

Taylor expansions and Transition Density Functions

This is a non-assessed problem sheet.

1. Expand $(2+x)^{-2}$ in ascending powers of x up to and including the term in x^3 , and state the set of values of x for which the expansion is valid. Hence find the coefficient of x^3 in the expansion of $\frac{1+x^2}{(2+x)^2}$.

Using a binomial expansion

$$\begin{aligned}(2+x)^{-2} &= 2^{-2} \left(1 + \frac{x}{2}\right)^{-2} = \frac{1}{4} \left(1 + (-2) \frac{x}{2} + \frac{(-2)(-3)}{2!} \left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!} \left(\frac{x}{2}\right)^3\right) \\ &= \frac{1}{4} \left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3\right) = \left(\frac{1}{4} - \frac{x}{4} + \frac{3}{16}x^2 - \frac{1}{8}x^3\right)\end{aligned}$$

The expansion is valid provided $\left|\frac{x}{2}\right| < 1$, i.e. $|x| < 2$ which is $-2 < x < 2$.
The next part relies on the earlier expansion. So

$$\begin{aligned}\frac{1+x^2}{(2+x)^2} &= (1+x^2)(2+x)^{-2} \\ &= (1+x^2) \left(\frac{1}{4} - \frac{x}{4} + \frac{3}{16}x^2 - \frac{1}{8}x^3\right) \\ &= \frac{1}{4} - \frac{x}{4} + \frac{3}{16}x^2 - \frac{1}{8}x^3 + x^2 \left(\frac{1}{4} - \frac{x}{4} + \frac{3}{16}x^2\right) \\ &= \frac{1}{4} - \frac{x}{4} + \frac{3}{16}x^2 - \frac{1}{8}x^3 + \frac{x^2}{4} - \frac{x^3}{4} + \frac{3}{16}x^4\end{aligned}$$

2. Find the Maclaurin series for $\ln(1+x)$ and hence that for $\ln\left(\frac{1+x}{1-x}\right)$.

We know $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$. The Maclaurin series for $\ln(1+x)$ is standard and given in my calculus books.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

so simply replace x by $-x$ to get

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$$

Hence subtracting gives $\ln(1+x) - \ln(1-x) =$

$$2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right) = 2 \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}.$$

3. Find the Taylor series expansions of the following functions about $x = 0$ (by first using a Binomial expansion in part a) and then considering how the function in part b) is related to that in part a)).

(a) $f(x) = \frac{1}{1+x}$.

(b) $g(x) = \ln(1+x)$.

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

Now

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt,$$

so we integrate each term in the series for $(1+x)^{-1}$ giving

$$\begin{aligned} \frac{1}{1+x} &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots + (-1)^{n+1} \frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}. \end{aligned}$$

4. Find the first 4 terms of the Taylor series for the following functions centered at $a = 1$. **Hint: The expansion will have powers of $(x-1)$:**

(a) $f(x) = \ln x$

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = -\frac{6}{x^4} \text{ and so}$$

$$\begin{aligned} \ln x &= \ln 1 + (x-1) \times 1 + \frac{(x-1)^2}{2!} \times (-1) + \frac{(x-1)^3}{3!} \times (2) + \frac{(x-1)^4}{4!} \times (-6) + \dots \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots \\ f(x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \end{aligned}$$

(b) $g(x) = \frac{1}{x}$

$$g'(x) = -\frac{1}{x^2}, \quad g''(x) = \frac{2}{x^3}, \quad g^{(3)}(x) = -\frac{6}{x^4}, \quad g^{(4)}(x) = \frac{24}{x^5} \text{ to give}$$

$$\begin{aligned} \frac{1}{x} &= 1 + (x-1) \times (-1) + \frac{(x-1)^2}{2!} \times (2) + \frac{(x-1)^3}{3!} \times (-6) + \frac{(x-1)^4}{4!} \times (24) + \dots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \dots \\ g(x) &= \sum_{n=1}^{\infty} (x-1)^{n-1} \end{aligned}$$

5. Find all first order partial derivatives

(a) $f(x, y) = 2x^4y^3 - xy^2 + 3y + 1$.

$$\begin{aligned} f_x &= 8x^3y^3 - y^2 \\ f_y &= 6x^4y^2 - 2xy + 3 \end{aligned}$$

(b) $f(x, y, z) = xyz e^{xyz}$.

$$\begin{aligned} f_x &= xyz(yz)e^{xyz} + (yz)e^{xyz} = yze^{xyz}(xyz + 1) \\ f_y &= xyz(xz)e^{xyz} + (xz)e^{xyz} = xze^{xyz}(xyz + 1) \\ f_z &= xyz(xy)e^{xyz} + (xy)e^{xyz} = xye^{xyz}(xyz + 1) \end{aligned}$$

(c) $f(x, y, z) = (y^2 + z^2)^x$. Hint: $\frac{d}{dx}a^x = a^x \ln a$; where $a > 0$.

$$\begin{aligned} f_x &= (y^2 + z^2)^x \ln(y^2 + z^2) \\ f_y &= x(y^2 + z^2)^{x-1} \times 2y = 2yx(y^2 + z^2)^{x-1} \\ f_z &= x(y^2 + z^2)^{x-1} \times 2z = 2xz(y^2 + z^2)^{x-1} \end{aligned}$$

6. Consider a **symmetric** random walk which starts with a marker placed at a point x at time s ; written (x, s) . Suppose at a later time $t > s$ the marker is at y ; the future state denoted (y, t) . The marker can move in step sizes of δy in a time step of δt . At the previous step the marker must have been at one of $(y - \delta y, t - \delta t)$ or $(y + \delta y, t - \delta t)$. The transition probability density function of the position y of the diffusion at a later time t , is written $p(x, s; y, t)$. Derive the Forward Equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial y^2}. \quad (6.1)$$

You may omit the dependence on (x, s) in your working as they will not change.

Assume a solution of (6.1) exists and takes the following form

$$p(y, t) = t^{-1/2} f(\eta); \quad \eta = \frac{y}{t^{1/2}}.$$

Solve (6.1) to show that a particular solution of this is

$$p(x, s; y, t) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right).$$

You may use the result $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$, in your working.

$$p(y', t') = \frac{1}{2} p(y' + \delta y, t' - \delta t) + \frac{1}{2} p(y' - \delta y, t' - \delta t)$$

Taylor series expansion gives

$$\begin{aligned} p(y' + \delta y, t' - \delta t) &= p(y', t') - \frac{\partial p}{\partial t'} \delta t + \frac{\partial p}{\partial y'} \delta y + \frac{1}{2} \frac{\partial^2 p}{\partial y'^2} \delta y^2 + \dots \\ p(y' - \delta y, t' - \delta t) &= p(y', t') - \frac{\partial p}{\partial t'} \delta t - \frac{\partial p}{\partial y'} \delta y + \frac{1}{2} \frac{\partial^2 p}{\partial y'^2} \delta y^2 + \dots \end{aligned}$$

Substituting into the above

$$\begin{aligned} p(y', t') &= \frac{1}{2} \left(p(y', t') - \frac{\partial p}{\partial t'} \delta t + \frac{\partial p}{\partial y'} \delta y + \frac{1}{2} \frac{\partial^2 p}{\partial y'^2} \delta y^2 \right) \\ &\quad + \frac{1}{2} \left(p(y', t') - \frac{\partial p}{\partial t'} \delta t - \frac{\partial p}{\partial y'} \delta y + \frac{1}{2} \frac{\partial^2 p}{\partial y'^2} \delta y^2 \right) \\ 0 &= -\frac{\partial p}{\partial t'} \delta t + \frac{1}{2} \frac{\partial^2 p}{\partial y'^2} \delta y^2 \\ \frac{\partial p}{\partial t'} &= \frac{1}{2} \frac{\delta y^2}{\delta t} \frac{\partial^2 p}{\partial y'^2} \end{aligned}$$

Now take limits. This only makes sense if $\frac{\delta y^2}{\delta t}$ is $O(1)$, i.e. $\delta y^2 \sim O(\delta t)$ and letting $\delta y, \delta t \rightarrow 0$ gives the equation

$$\frac{\partial p}{\partial t'} = \frac{1}{2} \frac{\partial^2 p}{\partial y'^2}$$

To solve, write

$$p(y, t) = t^{-1/2} f(\eta)$$

therefore

$$\begin{aligned} \frac{\partial p}{\partial y} &= \frac{\partial p}{\partial \eta} \frac{\partial \eta}{\partial y} = t^{-1/2} f'(\eta) \times t^{-1/2} = t^{-1} f'(\eta) \\ \frac{\partial^2 p}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial p}{\partial y} \right) = \frac{\partial}{\partial y} (t^{-1} f'(\eta)) = t^{-3/2} f''(\eta) \\ \frac{\partial p}{\partial t} &= t^{-1/2} \frac{\partial}{\partial t} f(\eta) - \frac{1}{2} t^{-3/2} f(\eta) \\ &= t^{-1/2} \left(-\frac{1}{2} \eta t^{-3/2} \right) f'(\eta) - \frac{1}{2} t^{-3/2} f(\eta) \\ &= -\frac{1}{2} \eta t^{-3/2} f'(\eta) - \frac{1}{2} t^{-3/2} f(\eta) \end{aligned}$$

and then substituting

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\frac{1}{2} t^{-3/2} (\eta f'(\eta) + f(\eta)) \\ \frac{\partial^2 p}{\partial y^2} &= t^{-3/2} f''(\eta) \end{aligned}$$

gives

$$-\frac{1}{2} t^{-3/2} (\eta f'(\eta) + f(\eta)) = \frac{1}{2} t^{-3/2} f''(\eta)$$

simplifying to the ODE

$$-(f + \eta f') = f''.$$

We have an exact derivative on the lhs, i.e. $\frac{d}{d\eta}(\eta f) = f + \eta f'$, hence

$$-\frac{d}{d\eta}(\eta f) = f''$$

and we can integrate once to get

$$-\eta f = f' + K.$$

We set $K = 0$ (see class notes for justification) in order to get the correct solution, i.e.

$$-\eta f = f'$$

which can be solved as a simple first order variable separable equation:

$$f(\eta) = A \exp\left(-\frac{1}{2}\eta^2\right)$$

A is a normalizing constant, so write

$$A \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\eta^2\right) d\eta = 1$$

put $x = \eta/\sqrt{2} \rightarrow \sqrt{2}dx = d\eta$

$$\underbrace{\sqrt{2}A \int_{\mathbb{R}} \exp(-x^2) d\eta}_{=\sqrt{\pi}} = 1 \rightarrow A = \frac{1}{\sqrt{2\pi}}$$

$$u(y, t) = t^{-1/2} f(\eta) \text{ becomes } u(y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right).$$

If the random variable y has value x at time s then we can generalize to

$$p(x, s; y, t) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-x)^2}{2(t-s)}\right)$$