# Computational Methods - Problems and Solutions

1. Consider the pricing equation for the value of a derivative security $V(S,t)$,

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, \tag{1.1}$$

where $S\geq 0$ is the spot price of the underlying equity, $0<t\leq T$ is the time, $r\geq 0$ the constant rate of interest, and $\sigma>0$ is the constant volatility of $S$.

The variables $(t,S)$ can be written as

$$\begin{aligned} t &= T-m\delta t \qquad 0\leq m\leq M, \\ S &= n\delta S \qquad 0\leq n\leq N, \end{aligned}$$

where $(\delta t,\delta S)$ are fixed step sizes in turn. $V(S,t)$ is written discretely as $V_n^m$. A Fully Implicit Finite Difference Method is to be developed to solve (1.1) using a *forward marching scheme*.

(a) Derive a difference equation in the form

$$\alpha_n V_{n-1}^{m+1}+\beta_n V_n^{m+1}+\gamma_n V_{n+1}^{m+1}=V_n^m, \tag{1.2}$$

where $\alpha_n$, $\beta_n,\gamma_n$ should be defined; you may use the following

$$\begin{aligned} \frac{\partial V}{\partial t} &\sim \frac{V_n^{m+1}-V_n^m}{\delta t}, \\ \frac{\partial V}{\partial S} &\sim \frac{V_{n+1}^{m+1}-V_{n-1}^{m+1}}{2\delta S}, \\ \frac{\partial^2 V}{\partial S^2} &\sim \frac{V_{n-1}^{m+1}-2V_n^{m+1}+V_{n+1}^{m+1}}{\delta S^2}. \end{aligned}$$

**SOLUTION** The PDE is (under the transformation)

$$-\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+(r-D)\,S\frac{\partial V}{\partial S}-rV=0.$$

Substituting in

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{V_n^{m+1}-V_n^m}{\delta t}+O(\delta t), \\ \frac{\partial V}{\partial S} &= \frac{V_{n+1}^{m+1}-V_{n-1}^{m+1}}{2\delta S}+O\left(\delta S^2\right), \\ \frac{\partial^2 V}{\partial S^2} &= \frac{V_{n-1}^{m+1}-2V_n^{m+1}+V_{n+1}^{m+1}}{\delta S^2}+O\left(\delta S^2\right) \end{aligned}$$

renders the PDE

$$\frac{V_n^m-V_n^{m+1}}{\delta t}+\frac{1}{2}\sigma^2n^2\left(V_{n-1}^{m+1}-2V_n^{m+1}+V_{n+1}^{m+1}\right)+\frac{(r-D)\,n}{2}\left(V_{n+1}^{m+1}-V_{n-1}^{m+1}\right)-rV_n^{m+1}=0.$$

The following finite difference equation is obtained

$$a_nV_{n-1}^{m+1}+b_nV_n^{m+1}+c_nV_{n+1}^{m+1}=V_n^m$$

where the coefficients are given by

$$\begin{aligned} a_n &= -\frac{1}{2}\left(\sigma^2n^2-n\,(r-D)\right)\delta t,\ b_n=1+\left(\sigma^2n^2+r\right)\delta t, \\ c_n &= -\frac{1}{2}\left(\sigma^2n^2+n\,(r-D)\right)\delta t \end{aligned}$$

(b) Obtain expressions for the final and boundary conditions in finite difference form for a European Call Option.

$$\left.\begin{aligned} V_n^M &= \max\left(n\delta S-E,0\right) \\ & 0\leq n\leq N; \end{aligned}\right\} \text{ Final Payoff Condition}$$

$$\left.\begin{aligned} (1+r\delta t)\,V_0^{m+1} &= V_0^m \\ M\geq m\geq 1 & \quad 1 \end{aligned}\right\} \text{Boundary condition at } (S=0)$$

$$\left.\begin{aligned} \widehat{a}_NV_{N-1}^{m+1}+\widehat{b}_NV_N^{m+1} &= V_N^m \\ M\geq m\geq 1;\ S &= N\delta S \end{aligned}\right\} \text{Boundary condition at } S^*$$

(c) The resulting matrix inversion problem is of the form

$$A\mathbf{x} = \mathbf{b}. \tag{1.3}$$

Give the form of the matrix $A$.
**SOLUTION**

$$\begin{pmatrix} b_0 & c_0 & 0 & \cdots & \cdots & 0 \\ a_1 & b_1 & c_1 & & & \vdots \\ 0 & a_2 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \cdots & \cdots & 0 & a_n & b_n \end{pmatrix}$$

2. (a) Given a vector $\mathbf{x} \in \mathbb{R}^n$, define the $l_p$−norm, written $\|\mathbf{x}\|_p$.
Hence for $p = 1, 2, \infty$ obtain the $l_p$ norms for $\mathbf{x} = (-2, 5, -7, 0)$.
**SOLUTION** The norm is defined as

$$\|\mathbf{x}\|_p = \left\{ \sum_{i=1}^{n} |x_i|^p \right\}^{1/p}$$

For the given $\mathbf{x} \in \mathbb{R}^4$

$$\|\mathbf{x}\|_1 = 14; \ \|\mathbf{x}\|_2 = \sqrt{4+25+49} = 8.83; \ \|\mathbf{x}\|_\infty = 7.$$

(b) Consider the linear system

$$\begin{aligned} 4x_1 + 3x_2 \qquad &= 24, \\ 3x_1 + 4x_2 - x_3 &= 30, \\ -x_2 + 4x_3 &= -24. \end{aligned}$$

Find the first two iterations $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, using the SOR (Gauss-Seidel) method with relaxation factor $\omega = 1.25$; **SOLUTION** You may take $\mathbf{x}^{(0)} = (1, 1, 1)$. Hence calculate

$$\frac{\left\|x^{(2)} - x^{(1)}\right\|_\infty}{\left\|x^{(1)}\right\|_\infty}.$$

| $k$ | 0 | 1 | 2 |
|---|---|---|---|
| $x_1^{(k)}$ | 1 | 6.3125 | 2.622 |
| $x_2^{(k)}$ | 1 | 3.5195 | 3.959 |
| $x_3^{(k)}$ | 1 | −6.6501 | −4.600 |

$$\frac{\left\|x^{(2)} - x^{(1)}\right\|_\infty}{\left\|x^{(1)}\right\|_\infty} = \frac{3.6905}{6.6501} = 0.555$$

(c) Solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 10 \\ 1 \\ 4 \end{pmatrix}$$

using **Doolittle's** method.
**SOLUTION** In Doolittle's method the matrix $L$ has a unit diagonal

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

$$
\begin{aligned}
A &= LU = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{pmatrix} \\
&= \begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}
\end{aligned}
$$

The solution is $\mathbf{x} = (1, 2, -3)$ .

(d) Given a square matrix $B$, explain what it means for $B$ to be strictly diagonally dominant. Hence test if the matrix

$$
B = \begin{pmatrix} 7 & 1 & -2 \\ 0 & 2 & 2 \\ 1 & 3 & 6 \end{pmatrix}
$$

is strictly diagonally dominant.

**SOLUTION** The matrix $B$ is strictly diagonally dominant if

$$
|\mathbf{B}_{ii}| > \sum_{j \neq i} |\mathbf{B}_{ij}| \, ,
$$

In the example given: in row 1- $7 > 1 + |-2|$ . in row 2- 2 is not greater than $0 + 2$. Hence it is not strictly diagonally dominant.