Black-Scholes Model - Exercises

Throughout this exercise you may use assume (where appropriate) the following results without proof

$$
\begin{aligned}
d_1 &= \frac{\log\left(S/E\right) + \left(r - D + \frac{1}{2}\sigma^2\right)\left(T-t\right)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{\log\left(S/E\right) + \left(r - D - \frac{1}{2}\sigma^2\right)\left(T-t\right)}{\sigma\sqrt{T-t}} \quad \text{and} \\
N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du
\end{aligned}
$$

where $S \geq 0$ is the spot price, $t \leq T$ is the time, $E > 0$ is the strike, $T > 0$ the expiry date, $r \geq 0$ the interest rate, D is the dividend yield and σ is the volatility of S.

1. The Black–Scholes formula for a European call option $C\left(S,t\right)$ is given by

$$C\left(S,t\right) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2).$$

By differentiating with respect to S and σ show that the delta and vega are given by

$$\Delta = e^{(-D(T-t))}N\left(d_1\right), \text{ and } \upsilon = \sqrt{\frac{T-t}{2\pi}}Se^{(-D(T-t))}\exp\left(\frac{-{d_1}^2}{2}\right).$$

You are given the following result to assist the first greek calculation

$$Se^{(-D(T-t))}\exp\left(-\frac{d_1^2}{2}\right) = Ee^{(-r(T-t))}\exp\left(-\frac{d_2^2}{2}\right).$$

2. Given that S is defined by the SDE

$$dS = a\left(S,t\right)dt + b\left(S,t\right)dW \tag{2.1}$$

where a and b are given functions of S and t, show **using** Itô's lemma that any function $V\left(S,t\right)$ satisfies the SDE

$$dV = \left(\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial S} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2}\right)dt + b\frac{\partial V}{\partial S}dW$$

where we have assumed that all partial derivatives exist.
Hence derive the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right) \tag{2.2}$$

for the fair price of an option based on a security S which satisfies (2.1) with r the risk-free interest rate.
Show (by substitution) that $V(S,t) = e^{-\alpha t} S^2$ is a solution of (2.2) provided

$$b^2 = (\alpha - r) S^2$$

and α is a constant. Write the PDE in terms of the greeks.

3. The Black–Scholes formula for a European call option $C(S,t)$ is

$$C(S,t) = S\exp(-D(T-t))N(d_1) - E\exp(-r(T-t))N(d_2)$$

From this expression, find the Black–Scholes value of the call option in the following limits:

a. (time tends to expiry) $t \to T^-$, $\sigma > 0$ (*this depends on* S/E);
b. (volatility tends to zero) $\sigma \to 0^+$, $t < T$; (*this depends on* $S\exp(-D(T-t))/E\exp(-r(T-t))$)

4. The value of an option $V(S,t)$ satisfies the Black–Scholes equation. Write the option value in the form

$$V(S,t) = \exp(-r(T-t))q(S,t). \tag{4.1}$$

Show that the function $q(S,t)$ satisfies the equation

$$\frac{\partial q}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 q}{\partial S^2} + (r-D)S\frac{\partial q}{\partial S} = 0.$$