The Black–Scholes Model

In this lecture. . .

- the assumptions that go into the Black–Scholes model
- foundations of options theory: delta hedging and no arbitrage
- the Black–Scholes partial differential equation
- the Black–Scholes formulæ for calls, puts and simple digitals
- the meaning and importance of the 'greeks,' delta, gamma, theta.

By the end of this lecture you will be able to

- derive the Black–Scholes partial differential equation

- quote formulæ for simple contracts

- understand the meaning of the common greeks

Introduction

The Black–Scholes equation was the biggest breakthrough in the pricing of options.

The theory is quite straightforward, using just the ideas from stochastic calculus that we have already seen.

The end result is a diffusion-type partial differential equation which can be used for the pricing of many different derivatives.

What determines the value of an option?

The value of an option is a function of the stock price S and time t.

The value of the option is also a function of parameters in the contract, such as the strike price E and the time to expiry $T - t$, T is the date of expiry.

The value will also depend on properties of the asset, such as its drift and its volatility, as well as the risk-free rate of interest:

$$V(\overbrace{S, t}^{\text{vars}}; \underbrace{\sigma, \mu}_{\text{GBM}}; \overbrace{E, T}^{\text{contract}}; r).$$

IR

Semi-colons separate different types of variables and parameters.

- S and t are variables;

- σ and μ are parameters associated with the asset price;

- E and T are parameters associated with the particular contract;

- r is a parameter associated with the currency.

For the moment just use $V(S,t)$ to denote the option value.

Black-Scholes Model

(1) Assumptions

(2) PDE

(3) Nobel prize winning formula

1973 Model Classic | Robust

The Black–Scholes assumptions

- The underlying follows a lognormal random walk with known volatility $\frac{dS}{S} = \mu\, dt + \sigma\, dW$ $\sigma, r \in \mathbb{R}$

- The risk-free interest rate is a known function of time $\sigma(t)$ $r(t)$

- There are no dividends on the underlying

- Delta hedging is done continuously

- There are no transaction costs on the underlying, no limits to trading, no taxes,

- There are no arbitrage opportunities

And more. . .

A very special portfolio

We assume that the asset evolves according to

$$\text{(*)} \quad dS = \mu S\, dt + \sigma S\, dX.$$

Then we imagine constructing a special portfolio.

Use Π to denote the value of a portfolio of one long option position and a short position in some quantity Δ, **delta**, of the underlying:

long

$$\Pi = V(S,t) - \Delta S. \tag{1}$$

short an amount of stock

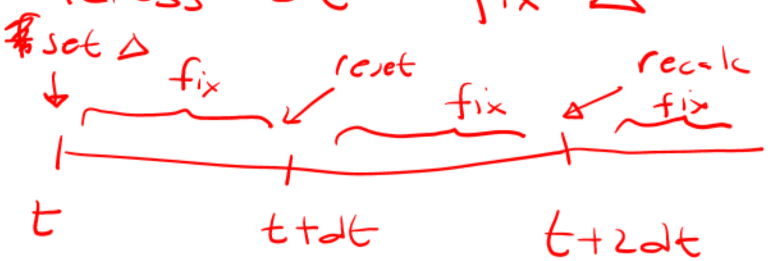
Intuition: Think of moves in S and accompanying move in V, for a call.

$t \longrightarrow t + dt$ $\qquad \Pi_{t+dt} - \Pi_t$

How does the value of the portfolio change? $\qquad \Delta(S, t)$

The change in the portfolio value is due partly to the change in the option value and partly to the change in the underlying:

across dt fix Δ $\qquad d(\Delta S) := \Delta\, dS$



$$d\underline{\Pi} = dV - \Delta\, dS.$$

rebalance

$\hookrightarrow \mu S\, dt + \sigma S\, dX$

- Notice that Δ has not changed during the time step.

We know that $V = V(S, t)$

Itô IV $\qquad dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S}\, dX$

From Itô we have

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt.$$

Thus the portfolio changes by

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\overset{(\mu S\,dt + \sigma S\,dX)}{dS} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta\, \overset{(\mu S\,dt + \sigma S\,dX)}{dS}. \qquad (2)$$

The right-hand side of (2) contains two types of terms, the deterministic and the random.

- The deterministic terms are those with the dt.
- The random terms are those with the dS. These random terms are the risk in our portfolio.

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S}\right)dt + \sigma S \frac{\partial V}{\partial S}\,dX - \Delta\left(\mu S\,dt + \sigma S\,dX\right)$$

$$\left(\sigma S \frac{\partial V}{\partial S} - \Delta \sigma S\right) dX$$

$$\underbrace{\qquad\qquad}_{=0} \qquad \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right) = 0$$

Elimination of risk: Delta hedging

Is there any way to reduce or even eliminate this risk? This can be done in theory by carefully choosing Δ.

If we choose

$$\Delta = \frac{\partial V}{\partial S} \qquad \Delta = \frac{V^+ - V^-}{S^+ - S^-} \tag{3}$$

then the randomness is reduced to zero.

* Any reduction in randomness is generally termed **hedging**. The perfect elimination of risk, by exploiting correlation between two instruments (in this case an option and its underlying) is generally called **Delta hedging**.

Delta hedging is an example of a **dynamic hedging** strategy. From one time step to the next the quantity $\frac{\partial V}{\partial S}$ changes, since it is, like V a function of the ever-changing variables S and t.

This means that the perfect hedge must be continually rebalanced.

$\Delta(S, t)$

$d\Pi$ riskless

$r \, d\Pi \, dt$

No arbitrage

After choosing the quantity Δ as suggested above, we hold a portfolio whose value changes by the amount

$$d\Pi = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \qquad (4)$$

- This change is completely *riskless*.

If we have a completely risk-free change $d\Pi$ in the value Π then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account:

$$d\Pi = r\Pi \, dt. \tag{5}$$

- This is an example of the **no arbitrage** principle.

$(4) \equiv (5)$

The Black–Scholes equation

$r\left(V - S\frac{\partial V}{\partial S}\right)dt$

Substituting (1), (3) and (4) into (5) we find that

$$\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt.$$

(4) (5)

On dividing by dt and rearranging we get

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{6}$$

This is the **Black–Scholes equation**. No μ

Observations:

$\frac{\partial V}{\partial t}$ $\frac{\partial^2 V}{\partial S^2}$

- The Black–Scholes equation equation is a **linear parabolic partial differential equation**

- The Black–Scholes equation contains all the obvious variables and parameters such as the underlying, time, and volatility, but there is no mention of the drift rate μ.

- This means that if two people agree on the volatility of an asset they will agree on the value of its derivatives *even if they have differing estimates of the drift*.

μ, σ

Linear

Introduce $\mathcal{L}$ as a linear operation. $\exists$ 2 functions f, g

$\exists$ a scalar $\lambda \in \mathbb{R}$

① $\mathcal{L}(\lambda f) = \lambda \mathcal{L}(f)$ Scalar multⁿ

② $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ vector addⁿ

Finance interpretation of above

① If an option is valued at $V \Longrightarrow$ 2 options cost $2 \times V$ & k options cost kV

② A portfolio of 2 options V_1, V_2 costs $(V_1 + V_2)$

2 Basic solutions of the B.S.E

① $\because$ Stock S is traded $V = S$ is a solⁿ of B.S.E

② Cash $V = S_0 e^{rt}$ is a solⁿ of B.S.E

Replication

$$\Pi = V - \Delta S$$
$$V = \Pi + \Delta S$$

Another way of looking at the hedging argument is to ask what happens if we hold a portfolio consisting of just the stock, in a quantity Δ, and cash.

If Δ is the partial derivative of some option value then such a portfolio will yield an amount at expiry that is simply that option's payoff.

In other words, we can use the same Black–Scholes argument to **replicate** an option just by buying and selling the underlying asset.

- This leads to the idea of a **complete market**. In a complete market an option can be replicated with the underlying, thus making options redundant.

Final conditions

The Black–Scholes equation knows nothing about what kind of option we are valuing.

This is dealt with by the **final condition**. We must specify the option value V as a function of the underlying at the expiry date T. That is, we must prescribe $V(S,T)$, the payoff.

For example, if we have a call option then we know that

$$V(S,T) = \max(S - E, 0).$$

Call

① Terminal Condition
(Payoff) $C(S,T) = \max(S-E,0)$

② Boundary Conditions

i) $S=0 \qquad C=0$

ii) $S \to \infty \qquad C \sim S$

Put

① Payoff
$P(S,T) = \max(E-S,0)$

② BCs

i) $S \to \infty \qquad P \to 0$

ii) $S=0$ use Put-Call parity

$$\underbrace{\cancel{C}}_{\to 0} - P = \underbrace{\cancel{S}}_{\to 0} - E e^{-r(T-t)}$$

$$P = E e^{-r(T-t)}$$

Again very ideal

Options on dividend-paying equities

Assume that the asset receives a continuous and constant dividend yield, D.

$DS\,dt$

- Thus in a time dt each asset receives an amount $DS\,dt$.

This must be built into the derivation of the Black–Scholes equation.

Take up the Black–Scholes argument at the point where we are looking at the change in the value of the portfolio:

$\Pi = V - \Delta S$

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta\, dS \underbrace{- D\Delta S\, dt}.$$

Set $\Delta = \partial V / \partial S$ to eliminate risk

The last term on the right-hand side is the amount of the dividend per asset, $DS\,dt$, multiplied by the number of the asset held, $-\Delta$.

No arb. $d\Pi = r\Pi\, dt$

The Δ must still be the rate of change of the option value with respect to the underlying for the elimination of risk.

$dS = (\mu - D)S\, dt + \sigma S\, dX$

End result:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0.$$

Two more interesting eqⁿs

① $$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f)\frac{\partial V}{\partial S} - rV = 0$$ FX

domestic   foreign

② $$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r + q)\frac{\partial V}{\partial S} - rV = 0$$ Commodities

cost of carry

Solving the equation and the greeks

The Black–Scholes equation has simple solutions for calls, puts and some other contracts. Now we are going to go quickly through the derivation of these formulæ.

The 'delta,' the first derivative of the option value with respect to the underlying, occurs as an important quantity in the derivation of the Black–Scholes equation. In this lecture we see the importance of other derivatives of the option price, with respect to the variables and with respect to some of the parameters.

- These derivatives are important in the hedging of an option position, playing key roles in risk management.

Derivation of the formulæ for calls, puts and simple digitals

The Black–Scholes equation is (again)

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \qquad (7)$$

This equation must be solved with final condition depending on the payoff: each contract will have a different functional form prescribed at expiry $t = T$, depending on whether it is a call, a put or something else.

i) Use transf$\hat{=}$s and subst$\hat{=}$s to reduce (7) to 1D heat eq$\hat{=}$

ii) Solve using similarity reduction

iii) Unwind the steps

The first step in the manipulation is to change from present value to future value terms.

Recalling that the payoff is received at time T but that we are valuing the option at time t this suggests that we write

Unknown

$$\frac{\partial V}{\partial S} = e^{-r(T-t)} \frac{\partial U}{\partial S}$$

- $$V(S,t) = \underbrace{e^{-r(T-t)}}_{PV} \underbrace{U(S,t)}.$$

$$\frac{\partial^2 V}{\partial S^2} = e^{-r(T-t)} \frac{\partial^2 U}{\partial S^2}$$

$$\frac{\partial V}{\partial t} = re^{-r(T-t)}U + e^{-r(T-t)}\frac{\partial U}{\partial t}.$$

$-rV$: PV

This takes our differential equation to

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0.$$

B.K.E

(Remember this result for later, present valuing means that one of the terms disappears.)

The second step is really trivial. Because we are solving a backward equation we'll write

$$\frac{\partial}{\partial t} : \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} \qquad - \frac{\partial}{\partial \tau}$$

- $\tau = T - t.$

This now takes our equation to

$$\frac{\partial U}{\partial \tau} = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S}.$$

F.K.E

When we first started modeling equity prices we used intuition about the asset price *return* to build up the stochastic differential equation model. Let's go back to examine the return and write

$\frac{\partial}{\partial S} \rightarrow \frac{d\xi}{dS}\frac{\partial}{\partial \xi} = \frac{1}{S}\frac{\partial}{\partial \xi}$ $\qquad \frac{d\xi}{dS} = \frac{1}{S}$

- $$\xi = \log S.$$

$\frac{\partial^2}{\partial S^2} \rightarrow \frac{1}{S^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi}\right)$ $\qquad S = e^{\xi}$ $\qquad \frac{d^2\xi}{dS^2} = -\frac{1}{S^2}$

With this as the new variable, we find that

$$\frac{\partial}{\partial S} = \underbrace{e^{-\xi}}_{\frac{1}{S}}\frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial^2}{\partial S^2} = \underbrace{e^{-2\xi}}_{\frac{1}{S^2}}\frac{\partial^2}{\partial \xi^2} - \underbrace{e^{-2\xi}}_{\frac{1}{S^2}}\frac{\partial}{\partial \xi}.$$

Now the Black–Scholes equation becomes

$$\longrightarrow \qquad \frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial \xi}.$$

$e^{(\mu - \frac{1}{2}\sigma^2)\,dt + \sigma\,dX}$

The last step is simple, but the motivation is not so obvious. Write

Use chain rule II for partial derivatives

$$\left\{ \begin{array}{l} x = \xi + \left(r - \frac{1}{2}\sigma^2\right)\tau \quad \text{and} \quad U = W(x, \tau). \\ \tau = \tau \end{array} \right.$$

This is just a 'translation' of the co-ordinate system. It's a bit like using the forward price of the asset instead of the spot price as a variable.

After this change of variables the Black–Scholes becomes the simpler

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2}. \tag{8}$$

Fwd eqn $\quad \dfrac{\partial p}{\partial t'} = c^2 \dfrac{\partial^2 p}{\partial y'^2}$ (Lecture 1.3)

And you've seen this equation before!

You've even solved it to find a special solution—out of the infinite number of possible solutions—and exactly the same solution will be needed here. (Lucky!)

Fundamental | Source Solution

To summarize, the transf$^{\text{ns}}$

$$V(S,t) = e^{-r(T-t)}U(S,t) = e^{-r\tau}U(S,T-\tau) = e^{-r\tau}U(e^{\xi},T-\tau)$$

$$= e^{-r\tau}U\left(e^{x-\left(r-\frac{1}{2}\sigma^2\right)\tau},T-\tau\right) = e^{-r\tau}W(x,\tau).$$

We are going to derive an expression for the value of any option whose payoff is a known function of the asset price at expiry.

This includes calls, puts and digitals. This expression will be in the form of an integral.

For special cases, we'll see how to rewrite this integral in terms of the cumulative distribution function for the Normal distribution. This is particularly useful since the function can be found on spreadsheets, calculators and in the backs of books.

But there are two steps before we can write down this integral.

- The first step is to find a special solution of (8), called the fundamental solution. This solution has useful properties.

- The second step is to use the linearity of the equation and the useful properties of the special solution to find the *general solution* of the equation.

Green's function

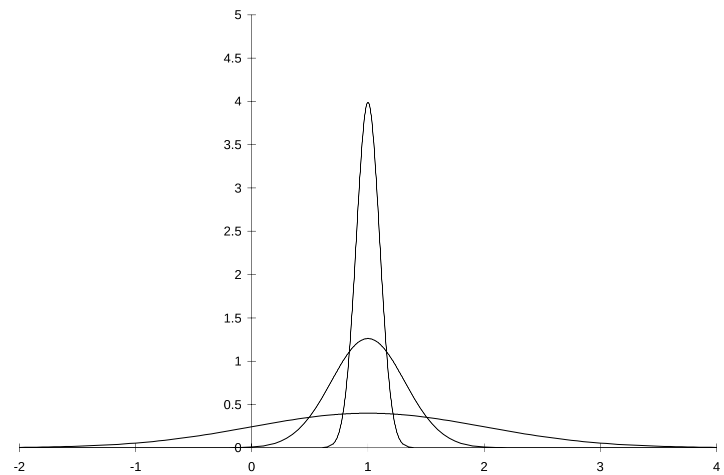
The first step is easy, just recall solving the equation from the earlier lecture. The solution we want is

- $$W_f(x,\tau;x') = \frac{1}{\sqrt{2\pi\tau}\,\sigma} e^{-\frac{(x-x')^2}{2\sigma^2\tau}}.$$

$$-\frac{1}{2} \qquad \frac{(x-x')^2}{\sigma^2\tau}$$

As you know, this is the probability density function for a Normal random variable x having mean of x' and standard deviation $\sigma\sqrt{\tau}$.

And this also strongly hints at a relationship between option values and probabilities. More anon!



Above is plotted W_f as a function of x' for several values of τ.

At $x' = x$ the function grows unboundedly, and away from this point the function decays to zero, as $\tau \to 0$.

Although the function is increasingly confined to a narrower and narrower region its area remains fixed at one.

- These properties of decay away from one point, unbounded growth at that point and constant area, result in a **Dirac delta function** $\delta(x' - x)$ as $\tau \to 0$.

The delta function has one important property, namely

$$* \quad \int \delta(x' - x) g(x')\, dx' = g(x)$$

where the integration is from any point below x to any point above x.

Thus the delta function 'picks out' the value of g at the point where the delta function is singular i.e. at $x' = x$.

In the limit as $\tau \to 0$ the function W becomes a delta function at $x = x'$. This means that

$$\lim_{\tau \to 0} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x'-x)^2}{2\sigma^2\tau}} g(x')\, dx' = g(x).$$

Whoa! This is tricky!

$$V(S,t) = e^{-r(T-t)}\, U(S,t)$$

I am going to 'cut to the chase' and quote the solution:

$(S,t) \longrightarrow (S',T)$

$\frac{dS}{S} = r\,dt + \sigma\,dX$

- $$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}$$

$$\int_0^\infty e^{-\left(\log(S/S')+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,\text{Payoff}(S')\frac{dS'}{S'}. \qquad (9)$$

This 'formula' works for any European, non path-dependent, option on a single lognormal underlying asset, all you need to know is the payoff function.

$V(S,t) = e^{-r(T-t)}\,\mathbb{E}\left[\ \text{Payoff}(S')\ \right]$

Observations

- This is a general formula (see above conditions on type of option)

- It is of the form of a) a discounting term multiplied by b) the integral of the payoff multiplied by c) another function

- This other 'function' is known as a Green's function

- This function can be interpreted as a probability

- The whole expression can be interpreted as the present value of the expected payoff under the
  i) risk-neutral density
  ii) " measure
  iii) " random walk

Let's look at special cases.

Define some option payoffs.

Formula for a call

The call option has the payoff function

$\int_0^\infty \to \int_E^\infty$

$$\text{Payoff}(S) = \max(S - E, 0).$$

Expression (9) can then be written as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^\infty e^{-\left(\log(S/S') + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2 / 2\sigma^2(T-t)} (S' - E) \frac{dS'}{S'}.$$

Return to the variable $x' = \log S'$, to write this as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\left(e^{x'}-E\right)dx'$$

$$= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,e^{x'}\,dx'$$

$$-E\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\left(-x'+\log S+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,dx'.$$

Both integrals in this expression can be written in the form

$$\int_d^\infty e^{-\frac{1}{2}x'^2}\,dx'$$

for some d (the second is just about in this form already, and the first just needs a completion of the square).

Thus the option price can be written as two separate terms involving the cumulative distribution function for a Normal distribution:

Full working to follow

$e^{-D(T-t)}$

$$\text{Call option value } = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

where

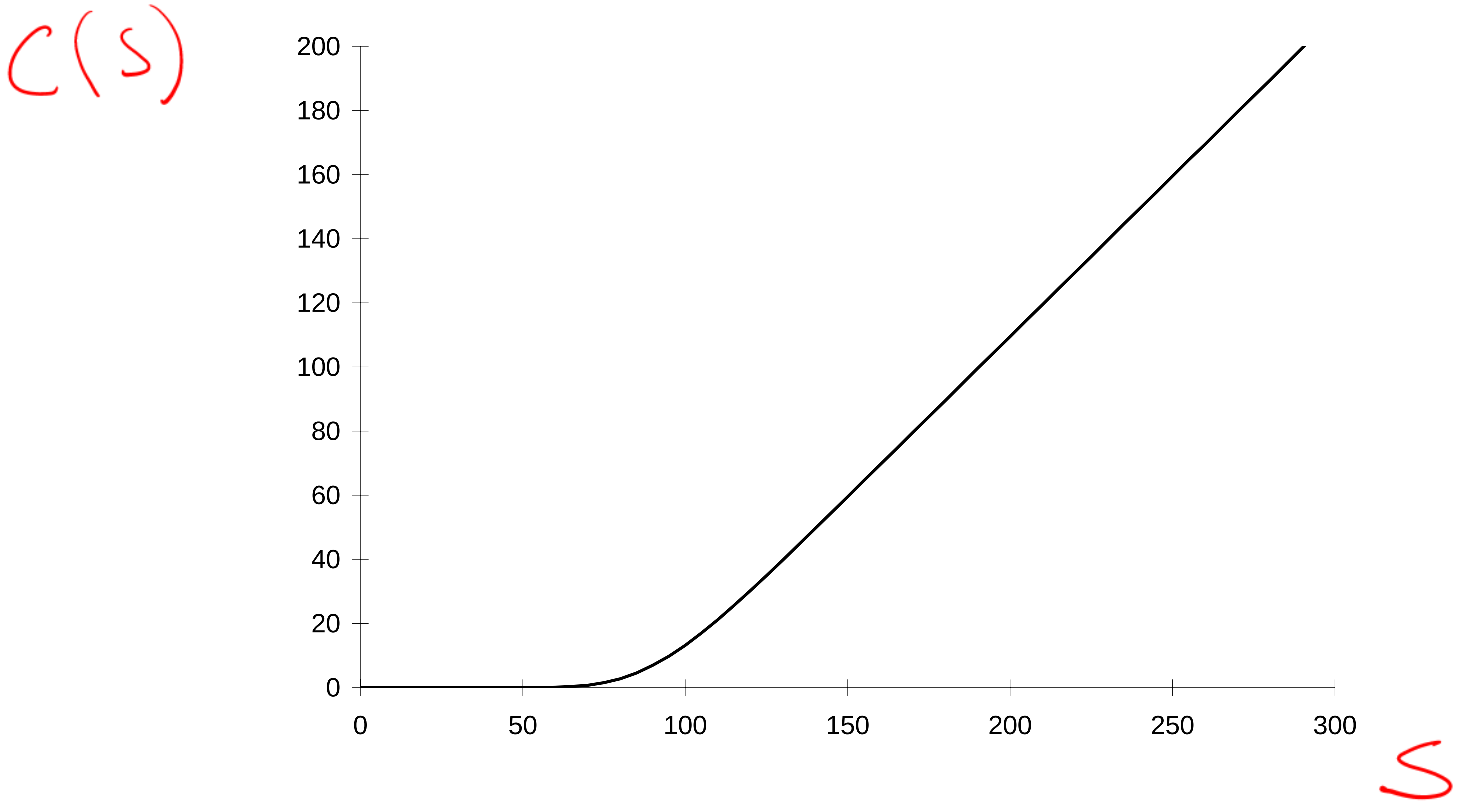
$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and}$$

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

$d_2 = d_1 - \sigma\sqrt{T-t}$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2}\,d\phi.$$

for some $t < T$

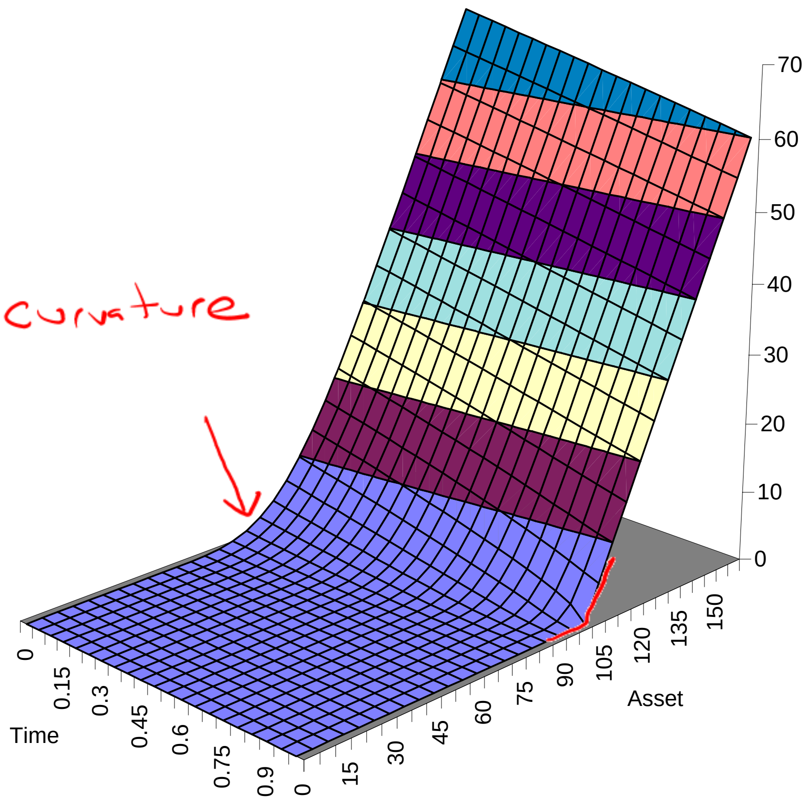


The value of a call option as a function of the underlying at a fixed time before expiry.

diffusion

$\sigma^2 S^2 \quad \underbrace{\frac{\partial^2 V}{\partial S^2}}_{\Gamma}$

curvature



$C(S,t)$

The value of a call option as a function of asset and time.

When there is continuous dividend yield on the underlying, or it is a currency, then

Call option value

$$C(S,t) = \underline{Se^{-D(T-t)}}N(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\log(S/E)+(r-D+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/E)+(r-D-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

$\tau = T-t$

When the asset is 'at-the-money forward,' i.e. $S = Ee^{-(r-D)(T-t)}$, and the option is close to expiration then there is a simple approximation for the call value (Brenner & Subrahmanyam, 1994):

$$\text{Call} \approx 0.4\, Se^{-D(T-t)}\sigma\sqrt{T-t}.$$

$E = Se^{(r-D)(T-t)}$

$$C = Se^{-D\tau}N(d_1) - Ee^{-r\tau}N(d_2) \qquad E = Se^{(r-D)\tau}$$

$$d_{12} = \frac{\log\left(\frac{S}{E}\right) + \left(r - D \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_{12} :: \frac{\cancel{\log} \frac{\cancel{S}\, \cancel{e^{-(r-D)\tau}}}{\cancel{S}\, e^{(r-D)\tau}} + \left(\cancel{r-D} \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_{12} = \pm \frac{1}{2}\sigma\sqrt{\tau}$$

$$C = Se^{-D\tau}N(d_1) - Se^{(r-D)\tau}e^{-\cancel{r}\tau}N(d_2) = Se^{-D\tau}\left(N(d_1) - N(d_2)\right)$$

Close to expiry means $t \to T$ or $\tau \to 0$

Kendall & Stuart $N(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(x - \frac{x^3}{6} + O(x^5)\right)$

however if $x \ll 1$ $N(x) \approx \frac{1}{2} + \frac{x}{\sqrt{2\pi}}$ $\qquad N(-x) \approx \frac{1}{2} - \frac{x}{\sqrt{2\pi}}$

$N(x) - N(-x) \approx \frac{2x}{\sqrt{2\pi}}$ but $x = \frac{1}{2}\sigma\sqrt{\tau}$ $\therefore$ $N(d_1) - N(d_2) = \frac{\cancel{2}}{\sqrt{2\pi}} \cdot \frac{1}{\cancel{2}}\sigma\sqrt{\tau} \approx 0.4\sigma\sqrt{T-t}$

$$C \approx 0.4\, Se^{-D(T-t)}\sigma\sqrt{T-t}$$

Formula for a put

The put option has payoff

$N(x) + N(-x) = 1$

$N(-x) = 1 - N(x)$

$$\text{Payoff}(S) = \max(E - S, 0).$$

The value of a put option can be found in the same way as above, or using put-call parity

$C - P = S - Ee^{-r(T-t)}$ $\quad$ $P = C - S + Ee^{-r(T-t)}$

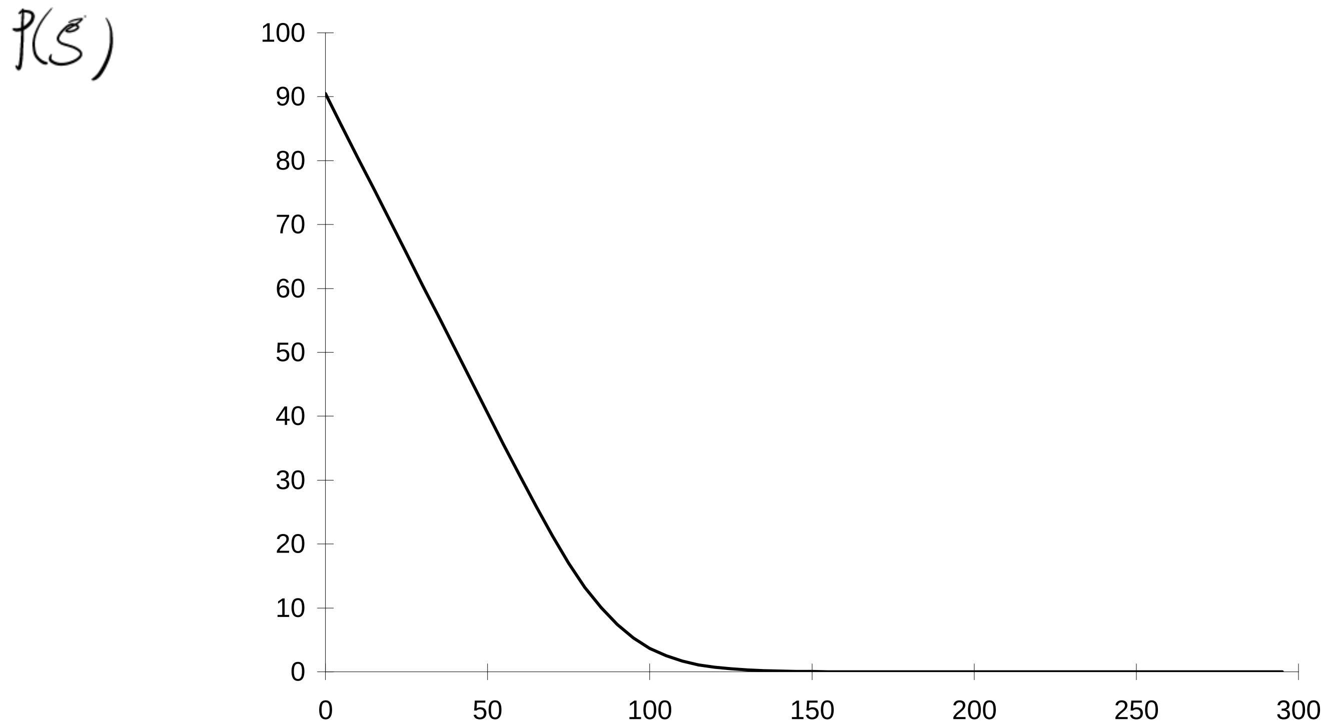
$$\text{Put option value } = -SN(-d_1) + Ee^{-r(T-t)}N(-d_2),$$

$P = SN(d_1) - Ee^{-r(T-t)}N(d_2) - S + Ee^{-r(T-t)}$

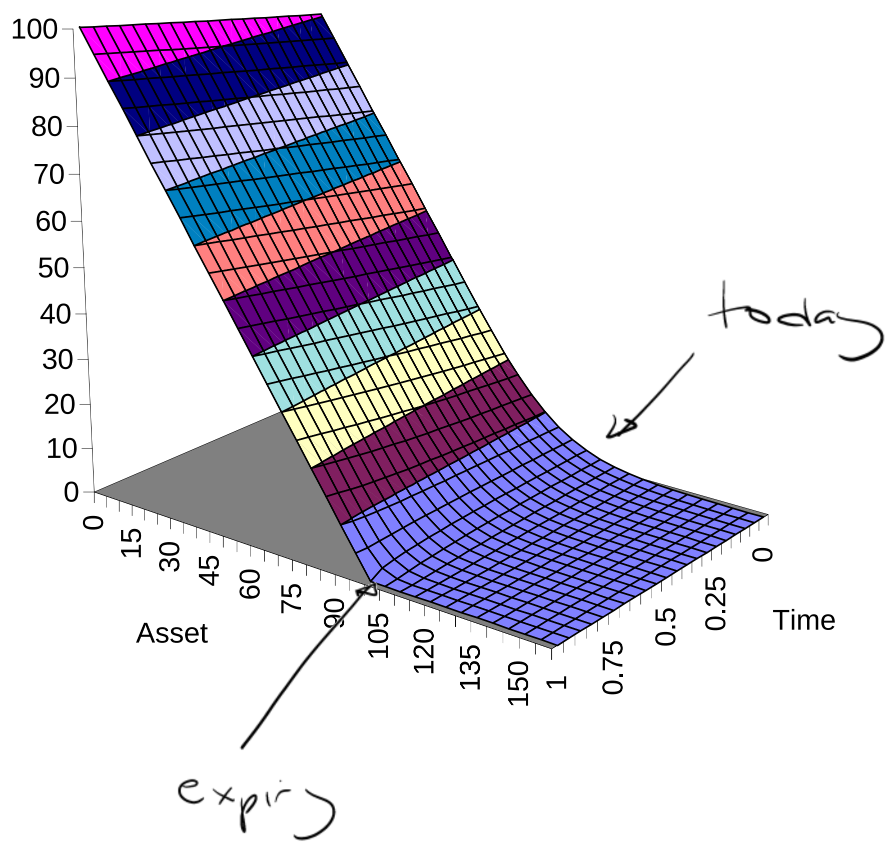
with the same d_1 and d_2.

$= -S(1 - N(d_1)) + Ee^{-r(T-t)}\left[1 - N(d_2)\right]$

$= -SN(-d_1) + Ee^{-r(T-t)}N(-d_2)$



The value of a put option as a function of the underlying at a fixed time to expiry.



The value of a put option as a function of the underlying and time to expiry.

When there is continuous dividend yield on the underlying, or it is a currency, then

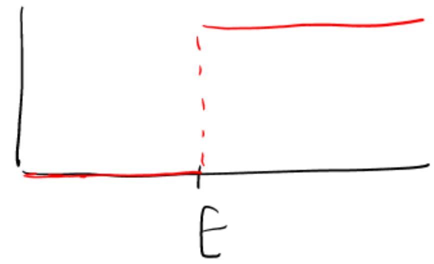
Put option value

$$-Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)$$

When the asset is at-the-money forward and the option is close to expiration the simple approximation for the put value (Brenner & Subrahmanyam, 1994) is

$$\text{Put} \approx 0.4\, Se^{-D(T-t)}\sigma\sqrt{T-t}.$$

Formula for a binary call



The binary call has payoff

$$\text{Payoff}(S) = \mathcal{H}(S-E), \quad = \begin{cases} 1 & S(T) > E \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{H}$ is the Heaviside function taking the value one when its argument is positive and zero otherwise.

Incorporating a dividend yield, we can write the option value as

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{\log E}^{\infty} e^{-\left(x'-\log S-\left(r-D-\frac{1}{2}\sigma^2\right)(T-t)\right)^2/2\sigma^2(T-t)}\,dx'.$$

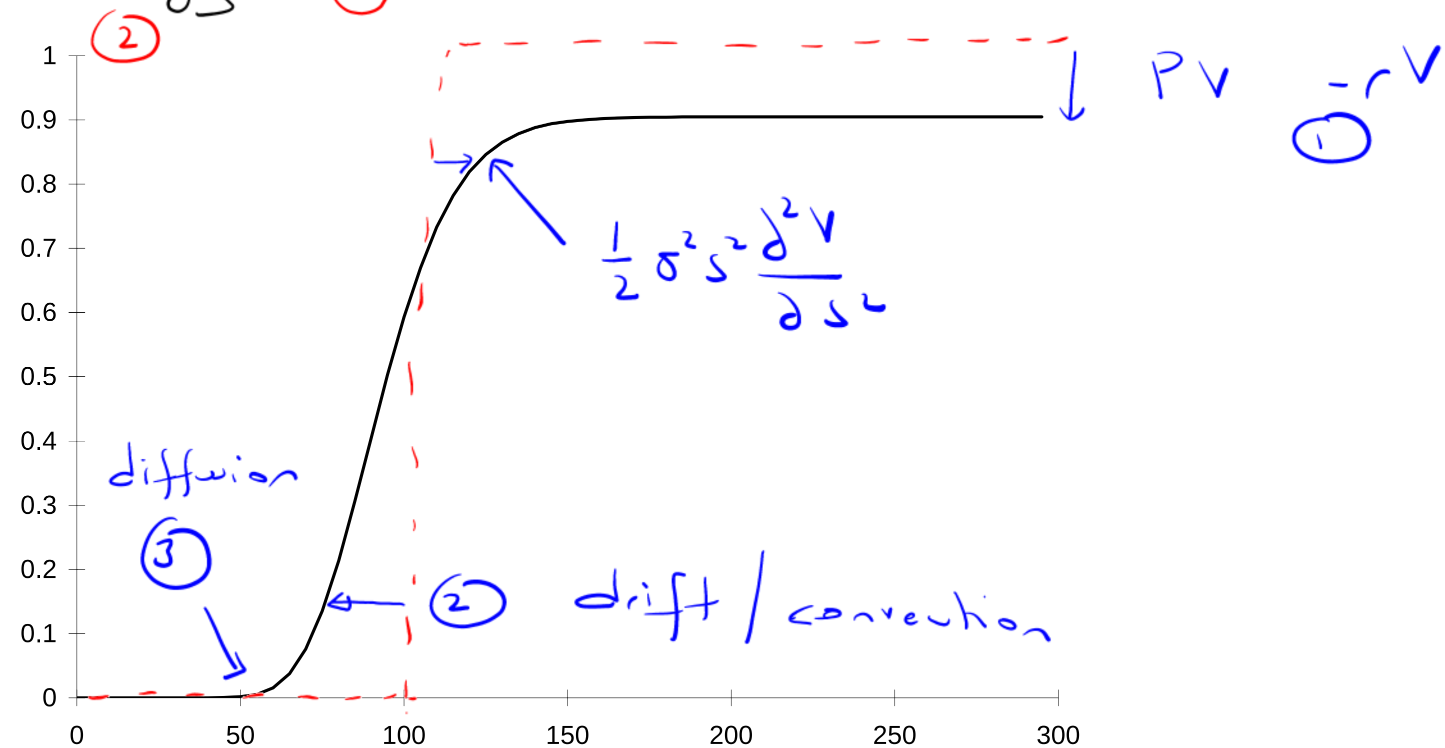
This term is just like the second term in the call option equation and so. . .

$$H(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$$

Binary call option value

$$B_c = e^{-r(T-t)} N(d_2)$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$



The value of a binary call option.

Formula for a binary put

The binary put has a payoff of one if $S < E$ at expiry. It has a value of

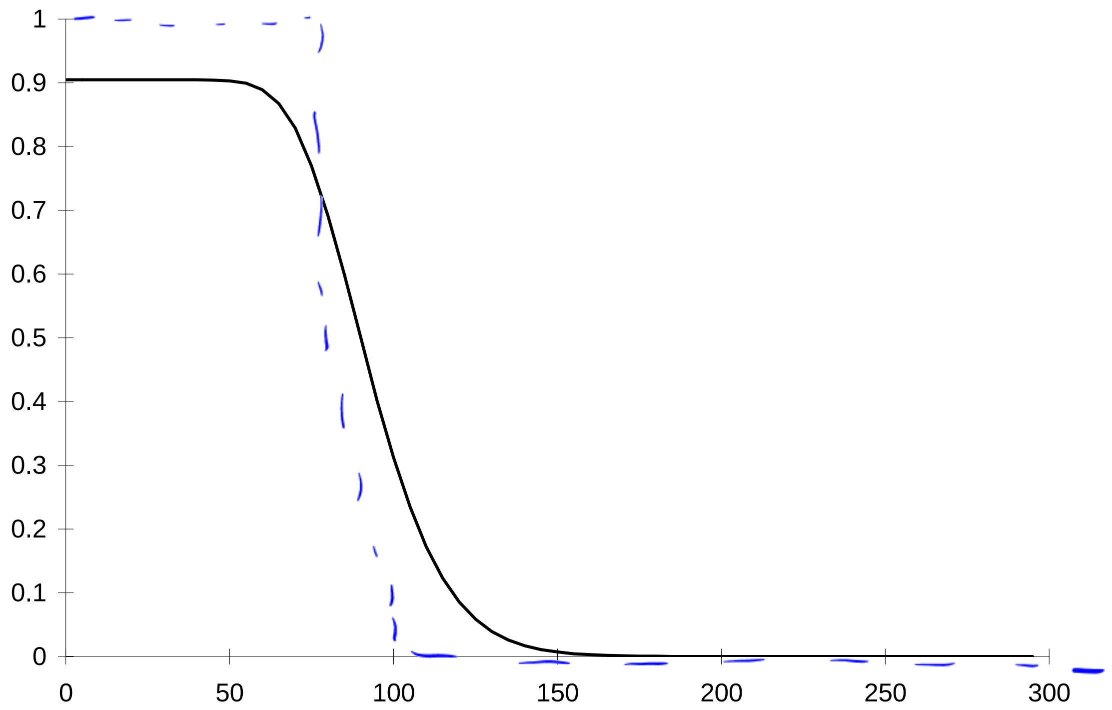
Binary put option value

$$B_p = e^{-r(T-t)}(1 - N(d_2))$$

A binary call and a binary put must add up to the present value of \$1 received at time T.

$$B_c + B_p = e^{-r(T-t)} N(d_2) + e^{-r(T-t)} (1 - N(d_2))$$

$$= e^{-r(T-t)} \times \underline{1}$$



The value of a binary put option.

Greeks.

Delta

The **delta** of an option or a portfolio of options is the sensitivity of the option or portfolio to the underlying. It is the rate of change of value with respect to the asset:

$$\boxed{\Delta = \frac{\partial V}{\partial S}}$$

$S \longrightarrow S + 1$

Here V can be the value of a single contract or of a whole portfolio of contracts. The delta of a portfolio of options is just the sum of the deltas of all the individual positions.

$$V(S+\delta S, t) = V(S,t) + \frac{\partial V}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\delta S^2 + O(\delta S^3)$$

- The theoretical device of delta hedging for eliminating risk is far more than that, it is a very important practical technique.

Delta hedging means holding one of the option and short a quantity Δ of the underlying.

Delta can be expressed as a function of S and t.

This function varies as S and t vary.

- This means that the number of assets held must be continuously changed to maintain a **delta neutral** position, this procedure is called **dynamic hedging**.

Changing the number of assets held requires the continual purchase and/or sale of the stock. This is called **rehedging** or **rebalancing** the portfolio.

Here are some formulæ for the deltas of common contracts (all formulæ assume that the underlying pays dividends or is a currency):

Deltas of common contracts

Call $e^{-D(T-t)}N(d_1)$

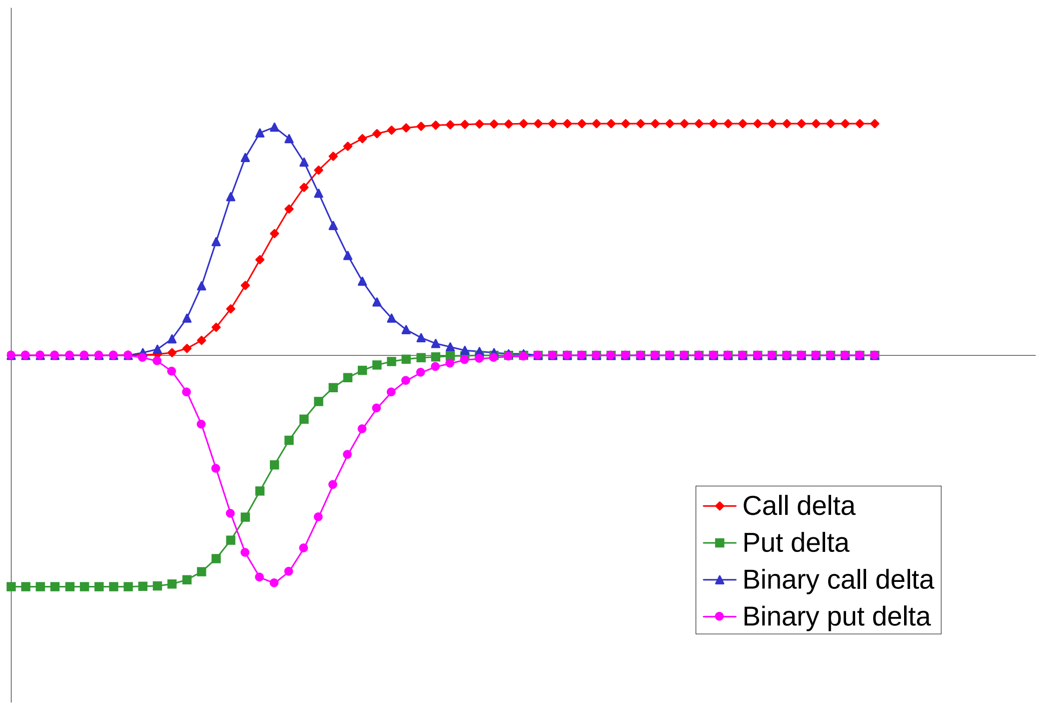
Put $e^{-D(T-t)}(N(d_1)-1)$

Binary call $\dfrac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$

Binary put $-\dfrac{e^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}$

$$\longrightarrow N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

Leibniz rule



The deltas of a call, put and binary options. (The deltas of the binaries have been scaled.)

Gamma

The **gamma**, Γ, of an option or a portfolio of options is the second derivative of the position with respect to the underlying:

$$\boxed{\Gamma = \frac{\partial^2 V}{\partial S^2}}$$

$$\frac{\partial^2 V}{\partial S^2}$$

This is, of course, just

$$\Gamma = \frac{\partial}{\partial S} \Delta$$

$$\frac{\partial \left(\frac{\partial V}{\partial S}\right)}{\partial S}.$$

Since gamma is the sensitivity of the delta to the underlying it is a measure of by how much or how often a position must be rehedged in order to maintain a delta-neutral position.

Because costs can be large and because one wants to reduce exposure to model error it is natural to try to minimize the need to rebalance the portfolio too frequently.

Since gamma is a measure of sensitivity of the hedge ratio Δ to the movement in the underlying, the hedging requirement can be decreased by a gamma-neutral strategy.

- This means buying or selling more *options*, not just the underlying.

$\log S - \log E \rightarrow \frac{1}{S}$

Because the gamma of the underlying (its second derivative) is zero, we cannot add gamma to our position just with the underlying.

$$\frac{\partial}{\partial S} d_{12} = \frac{\partial}{\partial S}\left[\frac{\log(\frac{S}{E})}{\sigma\sqrt{T-t}} + \frac{(r-D \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right]$$

- We can have as many options in our position as we want, we choose the quantities of each such that both delta and gamma are zero.

$$C = S N(d_1) - E e^{-r(T-t)} N(d_2)$$

$$\Delta_C = \frac{\partial}{\partial S} C \qquad \frac{\partial}{\partial S}\left(S N(d_1)\right) = N(d_1) + S \frac{\partial}{\partial S} N(d_1)$$

$$\frac{\partial}{\partial S} N(d_1) = \frac{d}{dd_1} N(d_1) \frac{\partial d_1}{\partial S}$$

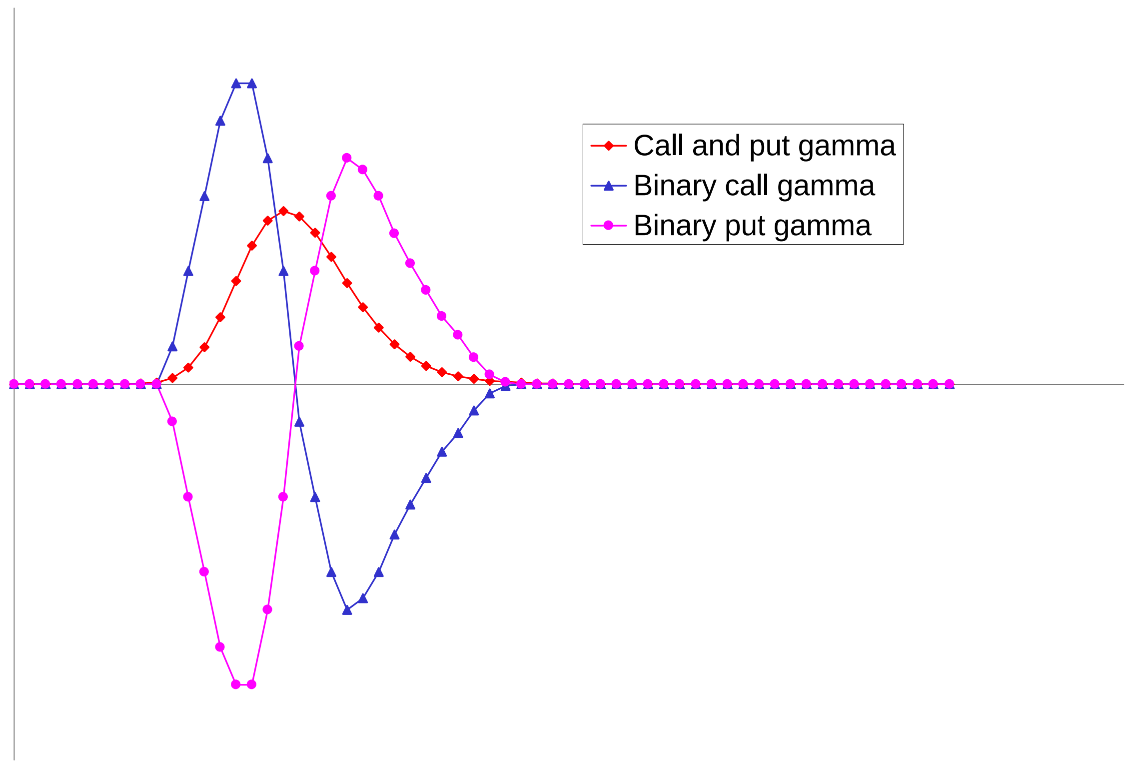
Δ

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds$$

$$\frac{dN}{dx} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad d_1$$

Here are some formulæ for the gammas of common contracts:

Gammas of common contracts	
Call	$\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Put	$\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}}$
Binary call	$-\frac{e^{-r(T-t)}d_1 N'(d_2)}{\sigma^2 S^2 (T-t)}$
Binary put	$\frac{e^{-r(T-t)}d_1 N'(d_2)}{\sigma^2 S^2 (T-t)}$



The gammas of a call, put and binary options.

Theta

$\tau = T - t$

Theta, Θ, is the rate of change of the option price with time.

$$\boxed{\Theta = \frac{\partial V}{\partial t}}$$

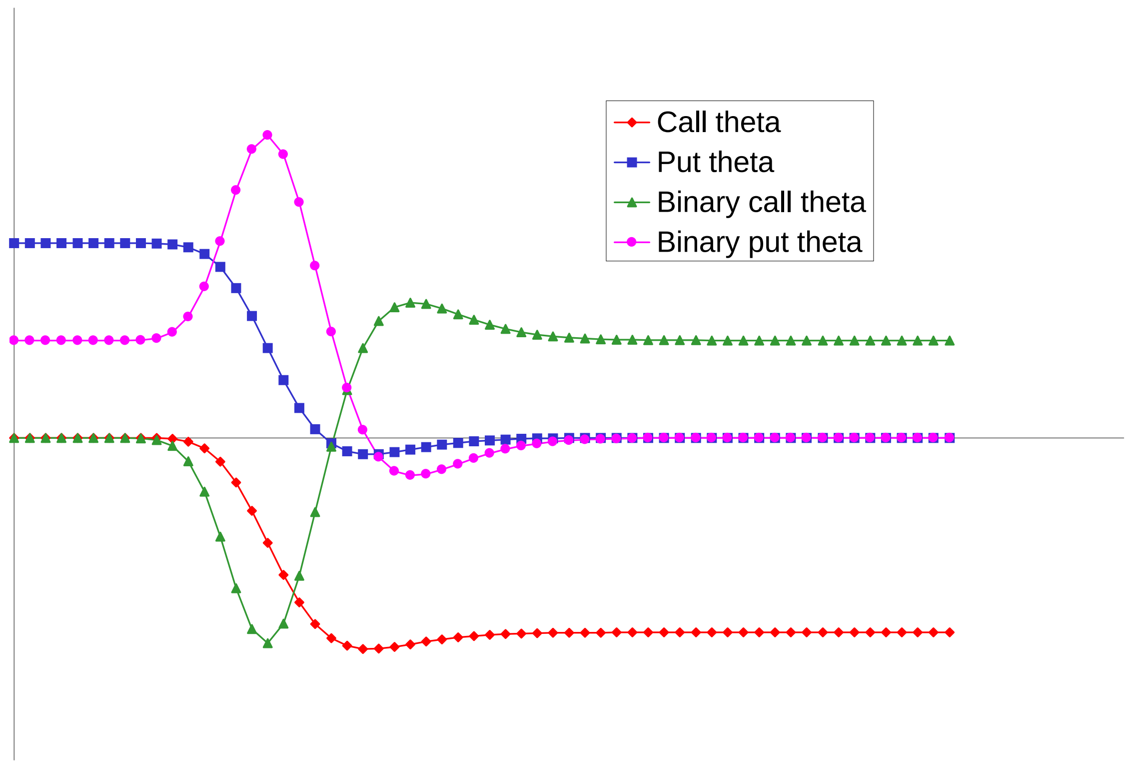
$\frac{\partial}{\partial \tau} V$

The theta is related to the option value, the delta and the gamma by the Black–Scholes equation. In a delta-hedged portfolio the theta contributes to ensuring that the portfolio earns the risk-free rate.

Here are some formulæ for the thetas of common contracts:

Thetas of common contracts

Call	$-\frac{\sigma S e^{-D(T-t)} N'(d_1)}{2\sqrt{T-t}} + DSN(d_1)e^{-D(T-t)} - rEe^{-r(T-t)}N(d_2)$
Put	$-\frac{\sigma S e^{-D(T-t)} N'(-d_1)}{2\sqrt{T-t}} - DSN(-d_1)e^{-D(T-t)} + rEe^{-r(T-t)}N(-d_2)$
Binary call	$re^{-r(T-t)}N(d_2) + e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$
Binary put	$re^{-r(T-t)}(1 - N(d_2)) - e^{-r(T-t)}N'(d_2)\left(\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}}\right)$



The thetas of a call, put and binary options.

Summary

Please take away the following important ideas

- Using tools from stochastic calculus we can build up an option pricing model from our lognormal asset price random walk model
- There are some 'simple' formulæ for the prices of simple contracts
- The greeks are important measures of the sensitivities of the option value to variables and parameters