Martingales and Stochastic Calculus Tools

In this lecture...

- Further stochastic calculus

- How to test a stochastic process is a martingale?

- The definition of the Itô integral

A Martingale is a Driftless Stochastic Process

Martingales are a key concept in probability and in mathematical finance. The term 'martingale' may refer to very different ideas...

... a stochastic process that has no drift.

Essentially, this is the idea of a fair (random) game.

This is the concept we explore in this session.

We encounter Martingales through three distinct, but closely connected ideas:

1. *Martingales* as a class of stochastic process;

2. *Exponential martingales*, which are a specific and extremely useful example of a martingale;

3. *Equivalent martingale measures*, where we look for a probability measure $\mathbb{Q}$ such that a given stochastic process $S(t)$ is a martingale under $\mathbb{Q}$ regardless of its nature under $\mathbb{P}$. The correspondence between the measures $\mathbb{P}$ and $\mathbb{Q}$ is done through a change of measure.

Most of this session focuses on the first concept, that is, martingales as a class of stochastic processes.

Conditional Expectations

What makes a conditional expectation different (from an unconditional one) is information (just as in the case of conditional probability). In our probability space, $(\Omega, \mathcal{F}, \mathbb{P})$ information is represented by the filtration $\mathcal{F}$; hence a conditional expectation with respect to the (usual information) filtration seems a natural choice.

$$Y = \mathbb{E}\left[X \mid \mathcal{F}\right]$$

is the expected value of the random variable conditional upon the filtration set $\mathcal{F}$. In general

- In general Y will be a random variable

- Y will be adapted to the filtration $\mathcal{F}$.

Conditional expectations have the following useful properties: If X, Y are integrable random variables and α, β are constants then

1. **Linearity**:

$$\mathbb{E}\left[\alpha X + \beta Y \,|\, \mathcal{F}\right] = \alpha \mathbb{E}\left[X \,|\, \mathcal{F}\right] + \beta \mathbb{E}\left[Y \,|\, \mathcal{F}\right]$$

2. **Tower Property (i.e. Iterated Expectations)**: if $\mathcal{F} \subset \mathcal{G}$

$$\mathbb{E}\left[\mathbb{E}\left[X \,|\, \mathcal{G}\right] \,|\, \mathcal{F}\right] = \mathbb{E}\left[X \,|\, \mathcal{F}\right]$$

   This property states that if taking iterated expectations with respect to several levels of information, we may as well take a single expectation subject to the smallest set of available information.

3. As a special case of the Tower property, we have

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{F}\right]\right] = \mathbb{E}\left[X\right]$$

since "no filtration" is always a smaller information set than any filtration

4. **Taking Out What Is Known**: if X is $\mathcal{F}$-measurable, then the value of X is known once we know $\mathcal{F}$. Therefore,

$$\mathbb{E}\left[X|\mathcal{F}\right] = X$$

5. **Taking Out What Is Known (2)**: by extension, if X is $\mathcal{F}$-measurable but not Y,

$$\mathbb{E}\left[XY|\mathcal{F}\right] = X\mathbb{E}\left[Y|\mathcal{F}\right]$$

6. **Independence**: if X is independent from $\mathcal{F}$, then knowing $\mathcal{F}$ is useless to predict the value of X. Hence,

$$\mathbb{E}\left[X|\mathcal{F}\right] = \mathbb{E}\left[X\right]$$

7. **Positivity:** if $X \geq 0$ then $\mathbb{E}[X|\mathcal{F}] \geq 0$.

8. **Jensen's Inequality**: let f be a convex function, then

$$f(\mathbb{E}[X|\mathcal{F}]) \leq \mathbb{E}[f(X)|\mathcal{F}]$$

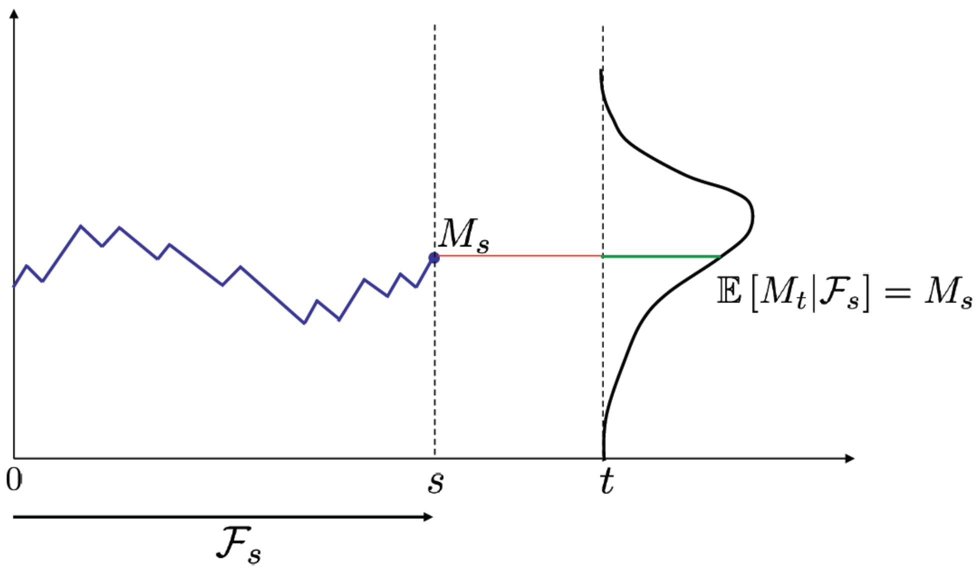
Adapted (Measurable) Process

A stochastic process S_t is said to be adapted to the filtration $\mathcal{F}_t$ (or measurable with respect to $\mathcal{F}_t$, or $\mathcal{F}_t$ -adapted) if the value of S_t at time t is known given the information set $\mathcal{F}_t$.

Discrete Time Martingales

A discrete time stochastic process $\{M_t : t = 0, \ldots, T\}$ such that M_t is $\mathcal{F}_t$-measurable for $\mathbb{T} = \{0, \ldots, T\}$ is a **martingale** if $\mathbb{E}\,|M_t| < \infty$ and

$$\mathbb{E}\left[M_{t+1}|\mathcal{F}_t\right] = M_t \tag{1}$$



The first equation represents a standard integrability condition.

The second equation tells you that the expected value of M at time $t+1$ conditional on all the information available up to time t is the value of M at time t. In short, a Martingale is a **driftless process**.

If we take expectation on both sides of eqn. 1, then

$$\mathbb{E}\left[M_{t+1}\right] = \mathbb{E}\left[M_t\right]$$

This is due to the **Tower Property** of conditional expectations.

Martingales are a very nice mathematical object. They "get rid of the drift" and enable us to focus on what probabilists consider is the most interesting part: the statistical properties of purely random processes.

In addition, Doob and Meyer have developed a powerful theory centred around martingales.

Continuous Time Martingales

Next, we generalize our definitions to continuous time: A continuous time stochastic process

$$\left\{ M_t : t \in \mathbb{R}^+ \right\}$$

such that M_t is $\mathcal{F}_t$-measurable for $t \in \mathbb{R}^+$ is a **martingale** if

$$\mathbb{E}\left|M_t\right| < \infty$$

and

$$\mathbb{E}\left[M_t | \mathcal{F}_s\right] = M_s, \quad 0 \leq s \leq t.$$

A Few More Things About Martingales...

1. Brownian motions are martingales;

2. Itô integrals are martingales;

3. Markov processes are not necessarily martingales (and vice versa).

A Brownian Motion is a Martingale

Lévy's Martingale Characterisation

Let $X_t,\ t > 0$ be a stochastic process and let $\mathcal{F}_t$ be the filtration generated by it. X_t is a Brownian motion iff the following conditions are satisfied:

1. $X_0 = 0$ a.s.;
2. the sample paths $t \mapsto X_t$ are continuous a.s.;
3. X_t is a martingale with respect to the filtration $\mathcal{F}_t$;
4. $|X_t|^2 - t$ is a martingale with respect to the filtration $\mathcal{F}_t$.

The Lévy characterization can be contrasted with the classical definition of a Brownian motion as a stochastic process X_t satisfying:

1. $X_0 = 0$ a.s.;

2. the sample paths $t \mapsto X(t)$ are continuous a.s.;

3. **independent increments**: for $t_1 < t_2 < t_3 < t_4$ the increments $X_{t_4} - X_{t_3}$, $X_{t_2} - X_{t_1}$ are independent;

4. **normally distributed increments**: $X_t - X_s \sim N(0, |t - s|)$.

Lévy's characterization neither mentions independent increments nor normally distributed increments.

Instead, Lévy introduces two easily verifiable martingale conditions.

Itô Integrals and Martingales

Next, we explore the link between Itô integration and martingales.

Consider the stochastic process $Y(t) = X^2(t)$. By Itô, we have

$$X^2(T) = T + \int_0^T 2X(t)dX(t)$$

Taking the expectation, we get

$$\mathbb{E}[X^2(T)] = T + \mathbb{E}\left[\int_0^T 2X(t)dX(t)\right]$$

Now, the quadratic variation property of Brownian motions implies that

$$\mathbb{E}[X^2(T)] = T$$

and hence

$$\mathbb{E}\left[\int_0^T 2X(t)dX(t)\right] = 0$$

Therefore, the Itô integral

$$\int_0^T 2X(t)dX(t)$$

is a martingale.

In fact, this property is shared by all Itô integrals.

The Itô integral is a martingale

Let $g(t, X_t)$ be a function on $[0, T]$ and satisfying the technical condition. Then the Itô integral

$$\int_0^T g(t, X_t) dX_t$$

is a martingale.

So, Itô integrals are martingales.

But does the converse hold? Can we represent any martingale as an Itô integral?

The answer is yes!

Martingale Representation Theorem

If M_t is a martingale, then there exists a function $g(t, X_t)$ satisfying the technical condition such that

$$M_T = M_0 + \int_0^T g(t, X_t) dX_t$$

Example We will show that

$$\mathbb{E}\left[X^2(T)\right] = T$$

using only Itô and the fact that Itô integrals are martingales.
Consider the function $F(t, X_t) = X_T^2$, then by Itô's lemma,

$$\begin{aligned} X_T^2 &= X_0^2 + \frac{1}{2}\int_0^T 2dt + \int_0^T 2X_t dX_t \\ &= \int_0^T dt + 2\int_0^T X_t dX_t \end{aligned}$$

since $X_0 = 0$
Taking the expectation,

$$\mathbb{E}\left[X_T^2\right] = \mathbb{E}\left[\int_0^T dt\right] + 2\mathbb{E}\left[\int_0^T X_t dX_t\right]$$

Now,

$$\int_0^T X_t dX_t$$

is an Itô integral and as a result $\mathbb{E}\left[\int_0^T X_t dX_t\right] = 0$
Moreover,

$$\mathbb{E}\left[\int_0^T dt\right] = \mathbb{E}\left[T\right] = T$$

We can conclude that

$$\mathbb{E}\left[X^2(T)\right] = T$$

As an aside, we can usually exchange the order of integration between the time integral and the expectation so that

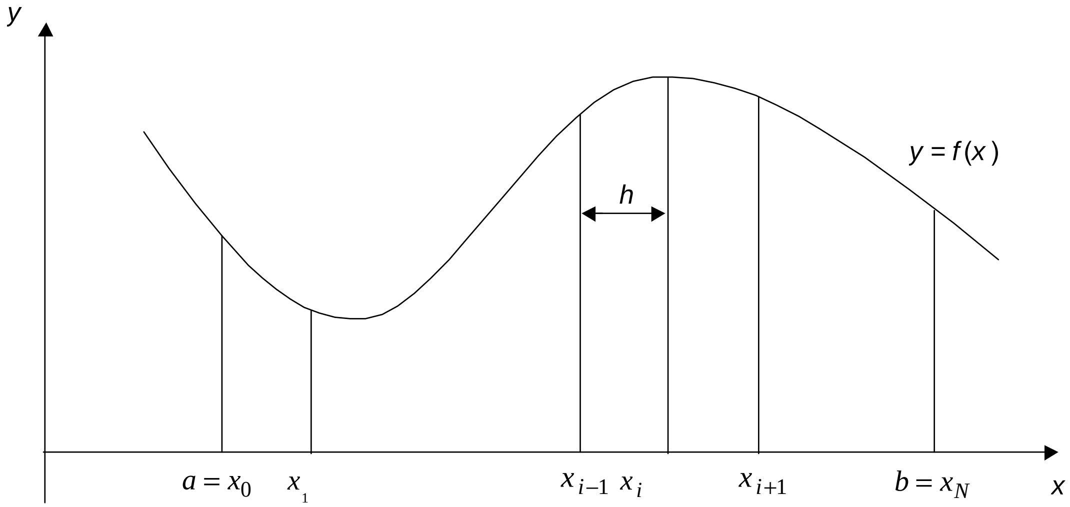
$$\mathbb{E}\left[\int_0^T f(X_t)dt\right] = \int_0^T \mathbb{E}\left[f(X_t)\right] dt$$

This is due to an analysis result known as **Fubini's Theorem**.

Itô Integral

Recall the usual Riemann definition of a definite integral

$$\int_a^b f(x)\,dx$$



which represents the area under the curve between $x = a$ and $x = b$, where the curve is the graph of $f(x)$ plotted against x.

Assuming f is a "well behaved" function on $[a, b]$, there are many different ways (which all lead to the same value for the definite integral).

Start by partitioning $[a, b]$ into N intervals with end points $x_0 = a < x_1 < x_2 < \ \ldots\ldots \ < x_{N-1} < x_N = b$, where the length of an interval $dx = x_i - x_{i+1}$ tends to zero as $N \to \infty$. So there are N intervals and $N + 1$ points x_i.

Discretising x gives

$$x_i = a + idx$$

Now consider the definite integral

$$\int_0^T f(t)\,dt.$$

With Riemann integration there are a number of ways we can approximate this:

1. left hand rectangle rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i)(t_{i+1} - t_i)$$

2. right hand rectangle rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1})(t_{i+1} - t_i)$$

3. trapezium rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{1}{2}\left(f(t_i) + f(t_{i+1})\right)(t_{i+1} - t_i)$$

4. midpoint rule

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(\tfrac{1}{2}(t_i + t_{i+1})\right)(t_{i+1} - t_i)$$

In the limit $N \to \infty$, $f(t)$ we get the same value for each definition of the definite integral, provided the function is integrable.

Now consider the stochastic integral of the form

$$\int_0^T f(t, X)\,dX = \int_0^T f(t, X(t))\,dX(t)$$

where $X(t)$ is a Brownian motion. We can define this integral as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$

where $X_i = X(t_i)$, or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(t_{i+\frac{1}{2}}, X_{i+\frac{1}{2}}\right)(X_{i+1} - X_i),$$

where $t_{i+\frac{1}{2}} = \frac{1}{2}(t_i + t_{i+1})$ and $X_{i+\frac{1}{2}} = X\left(t_{i+\frac{1}{2}}\right)$ or in many other ways. So clearly drawing parallels with the above Riemann form.

Very Important: In the case of a stochastic variable $dX(t)$ the value of the stochastic integral **does** depend on which definition we choose.

In the case of a stochastic integral, the definition

$$I = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$

is special. This definition results in the **Itô Integral**.

It is special because it is **non-anticipatory**; given that we are at time t_i we know $X_i = X(t_i)$ and therefore we know $f(t_i, X_i)$. The only uncertainty is in the $X_{i+1} - X_i$ term.

Compare this to a definition such as

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} f\left(t_{i+1}, X_{i+1}\right)\left(X_{i+1}-X_i\right),$$

which is **anticipatory**; given that at time t_i we know X_i but are uncertain about the future value of X_{i+1}. Thus we are uncertain about *both* the value of

$$f\left(t_{i+1}, X_{i+1}\right)$$

and the value of $\left(X_{i+1}-X_i\right)$ – there exists uncertainty in both of these quantities. That is, evaluation of this integral requires us to anticipate the future value of X_{i+1} so that we may evaluate $f\left(t_{i+1}, X_{i+1}\right)$.

The main thing to note about Itô integrals is that I is a random variable (unlike the deterministic case). Additionally, since I is essentially the limit of a sum of normal random variables, then by the CLT I is also normally distributed, and can be characterized by its mean and variance.

Example: Show that Itô's lemma implies that

$$3\int_0^T X^2 dX = X(T)^3 - X(0)^3 - 3\int_0^T X(t)\,dt.$$

Show that the result also can be found by writing the integral

$$3\int_0^T X^2 dX = \lim_{N\to\infty} 3\sum_{i=0}^{N-1} X_i{}^2 \left(X_{i+1} - X_i\right)$$

Hint: use $3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$.

The Itô integral here is defined as

$$\int_0^T 3X^2(t)\,dX(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} 3X_i^2 \left(X_{i+1} - X_i\right)$$

Now note the hint:

$$3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$$

hence

$$
\begin{aligned}
&\equiv 3X_i^2\left(X_{i+1}-X_i\right)\\
&= X_{i+1}^3 - X_i^3 - 3X_i\left(X_{i+1}-X_i\right)^2 - \left(X_{i+1}-X_i\right)^3,
\end{aligned}
$$

so that

$$
\sum_{i=0}^{N-1} 3X_i^2\left(X_{i+1}-X_i\right) =
$$

$$
\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 - \sum_{i=0}^{N-1} 3X_i\left(X_{i+1}-X_i\right)^2
$$

$$
-\sum_{i=0}^{N-1} \left(X_{i+1}-X_i\right)^3
$$

Now the first two expressions above give

$$\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 = X_N^3 - X_0^3$$
$$= X(T)^3 - X(0)^3 .$$

In the limit $N \to \infty$, i.e. $dt \to 0$, $(X_{i+1} - X_i)^2 \to dt$, so

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} 3X_i (X_{i+1} - X_i)^2 = \int_0^T 3X(t)\, dt$$

Finally $(X_{i+1} - X_i)^3 = (X_{i+1} - X_i)^2 \cdot (X_{i+1} - X_i)$ which when $N \to \infty$ behaves like $dX^2 dX \sim O\left(dt^{3/2}\right) \longrightarrow 0$.

Hence putting together gives

$$X(T)^3 - X(0)^3 - \int_0^T 3X(t)\, dt$$

which is consistent with Itô's lemma.

The other important property that the Itô integral has is that it is a martingale. We know that

$$X_{i+1} - X_i$$

is a martingale; i.e. in the context

$$\mathbb{E}\left[X_{i+1} - X_i\right] = 0.$$

Since

$$\mathbb{E}\left[\sum_{i=0}^{N-1} f\left(t_i, X_i\right)\left(X_{i+1} - X_i\right)\right] = \sum_{i=0}^{N-1} f\left(t_i, X_i\right)\mathbb{E}\left[X_{i+1} - X_i\right] = 0$$

Thus

$$\mathbb{E}\left[\int_0^T f\left(t, X\left(t\right)\right) dX\left(t\right)\right] = 0.$$

This is, essentially a consequence of the Itô integral being non-anticipatory, as discussed earlier. No other stochastic integral has this property.

Properties of Itô Integrals:

1. **Linearity**

$$\int_0^T (\alpha f(X_t) + \beta g(X_t))\, dX_t = \alpha \int_0^T f(X_t) dX_t + \beta \int_0^T g(X_t) dX_t$$

2. **Itô Isometry**

$$\mathbb{E}\left[\left(\int_0^T f_t dX_t\right)^2\right] = \mathbb{E}\left[\int_0^T f_t^2 dt\right]$$

The **linearity property** is carried over from the general definition of integrals as the limit of a sum.

The **isometry property** is used to extend the definition of the Itô integral to a very general class of functions. As a result, it is often mentioned as one of the key properties.

Proving that a Continuous Time Stochastic Process is a Martingale

Consider a stochastic process $Y(t)$ solving the following SDE:

$$dY(t) = f(Y_t, t)dt + g(Y_t, t)dX(t), \qquad Y(0) = Y_0$$

How can we tell whether $Y(t)$ is a martingale?

The answer has to do with the fact that Itô integrals are martingales.

$Y(t)$ is a martingale if and only if it satisfies the martingale condition

$$\mathbb{E}\left[Y_t | \mathcal{F}_s\right] = Y_s, \quad 0 \leq s \leq t$$

Let's start by integrating the SDE between s and t to get an exact form for $Y(t)$:

$$Y(t) = Y(s) + \int_s^t f(Y_u, u)du + \int_s^t g(Y_u, u)dX(u)$$

Taking the expectation conditional on the filtration at time s, we get

$$\begin{aligned}\mathbb{E}\left[Y_t|\mathcal{F}_s\right] &= \mathbb{E}\left[Y(s)+\int_s^t f(Y_u,u)du+\int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s\right]\\ &= Y(s)+\mathbb{E}\left[\int_s^t f(Y_u,u)du|\mathcal{F}_s\right]\end{aligned}$$

where the last line follows from the fact that a Itô integral is a martingale, $\therefore$

$$\mathbb{E}\left[\int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s\right]=\int_s^s g(Y_u,u)dX(u)=0.$$

So, $Y(t)$ is a martingale iff

$$\mathbb{E}\left[\int_s^t f(u)du|\mathcal{F}_s\right]=0$$

This condition is satisfied only if $f(Y_t,t)=0$ for all t. Returning to our SDE, we conclude that $Y(t)$ is a martingale iff it is of the form

$$dY(t)=g(Y_t,t)dX(t),\qquad Y(0)=Y_0$$

This is why we say that martingales are “driftless processes”

Example Determine which of the following processes are martingales.

1. $Y(t) = X(t) + 4t$

2. $Y(t) = X^2(t) + k$, where k is a given constant. Does the answer depend on the value of k?

3. $Y(t) = X_1(t)X_2(t)$ where $X_1(t)$ and $X_2(t)$ are two standard Brownian motions with correlation ρ so that $dX_1(t)dX_2(t) \rightarrow \rho dt$. Does the answer depend on the value of ρ?

(1) $Y(t) = X(t) + 4t$

Intuitively, this cannot be a Martingale since $X(t)$ is a martingale and $4t$ adds some drift.

Mathematically, the SDE for $Y(t)$ is:

$$dY(t) = 4dt + dX(t)$$

$Y(t)$ is a Brownian motion with drift. Hence, $Y(t)$ is **not a martingale**.

(2) $Y(t) = X^2(t) + k$

Intuitively, this cannot be a Martingale since $X^2(t) - t$ is a martingale (recalling the *quadratic variation* property of Brownian motions!).

Mathematically, by Itô applied to the function $f(x) = x^2 + k$, the SDE for the process $Y(t) = f(X(t))$ is given by

$$dY(t) = dt + 2X(t)dX(t)$$

The dynamics of $Y(t)$ has a drift: $Y(t)$ is **not a martingale** and this result is independent from the specific value of k.

(3) $Y(t) = X_1(t)X_2(t)$ where $X_1(t)$ and $X_2(t)$ are two independent standard Brownian motions

By the *Itô product rule*,

$$dY(t) = X_1(t)dX_2(t) + X_2(t)dX_1(t) + \rho dt$$

For $Y(t)$ to be a martingale, its dynamics must be driftless, i.e. we must have $\rho dt = 0$.

This is only the case when $\rho = 0$ and the two Brownian motions $X_1(t)$ and $X_2(t)$ are independent.

In the general case, when $\rho \neq 0$, $Y(t)$ is **not a martingale**.

Exponential Martingales

Let's start with a motivating example.

Consider the stochastic process $Y(t)$ satisfying the SDE

$$dY(t) = f(t)dt + g(t)dX(t), \qquad Y(0) = Y_0 \tag{2}$$

where $f(t)$ and $g(t)$ are two time-dependent functions and $X(t)$ is a standard Brownian motion.

Define a new process $Z(t) = e^{Y(t)}$

How should we choose $f(t)$ if we want the process $Z(t)$ to be a martingale?

Consider the process $Z(t) = e^{Y(t)}$. Applying Itô to the function $F(y) = e^y$ and the process $Y(t)$ given in eqn. 2, we obtain:

$$
\begin{aligned}
dZ(t) &= de^{Y(t)} \\
&= \frac{dZ}{dY} dY(t) + \frac{1}{2} \frac{d^2 F}{dY^2} dY^2(t) \\
&= \frac{dZ}{dY} \left(f(t)dt + g(t)dX(t) \right) + \frac{1}{2} \frac{d^2 Z}{dY^2} g^2(t)dt \\
&= e^{Y(t)} \left(f(t) + \frac{1}{2} g^2(t) \right) dt + e^{Y(t)} g(t)dX(t) \\
&= Z(t) \left[\left(f(t) + \frac{1}{2} g^2(t) \right) dt + g(t)dX(t) \right]
\end{aligned}
$$

$Z(t)$ is a martingale if and only if it is a driftless process.

Therefore for $Z(t)$ to be a martingale we must have

$$
f(t) + \frac{1}{2} g^2(t) = 0
$$

This is only possible if

$$f(t) = -\frac{1}{2} g^2(t)$$

Going back to the process $Y(t)$, we must have

$$dY(t) = -\frac{1}{2} g^2(t)dt + g(t)dX(t), \qquad Y(0) = Y_0$$

implying that

$$Y(T) = Y_0 - \frac{1}{2} \int_0^T g^2(t)dt + \int_0^T g(t)dX(t)$$

Hence, in terms of $Z(t)$:

$$dZ(t) = Z(t)g(t)dX(t).$$

Using the earlier relationship, we can write $Z(T) = e^{Y(T)}$.

Let's simplify this $Z(T) =$

$$\exp\left\{Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

to give

$$Z(T) = Z_0 \exp\left\{-\frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

Because the stochastic process $Z(t)$ is the exponential of another process (namely $Y(t)$) and because it is a martingale, we call $Z(t)$ an **exponential martingale**.

We have actually just stumbled upon a much more general and very important result.

Changing Probability Measure

You have seen in the Binomial Model lecture that there is more than just one probability measure.

Indeed, the lecture introduced you to the distinction between the "real" or "physical" probability measure, which we encounter every day on our Bloomberg or Reuters screen, and the so-called "risk-neutral" measure, which is used for pricing.

Probability measures are by no means unique. We will see in the next lecture that the powerful arsenal of martingale techniques enables us, under certain assumptions, to change measure and transpose our problem subject to the real world measure into an equivalent problem formulated as a martingale under a different measure.

For now, we just outline the rules that allow us to define equivalent measures.

Equivalent Measure

If two measures $\mathbb{P}$ and $\mathbb{Q}$ share the same sample space Ω and if $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$ for all subset A, we say that $\mathbb{Q}$ is **absolutely continuous** with respect to $\mathbb{P}$ and denote this by $\mathbb{Q} << \mathbb{P}$.

The key point is that all impossible events under $\mathbb{P}$ remain impossible under $\mathbb{Q}$. The probability mass of the possible events will be distributed differently under $\mathbb{P}$ and $\mathbb{Q}$. In short "it is alright to tinker with the probabilities as long as we do not tinker with the (im)possibilities"

If $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$ then the two measures are said to be **equivalent**, denoted by $\mathbb{P} \sim \mathbb{Q}$.

This extremely important result is formalized in the **Radon Nikodym Theorem**.

The main interest of a change of measure is to make difficult problems easier to solve. While some problems might be extremely difficult to tackle under the real-world measure $\mathbb{P}$, it might be possible to find an equivalent measure $\mathbb{Q}$ under which they are much easier to solve.

As a result, the change of measure techniques have become a cornerstone not only of modern probability but also of mathematical finance, where they are widely used in asset pricing.

Summary

- We have seen that a martingale is a driftless process
- We know how to show a process is a martingale
- Appreciate the importance of an exponential Martingale.