CQF Binomial Model Solutions

1. Binomial tree for share price S and for option price V is

$$\begin{array}{ccccc} & 84 & & & 5 \ \ (= \max(84-79,0)) \\ 80 & & & V & \\ & 76 & & & 0 \ \ (= \max(76-79,0) \end{array}$$

   Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{ccc} & & 5-84\Delta \\ V-80\Delta & & \\ & & -76\Delta \end{array}$$

   For risk-free portfolio choose Δ such that $5 - 84\Delta = -76\Delta \Rightarrow \Delta = \frac{5}{8}$. So in absence of arbitrage, $V - 80\Delta = -76\Delta$, and $V = 2.5$.

2. Binomial tree for share price and option respectively is

$$\begin{array}{ccccc} & 98 & & & 8 \ \ (= \max(98-90,0)) \\ 92 & & & V & \\ & 86 & & & 0 \ \ (= \max(86-90,0) \end{array}$$

   Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{ccc} & & 8-98\Delta \\ V-92\Delta & & \\ & & -86\Delta \end{array}$$

   For risk-free portfolio choose Δ such that $8 - 98\Delta = -86\Delta \Rightarrow \Delta = \frac{2}{3}$. So in absence of arbitrage, since portfolio is riskless, it must earn risk-free rate $r = 2\%$ and $V - 92\Delta = \exp(-0.02)(-86\Delta)$, then $V = \frac{2}{3}(92 - 86\exp(-0.02)) = 5.14$.

3. Binomial tree for share price and for option is

$$\begin{array}{ccccc} & 17 & & & 130 \ \ (= \max(17^2-159,0)) \\ 15 & & & V & \\ & 13 & & & 10 \ \ (= \max(13^2-159,0) \end{array}$$

   Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{ccc} & & 130-17\Delta \\ V-15\Delta & & \\ & & 10-13\Delta \end{array}$$

   For risk-free portfolio choose Δ such that $130 - 17\Delta = 10 - 13\Delta \Rightarrow \Delta = 30$. So in absence of arbitrage, $V - 15\Delta = 10 - 13\Delta$, and $V = 70$.

4. The Binomial tree for share price is

$$\begin{array}{ccc} & & 92 \\ & 75 & \\ & & 59 \end{array}$$

Time is 3 months - i.e. $\frac{1}{4}$ year, interest rate $r = 0$, so risk neutral probability that share price increases satisfies

$$92q + 59\,(1-q) = 75$$

Re-arranging gives

$$q = \frac{75-59}{33} = 0.485$$

So probability of a fall would be given by $1 - q = 0.515$.

5.

$$\begin{array}{ccc} \text{Stock} & & 84 \\ & \nearrow & \\ 80 & & \\ & \searrow & \\ & & 76 \end{array} \qquad \begin{array}{ccc} \text{Option:} & & \begin{array}{c} 1 \\ (84 > E) \end{array} \\ & \nearrow & \\ V & & \\ & \searrow & \\ & & \begin{array}{c} 0 \\ (76 < E) \end{array} \end{array}$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$\begin{array}{cc} & 1 - 84\Delta \\ V - 80\Delta & \\ & -76\Delta \end{array}$$

For risk-free portfolio choose Δ such that $1 - 84\Delta = -76\Delta \Rightarrow \Delta = 1/8$. So in absence of arbitrage, $V - 80\Delta = 1 - 84\Delta$, and $V = 0.5$. Note how the option price is significantly cheaper than the European style security in

6. See spreadsheet

7.

$$\begin{array}{ccccc} \multicolumn{5}{c}{\text{Binomial tree for stock}} \\ & & & & 69 \\ & & & \nearrow & \\ & & 66 & & \\ & \nearrow & & \searrow & \\ 63 & & & & 63 \\ & \searrow & & \nearrow & \\ & & 60 & & \\ & & & \searrow & \\ & & & & 57 \end{array} \qquad \begin{array}{ccccc} \multicolumn{5}{c}{\text{Binomial tree for option}} \\ & & & & 0 \\ & & & \nearrow & (= \max(61-69, 0)) \\ & & P_1 & & \\ & \nearrow & & \searrow & \\ P & & & & 0 \\ & \searrow & & \nearrow & (= \max(61-63, 0)) \\ & & P_2 & & \\ & & & \searrow & \\ & & & & 4 \\ & & & & (= \max(61-57, 0)) \end{array}$$

We must find the values of P_1 and P_2 before we can solve for P.
If we set up a Black-Scholes delta hedged portfolio $P_1 - \Delta_1 S$, for P_1, then the binomial tree for its

value is

$$\begin{array}{ccc} & & -69\Delta_1 \\ & \nearrow & \\ P_1 - 66\Delta_1 & & \\ & \searrow & \\ & & -63\Delta_1 \end{array}$$

For a risk-free portfolio, we choose Δ_1 s.t. $-69\Delta_1 = -63\Delta_1$, i.e. $\Delta_1 = 0$. Then in the absence of arbitrage it must earn at the risk-free interest-rate and

$$P_1 - 66\Delta_1 = e^{-0.04\times(3/12)}\left(-69\Delta_1\right) \rightarrow P_1 = 0.$$

If we set up a Black-Scholes delta hedged portfolio $P_2 - \Delta_2 S$, for P_2, then the binomial tree for its value is

$$\begin{array}{ccc} & & -63\Delta_2 \\ & \nearrow & \\ P_2 - 60\Delta_2 & & \\ & \searrow & \\ & & 4 - 57\Delta_2 \end{array}$$

For a risk-free portfolio, we choose Δ_2 s.t. $-63\Delta_2 = 4 - 57\Delta_2$, i.e. $\Delta_2 = -2/3$. Then in the absence of arbitrage it must earn at the risk-free interest-rate and

$$P_2 - 60\Delta_2 = e^{-0.04\times(3/12)}\left(4 - 57\Delta_2\right) \rightarrow P_2 = 1.58.$$

We can now set up a Black-Scholes hedged portfolio, $P - \Delta S$, for P. The binomial tree for its value is

$$\begin{array}{ccc} & & P_1 - 66\Delta \\ & \nearrow & \\ P - 63\Delta & & \\ & \searrow & \\ & & P_2 - 60\Delta \end{array}$$

For a risk-free portfolio, we choose Δ s.t.

$$P_1 - 66\Delta = P_2 - 60\Delta \rightarrow \Delta = \frac{P_1 - P_2}{6} = \frac{-P_2}{6} = -0.263.$$

Then in an arbitrage free market, the portfolio earns at the risk-free rate

$$\begin{aligned} P - 63\Delta &= e^{-0.04\times(3/12)}\left(-66\Delta\right) \\ P &= \Delta\left(63 - 65.34\right) = -0.263 \times -2.34 \\ &= 0.6154 \end{aligned}$$

Hence the value of the put option is £0.62.

8. To find V_1 from portfolio: $\Pi = V - \Delta S$. Then from T_1 to T we have

$$\Pi \longrightarrow \begin{cases} 15 - \Delta(\alpha + 20) \\ 0 - \Delta\alpha \end{cases}$$

so for risk free portfolio$\Rightarrow$ choose $15 - \Delta(\alpha + 20) = -\Delta\alpha \longrightarrow \Delta = 3/4$. For no arbitrage we want

$$V_1 - \Delta(\alpha + 10) = -\Delta\alpha$$

since $r = 0$. Solving gives $V_1 = 7.5$
For V_{-1} :

$$\Pi \longrightarrow \begin{cases} 0 - \Delta\alpha \\ 0 - \Delta(\alpha - 20) \end{cases} \Rightarrow \Delta = 0$$

Therefore $V_{-1} = 0$.
For V :

$$V - \Delta\alpha = \begin{cases} V_1 - \Delta(\alpha + 10) & \equiv \frac{15}{2} - \Delta(\alpha + 10) \\ V_{-1} - \Delta(\alpha - 10) & \equiv 0 - \Delta(\alpha - 10) \end{cases}$$

so $\frac{15}{2} - 20\Delta = 0 \longrightarrow \Delta = 3/8$.
Finally $V - \Delta\alpha = -(\alpha - 10) \longrightarrow V = 10\Delta = 15/4$. Hence

$$\begin{array}{ccc} & & 15 \\ & 15/2 & \\ V = 15/4 & & 0 \\ & 0 & \\ & & 0 \end{array}$$