{t_{i+1}} - W_{t_i})$ R.H Rule

③ $\int_0^T f(t, W_t)\,dW_t = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+\frac{1}{2}}, W_{t_{i+\frac{1}{2}}})(W_{t_{i+1}} - W_{t_i})$

$$t_{i+\frac{1}{2}} \equiv \tfrac{1}{2}(t_{i+1}+t_i)$$

① is special $\because$ it is a non-anticipating integral

②, ③ are examples of anticipating integrals

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} \mathbb{E}\left[f(t_i, W_{t_i})\,\mathbb{E}(W_{t_{i+1}} - W_{t_i})\right] = ?$$

$\sim N(0, |t_{i+1}-t_i|)$

$$\mathbb{E}\left[\int_0^T f(t, W_t)\,dW_t\right] = 0$$

Showing that a Continuous Time Stochastic Process is a Martingale

Consider a stochastic process $Y(t)$ solving the following SDE:

SDE for Y_t

$$dY(t) = f(Y_t, t)dt + g(Y_t, t)dW(t), \qquad Y(0) = Y_0 \quad \text{I.C.}$$

How can we tell whether $Y(t)$ is a martingale?

$\int_0^t$

The answer has to do with the fact that Itô integrals are martingales.

$Y(t)$ is a martingale if and only if it satisfies the martingale condition

$$* \quad \mathbb{E}\left[Y_t|\mathcal{F}_s\right] = Y_s, \quad 0 \leq s \leq t \qquad (*)$$

Let's start by integrating the SDE between s and t to get an exact form for $Y(t)$:

$$Y(t) = Y(s) + \int_s^t f(Y_u, u)du + \int_s^t g(Y_u, u)dW(u)$$

Taking the expectation conditional on the filtration at time s, we get $\mathbb{E}\left[Y_t|\mathcal{F}_s\right] =$

$$\mathbb{E}[Y_t] = \mathbb{E}\left[Y(s) + \int_s^t f(Y_u, u)du + \int_s^t g(Y_u, u)dW(u)|\mathcal{F}_s\right]$$

$$\mathbb{E}[Y_t] = Y(s) + \mathbb{E}\left[\int_s^t f(Y_u, u)du|\mathcal{F}_s\right]$$

$= 0$ ∵ Itô integral

where the last line follows from the fact that a Itô integral is a martingale, ∴

$$\mathbb{E}\left[\int_s^t g(Y_u, u)dW(u)|\mathcal{F}_s\right] = \int_s^s g(Y_u, u)dW(u) = 0.$$

So, $Y(t)$ is a martingale iff

$$\boxed{\mathbb{E}\left[\int_s^t f(u)du|\mathcal{F}_s\right] = 0} \implies f = 0$$

This condition is satisfied only if $f(Y_t, t) = 0$ for all t. Returning to our SDE, we conclude that $Y(t)$ is a martingale iff it is of the form

$$\boxed{dY(t) = g(Y_t, t)dW(t),} \qquad Y(0) = Y_0$$

This is why we say that martingales are "driftless processes"

Example Determine which of the following processes are martingales.

① , ② do Itô

1. $Y(t) = W(t) + 4t$ ← We know W_t is a Martingale. $4t$ adds drift.

2. $Y(t) = W^2(t) + k$, where k is a given constant. Does the answer depend on the value of k?

3. $Y(t) = W_1(t)W_2(t)$ where $W_1(t)$ and $W_2(t)$ are two standard Brownian motions with correlation ρ so that $dW_1(t)dW_2(t) \to \rho dt$. Does the answer depend on the value of ρ?

(1) $Y(t) = W(t) + 4t$

Intuitively, this cannot be a Martingale since $W(t)$ is a martingale and $4t$ adds some drift.

Mathematically, the SDE for $Y(t)$ is:

non-zero drift

$$dY(t) = 4dt + dW(t)$$

$Y(t)$ is a Brownian motion with drift. Hence, $Y(t)$ is **not a martingale**.

We know $W^2 - t$ is a Martingale

$\therefore$ W^2 + const cannot be a martingale.

(2) $Y(t) = W^2(t) + k$

Intuitively, this cannot be a Martingale since $W^2(t) - t$ is a martingale (recalling the *quadratic variation* property of Brownian motions!).

Itô

Mathematically, by Itô applied to the function $f(W) = W^2 + k$, the SDE for the process $Y(t) = f(W(t))$ is given by

drift $\Rightarrow$ not martingale.

$$dY(t) = dt + 2W(t)dW(t)$$

The dynamics of $Y(t)$ has a drift: $Y(t)$ is **not a martingale** and this result is independent from the specific value of k.

Let X, Y be a pair of stochastic processes e.g with SDEs $dX = a\,dt + b\,dW_1$, $dY = c\,dt + d\,dW_2$ — Mod 3, 6

$\overbrace{a(t,X)}$ $\overbrace{b(t,X)}$

$c(t,Y)$ $d(t,Y)$

$\mathbb{E}[dW_1 dW_2] = \rho\,dt$

Let $F = XY$. TSE for F gives

$$dF = \frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial Y} dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2} dX^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2} dY^2 + \frac{\partial^2 F}{\partial X \partial Y} dX dY$$

$$\frac{\partial F}{\partial X} = Y \quad \frac{\partial F}{\partial Y} = X \quad \frac{\partial^2 F}{\partial X^2} = 0 = \frac{\partial^2 F}{\partial Y^2} \quad \frac{\partial^2 F}{\partial X \partial Y} = 1 = \frac{\partial^2 F}{\partial Y \partial X}$$

$$d(XY) = Y\,dX + X\,dY + \boxed{dX\,dY}$$ Itô Product rule

$\delta x^2 << 1$

δ $\sqrt{dt}\sqrt{dt}$

$O(dt)$

$$d\left(\frac{X}{Y}\right)$$

$a=0 \qquad b=1$

$c=0 \qquad d=1$

(3) $Y(t) = W_1(t)W_2(t)$ where $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motions

By the *Itô product rule*,

$$dY(t) = W_1(t)dW_2(t) + W_2(t)dW_1(t) + \rho dt$$

$\boxed{\rho=0}$

For $Y(t)$ to be a martingale, its dynamics must be driftless, i.e. we must have $\rho dt = 0$.

This is only the case when $\rho = 0$ and the two Brownian motions $W_1(t)$ and $W_2(t)$ are independent.

In the general case, when $\rho \neq 0$, $Y(t)$ is **not a martingale**.

Exponential Martingales

Central to option pricing

Let's start with a motivating example.

Consider the stochastic process $Y(t)$ satisfying the SDE

$$dY(t) = f(t)dt + g(t)dW(t), \qquad Y(0) = Y_0 \quad (2)$$

where $f(t)$ and $g(t)$ are two time-dependent functions and $W(t)$ is a standard Brownian motion.

Define a new process $Z(t) = \boxed{e^{Y(t)}}$ (function)

How should we choose $f(t)$ if we want the process $Z(t)$ to be a martingale?

Consider the process $Z(t) = e^{Y(t)}$. Applying Itô to the function we obtain:

$f\,dt + g\,dW$ $\qquad g^2 dt$

$$\begin{aligned}
dZ(t) &= \frac{dZ}{dY}dY(t) + \frac{1}{2}\frac{d^2Z}{dY^2}dY^2(t) \\
&= \frac{dZ}{dY}\left(f(t)dt + g(t)dW(t)\right) + \frac{1}{2}\frac{d^2Z}{dY^2}g^2(t)dt \\
&= e^{Y(t)}\left(f(t) + \frac{1}{2}g^2(t)\right)dt + e^{Y(t)}g(t)dW(t) \\
dZ &= Z(t)\left[\left(f(t) + \frac{1}{2}g^2(t)\right)dt + g(t)dW(t)\right]
\end{aligned}$$

$= 0$

$dZ = g\,Z\,dW$

$Z(t)$ is a martingale if and only if it is a driftless process.

Therefore for $Z(t)$ to be a martingale we must have

drift

$$f(t) + \frac{1}{2}g^2(t) = 0$$

This is only possible if

$$f(t) = -\frac{1}{2}g^2(t)$$

and rewriting its SDE

Going back to the process $Y(t)$, we must have

$$dY(t) = -\frac{1}{2}g^2(t)dt + g(t)dW(t), \qquad Y(0) = Y_0$$

implying that

$$Y(T) = Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dW(t)$$

e e e

Hence, in terms of $Z(t)$:

$$dZ(t) = Z(t)g(t)dW(t).$$

Using the earlier relationship, we can write $Z(T) = e^{Y(T)}$.

$Z(T) = Z_0 e^{------}$

$V = e^S$ $\qquad \frac{\partial V}{\partial S} = e^S$ $\qquad \frac{\partial^2 V}{\partial S^2} = e^S$ $\qquad$ Now in Itô IV

$$d(e^S) = \left(\mu S e^S + \frac{1}{2}\sigma^2 S^2 e^S\right) dt + \sigma S e^S dW$$

fixed $\boxed{dS = \mu S dt + \sigma S dW}$

$$\frac{d(e^S)}{e^S} = \left(\mu S + \frac{1}{2}\sigma^2 S^2\right) dt + \sigma S dW$$

$$\frac{dV}{V} = (\quad) dt + \sigma S dW \qquad e^S$$

① Do Itô

② Is there drift $\rightarrow$ Yes — not Mart.

$\rightarrow$ No — Mart.

$E[dW^2] = t$ $\qquad \boxed{t - t = 0}$

dW^2 unlike dx^2 in everyday calculus cannot be ignored

Neftci . 5

$$\boxed{\mathbb{E}\left[dW^2\right] = dt}$$

Let's simplify this $Z(T) =$

$$\exp\left\{Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dW(t)\right\}$$

to give

$$\boxed{e^Y} \qquad Z(T) = Z_0 \exp\left\{-\frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dW(t)\right\}$$

Because the stochastic process $Z(t)$ is the exponential of another process (namely $Y(t)$) and because it is a martingale, we call $Z(t)$ an **exponential martingale**.

We have actually just stumbled upon a much more general and very important result.

Tomorrow

4 simulation exponent (Excel)

1. B.M
2. GBM
3. Excel
4. Comparision of E-M and Closed form

$$S_{t+dt} = S_t\, e^{(\mu - \frac{1}{2}\sigma^2)\delta t + \sigma\phi\sqrt{dt}}$$

Itô III on $V = e^S$

$V = \log S$