Further Stochastic Differential Equations and Stochastic Integration

W_t is a Brownian Motion (Wiener Process) and dW_t or $dW(t)$ is its increment. $W_0 = 0$.

1. The change in a share price $S(t)$ satisfies
$$dS = A(S,t)\,dW_t + B(S,t)\,dt,$$
for some functions A and B. If $f = f(S,t)$, then Itô's lemma gives the following SDE
$$df = \left(\frac{\partial f}{\partial t} + B\frac{\partial f}{\partial S} + \frac{1}{2}A^2\frac{\partial^2 f}{\partial S^2}\right)dt + A\frac{\partial f}{\partial S}dW_t.$$
Can A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.

2. Show that $F(W_t) = \arcsin(2aW_t + \sin F_0)$ is a solution of the SDE
$$dF = 2a^2(\tan F)\left(\sec^2 F\right)dt + 2a(\sec F)\,dW_t,$$
Ignore this problem
where F_0 and a is a constant. The following standard result may be used
$$\frac{d}{dx}\sin^{-1}ax = \frac{a}{\sqrt{1-a^2x^2}}$$

3. Show that
$$\int_0^t W_\tau\left(1 - e^{-W_\tau^2}\right)dW_\tau = \overline{F}(W_t) + \int_0^t G(W_\tau)\,d\tau.$$
where the functions $\overline{F}$ and G should be determined.

4. Consider the process
$$d(\log y) = (\alpha - \beta\log y)\,dt + \delta dW_t.$$
The parameters $\alpha,\ \beta,\ \delta$ are constant. Show that y satisfies
$$\frac{dy}{y} = \left(\alpha - \beta\log y + \frac{1}{2}\delta^2\right)dt + \delta dW_t.$$

5. Show that
$$G_t = e^{t + ae^{W_t}}$$
is a solution of the stochastic differential equation
$$dG_t = G_t\left(1 + \tfrac{1}{2}(\ln G_t - t) + \tfrac{1}{2}(\ln G_t - t)^2\right)dt + G_t(\ln G_t - t)\,dW,$$
where a is a constant.

6. A spot rate r_t, evolves according to the popular form
$$dr_t = u(r_t)\,dt + \nu r_t^\beta dW_t, \qquad (*)$$
where ν and β are constants. Suppose such a model has a **steady state transition probability density function** $p_\infty(r)$ that satisfies the forward Fokker Planck Equation. Show that this implies the drift structure of $(*)$ is given by
$$u(r_t) = \nu^2\beta r_t^{2\beta-1} + \frac{1}{2}\nu^2 r_t^{2\beta}\frac{d}{dr}(\log p_\infty).$$