Solutions

1. Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad t > 0, \quad 0 < x < L \tag{1.1}$$

where the unknown function $u = u(x,t)$; c^2 is a constant. To discretize the equation, take N and M steps for x and t respectively, so

$$\begin{aligned} x &= n\delta x \qquad 0 \le n \le N \\ t &= m\delta t \qquad 0 \le m \le M, \end{aligned}$$

where $\delta x = \frac{L}{N}$; $\delta t = \frac{T}{M}$. By using the following approximations

$$\begin{aligned} \frac{\partial u}{\partial t}(n\delta x, m\delta t) &\sim \frac{u_n^{m+1} - u_n^m}{\delta t}, \\ \frac{\partial^2 u}{\partial x^2}(n\delta x, m\delta t) &\sim \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{\delta x^2} \end{aligned}$$

and writing $r = c^2 \dfrac{\delta t}{\delta x^2}$, derive the following **forward marching scheme** for (1.1)

$$u_n^{m+1} = Au_{n-1}^m + Bu_n^m + Cu_{n+1}^m, \tag{1.2}$$

where A, B, C should be stated.
Assume an initial disturbance E_n^m given by

$$E_n^m = \overline{a}^m e^{in\omega}, \tag{1.3}$$

which is oscillatory of amplitude $\overline{a}$ and frequency ω; $i = \sqrt{-1}$. By substituting (1.3) into (1.2), obtain a stability condition for this scheme.
SOLUTION Start by substituting the given derivative approximations into (1.1)

$$\frac{u_n^{m+1} - u_n^m}{\delta t} = c^2 \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{\delta x^2}$$

Now rearrange this

$$\begin{aligned} u_n^{m+1} &= \frac{\delta t}{\delta x^2} c^2 \left(u_{n-1}^m - 2u_n^m + u_{n+1}^m\right) + u_n^m \\ &= ru_{n-1}^m - 2ru_n^m + ru_{n+1}^m + u_n^m \\ &= ru_{n-1}^m + (1-2r)\, u_n^m + ru_{n+1}^m \end{aligned}$$

Now substitute in the above $E_n^m = \overline{a}^m e^{in\omega}$

$$\begin{aligned} \overline{a}^{m+1} e^{in\omega} &= r\overline{a}^m e^{i(n-1)\omega} + (1-2r)\, \overline{a}^m e^{in\omega} + r\overline{a}^m e^{i(n+1)\omega} \\ \overline{a} &= re^{-i\omega} + (1-2r) + re^{i\omega} = 2r\left(\frac{e^{i\omega} + e^{-i\omega}}{2}\right) + 1 - 2r \\ &= 2r\cos\omega - 2r + 1 = 2r(\cos\omega - 1) + 1 = 1 - 2r(1 - \cos\omega) \\ &= 1 - 2r \times 2\sin^2\frac{\omega}{2} = 1 - 4r\sin^2\frac{\omega}{2} \end{aligned}$$

For stability we require

$$\left|1 - 4r\sin^2\frac{\omega}{2}\right| \le 1$$

We know $\left|\sin^2\dfrac{\omega}{2}\right| \le 1$, hence solve $|1-4r| \le 1$

$$\begin{aligned} -1 &\le 1 - 4r \le 1 \to -2 \le -4r \le 0 \\ 4r &\le 2 : r \le \frac{1}{2} \end{aligned}$$

i.e. $c^2 \dfrac{\delta t}{\delta x^2} \le \frac{1}{2}$ or $\delta t \le \frac{1}{2c^2}\delta x^2$.

2. Consider the pricing equation for the value of a derivative security $V(S,t)$,

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-D)\,S\frac{\partial V}{\partial S}-rV=0, \tag{2.1}$$

where $S\geq 0$ is the spot price of the underlying equity, $0<t\leq T$ is the time, $r\geq 0$ the constant rate of interest, and σ is the constant volatility of S. The variables (t,S) can be written as

$$t=m\delta t \quad 0\leq m\leq M; \quad S=n\delta S \qquad\qquad 0\leq n\leq N,$$

where $(\delta t,\delta S)$ are fixed step sizes in turn. $V(S,t)$ is written discretely as V_n^m. An Explicit Finite Difference Method is to be developed to solve (2.1) using a *backward marching scheme*. Derive a difference equation in the form

$$V_n^{m-1}=a_nV_{n-1}^m+b_nV_n^m+c_nV_{n+1}^m$$

where a_n, b_n, c_n should be defined; you may use the following as a starting point,

$$\begin{aligned}\frac{\partial V}{\partial t} &\sim \frac{V_n^m-V_n^{m-1}}{\delta t};\ \frac{\partial V}{\partial S}\sim\frac{V_{n+1}^m-V_{n-1}^m}{2\delta S};\\ \frac{\partial^2 V}{\partial S^2} &\sim \frac{V_{n-1}^m-2V_n^m+V_{n+1}^m}{\delta S^2}.\end{aligned}$$

SOLUTION Substituting the above in (2.1) yields the following

$$\begin{aligned}V_n^{m-1} &= V_n^m+\frac{1}{2}\sigma^2n^2\delta t\left(V_{n-1}^m-2V_n^m+V_{n+1}^m\right)+\\ &\quad \frac{(r-D)\,n\delta t}{2}\left(V_{n+1}^m-V_{n-1}^m\right)-r\delta tV^m\end{aligned}$$

Arranging so that we get a difference equation in the form $V_n^{m-1}=a_nV_{n-1}^m+b_nV_n^m+c_nV_{n+1}^m$ with

$$\begin{aligned}a_n &= \frac{1}{2}\left(n^2\sigma^2-n\,(r-D)\right)\delta t,\\ b_n &= 1-\left(r+n^2\sigma^2\right)\delta t,\\ c_n &= \frac{1}{2}\left(n^2\sigma^2+n\,(r-D)\right)\delta t.\end{aligned}$$

2. A binary call option is to be priced. Outline a Monte Carlo method to do this.
SOLUTION Simulate sample paths for the underlying stock over the relevant time horizon, according to the risk-neutral measure. Here we use the discretized SDE

$$S_{i+1}=S_i\left(1+r\delta t+\sigma\phi\sqrt{\delta t}\right).$$

Evaluate the discounted cashflows (using domestic rate of interest) of a derivative on each sample path, as determined by the structure of the security being priced. For a binary call this becomes

$$e^{(-r(T-t))}\mathcal{H}\left(S-E\right)$$

where

$$\mathcal{H}\left(S\left(T\right)-E\right)=\begin{cases}1 & S\left(T\right)>E\\ 0 & \text{otherwise}\end{cases}$$

Average the discounted cashflows over sample paths. So the option price becomes

$$e^{(-r(T-t))}\frac{1}{N}\sum_{n=1}^{N}\mathcal{H}\left(S\left(T\right)-E\right)$$