

Simulating and Manipulating Stochastic Differential Equations

In this lecture...

- Using Itô's lemma to manipulate stochastic differential equations
- Continuous-time stochastic differential equations as discrete-time processes
- Simple ways of generating random numbers in Excel
- Correlated random walks

By the end of this lecture you will be able to

- manipulate stochastic differential equations
- find transition probability density functions for arbitrary stochastic differential equations
- simulate stochastic differential equations

Introduction

In order to become comfortable with the kind of models commonly used in quantitative finance you must be able to manipulate stochastic differential equations and generate random walks numerically.

$$\{X_t : t \in \mathbb{R}^+\}$$

$$G := S$$

$$a := \mu$$

$$b := \sigma S$$

Manipulating stochastic differential equations

G_t stoch. process $t \rightarrow t+dt$
An equation of the form

SDE for G_t
RW for dG_t

$$dG_t = \underbrace{a(G_t, t) dt}_{\text{deterministic}} + \underbrace{b(G_t, t) dX_t}_{\text{random}}$$

drift diffusion

$G_0 = g \in \mathbb{R}$
I.C

is called a Stochastic Differential Equation (SDE) for G (or random walk for dG) and consists of two components:

1. $a(G, t) dt$ is deterministic – coefficient of dt is known as the **drift** or **growth**
2. $b(G, t) dX$ is random – coefficient of dX is known as the **diffusion** or **volatility**

and we say G evolves according to (or follows) this process. SDE/differential = 1

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Recall dG_t can be written as

$$G_t = G_0 + \int_0^t a(G_s, s) ds + \int_0^t b(G_s, s) dX_s$$

$$\begin{aligned} \mathbb{E}[dG_t] &= \mathbb{E}[a dt] + \mathbb{E}[b dX] \\ &= a dt + b \mathbb{E}[dX] = a dt \end{aligned}$$

$$\begin{aligned} \mathbb{V}[dG_t] &= \mathbb{V}[a dt] + b^2 \mathbb{V}[dX] \\ &= 0 + b^2 dt = b^2 dt \end{aligned}$$

So if for example we have a random walk

$$dS = \mu S dt + \sigma S dX \quad (1)$$

$$\frac{dS}{S} = \mu dt + \sigma dX$$

then the drift is $a(S, t) = \mu S$ and the diffusion is $b(S, t) = \sigma S$.

The process (1) is also called **Geometric Brownian Motion** (GMB) or **Exponential Brownian motion** (EMB) and is a popular model for a wide class of asset prices.

We have previously considered Itô's lemma to obtain the change in a function $f(X)$ when $X \rightarrow X + dX$, where X is a standard Brownian motion.

This jump $df = f(X + dX) - f(X)$ is given by

$$df = \frac{df}{dX}dX + \frac{1}{2} \frac{d^2f}{dX^2}dt \quad (2)$$

using the result

$$\lim_{dt \rightarrow 0} dX^2 = dt.$$

Thus for Itô I, Itô II

Suppose we now wish to extend the result (2) to consider the change in an option price $V(S)$ where the underlying variable S follows a geometric Brownian motion.

(Of course, you are not supposed to know anything about options yet. Just think of manipulating functions.)

$$V = V(S) \quad S \text{ stock price}$$
$$S \rightarrow \text{a GBM}$$

If we rewrite (1) as

$$\frac{dS}{S} = \mu dt + \sigma dX$$

then dS represents the change in asset price S in a small time interval dt .

This expression is the return on the asset.

μ is the average growth rate of the asset and σ the associated volatility (standard deviation) of the returns.

dX is an increment of a Brownian Motion, known as a Wiener process and is a Normally distributed random variable such that $dX \sim N(0, dt)$.

$$V = V(S) \quad S \rightarrow S + dS$$

✓ An obvious question we may ask is, what is the jump in $V(S + dS)$ when $S \rightarrow S + dS$? GBM

We begin (again) by using a Taylor series as in (2), but for $V(S + dS)$ to get

$$V(S + dS) = V(S) + \frac{dV}{dS} dS + \frac{1}{2} \frac{d^2V}{dS^2} dS^2$$

$$\boxed{dV} = \frac{dV}{dS} \underline{dS} + \frac{1}{2} \frac{d^2V}{dS^2} \underline{dS^2}.$$

It's mult + St

Multiply	dt	dX
dt	0	0 ($dt^{3/2}$)
dX	0	dt

$$dS^2 = \cancel{\mu^2 S^2 dt^2} + 2 \cancel{\mu \sigma S^2 dt dX} + \boxed{\sigma^2 S^2 dt}$$

$\mu S dt + \sigma S dX$

$$dS^2 = \sigma^2 S^2 dt$$

$$F(S_t) \quad S_t \neq X_t \quad F(X_t, t)$$

We can proceed further now as we have an expression for dS (and hence dS^2). As dt is very small, any terms in $dt^{\frac{3}{2}}$ or dt^2 are insignificant in comparison and can be ignored. So working to $O(dt)$

$$dS^2 = \sigma^2 S^2 dt.$$

If G_t satisfies $dG_t = a dt + b dX \Rightarrow dG_t^2 = b^2 dt$

If we substitute this into the previous expression for dV we get Itô's lemma as applied to $V(S)$:

$$\text{Hö III} \quad dV = \underbrace{\left(\mu S \frac{dV}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 V}{dS^2} \right)}_{\text{drift}} dt + \underbrace{\left(\sigma S \frac{dV}{dS} \right)}_{\text{diffusion}} dX. \quad (3)$$

Note that this is another stochastic differential equation!

It contains a predictable part and a random part.

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Very Important Finance Example:

Suppose that we had a formula for $V(S)$. Let's take a very special case, let's consider

Special function $V(S) = \log S$.
log of the stock price

Differentiating this once gives

$$\frac{dV}{dS} = \frac{1}{S}.$$

We do It's III on $V(S)$

$$dS = \mu S dt + \sigma S dX$$

$\mu, \sigma \in \mathbb{R}$

Differentiating this again gives

(Euler's $\hat{=}$)

$$\frac{d^2V}{dS^2} = -\frac{1}{S^2}.$$

Now sub in It's III

$$dV = d(\log S) = \left[\cancel{\mu S} \times \frac{1}{\cancel{S}} + \frac{1}{2} \sigma^2 \cancel{S^2} \left(-\frac{1}{\cancel{S^2}} \right) \right] dt + \sigma \cancel{S} \left(\frac{1}{\cancel{S}} \right) dX = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dX$$

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Now from (3) we have *We are after S*

$$d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dX.$$

Integrating both sides between 0 and *now* t *some time in future*

$$\int_0^t d(\log S) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) d\tau + \int_0^t \sigma dX \quad (t > 0)$$

$$\log \frac{S(t)}{S(0)} = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma (X(t) - \cancel{X(0)})$$

Take exp of both sides and rearrange

$$\int_0^t dX^2 = dt$$

Ⓔ general stock price,

Therefore

S - stock price

X (Paul)

$$S_t = S_0 e^{\mu t}$$

$$\log \left(\frac{S(t)}{S(0)} \right) = \underbrace{\left(\mu - \frac{1}{2} \sigma^2 \right)}_{\text{drift}} \underbrace{t}_{\text{time}} + \underbrace{\sigma}_{\text{volatility}} \underbrace{(X(t) - X(0))}_{\sqrt{t}}$$

$$\int_0^t d\tilde{X}_t$$

Assuming $X(0) = 0$ and $S(0) = S_0$, the exact solution becomes

$$\phi \sim N(0, 1)$$

$$S(t) = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma X(t) \right). \quad (4)$$

$$\int_0^t \uparrow: S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma \phi \sqrt{t}}$$

$$\int_t^T \downarrow: S_T = S_t e^{(\mu - \frac{1}{2} \sigma^2)(T-t) + \sigma \phi \sqrt{T-t}}$$

$$\int_t^{t+dt} \downarrow: S_{t+dt} = S_t e^{(\mu - \frac{1}{2} \sigma^2)dt + \sigma \phi \sqrt{dt}}$$

(4) very important result

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1 to IV

$$\frac{dS}{S} = \mu dt + \sigma dX$$

Now consider $V = V(t, S)$

$$\begin{aligned} t &\rightarrow t+dt \\ S &\rightarrow S+dS \end{aligned}$$

2D TSE $V(t+dt, S+dS) =$

$$V(S, t) + \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 \rightarrow \sigma^2 S^2 dt$$

$$dV = V(S+dS, t+dt) - V(S, t) = \underbrace{\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right)}_{\text{drift}} dt + \underbrace{\sigma S \frac{\partial V}{\partial S}}_{\text{diffusion}} dX$$

HS IV

Let us generalise. G_t satisfies $dG_t = A(t, G_t)dt + B(t, G_t)dX$

Consider $F = F(t, G_t)$

$$dF = \left(\frac{\partial F}{\partial t} + A(t, G_t) \frac{\partial F}{\partial G_t} + \frac{1}{2} B^2(t, G_t) \frac{\partial^2 F}{\partial G_t^2} \right) dt + B(t, G_t) \frac{\partial F}{\partial G_t} dX$$

1 to V

$$dS = \mu S_t dt + \sigma_t J_t dX_t^{(1)}$$

$$d\sigma = \dots$$

Another example:

Let's take a look at the Vasicek interest rate model for short-term interest rates, and try manipulating that.

$$dr = (\eta - \gamma r)dt + \sigma dX$$

$$dr = \gamma (\bar{r} - r) dt + \sigma dX.$$

γ refers to the **reversion rate** and \bar{r} denotes the **mean rate**.

$$dr_t = -\gamma(r_t - \bar{r})dt + \sigma dX_t$$

Consider a transform

By setting $\bar{u} = r - \bar{r}$, u is a solution of

$u(0) = I.C.$

$$du = -\gamma u dt + \sigma dX.$$

$$du = d(r - \bar{r})$$

but $\bar{r} \in \mathbb{R}$

People love Vasicek
 $\leftarrow \therefore$ it is an Ornstein-Uhlenbeck

An analytic solution for this equation exists. To see, this write the equation as

mult. both sides by an integrating factor $e^{\gamma t}$

$$e^{\gamma t} (du + \gamma u dt) = \sigma e^{\gamma t} dX$$

exact differential

$$d(u e^{\gamma t}) = \sigma e^{\gamma t} dX.$$

$$e^{\gamma t} du + u d(e^{\gamma t}) = \gamma e^{\gamma t} dt$$

$$\frac{d}{dt}(e^{\gamma t}) = \gamma e^{\gamma t}$$

$$de^{\gamma t} = \gamma e^{\gamma t} dt$$

stop at this point.

Integrating over from zero to t gives

$$\int_0^t d(u(s) e^{\gamma s}) = \sigma \int_0^t e^{\gamma s} dX_s$$

$$u(s) e^{\gamma s} \Big|_0^t = u(t) e^{\gamma t} - u(0)$$

$$u(t) = u(0) e^{-\gamma t} + \sigma \int_0^t e^{\gamma(s-t)} dX_s.$$

This can be **integrated by parts** to give

$$u(t) = u(0) e^{-\gamma t} + \sigma \left(X(t) - \gamma \int_0^t X(s) e^{\gamma(s-t)} ds \right).$$

or apply last week's stock integ. formula

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Transition probability density functions again

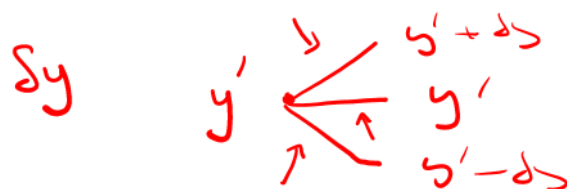
Let's look at the equations governing the probability distribution for an arbitrary random walk:

$$\mathbb{E}[dy] = A dt$$

$$\mathbb{V}[dy] = B^2 dt$$

for the variable y .

$$dy = A(y, t) dt + B(y, t) dX$$



cts time Random

walk:
 f^+ f^- $1 - f^+ - f^-$

Remember the **transition probability density function** $p(y, t; y', t')$ defined by

$$\text{Prob}(a < y' < b \text{ at time } t' | y \text{ at time } t) = \int_a^b p(y, t; y', t') dy'.$$

In words this is 'the probability that the random variable y lies between a and b at time t' in the future, given that it started out with value y at time t .'

Think of y and t as being current values with y' and t' being future values.

The transition probability density function can be used to answer questions such as

“What is the probability of the variable y being in a certain range at time t' given that it started out with value y at time t ?”

The transition probability density function $p(y, t; y', t')$ satisfies two equations.

One involves derivatives with respect to the future state and time (y' and t') and is called the **forward equation**.

The other involves derivatives with respect to the current state and time (y and t) and is called the **backward equation**.

These can be derived by the same trinomial idea we used before (but the details are a lot messier for the general stochastic differential equation).

The forward equation

R.V. has both ^A drift and diffusion ^B

Cutting to the chase, the transition probability density function satisfies the partial differential equation

$$\frac{\partial p}{\partial t'} = \frac{1}{2} \frac{\partial^2}{\partial y'^2} (B(y', t')^2 p) - \frac{\partial}{\partial y'} (A(y', t') p)$$

This is the **Fokker–Planck** or **forward Kolmogorov equation**.

Example: The most important example to us is that of the distribution of equity prices in the future. If we have the random walk

$$\xrightarrow{A(S', t') \quad B(S', t')} dS = \mu S dt + \sigma S dX$$

then the forward equation becomes

To solve

① Introduce $\xi = \log S$

② Eliminate the drift

FKE:

$$\frac{\partial p}{\partial t'} = \frac{1}{2} \frac{\partial^2}{\partial S'^2} (\sigma^2 S'^2 p) - \frac{\partial}{\partial S'} (\mu S' p)$$

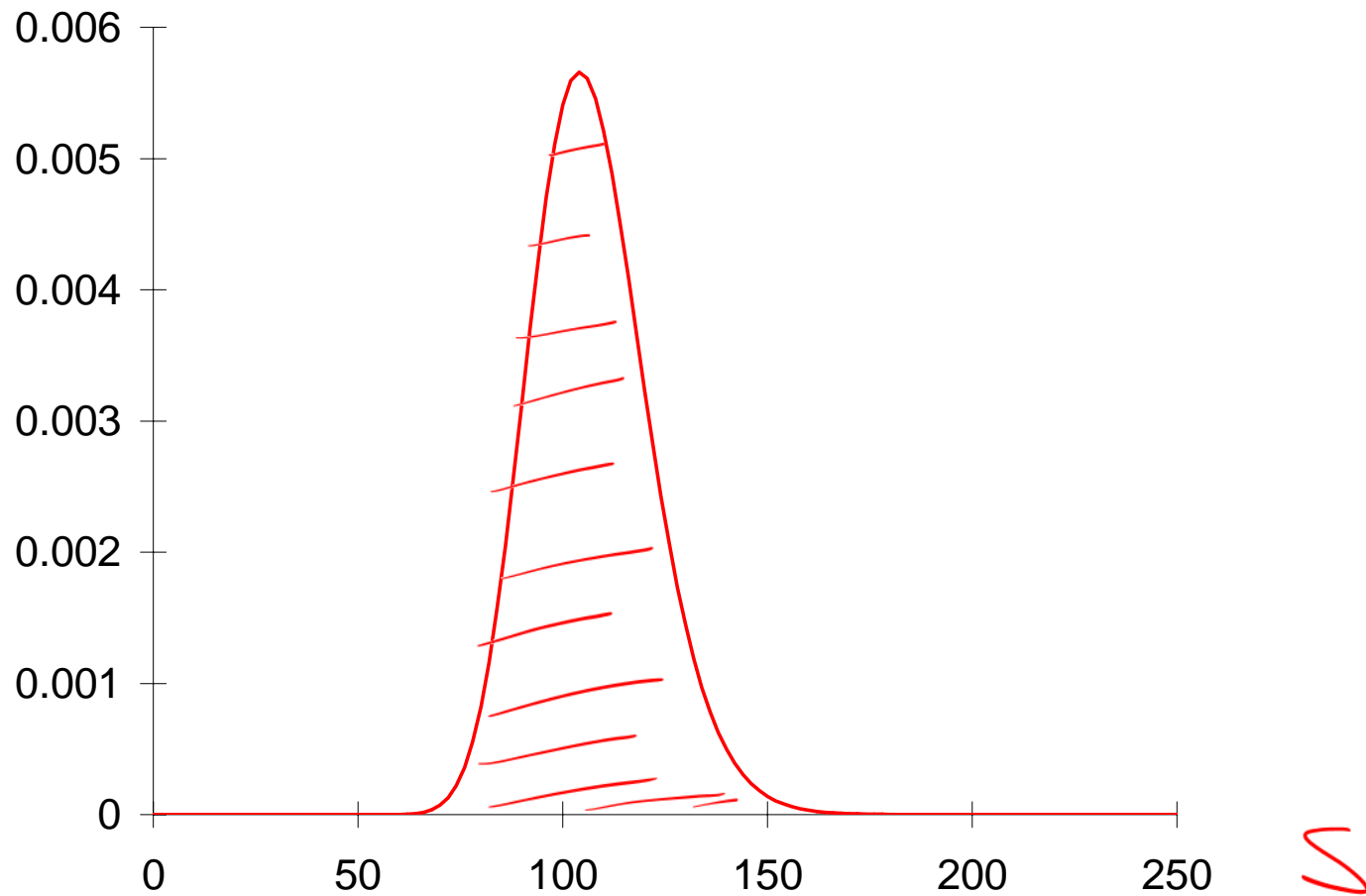
Reduce to 1D diffusion eqⁿ

The solution of this representing a stock price starting at $S' = S$ at $t' = t$ is

In mod. 2.1 (B.S. Model) we see ① full solⁿ

$$p(S, t; S', t') = \frac{1}{\sigma S' \sqrt{2\pi(t' - t)}} e^{-\left(\underbrace{\log(S/S')}_{\text{transition pdf}} + \underbrace{\left(\mu - \frac{1}{2}\sigma^2\right)(t' - t)}_{\text{for a stock price}}\right)^2 / 2\sigma^2(t' - t)}$$

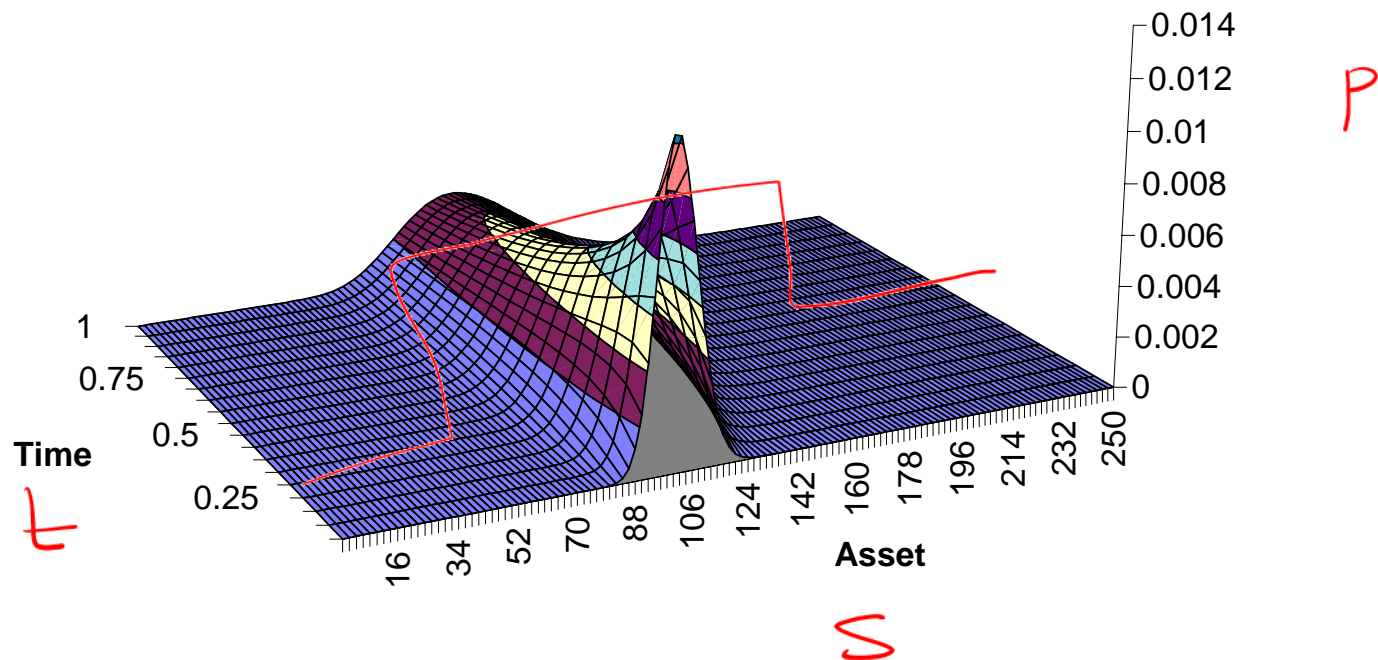
$P(S)$



The probability density function for the lognormal random walk, after a certain time.

at

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The probability density function for the lognormal random walk evolving through time.

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S.S.d

The steady-state distribution

Some random walks have a steady-state distribution.

That is, in the long run as $(t' \rightarrow \infty)$ the distribution $p(y, t; y', t')$ as a function of y' settles down to be independent of the starting state y and time t . Possible examples are stochastic differential equation models for interest rates, inflation, volatility.

Some random walks have no such steady state even though they have a time-independent equation. For example the lognormal random walk either grows without bound or decays to zero.

time indep. behaviour p no longer
dependent on
time t .

If there is a steady-state distribution $p_\infty(y')$ then it satisfies the ordinary differential equation

s.s.d (i.e. $t' \rightarrow \infty$)

FKE steady state dist.

$$\frac{1}{2} \frac{d^2}{dy'^2} (B^2 p_\infty) - \frac{d}{dy'} (A p_\infty) = 0.$$

subset of original P.P.E

Example: The Vasicek model

A

B

$$dr = \gamma (\bar{r} - r) dt + \sigma dX.$$

The steady-state distribution $p_\infty(r')$ satisfies

$$\frac{1}{2} \sigma^2 \frac{d^2 p_\infty}{dr'^2} + \gamma \frac{d}{dr'} ((\bar{r} - r') p_\infty) = 0.$$

$$\frac{1}{2} \sigma^2 p'' + \gamma \frac{d}{dr} ((r - \bar{r}) p) = 0$$

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$$\frac{1}{2} \sigma^2 \frac{d^2 p}{dr^2} = -\gamma \frac{d}{dr} ((r-\bar{r})p) \quad \text{Integrate once}$$

$$\frac{1}{2} \sigma^2 \frac{dp}{dr} = -\gamma (r-\bar{r})p + \text{const.} \quad \text{const} = 0 \quad \because \begin{cases} r \rightarrow \infty \\ p \rightarrow 0 \\ \frac{dp}{dr} \rightarrow 0 \end{cases}$$

$$\frac{1}{2} \sigma^2 \frac{dp}{dr} = -\gamma (r-\bar{r})p \quad \text{var. sep eqn}$$

$$\frac{dp}{dr} = -\frac{2\gamma}{\sigma^2} (r-\bar{r})p \quad \frac{dp}{p} = -\frac{2\gamma}{\sigma^2} (r-\bar{r}) dr$$

$$\int \frac{dp}{p} = -\frac{2\gamma}{\sigma^2} \int (r-\bar{r}) dr$$

$$\log p = -\frac{2\gamma}{\sigma^2} \int \frac{1}{2} \frac{d}{dr} (r-\bar{r})^2 dr = -\frac{\gamma}{\sigma^2} (r-\bar{r})^2 + \text{const.}$$

Now take exp. of both sides, $p(r) = A e^{-\frac{\gamma}{\sigma^2} (r-\bar{r})^2} \quad A = ?$

We know $\int_{\mathbb{R}} p(r) dr = 1$

$$A \int_{-\infty}^{\infty} e^{-\frac{\gamma}{\sigma^2} (r-\bar{r})^2} dr \quad x = \frac{\sqrt{\gamma}}{\sigma} (r-\bar{r})$$

$$\frac{\sigma}{\sqrt{\gamma}} dr = dx$$

$$A \underbrace{\frac{\sigma}{\sqrt{\gamma}} \int_{\mathbb{R}} e^{-x^2} dx}_{\sqrt{\pi}} = 1$$

$$A \sigma \sqrt{\frac{\pi}{\gamma}} = 1 \quad \therefore A = \frac{1}{\sigma} \sqrt{\frac{\gamma}{\pi}}$$

$$\therefore p(r) = \frac{1}{\sigma} \sqrt{\frac{\gamma}{\pi}} e^{-\frac{\gamma}{\sigma^2} (r-\bar{r})^2}$$

$$-\frac{\gamma}{\sigma^2} (r' - \bar{r})^2 \quad -\frac{1}{2} \frac{(r' - \bar{r})^2}{\underbrace{\sigma^2/2\gamma}_{\text{Variance of } r'}} \quad \bar{r} - \text{mean.}$$

The solution is

$$p_{\infty} = \frac{1}{\sigma} \sqrt{\frac{\gamma}{\pi}} e^{-\frac{\gamma(\bar{r}-r')^2}{\sigma^2}}.$$

In other words, the interest rate r is Normally distributed with mean \bar{r} and standard deviation $\sigma/\sqrt{2\gamma}$.

The backward equation

Now we come to the backward equation. This will be useful if we want to calculate probabilities of reaching a specified final state from various initial states.

The transition probability density function satisfies the **backward Kolmogorov equation**

$$\frac{\partial p}{\partial t} + \frac{1}{2}B(y, t)\frac{\partial^2 p}{\partial y^2} + A(y, t)\frac{\partial p}{\partial y} = 0.$$

Simulating the lognormal random walk

The lognormal random walk model for assets can be written in continuous time as

$$\phi \sim N(0,1)$$

cts. time

$$\rightarrow dS = \mu S dt + \sigma S dX.$$

discrete time

$$\delta S = \mu S \delta t + \sigma S \phi \sqrt{\delta t}$$

In discrete time this is

$$\begin{aligned} \delta S &= S(t+\delta t) - S(t) \\ &= S_{t+1} - S_t = S_{i+1} - S_i \end{aligned}$$

$$\rightarrow S_{i+1} - S_i = S_i (\mu \delta t + \sigma \phi \delta t^{1/2}).$$

We are at $t_i \Rightarrow$ we know S_i . What is S_{i+1} ?

To generate representative simulations of possible asset paths we must obviously work in discrete time.

The random walk on a spreadsheet

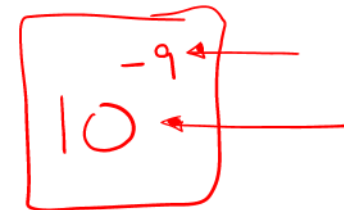
The random walk can be written as a 'recipe' for generating S_{i+1} from S_i :

$$S_{i+1} = S_i \left(1 + \underline{\underline{\mu}} \delta t + \underline{\underline{\sigma}} \phi \delta t^{1/2} \right).$$

$\delta t = \frac{1}{252}$

We can easily simulate the model using a spreadsheet.

The method is called the **Euler method**.



Start with an initial stock price, say, 100. S_0

And a couple of parameters, $\mu = 0.1$ and $\sigma = 0.2$, say, that best represent the asset in question. 10% 20%

Decide on a (small) time step, $\delta t = 0.01$, say. 10^{-2}

Now start picking random numbers! $\phi \sim N(0, 1)$

$$S_0 = 100$$

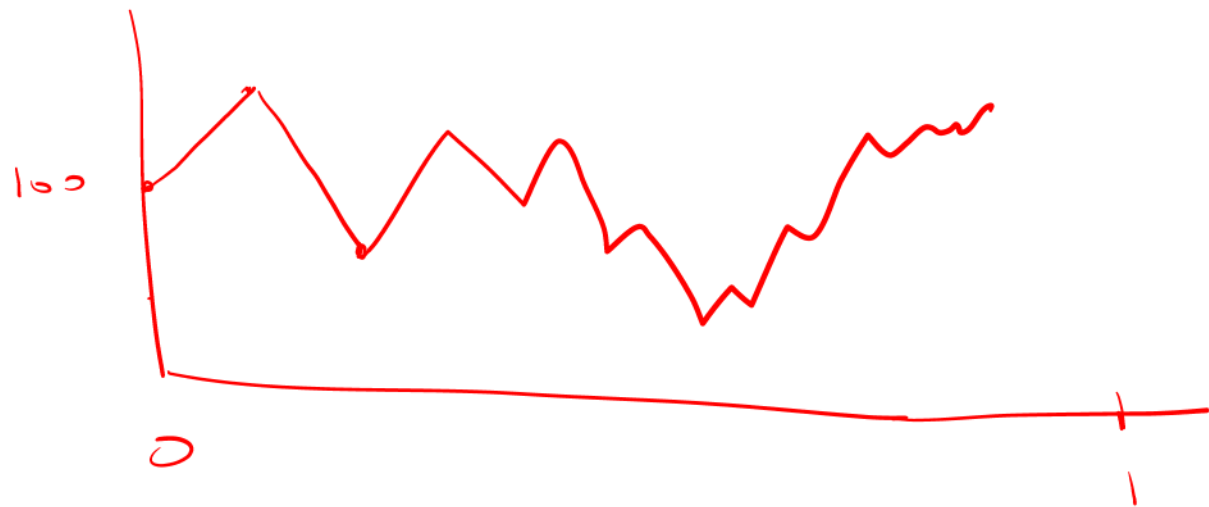
First time step: The random number is... 0.12. So

Put in S_0 $S_{i+1} = 100 (1 + 0.1 \times 0.01 + 0.2 \times 0.1 \times 0.12) = 100.34 = S_1$

Second time step: The random number is... -0.25. So

Put in S_1 $S_{i+1} = 100.34 (1 + 0.1 \times 0.01 + 0.2 \times 0.1 \times (-0.25)) = 99.94 = S_2$

And so on.



In this simulation there are several input parameters, which remain constant:

- a starting value for the asset
- a time step δt
- the drift rate μ
- the volatility σ
- the total number of time steps

Then, at each time step, we must choose a random number ϕ from a Normal distribution.

This can be done easily in Excel in several ways, we will see a couple now.

Slow but accurate

The Excel spreadsheet function RAND() gives a uniformly-distributed random variable.

$$Y \sim U(0,1) \quad \mathbb{E}[Y] = \frac{1}{2} \quad \text{var}[Y] = \frac{1}{12}$$

This can be used, together with the inverse cumulative distribution function NORMSINV to give a genuinely Normally distributed number:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}s^2} ds$$

$= P(X \leq x)$

$N(0,1)$

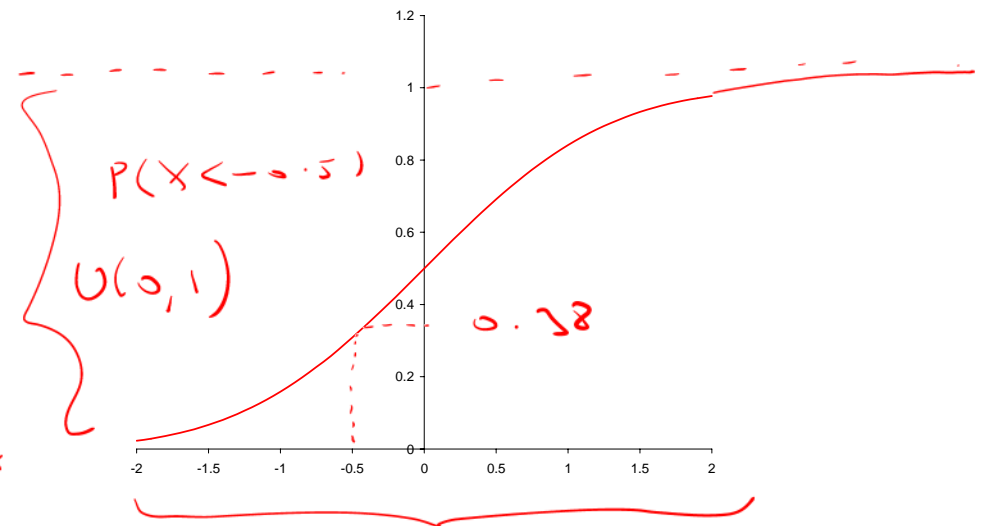
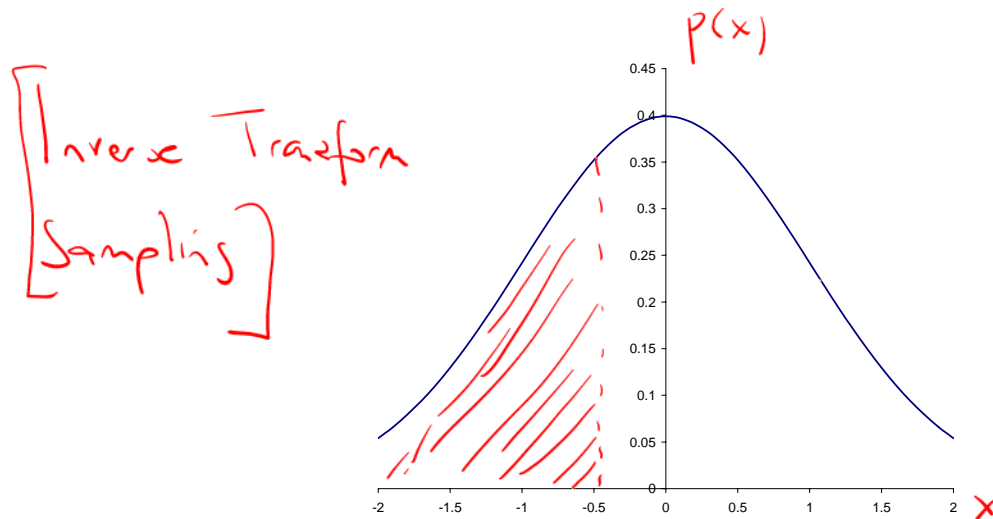
$\text{NORMSINV}(\text{RAND}()) \sim \phi$

$N^{-1}(U(0,1))$

Why does this work?

The pdf and cdf for the Normal distribution

CDF



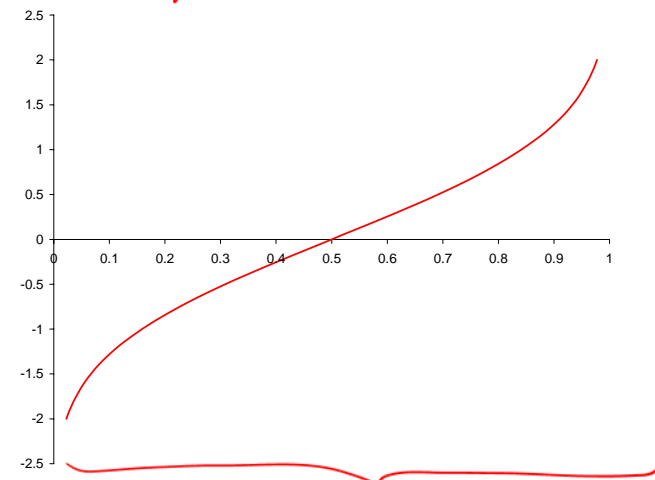
The inverse cumulative distribution function

$N(0,1)$

$N^{-1}(\dots)$

CDF $F(x)$

$F^{-1}(U(0,1)) \rightarrow \text{R.V. from dist. } F(x)$



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$U(0,1)$

RAND(),

Fast but inaccurate

where accuracy can be improved

An approximation to a Normal variable that is fast in a spreadsheet, and quite accurate, is simply to add up twelve random variables drawn from a uniform distribution over zero to one, and subtract six:

- $$\left(\sum_{i=1}^{12} \text{RAND}() \right) - 6. \sim N(0,1)$$

Why 12?

Any 'large' number will do. The larger the number, the closer the end result will be to being normal, but the slower it is.

Why subtract off 6?

The random number must have a mean of zero.

And the standard deviation?

Must be 1.

n - any arb. large number. Sum n lots of RAND()

$\sum_{i=1}^n \text{RAND()}$. Examine its mean.

$$E\left[\sum_{i=1}^n \text{RAND()}\right] = \sum_{i=1}^n E[\text{RAND()}] = n \times \frac{1}{2} \neq 0 \therefore \text{subtract off } \frac{n}{2}$$

• $\sum_{i=1}^n \text{RAND()} - \frac{n}{2}$ Examine its variance

$$V\left[\sum_{i=1}^n \text{RAND()} - \frac{n}{2}\right] = \sum_{i=1}^n V[\text{RAND()}] = n \times \frac{1}{12} \neq 1$$

Consider a normalising constant $V\left[A\left[\sum_{i=1}^n \text{RAND()} - \frac{n}{2}\right]\right] = 1$

$$A^2 V\left[\sum_{i=1}^n \text{RAND()} - \frac{n}{2}\right] = 1$$

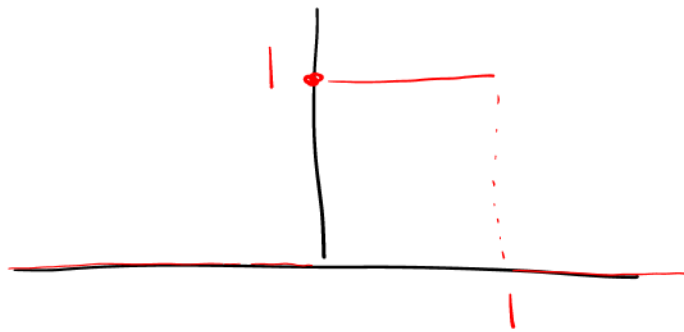
$$A^2 = \frac{12}{n} \therefore A = \sqrt{\frac{12}{n}}$$

Finally we update: $\sqrt{\frac{12}{n}} \left[\sum_{i=1}^n \text{RAND()} - \frac{n}{2} \right]$ Y_i iid

$$\frac{\sum_{i=1}^n \text{RAND()} - n \times \frac{1}{2}}{\sqrt{\frac{1}{12}} \sqrt{n}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n Y_i - n\mu}{\sigma \sqrt{n}}$$

$U(0,1)$ The pdf is



$$p(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y \sim U(0,1)$$

$$E[Y] = \int_{-\infty}^{\infty} y p(y) dy$$

$$= \int_{-\infty}^0 y \times 0 dy + \int_0^1 y \times 1 dy + \int_1^{\infty} y \times 0 dy$$

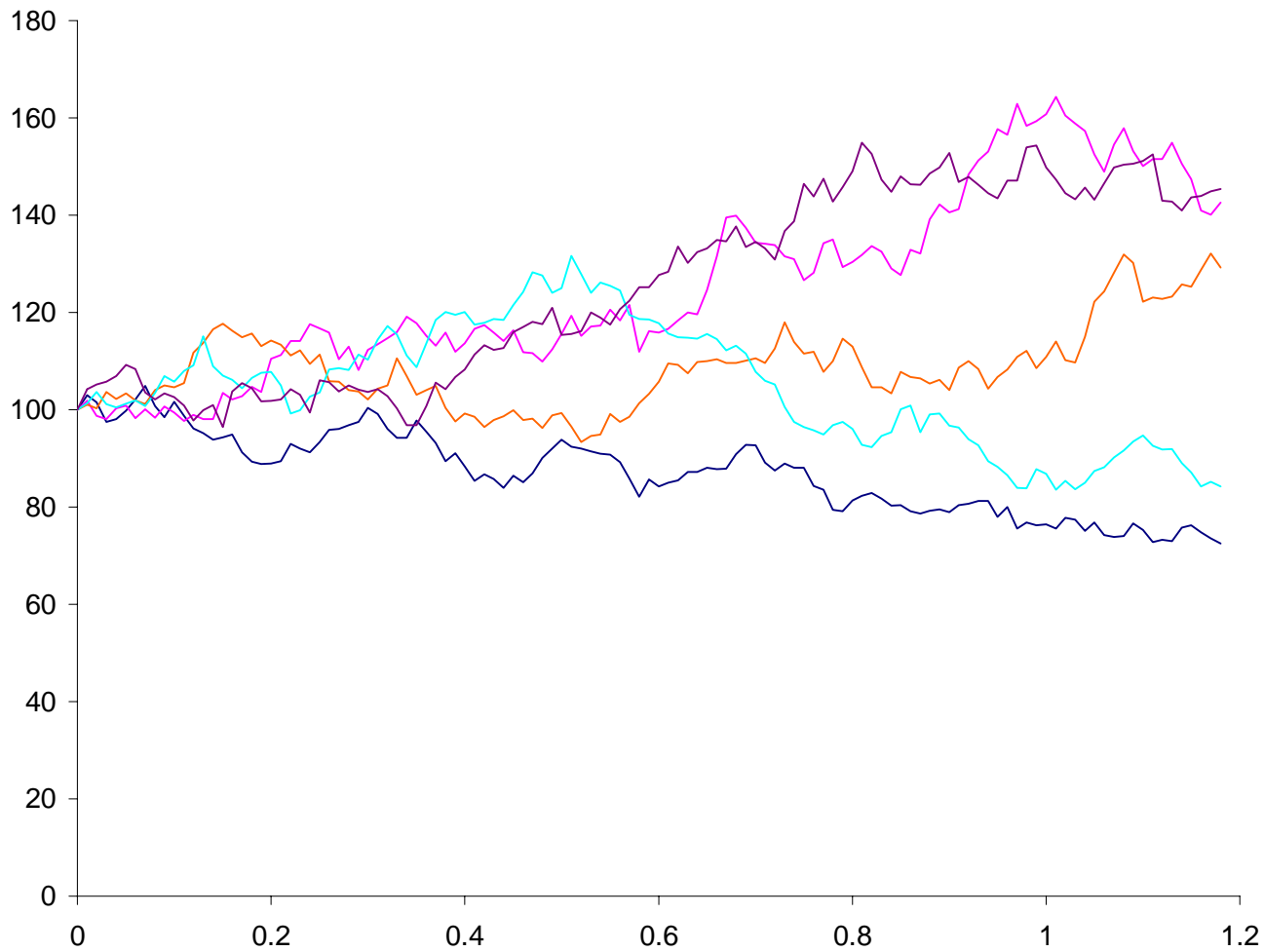
$$= \int_0^1 y dy = \left. \frac{y^2}{2} \right|_0^1 = \frac{1}{2}$$

$$V[Y] = E[Y^2] - E^2[Y] =$$

$$= \left. \frac{y^3}{3} \right|_0^1 - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\int_{\mathbb{R}} y^2 p(y) dy - \frac{1}{4} = \int_0^1 y^2 dy - \frac{1}{4}$$

	A	B	C	D	E	F	G
1	Asset	100		Time	Asset		
2	Drift	0.15		0	100		
3	Volatility	0.25		0.01	96.10692		
4	Timestep	0.01		0.02	96.99647		
5				0.03	94.76352		
6				0.04	91.46698		
7				0.05	88.83325		
8				0.06	88.42727		
9				0.07	90.62882		
10				0.08	88.80545		
11	=E7*(1+\$B\$2*\$B\$4+\$B\$3*SQRT(\$B\$4)*(RAND()+RAND()+RAND()+RAND()+ +RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()+RAND()-6))						
12							
13				0.11	84.93865		



Simulating other random walks

This method is not restricted to the lognormal random walk.

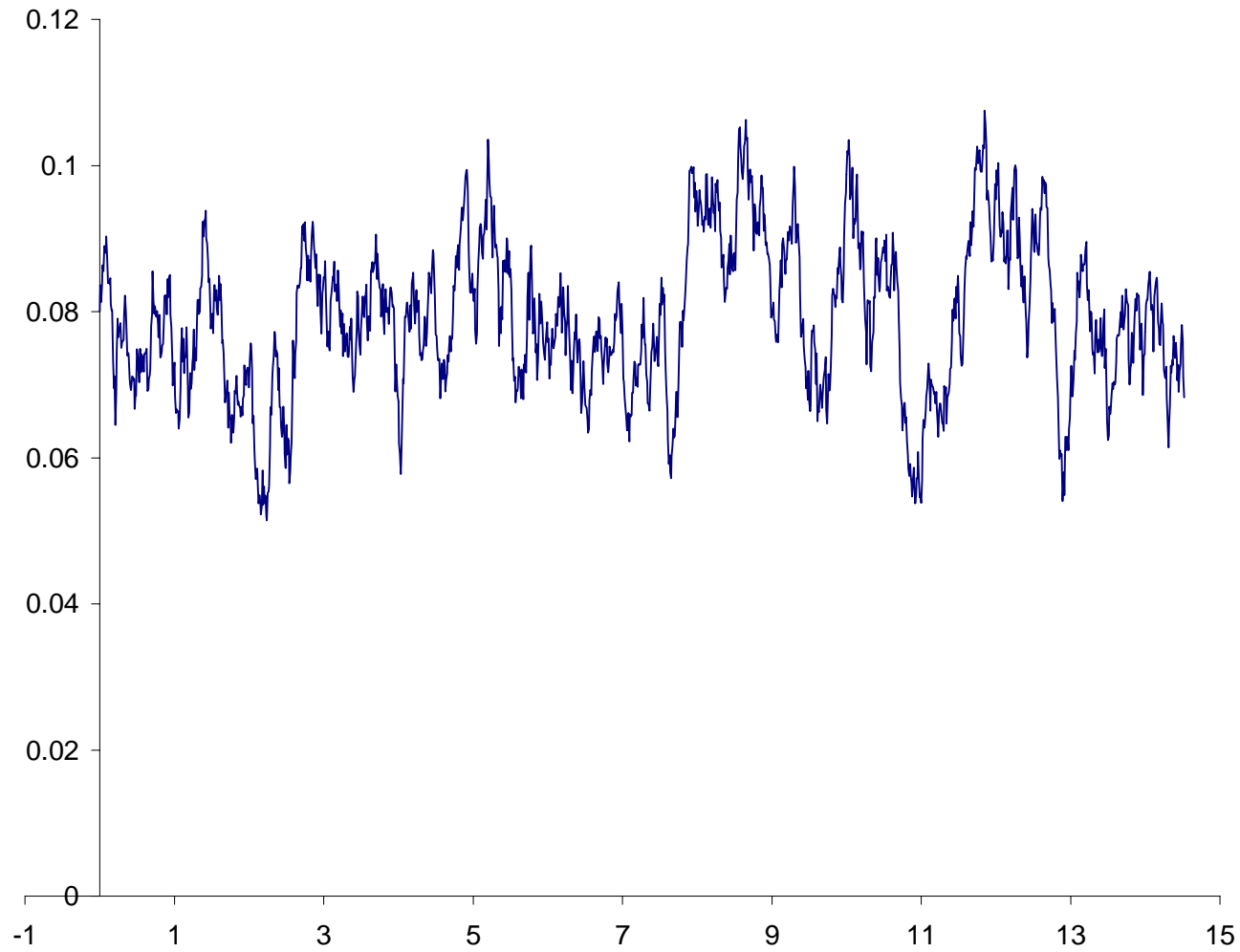
Later in the course we will be modeling interest rates as stochastic differential equations.

The following is a stochastic differential equation model for an interest rate, that goes by the name of an **Ornstein-Uhlenbeck process** (an example of a mean-reverting random walk), or when used in an interest rate context the **Vasicek model**:

$$r_{i+1} - r_i \quad \longleftarrow \quad dr = \gamma (\bar{r} - r) dt + \sigma dX.$$

In discrete time we can approximate this by

$$\longrightarrow \quad r_{i+1} = r_i + \gamma (\bar{r} - r_i) dt + \sigma \phi \delta t^{1/2}.$$



Producing correlated random numbers

We will often want to simulate paths of correlated random walks.

$$\mathbb{E}[\phi_1 \phi_2] = \rho$$

We may want to examine the statistical properties of a portfolio of stocks, or value a convertible bond under the assumption of random asset price and random interest rates.

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dX^{(1)}$$

$$\frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 dX^{(2)}$$

$$\mathbb{E}[dX_1 dX_2] = \rho dt$$

$$\phi_1 \sqrt{dt} \leftarrow \phi_2 \sqrt{dt}$$

$$V(t, S_1, S_2)$$

$$\exists \Delta T, \Delta t \in$$

$$t \rightarrow t + \Delta t \quad S_i \rightarrow S_i + \Delta S_i$$

$$i = 1, 2$$

$$V(S_1 + \Delta S_1, S_2 + \Delta S_2, t + \Delta t) = V(t, S_1, S_2) +$$

$$dS_1^2 = \sigma_1^2 S_1^2 dt$$

$$dS_2^2 = \sigma_2^2 S_2^2 dt$$

$$dS_1 dS_2 = \rho \sigma_1 \sigma_2 S_1 S_2 dt$$

$$\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_1^2} dS_1^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S_2^2} dS_2^2 + \frac{\partial^2 V}{\partial S_1 \partial S_2} dS_1 dS_2$$

$$dV = \left(\frac{\partial V}{\partial t} + \mu_1 S_1 \frac{\partial V}{\partial S_1} + \mu_2 S_2 \frac{\partial V}{\partial S_2} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} \right) dt + \sigma_1 S_1 \frac{\partial V}{\partial S_1} dX^{(1)} + \sigma_2 S_2 \frac{\partial V}{\partial S_2} dX^{(2)}$$

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Example:

Assets S_1 and S_2 both follow lognormal random walks with correlation ρ .

In continuous time we write

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dX_1,$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dX_2,$$

with

$$E[dX_1 dX_2] = \rho dt.$$

dX_t

In discrete time these become

$$S_{1_{i+1}} - S_{1_i} = S_{1_i} (\mu_1 \delta t + \sigma_1 \phi_1 \delta t^{1/2})$$

and

$$S_{2_{i+1}} - S_{2_i} = S_{2_i} (\mu_2 \delta t + \sigma_2 \phi_2 \delta t^{1/2})$$

with

$$\longrightarrow E[\phi_1 \phi_2] = \rho.$$

Recipe

$$\phi_i \sim N(0, 1) \quad \mathbb{E}[\phi_1] = 0 = \mathbb{E}[\phi_2]$$

$$\mathbb{E}[\phi_1^2] = 1 = \mathbb{E}[\phi_2^2]$$

Q: How can we choose a ϕ_1 and a ϕ_2 which are both Normally distributed, both have mean zero and standard deviation of one, and with a correlation of ρ between them? $\mathbb{E}[\phi_1 \phi_2] = \rho$

A: This can be done in two steps, first pick two *uncorrelated* Normally distributed random variables, and then combine them.

$$\varepsilon_i \sim N(0, 1)$$

$$\mathbb{E}[\varepsilon_1 \varepsilon_2] = 0$$

$$\mathbb{E}[\varepsilon_1] = 0 = \mathbb{E}[\varepsilon_2]$$

$$\mathbb{E}[\varepsilon_1^2] = 1 = \mathbb{E}[\varepsilon_2^2]$$

Step 1: Choose uncorrelated ϵ_1 and ϵ_2 , both Normally distributed with zero means and standard deviations of one.

Step 2: Convert these independent Normal numbers into correlated Normals by taking a linear combination.

$$\phi_1 = \epsilon_1$$

$$\phi_2 = \rho \epsilon_1 + \sqrt{1 - \rho^2} \epsilon_2.$$

Check:

$$E[\phi_1^2] = 1,$$

$$\begin{aligned} E[\phi_2^2] &= E\left[\rho^2\epsilon_1^2 + 2\rho\sqrt{1-\rho^2}\epsilon_1\epsilon_2 + (1-\rho^2)\epsilon_2^2\right] \\ &= \rho^2 + 0 + (1-\rho^2) = 1, \end{aligned}$$

and

$$E[\phi_1\phi_2] = E\left[\rho\epsilon_1^2 + \sqrt{1-\rho^2}\epsilon_1\epsilon_2\right] = \rho.$$

And Normality?

① Put $\phi_1 = \varepsilon_1$

② Create a linear combination of ε_i ($i=1,2$) and write
 $\phi_2 = \alpha \varepsilon_1 + \beta \varepsilon_2$ for some $\alpha, \beta \in \mathbb{R}$

Recipe:

i) $\mathbb{E}[\phi_1 \phi_2] = \rho = \mathbb{E}[\varepsilon_1 (\alpha \varepsilon_1 + \beta \varepsilon_2)] = \alpha \mathbb{E}[\cancel{\varepsilon_1^2}] + \beta \mathbb{E}[\cancel{\varepsilon_1 \varepsilon_2}]$

$\therefore \alpha = \rho$

ii) $\mathbb{E}[\phi_2^2] = 1 = \mathbb{E}[(\alpha \varepsilon_1 + \beta \varepsilon_2)^2] = \mathbb{E}[\alpha^2 \varepsilon_1^2 + 2\alpha\beta \varepsilon_1 \varepsilon_2 + \beta^2 \varepsilon_2^2]$
 $= \alpha^2 \mathbb{E}[\cancel{\varepsilon_1^2}] + 2\alpha\beta \mathbb{E}[\cancel{\varepsilon_1 \varepsilon_2}] + \beta^2 \mathbb{E}[\cancel{\varepsilon_2^2}] = 1 \rightarrow \rho^2 + \beta^2 = 1$

hence $\beta = \sqrt{1 - \rho^2}$

Finally

$\phi_1 = \varepsilon_1$

$\phi_2 = \rho \varepsilon_1 + \sqrt{1 - \rho^2} \varepsilon_2$

Weighted sums of Normally distributed numbers are themselves Normally distributed!

If $\underbrace{X_i \sim N(\mu_i, \sigma_i^2)}_{\text{for } i = 1, \dots, n}$ then

$$\sum_{i=1}^n w_i X_i \sim N \left(\underbrace{\sum_{i=1}^n w_i \mu_i}_{\text{mean}}, \underbrace{\sum_{i=1}^n w_i^2 \sigma_i^2}_{\text{variance}} \right).$$

Summary

Please take away the following important ideas

- With the right tool (Itô's lemma) you can examine functions of stochastic variables
- Partial differential equations can be used for finding probability density functions for arbitrary random walks
- Simulating random walks can be very easy indeed