# CQF Module 2
# Fundamentals of Optimization and Application to Portfolio Selection

## Exercises

1. Using the Lagrange multipliers optimization approach, parametrize the boundary of the opportunity set in terms of the expected portfolio returns, $m$, for the following set of assets:

| **Asset** | $\mu$ | $\sigma$ |
|---|---|---|
| A | 0.08 | 0.12 |
| B | 0.10 | 0.12 |
| C | 0.10 | 0.15 |
| D | 0.14 | 0.20 |

in the following three cases:

**a).** The correlation coefficients between the four assets are given by

$$\rho = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

**b).** The correlation coefficients between the four assets are given by

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**c).** The correlation coefficients between the four assets are given by

$$\rho = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

2. Consider a three asset risky economy where the covariance matrix of expected returns is given by

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 0 \\ 3 & 16 & 5 \\ 0 & 5 & 25 \end{pmatrix}$$

Show that the covariance matrix is strictly positive definite in the sense that

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} > 0$$

if $\begin{pmatrix} x & y & z \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$. [Hint: Show that the resulting quadratic form is a sum of perfect squares.]

Show by calculation that the inverse of the covariance matrix is

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}^{-1} = \frac{1}{210} \begin{pmatrix} 25 & -5 & 1 \\ -5 & 15 & -3 \\ 1 & -3 & 9 \end{pmatrix}$$

3. Where should we be on the efficient frontier in question 1.a. if we wish to minimise the chance of a return less than 0.05?

4. Consider a Markowitz world with a two asset risky economy. Assets $A$ and $B$ have an expected return over a one year time horizon, of 0.1 and 0.2 respectively. The standard deviation for asset $A$ is 0.2 and for asset $B$ is 0.3, over that year. The correlation between returns is 0.5. If $W$ is the weighting attached to asset $A$, calculate the *boundary of the opportunity set*, by obtaining the expected return $r(W)$ and the standard deviation $\sigma_{\Pi}(W)$, for this portfolio $\Pi$. By varying $W$ plot the boundary on a risk/reward diagram, and mark the *efficient frontier* (you may use Excel for the plot).

   Now introduce a risk free asset that has an annual return of 5%. Obtain the *Capital Market Line* and *Market Portfolio*.