Value at Risk and ES Solutions

CQF

1. **Portfolio Risk warm up.** Consider a position of £5 million in a single asset X with daily volatility of 1%. What are the annualised and 10-day standard deviations? Using the Normal factor calculate 99%/10day VaR in money terms.

   Solution: In order to annualise volatility we use the additivity of variance,

   $$\sigma_{1Y} = \sqrt{\sigma_{1D}^2 \times 252} = \sigma_{1D}\sqrt{252} = 0.01 \times \sqrt{252} \approx 0.16$$

   Notice that 1% daily volatility equates approximately to 16% volatility per annum.

   In order to calculate Value at Risk we need the value of Factor which corresponds to the $c = 99\%$ confidence. Using tables for the Normal Distribution we identify the factor value that cuts 1% on the left tail as $\Phi(-2.33) = 0.01$.

   $$\text{VaR}_{99\%/10D} = \Phi^{-1}(1-0.99) \times \sigma_{10D} \times \Pi = 2.33 \times 0.01 \times \sqrt{10} \times £5\,\text{million} = £368,405$$

   where Π is portfolio value (for one asset).

2. Now, consider a portfolio of two assets X and Y, £100,000 investment each. The daily volatilities of both assets are 1% and correlation between their returns is $\rho_{XY} = 0.3$. Calculate 99%/5day Analytical VaR (in money terms) for this portfolio.

   Solution:
   The standard deviation in money terms is $\sigma_X = \sigma_Y = £1000$, which is 1% from £100,000. The variance of the portfolio's daily change is

   $$\begin{aligned} \sigma_\Pi^2 &= \sigma_X^2 + 2\rho_{XY}\,\sigma_X\sigma_Y + \sigma_Y^2 \\ \sigma_\Pi^2 &= 1000^2 + 2 \times 0.3 \times 1000 \times 1000 + 1000^2 = 2.6 \times 10^6 \end{aligned}$$

   which gives the standard deviation for the portfolio (its daily change) $\sigma_\Pi = £1,612.45$.

   Scaling to 5 days and using the factor value for $c = 99\%$ confidence, the result is

   $$\text{VaR}_{99\%/5D} = 2.33 \times 1612.45 \times \sqrt{5} = £8,401.$$

   Question 1 and 2 calculations assume that portfolio value (its cumulative P&L) follows the Normal Distribution. 99% VaR risk measure represents any move beyond 2.33 standard deviations, however we do not know how worse the move (loss) can be.

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3. **Fully Analytical VaR for Efficient Markets** – Assume that P&L of an investment portfolio is a random variable that follows Normal distribution $X \sim N(\mu, \sigma^2)$. Use the definition of *VaR as a percentile*,

$$\Pr\left(x \leq \mathrm{VaR}(X)\right) = 1 - c$$

in order to derive analytical expression for VaR calculation.

$\boxed{\textbf{Solution:}}$ We start with probability for the P&L (loss) X exceeding VaR(X) threshold and convert X to a Standard Normal variable ϕ. The probability of loss $x < 0$ being worse than VaR < 0 is

$$\begin{aligned}
\Pr\left(x \leq \mathrm{VaR}(X)\right) &= 1 - c \\
\mathrm{VaR}_c(X) &= \inf\{x \mid \Pr(X > x) \leq 1 - c\} = \inf\{x \mid F_X(x) \geq c\}
\end{aligned}$$

for or 99% confidence, the probability that X above loss x is less than $(1 - 0.99) = 0.01$.

If P&L X is a random variable then VaR(X) is also a random variable. In order to use the well-known Normal Distribution functions, we have to work with the Standard Normal variable

$$\begin{aligned}
\Pr(\phi \leq \frac{\mathrm{VaR}(X) - \mu}{\sigma}) &= 1 - c \qquad \Longrightarrow \\
\mathrm{VaR}(X) &= \mu + \Phi^{-1}(1 - c) \times \sigma
\end{aligned}$$

Inverse CDF is a percentile function.

4. What about Expected Shortfall? The universal definition of ES in terms of expectations algebra is given as follows:

$$\begin{aligned}
\mathrm{ES}_c(X) &= \mathbb{E}[\, X \mid X \leq \mathrm{VaR}_c(X)] \\
\mathrm{ES}_c(X) &= \frac{1}{1 - c} \int_0^{1-c} \mathrm{VaR}_u(X) du
\end{aligned}$$

The actual ES calculation formula will vary depending on the distribution of P&L X, a random variable. Derive ES calculation formula for the case of Normal Distribution using the result $\mathrm{VaR}(X) = \mu + \Phi^{-1}(1 - c) \times \sigma$.

$\boxed{\textbf{Solution:}}$ ES universal definition (via integral above) means averaging over all VaR values, e.g., $99\%, 99.1\%, 99.2\%, \ldots$ – the tail percentile values.

$$
\begin{aligned}
\mathrm{ES}_c(X) &= \frac{1}{1-c}\int_c^1 \mathrm{VaR}_u(X)du \qquad \text{percentile changed to upper}\\
&= \frac{1}{1-c}\int_c^1 \left(\mu+\sigma\Phi^{-1}(1-u)\right)du\\
&= \mu+\frac{1}{1-c}\int_c^1 \sigma\Phi^{-1}(1-u)du
\end{aligned}
$$

To cancel out $\sigma\Phi^{-1}()$, it makes sense to choose $u=\Phi_Z(z)$. Respectively, $du=\phi_Z(z)dz$ and integration limits re-map $[c,1]\mapsto[\Phi^{-1}(c),\infty]$

$$
\begin{aligned}
&= \mu+\frac{1}{1-c}\int_{\Phi^{-1}(c)}^{\infty}\sigma\Phi^{-1}\left(1-\Phi_Z\right)\phi_Z dz\\
&= \mu-\frac{1}{1-c}\int_{\Phi^{-1}(c)}^{\infty}\sigma\, z\,\phi(z)\,dz\\
&= \text{using} \qquad \int ze^{-z^2/2}=-e^{-z^2/2}\\
&= \mu-\frac{1}{1-c}\sigma\left[-e^{-\infty^2/2}-\left(-e^{[\Phi^{-1}(c)]^2/2}\right)\right]\\
&= \mu-\sigma\,\frac{\phi(\Phi^{-1}(c))}{1-c}.
\end{aligned}
$$

The result has a quirk of ICDF being inside PDF but this is simply $\phi(-2.32635)$ for our $99\%th$ Standard Percentile (see numbers in the next exercise). For cases other than Normal, the average loss on the tail might have to be computed by Monte-Carlo than fully analytically.

5. Let's figure out a few numbers for these efficient, 'elliptical' markets – the term means asset (market index) returns are Normally distributed or close.

   - What percentage of returns are outside 2σ from the mean? Consider the left tail.
   - On that tail, what is the mean of standardised returns – that is, what is an average tail loss?

   Provide general solutions using PDF for Normal Distribution $N(\mu,\sigma^2)$ below, and assume standard Normal to obtain numerical results.

   Solution:
   The percentage of returns outside n standard deviations (in general) on the left tail is given by the cumulative density function , which is an integral over probability density

$$
\Phi(\mu-n\sigma)=\int_{-\infty}^{\mu-n\sigma}f(x)dx=\int_{-\infty}^{\mu-n\sigma}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx
$$

For the Standard Normal, $\mu = 0, \sigma^2 = 1$ with *pdf* $f(x) \equiv \phi(z)$ but variable standardised $z = \frac{x-\mu}{\sigma}$, the percentage of returns outside two standard deviations on the left tail is

$$\Phi(-2) = \int_{-\infty}^{-2} \phi(z) = \int_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = 0.02275$$

$\mathrm{VaR}_c(X)$ calculation is an inverse task to the above: we are given confidence level $c = 99\%$ and so ask which empirical value 'cuts' $(1-c)$ on the tail. That is, what is the lower percentile that cuts 1% of observations. Inverse CDF gives the percentile $\Phi^{-1}(1-0.99) \approx -2.32635$. In Excel use *=NORM.S.INV(0.01)*.

The standard percentile value ≈ -2.32635 is referred to as **Factor**. This value is the VaR on the Standard scale when all values x are Standarised into **Z-scores** $z = \frac{x-\mu}{\sigma}$.

Expected Shortfall $\mathrm{ES}_c(X) = \frac{1}{1-c}\int_0^{1-c} \mathrm{VaR}_u(X)du$ have the following names:

- the average tail loss
- the mean of standardised returns on the tail
- the moment on the tail divided by cumulative density

When computing the following expression, we really compute the first moment of the distribution $f(x)$ (just within $-\infty$ to $\mu - n\sigma$ limits).

$$\int_0^{1-c} \mathrm{VaR}_u(X)du \;\equiv\; \int_{-\infty}^{\mu-n\sigma} x\, f(x)dx \quad \dagger$$

Remember we need to find **an average**. Averaging done by $\frac{1}{1-c} = \frac{1}{\int_{-\infty}^{\mu-n\sigma} f(x)dx} \quad \ddagger$

We can finally form an expression for **the average** as follows $\ddagger \times \dagger$

$$\frac{\int_{-\infty}^{\mu-n\sigma} x\, f(x)dx}{\int_{-\infty}^{\mu-n\sigma} f(x)dx} \quad = \quad \frac{1}{\Phi(\mu - n\sigma)} \int_{-\infty}^{\mu-n\sigma} \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \quad \dagger$$

This is the most generalised way of expressing Normal **Expected Shortfall.**

The solution must be simple in terms of Standard Normal, and so we define the following change of variable $-z = (x-\mu)/\sigma$:

$$x = \mu - z\sigma, \quad dx = -\sigma dz, \quad -\frac{(x-\mu)^2}{2\sigma^2} = -\frac{(-z\sigma)^2}{2\sigma^2} = -z^2/2$$

$$
\begin{aligned}
&= \frac{1}{\Phi(x)} \int_{-\infty}^{x} (\mu - z\sigma)\, \frac{1}{\sigma\sqrt{2\pi}}\, e^{-z^2/2}\, (-\sigma dz) \qquad \dagger \text{ variable changed} \\
&= \text{cdf for Standard Normal} \qquad \Phi(\mathbf{x}) = \int_{-\infty}^{\mathbf{x}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= -\frac{1}{\Phi(x)} \left(\mu\, \Phi(x) - \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{x} z\, e^{-z^2/2}\, dz \right) \\
&= \text{using} \qquad \int z e^{-z^2/2} = -e^{-z^2/2} \qquad \text{and} \qquad e^{-\infty} = 1/e^{\infty} = 0 \\
&= -\left(\mu - \sigma\, \frac{1}{\Phi(x)}\, \underbrace{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}} \right)
\end{aligned}
$$

Underbrace highlights Standard Normal PDF $\phi(x)$ and the result can be re-expressed as follows, it gives a positive value but one has to remember that **ES is a loss**,

$$
\mathrm{ES}_c = -\left(\mu - \sigma \frac{\phi(\Phi^{-1}(1-c))}{1-c} \right)
$$

where $\frac{1}{1-c}$ equates to $\frac{1}{\Phi(x)}$ for Standard Normal.

$$
\begin{aligned}
\mathrm{ES}_{99\%} &= -\left(0 - \frac{\phi\left(\Phi^{-1}(0.01)\right)}{0.01} \right) \\
&= \frac{\phi\,(-2.32635)}{0.01} = 0.026652/0.01 = 2.6652.
\end{aligned}
$$

$$
-2.6652 < -2.32635, \qquad \text{loss } \mathrm{ES}_{99\%} < \mathrm{VaR}_{99\%}
$$

This ES value -2.6652 is below the percentile $\Phi^{-1}(0.01) = -2.32635$ standard deviations, a value cuts 1% on the lower tail.

Back to our problem of the left tail of 2 standard deviations,

$$
\begin{aligned}
\mathrm{ES}_{96.725\%} &= \frac{1}{\Phi(-2)} \int_{-\infty}^{-2} \frac{z}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz = -\frac{1}{0.02275} \frac{1}{\sqrt{2\pi}} e^{-2} \approx -2.37 \\
& \text{using} \qquad \int z e^{-z^2/2} = -e^{-z^2/2} \\
& -2.37 < -2, \qquad \mathrm{ES}_c < \mathrm{VaR}_c
\end{aligned}
$$

With this approach the result is negative and corresponds to $\mathrm{ES}_{96.725\%}$ since -2 standard deviations cut 2.275% on the left tail.

A note on **Portfolio VaR and ES** calculation. The inputs for portfolio optimisation have a long-term timescale, e.g., annual or monthly – VaR calculation over these horizons makes lesser sense and becomes challenging. However, the analytical formulae are available for portfolio return $\mu_\Pi = \boldsymbol{w}'\boldsymbol{\mu}$ (assumed positive) and risk $\sigma_\Pi = \sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}}$,

$$\begin{aligned}\mathrm{VaR}_c(\Pi) &= \boldsymbol{w}'\boldsymbol{\mu} + \text{Factor} \times \sqrt{\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}} \\ &= \mu_\Pi + \text{Factor} \times \sigma_\Pi\end{aligned}$$

$$\mathrm{ES}_c(\Pi) = \mu_\Pi - \sigma_\Pi \, \frac{\phi(\text{Factor})}{1-c}$$

where Factor $= \Phi^{-1}(1-c)$ is a standardised Normal percentile. For illustration, $\Phi^{-1}(0.01) \approx -2.32635$ but a calculation must rely on the exact factor value.

In practice, the mean μ is 'dropped' from VaR and ES computation. Think of the average daily return for a market index, it is 0.02% or so, a small quantity. Second, it is difficult to estimate such quantity with statistical robustness.

6. **The Tail Story...** Consider improvement to risk measures estimation by choosing a distribution with desired properties. Generalised Pareto Distribution utilised by the updated extreme value theory (EVT) to account for exceedances over threshold $X - u$.

   Why GPD is worth considering? It is useful because of the neat result: for any random variable X, its values over threshold u follow $\mathrm{GPD}(\xi, \mu, \sigma)$ probability density. The useful expectation result (any RV) below is known as **the excess function**

$$e(u) = \mathbb{E}[X - u | X > u] = \frac{\beta(u_0) + \xi(u - u_0)}{1 - \xi}$$

   It is the practitioner recipe to fit the tail data of market index returns to the Pareto Distribution. Then, VaR can be calculated using the percentile function $\mathrm{VaR}_c = F_X^{-1}$:

$$\mathrm{VaR}_c = u + \frac{\beta}{\xi}\left[\left(\frac{N}{N_u}(1-\alpha)\right)^{-\xi} - 1\right].$$

   N_u is the number of exceedances among N observations. ξ is a tail index from data. Now, formulate the EVT result for Expected Shortfall $\mathrm{ES}_c = \mathbb{E}[X | X \geq \mathrm{VaR}_c]$.

Solution:

$$\begin{aligned}
\mathrm{ES}_c = \mathbb{E}[X|X \geq \mathrm{VaR}_c] &= \mathrm{VaR}_c + \mathbb{E}[X - \mathrm{VaR}_c|X \geq \mathrm{VaR}_c] \\
&\quad \text{invoke excess function result, } \mathrm{VaR}_c > u \text{ (right tail)} \\
&= \mathrm{VaR}_c + \frac{\beta + \xi\left(\mathrm{VaR}_c - u\right)}{1-\xi} \\
&= \left(1 + \frac{\xi}{1-\xi}\right)\mathrm{VaR}_c + \frac{\beta - \xi u}{1-\xi} \\
&= \frac{\mathrm{VaR}_c}{1-\xi} + \frac{\beta - \xi u}{1-\xi} \\
&= \frac{1}{1-\xi}\left(u + \frac{\beta}{\xi}\left[\left(\frac{N}{N_u}(1-\alpha)\right)^{-\xi} - 1\right]\right) + \frac{\beta - \xi u}{1-\xi}.
\end{aligned}$$

Frequency of large returns and their magnitude are important. EVT defines the conditional excess distribution function (conditional tail distribution) for a random variable $X > u$, where $Y_i = X_i - u$, Y_i is excesses over the threshold u.

$$F_u(y) = \Pr\left((X-u) \leq y \,|X > u\right) = \frac{F(x) - F(u)}{1 - F(u)} \qquad 0 \leq y \leq (x-u)$$

Generalised Pareto Distribution is utilised to approximate this function – the distribution of exceedances on the tail.

7. **Back to ABC.** Assume three bonds A,B and C, each has a face value of £$1,000$ payable at maturity. The the independent probability of default is 0.5%.

   (a) For the portfolio equally invested in bonds A, B and C, the 99% VaR is £$1,000$. Explain this result.

   (b) Calculate the Expected Shortfall for the bond A only. Assume 1% tail and partial loss possible.

   (c) Calculate the Expected Shortfall of a portfolio *equally invested* in bonds A, B and C. Assume 1% tail and continuous distribution.

   (d) Compare results (b) and (c) to conclude whether ES is *sub-additive*.

   Solution:

   (a) Remember that VaR is value (loss) at the percentile. The probability of no default is $1 - 0.005 * 3 = 0.985$ and so 98.5th percentile loss is £0. Adding the probability of the next outcome (1 default) gives us $0.985 + 0.01485 = 0.99985$th percentile, so the minimum loss at $99th$ percentile is £$1,000$.

	Loss	Cumulative Density
No defaults	£0	$\approx 98.51\%$ $\quad 0.995 \times 0.995 \times 0.995$ $\quad$ for (not ABC)
1 default	£$1,000$	$\approx 1.485\%$ $\quad 3 \times 0.005 \times 0.995^2$ $\quad$ for A (not BC) + B (not AC) + C (not AB)
2 defaults	£$2,000$	0.0074625% $\quad 3 \times 0.005^2 \times 0.995$ $\quad$ for AB (not C) + BC (not A) + AC (not B)
3 defaults	£$3,000$	0.0000125% $\quad 0.005 \times 0.005 \times 0.005$ $\quad$ for ABC

Table 1: Loss Distribution (*cdf*)

Loss Distribution takes account of combinatorial outcome of all cases: 1 or 2 or 3 defaults have respectively, 2, 1, and 0 survivals. The large chunk of loss distribution's density assigned zero loss value, at $\Phi^{-1}(1 - 0.9851) = -2.17274$ Standard Percentile. Computing $\Phi^{-1}(1 - 0.9851 - 0.01485) = -3.8906$ Standard Percentile is too far to define a tail.
Here is our problem of discrete loss mapped onto continuous analytical Normal distribution.

   (b) We are in the tail situation (loss has to occur) but we retain the probability of default. Conditional probability of loss is $\frac{\text{Pr}}{1-c}$ and and we multiply it by Loss as usual (probability $\times$ expected value),

$$\text{ES} = \frac{\text{Pr}}{1-c} \times \text{Loss} = \frac{0.5\%}{1\%} \times 1000 = £500.$$

   (c) We already established in (a) that 99% VaR for the portfolio is £$1,000$. Because ES (CVaR) is an average of VaRs on the tail, it can't be less than this value.

$$\text{ES}_{1-c} = \frac{1}{1-c} \int_0^{1-c} \text{VaR}_\gamma(X) d\gamma$$

For ES for a continuously distributed variable we integrate, where VaR_γ is loss which in our case £3,000, £2,000 £1,000 and $d\gamma$ are 'chunks' of density that are unequal (see Loss Distribution).

ES ≡ Average loss on the tail ≡ EV over tail of loss distribution

Discrete case has summation instead of integration. Form Conditional Loss Distribution from the tail of Loss Distribution:

$$\mathbf{ES} = 0.992525 \times 1000 + 0.0074625 \times 2000 + 0.0000125 \times 3000 = \text{£}\mathbf{1,007.49}$$

How did we obtained those densities?

	Loss	Likelihood
3 defaults	£3,000	0.0000125%/1%
2 defaults	£2,000	0.0074625%/1%
1 default	£1,000	≈ 0.9925 †

Table 2: Conditional Loss Distribution

where Pr(1 default) = 1 - Pr(2 defaults) - Pr(3 defaults) so,

$$1 - 0.0074625 - 0.0000125 = 0.992525. \quad \dagger$$

We take this opportunity to illustrate a side of Bayesian rule: to recover the unconditional distribution, you have to integrate as follows:

$$f(y) = \int f(y|x) f(x) dx$$

Discrete implementation: take those conditional probabilities (Table 2) and multiply them by the respective marginal probabilities (Table 1, rounded).

$$\begin{aligned} 0.9925 \times 0.015 + 0.0074625 \times 0.0075 + 0.0000125 \times 0.00001 &\approx 0.0149 \\ 1 - \text{PrSurv} = 1 - \times 0.995^3 &\approx 0.0149. \end{aligned}$$

0.0149 is the total density allocated to the conditional tail from **Total** Loss Distribution. At least one default Pr= 0.0149, no defaults Pr= 0.9851 sum up to 1.

There is one disclaimer to make: given the first discrete loss occurs at 0.9851 it makes sense to use this percentile. Then numerical results will be different for $98.51th$ VaR and ES where the first default serves as 'a natural boundary'.

(d) ES for the portfolio of ABC is noticeably less than 3 × ES of each bond = £1,500 and so it is sub-additive. This holds for most cases, except a few that are only of academic interest.

8. **The practice** – What are two main numerical methods that support VaR Backtesting and Stress-testing in terms of generating rather than merely sampling asset returns? What are their drawbacks?

Discussion:

- **Monte Carlo** method requires generation of quasi-random numbers and relies on their low latency (evenness). Fractals reveal the problem of correlated random numbers occurs if number of asset paths (dimensions) is high.

  The assumption of a log-normal random walk is not as innocent because asset log-returns might not be Normal, and in times of stress are likely to be autocorrelated. Correlation among assets might give humongous covariance matrices that require factorisation (eg, by Cholesky decomposition) and simulation via the multi-factor PDEs. Monte-Carlo can be slow computationally.

- **Bootstrapping** (or Historic Simulation) method uses actual asset price movements taken from historical data. Common practice for VaR calculation is consider the last two years (this is still an arbitrary choice). We refer to bootstrapping as sampling from *the standardised historic residuals* – that is, we convert each log-return u_t into corresponding Standard Normal variable,

$$Z_t^* = \frac{u_{t,Hist}}{\sqrt{\sigma^2_{t,GARCH}}}$$

  Instead the sample standard deviation σ_t, RiskMetrics methodology suggests $\sigma_{t+1,GARCH}$, a prediction of volatility from a pre-calibrated GARCH model (or EWMA if predicting the short-term volatility, for the periods of 10-60 days it might not be feasible to expect reversion of variance to the average long-term level $\bar{\sigma}^2$).

  Bootstrapping method requires a large amount of clean data, including the periods of past crises, such as credit crunch 2008. It happens that exactly the data needed is not available due to absence of liquidity in markets – the data suffers from 'structural breaks', such as missing market prices.

9. **A useful risk modelling technique...** Covariance matrix can be decomposed as $\boldsymbol{\Sigma} = \boldsymbol{AA'}$ by Cholesky method (presented in Credit Risk module). The result is a lower triangular matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Let $X_1(t)$ and $X_2(t)$ are two uncorrelated Wiener processes (orthogonal). How would you use the Cholesky result in order to construct two correlated processes?

$\boxed{\textbf{Solution:}}$

In order to perform Cholesky decomposition, the matrix $\boldsymbol{\Sigma}$ must be symmetric and **positive definite**, this is a numerical condition.

$$\boldsymbol{x'\Sigma x} > 0 \qquad \text{for any vector} \qquad \boldsymbol{x} \in \mathbb{R}$$

If matrix eigenvalues are positive $\lambda_1 > \cdots > \lambda_n > 0$, then this condition is satisfied.

Correlation is imposed by $\boldsymbol{Y} = \boldsymbol{AX}$ (result below), so $Y_1(t)$ and $Y_2(t)$ are correlated.

$$\begin{aligned} Y_1 &= \sigma_1 X_1 \\ Y_2 &= \rho\,\sigma_2 X_1 + \sqrt{1-\rho^2}\,\sigma_2 X_2 \end{aligned}$$

$Y_1(t)$ and $Y_2(t)$ remain Brownian Motions albeit non-standardised. In particular, as a linear combination of two Brownian Motions, $Y_2(t)$ **keeps the properties of BM**.

Simplify $\sigma_1 = \sigma_2 = 1$, the increment of such standardised $Y_t - Y_s, \forall s < t$ follows $N(0, \tau)$, which is the Normal distribution closed under sum

$$N(0, \tau\rho^2) \qquad \text{and} \qquad N(0, \tau(1-\rho^2))$$

Alternatively, consider the variance of any random variable

$$\begin{aligned} \mathbb{V}\text{ar}[Y_2(t)] &= \mathbb{V}\text{ar}\left[\rho\, X_1(t) + \sqrt{1-\rho^2}\, X_2(t)\right] \\ &= \rho^2\, \mathbb{V}\text{ar}[X_1(t)] + (1-\rho^2)\, \mathbb{V}\text{ar}[X_2(t)] \\ &= \rho^2\tau + (1-\rho^2)\tau = \tau \quad \text{variance scales with time; Standard BM has } \sigma^2 = 1 \end{aligned}$$

It follows that the increment $Y_2(t) - Y_2(s)$ is distributed as $\sim N(0, t-s) \equiv N(0, \tau)$.