CQF Lecture on Understanding Volatility

Solutions

1. Explain what actual and implied volatilities are, and what is their relationship? Name three assumptions made in estimation of actual volatility from the market option prices.

   $\boxed{\textbf{Solution:}}$ *Actual volatility* is the measure of the amount of randomness in asset returns at any particular time. *Implied volatility* is the key input into the Black-Scholes formulae that gives the market price of an option:

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$P(S,t) = C(S,t) + Ee^{-r(T-t)} - S$$

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

   Implied volatility is often described as the market's view of the future actual volatility over the lifetime of an option. However, one must be aware that it can be influenced by other effects, such as expectation of a crush, supply and demand.

   Each implied volatility number has timescale associated with it. Actual volatility exists in a very instant, it is an instantaneous process. **Actual, not implied, volatility is supposed to be an input to all option pricing formulae.**

   If the actual volatility were known it would straightforward to figure out the implied, but the inverse does not hold. Calibrating the actual volatility from the market option prices requires making big assumptions:

   (a) Option prices today have full information about future volatility.

   (b) This inverse problem $\sigma(S,t) \mapsto \sigma(K,T)$ has multiple solutions (ways to fit actual/local volatility).

   (c) 'Piecewise constant' assumption is also common to fitting of an instantaneous quantity, such as the actual volatility.

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2. The market price for a European put with strike 100 is quoted at \$5.57 for the asset value at \$100. Option expiry is one year, and interest rate is 5% p.a. How do you find its implied volatility?

Solution: The Black-Scholes formula for the European put option on a non-dividend paying stock with $t = 0$ is

$$V_{BS} = K\, e^{-rT} N(-d_2) - S_0 N(-d_1)$$

$$\begin{aligned} d_1 &= \frac{\ln(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma_{BS}^2)T}{\sigma_{BS}\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

It is not possible to invert option price formilae in order to express the implied volatility σ_{BS} as a function of option price V.

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{\frac{-t^2}{2}} dt$$

since Normal *cdf* is defined via an integral, then $x = N^{-1}(N(x))$ would require solving **an integral equation**. Hint: use Mathematica software to look up the actual function for the Normal Inverse CDF.

The problem <u>has to</u> be solved numerically using **a root-finding method**, which implements an iterative search for such value of σ that

$$f(\sigma) = V(\sigma) - P_{Mkt} = 0$$

Define a scheme of Newton-Raphson kind as follows:

$$\begin{aligned} \frac{f(\sigma_{k+1}) - f(\sigma_k)}{\sigma_{k+1} - \sigma_k} &\approx f'(\sigma_k) \\ f(\sigma_{k+1}) &\approx f(\sigma_k) + f'(\sigma_k)(\sigma_{k+1} - \sigma_k) \ \text{ tangent line approximation} \\ \frac{f(\sigma_{k+1}) - f(\sigma_k)}{f'(\sigma_k)} &\approx \sigma_{k+1} - \sigma_k \\ \sigma_{k+1} &\approx \sigma_k + \left[\frac{f(\sigma_{k+1})}{f'(\sigma_k)}\right] - \frac{f(\sigma_k)}{f'(\sigma_k)} \quad \text{ at solution } f(\sigma_{k+1}) = 0 \\ \sigma_{k+1} &= \sigma_k - \frac{f(\sigma_k)}{f'(\sigma_k)} \quad \text{ or } \quad \sigma_{k+1} = \sigma_k - \frac{V(\sigma_k) - P_{Mkt}}{Vega} \end{aligned}$$

Start with realistic σ_0 and produce successively better estimates $\sigma_1, \sigma_2, \ldots$ by moving 'up' or 'down' the function. A unique solution relies on option price monotonically increasing *wrt* volatility parameter.

The iteration stops as soon as $f(x_n)$ reaches a pre-determined tolerance level,

$$(V(\sigma) - P_{Market}) < \text{Tolerance}$$

The numerical answer for the European put is $\sigma \approx 20\%$.

Root-finding methods are the bisection, Newton-Raphson, Secant with variations specific to Black-Scholes formulae, where calculation speed is improved by the choice of initial value σ_0.

Though it is possible to set up root-finding task using Goal Seek or Solver in Excel, it is a different task than optimisation or numerical pricing of option payoff for call $V = (S_T - K)^+$ or put $V = (K - S_T)^+$ by the finite differences scheme.

Extra assumption in the Newton-Raphson scheme: if σ_k is good approximation of true root σ_{BS}, then

$$\sigma_{BS} = \sigma_k + (\sigma_{k+1} - \sigma_k) \qquad\qquad (\sigma_{k+1} - \sigma_k) = \sigma_{BS} - \sigma_k \dagger$$

$(\sigma_{k+1} - \sigma_k)$ measures how far the estimate σ_k is from the true value.

At solution, $f(\sigma_{BS}) = 0 = f(\sigma_{k+1}) \approx f(\sigma_k) + f'(\sigma_k)(\sigma_{k+1} - \sigma_k)$

$$\begin{aligned} (\sigma_{k+1} - \sigma_k) &= -\frac{f(\sigma_k)}{f'(\sigma_k)} \\ \dagger\, \sigma_{BS} - \sigma_k &= -\frac{f(\sigma_k)}{f'(\sigma_k)} \\ \dagger\, \sigma_{BS} &= \sigma_k - \frac{f(\sigma_k)}{f'(\sigma_k)}. \end{aligned}$$

Numerically, $\frac{1}{f'(\sigma_k)} \approx \frac{\Delta\sigma}{\Delta f}$ gives small correction, since $\Delta\sigma \ll \Delta f$.

Pseudo code in the Newton-Raphson scheme in Python,

```
def f(x):
    return [insert function formula]
def df(x):
    return [insert derivative formula]

def dx(f, x):
    return abs(0-f(x))

def FindRoot(f, df, sigma_BS, tolerance):
    delta = dx(f, sigma_BS)
    while delta > tolerance:
        sigma_BS = sigma_BS - f(sigma_BS)/df(sigma_BS)
        delta = dx(f, sigma_BS)
```

3. Assume **a time-dependent** volatility function $\sigma(t)$. Consistent with Black-Scholes framework, the implied volatility $\sigma_i(t, T)$ measured at time t of an European option expiring at time T must satisfy

$$\sigma_i(t, T) = \sqrt{\frac{1}{T-t} \int_t^T \sigma^2(s)\ ds}$$

Solve the inverse problem (an integral equation) to show that, at calibration time t^*, the volatility function $\sigma(t)$ must be consistent with implied volatility σ_i

$$\sigma^2(t) = 2(t - t^*)\,\sigma_i(t^*, t)\,\frac{\partial \sigma_i(t^*, t)}{\partial t} + \sigma_i^2(t^*, t)$$

$\boxed{\textbf{Solution:}}$ We know that the implied volatility is the square root of variance, so

$$\sigma_i^2(t, T) = \frac{1}{T-t} \int_t^T \sigma^2(s)\ ds \tag{1}$$

We solve by differentiating both sides *wrt* T (use partial derivatives and chain rule). To make solution easier we leave the integral on the *rhs*:

$$\frac{d}{dT}\left((T-t)\sigma_i^2(t, T)\right) = \frac{d}{dT} \int_t^T \sigma^2(s)\ ds \tag{2}$$

$$\sigma_i^2(t, T) + (T-t)2\sigma_i(t, T)\frac{\partial \sigma_i(t, T)}{\partial T} = \sigma^2(T) \tag{3}$$

$$\underline{\sigma^2(T)} = \sigma_i^2(t, T) + 2(T-t)\sigma_i(t, T)\frac{\partial \sigma_i(t, T)}{\partial T}$$

But σ^2 is 'not allowed' to be a function of expiration T. At this stage, **variables became parameters**. Let t^* be the calibration time, then $t \to t^*$ and $T \to t$

$$\sigma^2(t) = \sigma_i^2(t^*, t) + 2(t - t^*)\,\sigma_i(t^*, t)\,\frac{\partial \sigma_i(t^*, t)}{\partial t}$$

In the inverse notation $\sigma_i(T; t^*)$, to mean the implied volatility measured at time t^* of a European option expiring at time $T \to t$, the solution is expressed as

$$\sigma(t) = \sqrt{\sigma_i^2(t; t^*) + 2(t - t^*)\,\sigma_i(t; t^*)\,\frac{\partial \sigma_i(t; t^*)}{\partial t}}$$

To formulate the inverse-time calibration solution, we had to change notation twice!

4. Suppose implied volatilities are observable at $T_i, i = 0, 1, 2, \cdots, n$, with $T_0 = t^*$ is the date of calibration (fitting). Assuming that the actual volatility function is **piecewise constant**, show that for $T_{i-1} < t < T_i$ the total variance is

$$\sigma^2(t) = \frac{(T_i - t^*)\,\sigma_i^2(t^*, T_i) - (T_{i-1} - t^*)\,\sigma_i^2\,(t^*, T_{i-1})}{T_i - T_{i-1}}$$

$\boxed{\textbf{Solution:}}$ The task is to solve a discretised integral equation.

$$\begin{aligned} \sigma_i^2(t^*, T_i) &= \frac{1}{T_i - t^*}\int_{T_0=t^*}^{T_i} \sigma^2(s)\,ds \quad \text{(splitting the integral)} & (4)\\ (T_i - t^*)\sigma_i^2(t^*, T_i) &= \int_{T_0=t^*}^{T_{i-1}} \sigma^2(s)\,ds + \int_{T_{i-1}}^{T_i} \sigma^2(s)\,ds & (5)\end{aligned}$$

The assumption of piecewise constant actual volatility $\forall\, t \in (T_{i-1}, T_i)$ means that the second integral on the *rhs* is over a constant

$$\int_{T_{i-1}}^{T_i} \sigma^2(s)\,ds = \sigma^2(t)\int_{T_{i-1}}^{T_i} ds = \underline{\sigma^2(t)(T_i - T_{i-1})} \qquad (6)$$

We saw the first integral on *rhs* as implied volatility. Plugging in result (6) gives

$$(T_i - t^*)\sigma_i^2(t^*, T_i) = (T_{i-1} - t^*)\sigma_i^2(t^*, T_{i-1}) + \underline{\sigma^2(t)(T_i - T_{i-1})} \qquad (7)$$

rearrange to obtain the expression for $\sigma^2(t)$

$$\sigma^2(t) = \frac{(T_i - t^*)\,\sigma_i^2(t^*, T_i) - (T_{i-1} - t^*)\,\sigma_i^2\,(t^*, T_{i-1})}{T_i - T_{i-1}} \qquad (8)$$

for $T_{i-1} < t < T_i$, where $t^* = T_0$ is earlier time of calibration $t^* < T_{i-1} < T_i$.

The implication from the interim result (7) is that we can decompose the implied volatility into a time-weighted average of actual (local) volatilities:

$$\begin{aligned} (T_i - t^*)\sigma_i^2(t^*, T_i) &= (T_{i-2} - t^*)\sigma_i^2(t^*, T_{i-2}) + \cdots + \sigma^2(t)(T_{i-1} - T_{i-2}) + \sigma^2(t)(T_i - T_{i-1})\\ &\approx (T_{i-2} - T_{i-3})\sigma_{1M}^2(t) + (T_{i-1} - T_{i-2})\sigma_{2M}^2(t) + (T_i - T_{i-1})\sigma_{3M}^2(t)\\ &\approx \quad \text{(as seen in the Lecture example with three-month term structure).}\end{aligned}$$

5. Suppose that we know the actual volatility σ_a to realise (it can be a good forecast of average volatility) and can trade options at the implied volatility σ_i. We have a choice of calculating a hedge $\Delta = N(d_1)$ using implied or actual. Assume the asset follows the GBM with continuous dividend rate D, and an option denoted by $V_i(S, t; \sigma)$.

   - Within the Black-Scholes framework, what is **the P&L from a replicated option** (Mark-to-Market value over dt) if one calculates Δ_a with actual volatility. What can we say about the total P&L?
   - What about replicating with the implied volatility?

   Solution: At time t, we construct the following hedging portfolio

   (a) Buy one unit of option: V_i

   (b) Short Δ_a unit of stock: $-\Delta_a S$

   (c) Leftover cash borrowed/received: $-(V_i - \Delta_a S)$

   At time $t + dt$, the portfolio value becomes

   (a) Option: $V_i + d\,V_i$

   (b) Stock: $-\Delta_a S - \Delta_a dS$

   (c) Cash: $-(V_i - \Delta_a S)(1 + r dt) - \Delta_a D S dt$

   The Mark-to-Market profit over time dt will be the difference of the portfolios

   $$dV_i - \Delta_a dS - (V_i - \Delta_a S) r dt - \Delta_a D S dt \quad \dagger$$

   Because the option would be correctly valued at V_a its MtM profit will be offset by delta hedging exactly, so we have the equality

   $$dV_a - \Delta_a dS - (V_a - \Delta_a S) r dt - \Delta_a D S dt = 0$$

   Subtracting term equal to zero from † gets cancellation of all terms multiplied by Δ_a. The **P&L over timestep** dt is

   $$dV_i - dV_a - r(V_i - V_a)dt \quad \Rightarrow \quad \boxed{e^{rt} d\left(e^{-rt}(V_i - V_a)\right)}$$

   $$\begin{aligned} d(e^{-rt}V) &= e^{-rt} dV - r e^{-rt} V dt \\ &= e^{-rt}(dV - rV dt) \end{aligned}$$

   We have used a Factor e^{-rt}.

To calculate the **total profit**, sum up all the small-step profits and apply a discount factor $e^{-r(t-t_0)}$.

$$
\begin{aligned}
& \int_{t_0}^{T} e^{-r(t-t_0)} e^{rt}\, d\left(e^{-rt}(V_i - V_a)\right) \\
= \quad & e^{rt_0} e^{-rT} \left(V_i(T) - V_a(T)\right) - e^{rt_0} e^{-rt_0} \left(V_i(0) - V_a(0)\right) \\
= \quad & V_a(0) - V_i(0) \qquad \text{because} \qquad V_i(T) - V_a(T) \equiv 0
\end{aligned}
$$

Conclusion: if we hedge with actual volatility, then the total profit is a guaranteed amount, equal to the difference between Black Scholes option values $V_a - V_i$.

To see how this guaranteed profit is realised on the MtM basis over dt, we now **invoke Itô lemma directly** to expand dV_i, the first term of † and write using familiar Greeks from $dV = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_a^2 S^2 \frac{\partial^2 V}{\partial S^2})dt + \frac{\partial V}{\partial S} dS$, where $\frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 \;\to\; \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_a^2 S^2 dt$.

$$
\begin{aligned}
& \Theta_i dt + \frac{1}{2}\sigma_a^2 S^2 \Gamma_i dt + \Delta_i dS \; - \Delta_a dS - (V_i - \Delta_a S) r dt - \Delta_a D S dt \\
& \text{insert our model } dS = \boldsymbol{\mu} S dt + \boldsymbol{\sigma_a} S dX \\
& \underbrace{(\Theta_i - rV_i)}\, dt + \frac{1}{2}\sigma_a^2 S^2 \Gamma_i dt + (\Delta_i - \Delta_a) S(\mu dt + \sigma_a dX) + (r - D)\Delta_a S dt
\end{aligned}
$$

$$
V_i \text{ satisfies BSE with } \sigma = \sigma_i \text{ for } \quad \underbrace{\Theta_i - rV_i} = -\left(\Delta_i (r - D) S + \frac{1}{2}\sigma_i^2 S^2 \Gamma_i\right)
$$

$$
\begin{aligned}
= \quad & -\Delta_i (r - D) S dt - \frac{1}{2}\sigma_i^2 S^2 \Gamma_i dt + \frac{1}{2}\sigma_a^2 S^2 \Gamma_i dt + (\Delta_i - \Delta_a) S(\mu dt + \sigma_a dX) \\
& + (r - D)\Delta_a S dt \\
= \quad & \frac{1}{2}(\sigma_a^2 - \sigma_i^2) S^2 \Gamma_i dt + (\Delta_i - \Delta_a) S(\mu dt + \sigma_a dX) + (\Delta_a - \Delta_i)(r - D) S dt \\
= \quad & \frac{1}{2}(\sigma_a^2 - \sigma_i^2) S^2 \Gamma_i dt + (\Delta_i - \Delta_a)\left[(\mu - r + D) S dt + \sigma_a S dX\right]
\end{aligned}
$$

With the asset following a risk-neutral drift $\mu = r$ and no dividends $D = 0$,

$$
\frac{1}{2}(\sigma_a^2 - \sigma_i^2) S^2 \Gamma_i dt + \underline{(\Delta_i - \Delta_a)\sigma_a S dX}
$$

how the profit is achieved is random due to the diffusion term. See Figure 1 below.

If hedging with the implied volatility, the Mark-to-Market profit over time dt will look similar to † with the obvious change $\Delta_a \to \Delta_i$

$$
\begin{aligned}
& \underline{dV_i - \Delta_i \, dS} - r(V_i - \Delta_i S) \, dt \\
& \text{using BSE first time with } \sigma = \sigma_a \\
= \; & \Theta_i dt + \frac{1}{2}\sigma_a^2 S^2 \Gamma_i \, dt - rV_i \, dt + r\Delta_i S \, dt \\
& \text{using BSE second time with } \sigma = \sigma_i \text{ for } \quad \Theta_i dt = rV_i dt - r\Delta_i S dt - \frac{1}{2}\sigma_i^2 S^2 \Gamma_i dt \\
= \; & \frac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2 \Gamma_i \, dt
\end{aligned}
$$

We understand this result as the gain from the curvature $\frac{1}{2}\sigma_a^2 S^2 \Gamma_i \, dt$ being cancelled by the loss from time decay $\frac{1}{2}\sigma_i^2 S^2 \Gamma_i \, dt$ because the simplified Black-Scholes is $\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} = 0 \quad \Rightarrow \quad \Theta_i = -\dfrac{1}{2}\sigma_i^2 S^2 \Gamma_i$ (remember the heat equation).

Add up the present value of all of the small-step profits to get the total profit of

$$\frac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) \int_0^T e^{-rt} S^2 \Gamma_i \, dt$$

This is always positive but path-dependent. Please see Figure 2.

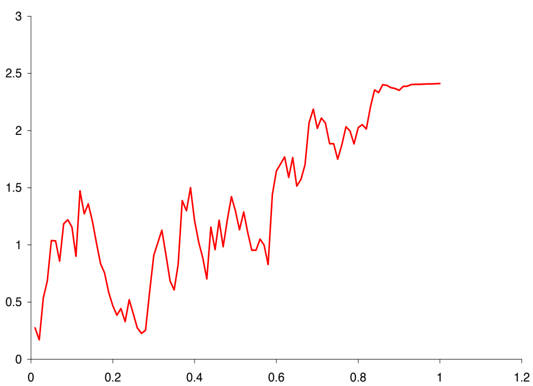

Figure 1: Hedging with σ_a

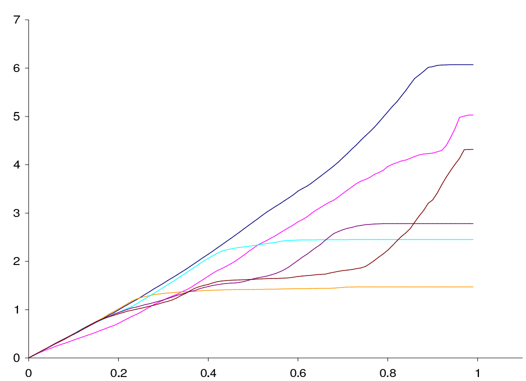

Figure 2: Hedging with σ_i

Advanced Notes

- Naive replication that assumes no discrepancy $\sigma_a = \sigma_i$ has $\frac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2\Gamma\, dt = 0$. Depending on a setup this term can turn negative because empirically $\sigma_i \geq \sigma_{Hist}$ most of the time.

- Delta-hedging can be visualised as removing the linear component from the convex payoff of an option, "_/" - "/". The resulting parabola $\frac{1}{2}\Gamma(\Delta S)^2$ payoff over $(\Delta S)^2$ is proportional to Gamma – the sign of asset move ΔS does not matter. This fully convex payoff is higher with volatility.

  If we adjust the convex payoff by $1/S_t^2$ then $\frac{1}{2}\Gamma(\frac{\Delta S}{S})^2 = \sigma^2\Delta t$. This means that a derivative with the constant Gamma will deliver the payoff equal to variance. Such derivative is known as *variance swap*.

- If σ_i the volatility that we fix at the time of entering a position then

$$\sigma_i^2 = \frac{\int_0^T \sigma_a^2 S^2\Gamma\, dt}{\int_0^T S^2\Gamma\, dt}$$

  meaning that implied volatility is a Gamma-weighted average over actual volatility. This is consistent with representation of variance via the log contract with constant dollar Gamma.

- Vega P&L 'sits' within Delta P&L – we need volatility number to calculate Delta, therefore, how Delta changes is, in fact, important. Higher volatility gives a smooth Delta function, while low volatility gives a step-like Delta function and peaking Gamma. If volatility is not a constant, 'the bastard Vega' factor appears and other Greeks become uncertain.

Regarding the use of Itô lemma when expanding dV_i, when deriving **the Black-Sholes PDE** you have seen

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt$$

also seen

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt$$

where

$$dV = rV dt \qquad \text{and so} \qquad \Delta\, dS = \frac{\partial V}{\partial S}\, rSdt.$$

Volatility information contained in ATM straddle

The straddle position is made up of a long call and a long put with the same strikes and expiries. Practitioners use the market prices of at-the-money straddles to deduce the at-the-money volatility.

The Black–Scholes value of a straddle is given below with C and P are the values of the call and the put respectively and we have used the put-call parity.

$$\begin{aligned} V_{\mathrm{S}} &= C + P = C + C - Se^{-D(T-t)} + Ee^{-r(T-t)} \\ &= 2C - Se^{-D(T-t)} + Ee^{-r(T-t)} \end{aligned}$$

We can therefore deduce the price of a single call and hence the implied volatility

$$C = \frac{1}{2}(V_{\mathrm{S}} + Se^{-D(T-t)} - Ee^{-r(T-t)})$$

Since the straddle is at the money we have $S = E = S^*$ and $t = t^*$.

Risk Reversal

The risk-reversal is a long call, with strike above the current spot, and a short put with a strike below the current spot. Both have the same expiry. Practitioners use the market price of the risk-reversal to deduce the volatility skew.

Assume that the strikes of the call and the put are a short distance ϵ away from the current spot: the strike of the call is thus $S^* + \epsilon$ and the strike of the put is $S^* - \epsilon$. With more informative notation, $C(E, \sigma_{\mathrm{imp}})$ means a call with strike E and implied volatility σ_{imp}, similarly for puts.

The Black–Scholes value of the risk-reversal in terms of the current spot and time is

$$\begin{aligned} V_{\mathrm{RR}} &= C(S^* + \epsilon, \sigma_{\mathrm{imp}}(S^* + \epsilon, T)) - P(S^* - \epsilon, \sigma_{\mathrm{imp}}(S^* - \epsilon, T)) \\ &= C(S^* + \epsilon, \sigma_{\mathrm{imp}}(S^* + \epsilon, T)) - C(S^* - \epsilon, \sigma_{\mathrm{imp}}(S^* - \epsilon, T)) \\ &\qquad + S^* e^{-D(T-t^*)} - (S^* - \epsilon)e^{-r(T-t^*)} \end{aligned}$$

If ϵ is small, that is if the two strikes are close together we can expand and find

$$V_{\mathrm{RR}} - S^*(e^{-D(T-t^*)} - e^{-r(T-t^*)}) =$$

$$\epsilon \left(e^{-r(T-t^*)} + 2\frac{\partial C}{\partial E}(S^*, b) + 2\frac{\partial C}{\partial \sigma_{\mathrm{imp}}}(S^*, b)a \right).$$

where $a(T)$ and $b(T)$ parametrise the implied volatility $\sigma\sqrt{T}$.