The Binomial Model - Discrete Time Finance

The model has made option pricing accessible to MBA students and finance practitioners preparing for the CFA. It is a very useful tool for conveying the ideas of delta hedging and no arbitrage, in addition to the subtle concept of risk neutrality and option pricing.

Here the model is considered in a slightly more mathematical way. The basic assumptions in option pricing theory consist of two forms, key:

- Short selling allowed
- No arbitrage opportunities

and relaxable

- Frictionless markets - no transaction costs, limits to trading or taxes
- Perfect liquidity
- Known volatility and interest rates
- No dividends on the underlying

The key assumptions underlying the binomial model are:

- an asset value changes only at discrete time intervals
- fractional trading is allowed
- an asset's worth can change to one of only two possible new values at each time step.

Suppose the share price in '**T**rump **I**ndependent **T**raders**'** today is 100p, on GSE (Genius's Stock Exchange). We write a call option on these shares with a strike of 100p that expires tomorrow. Tomorrow's share price will be either

- 101p with probability 0.6

- 99p with probability 0.4

Question: What is today's option value?

Method 1: Calculate the expected value of the call. If the share is 101p then we exercise the option for 100p to obtain a 1p profit. On the other hand if the share is 99p tomorrow then the option is worthless. The expected value is

$$C = 0.6 \times 1\text{p} + 0.4 \times 0\text{p} = 0.6\text{p}$$

Method 2: Set up a portfolio Π consisting of a long position in a call option and short the shares

$$\Pi = C - \Delta S.$$

C is the value of the call; Δ the number of shares; S is the stock price of **T**rump **I**nternational **T**raders.

The delta means there is one degree of freedom so arrange the above so that at T, the portfolio has the same value regardless if $S = 101$p or $S = 99$p.

$$\left.\begin{array}{ll} C = 1\text{p} & \text{if } S = 101\text{p} \\ C = 0\text{p} & \text{if } S = 99\text{p} \end{array}\right\} \Longrightarrow$$

the two scenarios

- If $S = 101$p at T then $C = 1$ therefore $\Pi = 1 - \Delta 101$

- If $S = 99$p at T then $C = 0$ therefore $\Pi = -\Delta 99$

thus equating the 2 expressions for Π gives

$$\begin{aligned} 1 - \Delta 101 &= -\Delta 99 \\ \Delta &= \tfrac{1}{2},\ \Pi = C - \Delta S = -49.50\text{p} \end{aligned}$$

So tomorrow the portfolio takes the value -49.50p regardless of whether the stock rises or falls. We have created a perfectly risk free portfolio. If the portfolio is worth -49.50p tomorrow, and interest rates are zero, how much is the portfolio worth today? It must also be worth this amount today - this is an example of **no arbitrage**. There are two ways to ensure we have -49.50p.

1. Buy one option and sell one half of the stock

2. Put the money under the mattress

Both of these portfolios must be worth the same today.

So we calculate the option value by writing

$$\begin{aligned} C - \frac{1}{2} \times 100 &= -\frac{1}{2} \times 99 \\ C &= \tfrac{1}{2} \end{aligned}$$

So clearly the part of the model we didn't need was the probability of the stock rising or falling. This is saying that when option pricing the stock growing or decreasing in value is irrelevant, i.e. the stock growth rate. This is because we have hedged the option with the stock.

Whilst we are not interested in the direction that the stock moves, we do care about the stock price range (101 and 99), as that is related to the volatility. The larger the vol, the greater that range is. Later we will see that the drift does not appear in the model when pricing derivatives.

Example: A share price is currently £80. At the end of three months, it will be either £84 or £76. Ignoring interest rates, calculate the value of a three-month European call option with exercise price £79.

Binomial tree for share price is

$$
\begin{array}{ccc}
 & & 84 \\
80 & & \\
 & & 76
\end{array}
$$

Binomial tree for option price V is

$$5 \;\; (= \max(84 - 79, 0))$$

$$V$$

$$0 \;\; (= \max(76 - 79, 0)$$

Now set up a Black-Scholes hedged portfolio, $V - \Delta S$, then binomial tree for its value is

$$5 - 84\Delta$$

$$V - 80\Delta$$

$$-76\Delta$$

For risk-free portfolio choose Δ such that $5 - 84\Delta = -76\Delta \Rightarrow \Delta = \frac{5}{8}$. So in absence of arbitrage, $V - 80\Delta = -76\Delta$, and $V = 2.5$.

The stock volatility is of key importance in the pricing of options.

Earlier we had

$$\begin{aligned} 1 - \Delta 101 &= -\Delta 99 \\ \Delta &= \frac{1-0}{101-99} = \frac{1}{2} \end{aligned}$$

Note it is purely a coincident in this example that delta has the same value as the option. Note

$$\Delta = \frac{V^+ - V^-}{S^+ - S^-} = \frac{\text{Range of option payoffs}}{\text{Range of stock prices}}$$

This model is discrete time, discrete stock. When we go to continuous time continuous stock delta Δ will become $\frac{\partial V}{\partial S}$.

- How does this change if interest rates are non-zero? Everything is as before but we now have a discount factor.
  Consider the earlier example but with $r = 10\%$ over one day, i.e.

$$\frac{1}{1+rt} = \frac{1}{1+0.1/252} \approx 0.9996$$

Now discount tomorrow's value to get to todays

$$\begin{aligned} V - 0.5 \times 100 &= -0.5 \times 99 \times 0.9996 \\ V &= 0.51963 \end{aligned}$$

So the portfolio value today must be the Present Value of the portfolio value tomorrow.

Consider a portfolio Π, long an option and short Δ assets. $V^{\pm}$ denotes the option value corresponding to asset price $S^{\pm}$:

$$
\begin{array}{ccccccc}
 & & S^{+} & & & & V^{+} \\
S & \nearrow & & & V & \nearrow & \\
 & \searrow & S^{-} & & & \searrow & V^{-}
\end{array}
$$

$$\Pi = V - \Delta S$$

At time T there are two possible outcomes:

$$
\begin{aligned}
\Pi_{-} &= V^{-} - \Delta S^{-} \\
\Pi_{+} &= V^{+} - \Delta S^{+}
\end{aligned}
$$

Choosing Δ so $\Pi_{-} = \Pi_{+}$ gives

$$\Delta = \frac{V^{+} - V^{-}}{S^{+} - S^{-}}$$

This choice of Δ makes Π risk-free, so no-arbitrage suggests that the return on Π equal the risk-free rate, i.e.

$$\begin{aligned}
e^{rT}\Pi &= \Pi_- = \Pi_+ \\
&= V^- - \Delta S^- = V^- - \left(\frac{V^+ - V^-}{S^+ - S^-}\right) S^- \\
&= \frac{V^-\left(S^+ - S^-\right) - S^-\left(V^+ - V^-\right)}{S^+ - S^-} \\
&= \frac{V^- S^+ - S^- V^+}{S^+ - S^-}.
\end{aligned}$$

Hence we can write

$$
\begin{aligned}
e^{rT}(V - \Delta S) &= \frac{V^- S^+ - S^- V^+}{S^+ - S^-} \\
e^{rT} V &= \frac{V^- S^+ - S^- V^+}{S^+ - S^-} + \left(\frac{V^+ - V^-}{S^+ - S^-}\right) S_0 e^{rT} \\
&= \frac{\left(e^{rT} S - S^-\right)}{S^+ - S^-} V^+ + \left(\frac{S^+ - e^{rT} S}{S^+ - S^-}\right) V^- \\
&= q V^+ + (1 - q) V^-
\end{aligned}
$$

and finally we can write

$$
V = e^{-rT}\left(q V^+ + (1 - q) V^-\right)
$$

where we define

$$
q = \frac{\left(e^{rT} S - S^-\right)}{S^+ - S^-}
$$

with $0 < q < 1$.

If compounding is discrete

$$q = \frac{\Big((1 + rT)\,S_0 - S^-\Big)}{S^+ - S^-}$$

and the option price becomes

$$V = (1 - rT)\left(qV^+ + (1-q)\,V^-\right).$$

q is a *risk-neutral probability* which come about from insistence on no arbitrage. It has nothing to do with the real probabilities of S^+ and S^- occurring. Pricing a call using the real probability, p, will probably make you a profit, but a loss is also possible. Pricing an option using the risk neutral probability, q, you will certainly make neither a profit nor a loss.

Example: A share price is currently £63. At the end of each three-month period, it will change by going up £3 or going down £3. Calculate the value of a **six-month** European **put** option with strike price £61. The risk-free interest rate is 4% per annum with continuous compounding.

Binomial tree for stock

$$\begin{tikzcd}
 & & 69 \\
 & 66 \arrow[ur] \arrow[dr] & \\
63 \arrow[ur] \arrow[dr] & & 63 \\
 & 60 \arrow[ur] \arrow[dr] & \\
 & & 57
\end{tikzcd}$$

Binomial tree for option

$$\begin{tikzcd}
 & & \begin{array}{c}\mathbf{0} \\ (= \max(61-69,0))\end{array} \\
 & P_1 \arrow[ur] \arrow[dr] & \\
P \arrow[ur] \arrow[dr] & & \begin{array}{c}\mathbf{0} \\ (= \max(61-63,0))\end{array} \\
 & P_2 \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{c}\mathbf{4} \\ (= \max(61-57,0))\end{array}
\end{tikzcd}$$

We must find the values of P_1 and P_2 before we can solve for P.

If we set up a Black-Scholes delta hedged portfolio $P_1 - \Delta_1 S$, for P_1, then the binomial tree for its value is

$$\begin{array}{ccc} & & -69\Delta_1 \\ & \nearrow & \\ P_1 - 66\Delta_1 & & \\ & \searrow & \\ & & -63\Delta_1 \end{array}$$

For a risk-free portfolio, we choose Δ_1 s.t. $-69\Delta_1 = -63\Delta_1$, i.e. $\Delta_1 = 0$. Then in the absence of arbitrage it must earn at the risk-free interest-rate and

$$P_1 - 66\Delta_1 = e^{-0.04\times(3/12)}(-69\Delta_1) \to P_1 = 0.$$

If we set up a Black-Scholes delta hedged portfolio $P_2 - \Delta_2 S$, for P_2, then the binomial tree for its value is

$$\begin{array}{ccc} & & -63\Delta_2 \\ & \nearrow & \\ P_2 - 60\Delta_2 & & \\ & \searrow & \\ & & 4 - 57\Delta_2 \end{array}$$

For a risk-free portfolio, we choose Δ_2 s.t. $-63\Delta_2 = 4 - 57\Delta_2$, i.e. $\Delta_2 = -2/3$. Then in the absence of arbitrage it must earn at the risk-free interest-rate and

$$P_2 - 60\Delta_2 = e^{-0.04\times(3/12)}(4 - 57\Delta_2) \to P_2 = 1.58.$$

We can now set up a Black-Scholes hedged portfolio, $P - \Delta S$, for P. The binomial tree for its value is

$$\begin{tikzcd}[row sep=small] & P_1 - 66\Delta \\ P - 63\Delta \arrow[ur] \arrow[dr] & \\ & P_2 - 60\Delta \end{tikzcd}$$

For a risk-free portfolio, we choose Δ s.t.

$$P_1 - 66\Delta = P_2 - 60\Delta \to \Delta = \frac{P_1 - P_2}{6} = \frac{-P_2}{6} = -0.263.$$

Then in an arbitrage free market, the portfolio earns at the risk-free rate

$$
\begin{aligned}
P - 63\Delta &= e^{-0.04\times(3/12)}\,(-66\Delta) \\
P &= \Delta\,(63 - 65.34) = -0.263 \times -2.34 \\
&= 0.6154
\end{aligned}
$$

Hence the value of the put option is £0.62.

The one period model - Replication

Another way of looking at the Binomial model is in terms of replication: we can replicate the option using only cash (or bonds) and the asset. That is, mathematically, simply a rearrangement of the earlier equations. It is, nevertheless a very important interpretation.
In one time step:

1. The asset moves from $S_0 = s$ to $S_1 = s_u$ or $S_1 = s_d$.

2. An option X pays off x_u if the asset price is s_u and x_d if the price is s_d.

$$
\begin{array}{lccc}
\text{Stock} & & & s_u \\
 & & \nearrow & \\
 & s & & \\
 & & \searrow & \\
 & & & s_d \\
 & t=0 & & t=1
\end{array}
\qquad\qquad
\begin{array}{lccc}
\text{Option:} & & & x_u \\
 & & \nearrow & \\
 & V & & \\
 & & \searrow & \\
 & & & x_d \\
 & t=0 & & t=1
\end{array}
$$

3. There is a bond market in which a pound invested today is continuously compounded at a constant (risk-free) rate r and becomes e^r, one time-step later. The dynamics of the risk-free asset B_t satisfies

$$dB_t = rB_t dt; \; B_0 = 1.$$

Integrating over $[t, T]$, where $t = 0, \; T = 1$ gives $B_1 = e^r$.

Now consider a portfolio of ψ bonds and ϕ assets which at time $t = 0$, will have an initial value of

$$\Pi = \phi s + \psi$$

Now with this money we can buy or sell bonds or stocks in order to obtain a new portfolio at time-step 1. At this new time-step there exist two possible outcomes

$$\begin{aligned}\Pi^+ &= \phi s_u + \psi e^r \\ \Pi^- &= \phi s_d + \psi e^r\end{aligned}$$

In order to replicate the option insist that

$$\Pi^+ = x_u; \quad \Pi^- = x_d$$

Can we construct a hedging strategy which will guarantee to pay off the option, whatever happens to the asset price?

The Hedging Strategy

We arrange the portfolio so that its value is exactly that of the required option pay-out at the terminal time regardless of whether the stock moves up or down. This is because having two unknowns ϕ, ψ, the amount of stock and bond, and we wish to match the two possible terminal values, x_u, x_d, the option payoffs. Thus we need to have

$$\begin{aligned} x_u &= \phi s_u + \psi e^r, \\ x_d &= \phi s_d + \psi e^r. \end{aligned}$$

Subtracting the two expressions and rearranging gives

$$\phi = \frac{x_u - x_d}{s_u - s_d}.$$

Then substituting for ϕ in either equation yields

$$\psi = e^{-r} \frac{x_d s_u - x_u s_d}{s_u - s_d}$$

This is a *hedging strategy*.

At time step 1, the value of the portfolio is

$$X = \left\{ \begin{array}{ll} x_u & \text{if } S_1 = s_u \\ x_d & \text{if } S_1 = s_d \end{array} \right\}$$

This is the option payoff. Thus, given $V = \phi s + \psi$ we can construct the above portfolio which has the same payoff as the option. Hence the price for the option must be V. Any other price would allow arbitrage as you could play this hedging strategy, either buying or selling the option, and make a guaranteed profit.

Thus the fair, arbitrage-free price for the option is given by

$$\begin{aligned} V &= (\phi s + \psi) \\ &= \frac{x_u - x_d}{s_u - s_d} s + e^{-r} \frac{x_d s_u - x_u s_d}{s_u - s_d} \\ &= e^{-r} \left(\frac{e^r s - s_d}{s_u - s_d} x_u + \frac{s_u - e^r s}{s_u - s_d} x_d \right). \end{aligned}$$

Define

$$q = \frac{e^r s - s_d}{s_u - s_d},$$

then we conclude that

$$V = e^{-r}\left(q x_u + (1-q)\, x_d\right)$$

where

$$0 < q < 1.$$

We can think of q as a probability induced by insistence on no-arbitrage, i.e. the so-called *risk-neutral probability.* It has nothing to do with the real probabilities of s_u and s_d occurring; these are p and $1-p$, in turn.

The option price can be viewed as the discounted expected value of the option pay-off X with respect to the probabilities q,

$$\begin{aligned} V &= e^{-r}\left(q x_u + (1-q)\, x_d\right) \\ &= \mathbb{E}_q\left[e^{-r} X\right]. \end{aligned}$$

The fact that the risk neutral/fair value (or $q-$value) of a call is less than the expected value of the call (under the real probability p), is not a puzzle.

Pricing a call using the real probability, p, you will probably make a profit, but you might also might make a loss. Pricing an option using the risk-neutral probability, q, you will certainly make neither a profit nor a loss.

Example: A stock is currently trading at 100. In one year it will have risen to 103 or fallen to 98. If interest rates are zero, use a replicating strategy to price a one year call option with strike $K = 100$.

Asset:

$$
\begin{array}{ccl}
 & & 103 \quad S(1) > 100 \\
 & \nearrow & \qquad\quad \text{Payoff } = 3 \\
S(0) = 100 & & \\
 & \searrow & \\
 & & 98 \quad S(1) < 100 \\
 & & \qquad\quad \text{Payoff } = 0
\end{array}
$$

Option:

$$
\begin{array}{ccl}
 & & V_1 = 3 \\
 & \nearrow & \\
V_0 & & \\
 & \searrow & \\
 & & V_2 = 0
\end{array}
$$

ϕ stocks and ψ units of bonds.

$$\left.\begin{array}{r} 103\phi + \psi = 3 \\ 98\phi + \psi = 0 \end{array}\right\} \quad \rightarrow \quad \phi = 3/5; \;\; \psi = -294/5$$

$$\therefore \quad V_0 = S(0) \times \phi + \psi \times 1 = \frac{300}{5} - \frac{294}{5} = 1.2$$

$$q(\text{up}) = \frac{e^{rt}s - s_d}{s_u - s_d}, \;\; q(\text{down}) = \frac{s_u - e^{rt}s}{s_u - s_d}$$

where $r = 0$.

ω_1 100 103

$$q(\omega_1) = \frac{100 - 98}{103 - 98} = \frac{2}{5}$$

ω_2 100 98

$$q(\omega_2) = \frac{103 - 100}{103 - 98} = \frac{3}{5}$$

So the risk neutral probabilities are

$$\left(\frac{2}{5}, \frac{3}{5}\right)$$

So the expected value (under the risk-neutral probabilities/measure) is

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}X\right]$$

where $r = 0, t = 0, T = 1$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[X] &= \sum_{\omega} q(\omega_i) X(\omega_i) \\
&= \frac{2}{5} \times 3 + \frac{3}{5} \times 0 = 1.2
\end{aligned}$$

Using Sample Paths

A *binary option* (also called a *digital option*) pays one dollar at time $t = T$ if the asset price is above a fixed strike level K and is worthless otherwise.

Consider the following model with $K = 5,\ r = 0$:

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	4	7	11
ω_2	4	7	6
ω_3	4	3	6
ω_4	4	3	1

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	s	u	u
ω_2	s	u	d
ω_3	s	d	u
ω_4	s	d	d

Asset:

$$
\begin{array}{ccccl}
 & & & & 11 \quad S(2) > K \quad \text{Payoff } = 1 \\
 & & & \nearrow & \\
 & & 7 & & \\
 & \nearrow & & \searrow & \\
S(0) = 4 & & & & 6 \quad S(2) > K \quad \text{Payoff } = 1 \\
 & \searrow & & \nearrow & \\
 & & 3 & & \\
 & & & \searrow & \\
 & & & & 1 \quad S(2) < K \quad \text{Payoff } = 0
\end{array}
$$

Option:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & 1 \\
 & V_1 \arrow[ur] \arrow[dr] & \\
V_0 \arrow[ur] \arrow[dr] & & 1 \\
 & V_2 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}$$

Replicate backwards over each time period 'sub-model':

Stock

$$\begin{tikzcd}[column sep=small, row sep=small]
 & 11 \\
7 \arrow[ur] \arrow[dr] & \\
 & 6
\end{tikzcd}$$

Option:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & 1 \\
V_1 \arrow[ur] \arrow[dr] & \\
 & 1
\end{tikzcd}$$

Replicate with ϕ units of stock and ψ units of bonds/riskless asset: $V = \phi S + \psi e^{rt}$

$$\left.\begin{array}{c} 11\phi + \psi = 1 \\ 6\phi + \psi = 1 \end{array}\right\} \rightarrow \phi = 0;\ \psi = 1$$

$$\therefore \quad V_1 = S(1) \times \phi + \psi \times 1 = 7 \times 0 + 1 \times 1 = 1$$

$$\begin{array}{ccccccc} \text{Stock} & & 6 & \qquad & \text{Option:} & & 1 \\ & \nearrow & & & & \nearrow & \\ 3 & & & & V_2 & & \\ & \searrow & & & & \searrow & \\ & & 1 & & & & 0 \end{array}$$

Replicate with ϕ units of stock and ψ units of bonds/riskless asset: $V = \phi S + \psi e^{rt}$

$$\left.\begin{array}{c} 6\phi + \psi = 1 \\ \phi + \psi = 0 \end{array}\right\} \rightarrow \phi = \tfrac{1}{5};\ \psi = -\tfrac{1}{5}$$

$$\therefore \quad V_2 = S(1) \times \phi + \psi \times 1 = 3 \times \tfrac{1}{5} - \tfrac{1}{5} \times 1 = \tfrac{2}{5}.$$

Stock
$$\begin{tikzcd}[row sep=small] & 7 \\ 4 \arrow[ur] \arrow[dr] & \\ & 3 \end{tikzcd}$$

Option:
$$\begin{tikzcd}[row sep=small] & 1 \\ V_0 \arrow[ur] \arrow[dr] & \\ & 0.4 \end{tikzcd}$$

Replicate with ϕ units of stock and ψ units of bonds/riskless asset: $V = \phi S + \psi e^{rt}$

$$\left.\begin{array}{r} 7\phi + \psi = 1 \\ 3\phi + \psi = 0.4 \end{array}\right\} \quad \longrightarrow \quad \phi = 0.15;\ \psi = -0.05$$

$$\therefore \quad V_0 = S(0) \times \phi + \psi \times 1 = 4 \times 0.15 - 0.05 \times 1 = 0.55.$$

So the binary call option struck at 5 is valued at 0.55

Now calculate the risk-neutral probabilities and use these to validate the option price calculated above.

$$q\,(\mathsf{up}) = \frac{e^{rt}s - s_d}{s_u - s_d},\;\; q\,(\mathsf{down}) = \frac{s_u - e^{rt}s}{s_u - s_d}$$

where $r = 0$.

ω_1	4	7	11

$$q\,(\omega_1) = \frac{4-3}{7-3} \times \frac{7-6}{11-6} = \frac{1}{20}$$

ω_2	4	7	6

$$q\,(\omega_2) = \frac{4-3}{7-3} \times \frac{11-7}{11-6} = \frac{1}{5}$$

ω_3	4	3	6

$$q\,(\omega_3) = \frac{7-4}{7-3} \times \frac{3-1}{6-1} = \frac{3}{10}$$

ω_4 4 3 1

$$q(\omega_4) = \frac{7-4}{7-3} \times \frac{6-3}{6-1} = \frac{9}{20}$$

So the expected value (under the risk-neutral probabilities/measure) is

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}X\right]$$

where $r = 0, t = 0, T = 1$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[X] &= \sum_{\omega} q(\omega_i) X(\omega_i) \\
&= \frac{1}{20} \times 1 + \frac{1}{5} \times 1 + \frac{3}{10} \times 1 + \frac{9}{20} \times 0 \\
&= 0.55
\end{aligned}$$

hence verified.

Binomial Model II

Assume an asset which has value S and during a time step δt can either rise to uS or fall to vS with $0 < v < 1 < u$. So as earlier probabilities of a rise and fall in turn are p and $1 - p$.

$$\underbrace{\begin{array}{ccc} & & uS \\ & \nearrow & \\ S & & \\ & \searrow & \\ & & vS \end{array}}_{\delta t}$$

$$\underbrace{\begin{array}{ccc} & & V^{+}\Delta uS \\ & \nearrow & \\ V - \Delta S & & \\ & \searrow & \\ & & V^{-}\Delta vS \end{array}}_{\delta t}$$

Also set $uv = 1$ so that after an up and down move, the asset returns to S.

To implement the Binomial model we need a model for asset price evolution to predict future possible spot prices. So use $\delta S = \mu S \delta t + \sigma S \phi \sqrt{\delta t}$. The 3 constants u, v, p are chosen to give the binomial model the same drift and diffusion as the SDE. For the correct drift, choose

$$pu + (1 - p)\, v = e^{\mu \delta t} \qquad \text{(a)}$$

and for the correct standard deviation set

$$pu^2 + (1 - p)\, v^2 = e^{\left(2\mu + \sigma^2\right)\delta t} \qquad \text{(b)}$$

$u \times (a) + v \times (a)$ gives

$$(u+v)\,e^{\mu\delta t} = pu^2 + uv - puv + pvu + v^2 - pv^2.$$

Rearrange to get

$$(u+v)\,e^{\mu\delta t} = pu^2 + (1-p)\,v^2 + uv$$

and we know from (b) that $pu^2 + (1-p)\,v^2 = e^{\left(2\mu+\sigma^2\right)\delta t}$ and $uv = 1$. Hence we have

$$\begin{aligned} (u+v)\,e^{\mu\delta t} &= e^{\left(2\mu+\sigma^2\right)\delta t} + 1 \Rightarrow \\ (u+v) &= e^{-\mu\delta t} + e^{\left(\mu+\sigma^2\right)\delta t}. \end{aligned}$$

Now recall that the quadratic equation $ax^2 + bx + c = 0$ with roots α and β has

$$\alpha + \beta = -\frac{b}{a}; \quad \alpha\beta = \frac{c}{a}.$$

We have

$$
\begin{aligned}
(u+v) &= e^{-\mu\delta t} + e^{\left(\mu+\sigma^2\right)\delta t} \equiv -\frac{b}{a} \\
uv &= 1 \equiv \frac{c}{a}
\end{aligned}
$$

hence u and v satisfy

$$(x-u)(x-v) = 0$$

to give the quadratic

$$x^2 - (u+v)x + uv = 0 \Rightarrow$$

$$x = \frac{(u+v) \pm \sqrt{(u+v)^2 - 4uv}}{2}$$

so with $u > 1$

$$u = \frac{1}{2}\left(e^{-\mu\delta t} + e^{\left(\mu+\sigma^2\right)\delta t}\right) + \frac{1}{2}\sqrt{\left(e^{-\mu\delta t} + e^{\left(\mu+\sigma^2\right)\delta t}\right)^2 - 4}$$

In this model, the hedging argument gives

$$V^{+} - \Delta u S = V^{-} - \Delta v S$$

which leads to $\Delta = \dfrac{V^{+} - V^{-}}{(u-v)\,S}$. Because all other terms are known choose Δ to eliminate risk.

We know tomorrow's option value therefore price today is tomorrow's value discounted for interest rates

$$V - \Delta S = \frac{1}{1 + r\delta t}\left(V^{+} - \Delta u S\right)$$

so $(1 + r\delta t)\,(V - \Delta S) = V^{+} - \Delta u S$ and replace using the definition of Δ above

$$(1 + r\delta t)\,V = V^{+}\left(\frac{-v + 1 + r\delta t}{(u-v)}\right) + V^{-}\left(\frac{u - 1 - r\delta t}{(u-v)}\right)$$

where the risk-neutral probabilities are

$$\begin{aligned} q &= \frac{-v+1+r\delta t}{(u-v)} \\ 1-q &= \frac{u-1-r\delta t}{(u-v)}. \end{aligned}$$

So $(1+r\delta t)\,V = V^{+}q + V^{-}\,(1-q)\,.$

Finally we have

$$V = \frac{V^{+}-V^{-}}{(u-v)} + \frac{uV^{-}-vV^{+}}{(1+r\delta t)\,(u-v)}$$

With continuous compounding we have

$$\begin{aligned} q &= \frac{e^{r\delta t}-v}{(u-v)} \\ 1-q &= \frac{u-e^{r\delta t}}{(u-v)} \end{aligned}$$

The earlier model had risk-neutral probabilities

$$\begin{aligned} q &= \frac{e^r s - s_d}{s_u - s_d} \\ 1 - q &= \frac{s_u - e^r s}{s_u - s_d} \end{aligned}$$

But we now have $s_u = us, \; s_d = ds$, hence the above simplifies to

$$\begin{aligned} q &= \frac{e^r s - ds}{us - ds} = \frac{e^r - d}{u - d} \\ 1 - q &= \frac{us - e^r s}{us - ds} = \frac{u - e^r}{u - d} \end{aligned}$$

Example: A non-dividend paying stock is currently trading at $S = 20$. At the end of each year, the stock can rise to uS or go down to dS; where $u = 1.2840$ and $d = 0.8607$. The continuously compounded risk-free interest rate is 5% per annum. Using **risk-neutral probabilities**, calculate the price of a two year European call option with a strike price of 22.

Asset tree and payoff:

$$
\begin{tikzcd}
 & & 32.9731 \quad \text{Payoff}=10.9731 \\
 & 25.68 \arrow[ur] \arrow[dr] & \\
S(0) = 20 \arrow[ur] \arrow[dr] & & 22.1028 \quad \text{Payoff}=0.1028 \\
 & 17.214 \arrow[ur] \arrow[dr] & \\
 & & 14.8161 \quad \text{Payoff}=0
\end{tikzcd}
$$

The risk-neutral probability for the stock price to go up is

$$q_u = \frac{e^r S - dS}{uS - dS} = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 0.8607}{1.2840 - 0.8607} = 0.4502,$$

and the probability to go down is $q_d = 1 - q_u = 0.5498$

$\omega_1 \quad u \quad u$

$$q_{uu} = 0.4502 \times 0.4502 = 0.2027$$

$\omega_2 \quad u \quad d$

$$q_{ud} = 0.4502 \times 0.5498 = 0.2475$$

$\omega_3 \quad d \quad u$

$$q_{du} = 0.5498 \times 0.4502 = 0.2475$$

$\omega_4 \quad d \quad d$

$$q_{dd} = 0.5498 \times 0.5498 = 0.3023$$

So the expected value (under the risk-neutral probabilities/measure) is

$$e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[X]$$

where $r = 0.05, t = 0, T = 2,$

$$\begin{aligned} &= e^{-0.1}(0.2027 \times 10.9731 + 2 \times 0.2475 \times 0.1028 + 0.3023 \times 0) \\ &= e^{-0.1} \times 2.2751 = 2.059 \end{aligned}$$

The Continuous Time Limit

Performing a Taylor expansion around $\delta t = 0$ we have

$$u \sim \frac{1}{2}\left((1 - \mu\delta t + \cdots) + \left(1 + \left(\mu + \sigma^2\right)\delta t + \cdots\right)\right) + \frac{1}{2}\left(e^{-2\mu\delta t} + 2e^{\sigma^2\delta t} + e^{2\left(\mu+\sigma^2\right)\delta t} - 4\right)^{\frac{1}{2}}$$

$$= \left(1 + \frac{1}{2}\sigma^2\delta t + \cdots\right) + \frac{1}{2}\left(1 - 2\mu\delta t + 2 + 2\sigma^2\delta t + 1 + 2\mu\delta t + 2\sigma^2\delta t - 4 + \cdots\right)$$

$$= \left(1 + \frac{1}{2}\sigma^2\delta t + \cdots\right) + \frac{1}{2}\left(4\sigma^2\delta t + \cdots\right)^{\frac{1}{2}}$$

Ignoring the terms of order $\delta t^{\frac{3}{2}}$ and higher we get the result

$$= \left(1 + \sigma\delta t^{\frac{1}{2}} + \frac{1}{2}\sigma^2\delta t\right) + \cdots$$

Since $uv = 1$ this implies that $v = u^{-1}$. Using the expansion for u obtained earlier we have

$$v = \left(1 + \sigma\delta t^{\frac{1}{2}} + \frac{1}{2}\sigma^2\delta t + \cdots\right)^{-1}$$

$$= \left(1 + \sigma\delta t^{\frac{1}{2}}\left(1 + \frac{1}{2}\sigma\delta t^{\frac{1}{2}}\right)\cdots\right)^{-1}$$

$$= \left(1 - \sigma\delta t^{\frac{1}{2}}\left(1 + \frac{1}{2}\sigma\delta t^{\frac{1}{2}}\right) + \left(\sigma\delta t^{\frac{1}{2}}\left(1 + \frac{1}{2}\sigma\delta t^{\frac{1}{2}}\right)^2 + \cdots\right)\right)$$

$$= 1 - \sigma\delta t^{\frac{1}{2}} - \frac{1}{2}\sigma^2\delta t + \sigma^2\delta t + \cdots$$

$$= 1 - \sigma\delta t^{\frac{1}{2}} + \frac{1}{2}\sigma^2\delta t$$

So we have

$$
\begin{aligned}
u &\sim 1+\sigma\sqrt{\delta t}+\frac{1}{2}\sigma^2\delta t \\
v &\sim 1-\sigma\sqrt{\delta t}+\frac{1}{2}\sigma^2\delta t
\end{aligned}
$$

So to summarise we can write

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\delta t}} \\
v &= e^{-\sigma\sqrt{\delta t}} \\
q &= \frac{e^{r\delta t}-v}{(u-v)}
\end{aligned}
$$

and use these to build the asset price tree using u and v, and then value the option backwards from T using

$$
e^{r\delta t}V(S,t)=qV(uS,t+\delta t)+(1-q)V(vS,t+\delta t)
$$

and at each stage the hedge ratio Δ is obtained using

$$
\Delta=\frac{V^{+}-V^{-}}{(u-v)S}=\frac{V(uS,t+\delta t)-V(vS,t+\delta t)}{(u-v)S}
$$

Now expand

$$
\begin{aligned}
V^{+} &= V\left(uS, t+\delta t\right) \sim V + \delta t\frac{\partial V}{\partial t} + \sigma\sqrt{\delta t}S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^{2}\delta t S^{2}\frac{\partial^{2} V}{\partial S^{2}},\\
V^{-} &= V\left(vS, t+\delta t\right) \sim V + \delta t\frac{\partial V}{\partial t} - \sigma\sqrt{\delta t}S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^{2}\delta t S^{2}\frac{\partial^{2} V}{\partial S^{2}}.
\end{aligned}
$$

Note that

$$
\Delta = \frac{V^{+} - V^{-}}{\left(u - v\right)S} \sim \frac{2\sigma\sqrt{\delta t}S\frac{\partial V}{\partial S}}{2\sigma\sqrt{\delta t}S} = \frac{\partial V}{\partial S}
$$

Then

$$
\begin{aligned}
V &= \frac{V^{+} - V^{-}}{\left(u - v\right)} + \frac{uV^{-} - vV^{+}}{\left(1 + r\delta t\right)\left(u - v\right)}\\
&= \frac{2\sigma\sqrt{\delta t}S}{2\sigma\sqrt{\delta t}}\frac{\partial V}{\partial S} + \frac{\left(1 + \sigma\sqrt{\delta t}\right)V^{-} - \left(1 - \sigma\sqrt{\delta t}\right)V^{+}}{\left(1 + r\delta t\right)2\sigma\sqrt{\delta t}}
\end{aligned}
$$

Rearranging to give

$$(1+r\delta t)\,2\sigma\sqrt{\delta t}V = 2\sigma\sqrt{\delta t}S\,(1+r\delta t)\,\frac{\partial V}{\partial S}+\left(V^- - V^+\right)+\sigma\sqrt{\delta t}\,\left(V^- + V^+\right),$$

and so

$$\begin{aligned}(1+r\delta t)\,2\sigma\sqrt{\delta t}V &= 2\sigma\sqrt{\delta t}S\,(1+r\delta t)\,\frac{\partial V}{\partial S} - 2\sigma\sqrt{\delta t}S\frac{\partial V}{\partial S} + \\ &\quad 2\sigma\sqrt{\delta t}\left(V+\tfrac{1}{2}\sigma^2\delta t S^2\frac{\partial^2 V}{\partial S^2}+\delta t\frac{\partial V}{\partial t}\right),\end{aligned}$$

$$(1+r\delta t)\,V = S\,(1+r\delta t)\,\frac{\partial V}{\partial S} - S\frac{\partial V}{\partial S} + \left(V+\tfrac{1}{2}\sigma^2\delta t S^2\frac{\partial^2 V}{\partial S^2}+\delta t\frac{\partial V}{\partial t}\right),$$

divide through by δt and allow $\delta t \longrightarrow 0$

$$rV = rS\frac{\partial V}{\partial S}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial t}$$

and hence the Black-Scholes Equation.