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STAT3500

1. $f(x) = \frac{1}{\theta}$

$L(\theta; x) = \left(\frac{1}{\theta}\right)^3$

$= \frac{1}{\theta^3}$

$\theta \le X_{(1)} \le X_{(2)} \le X_{(3)} \le 2\theta$

$\Rightarrow \frac{X_{(3)}}{2} \le X_{(1)}$

The likelihood function is a decreasing function, therefore MLE, $\hat{\theta} = \frac{X_{(3)}}{2}$

WTF: $k$ s.t $E_\theta(k\hat{\theta}) = \theta$

Let $Y = \max(X_1, X_2, X_3) = X_{(3)}$

$P(Y \le y) = P(\max(X_1, X_2, X_3) \le y)$

$= P(X_1 \le y, X_2 \le y, X_3 \le y)$

$= P(X_1 \le y) P(X_2 \le y) P(X_3 \le y)$

$= \left(\frac{y - \theta}{2\theta - \theta}\right)^3$

$= \left(\frac{y - \theta}{\theta}\right)^3$

$P(Y = y) = \frac{d}{dy} P(Y \le y)$

$= \frac{d}{dy}\left(\frac{y-\theta}{\theta}\right)^3$

$= 3\left(\frac{y-\theta}{\theta}\right)^2\left(\frac{1}{\theta}\right)$

$= \frac{3}{\theta^3}(y-\theta)^2$

$$E(Y) = \int_0^{2\theta} y \cdot \frac{3}{\theta^3}(y-\theta)^2 \, dy$$

$$= \frac{3}{\theta^3}\int_0^{2\theta} y \cdot (y^2 - 2y\theta + \theta^2) \, dy$$

$$= \frac{3}{\theta^3}\left[\frac{1}{4}y^4 - \frac{2}{3}y^3\theta + \frac{1}{2}y^2\theta^2\right]_0^{2\theta}$$

$$= \frac{3}{\theta^3}\left[\left(4\theta^4 - \frac{2}{3}(8\theta^3)\theta + \frac{1}{2}(4\theta^2)\theta^2\right) - \left(\frac{1}{4}\theta^4 - \frac{2}{3}\theta^4 + \frac{1}{2}\theta^4\right)\right]$$

$$= \frac{3}{\theta^3}\left(4\theta^4 - \frac{16}{3}\theta^4 + 2\theta^4 - \frac{1}{4}\theta^4 + \frac{2}{3}\theta^4 - \frac{1}{2}\theta^4\right)$$

$$= \frac{3}{\theta^3}\left(\frac{7}{12}\theta^4\right)$$

$$= \frac{21}{12}\theta$$

$$= \frac{7}{4}\theta$$

Since $Y = X_{(3)} = 2\hat{\theta}$,

$$E(Y) = E(2\hat{\theta})$$
$$= \frac{7}{4}\theta$$

$$\Rightarrow \frac{4}{7}E(Y) = \frac{4}{7}E(2\hat{\theta})$$
$$= E\left(\frac{8}{7}\hat{\theta}\right)$$
$$= \theta$$

$\therefore k = \frac{8}{7}$

2. $f(y;\theta) = \exp\left\{ \frac{a(y)b(\theta)+c(\theta)}{\ell(\phi)} + d(y;\phi) \right\}$

a) $\int_{-\infty}^{\infty} f(y;\theta)\, dy = 1$ since $f(y;\theta)$ is a pdf

$\Rightarrow \frac{d}{d\theta}\left(\int_{-\infty}^{\infty} f(y;\theta)\, dy\right) = \frac{d}{d\theta}(1)$

$\Rightarrow \int_{-\infty}^{\infty} \frac{d}{d\theta} f(y;\theta)\, dy = 0$ — ①

$$\frac{d}{d\theta} f(y;\theta) = \left(\frac{a(y)b'(\theta)+c'(\theta)}{\ell(\phi)}\right) f(y;\theta) \quad \text{— ②}$$

② -> ①, $\int_{-\infty}^{\infty} \left(\frac{a(y)b'(\theta)+c'(\theta)}{\ell(\phi)}\right) f(y;\theta)\, dy = 0$

$\Rightarrow \frac{b'(\theta)}{\ell(\phi)} \int_{-\infty}^{\infty} a(y) f(y;\theta)\, dy + \frac{c'(\theta)}{\ell(\phi)} \int_{-\infty}^{\infty} f(y;\theta)\, dy = 0$

$\Rightarrow \frac{b'(\theta)}{\ell(\phi)} E(a(y)) + \frac{c'(\theta)}{\ell(\phi)}(1) = 0$

$\Rightarrow E(a(y)) = -\frac{c'(\theta)}{b'(\theta)}$ #

b) From ①, take the derivative with respect to $\theta$ of both sides

$\int_{-\infty}^{\infty} \frac{d^2}{d\theta^2} f(y;\theta)\, dy = 0$ — ③

From ②, we have

$$\frac{d^2}{d\theta^2} f(y;\theta) = \left(\frac{a(y)b'(\theta)+c'(\theta)}{\ell(\phi)}\right)^2 f(y;\theta) + \left(\frac{a(y)b''(\theta)+c''(\theta)}{\ell(\phi)}\right) f(y;\theta)$$

$\cancel{\int \frac{d^2}{d\theta^2} f(y;\theta)\, dy = \frac{b'(\theta)^2}{\ell(\phi)^2} \int_{-\infty}^{\infty} a(y)^2 f(y;\theta)\, dy + \frac{c(\theta)^2}{\ell(\phi)^2} \int_{-\infty}^{\infty} f(y;\theta)\, dy}$

$\cancel{+ \frac{b''(\theta)}{\ell(\phi)} \int^{\infty} a(y) f(y;\theta)\, dy + c''(\theta) \int^{\infty}}$

$$= f(y;\theta)\left(\frac{a(y)^2 b'(\theta)^2}{\ell(\phi)^2} + 2\left(\frac{a(y)b'(\theta)c'(\theta)}{\ell(\phi)^2}\right) + \frac{c'(\theta)^2}{\ell(\phi)^2} + \frac{a(y)b''(\theta) + c''(\theta)}{\ell(\phi)}\right)$$

$$\int_{-\infty}^{\infty}\frac{d^2}{d\theta^2} f(y;\theta)\,dy = \frac{b'(\theta)^2}{\ell(\phi)^2}\int_{-\infty}^{\infty} a(y)^2 f(y;\theta)\,dy + 2\left(\frac{b'(\theta)c'(\theta)}{\ell(\phi)^2}\right)\int_{-\infty}^{\infty} a(y) f(y;\theta)\,dy$$

$$+ \frac{c'(\theta)^2}{\ell(\phi)^2}\int_{-\infty}^{\infty} f(y;\theta)\,dy + \frac{b''(\theta)}{\ell(\phi)}\int_{-\infty}^{\infty} a(y) f(y;\theta)\,dy$$

$$+ \frac{c''(\theta)}{\ell(\phi)}\int_{-\infty}^{\infty} f(y;\theta)\,dy$$

$$= \frac{b'(\theta)^2}{\ell(\phi)^2} E(a(y)^2) + 2\left(\frac{b'(\theta)c'(\theta)}{\ell(\phi)^2}\right)\left(-\frac{c'(\theta)}{b'(\theta)}\right)$$

$$+ \frac{c'(\theta)^2}{\ell(\phi)^2}(1) + \frac{b''(\theta)}{\ell(\phi)}\left(-\frac{c'(\theta)}{b'(\theta)}\right) + \frac{c''(\theta)}{\ell(\phi)}(1)$$

$$= \frac{b'(\theta)^2}{\ell(\phi)^2} E(a(y)^2) - \frac{c'(\theta)^2}{\ell(\phi)^2} - \frac{b''(\theta)}{\ell(\phi)}\frac{c'(\theta)}{b'(\theta)} + \frac{c''(\theta)}{\ell(\phi)} \quad \text{— (4)}$$

From (3) and (4),

$$b'(\theta)^2 E(a(y)^2) - c'(\theta)^2 - \frac{b''(\theta)c'(\theta)\ell(\phi)}{b'(\theta)} + c''(\theta)\ell(\phi) = 0$$

$$\Rightarrow E(a(y)^2) - \frac{c'(\theta)^2}{b'(\theta)^2} - \frac{b''(\theta)c'(\theta)\ell(\phi)}{b'(\theta)^3} + \frac{c''(\theta)\ell(\phi)}{b'(\theta)^2} = 0$$

$$\Rightarrow E(a(y)^2) = \frac{c'(\theta)^2}{b'(\theta)^2} + \frac{b''(\theta)c'(\theta)\ell(\phi)}{b'(\theta)^3} - \frac{c''(\theta)\ell(\phi)}{b'(\theta)^2}$$

$$Var(a(y)) = E(a(y)^2) - E(a(y))^2$$

$$= \frac{c'(\theta)^2 b'(\theta) + b''(\theta)c'(\theta)\ell(\phi) - c''(\theta)\ell(\phi)b'(\theta) + c'(\theta)^2 b'(\theta)}{b'(\theta)^3}$$

$$= \frac{2c'(\theta)^2 b'(\theta) + b''(\theta)c'(\theta)\ell(\phi) - c''(\theta)\ell(\phi)b'(\theta)}{b'(\theta)^3}$$

3. a)

$$f_Y(y) = \theta y^{-\theta-1}$$

$$= \exp\left(\log\left(\theta y^{-\theta-1}\right)\right)$$

$$= \exp\left(\log\theta + \log y^{-\theta-1}\right)$$

$$= \exp\left(\log\theta - (\theta+1)\log y\right)$$

$$= \exp\left(\log\theta - \theta\log y - \log y\right)$$

Using the exp family form from Q.2,

$a(y) = -\log y$
$b(\theta) = \theta$
$c(\theta) = \log\theta$
$e(\phi) = \phi = 1$
$d(y;\phi) = -\log y$

$\therefore$ The pareto distribution is a member of the exponential family with natural parameter $\theta$

b)

$$E(\log y) = E(-a(y))$$

$$= -E(a(y))$$

$$= \frac{c'(\theta)}{b'(\theta)}$$

$$= \frac{(1/\theta)}{(1)}$$

$$= 1/\theta$$

$$\mathrm{Var}(\log y) = \mathrm{var}(-a(y))$$

$$= \mathrm{var}(a(y))$$

From part 2

$c'(\theta) = \frac{1}{\theta}$
$c''(\theta) = -\frac{1}{\theta^2}$
$b'(\theta) = 1$
$b''(\theta) = 0$

$$= \frac{2\left(\frac{1}{\theta}\right)^2(1) + 0 - \left(-\frac{1}{\theta^2}\right)(1)(1)}{(1)^3}$$

$$= \frac{3}{\theta^2}$$

3. c) $E(Y) = \int_1^\infty y\, f(y)\, dy$

$= \int_1^\infty y\, \theta y^{-\theta-1}$

$= \theta \int_1^\infty y^{-\theta}\, dy$

$= \theta \left[ \frac{y^{-\theta+1}}{-\theta+1} \right]_1^\infty$

$= \frac{\theta}{-\theta+1} \left[ y^{-\theta+1} \right]_1^\infty$

$= \frac{\theta}{1-\theta} (0 - 1)$

$= \underline{\frac{\theta}{\theta-1}}$

d) $E(Y^2) = \int_1^\infty y^2 f(y)\, dy$

$= \theta \int_1^\infty y^2 y^{-\theta-1}\, dy$

$= \theta \int_1^\infty y^{1-\theta}\, dy$

$= \theta \left[ \frac{y^{2-\theta}}{2-\theta} \right]_1^\infty$

$= \frac{\theta}{2-\theta} (0 - 1)$

$= \frac{\theta}{\theta-2}$

$Var(Y) = E(Y^2) - (E(Y))^2$

$= \frac{\theta}{\theta-2} - \left(\frac{\theta}{\theta-1}\right)^2$

$= \underline{\frac{\theta}{\theta-2} - \frac{\theta^2}{(\theta-1)^2}}$