Further Stochastic Differential Equations and Stochastic Integration

W_t is a Brownian Motion (Wiener Process) and dW_t or $dW(t)$ is its increment. $W_0 = 0$.

1. The change in a share price $S(t)$ satisfies

$$dS = A(S, t)\, dW_t + B(S, t)\, dt,$$

for some functions A and B. If $f = f(S, t)$, then Itô's lemma gives the following SDE

$$df = \left(\frac{\partial f}{\partial t} + B \frac{\partial f}{\partial S} + \frac{1}{2} A^2 \frac{\partial^2 f}{\partial S^2} \right) dt + A \frac{\partial f}{\partial S} dW_t.$$

Can (non-zero) A and B be chosen so that a function $g = g(S)$ has a change which has zero drift, but non-zero diffusion? State any appropriate conditions.

A function $g(S)$ will satisfy the shorter SDE

$$dg = \left(B \frac{dg}{dS} + \frac{1}{2} A^2 \frac{d^2 g}{dS^2} \right) dt + A \frac{dg}{dS} dW_t.$$

For $g(S)$ to have a zero drift but non-zero diffusion, we require the condition

$$B \frac{dg}{dS} + \frac{1}{2} A^2 \frac{d^2 g}{dS^2} = 0.$$

i.e.

$$\frac{dg}{dS} + \frac{1}{2} \frac{A^2}{B} \frac{d^2 g}{dS^2} = 0$$

We can find a solution to this problem if $\frac{A^2}{B}$ is independent of time.

3. Show that

$$\int_0^t W_\tau \left(1 - e^{-W_\tau^2}\right) dW_\tau = \overline{F}(W_t) + \int_0^t G(W_\tau) d\tau.$$

where the functions $\overline{F}$ and G should be determined.
We can do this two ways. Perform Itô's lemma and then integrate or use the stochastic integral version as given in the first question. Comparing

$$\int_0^t W(\tau)\left(1 - e^{-W^2(\tau)}\right) dW(\tau) = \overline{F}(W(t)) + \int_0^t G(W(t)) d\tau$$

with

$$\int_0^t \frac{\partial F}{\partial W} dW(\tau) = F(W(t), t) - F(W(0), 0) + \int_0^t -\left(\frac{\partial F}{\partial \tau} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}\right) d\tau$$

suggests that

$$\frac{\partial F}{\partial W} = W(\tau)\left(1 - e^{-W^2(\tau)}\right)$$

so integrating over $[0, t]$ gives $\overline{F}(W(t), t)$, which we will do by substitution, i.e. put $u = W^2$ which gives

$$F(W(t), t) - F(W(0), 0) = \tfrac{1}{2}W^2(t) + \tfrac{1}{2}e^{-W^2(t)} - \tfrac{1}{2}.$$

Also knowing $\frac{\partial F}{\partial W}$ allows us to easily obtain $\frac{\partial^2 F}{\partial W^2} = 2W^2(t)e^{-W^2(t)} - e^{-W^2(t)} + 1$. Hence

$$G(W(t)) = -\tfrac{1}{2}\frac{\partial^2 F}{\partial W^2} = -\tfrac{1}{2}\left(1 - e^{-W^2(t)}\right) - W^2(t)e^{-W^2(t)}$$

and we have shown

$$\int_0^t W(\tau)\left(1 - e^{-W^2(\tau)}\right) dW(\tau) = \overline{F}(W(t)) + \int_0^t G(W(t)) d\tau$$

where

$$
\begin{aligned}
\overline{F}(W(t), t) &= \tfrac{1}{2}W^2(t) + \tfrac{1}{2}e^{-W^2(t)} - \tfrac{1}{2} \\
G(W(t)) &= -\tfrac{1}{2}\left(1 - e^{-W^2(t)}\right) - W^2(t)e^{-W^2(t)}.
\end{aligned}
$$

4. Consider the process

$$d(\log y) = (\alpha - \beta \log y) dt + \delta dW_t.$$

The parameters α, β, δ are constant. Show that y satisfies

$$\frac{dy}{y} = \left(\alpha - \beta \log y + \frac{1}{2}\delta^2\right) dt + \delta dW_t.$$

By Ito's lemma if $dZ = a(Z,t)dt + b(Z,t)dW_t$ and $Y = f(Z)$ then

$$dY = \left(a\frac{\partial Y}{\partial Z} + \frac{1}{2}b^2\frac{\partial^2 Y}{\partial Z^2} \right) dt + b\frac{\partial Y}{\partial Z}dW_t$$

here $Z \equiv \log y_t$, $a \equiv (\alpha - \beta Z)$, $b \equiv \delta$, , $Y = e^Z = y$, $\frac{\partial Y}{\partial Z} = e^Z = \frac{\partial^2 f}{\partial Z^2}$, putting all these in Ito's lemma we have

$$dY \equiv dy_t = \left((\alpha - \beta \log y_t) y_t + \frac{1}{2}\delta^2 y_t \right) dt + \delta y_t dW_t$$

hence

$$\frac{dy_t}{y_t} = \left(\alpha - \beta \log y_t + \frac{1}{2}\delta^2 \right) dt + \delta dW_t$$

5. Show that

$$G = e^{t+ae^{W_t}}$$

is a solution of the stochastic differential equation

$$dG_t = G_t \left(1 + \tfrac{1}{2}\left(\ln G_t - t \right) + \tfrac{1}{2}\left(\ln G_t - t \right)^2 \right) dt + G_t \left(\ln G_t - t \right) dW_t,$$

where a is a constant.

$$\frac{\partial G_t}{\partial t} = G_t, \quad \frac{\partial G_t}{\partial W_t} = aG_t e^{W_t}, \quad \frac{\partial^2 G_t}{\partial W_t^2} = ae^{W_t}G_t + ae^{W_t}\frac{\partial G_t}{\partial W_t} = ae^{W_t}G_t + a^2 e^{2W_t}G_t$$

In Itô, i.e.

$$\begin{aligned}
dG_t &= \left(\frac{\partial G_t}{\partial t} + \frac{1}{2}\frac{\partial^2 G_t}{\partial W_t^2} \right) dt + \frac{\partial G_t}{\partial W_t}dW_t \\
&= \left(G_t + \tfrac{1}{2}ae^{W_t}G_t + \tfrac{1}{2}a^2 e^{2W_t}G_t \right) dt + ae^{W_t}G_t dW_t
\end{aligned}$$

From $G_t = e^{t+ae^{W_t}}$ we have

$$ae^{W_t} + t = \ln G_t \implies ae^{W_t} = \ln G_t - t$$

so we can write the SDE in terms of the process G_t

$$dG_t = G_t \left(1 + \tfrac{1}{2}ae^{W_t} + \tfrac{1}{2}a^2 e^{2W_t} \right) dt + ae^{W_t}G_t dW_t$$

So

$$dG_t = G_t \left(1 + \tfrac{1}{2}\left(\ln G_t - t \right) + \tfrac{1}{2}\left(\ln G_t - t \right)^2 \right) dt + G_t \left(\ln G_t - t \right) dW_t.$$

6. A spot rate r_t, evolves according to the popular form

$$dr_t = u(r_t) dt + vr_t^\beta dW_t, \qquad (*)$$

where v and β are constants. Suppose such a model has a **steady state transition probability density function** $p_\infty(r)$ that satisfies the forward Fokker Planck Equation.
Show that this implies the drift structure of $(*)$ is given by

$$u(r_t) = v^2 \beta r_t^{2\beta-1} + \frac{1}{2}v^2 r_t^{2\beta}\frac{d}{dr}\left(\log p_\infty \right).$$

The forward F.P equation for $dr = u(r,t)\,dt + w(r,t)\,dW_t$ is

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial r^2}\left(w^2(r,t)\,p(r,t)\right) - \frac{\partial}{\partial r}\left(u(r,t)\,p(r,t)\right)$$

for the probability density $p(r,t)$. The steady state equation for our model becomes

$$\frac{1}{2}\nu^2\frac{d^2}{dr^2}\left(r^{2\beta}p_\infty(r)\right) - \frac{d}{dr}\left(u(r)\,p_\infty(r)\right) = 0$$

This can be simply integrated once to give

$$\frac{1}{2}\nu^2\frac{d}{dr}\left(r^{2\beta}p_\infty(r)\right) - \left(u(r)\,p_\infty(r)\right) = \text{const}$$

$$\frac{1}{2}\nu^2\left(r^{2\beta}\frac{dp_\infty}{dr}\right) + \nu^2\beta r^{2\beta-1}p_\infty(r) - \left(u(r)\,p_\infty(r)\right) = \text{const}$$

The constant of integration is zero because as r becomes large

$$\left.\begin{array}{c} p_\infty(r) \\ \frac{dp_\infty}{dr} \end{array}\right\} \longrightarrow 0$$

$$u(r)\,p_\infty(r) = \frac{1}{2}\nu^2 r^{2\beta}\frac{dp_\infty}{dr} + \nu^2\beta r^{2\beta-1}p_\infty(r)$$

$$u(r) = \frac{1}{2}\nu^2 r^{2\beta}\frac{1}{p_\infty(r)}\frac{dp_\infty}{dr} + \nu^2\beta r^{2\beta-1}$$

We can write $\dfrac{1}{p_\infty}\dfrac{dp_\infty}{dr}$ as $\dfrac{d}{dr}(\log p_\infty)$

$$u(r) = \frac{1}{2}\nu^2 r^{2\beta}\frac{d}{dr}(\log p_\infty) + \nu^2\beta r^{2\beta-1}$$