Martingales

1. X_t and Y_t are two stochastic processes.

   a. Using the Itô rule for products (X_tY_t) deduce the following *integration by parts formula*

   $$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s.$$

   b. Derive the Itô rule for quotients $\left(\frac{X_t}{Y_t}\right)$

2. In this question $t \geq 0$.

   a. For which values of k is the process

   $$Y_t = W_t^4 - 6tW_t^2 + kt^2,$$

   a martingale?

   b. Is $X_t = \cosh(\theta W_t)\, e^{-\theta^2 t/4};\ \theta \in \mathbb{R}$, a martingale?

3. Consider the Vasicek model

   $$dr_t = \kappa(\theta - r_t)\,dt + \sigma dW_t,$$

   where $\kappa,\ \theta, \sigma \in \mathbb{R}$. We are familar with the following solution for $s < t$

   $$r_t = r_s e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma\int_s^t e^{-\kappa(t-u)} dW_u.$$

   Show that as $t \to \infty$, the mean and variance become in turn

   $$\begin{aligned} \mathbb{E}[r_t | r_s] &= \theta \\ \mathbb{V}[r_t | r_s] &= \frac{\sigma^2}{2\kappa} \end{aligned}$$

   Hint: First calculate both mean and variance at time t. For the latter you can use the Itô isometry.