Important Cases - Equities and Interest Rates

If we now consider S which follows a lognormal random walk, i.e. $V = \log(S)$ then substituting into (6) gives

$$d\left((\log S)\right) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW$$

Integrating both sides over a given time horizon (between t_0 and T)

$$\int_{t_0}^{T} d\left((\log S)\right) = \int_{t_0}^{T}\left(\mu - \frac{1}{2}\sigma^2\right)dt + \int_{t_0}^{T}\sigma dW \quad (T > t_0)$$

we obtain

$$\log\frac{S(T)}{S(t_0)} = \left(\mu - \frac{1}{2}\sigma^2\right)(T - t_0) + \sigma\left(W(T) - W(t_0)\right)$$

Assuming at $t_0 = 0$, $W(0) = 0$ and $S(0) = S_0$ the exact solution becomes

$$S_T = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\phi\sqrt{T}\right\}. \tag{7}$$

(7) is of particular interest when considering the pricing of a simple European option due to its non path dependence. Stock prices cannot become negative, so we allow S, a non-dividend paying stock to evolve according to the lognormal process given above - and acts as the starting point for the Black-Scholes framework.

However μ is replaced by the risk-free interest rate r in (7) and the introduction of the risk-neutral measure - in particular the Monte Carlo method for option pricing.

Interest rates exhibit a variety of dynamics that are distinct from stock prices, requiring the development of specific models to include behaviour such as return to equilibrium, boundedness and positivity. The earliest interest rate models took as their starting point a stochastic model for the short rate, or instantaneous interest rate denoted r_t. These were one factor models meaning the single source of randomnessThe interest rate for the shortest possible deposit is commonly called the spot interest rate, or simply the **spot rate**.This is defined as the rate of interest for the infinitessimal interval $[t, t + dt]$, with

$$r_t dt = \text{total interest gained in } [t, t + dt].$$

The spot rate was introduced specifically for the purpose of efficient interest rate modelling and is not traded in the markets. In practice one takes yield on a liquid finite maturity bond e.g. a one month US Treasury bill. Unlike equities, there are numerous models for capturing the dynamics of interest rates. These short rate models for r_t are expressed as a SDE

$$dr_t = u(r_t, t)\,dt + w(r_t, t)\,dW_t$$

for given coefficients $u(r_t, t)$ and $w(r_t, t)$.

Here we consider another important example of a stochastic differential equation, put forward by Vasicek in 1977. This model has a mean reverting Ornstein-Uhlenbeck process for the short rate and is used for generating interest rates, given by

$$dr_t = (\eta - \gamma r_t)\,dt + \sigma dW_t. \tag{8}$$

So drift is $(\eta - \gamma r_t)$ and volatility given by σ.

γ refers to the *speed of reversion* or simply the *speed*. $\frac{\eta}{\gamma}(= \overline{r})$ denotes the mean (equlibrium) rate, and we can rewrite this random walk (7) for dr_t in a more popular form as

$$dr_t = -\gamma\left(r_t - \overline{r}\right)dt + \sigma dW_t.$$

The dimensions of γ are 1/time, hence $1/\gamma$ has the dimensions of time (years). For example a rate that has speed $\gamma = 3$ takes one third of a year to revert back to the mean, i.e. 4 months. $\gamma = 52$ means $1/\gamma = 1/52$ years i.e. 1 week to mean revert (hence very rapid). The mean reverting behaviour is consistent with economics theory supporting the idea that interest rates should fluctuate along a long term mean equlibrium rate, determined by economic equlilibrium between demand and supply - a comon feature of all short rate models. The one disadvantage of the Vasicek model is that interest rates can become negative, which is undesirable in economics and highly disliked by economists - they below in a zero lower bound. This bizarre concept in its simplest form means a lender pays another party to borrow its money. That is, the bank will charge its customers for having accounts. However, it is not as odd as it seems given the situation of negative interest rates in Switzerland in the 60s and more recently in Japan. That said, you and I do not need to worry as this is not a concept about to hit the high street.
Returning to the mathematics! By setting $X_t = r_t - \overline{r}$, X_t is a solution of

$$dX_t = -\gamma X_t dt + \sigma dW_t;\ X_0 = \alpha, \tag{9}$$

hence it follows that X_t is an Ornstein-Uhlenbeck process and an analytic solution for this equation exists. (9) can be written as $dX_t + \gamma X_t dt = \sigma dW_t$.

Multiply both sides by an integrating factor $e^{\gamma t}$

$$\begin{aligned} e^{\gamma t}\left(dX_t + \gamma X_t dt\right) &= \sigma e^{\gamma t} dW_t \\ d\left(e^{\gamma t} X_t\right) &= \sigma e^{\gamma t} dW_t \end{aligned}$$

Integrating over $[0,t]$ gives

$$\begin{aligned} \int_0^t d\left(e^{\gamma s} X_s\right) &= \int_0^t \sigma e^{\gamma s} dW_s \\ e^{\gamma s} X_s\big|_0^t &= \int_0^t \sigma e^{\gamma s} dW_s \rightarrow e^{\gamma t} X_t - X_0 = \int_0^t \sigma e^{\gamma s} dW_s \end{aligned}$$

$$X_t = \alpha e^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)} dW_s. \tag{10}$$

By using integration by parts, i.e. $\int v\ du = uv - \int u\ dv$ we can simplify (10).

$$\begin{aligned} &u = W_s \\ &v = e^{\gamma(s-t)} \rightarrow dv = \gamma e^{\gamma(s-t)} ds \end{aligned}$$

Therefore

$$\int_0^t e^{\gamma(s-t)} dW_s = W_t - \gamma \int_0^t e^{\gamma(s-t)} W_s\ ds$$

and we can write (10) as

$$X_t = \alpha e^{-\gamma t} + \sigma\left(W_t - \gamma \int_0^t e^{\gamma(s-t)} W_s\ ds\right)$$

allowing numerical treatment for the integral term.

Returning to the integral in (10)

$$\int_0^t e^{\gamma(s-t)} dW_s$$

let's use the stochatsic integral formula to verify the result. Recall

$$\int_0^t \tfrac{\partial f}{\partial W} dW = f(t, W_t) - f(0, W_0) - \int_0^t \left(\tfrac{\partial f}{\partial s} + \tfrac{1}{2} \tfrac{\partial^2 f}{\partial W^2} \right) ds$$

so $\frac{\partial f}{\partial W} \equiv e^{\gamma(s-t)} \Longrightarrow f = e^{\gamma(s-t)} W_s,\ \frac{\partial f}{\partial s} = \gamma e^{\gamma(s-t)} W_s,\ \frac{\partial^2 f}{\partial W^2} = 0$

$$\begin{aligned}\int_0^t e^{\gamma(s-t)} dW_s &= W_t - 0 - \int_0^t \left(\gamma e^{\gamma(s-t)} W_s + \tfrac{1}{2} \times 0 \right) ds \\ &= W_t - \gamma \int_0^t e^{\gamma(s-t)} W_s ds.\end{aligned}$$

We have used an integrating factor to obtain a solution of the Ornstein Uhlenbeck process. Let's look at $d(e^{\gamma t} U_t)$ by using Itô. Consider a function $V(t, U_t)$ where $dU_t = -\gamma U_t dt + \sigma dW_t$, then

$$\begin{aligned}dV &= \left(\frac{\partial V}{\partial t} - \gamma U \frac{\partial V}{\partial U} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial U^2} \right) dt + \sigma \frac{\partial V}{\partial U} dW \\ d\left(e^{\gamma t} U\right) &= \left(\frac{\partial}{\partial t} \left(e^{\gamma t} U\right) - \gamma U \frac{\partial}{\partial U} \left(e^{\gamma t} U\right) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial U^2} \left(e^{\gamma t} U\right) \right) dt + \\ & \quad \sigma \frac{\partial}{\partial U} \left(e^{\gamma t} U\right) dW \\ &= \left(\gamma e^{\gamma t} U - \gamma U e^{\gamma t} \right) dt + \sigma e^{\gamma t} dW \\ &= \sigma e^{\gamma t} dW\end{aligned}$$

Example: The Ornstein-Uhlenbeck process satisfies the spot rate SDE given by

$$dX_t = \kappa(\theta - X_t) dt + \sigma dW_t,\ X_0 = x,$$

where κ, θ and σ are constants. Solve this SDE by setting $Y_t = e^{\kappa t} X_t$ and using Itô's lemma to show that

$$X_t = \theta + (x - \theta) e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dW_s.$$

First write Itô for Y_t given $dX_t = A(X_t, t) dt + B(X_t, t) dW_t$

$$\begin{aligned}dY_t &= \left(\frac{\partial Y_t}{\partial t} + A(X_t, t) \frac{\partial Y_t}{\partial X_t} + \tfrac{1}{2} B^2(X_t, t) \frac{\partial^2 Y_t}{\partial X_t^2} \right) dt + B(X_t, t) \frac{\partial Y_t}{\partial X_t} dW_t \\ &= \left(\frac{\partial Y_t}{\partial t} + \kappa(\theta - X_t) \frac{\partial Y_t}{\partial X_t} + \tfrac{1}{2} \sigma^2 \frac{\partial^2 Y_t}{\partial X_t^2} \right) dt + \sigma \frac{\partial Y_t}{\partial X_t} dW_t\end{aligned}$$

$$\frac{\partial Y_t}{\partial t} = \kappa e^{\kappa t} X_t;\ \frac{\partial Y_t}{\partial X_t} = e^{\kappa t};\ \frac{\partial^2 Y_t}{\partial X_t^2} = 0.$$

$$\begin{aligned}d\left(e^{\kappa t} X_t\right) &= \left(\kappa e^{\kappa t} X_t + \kappa(\theta - X_t) e^{\kappa t} \right) dt + \sigma e^{\kappa t} dW_t \\ &= \kappa \theta e^{\kappa t} dt + \sigma e^{\kappa t} dW_t\end{aligned}$$

$$\begin{aligned}\int_0^t d\left(e^{\kappa s} X_s\right) &= \kappa \theta \int_0^t e^{\kappa s} ds + \sigma \int_0^t e^{\kappa s} dW_s \\ e^{\kappa t} X_t - x &= \theta e^{\kappa t} - \theta + \sigma \int_0^t e^{\kappa s} dW_s \\ X_t &= x e^{-\kappa t} + \theta - \theta e^{-\kappa t} + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} dW_s \\ X_t &= \theta + (x - \theta) e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dW_s.\end{aligned}$$

Consider

$$dr_t = \kappa\left(\theta - r_t\right)dt + \sigma dW_t,$$

and show by suitable integration for $s < t$

$$r_t = r_s e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma\int_s^t e^{-\kappa(t-u)}dW_u.$$

The lower limit gives us an initial condition at time $s < t$. Expand $d\left(e^{\kappa t}r_t\right)$

$$\begin{aligned} d\left(e^{\kappa t}r_t\right) &= \left(\kappa e^{\kappa t}r_t dt + e^{\kappa t}dr_t\right) \\ &= e^{\kappa t}\left(\kappa\theta dt + \sigma dW_t\right) \end{aligned}$$

Now integrate both sides over $[s,t]$ to give for each $s < t$

$$\begin{aligned} \int_s^t d\left(e^{\kappa u}r_u\right) &= \kappa\theta\int_s^t e^{\kappa u}du + \sigma\int_s^t e^{\kappa u}dW_u \\ e^{\kappa t}r_t - e^{\kappa s}r_s &= \theta e^{\kappa t} - \theta e^{\kappa s} + \sigma\int_s^t e^{\kappa u}dW_u \end{aligned}$$

rearranging and dividing through by $e^{\kappa t}$

$$\begin{aligned} r_t &= e^{-\kappa(t-s)}r_s + \theta - \theta e^{-\kappa(t-s)} + \sigma e^{-\kappa t}\int_s^t e^{\kappa s}dW_u \\ r_t &= e^{-\kappa(t-s)}r_s + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma\int_s^t e^{-\kappa(t-u)}dW_u. \end{aligned}$$

Earlier the disadvantage of the Vasicek model was mentioned. The **Cox Ingersoll Ross** (CIR) model is similar to Vasicek but its diffusion is scaled with the square root of the interest rate:

$$dr_t = \gamma\left(\overline{r} - r_t\right)dt + \sigma\sqrt{r_t}dW_t.$$

If r_t ever gets close to zero, the amount of randomness decreases, i.e. diffusion $\longrightarrow 0$, therefore the drift dominates, in particular the mean rate. Hence the short rate will stay positive with probability 1.

Another short rate model is the **Longstaff** model also known as the double square root process

$$dr_t = \gamma\left(\overline{r} - \sqrt{r_t}\right)dt + \sigma\sqrt{r_t}dW_t.$$

These models and more will be studied in greater detail in the fixed-income modelling section.

Example: Given $U = \log Y$, where Y satsfies the diffusion process

$$\begin{aligned} dY &= \frac{1}{2Y}dt + dW \\ Y(0) &= Y_0 \end{aligned}$$

use Itô's lemma to find the SDE satsfied by U.

Since $U = U(Y)$ with $dY = a(Y,t)\,dt + b(Y,t)\,dW$, we can write

$$dU = \left(a(Y,t)\frac{dU}{dY} + \frac{1}{2}b^2(Y,t)\frac{d^2U}{dY^2}\right)dt + b(Y,t)\frac{dU}{dY}dW$$

Now $U = \log(Y)$ so $\frac{dU}{dY} = \frac{1}{Y}, \frac{d^2U}{dY^2} = -\frac{1}{Y^2}$ and substituting in

$$\begin{aligned} dU &= \left(\frac{1}{2Y}\left(\frac{1}{Y}\right) + \frac{1}{2}(1)^2\left(-\frac{1}{Y^2}\right)\right)dt + \frac{1}{Y}dW \\ &= \frac{1}{Y}dW \end{aligned}$$

$$dU = e^{-U}dW$$

Example: Consider the stochastic volatility model

$$d\sqrt{v} = \left(\alpha - \beta\sqrt{v}\right)dt + \delta dW$$

where v is the variance. Show that

$$dv = \left(\delta^2 + 2\alpha\sqrt{v} - 2\beta v\right)dt + 2\delta\sqrt{v}dW$$

Setting the variable $X = \sqrt{v}$ giving $dX = \underbrace{(\alpha - \beta X)}_{A}dt + \underbrace{\delta}_{B}dW$. We now require a SDE for dY, where $Y = X^2$. So

$$\begin{aligned} dY &= \left(A\frac{dY}{dX} + \frac{1}{2}B^2\frac{d^2Y}{dX^2}\right)dt + B\frac{dY}{dX}dW \\ &= \left((\alpha - \beta X)(2X) + \frac{1}{2}\delta^2.2\right)dt + \delta.2X\,dW \\ &= \left(2\alpha X - 2\beta X^2 + \delta^2\right)dt + 2\delta\sqrt{v}dW \\ &= \left(\delta^2 + 2\alpha\sqrt{v} - 2\beta v\right)dt + 2\delta\sqrt{v}dW \end{aligned}$$

(Harder) Example: Consider the dynamics of a non-traded asset S_t given by

$$\frac{dS_t}{S_t} = \alpha(\theta - \log S_t)\,dt + \sigma dW_t$$

where the constants $\sigma, \alpha > 0$. If $T > t$, show that

$$\log S_T = e^{-\alpha(T-t)}\log S_t + \left(\theta - \frac{1}{2\alpha}\sigma^2\right)\left(1 - e^{-\alpha(T-t)}\right) + \sigma\int_t^T e^{-\alpha(T-s)}dW_s.$$

Hence show that

$$\log S_T \sim N\left(e^{-\alpha(T-t)}\log S_t + \left(\theta - \frac{1}{2\alpha}\sigma^2\right)\left(1 - e^{-\alpha(T-t)}\right), \sigma^2\left(\frac{1 - e^{-2\alpha(T-t)}}{2\alpha}\right)\right)$$

Writing Itô for the SDE where $f = f(S_t)$ gives

$$df = \left(\alpha(\theta - \log S_t) S_t \frac{df}{dS} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{d^2 f}{dS^2} \right) dt + \sigma S_t \frac{df}{dS} dW_t.$$

Hence if $f(S_t) = \log S_t$ then

$$\begin{aligned} d(\log S_t) &= \left(\alpha(\theta - \log S_t) - \tfrac{1}{2}\sigma^2\right) dt + \sigma dW_t \\ &= \alpha\left(\theta - \frac{1}{2\alpha}\sigma^2 - \log S_t\right) dt + \sigma dW_t \\ &= -\alpha(\log S_t - \mu) dt + \sigma dW_t \end{aligned}$$

where $\mu = \theta - \frac{1}{2\alpha}\sigma^2$. Going back to

$$df = -\alpha(f - \mu) dt + \sigma dW_t$$

and now write $x_t = f - \mu$ which gives $dx_t = df$ and we are left with an Ornstein-Uhlenbeck process

$$dx_t = -\alpha x_t dt + \sigma dW_t.$$

Following the earlier integrating factor method gives

$$\begin{aligned} d\left(e^{\alpha t} x_t\right) &= \sigma e^{\alpha t} dW_t \\ \int_t^T d\left(e^{\alpha s} x_s\right) &= \sigma \int_t^T e^{\alpha s} dW_s \\ x_T &= e^{-\alpha(T-t)} x_t + \sigma \int_t^T e^{-\alpha(T-s)} dW_s. \end{aligned}$$

Now replace these terms with the original variables and parameters

$$\log S_T - \left(\theta - \frac{1}{2\alpha}\sigma^2\right) = e^{-\alpha(T-t)}\left(\log S_T - \left(\theta - \frac{1}{2\alpha}\sigma^2\right)\right) + \sigma \int_t^T e^{-\alpha(T-s)} dW_s,$$

which upon rearranging and factorising gives

$$\log S_T = e^{-\alpha(T-t)} \log S_T + \left(\theta - \frac{1}{2\alpha}\sigma^2\right)\left(1 - e^{-\alpha(T-t)}\right) + \sigma \int_t^T e^{-\alpha(T-s)} dW_s.$$

Example: Consider the SDE for the variance process v

$$dv = \varepsilon(m - \sigma) dt + \xi \sigma dW_t,$$

where $v = \sigma^2$. ε, ξ, m are constants. Using Itô's lemma, show that the volatility σ satisfies the SDE

$$d\sigma = a(\sigma, t) dt + b(\sigma, t) dW_t,$$

where the precise form of $a(\sigma, t)$ and $b(\sigma, t)$ should be given.

Consider the stochastic volatility model

$$dv = \varepsilon\left(m - \sqrt{v}\right) dt + \xi \sqrt{v} dW_t$$

If $F = F(v)$ then Itô gives

$$dF = \left(\varepsilon(m-\sigma)\frac{dF}{dv} + \frac{1}{2}\xi^2 v \frac{d^2F}{dv^2}\right)dt + \xi\sqrt{v}\frac{dF}{dv}dW_t.$$

For $F(v) = v^{1/2}$; $\frac{dF}{dv} = \frac{1}{2}v^{-1/2}$, $\frac{d^2F}{dv^2} = -\frac{1}{4}v^{-3/2}$

$$\begin{aligned} dF &= d\sigma = \left(\frac{\varepsilon}{2}(m-\sigma)v^{-1/2} - \frac{1}{8}\xi^2 v^{-1}\right)dt + \frac{\xi}{2}dW_t \\ &= \left(\frac{\varepsilon}{2\sigma}(m-\sigma) - \frac{1}{8\sigma}\xi^2\right)dt + \frac{\xi}{2}dW_t \end{aligned}$$

$$a(\sigma,t) = \left(\frac{\varepsilon}{2\sigma}(m-\sigma) - \frac{1}{8\sigma}\xi^2\right); \quad b(\sigma,t) = \frac{\xi}{2}$$

Higher Dimensional Itô

There is a multi-dimensional form of Itô's lemma. Let us consider the two-dimensional version initially, as this can be generalised nicely to the $N-$dimensional case, driven by a Brownian motion of any number (not necessarily the same number) of dimensions. Let

$$W_t := \left(W_t^{(1)}, W_t^{(2)} \right)$$

be a two-dimensional Brownian motion, where $W_t^{(1)}, W_t^{(2)}$ are independent Brownian motions, and define the two-dimensional Itô process

$$X_t := \left(X_t^{(1)}, X_t^{(2)} \right)$$

such that

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i$$

Consider the case where N shares follow the usual Geometric Brownian Motions, i.e.

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i,$$

for $1 \le i \le N$. The share price changes are correlated with correlation coefficient ρ_{ij}. By starting with a Taylor series expansion

$$V\left(t + \delta t, S_1 + \delta S_1, S_2 + \delta S_2, ..., S_N + \delta S_N\right) =$$

$$V\left(t, S_1, S_2, ..., S_N\right) + \frac{\partial V}{\partial t} dt + \sum_{i=1}^{N} \frac{\partial V}{\partial S_i} dS_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=i}^{N} \frac{\partial^2 V}{\partial S_i \partial S_j} dS_i dS_j + ...$$

which becomes, using $dW_i dW_j = \rho_{ij} dt$

$$dV = \left(\frac{\partial V}{\partial t} + \sum_{i=1}^{N} \mu_i S_i \frac{\partial V}{\partial S_i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right) dt + \sum_{i=1}^{N} \sigma_i S_i \frac{\partial V}{\partial S_i} dW_i.$$

We can integrate both sides over 0 and t to give

$$\begin{aligned} V\left(t, S_1, S_2, ..., S_N\right) &= V\left(0, S_1, S_2, ..., S_N\right) + \\ &\int_0^t \left(\frac{\partial V}{\partial \tau} + \sum_{i=1}^{N} \mu_i S_i \frac{\partial V}{\partial S_i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right) d\tau \\ &+ \int_0^t \sum_{i=1}^{N} \sigma_i S_i \frac{\partial V}{\partial S_i} dW_i. \end{aligned}$$

Discrete Time Random Walks

When simulating a random walk we write the SDE given by (6) in discrete form

$$\delta S = S_{i+1} - S_i = rS_i\delta t + \sigma S_i \phi \sqrt{\delta t}$$

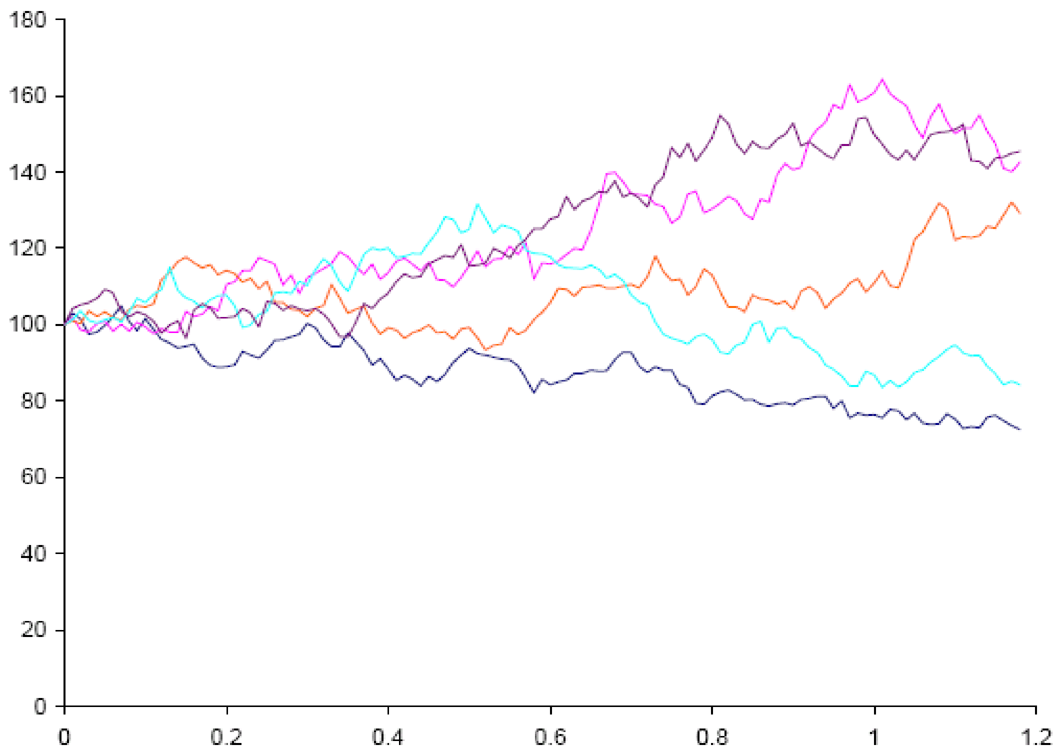
which becomes

$$S_{i+1} = S_i \left(1 + r\delta t + \sigma\phi\sqrt{\delta t}\right). \tag{11}$$

This gives us a time-stepping scheme for generating an asset price realization if we know S_0, i.e. $S(t)$ at $t = 0$. $\phi \sim N(0,1)$ is a random variable with a standard Normal distribution.

Alternatively we can use discrete form of the analytical expression (7)

$$S_{i+1} = S_i \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t}\right\}.$$



So we now start generating random numbers. In C++ we produce uniformly distributed random variables and then use the Box Muller transformation (Polar Marsaglia method) to convert them to Gaussians.

This can also be generated on an Excel spreadsheet using the in-built random generator function RAND(). A crude (but useful) approximation for ϕ can be obtained from

$$\sum_{i=1}^{12} \text{RAND}() - 6$$

where RAND() $\sim U[0,1]$. A more general result is now derived.

Consider the RAND() function in Excel that produces a uniformly distributed random number over 0 and 1, written $\mathbf{U}_i$. We can show that for a large number n,

$$\lim_{n\to\infty} \sqrt{\frac{12}{N}} \left(\sum_{1}^{n} \mathbf{U}_i - \tfrac{n}{2}\right) \sim N(0,1).$$

Introduce $\mathbf{U}_i$ to denote a uniformly distributed random variable over $[0,1]$ and sum up. Recall that

$$\begin{aligned}\mathbb{E}\left[\mathbf{U}_i\right] &= \tfrac{1}{2}\\ \mathbb{V}\left[\mathbf{U}_i\right] &= \tfrac{1}{12}\end{aligned}$$

The mean is then

$$\mathbb{E}\left[\sum_{i=1}^{n}\mathbf{U}_i\right] = n/2$$

so subtract off $n/2$, so we examine the variance of $\left(\sum_1^n \mathbf{U}_i - \frac{n}{2}\right)$

$$\begin{aligned}\mathbb{V}\left[\sum_1^n \mathbf{U}_i - \tfrac{n}{2}\right] &= \sum_1^n \mathbb{V}\left[\mathbf{U}_i\right]\\ &= n/12\end{aligned}$$

As the variance is not 1, write

$$\mathbb{V}\left[\alpha\left(\sum_1^n \mathbf{U}_i - \tfrac{n}{2}\right)\right]$$

for some $\alpha \in \mathbb{R}$. Hence $\alpha^2 \frac{n}{12} = 1$ which gives $\alpha = \sqrt{12/n}$ which normalises the variance. Then we achieve the result

$$\sqrt{\frac{12}{n}}\left(\sum_1^n \mathbf{U}_i - \tfrac{n}{2}\right).$$

Rewrite as

$$\frac{\left(\sum_1^n \mathbf{U}_i - n \times \frac{1}{2}\right)}{\sqrt{\frac{1}{12}}\sqrt{n}}.$$

and for $n \to \infty$ by the Central Limit Theorem we get $N(0,1)$.

A more accurate (but slower) $\phi \sim N(0,1)$ can be computed using $\text{NORMSINV}(\text{RAND}())$. This relies on the inverse cumulative distribution function.

Dynamics of Vasicek Model

Returning to the Vasicek model, a reminder that

$$dr_t = \gamma(\overline{r} - r_t)\,dt + \sigma dW_t$$

is an example of a *Mean Reverting Process* - an important property of interest rates. γ refers to the *reversion rate* (also called the speed of reversion) and $\overline{r}$ denotes the *mean rate*.

γ acts like a "spring". Mean reversion means that a process which increases has a negative trend (γ pulls it down to a mean level $\overline{r}$), and when r_t decreases on average γ pulls it back up to $\overline{r}$.

In discrete time we can approximate this by writing (as earlier)

$$r_{i+1} = r_i + \gamma(\overline{r} - r_i)\,\delta t + \sigma\phi\sqrt{\delta t}$$

To gain an understanding of the properties of this model, look at dr in the absence of randomness

$$\begin{aligned} dr &= -\gamma(r - \overline{r})\,dt \\ \int \frac{dr}{(r-\overline{r})} &= -\gamma \int dt \\ r(t) &= \overline{r} + k\exp(-\gamma t) \end{aligned}$$

So γ controls the rate of exponential decay.

Generating Correlated Normal Variables

Consider two uncorrelated standard Normal variables ε_1 and ε_2 from which we wish to form a correlated pair ϕ_1, & ϕ_2 $(\sim N(0,\ 1))$, such that $\mathbb{E}[\phi_1\phi_2] = \rho$. The following scheme can be used

1. $\mathbb{E}[\varepsilon_1] = \mathbb{E}[\varepsilon_2] = 0$; $\mathbb{E}[\varepsilon_1^2] = \mathbb{E}[\varepsilon_2^2] = 1$ and $\mathbb{E}[\varepsilon_1\varepsilon_2] = 0$ $(\because\ \varepsilon_1,\ \varepsilon_2 \text{ are uncorrelated})$.
2. Set $\phi_1 = \varepsilon_1$ and $\phi_2 = \alpha\varepsilon_1 + \beta\varepsilon_2$ (i.e. a linear combination).
3. Now

$$\begin{aligned}
\mathbb{E}[\phi_1\phi_2] &= \rho = \mathbb{E}[\varepsilon_1(\alpha\varepsilon_1 + \beta\varepsilon_2)] \\
\mathbb{E}[\varepsilon_1(\alpha\varepsilon_1 + \beta\varepsilon_2)] &= \rho \\
\alpha\mathbb{E}\left[\varepsilon_1^2\right] + \beta\mathbb{E}[\varepsilon_1\varepsilon_2] &= \rho \rightarrow \alpha = \rho
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\phi_2^2\right] &= 1 = \mathbb{E}\left[(\alpha\varepsilon_1 + \beta\varepsilon_2)^2\right] \\
&= \mathbb{E}\left[\alpha^2\varepsilon_1^2 + \beta^2\varepsilon_2^2 + 2\alpha\beta\varepsilon_1\varepsilon_2\right] \\
&= \alpha^2\mathbb{E}\left[\varepsilon_1^2\right] + \beta^2\mathbb{E}\left[\varepsilon_2^2\right] + 2\alpha\beta\mathbb{E}[\varepsilon_1\varepsilon_2] = 1 \\
\rho^2 + \beta^2 &= 1 \rightarrow \beta = \sqrt{1-\rho^2}
\end{aligned}$$

4. This gives $\phi_1 = \varepsilon_1$ and $\phi_2 = \rho\varepsilon_1 + \left(\sqrt{1-\rho^2}\right)\varepsilon_2$ which are correlated standardized Normal variables.

Transition Probability Density Functions for Stochastic Differential Equations

To match the mean and standard deviation of the trinomial model with the continuous-time random walk we choose the following definitions for the probabilities

$$\begin{aligned}\phi^{+}(y,t) &= \frac{1}{2}\frac{\delta t}{\delta y^{2}}\left(B^{2}(y,t)+A(y,t)\,\delta y\right),\\ \phi^{-}(y,t) &= \frac{1}{2}\frac{\delta t}{\delta y^{2}}\left(B^{2}(y,t)-A(y,t)\,\delta y\right)\end{aligned}$$

We first note that the expected value is

$$\begin{aligned}& \phi^{+}(\delta y)+\phi^{-}(-\delta y)+\left(1-\phi^{+}-\phi^{-}\right)(0)\\ =\ & \left(\phi^{+}-\phi^{-}\right)\delta y\end{aligned}$$

We already know that the mean and variance of the continuous time random walk given by

$$dy = A(y,t)\,dt + b(y,t)\,dW$$

is, in turn,

$$\begin{aligned}\mathbb{E}[dy] &= Adt\\ \mathbb{V}[dy] &= B^{2}dt.\end{aligned}$$

So to match the mean requires

$$\left(\phi^{+}-\phi^{-}\right)\delta y = A\delta t$$

The variance of the trinomial model is $\mathbb{E}[u^{2}]-\mathbb{E}^{2}[u]$ and hence becomes

$$\begin{aligned}& (\delta y)^{2}\left(\phi^{+}+\phi^{-}\right)-\left(\phi^{+}-\phi^{-}\right)^{2}(\delta y)^{2}\\ =\ & (\delta y)^{2}\left(\phi^{+}+\phi^{-}-\left(\phi^{+}-\phi^{-}\right)^{2}\right).\end{aligned}$$

We now match the variances to get

$$(\delta y)^{2}\left(\phi^{+}+\phi^{-}-\left(\phi^{+}-\phi^{-}\right)^{2}\right) = B^{2}\delta t$$

First equation gives

$$\phi^{+} = \phi^{-} + A\tfrac{\delta t}{\delta y}$$

which upon substituting into the second equation gives

$$(\delta y)^{2}\left(\phi^{-}+\alpha+\phi^{-}-\left(\phi^{-}+\alpha-\phi^{-}\right)^{2}\right) = B^{2}\delta t$$

where $\alpha = A\frac{\delta t}{\delta y}$. This simplifies to

$$2\phi^{-}+\alpha-\alpha^{2} = B^{2}\tfrac{\delta t}{(\delta y)^{2}}$$

which rearranges to give

$$\begin{aligned}\phi^{-} &= \frac{1}{2}\left(B^{2}\tfrac{\delta t}{(\delta y)^{2}}+\alpha^{2}-\alpha\right)\\ &= \frac{1}{2}\left(B^{2}\tfrac{\delta t}{(\delta y)^{2}}+\left(A\tfrac{\delta t}{\delta y}\right)^{2}-A\tfrac{\delta t}{\delta y}\right)\\ &= \frac{1}{2}\tfrac{\delta t}{(\delta y)^{2}}\left(B^{2}+A^{2}\delta t-A\delta y\right)\end{aligned}$$

δt is small compared with δy and so

$$\phi^- = \frac{1}{2}\frac{\delta t}{(\delta y)^2}\left(B^2 - A\delta y\right).$$

Then

$$\phi^+ = \phi^- + A\frac{\delta t}{\delta y} = \frac{1}{2}\frac{\delta t}{(\delta y)^2}\left(B^2 + A\delta y\right).$$

Note

$$\left(\phi^+ + \phi^-\right)(\delta y)^2 = B^2\delta t$$

Derivation of the Fokker-Planck/Forward Kolmogorov Equation

Recall that $y',\ t'$ are futures states.

We have $p\left(y,t;y',t'\right) =$

$$\begin{aligned}
&\phi^-\left(y'+\delta y, t'-\delta t\right) p\left(y,t;y'+\delta y,t'-\delta t\right)\\
&+\left(1-\phi^-\left(y',t'-\delta t\right)-\phi^+\left(y',t'-\delta t\right)\right) p\left(y,t;y',t'-\delta t\right)\\
&+\phi^+\left(y'-\delta y, t'-\delta t\right) p\left(y,t;y'-\delta y,t'-\delta t\right)
\end{aligned}$$

Expand each of the terms in Taylor series about the point y', t' to find

$$p\left(y,t;y'+\delta y,t'-\delta t\right) = p\left(y,t;y',t'\right) + \delta y\frac{\partial p}{\partial y'} + \frac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} - \delta t\frac{\partial p}{\partial t'} + ...,$$

$$p\left(y,t;y',t'-\delta t\right) = p\left(y,t;y',t'\right) - \delta t\frac{\partial p}{\partial t'} + ...,$$

$$p\left(y,t;y'-\delta y,t'-\delta t\right) = p\left(y,t;y',t'\right) - \delta y\frac{\partial p}{\partial y'} + \frac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} - \delta t\frac{\partial p}{\partial t'} + ...,$$

$$\phi^+\left(y'-\delta y,t'-\delta t\right) = \phi^+\left(y',t'\right) - \delta y\frac{\partial \phi^+}{\partial y'} + \frac{1}{2}\delta y^2\frac{\partial^2 \phi^+}{\partial y'^2} - \delta t\frac{\partial \phi^+}{\partial t'} + ...,$$

$$\phi^+\left(y',t'-\delta t\right) = \phi^+\left(y',t'\right) - \delta t\frac{\partial \phi^+}{\partial t'} + ...,$$

$$\phi^-\left(y'+\delta y,t'-\delta t\right) = \phi^-\left(y',t'\right) + \delta y\frac{\partial \phi^-}{\partial y'} + \frac{1}{2}\delta y^2\frac{\partial^2 \phi^-}{\partial y'^2} - \delta t\frac{\partial \phi^-}{\partial t'} + ...,$$

$$\phi^-\left(y',t'-\delta t\right) = \phi^-\left(y',t'\right) - \delta t\frac{\partial \phi^-}{\partial t'} + ...,$$

Substituting in our equation for $p\left(y,t;y',t'\right)$, ignoring terms smaller than δt, noting that $\delta y \sim O\left(\sqrt{\delta t}\right)$, gives

$$\frac{\partial p}{\partial t'} = -\frac{\partial}{\partial y'}\left(\frac{1}{\delta y}\left(\phi^+ - \phi^-\right)p\right) + \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(\left(\phi^+ - \phi^-\right)p\right).$$

Noting the earlier results

$$\begin{aligned}
A &= \frac{(\delta y)^2}{\delta t}\left(\frac{1}{\delta y}\left(\phi^+ - \phi^-\right)\right),\\
B^2 &= \frac{(\delta y)^2}{\delta t}\left(\phi^+ + \phi^-\right)
\end{aligned}$$

gives the *forward equation*

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(B^2\left(y',t'\right)p\right) - \frac{\partial}{\partial y'}\left(A\left(y',t'\right)p\right)$$

The initial condition used is

$$p\left(y,t;y',t'\right) = \delta\left(y'-y\right)$$

As an example consider the important case of the distribution of stock prices. Given the random walk for equities, i.e. Geometric Brownian Motion

$$\frac{dS}{S} = \mu dt + \sigma dW.$$

So $A\left(S',t'\right) = \mu S'$ and $B\left(S',t'\right) = \sigma S'$. Hence the forward equation becomes

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2 S'^2 p\right) - \frac{\partial}{\partial S'}\left(\mu S' p\right).$$

More on this and solution technique later, but note that a transformation reduces this to the one dimensional heat equation and the *similarity reduction method* which follows is used.

The Steady-State Distribution

As the name suggests 'steady state' refers to time independent. Random walks for interest rates and volatility can be modelled with stochastic differential equations which have steady-state distributions. So in the long run, i.e. as $t' \longrightarrow \infty$ the distribution $p\left(y,t;y',t'\right)$ settles down and becomes independent of the starting state y and t. The partial derivatives in the forward equation now become ordinary ones and the unsteady term $\frac{\partial p}{\partial t'}$ vanishes.

The resulting forward equation for the steady-state distribution $p_\infty\left(y'\right)$ is governed by the ordinary differential equation

$$\frac{1}{2}\frac{d^2}{dy'^2}\left(B^2 p_\infty\right) - \frac{d}{dy'}\left(A p_\infty\right) = 0.$$

Example: The Vasicek model for the spot rate r evolves according to the stochastic differential equation

$$dr = \gamma\left(\overline{r}-r\right)dt + \sigma dW$$

Write down the Fokker-Planck equation for the transition probability density function for the interest rate r in this model.

Now using the steady-state version for the forward equation, solve this to find the **steady state** probability distribution $p_\infty\left(r'\right)$, given by

$$p_\infty = \tfrac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}\exp\left(-\tfrac{\gamma}{\sigma^2}\left(r'-\overline{r}\right)^2\right).$$

Solution:

For the SDE $dr = \gamma\left(\overline{r}-r\right)dt + \sigma dW$ where drift $= \gamma\left(\overline{r}-r\right)$ and diffusion is σ the Fokker Planck equation becomes

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial r'^2} - \gamma\frac{\partial}{\partial r'}\left(\left(\overline{r}-r'\right)p\right)$$

where $p = p\left(r',\ t'\right)$ is the transition PDF and the variables refer to future states. In the steady state case, there is no time dependency, hence the Fokker Planck PDE becomes an ODE with

$$\frac{1}{2}\sigma^2\frac{d^2 p_\infty}{dr^2} - \gamma\frac{d}{dr}\left(\left(\overline{r}-r\right)p_\infty\right) = 0$$

$p_\infty = p_\infty(r)$. The prime notation and subscript have been dropped simply for convenience at this stage. To solve the steady-state equation:

Integrate wrt r

$$\frac{1}{2}\sigma^2\frac{dp}{dr} - \gamma\left((\overline{r} - r)\,p\right) = k$$

where k is a constant of integration and can be calculated from the conditions, that as $r \to \infty$

$$\left\{\begin{array}{c} \dfrac{dp}{dr} \to 0 \\ p \to 0 \end{array}\right. \Rightarrow k = 0$$

which gives

$$\frac{1}{2}\sigma^2\frac{dp}{dr} = -\gamma\left((r - \overline{r})\,p\right),$$

a first order variable separable equation. So

$$\begin{aligned} \frac{1}{2}\sigma^2\int\frac{dp}{p} &= -\gamma\int\left((r-\overline{r})\right)dr \to \\ \frac{1}{2}\sigma^2\ln p &= -\gamma\left(\frac{r^2}{2} - \overline{r}r\right) + C, \qquad C \text{ is arbitrary.} \end{aligned}$$

Rearranging and taking exponentials of both sides to give

$$p = \exp\left(-\tfrac{2\gamma}{\sigma^2}\left(\frac{r^2}{2} - \overline{r}r\right) + D\right) = E\exp\left(-\tfrac{2\gamma}{\sigma^2}\left(\frac{r^2}{2} - \overline{r}r\right)\right)$$

Complete the square to get

$$\begin{aligned} p &= E\exp\left(-\tfrac{\gamma}{\sigma^2}\left[(r-\overline{r})^2 - \overline{r}^2\right]\right) \\ p_\infty &= A\exp\left(-\tfrac{\gamma}{\sigma^2}\left(r'-\overline{r}\right)^2\right). \end{aligned}$$

There is another way of performing the integration on the rhs. If we go back to $-\gamma\int(r-\overline{r})\,dr$ and write as

$$-\gamma\int\tfrac{1}{2}\tfrac{d}{dr}\left(r-\overline{r}\right)^2 dr = \frac{-\gamma}{2}\left(r-\overline{r}\right)^2$$

to give

$$\frac{1}{2}\sigma^2\ln p = \frac{-\gamma}{2}\left(r-\overline{r}\right)^2 + C.$$

Now we know as p_∞ is a PDF

$$\begin{aligned} \int_{-\infty}^{\infty} p_\infty\, dr' &= 1 \to \\ A\int_{-\infty}^{\infty}\exp -\left(\tfrac{\gamma}{\sigma^2}\left(r'-\overline{r}\right)^2\right)\, dr' &= 1 \end{aligned}$$

A few (related) ways to calculate A. Now use the error function, i.e.

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

So put

$$x = \sqrt{\tfrac{\gamma}{\sigma^2}}\left(r'-\overline{r}\right) \to dx = \sqrt{\tfrac{\gamma}{\sigma^2}}dr'$$

which transforms the integral above

$$\frac{A\sigma}{\sqrt{\gamma}}\int_{-\infty}^{\infty} e^{-x^2}\ dx = 1 \rightarrow A\sigma\sqrt{\frac{\pi}{\gamma}} = 1$$

therefore

$$A = \tfrac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}.$$

This allows us to finally write the steady-state transition PDF as

$$p_\infty = \tfrac{1}{\sigma}\sqrt{\frac{\gamma}{\pi}}\exp\left(-\tfrac{\gamma}{\sigma^2}\left(r' - \overline{r}\right)^2\right).$$

The *backward equation* is obtained in a similar way to the forward

$p\left(y, t; y', t'\right) =$

$$\begin{aligned}&\phi^+\left(y,t\right)p\left(y+\delta y, t+\delta t; y', t'\right)\\&+\left(1-\phi^-\left(y,t\right)-\phi^+\left(y,t\right)\right)p\left(y, t+\delta t; y', t'\right)\\&+\phi^-\left(y,t\right)p\left(y-\delta y, t+\delta t; y', t'\right)\end{aligned}$$

and expand using Taylor. The resulting PDE is

$$\frac{\partial p}{\partial t} + \frac{1}{2}B^2\left(y,t\right)\frac{\partial^2 p}{\partial y^2} + A\left(y,t\right)\frac{\partial p}{\partial y} = 0.$$

So the forward equation can be obtained from the backward equation using the transformation $t' = T - t$,

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 S^2\frac{\partial^2 p}{\partial S^2} + \mu S\frac{\partial p}{\partial S}.$$

Write $p = p\left(S', t\right)$ as $p = p\left(\xi, t\right)$ where $\xi = \log S'$

$$\frac{\partial p}{\partial S'} = \frac{1}{S'}\frac{\partial p}{\partial \xi};\ \frac{\partial^2 p}{\partial S'^2} = \frac{1}{S'^2}\left(\frac{\partial^2 p}{\partial \xi^2} - \frac{\partial p}{\partial \xi}\right).$$

To solve, reduce to a 1D heat equation initially.

This can be solved with a starting condition of $S' = S$ at $t' = t$ to give the transition pdf

$$p\left(S, t; S', T\right) = \frac{1}{\sigma S'\sqrt{2\pi(t'-t)}}e^{-\left(\log(S/S')+\left(\mu-\frac{1}{2}\sigma^2\right)(t'-t)\right)^2/2\sigma^2(t'-t)}.$$