Changing Probability Measure

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You saw in the Binomial Model lecture that there is more than just one probability measure.

Indeed, the lecture introduced you to the distinction between the 'real' or 'physical' probability measure, which we encounter every day on our Bloomberg screen, and the 'risk-neutral' measure, which is used for pricing.

Probability measures are by no means unique. We saw in our Black-Scholes lectures that the powerful arsenal of martingale techniques enables us, under certain assumptions, to change measure and transpose our problem set in the real world measure into an equivalent problem formulated as a martingale under a different measure.

To begin, we outline the rules that allow us to define equivalent measures. Then, we will introduce the main result: the Radon-Nikodym Theorem.

1. Equivalent Measure

If two measures $\mathbb{P}$ and $\mathbb{Q}$ share the same sample space Ω and if $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$ for all subset A, we say that $\mathbb{Q}$ is **absolutely continuous** with respect to $\mathbb{P}$ and denote this by $\mathbb{Q} << \mathbb{P}$.

The key point is that all impossible events under $\mathbb{P}$ remain impossible under $\mathbb{Q}$. The probability mass of the possible events will be distributed differently under $\mathbb{P}$ and $\mathbb{Q}$. In short 'it is alright to tinker with the probabilities as long as we do not tinker with the (im)possibilities.'

If $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$ then the two measures are said to be **equivalent**, denoted by $\mathbb{P} \sim \mathbb{Q}$.

This extremely important result is formalized in the **Radon Nikodym Theorem**.

2. The Radon-Nikodym Theorem

Key Fact: The Radon-Nikodym Theorem *If the measures $\mathbb{P}$ and $\mathbb{Q}$ are absolutely continuous, then, there exists a random variable Λ such that for all subsets $A \subset \Omega$*

$$\mathbb{Q}(A) = \int_A \Lambda d\mathbb{P}$$

where

$$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

is called the **Radon-Nikodym derivative**.

This formulation is general: it applies to both continuous and discrete distributions.

However, the Radon-Nikodym theorem simplifies considerably when we deal with *discrete distributions*, such as coin toss, throw of a dice or the binomial model. In particular, we have for all subsets $A \subset \Omega$

$$\mathbb{Q}(A) = \Lambda(A)\mathbb{P}(A)$$

where the Radon-Nikodym derivative is given by

$$\Lambda(\cdot) = \frac{\mathbb{Q}}{\mathbb{P}}(\cdot)$$

3. Example of Change of Measure

Take the coin toss game in which you gain GBP 1 if the toss produces a Head and lose GBP 1 if the toss produces a Tail.

The sample space Ω has two events:

- ω_1 = Head;
- ω_2 = Tail.

so $\Omega = \{\omega_1, \omega_2\}$.

In the general case when the coin is not 'fair,' the probability measure $\mathbb{P}$ is
$\mathbb{P}(\omega_1) = p; \mathbb{P}(\omega_2) = q = 1 - p;$

with $0 < p < 1$.

Suppose that we want to evaluate our expected P\&L in a world where all coins are fair.

To do the, we introduce a new probability measure $\bar{\mathbb{P}}$ such that

- $\bar{\mathbb{P}}(\omega_1) = \frac{1}{2};$
- $\bar{\mathbb{P}}(\omega_2) = \frac{1}{2}.$

This implies that we must *apply extra weight* to each of the two events to travel from the real world $(\Omega, \mathcal{F}, \mathbb{P})$ to the ` ` fair'' world $(\Omega, \mathcal{F}, \bar{\mathbb{P}})$. Namely,

- we reweigth the likelihood of event ω_1 by $\frac{\bar{\mathbb{P}}}{\mathbb{P}}(\omega_1) = \frac{1}{2p};$
- we reweigth the likelihood of event ω_2 by $\frac{\bar{\mathbb{P}}}{\mathbb{P}}(\omega_2) = \frac{1}{2q};$

so that

$$\bar{\mathbb{P}}(\cdot) = \frac{\bar{\mathbb{P}}}{\mathbb{P}}(\cdot)\mathbb{P}(\cdot)$$

This last formula is the **Radon-Nikodym formula** for discrete processes!

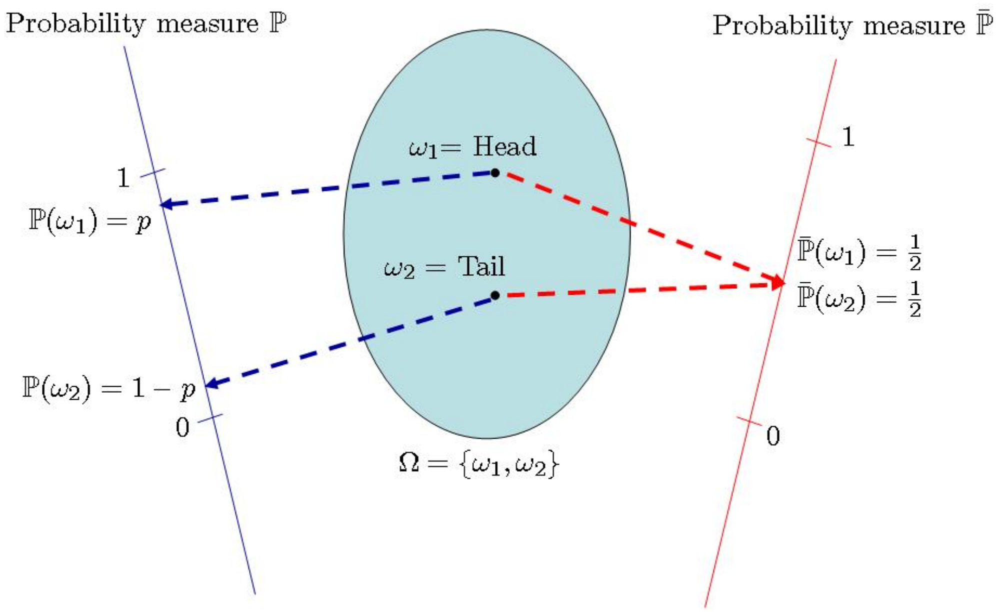

Figure: probability measures for the coin toss example

Note that since $0 < p < 1$, we have

- $\mathbb{P}(\omega_1) \neq 0$;
- $\mathbb{P}(\omega_2) \neq 0$.

Hence the measure $\mathbb{P}$ and $\bar{\mathbb{P}}$ are **equivalent**.

This example shows that the Radon-Nikodym derivative is just weighing scheme to transform the probabilities in the $\mathbb{P}$ measure into probabilities in the $\bar{\mathbb{P}}$ measure.

4. What's The Use?

The main interest of a change of measure is to make difficult problems easier to solve. While some problems might be extremely difficult to tackle under the real-world measure $\mathbb{P}$, it might be possible to find an equivalent measure $\mathbb{Q}$ under which they are much easier to solve.

As a result, the change of measure techniques have become a cornerstone not only of modern probability but also of mathematical finance, where they are widely used in asset pricing (as in our Lectures on **Black-Scholes via Martingales** and on **PDE vs. Martingales**).